Network formation among rivals*

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November 19, 2015

Abstract

We study the formation of bilateral agreements among farsighted agents in which their payoffs increase in their own number of partners and decrease in their rivals' number of partners. When more cooperation among equals is profitable, and when the payoff of agents in a small clique increases in the size of the clique, we show existence of and characterize a singleton von-Neumann-Morgenstern farsighted stable set. The set contains either two-clique networks, or dominant group networks in which only connected agents are active competitors. Network formation may thus endogenously create a barrier to entry. If the sum of payoffs increases when the connections are more unequally distributed among rivals, the efficient networks are either nested split graphs when there are no barriers to entry, or may also have a core-periphery structure when barriers to entry exist. The farsighted networks between rivals we characterize are not efficient. We show that standard economic models of network formation among competitors belong to our generalized framework.

JEL classification: C70, D20, D40.

Keywords: Network formation, Competition, Rivalry, Farsightedness, Efficiency.

*Gilles Grandjean and Wouter Vergote acknowledge financial support from the FNRS. We are indebted to Francis Bloch, Pascal Billand, Fréderic Deroïan, Françoise Forges and Timo Hiller for valuable comments. This paper benefitted from comments of participants to seminars and conferences in Antwerp (UA, 2014), Liège (ULG, 2014), Brussels (USL-B, 2014), Stony Brook (2014), Aix-en-Provence (ASSET, 2014), Munich (EARIE, 2015), Venice (CTN, 2015), Paris (Paris 1, 2014), Paris (ADRES, 2015), Lisbon (UECE, 2014), Louvain-la-Neuve (CORE, 2014). All remaining errors are ours.

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1 Introduction

Examples of cooperation between rivals are abundant. Firms which are competitors on a final market jointly invest in R&D to share its costs and benefits, they also share customer databases or engage in cross-licensing agreements. Countries sign bilateral trade agreements, colleagues competing for a promotion work in teams, etc. In this paper, we propose a general class of games to analyze these environments. In a *game of network formation among rivals*, ex ante symmetric agents first engage in bilateral cooperation and then compete. Agents' payoffs increase in their own number of partners (degree monotonicity) and decrease in their rivals' number of partners (negative externalities).[1] In this setup, we analyze networks formed by farsighted agents and contrast these to efficient networks, i.e. those leading to the highest sum of payoffs.

Farsighted agents forecast how other agents may react to their choice of partners, and make a decision by comparing the current network to the end network which is formed when other agents have further deviated. Farsightedness in network formation has received increasing attention over the past few years.[2] In his survey on network formation, Jackson (2005) has stated that:

"...in large networks it might be that players have very little ability to forecast how the network might change in reaction to the addition or deletion of a link. In such situations the myopic solutions are quite reasonable. However, if players have very good information about how others might react to changes in the network, then these are things that one wants to allow for either in the specification of the game or in the definition of the stability concept".

We believe that farsightedness is an appropriate assumption when studying cooperation between competitors, as the number of competitors is usually rather small and the stakes are high. Rivals then have the opportunity and the incentives to foresee how others might react to changes in the network. We capture this through the notion of indirect dominance (Harsanyi 1974). A final network indirectly dominates an initial network if there exists a sequence of networks that implements the final network from the initial network such that at any step of the sequence all agents who deviate have a higher payoff in the final network than in the current network. We use the stable set (von Neumann Morgenstern, 1944), based on indirect dominance as a solution concept. The farsighted stable set is both internally stable - no network in the set indirectly dominates another network in the set - and externally stable - every network outside the set is indirectly dominated by a network belonging to the set. The farsighted stable set can then be interpreted as a standard of behavior when agents are farsighted.

We show that there always exists a farsighted stable set in a game of network formation

[1] See for instance Goyal and Moraga (2001), Goyal and Joshi (2003), Goyal and Joshi (2006a), Goyal and Joshi (2006b), Marinucci and Vergote (2011), Grandjean et al. (2013) for models of competition in networks competition lying in this class of games.

[2] Approaches to farsightedness in network formation are suggested by the work of Chwe (1994), Xue (1998), Herings, Mauleon, and Vannetelbosch (2004), Mauleon and Vannetelbosch (2004), Page, Wooders and Kamat (2005), Dutta, Ghosal, and Ray (2005), Herings, Mauleon, and Vannetelbosch (2004), Page and Wooders (2009), Herings, Mauleon, and Vannetelbosch (2014), and Ray and Vohra (2015).

among rivals satisfying *strong degree monotonicity* and *minority economies to scale.* It is either composed of dominant group networks, or of 2-clique networks. In a dominant group network, each member of the group is connected to the other group members while the remaining agents are not connected and do not take part in the competition. Networking then endogenously creates a barrier to entry. In a network composed of two asymmetric cliques, each agent belongs either to a large or to a small group of connected agents, and each agent is an active competitor. The first property needed to establish this result, strong degree monotonicity, implies that agents who have the same degree find increasing their degree worthwhile. The second one, minority economies to scale, imposes that the payoff of agents in a small clique increases in the size of the clique when they are facing another clique with the majority of agents.

We then analyze the efficient networks in a game of network formation among rivals when two properties hold. The first property, *welfare improving switch*es, imposes that the sum of payoffs increases after a switch - by which one agent's degree increases while that of a less connected agent decreases - when the agents whose degree decreases remains active in the competition. The second one, *switch externalities*, imposes that agents who are not involved in the switch are not hurt by it. We then show that the networks that maximize the sum of payoffs are nested split graphs when agents are active in every network.[3] Otherwise, when poorly connected agents may decide not to participate in the competition, a switch may no longer be welfare improving if it leads to the exclusion of the agent hurt by the switch. We then find that the efficient network is either a core-periphery network or a (quasi-)nested split graph.[4]

The four properties we impose are satisfied in many models of network formation among rivals. We show that they are satisfied in a model of bilateral R&D agreements among differentiated firms and in Grandjean et al. (2013)'s model of cooperation among rivals in a Tullock contest. The patent races' model of Goyal and Joshi (2006) satisfies strong degree monotonicity and minority economies to scale but violates welfare improving switch.

The structure of stable and efficient networks is in general different. There is a tension between the networks that are formed by agents and those that would produce the highest sum of payoffs. In a stable network, competitors cooperate with equally connected agents while the sum of payoffs would be higher if the links were more unequally distributed. In Goyal and Joshi (2003)'s model of R&D network formation for example, firms with more partners produce more since they have a smaller marginal cost. The benefit of a new partnership is thus increasing in a firm's degree since it affects a larger volume of production. Firms in the large clique do not cooperate with those in the small one, and as such do not completely exploit R&D network benefits, leading to the aforementioned inefficiencies.

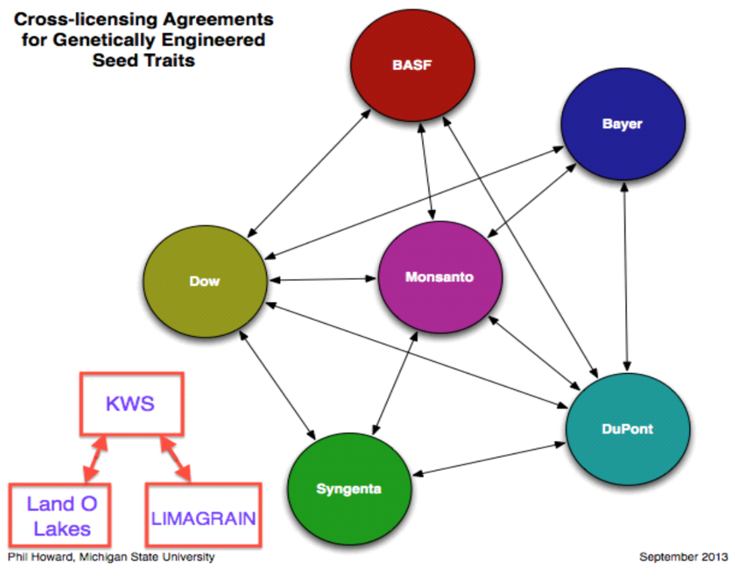
Our theoretical predictions mirror empirical findings on and policy concerns about cooper-

[3] Nested split graphs are networks such that each agent is connected to other agents with fewer links. The network structures are presented in Figure 2.

[4] Core-periphery networks are networks in which some agents in the core are connected to every other agent, while agents in the periphery are only connected to agents in the core.

ation among rivals. Hochberg et al. (2010) show that networking may create barriers to entry for the supply of venture capital. Regibeau and Rockett (2011) indicate that cross-licensing agreements may warrant antitrust scrutiny. Bekkers et al. (2002) have documented Motorola's successful attempt in the eighties at creating a group of 5 dominant firms in the GSM industry by forming cross-licensing agreements with these firms and refusing agreements with outsiders. Motorola and its competitors influenced the market structure and ended up dominating the GSM industry. Howard (2009 and 2013) provides another striking example in the seed industry, where six of the nine largest firms have closely cooperated through cross-licensing agreements while the other three have formed joint ventures to share research output and expertise. Figure 1 (Howard, 2013) illustrates[5] that, by September 2013, two collaborating "cliques" had formed.

Figure 1. R&D collaboration in the seed industry (Howard, 2013)



Cooperation among rivals has been studied in a coalition formation setting.[6] Bloch (1995) shows that firms form two asymmetric coalitions in the cost reducing R&D Cournot model, in which the largest group comprises 3/4 of the firms. Yi (1997) identifies conditions leading to the formation of two asymmetric coalitions in the coalitional unanimity game of Bloch (1996).

Our properties, restricted to networks composed of strongly connected components imply that the conditions in Yi (1997) are satisfied. Thus, forward looking agents forming coalitions according to the rules of Bloch (1996)'s coalitional unanimity game would form two coalitions. We find that the farsighted stable set is composed of networks featuring two groups, a strongly connected component among a majority of the agents, and another group of agents that are either strongly connected or not connected. Furthermore, the size of the groups is equivalent in the two approaches. We have identified sufficient conditions for establishing an equivalence

[5] Black arrows represent cross-licensing agreements while red arrows represent joint ventures.
[6] See Bloch (2005) for a survey of this literature.

between the networks formed by farsighted agents and the coalitions formed among forward looking agents. These conditions are also necessary. By means of example, we show that the equivalence no longer holds when minority economies to scale is violated.

In a network formation setting, farsightedness has been shown to lead to an asymmetric partition of agents in the work of Roketskiy (2012) and of Mauleon et al. (2014). In Roketskiy 's (2012) model, the agents' payoffs are the sum of two terms. The first one is their production, which is increasing in degree, and the second one is a bonus shared among the agents with the highest degree. Mauleon et al. (2014) study cost reducing R&D agreements, assuming that R&D externalities perfectly spread across the network so that each member of a component has the same marginal cost, like in a coalition.

Westbrock (2010) studies efficient networks by extending the R&D collaboration model in Goyal and Joshi (2003) to a network game of differentiated oligopoly and finds that when the participation constraints are not binding, efficient and profit maximizing networks are interlinked stars.[7] Our focus is on a class of games that includes the model in Goyal and Joshi (2003). Our predictions are more precise than those of Westbrock (2010), and we also analyze the case where participation constraints are binding. König et al. (2012) study R&D collaborations with network dependent indirect spillovers and show that the efficient network structure is a nested split graph. In a standard linear quadratic utility function with local synergies (Ballester et al., 2006), Belhadj et al. (2015) show that an efficient network must be a nested split graph in network games with strategic local complementarity.

The paper is organized as follows. In Section 2 we present our framework and introduce the notation. In Section 3, we provide three motivating examples. In Sections 4 and 5, we study respectively pairwise and farsighted stability. Section 6 characterizes the efficient network. Section 7 concludes.

2 Notation and framework

2.1 Networks

Let $N = \{1, 2, ..., n\}$ be a finite set of agents. We write $g_{i,j} = 1$ when a link between i and j exists and $g_{i,j} = 0$ otherwise. A network $g = \{(g_{i,j})_{i,j \in N}\}$ is the list of pairs of individuals who are linked to each other. Let g^N be the collection of all subsets of N with cardinality 2, so g^N is the complete network. The set of all possible networks on N is denoted by $\mathbb{G}$ and consists of all subsets of g^N. The network obtained by adding the link ij to an existing network g is denoted $g + ij$ and the network that results from deleting the link ij from an existing network g is denoted $g - ij$. For any network g, let $N(g) = \{i \in N \mid \exists j \text{ such that } ij \in g\}$ be the

[7] Interlinked star networks are such that each agent with the maximal number of links is connected to each connected agent.

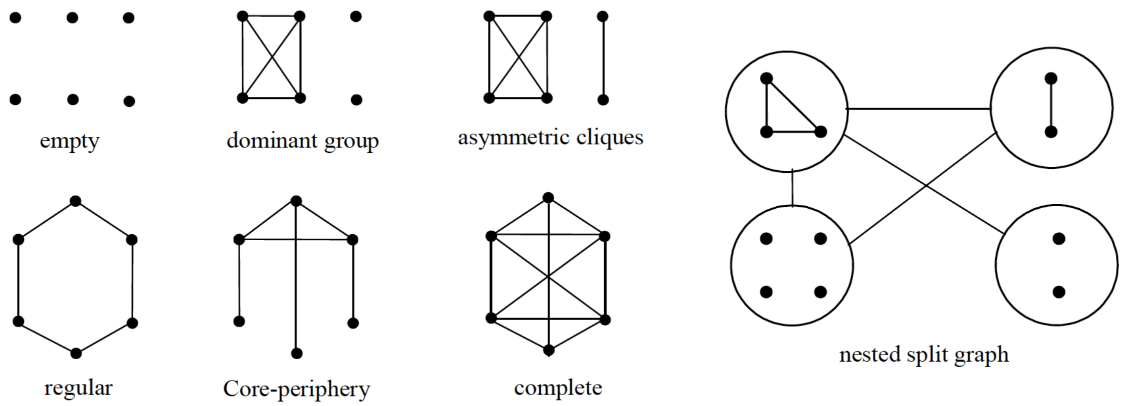
set of agents who have at least one link in the network g. Let $N_i(g)$ be the set of agents that are linked to $i : N_i(g) = \{j \in N \mid ij \in g\}$. Agent $i's$ degree in a network g is the number of links which involve this agent: $n_i(g) = \#N_i(g)$.[8] A path in a network $g \in \mathbb{G}$ between i and j is a sequence of agents $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = i$ and $i_K = j$. A network g is connected if for each pair of agents i and j such that $i \neq j$ there exists a path in g between i and j. A component h of a network g is a nonempty subnetwork $h \subseteq g$ satisfying (i) for all $i \in N(h)$ and $j \in N(h) \backslash \{i\}$, there exists a path in h connecting i and j, and (ii) for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$. Given a network g, let $K_1(g) = \{i \in N \mid n_i(g) \geq n_j(g) \text{ for all } j \in N\}$ be the set of agents with the highest degree. For all $t \geq 2$, let $K_t(g) = \{i \in N \mid n_i(g) \geq n_j(g) \text{ for all } j \in N \backslash (K_s(g))_{s<t}\}$ be the set of agents with the highest degree among the agents that are not in $K_1(g), ..., K_{t-1}(g)$. We write $X \leftrightarrow_g Y$ if there is at least a link between one agent from the agent set X and one agent from the agent set Y in the network g. Similarly, we write $X \top_g Y$ if each agent in X is connected to each agent in Y in g, and $X \perp_g Y$ if there are no links among agents from those sets in g. Let $N^-(g) = \{i \in N(g) \mid n_i(g) \leq n_j(g) \text{ for all } j \in N(g)\}$ be the set of agents in g with the smallest degree among those that are connected, and let $N^0(g)$ be the set of agents that are not connected in g. For $S \subseteq N$, let $g_{-S} = \{jk \in g \mid j \notin S \text{ and } k \notin S\}$ be the set of links among the agents outside S in the network g.

We now define some networks that play an important role in our analysis (see Figure 2). Given a set of agents $S \subsetneq N$, a *dominant group network* g^S is such that the agents in S are connected to each other in S while the agents in $N \backslash S$ have no links. In a *k-clique network* $g = g^{S_1} \cup ... \cup g^{S_k}$, the agents are partitioned into k groups such that there is a link between every pair of agents in the same group and no link between any two agents in different groups.[9] We write a *2-clique network* with a clique S involving the majority of agentq by $\widetilde{g}^S = g^S \cup g^{N \backslash S}$. A network g is a *nested split graph* with t classes if $K_s(g) \top_g K_r(g)$ for all $r \leq t-s+1$. The agents in class 1 are connected to every connected agent, while the agents in class t are only connected to the agents in class 1. Similarly, the agents in class 2 are connected to every connected agent other than those in class t while the agents in class $t-1$ are only connected to the agents in classes 1 and 2, etc. In the nested split graph depicted in Figure 2, there are four classes of agents. A line between two groups indicates that each agent from one group is connected to every agent from the other group. In a *core-periphery network*, each agent in the core is connected to every other agent while agents in the periphery are only connected to agents in the core. Finally, each agent has the same degree in a *regular network*.

[8]Throughout the paper we note the cardinality of a set X by the lower case letter x.

[9]A clique is a set of agents $S \subseteq N$ such that there is a link between each pair of agents in S.

Figure 2. Some network structures.



2.2 Framework

The games of network formation among rivals $\mathcal{G}$ which we consider involve n ex-ante identical agents playing a two-stage game. Agents first form bilateral agreements in the network formation stage. A network g induces a degree distribution of the agents $(n_1(g), n_2(g), ..., n_n(g))$, where the degree represents an agent's strength, and is the only payoff relevant network statistic. Each agent i then chooses a strategy $\sigma_i \in \sum$ in the competition stage. The strategy set $\sum$ is identical for each agent in every network, it does not depend on the network and it contains the strategy "out". An agent choosing the strategy "out" is guaranteed to get a payoff of 0. We assume that for each network g, there is a unique Nash equilibrium of the game in the second stage $\sigma^*(g) = (\sigma_1^*(g), \sigma_2^*(g), ..., \sigma_n^*(g))$, and we denote agent's i Nash equilibrium payoff, gross of linking costs, by$\pi_i(g)$. In a network g, the agents playing a strategy other than "out" are the active (or participating) agents $K(g)$. Participating agents obtain a non-negative payoff at the Nash equilibrium since they would otherwise have a profitable deviation through option out: $\pi_i(g) \geq 0$ iff $i \in K(g)$. Uniqueness of the Nash equilibrium then implies that $\pi_i(g) > 0$ iff $i \in K(g)$.

Two properties relate an agent's payoff to specific network configurations. The first one imposes that the stronger an active agent, the higher her payoff, a property which is called degree monotonicity.

Property 1. Degree monotonicity (P1): $\pi_i(g') > \pi_i(g)$ if $n_i(g') > n_i(g)$, $n_k(g') = n_k(g)$ for all $k \in N \backslash \{i\}$ and $\pi_i(g') > 0$.

The second one imposes that the stronger an active agent, the smaller her rivals' payoffs, a property which is called negative externalities.

Property 2. Negative externalities (P2): $\pi_j(g') < \pi_j(g)$ if $n_i(g') > n_i(g)$ for some $i \neq j$, $n_k(g') = n_k(g)$ for all $k \in N \backslash \{i\}$, $\pi_j(g) > 0$ and $\pi_i(g') > 0$.

These two properties imply that agent i's payoff is higher than agent j's payoff in network g if $i's$ degree is higher than $j's$ degree, $n_i(g) > n_j(g) \implies \pi_i(g) \geq \pi_j(g)$. Indeed let g' be such that $n_i(g') = n_j(g)$ and $n_k(g') = n_k(g)$ for $k \in N \backslash \{i\}$. Then, we have $\pi_i(g) > \pi_i(g')$ by degree monotonicity, $\pi_i(g') = \pi_j(g')$ by symmetry and uniqueness of the Nash equilibrium, and $\pi_j(g') > \pi_j(g)$ by negative externalities. Degree monotonicity and negative externalities imply that the non-participating agents' degree is smaller than participating agents' degree. We let $K_m(g)$ be the set of weak agents, i.e. the set of participating agents with the smallest degree in the network g, $K_m(g) = \{i \in K(g) \mid n_i(g) \leq n_j(g) \text{ for all } j \in K(g)\}$. The set of non-participating agents in g is $E(g) = N \backslash K(g)$ and the set of strong agents, i.e. agents who have strictly more connections than the weak agents is $K^+(g) = K(g) \backslash K_m(g)$.

We assume that each link costs $c \geq 0$ to the two agents involved in the link. Agent $i's$ net payoff is then given by $\Pi_i(g) = \pi_i(g) - cn_i(g)$. We analyze the structure of stable and efficient networks in the class of games $\mathcal{G}$ when some additional properties are satisfied.[10] The first two properties determine the effect of having more collaborations on payoff.

Property 3, *Strong degree monotonicity,* imposes that participating agents who have the same degree find it worthwhile to see their degree increase.

Property 3. Strong degree monotonicity (P3): $\Pi_i(g) < \Pi_i(g')$ for $i \in S \subseteq K_l(g) \subseteq K(g)$ where g' is such that $n_j(g') = n_j(g) + 1$ for all $j \in S$ and $n_j(g') = n_j(g)$ for all $j \in N \setminus S$.

Strong degree monotonicity is stronger than degree monotonicity since it requires that the benefit of increasing one's degree outweigh the additional linking cost and the cost of facing stronger competitors. This property implies that, all else being equal, more cooperation among equals is better for them. Property 4, *Minority economies to scale,* imposes that the payoff of agents in a small clique must increase in the size of the clique when they are facing another clique with the majority of agents.

Property 4. Minority economies to scale (P4): $\Pi_i(g^S \cup g^T) < \Pi_i(g^S \cup g^{T \cup \{j\}})$ for $i \in T$ if $T \subseteq K(g^S \cup g^T)$, where $j \notin \{S \cup T\}$, $T \cap S = \emptyset$ and $s \geq n/2$.

Properties 5 and 6 determine the effect of a reallocation of links leading to an increase in one agent's degree at the expense of a less connected agent. When a network g' can be obtained from a network g by a mean preserving spread in the distribution of links favoring i at the expense of j, we say that g' is obtained from g by a switch in favor of i relative to j, and write it $g' \in S(g, i, j)$.

Definition 1. A network g' is obtained from g by a **switch** in favor of i relative to j -$g' \in S(g, i, j)$- if $n_i(g') = n_i(g) + 1$, $n_j(g') = n_j(g) - 1$ where $n_i(g) \geq n_j(g)$ while $n_k(g) = n_k(g')$ for all $k \in N \backslash \{i, j\}$.

[10] Goyal and Joshi (2006a) and Hellman and Landwher (2014) also propose properties on the payoff function in adjacent networks, and relate these to the structure of pairwise stable networks.

A switch leads to a new pattern of collaboration in which the number of partners of agents are less equally distributed. Property 5, *welfare improving switch,* imposes that the sum of payoffs in a network must increase after a switch if the agent whose degree decreases remains active.

Property 5. Welfare improving switches (P5): $\sum_{i\in N}\Pi_i(g') > \sum_{i\in N}\Pi_i(g)$ if $g' \in S(g,i,j)$ and $j \in K(g)\cap K(g')$.

Finally Property 6, *switch externality*, imposes that the payoff of an agent not involved in a switch among strong agents must not decrease.

Property 6. Switch externalities (P6): $\pi_l(g') \geq \pi_l(g)$ for $g' \in S(g,i,j)$ if $j \in K^+(g)$ and $l \neq j$.

In the rest of the paper, we show that Properties 1-6 are satisfied in standard models of bilateral cooperation among rivals, and we analyze how they shape the farsighted stable set of networks, and the set of efficient networks.

3 Motivating examples

We show in this section that Properties 1-6 are satisfied in a class of models of bilateral R&D agreements among differentiated firms encompassing the standard model of Goyal and Joshi (2003) and in Grandjean et al. (2013)'s model of cooperation among rivals in a Tullock contest. The patent races' model of Goyal and Joshi (2006) satisfies Properties 1-4 but does not satisfy Property 5.

3.1 R&D cooperation in differentiated oligopoly

Consider the following two-stage game in which n firms first form bilateral R&D agreements to reduce their marginal cost, and then compete either in quantity or in price. Following Bloch (1995), Goyal and Joshi (2003) and Westbrock (2010), assume that the marginal cost of a firm i depends linearly on its degree $c_i(g) = \lambda - \mu n_i(g)$, where λ is the marginal cost of an isolated firm, and μ measures the effect of links on marginal cost. Competition follows the paper by Singh and Vives (1984). Each firm may sell a single product to a continuum of consumers, who optimally choose their consumption levels of each good including a numeraire good m. The utility function of the representative consumer is given by:

$$U(q_1,...,q_n,m) = m + \alpha\sum_{i\in N} q_i - \frac{1}{2}\sum_{i\in N} q_i^2 - \frac{\beta}{2}\sum_{i\in N}\sum_{j\neq i} q_i q_j$$

The parameter $\beta \in (0,1)$ represents the substitutability between products. Products are perfect substitutes if $\beta = 1$ and are independent if $\beta = 0$. Consumer utility maximization yields a system of linear inverse demand functions which enter the firm profit maximization problem:

$$p_i(q = \sum_{i \in N} q_i) = \alpha - q_i - \beta \sum_{j \neq i} q_j$$

Firms either compete in price or quantity. Ledvina and Sircar (2012) provide a full characterization of the Nash equilibria of quantity and price competition in the face of participation constraints. The Nash equilibrium quantity in the second stage *quantity competition,* $q_i^Q(g)$, is uniquely given by

$$q_i^Q(g) = \frac{1}{2 + (k(g) - 1)\beta} \max\{0, \alpha - \lambda + \frac{\mu}{2 - \beta} \left[(2 + (k(g) - 2)\beta) n_i(g) - \beta \sum_{j \in K(g) \setminus \{i\}} n_j(g) \right] \}$$

and the Nash equilibrium quantity in the second stage *price competition,* $q_i^P(g)$, is uniquely given by[11]

$$q_i^P(g) = \Psi(k(g), \beta) \max\{0, (1 - \beta)(\alpha - \lambda) + \mu \left(\zeta(k(g), \beta) n_i(g) - \delta(k(g), \beta) \sum_{j \in K(g) \setminus \{i\}} n_j(g) \right) \}$$

In both cases, the gross payoffs are $\pi_i(g) = q_i(g)^2$.[12]

For sufficiently small linking costs, we show that Properties 1-6 are satisfied.[13]

Lemma 1. *The differentiated oligopoly model with linear cost reducing R&D* $c_i(g) = \lambda - \mu n_i(g)$ *satisfies* $P1 - P6$ *when linking costs are small.*

All proofs are in the appendix. Only own degree and the sum of firms' degrees are payoff relevant in this game. When participation constraints are not binding, the distribution of firms' degrees affects the allocation of production among the competitors but does not affect the total output. After a switch, the output of the firms whose degree remains constant are thus unchanged (P6). It follows that a switch increases the industry profits since the production does not change but the total costs of production have decreased as some units are transferred from the firm whose degree decreases to the firm whose degree increases, and the latter produces the good at smaller marginal costs (P5). When more collaborations are formed, we show that a firm's profit increases when its degree increases in the same proportion as the degree of others. Thus, when the number of links of each member of a group increases by the same amount, the payoff of each agent of the group increases (P3). Similarly, an isolated agent and each member of a minority group get a higher payoff when the former creates a link to each agent in the

[11] where $\Psi(k(g), \beta) = \frac{1 + (k(g) - 2)\beta}{(1 - \beta)(1 + (k(g) - 1)\beta)(2 + (k(g) - 3)\beta)}$, $\zeta(k(g), \beta) = 2 + (3k(g) - 6)\beta + (k(g)^2 - 5k(g) + 5)\beta^2$ and $\delta(k(g), \beta) = (1 + (k(g) - 2)\beta)\,\beta$.

[12] Goyal and Joshi (2003) assume that firms produce homogeneous goods and always produce positive quantities, that is they assume that $\beta = 1$ and $(\alpha - \lambda) > (n - 1)(n - 2)\mu$. Goyal and Moraga (2001), Deroian and Gannon (2008), Westbrock (2010), Mauleon et al. (2014) also analyze this model, and rule out the issue of participation.

[13] We show that the claim holds when there are no linking costs. If Properties 1, 3 and 4 are satisfied for some linking costs c, they are satisfied for every linking cost c' smaller than c.

minority group since the total number of new links is then $2s - 1$, and $s < n/2$ is the size of the minority clique (P4). We omit here the intuition of the proof for the case where participation constraints matters. It follows the same logic.

3.2 Bilateral agreements in the Tullock contest

Grandjean et al. (2015) consider n agents involved in a contest to win a prize. Each agent i chooses a level of effort e_i. The cost of effort is the effort itself $C(e_i) = e_i$. The profile of efforts determines the probability that each agent gets the prize, according to the Tullock contest success function $p_i(e_i, e_{-i}) = e_i / \sum_{j \in N} e_j$. The valuation of an agent for the prize is decomposed into a fixed component, and a variable component that depends on the degree of the agent : $v_i(g) = v + n_i(g)\beta$. The Nash equilibrium choice of effort in a given network g is given by $e_i^*(g) = \max\{0, \frac{k(g)-1}{k(g)} h_{k(g)}(g)(1 - \frac{k(g)-1}{v_i(g)} \frac{h_{k(g)}(g)}{k(g)})\}$, where $h_{k(g)}(g) = k(g)/(\sum_{j \in K(g)} 1/v_j(g))$ is the harmonic mean of the valuations of the participating agents.[14] It follows that gross payoffs are given by $v_i(g)(e_i^*(g)/\sum_{j \in N} e_j^*(g))^2$. From these expressions, one may show that effort and payoff are increasing in own degrees and decreasing in others' degrees (P1 and P2). For sufficiently small linking costs, we show that it also satisfies strong degree monotonicity, minority economies to scale, welfare improving switch and switch externality.

Lemma 2. *The Tullock contest with contest success function $p_i(e_i, e_{-i}) = e_i / \sum_{j \in N} e_j$, linear cost of effort $C(e_i) = e_i$, and valuation $v_i(g) = v + n_i(g)\beta$ satisfies Properties $P1 - P6$ when linking costs are small.*

When participation constraints are not binding, the intuition for this result is as follows. Given some number of links in a network, the aggregate wasted efforts are higher when the links are allocated more equally among agents. The payoff of an agent not involved in a switch thus increases after a switch since the same effort leads to higher chances of winning the prize (P6). The sum of efforts is lower after a switch, and the expected valuation of the agent getting the prize increases. These two effects lead to a higher sum of payoffs (P5). When equal contestants form new collaborations, their valuation and their chance of getting the prize increases, so that their payoff is unambiguously higher (P3). When the size of a minority clique increases, we show that each member of the clique gets the prize with higher probability even if the new clique members increase their effort more than others do. They get the prize with higher probability and have a greater valuation for it, so that their payoff is higher (P4).

3.3 Patent Races

The following example is taken from Goyal and Joshi (2006). Consider n innovators who want to be the first to invent a new product which has a patent value equal to 1. Time is continuous and all innovators discount the future at rate δ. Innovators are endowed with one unit of

[14] Hillman and Riley (1989) are the first to show that the participation of all agents is not guaranteed when the valuations of agents is too asymmetric. See also Stein (2002), Cornes and Hartley (2005), and Ryvkin (2013).

R&D capability and can share their capability with others to speed up their innovation process. Denote $\tau(n_i(g))$ the random time at which innovator i develops a new product in network g and assume that τ has an exponential distribution:

$$Pr\left\{\tau(n_i(g)) \leq t\right\} = 1 - e^{-n_i(g)t}$$

Having more links thus shortens the expected innovation time. However, other innovators may obtain the new product before innovator i does. Stochastic independence of the distributions of the time of innovation implies that the expected profit of innovator i in network g is equal to:

$$\pi_i(g) = \frac{n_i(g)}{\delta + \sum_{k \in N} n_k(g)}$$

Lemma 3. *The patent race game of Goyal and Joshi (2006) satisfies Properties P1-P4 when linking costs are small. It does not satisfy P5.*

The benefit a firm obtain by forming a new collaboration, as specified in P3 and P4, always outweighs the cost of facing other firms who also become faster innovators. This arises as the modification of the network is such that the degree of a firm who has an additional partner increases relatively more than the total number of collaborations. In contrast to the previous models, the firms always participate in this model and the allocation of links among firms has no impact on the sum of payoffs. A switch does not affect the payoff of the firms not involved in it, and the gain of the firm whose degree increases is exactly compensated by the loss of the firm whose degree decreases. P5 is not satisfied.

4 Pairwise stable networks

Jackson and Wolinsky (1996) have introduced the notion of pairwise stability to characterize the networks immune to a single link addition or deletion. A network is pairwise stable if no agent benefits from severing one of his links and no two agents benefit from adding a link between them, with one benefiting strictly and the other at least weakly.

Definition 2. A network g is pairwise stable if
(i) $\Pi_i(g) \geq \Pi_i(g - ij)$ and $\Pi_j(g) \geq \Pi_j(g - ij)$ for each $ij \in g$,
(ii) $\Pi_i(g + ij) > \Pi_i(g)$, then $\Pi_j(g + ij) < \Pi_j(g)$ for each $ij \notin g$.

The set of pairwise stable networks of a game $\Gamma \in \mathcal{G}$ is such that connected agents participate, since otherwise they would get a negative payoff and could profitably delete links. If the game also satisfies strong degree monotonicity (P3), every connected agent is linked to every other agent agent with the same number of links in a pairwise stable network, since otherwise their would exist a pair of agents that could profitable add a link. Agents without links may be unconnected in a pairwise stable network if they do not participate. Let G^{PS} be the set of pairwise stable networks. Let $G^* = \{g \in \mathbb{G} \,|ij \in g$ if $i,j \in K_t(g) \subseteq K(g)$, and $ij \notin g$ if $\{i,j\} \nsubseteq K(g)\}$ be the set of networks such that each participating agent with the same degree is connected, and where non-participating agents are not connected. A pairwise stable network belongs to this set.

Proposition 1. *Let* $\Gamma \in \mathcal{G}$*. We have* $G^{PS} \subseteq G^*$ *if Property* 3 *holds.*

A pairwise stable network is composed of cliques with agents having the same degree. Agents in different cliques do not have the same degree. Also, each agent in a clique has the same number of links towards agents in other cliques. The complete network is always pairwise stable. If agents deviate from the complete network by cutting a link, they either reach a network g' where he is not participating or where he is participating but not connected to agents with the same degree. In both cases, they are better off by maintaining their links. Dominant group networks fall into this class provided isolated agents do not participate. A dominant group network is pairwise stable if in addition two isolated agents do not participate by forming a link. Network formation can thus endogenously create a barrier to entry for ex ante symmetric agents. Dominant group networks are the only pairwise stable networks in the product differentiation or Tullock models since any two participating agents are better off by forming a link in these games, not only those with the same degree. A network composed of completely connected components of different sizes is also in G^*. Such a network is pairwise stable if agents in different components are not better off when they form a link.

5 von Neumann-Morgenstern farsighted stability

In this section, we analyze the formation of networks among rivals when agents are farsighted, i.e. when they anticipate how other agents would react to their choice of partners. We use the notion of indirect dominance in Harsanyi (1974) to account for farsighted behavior. A network g *indirectly dominates* a network g' if there exists a sequence of networks that implements g over g' such that in every network in the sequence g_k, all deviating agents have a higher payoff in the end network g than in the current network g_k. In a network g_k in the sequence from g' to g, any group of agents $S \subseteq N$ may *enforce* the network g_{k+1} over g_k if the links that are created involve two agents from S while those that are deleted involve at least an agent from S.

Formally, enforceability and indirect dominance are defined as follows.

Definition 3. Given a network g, a coalition $S \subseteq N$ is said to be able to **enforce** a network g' if

(i) $ij \in g$ but $ij \notin g' \Longrightarrow \{i,j\} \cap S \neq \emptyset$
(ii) $ij \notin g$ but $ij \in g' \Longrightarrow \{i,j\} \subseteq S$

We then have;

Definition 4. A network g is **indirectly dominated** by a network g', or $g \ll g'$, if there exists a sequence of networks $g_0, g_1, ..., g_T$ (where $g_0 = g$ and $g_T = g'$) and a sequence of coalitions $S_0, S_1, ..., S_{T-1}$ such that for any $t \in \{1, 2, ..., T\}$,
(i) $\pi_i(g_T) > \pi_i(g_{t-1})$ for all $i \in S_{t-1}$, and
(ii) coalition S_{t-1} can enforce the network g_t over g_{t-1}.

We use the notion of indirect dominance in the stable set of von Neumann and Morgenstern (1944). A farsighted stable set of networks is such that no network in the set indirectly dominates another network in the set (internal stability) and each network which is not in the set is indirectly dominated by a network in the set (external stability). A deviation from a stable network leading to a network outside the set is deterred since at least one deviator is worse off in a stable network that indirectly dominates this network.

Definition 5. A set of networks $G \subseteq \mathbb{G}$ is a **von Neumann-Morgenstern farsighted stable set** of a game $\Gamma \in \mathcal{G}$ if
(i) for all $g \in G$, there does not exist $g' \in G$ such that $g \ll g'$, and
(ii) for all $g' \notin G$, there exists $g \in G$ such that $g' \ll g$.

We show that a von Neumann-Morgenstern farsighted stable set always exists in a network formation game among rivals satisfying strong degree monotonicity (P3) and minority economies to scale (P4). We use three thresold values in the characterization of a farsighted stable set. Let $\widetilde{s} \in \{int(n+1)/2, ..., n\}$ be the smallest size of a clique S such that the remaining agents do not participate in the 2-clique network $\widetilde{g}^S$: $K(\widetilde{g}^S) = S$ and $K(\widetilde{g}^{S\setminus\{k\}}) = N$ for $k \in S$. We denote by $\widehat{s} \in \{\widetilde{s}, ..., n\}$ the size of the clique that maximizes the per capita value $\widehat{\pi}$ of its members, among the cliques which ensure that the remaining agents do not participate ($\widehat{s} \geq \widetilde{s}$). Finally, we let $\overline{s} \in \{int(n+1)/2, ..., \widetilde{s}-1\}$ be the size of the large clique that maximizes the per capita payoff of its members $\overline{\pi}$ when the remaining agents form another clique and participate. For $i \in S$, we have $\widehat{s} \in \arg\max_{s\in\{\widetilde{s},...,n\}} \Pi_i(g^S)$, $\widehat{\pi} = \max_{s\in\{\widetilde{s},...,n\}} \Pi_i(g^S)$, $\overline{s} \in \arg\max_{s\in\{1,...,\widetilde{s}-1\}} \Pi_i(\widetilde{g}^S)$ and $\overline{\pi} = \max_{s\in\{1,...,\widetilde{s}-1\}} \Pi_i(\widetilde{g}^S)$.[15] Farsighted agents either form one large clique to drive the remaining agents out of the market, or they form a smaller clique to reduce the number of strong competitors and accommodate participation from the entire set of agents.

In Proposition 2, we show that the set of dominant group networks of size $\widehat{s}$ is a von Neumann-Morgenstern farsighted stable set if and only if $\widehat{\pi} > \overline{\pi}$ when the game satisfies strong degree monotonicity.

Proposition 2. *Let* $\Gamma \in \mathcal{G}$ *satisfy Property* 3. *Then,* $G^{FS1} = \{g^S \mid s = \widehat{s}\}$ *is a vNM farsighted stable set iff* $\widehat{\pi} > \overline{\pi}$.

The intuition of the proof of Proposition 2 is as follows. A network g^T in the set G^{FS1} is not indirectly dominated by another $g^{T'}$ since on every path from g^T to $g^{T'}$, each member of T has a payoff of $\widehat{\pi}$ in the first network g' where some members of T modify the network, and thus do not

[15]For simplicity, we assume that $\arg\max_{s\geq\widetilde{s}} \Pi_i(g^S)$ and $\arg\max_{s\leq\widetilde{s}-1} \Pi_i(\widetilde{g}^S)$ are singleton sets.

benefit from the deviation. The set G^{FS1} thus satisfies internal stability. We then propose an algorithm to generate a path from a network g' outside the set leading to some network g in the set such that g' is indirectly dominated by g. We briefly describe the steps of the algorithm. In the first step, agents delete links in order to reach a network where at least $\widehat{s}$ agents are isolated. In the second step, the isolated agents form a clique, leading to the exclusion of the remaining agents. The first step is trickier and we explain it in detail here. From the initial network, we let the agents with a payoff smaller than $\widehat{\pi}$ successively delete their links until a network g'' is reached where either $\widehat{s}$ agents are isolated and then jump directly to the second step, or where each connected agent has a payoff greater than $\widehat{\pi}$. In this last case, there should be less than $\widetilde{s}$ connected agents in the network g''.[16] We then let the unconnected agents create links until they all have the same degree as an agent i with the lowest degree among those that are connected in g'', or until they are completely connected. The network reached this way is g''' and the payoff of agent i in g''' is smaller than $\widehat{\pi}$.[17] The agents with no links in g'' and agent i then delete their links. The path proposed is such that at each step, at least one additional agent cuts her links. After a finite number of iterations, $\widehat{s}$ agents are isolated so that the second step is reached with probability 1. In the second step, isolated agents form a complete component. The remaining agents do not participate and delete their useless links. By construction, the path does not reach a dominant group network of size $\widehat{s}$ in an intermediate step, as there would be agents who delete their links in a network where their payoff is positive and who end up not participating in the final network. It follows that the condition $\widehat{\pi} > \overline{\pi}$ is sufficient for the set G^{FS1} to be a *vNM* farsighted stable set. It is also necessary since a network composed of two cliques of size $\overline{s}$ and $n - \overline{s}$ is not indirectly dominated by a network in the set when $\widehat{\pi} \leq \overline{\pi}$.

In Proposition 3, we show that the set of networks composed of two cliques of size $\overline{s}$ and $n - \overline{s}$ is a von Neumann-Morgenstern farsighted stable set if and only if $\widehat{\pi} > \overline{\pi}$ when the game satisfies strong link monotonicity (P3) and minority economies to scale (P4).

Proposition 3. *Let* $\Gamma \in \mathcal{G}$ *satisfy Property* 3 *and Property* 4. *The set* $G^{FS2} = \{g \subseteq g^N \mid g = \widetilde{g}^S$ *such that* $\#S = \overline{s}\}$ *is a vNM farsighted stable set if and only if* $\overline{\pi} > \widehat{\pi}$.

The intuition for Proposition 3 is as follows. No network $\widetilde{g}^T$ in the set G^{FS2} is indirectly dominated by another network in that set, say $\widetilde{g}^{T'}$, since in every path from $\widetilde{g}^T$ to $\widetilde{g}^{T'}$, each member of T has a payoff greater than $\overline{\pi}$ in the first network g' where some members of T modify the network. The set G^{FS2} thus satisfies internal stability. External stability is also satisfied. To show this, we propose an algorithm that generates a path from any network g' not in the set G^{FS2} to some network g in the set satisfying indirect dominance. At each step on the path, agents with a payoff smaller than $\overline{\pi}$ delete their links until a network g'' is reached where either

[16] The payoff of a connected agent with the smallest degree in a network with $s \geq \widetilde{s}$ connected agents is smaller than in a clique among s agents, and is thus smaller than $\widehat{\pi}$.

[17] Each agent participates in the network g'''. In the first case, agent i has the lowest degree in the current network, and would be better off in the complete network, and thus in the end network. In the other case, we have $\Pi_i(g''') \leq \overline{\pi} < \widehat{\pi}$.

$\overline{s}$ agents are isolated, or where each connected agent has a payoff greater than $\overline{\pi}$. In the first case, let the unconnected agents form a clique, then among the remaining agents, let the agent with the smallest degree successively delete his links and when they are all unconnected, they form the second clique.[18] In the second case, unconnected agents create links until they all have the same degree as an agent i with the lowest degree among those that are connected in g'', or until they are completely connected. The network reached this way is g''' and the payoff of agent i in g''' is smaller than $\overline{\pi}$. The agents with no links in g'' and agent i then delete their links. The path proposed is such that at each step, at least one additional agent cuts his links. After a finite number of iterations, $\overline{s}$ agents are isolated and may form the large clique. It follows that the condition $\overline{\pi} > \widehat{\pi}$ is sufficient for the set G^{FS2} to be a vNM farsighted stable set. It is also necessary since a network composed of one clique of size $\widehat{s}$ is not indirectly dominated by a network in the set when $\widehat{\pi} \geq \overline{\pi}$.

Propositions 2 and 3 thus establish the existence of a singleton (up to all permutations) von Neumann-Morgenstern farsighted stable set of networks G^{FS}, where $G^{FS} = G^{FS1}$ if $\widehat{\pi} > \overline{\pi}$ while $G^{FS} = G^{FS2}$ if $\widehat{\pi} < \overline{\pi}$. The agents are partitioned into two groups $\{S^*, N \setminus S^*\}$ where the size of the large group S^* is $\widehat{s}$ if $\widehat{\pi} > \overline{\pi}$ or $\overline{s}$ if $\widehat{\pi} < \overline{\pi}$. When cooperation only matters through the number of partners, one could draw a parallel between a coalition in the coalition formation approach and a clique in the network formation approach. For instance Bloch's (1995) model of group formation and Goyal and Joshi's (2003) model of network formation among agents competing in quantity both lead to the same profile of marginal costs, and thus to the same second stage equilibrium payoff, when a clique structure in Goyal and Joshi (2003) mirrors a group structure in Bloch (1995). Yi (1997) shows that ex ante symmetric agents form the partition $\{S^*, N \setminus S^*\}$ in the coalition unanimity game[19] provided conditions (C1)-(C4) are satisfied: (C1) when two coalitions merge, the remaining agents are worse off, (C2) a member of a coalition is better off if her coalition merges with a larger coalition, (C3) a member of a coalition is better off if she leaves a coalition to join another larger one, and (C4) members of any coalition of size $s \leq n/2$ do not want to exclude a member. The partition of the agents in a farsighted stable network is equivalent to the subgame perfect equilibrium in Bloch's (1996) coalition unanimity game. When minority economies to scale is not satisfied, this equivalence does no longer hold, as illustrated by the following example.

[18]The payoff of an agent who is in the small clique in the end network is smaller than $\underline{\pi}$ in a network where she deviates, either by minority economies to scale if she deletes links or by strong degree monotonicity when the second clique is formed.

[19]The rules of the coalition unanimity game are as follows. Agents are ranked according to an exogenous rule of order. The first agent proposes the formation of a coalition. If all members of this proposed coalition agree, then the coalition is formed and can no longer be dissolved and the game continues. In this game, the first agent in the updated ranking after removing the first coalition, makes the next proposal. If one agent rejects the proposal, she becomes the initiator in the next round. The proposer of a coalition and its potential members must thus foresee the coalition structure which will eventually prevail in order to decide on the current coalitional proposal.

Example 1. *Consider* $N = \{1, 2, ..., 6\}$ *firms competing in quantities. The marginal cost of an agent* i *depends on her degree in the network* g *in the following way:* $c_i(g) = c(n_i(g))$ *where* $c(0) = 0.25, c(1) = 0,15, c(2) = 0,126, c(3) = 0,116, c(4) = 0,106$ *and* $c(5) = 0,1$*. The linear inverse demand curve is* $p = 1 - \sum_{i \in N} q_i$*. The participation constraints never bind and hence:*

$$q_i = \frac{1}{n+1}\left[1 - nc(n_i(g)) + \sum_{j \in N \backslash \{i\}} c(n_j(g))\right]$$

The cost of forming a link is equal to $\varepsilon > 0$*, where* ε *is arbitrarily small. This game satisfies negative externalities and strong degree monotonicity. Minority economies to scale are violated since agents in the group with two firms prefer to keep the isolated agent without connections rather than to form a clique with her. Assuming that the marginal cost of a firm in a coalition of size* s *is given by* $c(s-1)$*, the equilibrium coalition structure of the coalition unanimity game is a partition of the six agents into a group of* 3 *agents, another of* 2 *agents and a singleton. Let* $g_1 = g^{S_1} \cup g^{S_2} \cup g^{S_3}$ *be a network composed of three cliques* $S_1 = \{1, 2, 3\}, S_2 = \{4, 5\}, S_3 = \{6\}$*. With a slight abuse of notation, we write* $g_1 = \{123, 45, 6\}$*. The set of permutations of* g_1 *which do not mutually indirectly dominate each other is given by* $G_1 = \{g_1, g_2, g_3, g_4, g_5, g_6\}$ *where:*

$$\begin{aligned} g_1 &= \{123, 45, 6\} \\ g_2 &= \{123, 46, 5\} \\ g_3 &= \{123, 56, 4\} \\ g_4 &= \{126, 45, 3\} \\ g_5 &= \{136, 45, 2\} \\ g_6 &= \{236, 45, 1\} \end{aligned}$$

Set G_1 *is internally stable. One can easily verify that every other permutation of* g_1 *is indirectly dominated by a network in* G_1*since in such network at least two agents who can improve themselves in a network in* G_1 *are not linked and can start a farsighted improving path by forming a link. It follows that every von Neumann Morgenstern farsighted stable set* G *containing solely* g_1 *and permutations of it must be* G_1*. However,* G_1 *does not satisfy external stability as no network in* G_1 *indirectly dominates* $g_7 = \{12, 13, 24, 35\}$ *for instance.*

When the farsighted stable set of networks is composed of dominant group networks ($G^{FS} = G^{FS1}$), a farsighted stable network g is also pairwise stable. Indeed, an agent does not find it profitable to delete a link from g by strong degree monotonicity, while an isolated agent does not gain by adding a link since she would remain inactive. When the farsighted stable set of networks is composed of asymmetric cliques ($G^{FS} = G^{FS2}$), a farsighted stable network is pairwise stable if two agents from different cliques do not find it profitable to add a link.

6 Efficiency

In this section, we analyze the relationship between the network architecture and the sum of payoffs to the agents in a game $\Gamma \in \mathcal{G}$ satisfying Properties 5 and 6. A network is efficient if no other network generates a higher sum of payoffs $W(g) = \sum_{i \in N} \Pi_i(g)$.

Definition 6. A network $g \in \mathbb{G}$ is efficient if $W(g) \geq W(g')$ for all $g' \in \mathbb{G}$.

Switching from a network g to another network g' plays a crucial role in our discussion. There is a switch from g to g' if the degree of one agent increases by one unit and that of a less connected agent decreases by one unit, while the number of partners of the remaining agents does not change. A switch leads to a mean-preserving spread in the distribution of links. The number of links and thus the total linking costs do not change after a switch. A welfare-improving switch (P5) imposes that the sum of payoffs increases after a switch if the agent whose degree decreases remains active. In the product differentiation model for example, such a switch leads to a reallocation of some units of production from one firm to another whose cost of production is smaller, reducing the total production cost for a fixed aggregate quantity. A switch leading to the exclusion of the agent whose degree decreases could reduce the sum of payoffs in some applications, as illustrated in Example 2.

Example 2. *Let Γ be the Tullock contest model of Grandjean et al. (2015) presented in Section 3.2 with $n = 11$, $v = 0$ and $\beta = 1$. Let g be such that agent 1 is connected to all the other agents, agent 2 is connected to 1, 3, 4 and 5, and agent 3 is connected to 1, 2 and 6, that is $g = \{i_1i_2, i_1i_3, ..., i_1i_{11}, i_2i_3, i_2i_4, i_2i_5, i_3i_6\}$. The valuation of the contestants for the prize in network g is then given by $v_1(g) = 10, v_2(g) = 4, v_3(g) = 3$ and $v_k(g) \leq 2$ for $k \geq 4$. Agents $1, 2$ and 3 participate and respectively get a payoff of $\pi_1(g) = 5.003$, $\pi_2(g) = 0.288$, and $\pi_3(g) = 0.002$. Let network g' be obtained from g by replacing the link i_3i_6 by the link i_2i_6. Only agents 1 and 2 participate in g', and respectively get a payoff of $\pi_1(g') = 4.444$ and $\pi_2(g') = 0.555$. The sum of payoffs is smaller in g' than in g even though the distribution of links under g' is a mean preserving spread of the distribution of links under g.*

Switch externalities (P6) impose that the payoff of an agent not involved in a switch among strong agents cannot decrease. Jointly with P5, it implies that the set of participating agents does not shrink after a switch among strong agents from g to g', so that the sum of payoffs increases.

Nested split graphs were introduced by Cvetkovic and Rowlinson (1990) and Mahadev and Peled (1995). Agents in a nested split graph can be decomposed into t classes such that an agent in class s is connected to each agent in class 1 to $t-s+1$. The agents in class 1 are connected to every connected agent. A network is immune to switches if and only if it is a nested split graph. By P5, it follows that the efficient network is a nested split graph if agents are active in every network configuration.

Proposition 4. *Let Γ satisfy Property 5. The efficient network of the game Γ is a nested split graph if $K(g) = N$ for all $g \in \mathbb{G}$.*

Links are costly to establish. It follows that links cannot be reallocated from an efficient network so that the total number of links is reduced, while the set of participating agents and their degree are unchanged. For a given network g, let $C^-(g)$ be the set of networks g' where (i) the degree distribution of the participating agents is as in g, (ii) the total number of links is smaller than in g, and (iii) each non-participating agent in g has at most k links in g', where k is the number of links of the non-participating agent with the highest degree in network g. The third condition is a sufficient condition to ensure that the set of participating agents is the same in both networks. Formally, $C^-(g) = \{g' \in \mathbb{G}$ |(i) $n_l(g) = n_l(g')$ for all $l \in K(g)$, (ii) $\sum_{k \in N} n_k(g) > \sum_{k \in N} n_k(g')$, and (iii) $\max_{l \in E(g)} n_l(g) \geq \max_{l \in E(g)} n_l(g')$.

We have seen that a switch involving a weak agent may reduce the set of participating agents and hence the sum of payoffs. On the other hand, P5 and P6 ensure that a switch among strong agents increases the sum of payoffs. In Lemma 4, we show that the sum of payoffs also increases by moving from g to g' if (i) one strong agent has one more partner in g' than in g, at the expense of another strong agent with fewer partners in g, (ii) every other participating agent has the same degree in both networks, (iii) every non-participating agent in g has at most k links in g' where k is the number of links of the non-participating agent with the highest degree in network g, and (iv) the total number of links is the same in both networks. We denote the set of networks g' obtained in this way from g by $S^*(g, i, j)$, where i is the strong agent whose degree has increased at the expense of j. Formally, for $g \in G$ and $i, j \in K^+(g)$ such that $n_i(g) \geq n_j(g)$, let $S^*(g, i, j) = \{g' \in \mathbb{G}$ |(i) $n_i(g') = n_i(g) + 1$, $n_j(g') = n_j(g) - 1$, (ii) $n_k(g') = n_k(g)$ for all $k \in K(g) \backslash \{i, j\}$, (iii) $n_k(g') \leq \max_{l \in E(g)} n_l(g)$ for all $k \in E(g)$, and (iv) $\sum n_i(g') = \sum n_i(g)\}$.

Lemma 4. *Let $\Gamma \in \mathcal{G}$ satisfy Property 5 and Property 6. Let $g, g' \in \mathbb{G}$ be such that $g' \in S^*(g, i, j)$. We have $W(g') > W(g)$.*

To prove this result, we construct a network g'' where the degree of each agent who participates in g does not change g, while the degree of each agent who does not participate in g does not change g'. The set of participating agents, their degree distribution, and their payoff are then equal in g and g''. Also, the sum of payoffs in g' is higher than in g'', and thus higher than in g, since g' is obtained by a switch among strong agents from g''. In what follows, with a slight abuse of notation, we say that g' is obtained from a switch among strong agents from g if $g' \in S^*(g, i, j)$.

When agents do not participate in some type of network configuration, an efficient network necessarily minimizes the linking costs given the degree distribution of the participating agents, and is immune to switches among strong agents.

Let $\overline{G}$ be the set of networks satisfying these constraints. Formally,

$$\overline{G} = \{g \in \mathbb{G} \mid \text{(i) } C^-(g) = \{\emptyset\} \text{ and (ii) } \nexists g' \in \mathbb{G} \text{ such that } g' \in S^*(g, i, j)\}$$

We decompose the set $\overline{G}$ into three subsets. Let $\overline{G}_1 = \{g \in \overline{G} \mid K_1(g) = K(g)\}$ be the set of networks that are in $\overline{G}$ such that the set of participating agents have the same degree. Let $\overline{G}_2 = \{g \in \overline{G} \backslash \overline{G}_1 \mid g^{K^+(g)} \subseteq g\}$ be the set of networks that are in $\overline{G} \backslash \overline{G}_1$ such that strong agents form a clique. Finally, let $\overline{G}_3 = \{g \in \overline{G} \backslash \overline{G}_1 \mid g^{K^+(g)} \not\subseteq g\}$ be the set of networks that are in $\overline{G} \backslash \overline{G}_1$ such that strong agents are not completely connected. From the definition of $\overline{G}_1$,$\overline{G}_2$ and $\overline{G}_3$, their intersection is empty and their union is $\overline{G}$. We now analyze how the two defining conditions of $\overline{G}$ shape the sets $\overline{G}_1$, $\overline{G}_2$ and $\overline{G}_3$.

Proposition 5 shows that a network g in $\overline{G}_1$ is either a quasi-regular network on a set of agents, or a core-periphery network.[20] Quasi-regular networks on the set of participating agents range from empty to dominant group networks, and core-periphery networks range from dominant group networks to nested split graphs with two groups.

Proposition 5. *A network $g \in \overline{G}_1$ is either a quasi-regular network on a set of agents $K \subseteq N$, or a core-periphery network.*

The intuition for the proof of Proposition 5 is as follows. Let a network g in $\overline{G}_1$ be such that the participating agents are not entirely connected to each other. There is in this case at most one link from a participating agent to a non-participating agent since it would otherwise be possible to replace two links by one and keep the degree distribution of participating agents unaffected. Network g is then a quasi-regular network on the set of participating agents. Since non-participating agents are not connected to each other in a network $g \in \overline{G}$, the network is a core-periphery network if the participating agents are completely connected to each other.

In a network g in $\overline{G}_2$, the strong agents are completely connected to each other. We show in Proposition 6 that network g is a nested split graph if weak agents are only connected to strong agents. Otherwise, all strong agents but one are connected to each participating agent. In addition, if there is a link between a weak agent and a non-participating agent, then all strong agents but one are connected to the entire population.

Proposition 6. *Let $g \in \overline{G}_2$, then*

(i) If $(E(g) \cup K_m(g)) \perp_g K_m(g)$, then g is a nested split graph
(ii) If $K_m(g) \leftrightarrow_g K_m(g)$, then $K(g) \top_g K^+(g) \backslash \{i_1\}$
(iii) If $K_m(g) \leftrightarrow_g E(g)$, then $N \top_g K^+(g) \backslash \{i_1\}$

The intuition for the proof of Proposition 6 is as follows. If agents are only connected to strong agents, they should be connected to the strong agents with the highest degree since otherwise there could be a switch among strong agents. If weak agents are only connected to strong agents, each link in the network involves at least one strong agent so that the network

[20] A quasi-regular network on a set of agents $K \subseteq N$ is a network g such that $n_i(g) = n_j(g)$ for all $i, j \in K$, and $\sum_{j \notin K} n_j(g) \leq 1$.

should be a nested split graph. If network g in $\overline{G}_2$ involves connections among weak agents, all weak agents but the one with the smallest number of strong partners should be connected to all strong agents but the one with the smallest number of partners i_1. Indeed, if two weak agents are not connected to two strong agents, a switch could be obtained, replacing a link between two weak agents and another link between two strong agents by a link between the two weak agents and the same strong agent. It follows that i_1 should have some connections towards weak or non-participating agents since she has more links than weak agents. Thus, all weak agents should be connected to all strong agents but i_1. Indeed, otherwise a link involving i_1 could be replaced by another involving another strong agent. If there is a link between weak agents and non-participating agents in network g, non-participating agents should then be connected to all strong agents but i_1. For example, if participating agents form a clique, then i_1 should be connected to non-participating agents, say e_1, since she has more links than weak agents. Thus, if a non-participating agent is not connected to a strong agent other than i_1, the link i_1e_1 could be replaced by this missing link. If a network $g \in \overline{G}_2$ is not a nested split graph, it is a quasi-nested split graph. In a quasi-nested split graph, a switch is possible but only at the expense of a weak agent. For example, let $N = \{i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4\}$ and let g be such that $\{i_2, i_3, i_4\} \top_g N$, and $i_1j_1, i_1j_2, j_3j_4 \in g$. If $K(g) = N$, then $g \in \overline{G}_2$. The network g is a quasi-nested split graph.

Proposition 7 shows that networks in $\overline{G}_3$ are nested split graphs.

Proposition 7. *Let $g \in \overline{G}_3$. Then, g is a nested-split graph.*

The intuition for the proof of Proposition 7 is as follows. Let i_1 be the strong agent with the lowest degree and i_2 be the one with fewer partners among the remaining strong agents in a network g in $\overline{G}_3$. These agents are not connected to each other by definition of $\overline{G}_3$. In a first step, we show that i_1 is not connected to non-participating agents. Indeed, if it was the case, say i_1 and the non-participating agent e_1 were connected, e_1 should be connected to each strong agent to avoid switches and it would be possible to save on linking costs by replacing the links i_1e_1 and i_2e_1 by the link i_1i_2. In a second step, we further show that i_1 cannot be connected to a weak agent, say j. To see this, notice that if it was the case, j should be connected to all strong agents to avoid switches. As a consequence i_1 should have some other connections to weak agents and weak agents should not be completely connected since the degree of i_1 is higher than that of a weak agent. Then, the degree of i_2 at the expense of i_1 could be increased by forming the link between these agents and cutting two links between i_1 and weak agents, while creating a link between two weak agents and rearranging these links so that they keep the same degree. In a third step, we show that weak agents are only connected to strong agents. If it was not the case, then i_1 should be connected to two strong agents to whom a weak agent is not connected since her degree is higher and the degree of one of these two strong agents could be increased at the expense of the other. Since each link in the network involves at least one strong agent, the network should be a nested split graph as a switch among strong agents would otherwise be feasible.

The efficient networks in a game satisfying Properties 5 and 6 belong to the set of networks $\overline{G}$. Networks in $\overline{G}$ that are not nested split graphs are not immune to switches at the expense of a weak agent. They are thus not efficient if the agent whose degree is reduced by a switch remains active. Otherwise, a network in $\overline{G}$ may not be efficient because another network in $\overline{G}$ generates a higher welfare.

Asymmetric cliques are never efficient. Indeed, the sum of payoffs could be increased by cutting a link in the two cliques, and adding two links between the agents in the small clique who are not connected and an agent in the large clique.

7 Conclusion

In this paper, we have studied the formation of bilateral agreements when cooperation between pairs of agents creates negative externalities for the remaining agents. This occurs for example when firms share patents through cross-licensing agreements or share the cost of joint R&D projects, or when countries sign bilateral trade agreements. In these applications, the number of competitors is usually rather small and the stakes are high. This motivates us to depart from the standard stability notions in network formation which assume that agents are myopic. Rather, we analyze networks formed by farsighted agents, that is by agents who forecast how other agents would react to their choice of partners, and make a decision by comparing the current network to the end network which is formed when other agents have further deviated. We use the notion of von Neumann-Morgenstern farsighted stable set, which can be interpreted as a standard of behavior when agents are farsighted.

We show that there always exists a farsighted stable set in a game of network formation among rivals satisfying *strong degree monotonicity* and *minority economies to scale*. It is either composed of dominant group networks, where isolated agents are excluded from the market, or of networks composed of two asymmetric cliques. Our results thus support two empirically relevant properties of observed R&D and cross-licensing networks: barriers to entry and clustering.

We then show that the efficient network is a nested split graph when the game satisfies *welfare-improving switching* if agents are active in every network. Otherwise, if agents prefer to leave the market in some network configurations, the efficient networks are (quasi-)nested split graphs, quasi-regular networks or core-periphery networks when *welfare-improving switches and switch externalities* are satisfied. As a result, the structure of stable and efficient networks is in general different, resulting in a tension between networks which are formed by agents and those which would produce the highest sum of payoffs.

The four properties we impose are satisfied in many models of network formation among rivals. We show it is the case in a model of bilateral R&D agreements among differentiated firms, in Grandjean et al. (2013)'s model of cooperation among rivals in a Tullock contest, and in Goyal and Joshi (2006)'s model of patent races.

We conclude this paper with some directions for future research. First, we have identified one farsighted stable set out of possibly many. We do not know at this stage whether other candidates exist, and if some exist, identifying all the candidates is probably not a realistic task. The candidates to consider could be restricted, for example by only considering the sets composed of one network and its permutations. One could also analyze whether our properties could be strengthened to guarantee that our candidate is unique.

Second, one could go in the other direction and study which networks would form if our properties were weakened. In particular, one could ask whether a set composed of a k-clique network and its permutations could be farsighted stable if minority economies to scale were not satisfied.

Third, it would be interesting to analyze the case of positive externalities, where the formation of an agreement between two agents benefits the other agents. This occurs for instance in Belleflamme and Bloch (2004)'s model of market-sharing agreements, where firms may commit not to compete in each other's markets, thereby reducing competition in these markets and increasing the profit of outsiders.

References

[1] Ballester, C., Calvo-Armengol, A. and Y. Zenou, 2006. Who's who in networks. Wanted: The key player. Econometrica 74(5), 1403-1417.

[2] Bekkers, R., Duysters, G. and B. Verspagen, 2002.Intellectual property rights, strategic technology agreements and market structure: the case of GSM. Research Policy 31 (7), 1141-1161.

[3] Belhadj, M., Bervoets, S. and F. Deroïan, 2015. Network design under local complementarities, Forthcoming, Theoretical Economics.

[4] Belleflamme, P., and F. Bloch, 2004.Market sharing agreements and collusive networks. International Economic Review 45, 387-411.

[5] Bloch, F., 1995. Endogenous Structures of Association in Oligopolies. RAND Journal of Economics, Vol. 26, 537–556.

[6] Bloch, F., 1996. Sequential formation of coalitions in games with externalities and fixed payoff division, Games and Economic Behavior 14, 90-123.

[7] Bloch, F., 2005. Group and network formation in industrial organisation: a survey. In Demange, G., Wooders, M. (Eds) A Short Handbook on Group and Network Formation. Cambridge University Press, Cambridge.

[8] Chwe, M., 1994. Farsighted coalitional stability. Journal of Economic Theory 63, 299–325.

[9] Cornes, R. and R. Hartley, 2005. Asymmetric contests with general technologies. Economic Theory, 26, 923–946.

[10] Cvetkovic, D. and P. Rowlinson, 1990. The largest eigenvalue of a graph: A survey. Linear and multilinear algebra 28(1-2), 3–33.

[11] Dutta, B., Ghosal, S. and D. Ray, 2005. Farsighted network formation. Journal of Economic Theory 122, 143-164.

[12] Diamantoudi, E. and L. Xue, 2003. Farsighted stability in hedonic games. Social Choice and Welfare 21, 39-61.

[13] Goyal, S. and S. Joshi, 2003. Networks of Collaboration in Oligopoly. Games and Economic Behavior 43, 57-85.

[14] Goyal S. and S. Joshi, 2006a. Unequal connections. International Journal of Game Theory 34 (3), 319-349.

[15] Goyal, S. and Joshi, S. 2006b. Bilateralism and free trade. International Economic Review 47(3), 749–778.

[16] Goyal, S. and Moraga, J.L., 2001. R&D networks. RAND Journal of Economics, 32 (4), 686-707.

[17] Harsanyi, J., 1974. An equilibrium-point interpretation of stable sets and a proposed alternative definition. Management Science 20, 1472-1495.

[18] Hellmann, T. and J. Landwehr, 2014. Stable networks in homogeneous societies. IME Working Paper 517.

[19] Herings, J-J., Mauleon, A. and V. Vannetelbosch, 2004. Rationalizability for social environments. Games and Economic Behavior 49, 135-156.

[20] Herings, J-J., Mauleon, A. and V. Vannetelbosch, 2009. Farsightedly stable networks. Games and Economic Behavior 67, 526-541.

[21] Herings, J-J., Mauleon, A. and V. Vannetelbosch, 2014. Stability of neworks under level-K farisghtedness. CORE Discussion Paper 2014/32.

[22] Hochberg, Y.V., Ljungqvist, A. and Y. Lu, 2010. Networking as a barrier to entry and the competitive supply of venture capital. Journal of Finance 65(3), 829–859.

[23] Howard, Philip, 2009. Visualizing consolidation in the global seed industry. Sustainability 1, 1266-1287. Figure from https://msu.edu/~howardp/seedindustry.html, September 2013, accessed on May 21, 2014.

[24] Jackson M. and A. Wolinsky, 1996. A strategic model of social and economic networks. Journal of Economic Theory 71 (1), 44-74.

[25] König, M., Battiston, S., Napoletano, M. and F. Schweitzer, 2012. The efficiency and stability of R&D networks. Games and Economic Behavior 75, 694-713.

[26] Mahadev, N. and U. Peled, 1995. Threshold graphs and related topics. North Holland.

[27] Marinucci, M. and W. Vergote, 2011. Endogenous network formation in patent contests and its role as a barrier to entry. The Journal of Industrial Economics 59(4), 529-551.

[28] Mauleon, A. and V. Vannetelbosch, 2004. Farsightedness and cautiousness in coalition formation games with positive spillovers. Theory and Decision 56, 291-324.

[29] Mauleon, A., Sempere-Monerris J. and V. Vannetelbosch, 2014. Farsighted R&D networks, Economics Letters 12(3), 471-500.

[30] Page, F.H., Jr., Wooders, M. and S. Kamat, 2005. Networks and farsighted stability. Journal of Economic Theory 120, 257-269.

[31] Page, F.H., Jr. and M. Wooders, 2009. Strategic basins of attraction, the path dominance core, and network formation games. Games and Economic Behavior 66, 462-487.

[32] Ray, D., and R. Vohra, 2014. The farsighted stable set. Econometrica 83, 977-1011, 2015.

[33] Regibeau, P. and K. Rockett, 2011. Assessment of potential anticompetitive conduct in the field of intellectual property rights and assessment of the interplay between competition policy and IPR protection. European Commission Report.

[34] Ryvkin, D., 2013. Heterogeneity of Players and Aggregate Effort in Contests. Journal of Economics & Management Strategy, Wiley Blackwell, vol. 22(4), 728-743.

[35] Roketskiy, N., 2012. Competition and networks of collaboration, mimeo.

[36] Stein, W. E, 2002.Asymmetric rent-seeking with more than two contestants. Public Choice, 113 (3-4), 325-36.

[37] Von Neumann, J. and O. Morgenstern, 1944. Theory of games and economic behavior. Princeton University Press, Princeton.

[38] Westbrock, B., 2010. Natural concentration in industrial research collaboration. The RAND Journal of

Economics 41(2), 351–371.

[39] Xue, L., 1998. Coalitional stability under perfect foresight. Economic Theory 11, 603-627.

[40] Yi, S.S. (1997), Stable Coalition Structures with Externalities, Games and Economic Behavior 20, 201-237.

8 Appendix

Appendix A. Proofs of Section 3

Proof of Lemma 1.

Property 3. Strong degree monotonicity:

Let $g \in \mathbb{G}$. Let $S \subseteq K_l(g) \subseteq K(g)$. For all $i \in S$, let $n_i(g') = n_i(g) + 1$ and for all $i \notin S$, let $n_i(g') = n_i(g)$. We show that $\pi_i(g') - \pi_i(g) = q_i(g')^2 - q_i(g)^2 > 0$ for $i \in S$.

Let us write $q_i(g')^2 - q_i(g)^2 = (q_i(g') - q_i(g))(q_i(g') + q_i(g))$. We show that $q_i(g') - q_i(g) > 0$. Notice that $K(g') \subseteq K(g)$

by negative externalities.

(i) Suppose $K(g) = K(g')$.

(i.1) In the quantity competition version of the model, we have

$$q_i^Q(g') - q_i^Q(g) = \frac{1}{2 + (k(g) - 1)\beta} \frac{\mu}{2 - \beta} \left[(2 + (k(g) - 2)\beta) - \beta(s - 1)\right] > 0,$$

where the inequality comes from $k(g) \geq s$ and $\beta \in (0, 1)$.

(i.2) In the price competition version of the model, we have

$$q_i^P(g') - q_i^P(g) = \Psi(k(g), \beta, \mu) \left[2 + (3k(g) - 6)\beta + (k(g)^2 - 5k(g) + 5)\beta^2 - (1 + (k(g) - 2)\beta)\, \beta(s - 1)\right]$$

Since $s \leq k(g)$ and $q_i^P(g') - q_i^P(g)$ is decreasing in s, we have

$$q_i^P(g') - q_i^P(g) \geq \Psi(k(g), \beta, \mu) \left[2 + (3k(g) - 6)\beta + (k(g)^2 - 5k(g) + 5)\beta^2 - (1 + (k(g) - 2)\beta)\, \beta(k(g) - 1)\right]$$

$$q_i^P(g') - q_i^P(g) \geq \Psi(k(g), \beta, \mu)[2 + (2k(g) - 5)\beta - (2k(g) - 3)\beta^2] \geq 0,$$

where the last inequality holds strictly if $\beta < 1$.

(ii) Now suppose that $K(g') \subsetneq K(g)$. Let π_i be the Nash equilibrium payoff of agent i in an auxiliary game among the agents that are active in g' when they have the same marginal cost as in the network g. By negative externalities, $\pi_i > \pi_i(g)$. From step (i), we know that $\pi_i(g') > \pi_i$.

Property 4. Minority economies to scale:

Let $g = g^T \cup g^S$, where $T \cap S = \emptyset$, $T \cup S \subsetneq N$, and $s \leq (n-1)/2 < t$. We show that if $S \subseteq K(g)$, then $\pi_i(g') > \pi_i(g)$ where $g' = g^T \cup g^{S \cup \{j\}}$ for $i \in S$, $j \in N \backslash \{S \cup T\}$.

(i) Suppose that $K(g) = K(g')$. Notice that $n_i(g') - n_i(g) = 1$ and $\sum_{j \in K(g) \backslash \{i\}} n_j(g') - n_j(g) = 2s - 1$.

(i.1) In the quantity competition version of the model, we have

$$q_i^Q(g') - q_i^Q(g) = \frac{1}{2+(k(g)-1)\beta}\frac{\mu}{2-\beta}\left[(2+(k(g)-2)\beta) - (2s-1)\beta\right] > 0,$$

where the inequality comes from $s < \frac{k(g)}{2}$.

(i.2) In the price competition version of the model, we have

$$q_i^P(g') - q_i^P(g) = \Psi(k(g),\beta,\mu)\left[(2+(3k(g)-6)\beta + (k(g)^2-5k(g)+5)\beta^2 - (1+(k(g)-2)\beta)\,\beta(2s-1)\right]$$

Since $s < k(g)/2$ and $q_i^P(g') - q_i^P(g)$ is decreasing in s, we have

$$\begin{aligned} q_i^P(g') - q_i^P(g) &> \Psi(k(g),\beta,\mu)\left[2+(3k(g)-6)\beta + (k(g)^2-5k(g)+5)\beta^2 - (1+(k(g)-2)\beta)\,\beta(k(g)-1)\right] \\ q_i^P(g') - q_i^P(g) &> \Psi(k(g),\beta,\mu)[2+(2k(g)-5)\beta - (2k(g)-3)\beta^2] \geq 0, \end{aligned}$$

(ii) Suppose that $K(g) \supseteq K(g')$. Then $K(g) = N$ and $K(g') = T \cup S \cup \{j\}$. Let π_i be the Nash equilibrium payoff of agent i in an auxiliary game among the agents that are active in g' when they have the same marginal cost as in the network g. By negative externalities, $\pi_i > \pi_i(g)$. From step (i), we know that $\pi_i(g') > \pi_i$.

(iii) Suppose that $K(g) \subseteq K(g')$. Then $K(g) = S \cup T$ and $K(g') = T \cup S \cup \{j\}$. Consider the auxiliary game $\Gamma(\varepsilon)$ among the agents in $T \cup S \cup \{j\}$ where the marginal costs of agents are given by $c_k = c_k(g)$ for all $k \in S \cup T$ and $c_j(\varepsilon) < c_j(g)$ is such that all agents produce positive amounts and agent j produces ε at the Nash equilibirum of Γ. Note by $\overline{\pi}_k(\varepsilon)$ the Nash equilibrium payoff of agent k in $\Gamma(\varepsilon)$. By (i) and by negative externalities, we then have $\pi_i(g') > \overline{\pi}_i(\varepsilon)$. This, for ε approaching 0, we have $\pi_i(g') > \overline{\pi}_i(\varepsilon) \simeq \pi_i(g)$.

Property 5. Welfare improving switch:

We show that $\sum_{i\in N}\pi_i(g') > \sum_{i\in N}\pi_i(g)$ if $g' \in S(g,i,j)$, $n_i(g) \geq n_j(g)$ and $j \in K(g)\cap K(g')$.

(i) We first show that $K(g) = K(g')$.

Let $k_l^+(g) = \#\{k \in N : n_k(g) \geq n_l(g)\}$. In the quantity competition model, the participation constraint of an agent l in the network g is:

$$q_l^Q(g) > 0 \Leftrightarrow n_l(g) > \frac{\beta\sum_{k\in N:n_k(g)\geq n_l(g)} n_k(g) - \frac{\alpha-\lambda}{\mu}(2-\beta)}{2+\left[k_l^+(g)-2\right]\beta}$$

In the price competition model, the participation constraint of an agent l in the network g is:

$$q_l^P(g) > 0 \Leftrightarrow n_l(g) > \frac{\left(1+(k_l^+(g)-2)\beta\right)\beta\sum_{k\in N:n_k(g)\geq n_l(g)} n_k(g) - \frac{\alpha-\lambda}{\mu}(1-\beta)}{(2+(3k_l^+(g)-6)\beta + (k_l^+(g)^2-5k_l^+(g)+5)\beta^2}$$

In the two models, the participation constraint of an agent l in a network g depends on the number of agents with at least the same number of links $k_l^+(g)$ and the sum of their links $\sum_{k\in N:n_k(g)\geq n_l(g)} n_k(g)$. Let $l \in E(g)$. We have $k_l^+(g) = k_l^+(g')$ and $\sum_{k\in N:n_k(g)\geq n_l(g)} n_k(g) =$

$\sum_{k\in N:n_k(g)\geq n_l(g)} n_k(g')$ so that $l \in E(g')$. Thus, $K(g') \subseteq K(g)$. If $j \in K_m(g)$, then the assumption $j \in K(g')$ implies $K(g) \subseteq K(g')$ since $n_j(g') < n_k(g')$ for all $k \in K(g)\backslash\{j\}$. Finally, if $j \in K^+(g)$, let $l \in K_m(g)$. We have $k_l^+(g) = k_l^+(g')$ and $\sum_{k\in N:n_k(g)\geq n_l(g)} n_k(g) = \sum_{k\in N:n_k(g)\geq n_l(g)} n_k(g')$ so that $l \in K(g')$, implying $K(g) \subseteq K(g')$ since $n_l(g') \leq n_k(g')$ for all $k \in K(g)$. We have $K(g) \subseteq K(g')$ and $K(g') \subseteq K(g)$ so that $K(g) = K(g')$.

(ii) We then show that $\sum_{i\in N} \pi_i(g') > \sum_{i\in N} \pi_i(g)$.

Since $K(g) = K(g')$, we have $q_i(g') - q_i(g) = q_j(g) - q_j(g') = x > 0$ and $q_l(g) = q_l(g')$ for $l \neq i, j$ in the two models. It follows that some units of production are transfered from agent j to agent i when moving from g to g'. We have $\sum_{l\in K(g)} c_l(g)q_l(g) - \sum_{l\in K(g')} c_l(g')q_l(g') = c_i(g)q_i(g) + c_j(g)q_j(g) - c_i(g')q_i(g') - c_j(g')q_j(g') = \mu(q_i(g') - q_j(g')) + x(c_j(g) - c_i(g)) > 0$. The total output remains unchanged but the total production costs are reduced.

Property 6. Switch externality:

We show that $\pi_l(g') = \pi_l(g)$ for $g' \in S(g, i, j)$ if $i, j \in K^+(g)$, $n_i(g) \geq n_j(g)$ and $l \neq i, j$.

This results has already been established in P5 where we have shown that $K(g) = K(g')$, $q_i(g') - q_i(g) = q_j(g) - q_j(g')$ and $q_k(g) = q_k(g')$ for all $k \neq i, j$. $\square$

$\square$

Proof of Lemma 2.

Property 3. Strong degree monotonicity:

Let $g \in \mathbb{G}$. Let $S \subseteq K_l(g) \subseteq K(g)$. For all $i \in S$, let $n_i(g') = n_i(g) + 1$ and for all $i \notin S$, let $n_i(g') = n_i(g)$. We show that $\pi_i(g') > \pi_i(g)$ for $i \in S$.

By negative externalities, we have $K(g') \subseteq K(g)$.

(i) If $K(g) = K(g')$, then $p_i(g') = 1 - (k(g) - 1)/(s + \sum_{j\in K(g)}(v_i(g) + \beta)/v_j(g)) > p_i(g) = 1 - (k(g) - 1)/(s + \sum_{j\in K(g)} v_i(g)/v_j(g))$.

(ii) If $K(g') \subsetneq K(g)$, then let π_i be the unique Nash equilibrium payoff of agent i in a game among the agents in $K(g')$ who have the same valuation as in the network g. By negative externalities, $\pi_i > \pi_i(g)$ for all $i \in K(g')$. From step (i), we know that $\pi_i(g') > \pi_i$.

Property 4. Minority economies to scale:

Let $g = g^T \cup g^S$, where $T \cap S = \emptyset$, $T \cup S \subsetneq N$, and $s \leq (n-1)/2 < t$. We show that if $S \subseteq K(g)$, then $\pi_i(g^T \cup g^{S\cup\{j\}}) > \pi_i(g^T \cup g^{S\cup\{j\}})$ for $i \in S$, $j \in N\backslash\{S \cup T\}$.

(i) Suppose that $K(g) = N$. Then, for $i \in S$, we have

$$\pi_i(g) = (v + (s-1)\beta)\left[1 - \frac{n-1}{s + (v + (s-1)\beta)(y + \frac{1}{v})}\right]^2,$$

where $y = \frac{t}{v+(t-1)\beta} + \frac{n-t-s-1}{v}$.

(i.a) If $K(g') = N$. Then

$$\pi_i(g') = (v + s\beta)\left[1 - \frac{n-1}{s+1+(v+s\beta)y}\right]^2.$$

We then have that $\pi_i(g') > \pi_i(g)$ when $yv + 1 > s$.

Since $K(g) = N$, we have

$$\begin{aligned} v &\geq \frac{n-2}{\frac{s}{v+(n-1)\beta} + y} > \frac{n-2}{\frac{s}{v} + y}; \\ yv &> n - 2 - s. \end{aligned}$$

This implies that $yv + 1 > s$ whenever $n - s - 1 > s$, which is satisfied when $s \leq \frac{n-1}{2}$.

(i.b) If $K(g') = T \cup S \cup \{j\} \subsetneq N$, then let π_i be the unique Nash equilibrium payoff of agent i in a game among the agents in $K(g')$ who have the same valuation as in the network g. By negative externalities, $\pi_i > \pi_i(g)$ for all $i \in K(g')$. From step (i.a), we know that $\pi_i(g') > \pi_i$.

(ii) Suppose that $K(g) = T \cup S$. Then, for $i \in S$ we have:

$$\pi_i(g) = (v + (s-1)\beta)\left[1 - \frac{t+s}{(v+(s-1)\beta)\left(\frac{s}{v+(s-1)\beta} + \frac{1}{\overline{v}} + \frac{t}{v+(t-1)\beta}\right)}\right]^2.$$

where $\overline{v} = \frac{t+s-1}{\frac{t}{v+(t-1)\beta}+\frac{s}{v+(s-1)\beta}} > v$. We also have that

$$\pi_i(g') = (v + s\beta)\left[1 - \frac{t+s}{(v+s\beta)\left(\frac{s+1}{v+s\beta} + \frac{t}{v+(t-1)\beta}\right)}\right]^2.$$

Hence, $\pi_i(g') > \pi_i(g)$ if $v < (t-1)^2\beta + st\beta$.

Since $j \notin K(g)$, we have $v < (t-1)^2\beta < (t-1)^2\beta + st\beta$.

Property 5. Welfare improving switch:

We show that $\pi_l(g') \geq \pi_l(g)$ for $g' \in S(g, i, j)$ if $i, j \in K^+(g)$, $n_i(g) \geq n_j(g)$ and $l \neq j$.

(i) We first show that $K(g) \subseteq K(g')$.

Let $k_l^+(g) = \#\{k \in N : n_k(g) \geq n_l(g)\}$. The participation constraint of an agent l in the network g is

$e_l(g) > 0 \Longleftrightarrow v_l(g) > \frac{k_l^+(g)-1}{\sum_{k\in N:n_k(g)\geq n_l(g)} 1/v_k(g)}$. Let $i \in K_m(g)$. We have $v_i(g) > \frac{k_i^+(g)-1}{\sum_{k\in N:n_k(g)\geq n_i(g)} 1/v_k(g)} > \frac{k_i^+(g)-1}{\sum_{k\in N:n_k(g)\geq n_i(g)} 1/v_k(g')}$, where the first inequality holds since agent i participates in the contest under the network g and the second holds since the vector of the valuation of the agents with more links than i under g' is a mean preserving spread on the vector of the valuation of these agents under g. We thus have $i \in K(g')$, implying $K(g) \subseteq K(g')$.

(ii) $\pi_l(g) < \pi_l(g')$ since $(1 - \frac{(k(g)-1)}{\sum_{k\in K(g)} v_l(g)/v_k(g)}) < (1 - \frac{(k(g)-1)}{\sum_{k\in K(g)} v_l(g)/v_k(g')}) < (1 - \frac{(k(g')-1)}{\sum_{k\in K(g')} v_l(g)/v_k(g')})$ where the second inequality holds by application of Lemma 1 in Grandjean et al. (2014).

Property 6. Switch externality:

We show that $\sum_{i\in N} \pi_i(g') > \sum_{i\in N} \pi_i(g)$ if $g' \in S(g,i,j)$, $n_i(g) \geq n_j(g)$ and $j \in K(g')$. In Grandjean et al. (2014), it is shown that $\sum_{i\in N} \pi_i(g) = \sum v_i(g) - (k(g)-1)h_{k(g)}(g) - \sum e_i^*(g)$ and that $\sum e_i^*(g) = \frac{k(g)-1}{k(g)} h_{k(g)}(g)$.

(i) $K(g) \subseteq K(g')$.

Let $l \in K_m(g)$. We show that $l \in K(g')$. If $l \in \{i,j\}$, the result holds by assumption since $i,j \in K(g')$. Suppose $l \neq i,j$, we have $v_l(g) = v_l(g') > \frac{k(g)-1}{\sum_{k\in K(g)} 1/v_k(g)} > \frac{k(g)-1}{\sum_{k\in K(g)} 1/v_k(g')}$, where the first inequality is the participation constraint of l in g, and the second holds since the vector of the valuation of the agents with more links than l under g' is a mean preserving spread on the vector of the valuation of these agents under g. We thus have $l \in K(g')$, implying $K(g) \subseteq K(g')$.

(ii) $\sum_{i\in N} e_i^*(g) > \sum_{i\in N} e_i^*(g')$

(ii.a) If $K(g) = K(g')$, we have $h_{k(g)}(g) > h_{k(g')}(g')$, leading to $\sum_{i\in N} e_i^*(g) > \sum_{i\in N} e_i^*(g')$.

(ii.b) If $K(g) \subsetneq K(g')$, the participation constraints of agent $j \in K(g')\backslash K(g)$ in g and g' are $\sum_{i\in N} e_i^*(g) \geq v_j(g)$ and $v_j(g') > \sum_{i\in N} e_i^*(g')$. We thus have $\sum_{i\in N} e_i^*(g) > \sum_{i\in N} e_i^*(g')$ since $v_j(g) = v_j(g')$.

(iii) $W(g') > W(g)$

(iii.a) If $K(g) = K(g')$, then this holds by (ii).

(iii.b) If $K(g) \subseteq K(g')$, then
$W(g') - W(g) = \sum_{j\in K(g')\backslash K(g)} v_j(g') - (k(g')+1)\sum_{i\in N} e_i^*(g') + (k(g)+1)\sum_{i\in N} e_i^*(g).$
$= \sum_{j\in K(g')\backslash K(g)} v_j(g') - (k(g')-k(g))\sum_{i\in N} e_i^*(g') + (k(g)+1)(\sum_{i\in N} e_i^*(g) - \sum_{i\in N} e_i^*(g'))$

The participation constraint of $j \in K(g')\backslash K(g)$ in g' is $v_j(g') > \sum_{i\in N} e_i^*(g')$. Adding these inequalities over all $j \in K(g')\backslash K(g)$, we obtain $\sum_{j\in K(g')\backslash K(g)} v_j(g') - (k(g')-k(g))\sum_{i\in N} e_i^*(g') > 0$. From (ii), we have $\sum_{i\in N} e_i^*(g) > \sum_{i\in N} e_i^*(g')$. We thus find that $W(g') - W(g)$ can be expressed as the sum of two positive numbers.

□

Proof of Lemma 3

Property 3. Strong degree monotonicity: $\pi_i(g) < \pi_i(g')$ for $i \in S \subseteq K_l(g) \subseteq K(g)$ where g' is such that $n_j(g') = n_j(g) + 1$ for all $j \in S$ and $n_j(g') = n_j(g)$ for all $j \in N \setminus S$.

Note that $\sum_{k\in N} n_k(g') = \sum_{k\in N} n_k(g') + s$

$$\pi_i(g') - \pi_i(g) = \frac{n_i(g')}{\delta + \sum_{k\in N} n_k(g')} - \frac{n_i(g)}{\delta + \sum_{k\in N} n_k(g)}$$

$$\pi_i(g') - \pi_i(g) = \frac{n_i(g)+1}{\delta + \sum_{k\in N} n_k(g) + s} - \frac{n_i(g)}{\delta + \sum_{k\in N} n_k(g)}$$

$$\pi_i(g') - \pi_i(g) = \frac{\delta + \sum_{k\in N} n_k(g) - n_i(g)s}{(\delta + \sum_{k\in N} n_k(g) + s)(\delta + \sum_{k\in N} n_k(g))} > 0$$

The latter is obtained by noting that $\sum_{k\in N} n_k(g) \geq \sum_{k\in S} n_k(g) = n_i(g)s$.

Property 4. Minority economies to scale: Let $g = g^S \cup g^T$, $T \cap S = \emptyset$ and $s \geq n/2$. Let $i \in T$ and $j \notin \{S \cup T\}$. We show that $\pi_i(g) < \pi_i(g')$ where $g' = g^S \cup g^{T\cup\{j\}}$.

$$\pi_i(g') - \pi_i(g) = \frac{n_i(g')}{\delta + \sum_{k\in N} n_k(g')} - \frac{n_i(g)}{\delta + \sum_{k\in N} n_k(g)}$$

$$\pi_i(g') - \pi_i(g) = \frac{n_i(g)+1}{\delta + \sum_{k\in N} n_k(g) + 2t} - \frac{n_i(g)}{\delta + \sum_{k\in N} n_k(g)}$$

$$\pi_i(g') - \pi_i(g) = \frac{\delta + \sum_{k\in N} n_k(g) - n_i(g)2t}{(\delta + \sum_{k\in N} n_k(g) + 2t)(\delta + \sum_{k\in N} n_k(g))} > 0$$

since $\sum_{k\in N} n_k(g) > n_i(g)2t$.

Property 5. Welfare neutral switch:

Let $g \in \mathbb{G}$ and $g' \in S(g,i,j)$. We show that $\sum_{k\in N} \pi_k(g') = \sum_{k\in N} \pi_k(g)$. Notice that $\pi_k(g') = \pi_k(g)$ for all $k \neq i,j$. It follows that $\sum_{k\in N} \pi_k(g') - \pi_k(g) = \pi_i(g') - \pi_i(g) + \pi_j(g') - \pi_j(g)$. Since $\sum_{k\in N} n_k(g') = \sum_{k\in N} n_k(g)$, we have:

$$\sum_{k\in N} (\pi_k(g') - \pi_k(g)) = \frac{n_i(g') - n_i(g) + n_j(g') - n_j(g)}{\delta + \sum_{k\in N} n_k(g')} = 0.$$

Property 6. Switch externality: Let $g \in \mathbb{G}$ and $g' \in S(g,i,j)$, we have $\pi_k(g') = \pi_k(g)$ for all $k \neq i,j$.

□

Appendix B. Proofs of Section 5

In the proof of Propositions 2 and 3, we note by $S_1(g) = \{i \in N \mid \pi_i(g) \geq \max\{\widehat{\Pi}, \overline{\Pi}\}\}$ the set of agents whose payoff in the network g is greater than the maximal per capita payoff in a 2-clique network, and by $S_2(g) = \{i \in N \mid \pi_i(g) < \max\{\widehat{\Pi}, \overline{\Pi}\}\}$ the remaining agents. We first introduce some lemmas.

1. *Let $\Gamma \in \mathcal{G}$ be such that P2 and P3 are satisfied. Let $g \in \mathbb{G}$. We have $\Pi_i(g) \leq \Pi_i(g^{N(g)})$ for $i \in N^-(g)$.*

Let g' be such that $n_j(g') \in \{n_i(g), n_i(g)+1\}$ and $n_k(g') = n_i(g)$for all $k \in N(g)\backslash\{j\}$. We have $\Pi_i(g) \leq \Pi_i(g') \leq \Pi_i(g^{N(g)})$, where the first inequality holds by $P2$ and the second by $P3$.

2. *Let $\Gamma \in \mathcal{G}$ be such that P2 and P3 are satisfied. Let $\widehat{\Pi} > \overline{\Pi}$. Let $g \in \mathbb{G}$ be such that $s_1(g) \geq \widehat{s}$. Then*

(i) $n_i(g) \geq s_1(g) - 1$ for all $i \in S_1(g)$

(ii) $n_i(g) \geq s_1(g)$ for all $i \in S_1(g)$ if $s_1(g) > \widehat{s}$ or $k(g) > s_1(g)$.

(i) By contradiction, suppose that $n_i(g) < s_1(g) - 1$ for some $i \in S_1(g)$. Without loss of generality, suppose $i \in \arg\min_{j \in S_1(g)} n_j(g)$. Let g' be such that $n_k(g') \in \{n_i(g), n_i(g)+1\}$ for $k \in S_1(g)$, $n_l(g') = n_i(g)$ for all $l \in S_1(g)\backslash\{k\}$ and $n_l(g') = 0$ for all $l \in S_2(g)$. We have $\Pi_i(g) \leq \Pi_i(g') < \Pi_i(g^{S_1(g)}) \leq \widehat{\Pi}$, where the first inequality holds by $P2$ and the second by $P3$. This contradicts $i \in S_1(g)$. Thus, $n_j(g) \geq s_1(g) - 1$ for all $j \in S_1(g)$.

(ii) By contradiction, suppose that $n_i(g) < s_1(g)$ for some $i \in S_1(g)$, which by (i) implies $n_i(g) = s_1(g) - 1$, and $i \in \arg\min_{j \in S_1(g)} n_i(g)$. We have $\Pi_i(g) \leq \Pi_i(g^{S_1(g)}) \leq \widehat{\Pi}$. The first inequality holds by $P2$, strictly if $k(g) > s_1(g)$, while the second inequality holds by definition of $\widehat{\Pi}$, strictly if $s_1(g) > \widehat{s}$. This contradicts $i \in S_1(g)$. Thus, $n_j(g) \geq s_1(g)$ for all $j \in S_1(g)$.

Proof of Proposition 2.

Let $G = \{g^S \mid s = \widehat{s}\}$.

($\Longleftarrow$) Suppose $\widehat{\Pi} > \overline{\Pi}$. We show that G satisfies internal and external stability.

Internal Stability

Let $g, g' \in G$. By contradiction, suppose $g \ll g'$. Let $g_0, g_1, ..., g_K$ be a sequence of networks going from $g_0 = g$ to $g_K = g'$ such that for each $t = 1, 2, ..., K$, coalition S_{t-1} can enforce the network g_t over g_{t-1}. Since $g' \neq g \cup h$ for some $h \subseteq g^{N\backslash N(g)}$, agents from $N(g)$ modify the current network at some point in the sequence. Let g_k be the first network in the sequence where $N(g) \cap S_k \neq \emptyset$. We have $\Pi_i(g_K) \leq \Pi_i(g_k) = \widehat{\Pi}$ for all $i \in N(g) \cap S_k$ since $K(g_k) = K(g)$, contradicting $g \ll g'$.

External Stability

Let $g' \notin G$. Let $g_0 = g'$. In g_0 and in the successive networks, let the agents who are not participating in the current network delete their links. Let $\widehat{g}$ be the network reached this way. Formally, for all $k \geq 0$, let $g_{k+1} = g_{k_{-N(g)\backslash K(g)}}$. Let $\widehat{g} = g_K$ where g_K satisfies $g_K = g_{K+1}$. By construction, $N(\widehat{g}) = K(\widehat{g})$. If $\widehat{g} \in G$, the proof stops here. Otherwise, let $g'' = \widehat{g}$ and go to the initial step.

Initial step: If $n^0(g'') \leq n - \widehat{s}$, go to step (i); if $n - \widehat{s} < n^0(g'') \leq n - \widetilde{s}$, go to step (ii) ; if $n - \widetilde{s} + 1 \leq n^0(g'') \leq \widehat{s}$, go to step (iii) and if $n^0(g'') \geq \widehat{s}$ go to step (iv).

Step (i): $n^0(g'') \leq n - \widehat{s}$

(i.a) If $s_1(g'') < \widehat{s}$, then let the agents in $T \subseteq S_2(g'') \cap N(g'')$ with $t = n - \widehat{s} + 1 - n^0(g'')$ delete their links, leading to $g''' = g''_{-T}$. We have $n^0(g''') \geq n - \widehat{s} + 1$.[21] Let $g'' = g'''$, and go to the initial step.

(i.b) If $s_1(g'') \geq \widehat{s}$, then $\{i\} \nleftrightarrow_{g''} S_2(g'')$ for each $i \in S_1(g'')$ by Lemma 2 since $g'' \notin G$. An agent from $S_1(g'')$ has at least one link with an agent in $S_2(g'')$. Let $g = g''_{-S_2(g'')}$.

(i.b.1) If $s_1(g) = \widehat{s}$ and $g^{S_1(g)} \nsubseteq g$, then in g'' let the agents from $S_2(g'')$ delete their links leading to the network g. In g, let the agents in $S_2(g)$ delete their links to reach the network $g''' = g_{-S_2(g)}$. Notice that $n_i(g''') < \widehat{s} - 1$ for $i \in \arg\min_{j \in S_1(g)} n_j(g''')$ since $N(g''') \subseteq S_1(g)$ and $g^{S_1(g)} \nsubseteq g$. It follows that $i \in S_2(g''')$. Let agent i delete his links to reach $g'''' = g'''_{-i}$. Each agent who deviates in a network in the sequence from g'' to g'''' cuts all his links. There are at least $n - \widehat{s} + 1$ unconnected agents in the network reached. Let $g'' = g''''$ and go to the initial step.

(i.b.2) If $s_1(g) = \widehat{s}$ and $g^{S_1(g)} \subseteq g$, then in g'' let the agents from $S_2(g'')$ delete their links but the link $i_1 i_2$ where $i_1 \in S_1(g)$ and $i_2 \in S_2(g'') \cap N(g'')$ leading to the network $g''' = g + i_1 i_2$. Then, let the agents from $S_2(g) \backslash \{i_2\}$ delete their links in order to reach the network $g'''' = g^{S_1(g)} + i_1 i_2$. Notice that $S_1(g'''') = \{i_1\}$. Then, let i_2 and $j \in S_1(g) \backslash \{i_1\}$ delete their links. The network reached this way is $g''''' = g^{S_1(g) \backslash \{j\}}$. Notice that $S_1(g''') \subseteq S_1(g) \cup \{i_2\}$ so that $S_2(g) \backslash \{i_2\} \subseteq S_2(g''') \backslash \{i_2\}$. Thus, the agents deleting a link in g''' have a payoff smaller than $\widehat{\Pi}$. Each agent who deviates in a network in the sequence from g'' to g''''' cuts all his links. There are $n - \widehat{s} + 1$ unconnected agents in the network reached. Let $g'' = g'''''$ and go to the initial step.

(i.b.3) If $s_1(g) \neq \widehat{s}$, then in g'' let the agents from $S_2(g'')$ delete their links leading to the network g. Let $g'' = g$ and go to the initial step.

Step (ii): $n - \widehat{s} < n^0(g'') \leq n - \widetilde{s}$

Let $i \in N^-(g'')$. We have $i \in S_2(g'')$ since $\Pi_i(g'') \leq \Pi_i(g^{N(g'')}) < \widehat{\Pi}$ where the first inequality holds by Lemma 1, and the second by definition of $\widehat{\Pi}$. Let agent i delete all his links leading to $g''' = g''_{-i}$. Let $g'' = g'''$ and go to the initial step.

Step (iii): $n - \widetilde{s} + 1 \leq n^0(g'') < \widehat{s}$

Let the agents from $N^0(g'')$ form a component where agent $k \in N^0(g'')$ has either $d-1$ or d links while each agent in $N^0(g'') \backslash \{k\}$ has d links, where $d = \min\{n_l(g''), n^0(g'') - 1\}$ for $l \in N^-(g'')$. Let g''' be the network reached this way.

(ii.a) If $\min\{n_l(g''), n^0(g'') - 1\} = n_l(g'')$, then $\Pi_l(g''') < \Pi_l(g^{N \backslash \{k\}}) \leq \widehat{\Pi}$. The first inequality holds by Lemma 1, and the second by definition of $\widehat{\Pi}$. In g''', let $\{l\} \cup N^0(g'')$ delete their links leading to $g'''' = g''_{-l}$. Thus, $n^0(g'''') \geq n^0(g'') + 1$. Then, let $g'' = g''''$ and go to the initial step.

[21] We have $n^0(g''') = n - \widehat{s} + 1$ if $N_i(g'') \nsubseteq T$ for all $i \in N(g'') \backslash T$. Otherwise, $n^0(g''') > n - \widehat{s} + 1$.

(ii.b) If $\min\{n_l(g''), n^0(g'') - 1\} = n^0(g'') - 1$, we have $\Pi_i(g''') \leq \Pi_i(\widetilde{g}^{N(g'')}) \leq \overline{\Pi} < \widehat{\Pi}$ for $i \in N^-(g'')$, where the first inequality holds by P2 and P3. In g''', let $\{i\} \cup N^0(g'')$ delete their links leading to $g'''' = g''_{-i}$. Thus, $n^0(g'''') \geq n^0(g'') + 1$. Let $g'' = g''''$ and go to the initial step.

Step (iv): $n^0(g'') \geq \widehat{s}$

Let $g^* = g''$. Let D be the set of agents who deviate in a network in the sequence from $\widehat{g}$ to g^*. By construction, $d \leq \widehat{s}$. In g^*, let the agents from D and $\widehat{s} - d$ agents from $N^0(g'')\backslash D$ form a completely connected component in order to reach g'''. In g''', the agents with less than $\widehat{s} - 1$ links do not participate. They delete their links, leading to the network $g'''' \in G$.

End: We have constructed a path from the network g' to some network in the set G satisfying indirect dominance. The set of unconnected agents is strictly larger after implementing the modifications of steps (i), (ii), or (iii). As a consequence, the algorithm reaches step (iv) with probability 1 if it passes through the initial step. If an agent modifies the network at some point in the sequence, he is either deleting links in a network in which he is not participating, or he has a payoff strictly smaller than $\widehat{\Pi}$ in the current network and is looking forward to get $\widehat{\Pi}$ in the end network.

($\Rightarrow$) Suppose $\widehat{\Pi} \leq \overline{\Pi}$. We show that G does not satisfy external stability. Take $g' = \widetilde{g}^S$ such that $\#S = \overline{s}$. By contradiction, suppose $g' \ll g$ for some $g \in G$. Let $g_0, g_1, ..., g_K$ be a sequence of networks going from $g_0 = g'$ to $g_K = g$ such that for each $t = 1, 2, ..., K$, coalition S_{t-1} can enforce the network g_t over g_{t-1}. Since $g' \nsubseteq g$, agents from S modify the current network at some point in the sequence. Let g_k be the first network in the sequence where $S \cap S_k \neq \emptyset$. We have $\Pi_i(g_K) = \widehat{\Pi} \leq \overline{\Pi} \leq \Pi_i(g_k)$ for all $i \in N(g) \cap S_k$ where the last inequality holds by negative externalities, contradicting $g' \ll g$.

□

3. *Let $\Gamma \in \mathcal{G}$ be such that P2 and P3 are satisfied. Let $g \in \mathbb{G}$. Let $i \in N^-(g)$. If $n_i(g) \geq n^0(g) - 1$, let $g' = g \cup g^{N^0(g)}$. If $n_i(g) < n^0(g) - 1$, let $g' = g \cup h$ where $h \subseteq g^{N^0(g)}$ such that $n_j(g') \in \{n_i(g) - 1, n_i(g)\}$ for some $j \in N^0(g)$ while $n_k(g') = n_i(g)$ for all $k \in N^0(g)\backslash\{j\}$. Then $\Pi_i(g') \leq \overline{\Pi}$, with strict inequality if $g' \neq \widetilde{g}^S$ for $s = \overline{s}$.*

(i) If $n_i(g) \geq n^0(g) - 1$, then $\Pi_i(g') \leq \Pi_i(g'') \leq \Pi_i(\widetilde{g}^{N(g)}) \leq \overline{\Pi}$, where $g'' = h'' \cup g^{N^0(g)}$, and $h'' \subseteq g^{N(g)}$ such that $n_j(g'') \in \{n_i(g) + 1, n_i(g)\}$ for some $j \in N(g)$ while $n_k(g'') = n_i(g)$ for all $k \in N(g)\backslash\{j\}$. The first inequality holds by $P2$ and the second by $P3$. If $n(g) \neq \overline{s}$, the last inequality holds strictly, while if $n(g) = \overline{s}$ and $g \neq g^{N(g)}$, then $n_i(g') < n(g) - 1$ and the second inequality holds strictly.

(ii) If $n_i(g) < n^0(g) - 1$, then $\Pi_i(g') \leq \Pi_i(g'') < \Pi_i(g^{N\backslash\{k\}}) \leq \overline{\Pi}$ for $k \neq i$, where $g'' = h'' \cup h$, and $h'' \subseteq g^{N(g)}$ such that $n_j(g'') \in \{n_i(g) + 1, n_i(g)\}$ for some $j \in N(g)$ while $n_k(g'') = n_i(g)$ for all $k \in N(g)\backslash\{j\}$. The first inequality holds by P2 and the second by $P2$ and $P3$.

4. *Let* $\Gamma \in \mathcal{G}$ *be such that* $P2$, $P3$ *and* $P4$ *are satisfied. Let* $g = g^S \cup h$ *where* $s = \overline{s}$ *and* $h \subsetneq g^{N\backslash S}$*. Then* $\Pi_i(g) < \underline{\Pi}$ *for* $i \in N^-(g)$*.*

Let $h' \subseteq g^{N(g)\backslash S}$ be such that $n_k(h') \in \{n_i(g), n_i(g)+1\}$ for some $k \in N(g)\backslash S$ and $n_j(h') = n_i(g)$ for all $j \in N(g)\backslash(S \cup k)$. Then $\Pi_i(g) \leq \Pi_i(g^S \cup h') \leq \Pi_i(g^S \cup g^{N(g)\backslash S}) \leq \Pi_i(g^S \cup g^{N\backslash S})$, where the first inequality holds by $P2$, the second by $P3$ and the third by $P4$. Since $h \subsetneq g^{N\backslash S}$, at least one inequality holds strictly.

Proof of Proposition 3

Let $G = \{g \subseteq g^N \mid g = \widetilde{g}^S \text{ such that } \#S = \overline{s}\}$.

($\Longleftarrow$) Suppose $\overline{\Pi} > \widehat{\Pi}$. We show that G satisfies internal and external stability.

Internal Stability

Let $g, g' \in G$. Let $g = \widetilde{g}^S$ with $s = \overline{s}$. By contradiction, suppose $g \ll g'$. Let $g_0, g_1, ..., g_K$ be a sequence of networks going from $g_0 = g$ to $g_K = g'$ such that for each $t = 1, 2, ..., K$, coalition S_{t-1} can enforce the network g_t over g_{t-1}. Since $N_i(g') \nsubseteq N_i(g)$ for some $i \in S$, agents from S modify the network at some point in the sequence. Let g_k be the first network in the sequence where $S \cap S_k \neq \emptyset$. We have $g_k = g^S \cup h$ where $h \subseteq g^{N\backslash S}$. Thus, $\Pi_i(g_K) \leq \Pi_i(g_k)$ for all $i \in S \cap S_k$ by P2, contradicting $g \ll g'$.

External Stability

Let $g' \notin G$. We construct a sequence of networks going from $g_0 = g'$ to $g_K = g \in G$ such that for each $t = 1, 2, ..., K$, coalition S_{t-1} can enforce the network g_t over g_{t-1} and $\Pi_i(g_T) > \Pi_i(g_{t-1})$ for all $i \in S_{t-1}$. Let $\gamma(g) \in \mathbb{G}$ be the unique network reached from a given network g by successively deleting all the links of the agents with a payoff strictly smaller than $\overline{\Pi}$. Formally, let $g_0 = g$ and for all $k \geq 0$, let $g_{k+1} = g_{k_{-S_2(g_k)}}$. We have $\gamma(g) = g_K$ where g_K satisfies $g_K = g_{K+1}$. If $s_1(\gamma(g')) \leq n - \overline{s}$ go to the step (i), if $n - \overline{s} < s_1(\gamma(g')) \leq \overline{s}$ go to the step (ii), if $s_1(\gamma(g')) = \overline{s}$ go to the step (iii), (iv) or (v), and if $s_1(\gamma(g')) > \overline{s}$ go to the step (vi).

(i) $s_1(\gamma(g')) \leq n - \overline{s}$. Let $g_0 = g'$. For all $k \geq 0$, let $g_{k+1} = g_{k_{-S_2(g_k)}}$. Let L be the smallest integer such that $s_1(g_{L+1}) \leq n - \overline{s}$. For $k = 0, 1, ..., L-1$, let the agents in $S_2(g_k)$ successively delete their links, leading to the network g_L. Let the agents from $T \subseteq S_2(g_{L+1})\backslash S_2(g_L)$, where $t = \overline{s} - s_2(g_L)$ delete their links in g_L. Then let the agents from T and $S_2(g_L)$ form a strongly connected component, leading to the network $g'' = g^S \cup g'_{-S}$ where $S = T \cup S_2(g_L)$. Let $g' = g''$ and go to step (iii).

(ii) $n - \overline{s} < s_1(\gamma(g')) < \overline{s}$. For $k = 0, 1, 2, ...$, let the agents in $S_2(g_k)$ successively delete their links, leading to the network $g'' = \gamma(g')$. Then add links between agents in $N^0(g'')$ in order to build the network g''' where $n_j(g''') \in \{d-1, d\}$ for $j \in N^0(g'')$ while $n_k(g''') = d$ for all $k \in N^0(g'')\backslash\{j\}$, where $d = min\{n^0(g'') - 1, n_l(g'')\}$ for $l \in N^-(g'')$. We have $\Pi_l(g''') < \overline{\Pi}$ by

Lemma 3. Since $n_k(g''') \leq n_l(g''')$ for all $k \in N^0(g'')$, we have $\{i\} \cup N^0(g'') \subseteq S_2(g''')$. Let agent i and those in $N^0(g'')$ delete their links to reach $g'''' = g''_{-i}$. Let $g' = g''''$. If $s_1(\gamma(g')) \leq n - \overline{s}$, go to step (i) while if $n - \overline{s} < s_1(\gamma(g')) < \overline{s}$, repeat step (ii).

(iii) $s_1(\gamma(g')) = \overline{s}$, $g' = g^S \cup h$ where $s = \overline{s}$ and $h \subseteq g^{N\setminus S}$. In g' and in the successive networks, the agent with fewer links delete his links until g^S is reached. Let $g_0 = g'$. Let $g_{k+1} = g_{k_{-i}}$ where $i \in N^-(g_k)\setminus S$. We have $\Pi_i(g_k) < \underline{\Pi}$ by Lemma 4. Let g_L be such that $g_L = g_{L+1}$. By construction, $g_L = g^S$. Then, the agents in $N\setminus S$ add each link between them leading to the end network $\widetilde{g}^S$, and go to the end step (vii).

(iv) $s_1(\gamma(g')) = \overline{s}$, $g' = g^S \cup h$ where $s = \overline{s}$ and $h \nsubseteq g^{N\setminus S}$. Then $i_1 i_2 \in g'$ for $i_1 \in S$, $i_2 \in N\setminus S$. In g' and in the successive networks, the agents with a payoff smaller than $\overline{\Pi}$ delete their links but the link $i_1 i_2$ in order to reach $g'' = g^S + i_1 i_2$. Let $g_0 = g'$. Let $g_{k+1} = g_{k_{-S_2(g_k)}} + i_1 i_2$. Let L be such that $g_{L+1} = g_L$. By construction, $g_L = g^S + i_1 i_2$, and each agent cutting a link in the path from g' to g_L has a payoff smaller than $\overline{\Pi}$. In $g'' = g^S + i_1 i_2$, agents from $N\setminus S$ add each link between them leading to $g''' = \widetilde{g}^S + i_1 i_2$. By negative externalities, we have $N\setminus\{i_1\} \subseteq S_2(g''')$. Let the agents from $N\setminus S$ and those from $T \subseteq S_2(g''')\setminus(N\setminus S)$ where $t = 2\overline{s} - n$ delete their links, and then add each link between them, leading to the network $g'''' = g^{(N\setminus S)\cup T} \cup g'_{-(N\setminus S)\cup T}$. Let $g' = g''''$ and go to step (iii).

(v) $s_1(\gamma(g')) = \overline{s}$ and $\gamma(g') \neq g^S$. In g' and in the successive networks, let the agents with a payoff smaller than $\overline{\Pi}$ delete their links, leading to the formation of the network $g'' = \gamma(g')$. Notice that $n_l(g'') < \overline{s} - 1$ for $l \in N^-(g'')$ since $g'' \neq g^S$ and $n(g'') = \overline{s}$. Then, let the agents from $S_2(g'')$ add links between them in order to build the network g''' where $n_j(g''') \in \{d-1, d\}$ for $j \in S_2(g'')$ while $n_k(g''') = d$ for all $k \in S_2(g'')\setminus\{j\}$, where $d = min\{s_2(g'') - 1, n_l(g'')\}$. We have $\Pi_l(g''') < \overline{\Pi}$ by Lemma 3. Let $g' = g'''$. If $s_1(\gamma(g')) \leq n - \overline{s}$, go to step (i) while if $n - \overline{s} < s_1(\gamma(g')) < \overline{s}$, go to step (ii).

(vi) $s_1(\gamma(g')) > \overline{s}$. In g' and in the successive networks, let the agents with a payoff smaller than $\overline{\Pi}$ delete their links, leading to the formation of the network $g'' = \gamma(g')$. Let $l \in N^-(g'')$. Then, let the agents from $S_2(g'')$ add links between them in order to build the network g''' where $n_j(g''') \in \{d-1, d\}$ for $j \in S_2(g'')$ while $n_k(g''') = d$ for all $k \in S_2(g'')\setminus\{j\}$, where $d = min\{s_2(g'') - 1, n_l(g'')\}$. We have $\Pi_l(g''') < \overline{\Pi}$ by Lemma 3.

(vi.a) If $s_1(\gamma(g''')) > \overline{s}$, in g''' and in the successive networks, let the agents with a payoff smaller than $\overline{\Pi}$ delete their links, leading to the formation of the network $\gamma(g''')$. Let $g' = \gamma(g''')$ and repeat step (vi).

(vi.b) If $s_1(\gamma(g''')) = \overline{s}$ and $\gamma(g''') = g^S$ for $s = \overline{s}$, then $lj \in g'''$ for $j \in S$.[22] In g''' and in the successive networks, let the agents with a payoff smaller than $\overline{\Pi}$ delete their links but the link

[22]Notice that $n_k(g'') \geq \overline{s} - 1$ for all $k \in S$ since $\gamma(g''') = g^S$ and $d_l(g'') \leq d_j(g'')$ for all $j \in S_1(g'')$ since $l \in N^-(g'')$. If we instead had $N_j(g'') \cap S = \emptyset$, we would have $n_l(g'') \leq n - \overline{s} - 1 - n^0(g'')$. But then $\Pi_l(g'') \leq \Pi_l(g^S \cup g^{N\setminus(S\cup N^0(g''))}) \leq \pi_l(\widetilde{g}^S) \leq \underline{\Pi}$, where the first inequality holds by P2 and the second holds by P4. This then contradicts $i \in S_1(g'')$.

lj, leading to the formation of the network $g'''' = g^S + ij$. Then let $g' = g''''$ and go to step (iv).

(vi.c) If $s_1(\gamma(g''')) = \overline{s}$ and $\gamma(g''') \neq g^S$ for $s = \overline{s}$. In g''' and in the successive networks, let the agents with a payoff smaller than $\overline{\Pi}$ delete their links, leading to the formation of the network $\gamma(g''')$. Then let $g' = \gamma(g''')$ and go to step (v).

(vi.d) If $s_1(\gamma(g''')) < \overline{s}$, let $g' = g'''$ and go to step (i) if $s_1(\gamma(g')) \leq n - \overline{s}$, or to step (ii) if $n - \overline{s} < s_1(\gamma(g')) < \overline{s}$.

(vii) End: The algorithm describes a sequence by which the network g' is indirectly dominated by the network $\widetilde{g}^S \in G$. With probability one, the algorithm reaches the end step (vii). In step (ii), each deviating agent has a payoff smaller than $\underline{\Pi}$ at the network where he deviates, and get $\underline{\Pi}$ in $\widetilde{g}^S$. In the others steps, the deviating agents have strictly less than $\overline{\Pi}$ when they modify the network, and get $\overline{\Pi}$ in $\widetilde{g}^S$. We thus have $g' \ll \widetilde{g}^S$.

($\Rightarrow$) Suppose $\overline{\Pi} \leq \widehat{\Pi}$. We show that G does not satisfy external stability. Take $g' = g^S$ such that $\#S = \widetilde{s}$. By contradiction, suppose $g' \ll g$ for some $g \in G$. Let $g_0, g_1, ..., g_K$ be a sequence of networks going from $g_0 = g'$ to $g_K = g$ such that for each $t = 1, 2, ..., K$, coalition S_{t-1} can enforce the network g_t over g_{t-1}. Since $g' \nsubseteq g$, agents from T modify the current network at some point in the sequence. Let g_k be the first network in the sequence where $S \cap S_k \neq \emptyset$. We have $\Pi_i(g_K) = \overline{\Pi} \leq \widehat{\Pi} = \Pi_i(g_k)$ for all $i \in N(g) \cap S_k$ where the last equality holds since $K(g_k) = S$, contradicting $g' \ll g$.

□

Appendix C - Proofs of Section 6

Proof of Lemma 4.

Let $g' \in S^*(g, i, j)$. Let g'' be such that $n_i(g'') = n_i(g)$ for all $i \in K(g)$ and $n_j(g'') = n_j(g')$ for all $j \in N \backslash K(g)$. We have $W(g) = W(g'')$ since $K(g) = K(g'')$ and $n_i(g) = n_i(g'')$ for all $i \in K(g)$. By P5 and P6, we have $W(g') > W(g'')$. Thus, $W(g') > W(g)$.

□

We now introduce some lemmas that are useful to establish the proofs of Proposition 5, 6, and 7.

Lemma 5 shows that when two agents do not have the same number of links in a network, there is another network where the agent with more links in the initial network looses one partner in favour of the less connected agent.

Lemma 5. *Let $g \in \mathbb{G}$ be such that $n_i(g) > n_j(g)$. Then, $\exists g' \in \mathbb{G}$ such that $n_i(g') = n_i(g) - 1$, $n_j(g') = n_j(g) + 1$ and $n_k(g') = n_k(g)$ for all $k \neq i, j$.*

Proof. We have $l \in N$ such that $il \in g$ and $jl \notin g$ since $n_i(g) > n_j(g)$. Let $g' = g - il + jl$.

□

As a corollary of Lemma 5, when the sum of the number of partners of a group of agents is a multiple of the number of those agents in a given network, then there is another network where they all have the same number of partners, while the number of partners of the remaining agents remains unchanged.

Corollary 1. *Let* $g \in \mathbb{G}$ *be such that* $\sum_{i\in K} n_i(g) = ak$ *for some* $a \in \mathbb{N}$ *and* $K \subseteq N$*. Then, there is* $g' \in \mathbb{G}$ *such that (i)* $n_i(g') = a$ *for all* $i \in K$ *and (ii)* $n_i(g') = n_i(g)$ *for all* $i \in N \backslash K$*.*

Lemma 6 shows that the neighborhood of strong agents is nested in a network $g \in \overline{G}$.

Lemma 6. *Let* $g \in \overline{G}$*. Let* $i, j \in K^+(g)$ *with* $n_i(g) \geq n_j(g)$*. Then,* $N_j(g) \subseteq N_i(g)$*.*

Proof. Suppose on the contrary that $N_j(g) \nsubseteq N_i(g)$. Then, there exists an agent $k \in N$ such that $jk \in g$ but $ik \notin g$. It follows that $g + ik - jk \in S^*(g, i, j)$, contradicting $g \in \overline{G}$.

□

As a Corollary of Lemma 6, an agent connected to a strong agent in a network $g \in \overline{G}$ is also connected to each other agent with at least the same number of links.

Corollary 2. *Let* $g \in \overline{G}$*. Let* $ij \in g$ *where* $i \in K^+(g)$ *and* $j \in N$*. Then,* $jk \in g$ *for all* $k \in N$ *such that* $n_k(g) \geq n_i(g)$

Lemma 7 shows that if two participating agents i, j are connected to non-participating agents in a network $g \in \overline{G}$, then each pair of agents k, l where the degree of k is higher than that of i and the degree of l is higher than that of j is connected in the network g.

Lemma 7. *Let* $g \in \overline{G}$ *such that* $e_1 i_1, e_2 i_2 \in g$ *for* $e_1, e_2 \in E(g)$*,* $i_1, i_2 \in K(g)$*. Then,* $ij \in g$ *if* $n_i(g) \geq n_{i_1}(g)$ *and* $n_j(g) \geq n_{i_2}(g)$*.*

Proof. Suppose $e_1 i_1, e_2 i_2 \in g$ but $ij \notin g$ for some $i, j \in N$ such that $n_i(g) \geq n_{i_1}(g)$ and $n_j(g) \geq n_{i_2}(g)$. Let $g' = g + ij - e_1 i_1 - e_2 i_2$. By Lemma 5, $\exists g''$ such that $n_k(g'') = n_k(g)$ for all $k \in K(g)$ and $n_k(g'') = n_k(g')$ for all $k \in E(g)$. It follows that $g'' \in C^-(g)$, contradicting $g \in \overline{G}$.

□

Proof of Proposition 5

Let $g \in \overline{G}_1$. By definition of $\overline{G}_1$, we have $n_i(g) = n_j(g)$ for all $i, j \in K(g)$. We show that either $K(g) \top_g K(g)$ and $E(g) \bot_g E(g)$ so that g is a core-periphery network, or $\#\{ie \in g \mid i \in K(g) \text{ and } e \in E(g)\} \leq 1$ so that g is a quasi regular network. We discuss two cases: one where the degree of each participating agent is greater than $\#K(g) - 1$, and the other where it is smaller.

Case 1: $n_i(g) \geq \#K(g) - 1$ for $i \in K(g)$.

We show that in this case $K(g) \top_g K(g)$. Suppose on the contrary that $i_1 i_2 \notin g$ for some $i_1, i_2 \in K(g)$. Since $n_{i_1}(g) = n_{i_2}(g) \geq \#K(g) - 1$, we have $i_1 e_1, i_2 e_2 \in g$ for some $e_1, e_2 \in E(g)$. Thus, $g + i_1 i_2 - i_1 e_1 - i_2 e_2 \in C^-(g)$, contradicting $g \in \overline{G}$.

Case 2: $n_i(g) < \#K(g) - 1$ for $i \in K(g)$

We show that in this case $\#\{ie \in g \mid i \in K(g) \text{ and } e \in E(g)\} \leq 1$. On the contrary, suppose $i_1 e_1, i_2 e_2 \in g$ where $i_1, i_2 \in K(g)$ and $e_1, e_2 \in E(g)$. Since $n_{i_1}(g) < \#K(g) - 1$, $i_1 i_3 \notin g$ for some $i_3 \in K(g)$. Let $g' = g + i_1 i_3 - i_1 e_1 - i_2 e_2$. By Corollary 1, $\exists g''$ such that $n_i(g'') = n_i(g)$ for all $i \in K(g)$ and $n_i(g) = n_i(g')$ for all $i \in E(g)$. We have $g'' \in C^-(g)$, contradicting $g \in \overline{G}$.

□

Proof of Proposition 6.

Let $g \in \overline{G}_2$. Let $i_1 \in K^+(g)$ be such that $n_{i_1}(g) \leq n_i(g)$ for all $i \in K^+(g)$. Let $j_1 \in K_m(g)$ be such that $\#(N_{j_1}(g) \cap K^+(g)) \leq \#(N_{j_k}(g) \cap K^+(g))$. We show that (i) g is a nested split graph if $K_m(g) \perp_g N \backslash K^+(g)$, (ii) $K^+(g) \backslash \{i_1\} \top_g K(g)$ if $K_m(g) \leftrightarrow K_m(g)$, and (iii) $K^+(g) \backslash \{i_1\} \top_g N$ if $K_m(g) \leftrightarrow E(g)$.

(i) Suppose $K_m(g) \perp_g N \backslash K^+(g)$. We thus have $N \backslash K^+(g) \perp_g N \backslash K^+(g)$ since $E(g) \perp_g E(g)$. By Corollary 2, if $k \in N_i(g)$ for some $i \in N \backslash K^+(g)$, then $k' \in N_i(g)$ for all k' such that $n_{k'}(g) \geq n_k(g)$. Thus, g is a nested split graph.

(ii) Suppose $j_3 j_4 \in g$ with $j_3, j_4 \in K_m(g)$. We show that $K_m(g) \top_g K^+(g) \backslash \{i_1\}$.

(ii.2.a) $K_m(g) \backslash \{j_1\} \top_g K^+(g) \backslash \{i_1\}$

Suppose on the contrary that $i_2 j_2 \notin g$ for some $i_2 \in K^+(g) \backslash \{i_1\}$ and $j_2 \in K_m(g) \backslash \{j_1\}$. Then $i_2 j_1 \notin g$ by Corollary 2. Let $g' = g + i_2 j_1 + i_2 j_2 - i_1 i_2 - j_3 j_4$. By Lemma 5, $\exists g''$ such that $n_k(g'') = n_k(g')$ for all $k \in K^+(g)$, and $n_k(g'') = n_k(g)$ for all $k \in N \backslash K^+(g)$. We have $g'' \in S(g, i_2, i_1)$, contradicting $g \in \overline{G}_1$.

(ii.2.b) $\{j_1\} \top_g K^+(g) \backslash \{i_1\}$.

Notice that $i_1 x_1 \in g$ for some $x_1 \in N \backslash K^+(g)$ since $n_{i_1}(g) > n_j(g) \geq \#K^+(g) - 1$. Suppose on the contrary that $i_2 j_1 \notin g$ for some $i_2 \in K^+(g) \backslash \{i_1\}$. Let $g' = g + i_2 j_1 - i_1 j_4$. By Lemma 5, there is g'' such that $n_k(g'') = n_k(g')$ for $k \in K^+(g)$ and $n_k(g'') = n_k(g)$ for $k \in N \backslash K^+(g)$. We have $g'' \in S(g, i_2, i_1)$, contradicting $g \in \overline{G}_1$.

(iii) Suppose $j_4 e_4 \in g$ with $j_4 \in K_m(g)$ and $e_4 \in E(g)$. We show that $K^+(g) \backslash \{i_1\} \top_g N$.

We consider first the case where $\#K_m(g) = 1$ and then $\#K_m(g) > 1$.

Case (iii.1). $K_m(g) = \{j_4\}$.

We show that $K(g) \top_g K(g)$ and $E(g) \top_g K^+(g) \backslash \{i_1\}$ for some $i_1 \in K^+(g)$.

(iii.1.a) $\{j_4\} \top_g K^+(g) \backslash \{i_1\}$.

Suppose on the contrary that $i_2 j_4 \notin g$ for some $i_2 \in K^+(g) \backslash \{i_1\}$. Then $i_1 j_4 \notin g$ by Corollary 2. It follows that $\{i_1, i_2\} \perp_g E(g)$. Indeed, if we had $\{i_1, i_2\} \leftrightarrow_g E(g)$, say $i_1 e_1 \in g$ for some $e_1 \in E(g)$, we would have $g - i_1 e_1 - j_4 e_4 + i_1 j_4 \in C^-(g)$ a contradiction. Let $g' = g + i_2 j_4 + i_2 e_4 - i_1 i_2 - j_4 e_4$. We have $g' \in S(g, j_1, i_1)$, contradicting $g \in \overline{G}$.

(iii.1.b) $i_1 j_4 \in g$

Suppose on the contrary that $i_1 j_4 \notin g$. It follows that $i_1 e_1 \in g$ for some $e_1 \in E(g)$, as we would otherwise have $n_{i_1}(g) = \#K^+(g) - 1 < n_j(g)$. Let $g' = g + i_1 j_4 - i_1 e_1 - j_4 e_4$. We have $g' \in C^-(g)$, contradicting $g \in \overline{G}$.

(iii.1.c) $E(g) \top_g K^+(g) \backslash \{i_1\}$

We have $n_j(g) \geq \#K^+(g) + 1$ since $\{j_4\} \top_g K^+(g)$ and $j_4 e_4 \in g$. Since $n_{i_1}(g) > n_{j_4}(g)$, we have $i_1 e_1 \in g$ for some $e_1 \in E(g)$, which in turn implies $\{e_1\} \top_g K^+(g)$ by Corollary 2. We thus have $E(g) \top_g K^+(g) \backslash \{i_1\}$. To see this, suppose on the contrary that $i_2 e_2 \notin g$ for some $i_2 \in K^+(g) \backslash \{i_1\}$, $e_2 \in E(g)$. As $n_{e_1}(g) > n_{e_2}(g)$, we have $g + i_2 e_2 - i_1 e_1 \in S^*(g, i_2, i_1)$, contradicting $g \in \overline{G}$.

Case (iii.2). Suppose $\#K_m(g) > 1$.

(iii.2.a) $K_m(g) \top_g K^+(g) \backslash \{i_1\}$

The proof of this step is similar to the proof of step (ii.2) by replacing j_3 by e_4.

(iii.2.b) $E(g) \top_g K^+(g) \backslash \{i_1\}$.

Suppose on the contrary that $i_2 e_2 \notin g$ for some $i_2 \in K^+(g) \backslash \{i_1\}$ and $e_2 \in E(g)$. We show in cases (iii.2.b.1) and (iii.2.b.2) that it leads to a contradiction when $\{i_1\} \top_g K_m(g)$ holds or not respectively.

Case (iii.2.b.1) Suppose $K^+(g) \top_g K_m(g)$. We then consider the cases where $K_m(g) \top_g K_m(g)$ holds or not.

Case (iii.2.b.1.1°) Suppose $K_m(g) \top_g K_m(g)$.

Notice that $n_{i_1}(g) > n_{j_4}(g) \geq \#K(g)$ implies $i_1 e_1 \in g$ for some $e_1 \in E(g)$. Let $g' = g + i_2 e_2 - i_1 e_1$. If $n_{e_2}(g') \leq n_{e_1}(g')$, let $g'' = g'$. Otherwise if $n_{e_2}(g') > n_{e_1}(g')$, $\exists g''$ such that $n_k(g'') = n_k(g')$ for $k \in K^+(g)$ and $n_k(g'') = n_k(g)$ for $k \in N \backslash K^+(g)$ by Lemma 5. We have $g'' \in S^*(g, i_2, i_1)$, contradicting $g \in \overline{G}$.

Case (iii.2.b.1.2°) Suppose $j_2 j_3 \notin g$ for some $j_2, j_3 \in K_m(g)$

We have $i_1 j_2 \in g$ since $K^+(g) \top_g K_m(g)$. Let $g' = g - i_1 j_2 + i_2 e_2 - j_4 e_4 + j_2 j_3$. If $n_{e_2}(g') \leq n_{e_4}(g')$, let $g'' = g'$. Otherwise if $n_{e_2}(g') > n_{e_4}(g')$, $\exists g''$ such that $n_k(g'') = n_k(g')$ for all $k \in K^+(g)$ and $n_k(g'') = n_k(g)$ for all $k \in N \backslash K^+(g)$ by Lemma 5. We have $g'' \in S^*(g, i_2, i_1)$, contradicting $g \in \overline{G}$.

Case (iii.2.b.2) Suppose $i_1 j_2 \notin g$ for some $j_2 \in K_m(g)$.

(iii.2.b.2.1°) $\{i_1\}\bot_g E(g)$

Suppose on the contrary that $i_1e_1 \in g$ for some $e_1 \in E(g)$. Then, let $g' = g - i_1e_1 - j_4e_4 + i_1j_2$. By Lemma 5, $\exists g''$ such that $n_k(g'') = n_k(g')$ for all $k \in E(g)$ and $n_k(g'') = n_k(g)$ for all $k \in K(g)$. We have $g'' \in C^-(g)$, contradicting $g \in \overline{G}$.

(iii.2.b.2.2°) $i_1 \leftrightarrow K_m(g)$ and we do not have $K_m(g)\top K_m(g)$ since we would otherwise have $n_{i_1}(g) \leq n_{j_4}(g)$.

(iii.2.b.2.3°) $E(g)\top_g K^+(g)\backslash\{i_1\}$

By assumption, we have $j_4e_4 \in g$. By contradiction, we have assumed $i_2e_2 \notin g$. From (iii.2.b.2.2°), we have $i_1j_2 \in g$, and $j_3j_5 \notin g$ for some $j_2, j_3, j_5 \in K_m(g)$ Let $g' = g - i_1j_2 + i_2e_2 - j_4e_4 + j_3j_5$. If $n_{e_2}(g') \leq n_{e_4}(g')$, let $g'' = g'$. Otherwise, if $n_{e_2}(g') > n_{e_4}(g')$, $\exists g''$ such that $n_k(g'') = n_k(g')$ for all $k \in K^+(g)$ and $n_k(g'') = n_k(g)$ for all $k \in N\backslash K^+(g)$ by Lemma 5. We have $g'' \in S^*(g, i_2, i_1)$, contradicting $g \in \overline{G}$.

□

Proof of Proposition 7.

Let $g \in \overline{G}_3$. Let $K^+(g) = \{i_1, i_2, ..., i_a\}$ be such that $n_{i_1}(g) \geq n_{i_2}(g) \geq ... \geq n_{i_a}(g)$. Let $K_m(g) = \{j_1, j_2, ..., j_b\}$ and let $E(g) = \{e_1, e_2, ..., e_c\}$ be such that $n_{e_1}(g) \geq n_{e_2}(g) \geq ... \geq n_{e_c}(g)$. Notice that $i_ai_{a-1} \notin g$ since $g \in \overline{G}_3$. We decompose the proof in four steps. We show that (i) $\{i_a\}\bot_g E(g)$, (ii) $\{i_a\}\bot_g K_m(g)$, (iii) $K_m(g)\bot_g(K_m(g) \cup E(g))$, and we conclude in step (iv) that g is a nested split graph: $N_{e_c}(g) \subseteq N_{e_{c-1}}(g) \subseteq .... \subseteq N_{e_1}(g) \subseteq N_{j_1}(g) = N_{j_2}(g) = ... = N_{j_b}(g) \subseteq N_{i_a}(g) \subseteq N_{i_{a-1}}(g) \subseteq ... \subseteq N_{i_1}(g)$.

(i) $\{i_a\}\bot_g E(g)$

Suppose on the contrary that $i_ae_k \in g$ for some $e_k \in E(g)$. We would then have $i_{a-1}e_k \in g$ by Corollary 2. Let $g' = g + i_ai_{a-1} - i_ae_k - i_{a-1}e_k$. We have $g' \in C^-(g)$, contradicting $g \in \overline{G}$.

(ii) $\{i_a\}\bot_g K_m(g)$

Suppose on the contrary that $i_aj_x \in g$ for some $j_x \in K_m(g)$. Then $\{j_x\}\top_g K^+(g)$ by Corollary 2. Since $n_{i_a}(g) > n_{j_x}(g)$ and $\{i_a\}\bot_g E(g)$ by (i), we have $i_aj_y \in g$ and $j_xj_z \notin g$ where $j_y, j_z \in K_m(g)$, $j_y \neq j_x$. Let $g' = g + i_ai_{a-1} - i_aj_x - i_aj_y + j_yj_z$. By Corollary 1, $\exists g''$ such that $n_i(g'') = n_i(g')$ for all $i \in N\backslash K_m(g)$, and $n_i(g'') = n_i(g)$ for all $i \in K_m(g)$. We have $g'' \in S(g, i_{a-1}, i_a)$, contradicting $g \in \overline{G}$.

(iii) $K_m(g)\bot_g(K_m(g) \cup E(g))$

Suppose on the contrary that $jx \in g$ for some $j \in K_m(g)$, $x \in K_m(g) \cup E(g)$. Since $n_{i_a}(g) > n_j(g)$ and $i_a\bot_g K_m(g) \cup E(g)$ by (i) and (ii), we have $i, i' \in K^+(g)$ such that $i_ai, i_ai' \in g$ and $ji, ji' \notin g$. Suppose without loss of generality that $n_i(g) \geq n_{i'}(g)$. Let $g' = g - jx - i_ai' + i_ax + ji$. We have $g' \in S(g, i, i')$, contradicting $g \in \overline{G}$.

(iv) $N_{k_c}(g) \subseteq N_{k_{c-1}}(g) \subseteq .... \subseteq N_{k_1}(g) \subsetneq N_{j_1}(g) = N_{j_2}(g) = ... = N_{j_b}(g) \subsetneq N_{i_a}(g) \subseteq N_{i_{a-1}}(g) \subseteq ... \subseteq N_{i_1}(g)$

We have shown that $N \backslash K^+(g) \perp_g N \backslash K^+(g)$. The result thus follows by Corollary 2 since $n_{i_1}(g) \geq n_{i_2}(g) \geq ... \geq n_{i_a}(g) > n_{j_1}(g) = ... = n_{j_b}(g) > n_{e_1}(g) \geq n_{e_2}(g) \geq ... \geq n_{e_c}(g)$.

□