Université Catholique de Louvain
Faculté des Sciences Economiques, Sociales et Politiques
Départementes des Sciences Economiques
Center for Operations Research and Econometrics

Essays on Nash Equilibrium Refinements

Federico De Stefano Grigis

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Composition du jury:

Promoteurs:
Prof. J.F. Mertens (Université Catholique de Louvain)
Prof. Vincent Vannetelbosch (Université Catholique de Louvain)
Lecteurs:
Prof. Claude D'Aspremont (Université Catholique de Louvain)
Prof. Pierre Dehez (Université Catholique de Louvain)
Prof. Enrico Minelli (Universitá degli Studi di Brescia)

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To my family

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A volte il fumo
é meglio dell'arrosto.

R. F. Antoni

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General introduction

Since the seminal contributions by Selten [1975] and Kreps and Wilson [1982], the theory of Nash equilibrium refinements considered just single valued solution concepts.

Both perfectness and sequentiality focused just on equilibria that can be obtained by a process of backwards induction that is by requiring that, at every point in any play of the game, each player believes that his equilibrium strategy will maximize his expected payoff in the reminder of the game. Consistency with backwards induction is thus a natural requirement for strategic stability. Unfortunately it is not a sufficient condition since it is not always invariant with respect to arbitrary details of the extensive form of the game. This flaw would lead naturally to properness (Myerson [1978]) since a proper equilibrium of a normal form game is sequential in any tree with that normal form. However, Kohlberg and Mertens [1986] proved that not even properness is immune to a forward induction argument or to deletion of strictly dominated strategies. Even more interestingly Kohlberg and Mertens [1986] proved that no single valued equilibrium refinement could conform with a natural definition of strategic stability. This result led the authors to modify the traditional approach and to consider set valued instead of single valued solution concepts.

The first chapter of the thesis is inspired by this final result and tries to investigate the relation between single and set valued solution concepts. The first aim is to evaluate whether properness could really be considered the most advanced single valued Nash equilibrium refinement. Then the more ambitious goal is to improve on properness and define a new single valued refinement that, even if necessarily fails to satisfy one or more desirable properties, at least succeeds in being included in a strategically stable set of equilibria. Unfortunately, while the new concept of *refined equilibria* amend some flaws of proper equilibria, the desired goal has not been reached.

It is interesting to note that the main result in Kohlberg and Mertens [1986] coincides with the *pars destruens* of their general argumentation: they proved that no single valued equilibrium could be considered strategically stable but none of the proposed definitions of stable sets could actually be regarded as strategically stable. Just Mertens [1989] and Mertens [1991] devised the *pars construens* that is a solution concept that satisfies all the desirable properties that define strategic stability. To get the result Mertens [1989] preserved the family of perturbations proposed in Kohlberg and Mertens [1986] to define stable sets but imposed a specific geometric requirement a set of equilibria should satisfy to be defined M-stable. Therefore it would be interesting to evaluate if Mertens' approach was necessary or if a slight modification of the setting proposed in Kohlberg and Mertens [1986] were enough to define a stable set. This is the core of chapter 2 that proposes a new approach that adopts a wider collection of perturbations as the main tool to get a satisfactory definition. This represents the main contribution of this thesis since it proves that the model proposed in Kohlberg and Mertens [1986] just needed the definition of an additional set of perturbations to get strategic stability.

Finally Chapter 3 is devoted to an extension of strategic stability to correlated equilibria in cooperative games. This development was part of Mertens' program since his seminal contribution to the theory of refinements in cooperative games (Mertens and Dhillon [1996]).

However the approach adopted in this thesis is technically closer to the one proposed in Myerson [1986] and the connection with Mertens' stability is confined to the attempt of devising a model consistent with the properties defining strategic stability as defined in Mertens [1989]. Also in this case, the devised definition of stable correlated equilibria improves on existing refinements but does not satisfy all the desirata.

Chapter 1

Is properness the last word in single valued equilibria?

1.1 Introduction

In their seminal paper Kohlberg and Mertens [1986] listed some desirable properties defining, albeit incompletely, the concept of strategic stability:

Existence: every game has at least one strategically stable set;

Connectedness: every strategically stable set is connected;

Backwards induction: a strategically stable set of a tree contains a backwards induction equilibrium of the tree;

Invariance: a strategically stable set of a game is also a strategically stable set of any game with the same reduced normal form;

Admissibility: the players' strategies are undominated at any point in a strategically stable set;

Iterated dominance and forward induction: a strategically stable set of the game Γ contains the strategically stable set of any game $\tilde{\Gamma}$ obtained from Γ by deleting a strategy that is either dominated or that is always an inferior response in every equilibrium of the set.

These properties, while defined for a set valued solution, would apply to a single valued Nash equilibrium refinement except that a strategically stable equilibrium should always conform with backwards induction and should be invariant with respect to iterated deletion of dominated strategies.

For generic games, the dichotomy between single-valued and set-valued refinements is recognized to be immaterial since any equilibrium of a generic normal form game is a singleton and, for a generic extensive form game, all the equilibria that belong to the same connected component determine an identical probability distribution over its final nodes. Thus the focus is on non generic games.

Sequential equilibria seem a natural candidate for strategic stability since, quoting Kohlberg and Mertens [1986], in every sequential equilibrium "$(\ldots)$ at every point during any play of the game, each player must believe that his prescribed strategy will maximize his expected payoff in the reminder of the game". Unfortunately, sequentiality is not always invariant with respect to arbitrary details of the extensive form of the game. Consider the game tree $A_{1.1}$ represented in Figure 1.1: given $1 < x \leq 2$ the equilibrium $\sigma = \{[T]\,;[R]\}$ is sequential. However it is no longer sequential in the extensive form game $B_{1.1}$ represented in Figure 1.2: at the second information set of player I strategy $[M]$ dominates strategy $[B]$ and should be part of every sequential equilibrium. Therefore player II should assign probability one to node α and choose $[L]$. Note that games $A_{1.1}$ and $B_{1.1}$ have the same normal form.

This flaw would naturally lead to properness since a proper equilibrium of a normal form game is sequential in any tree with that normal form. However, not even properness is immune to a forward induction argument. Consider the proper equilibrium $\sigma = \{[T]\,;[R]\}$ of game $A_{1.1}$ in Figure 1.1. Remind that in an ϵ-proper equilibrium σ^{ϵ} while each player i takes as given his opponents' strategies σ^{ϵ}_{-i}, he could make a mistake when choosing his own strategy. Still, his strategic choice is driven by an invisible hand: given σ^{ϵ}_{-i}, the worse the mistake for player i, the less likely it is (see Myerson [1978]). Thus $\sigma^{\epsilon} = \{(1-\epsilon-\epsilon^2)\,[T] + \epsilon\,[B] + \epsilon^2\,[M]\,;(1-\epsilon)\,[R] + \epsilon\,[L]\}$ is an ϵ-proper equilibrium since, once player II has to move, he'd assume that his opponent's most likely mistake is $[B]$. However, if player II had to move, he should infer that his opponent had played $[M]$ betting on a chance to get more than he could by playing $[T]$. Therefore player II should play $[L]$ and, consequently, player I should choose $[M]$ deviating from the equilibrium strategy $[T]$.

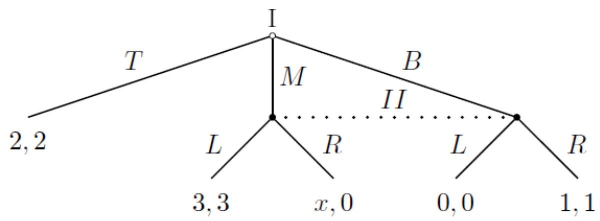

Figure 1.1: Game $A_{1.1}$

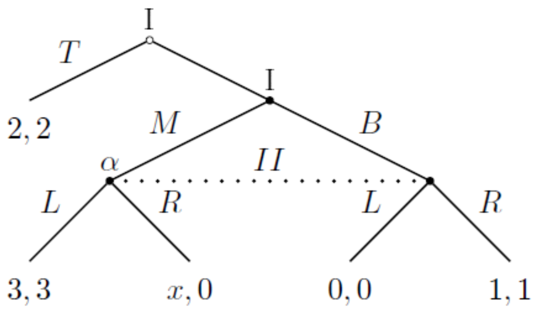

Figure 1.2: Game $B_{1.1}$

Note that if the dominated strategy $[B]$ were deleted $\sigma = \{[T]\,;[R]\}$ would no longer be a proper equilibrium. More generally, Kohlberg and Mertens [1986] conjecture that backwards induction solutions fail to imply strategic stability since they don't satisfy iterated dominance. Interestingly, when properness is considered, deletion of dominated

	L	R
T	$2,2$	$2,2$
M	$3,3$	$x,0$
B	$0,0$	$1,1$

Figure 1.3: Game $A_{1.1}$ in normal form

strategies can be mimicked by the introduction of a randomly redundant strategy that is a new strategy $\hat{s}_i$ payoff equivalent to a randomization among a finite number of pure strategies in player i's strategy set. In the normal form game $A_{1.1}$ represented in Figure 1.3 introduce a randomized strategy $(1-\alpha)[T]+\alpha[M]$ with $\alpha > \frac{1}{2}$ and $x = 0$. Given the ϵ-proper equilibrium strategy $\sigma_2^\epsilon = (1-\epsilon)[R] + \epsilon[L]$, the new strategy would represent the most profitable mistake by player 1 inducing player 2 to deviate from σ_2^ϵ.

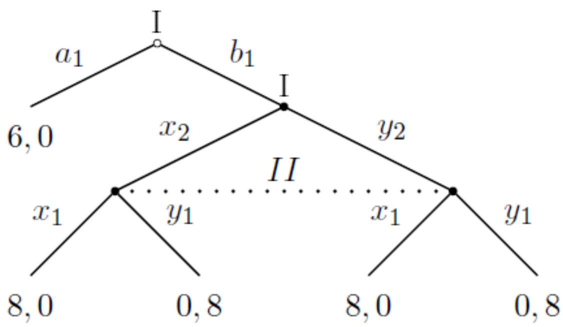

Figure 1.4: Game $C_{1.1}$

More generally, the introduction of a randomly redundant strategy could change dramatically the set of proper equilibria of a normal form game. The games represented in Figure 1.4 and Figure 1.5 differ just for strategy $[z_1]$ that is a randomization between strategies $[x_1]$ and $[a_1]$ with equal weight $[\frac{1}{2}]$. However each of the two extensive form games has a unique sequential equilibrium: subgame perfection requires that player 2 plays $\frac{1}{2}[x_2] + \frac{1}{2}[y_2]$ in game $C_{1.1}$ and $\frac{7}{12}[x_2] + \frac{5}{12}[y_2]$ in game $D_{1.1}$. This is an unfortunate feature of properness: since a mixed strategy could always be played even if not explicitly included among pure strategies, the introduction of a randomly redundant strategy should be regarded as an inessential transformation of the game to be added to the ones proposed by Thompson [1952b], Thompson [1952a] and Dalkey [1953] (coalescing of moves, inflation deflation, addition of superfluous moves and interchange of simultaneous moves).

Provided that properness does not satisfy neither iterated dominance nor invariance properties, a single valued solution concept improving on proper equilibria cannot be a refinement of properness itself.

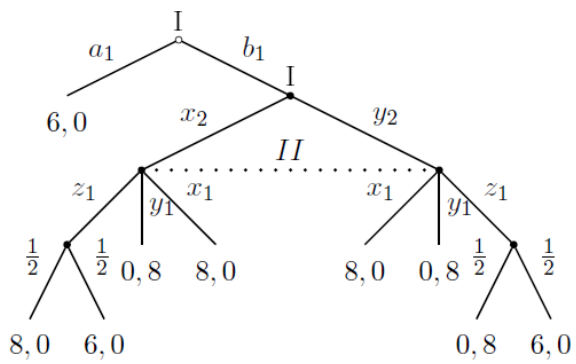

Figure 1.5: Game $D_{1.1}$

A slight modification of the concept of proper equilibria, namely refined equilibria, is then proposed. Unfortunately, while refined equilibria satisfy invariance, they fail to verify even a mild formulation of the property of iterated dominance. This implies that they do not always belong to a stable set whatever definition of stability were adopted.

1.1.1 Introductory examples

The idea underlying the definition of refined equilibria is rather intuitive: no player *i* trembles when choosing his own strategy, while he might have doubts about the beliefs that actually drive his opponents' strategic choices. This uncertainty is reflected by a probability distribution over his opponents' pure strategies. Consider the game represented in Figure 1.3 with $x = 0$. Player 1 chooses the (equilibrium) strategy [T] provided that he believes that player 2 is going to choose strategy [R]; player 1 doesn't need to be sure about his opponent's choice: in order to choose [T] he has to believe that his opponent is going to choose the equilibrium strategy [R] with some probability $\delta \in \left[\bar{\delta}, 1\right]$ where $\bar{\delta}$ is the solution of the following system of linear inequalities:

$$\begin{cases} 2 \geq 3\left(1-\delta\right) \\ 2 \geq \delta \end{cases} \Rightarrow \bar{\delta} = \frac{1}{3} \tag{1.1}$$

If player 2 thought that player 1's belief δ actually belongs to the interval $\left[\frac{1}{3}, 1\right]$, he would choose strategy [R] and the equilibrium $\{[T]\,;[R]\}$ would take place. Suppose, however, that player 2 is not perfectly safe about his opponent's beliefs. Assume, in particular, that he thinks that the true value of δ might be lower than $\bar{\delta}$ with an arbitrarily small probability $\epsilon > 0$. If the expected value of δ were strictly lower than $\bar{\delta}$, player 1 would choose [M] instead of the equilibrium strategy [T]. Therefore, from player 2's perspective, [M] seems, quite reasonably, more likely to be played than [B]. This would finally imply the choice of [L], instead of [R], by player 2 since:

$$2\left(1-\epsilon-\epsilon^2\right)+1\epsilon > 2\left(1-\epsilon-\epsilon^2\right)+1\epsilon^2 \tag{1.2}$$

Consider the second proper equilibrium of the game $\{[M]\,;[L]\}$ and apply the same way of reasoning. The limit value of δ is now given by the solution of the following system of linear inequalities:

$$\begin{cases} 3(1-\delta) \geq 2 \\ 3(1-\delta) \geq \delta \end{cases} \Rightarrow \bar{\delta} = \frac{1}{3} \tag{1.3}$$

Player 1 will play [M] for any value of $\delta \in \left[0, \bar{\delta}\right]$. Now if player 2 had a doubt about player 1's beliefs, with an expected value of δ in a neighborhood of $\bar{\delta}$ he would consider [T] more likely than [B]. This would confirm the choice of [L] by player 2 since:

$$1\left(1-\epsilon-\epsilon^2\right) + 2\epsilon > 2\epsilon + 1\epsilon^2 \tag{1.4}$$

The equilibrium $\{[M]\,;[L]\}$ is then a refined equilibrium. While the introduction of a randomly redundant strategy for player 1 might exclude $\{[T]\,;[R]\}$ from the set of proper equilibria, such an inessential transformation of the game would not affect the set of refined equilibria. Moreover the proposed procedure is not equivalent to the procedure of (iterated) deletion of (weakly) dominated strategies. As to make these points clear a second example[1] is proposed in Figure 2 where no strategy is (weakly) dominated.

	x_2	y_2
$a_1\cdot$	$6,0$	$6,0$
b_1x_1	$8,0$	$0,8$
b_1y_1	$0,8$	$8,0$

Figure 1.6: Game $C_{1.1}$ in normal form

Given the strategy s_i for player i, define its compatible set G_{s_i} as the set of player i's beliefs in Σ_{-i} consistent with the choice of s_i. Consider the proper equilibrium $\left\{[a_1\cdot]\,;\frac{1}{2}[x_2]+\frac{1}{2}[y_2]\right\}$. Strategy $[a_1\cdot]$ is a best response or, equivalently, a *first choice* strategy for player 1 if and only

[1] Myerson [1991].

if the probability δ of x_2 lies in $G_{[a_1\cdot]} = \left[\frac{1}{4}, \frac{3}{4}\right]$. If player 2 were unsafe about his opponent's beliefs, he could assume that the expected value of δ might be either lower than $\frac{1}{4}$ or higher than $\frac{3}{4}$. Since, in principle, there is no reason to prefer one of these two hypothesis, the choice, by player 1, of any of the two strategies b_1x_1 and b_1y_1 is equally likely from player 2's point of view. Therefore, in any ϵ-approximation of the equilibrium they could receive a vanishing probability of the same order. Strategies b_1x_1 and b_1y_1 can be regarded as *second choice*-strategies for player 1, being both a best response given beliefs just outside $G_{[a_1\cdot]}$. In any ϵ-approximation of an equilibrium a *second choice* strategy will receive less probability than a *first choice* strategy by a multiplicative factor ϵ strictly positive and arbitrarily close to zero. Therefore player 2 is willing to randomize between $[x_2]$ and $[y_2]$ (with equal probability $\frac{1}{2}$ in the proposed equilibrium) since:

$$0\,(1-2\epsilon) + 0\epsilon + 8\epsilon = 0\,(1-2\epsilon) + 0\epsilon + 8\epsilon \tag{1.5}$$

Consider, in this setting, the introduction of a randomly redundant strategy $[b_1z_1]$ for player 1 defined as $[b_1z_1] = \frac{1}{2}\,[a_1\cdot] + \frac{1}{2}\,[b_1y_1]$.

	x_2	y_2
$a_1\cdot$	$6,0$	$6,0$
b_1x_1	$8,0$	$0,8$
b_1y_1	$0,8$	$8,0$
b_1z_1	$3,4$	$7,0$

Figure 1.7: Game $D_{1.1}$ in normal form

In order to support any equilibrium σ in which player 1 plays $[a_1\cdot]$, player 2 must be willing to randomize between $[x_2]$ and $[y_2]$ in any ϵ-approximation σ^ϵ of the equilibrium itself. This in turn implies that the following condition has to be satisfied:

$$8\sigma_1^\epsilon(b_1y_1) + 4\sigma_1^\epsilon(b_1z_1) = 8\sigma_1^\epsilon(b_1x_1) \tag{1.6}$$

Given the normal form game represented in Figure 1.7, if player 2 played the equilibrium strategy $\left\{\frac{1}{2}[x_2] + \frac{1}{2}[y_2]\right\}$, then $[b_1z_1]$ would be a better response for player 1 than both $[b_1x_1]$ and $[b_1y_1]$ and, therefore, in any ϵ-proper equilibrium approximating $\left\{[a_1\cdot]\,; \frac{1}{2}[x_2] + \frac{1}{2}[y_2]\right\}$, it would receive a vanishing probability $\sigma_1^\epsilon(b_1z_1)$ of higher order, violating condition (1.6). Therefore the introduction of the randomly redundant strategy $[b_1z_1]$ for player 1 changes player 2's strategic behavior: now $[x_2]$ is a better response than $[y_2]$ to σ_1^ϵ and the equilibrium $\left\{[a_1\cdot]\,; \frac{1}{2}[x_2] + \frac{1}{2}[y_2]\right\}$ is not a proper equilibrium of the modified game.[2] Note, however, that the strategy $[b_1z_1]$ would be preferred by player 1 to neither $[b_1x_1]$ nor $[b_1y_1]$ if his belief δ about the choice of $[x_2]$ lied outside the compatible set $\left[\frac{1}{4}, \frac{3}{4}\right]$[3]. Therefore player 2 should regard $[b_1z_1]$ as a *third choice* strategy for player 1 and assign to it, in any ϵ-approximation of the equilibrium, a vanishing probability $\sigma_1^\epsilon(b_1z_1)$ lower (by a multiplicative factor ϵ strictly positive and arbitrarily close to zero) than both $\sigma_1^\epsilon(b_1x_1)$ and $\sigma_1^\epsilon(b_1y_1)$. Therefore, the vanishing probabilities of the two strategies $[b_1x_1]$ and $[b_1y_1]$ can be tied independently of the strategy σ_2 actually chosen by player 2 within $G_{[a_1\cdot]}$ and the introduction of the randomly redundant strategy $[b_1z_1]$ does not imply the violation of condition (1.6).

[2]In Myerson's approach the choice of a specific totally mixed strategy σ_2^ϵ by player 2 plays a crucial role, since in an ϵ-proper equilibrium each player 1's pure strategy is assigned a probability *given* σ_2^ϵ.

[3]The compatible set is closed and convex being the solution of a finite system of linear inequalities. Therefore, the other sets of beliefs are $\left(\frac{3}{4}; 1\right]$ and $\left[0, \frac{1}{4}\right)$.

1.2 Refined equilibria

Consider a finite n player game $\Gamma = \left\{I, \{\Sigma_i\}_{i\in I}, \{u_i\}_{i\in I}\right\}$, where I is the finite set of players indexed by i, Σ_i is player i's compact, convex strategy-polyhedron (in Euclidean space) being S_i his pure strategy set and u_i his multilinear payoff function defined on $\Sigma = \prod_{i\in I} \Sigma_i$.

Let $\mathcal{A}_i^0 = \left\{G_{s_i}^0\right\}_{s_i \in S_i}$ be the collection of all convex and compact sets[4] $G_{s_i}^0 = BR_i^{-1}(s_i) := \{\sigma_{-i} \in \Sigma_{-i} | BR_i(\sigma_{-i}) \cap s_i \neq \varnothing\}$ with $s_i \in S_i$ and $G_{s_i}^0 \subseteq \Sigma_{-i}$.

Given the minimal semigroup Θ_i^0 generated by $\mathcal{A}_i^0$ with set intersection $\cap$ as binary operation, define Ψ_i^0 as the collection of all orderings over the sets $A_i^0 \in \Theta_i^0$ such that for any pair $(A_i^0, \hat{A}_i^0) \in \Theta_i^0 \times \Theta_i^0$, if $A_i^0 \subseteq \hat{A}_i^0$ then $\hat{A}_i^0$ is ranked above A_i^0.

For every $A_i^0 \in \Theta_i^0$ let $A_i^1 = BR_i^{-1}(\bigcap_{j\neq i} \pi_i \left(BR_j^{-1}(\pi_j(A_i^0))\right))$ with $\pi_j : \Sigma_{-i} \to \Sigma_j$ and $\pi_i : \Sigma_{-j} \to \Sigma_i$ the canonical projection functions. The sets A_i^n are defined recursively for every $n \in \mathbb{N}$.

Indicate with Ψ_i^n the n^{th} collection of all orderings over the sets $A_i^0 \in \Theta_i^0$ such that for any pair $(A_i^0, \hat{A}_i^0) \in \Theta_i^0 \times \Theta_i^0$, if $A_i^n \subseteq \hat{A}_i^n$ then $\hat{A}_i^0$ is ranked above A_i^0.

Finally introduce, for every $n \in \mathbb{N}$, a correspondence $F_i^n : \Psi_i^n \to \Omega_i$ where Ω_i is the set of all orderings ω_i of any set of $K = |\Theta_i^0|$, not necessarily distinct, pure strategies in S_i. For each ordering $\psi_i \in \Psi_i^n$, define $F_i^n(\psi_i)$ as the set of all orderings in Ω_i such that A_i^0 in ψ_i and s_i in ω_i are identically ranked only if $A_i^0 \subseteq G_{s_i}^0$.

For every ordering ω_i define a mixed strategy σ_{ω_i} such that each strategy s_i in ω_i is played with a probability equal to $\frac{(1-\epsilon)\epsilon^{k-1}}{1-\epsilon^K}$ where k is its position in the ordering ω_i and ϵ a strictly positive constant arbitrarily close to zero.

[4]To define $G_{s_i}^0$ the lower inverse of a correspondence is adopted. If the upper inverse were considered then $G_{s_i}^0 = BR_i^{-1}(s_i) := \{\sigma_{-i} \in \Sigma_{-i} | BR_i(\sigma_{-i}) = s_i\}$.

Define $\bar{S}_i$ as the collection of all (mixed) strategies in Σ_i each corresponding to an ordering in $\bigcap_{n\in\mathbb{N}} F_i^n(\Psi_i^n)$.

LEMMA 1.2.1. *For any pair* $(A_i^n, \hat{A}_i^n)$, *if* $A_i^n \subseteq \hat{A}_i^n$ *for some* $n \in \mathbb{N}$, *then there does not exist an integer* $m > n$ *such that* $\hat{A}_i^m \subset A_i^m$. *Thus the set* $\bigcap_{n\in\mathbb{N}} F_i^n(\Psi_i^n)$ *is not empty.*

Proof. If $A_i^n \subseteq \hat{A}_i^n$ then for every player $j \neq i$ the canonical projection $\pi_j(A_i^n)$ is a subset of $\pi_j(\hat{A}_i^n)$. It comes easily that $BR_j^{-1}(\pi_j(A_i^n)) \subseteq BR_j^{-1}(\pi_j(\hat{A}_i^n))$ for every player $j \neq i$ given $BR_j^{-1}\left(\pi_j\left(\tilde{A}_i^n\right)\right) = \{\sigma_{-j} \in \Sigma_{-j} \,|BR_j(\sigma_{-j}) \cap \pi_j\left(\tilde{A}_i^n\right) \neq \varnothing\Big\}$. Finally, given any two pairs of sets (X, Y) and $(\hat{X}, \hat{Y})$ with $X \subseteq \hat{X}$ and $Y \subseteq \hat{Y}$ we have $X \cap Y \subseteq \hat{X} \cap \hat{Y}$. Therefore $A_i^{n+1} \subseteq \hat{A}_i^{n+1}$. □

Definition 1. *An ϵ-refined equilibrium is any equilibrium of the fictitious game* $\tilde{\Gamma} = \left\{I, \left\{\bar{S}_i\right\}_{i\in I}, \{u_i\}_{i\in I}\right\}$.

Definition 2. *A strategy profile* $(\sigma_1, \cdots, \sigma_n) \in \Delta S$ *is a refined equilibrium iff it is the limit of a sequence of ϵ-refined equilibria.*

This definition refines the intuition presented in the introductory examples: when a player considers the orderings over his opponent's beliefs he should restrict himself to the ones maximizing his expected utility given his own strategy. This setting seems a natural choice since backwards induction is one of the requirements to be verified.

1.2.1 Explanatory example

Consider the normal form game represented in Figure 1.6 and, given $\alpha_2 = \Pr(x_2)$, define the minimal semigroup Θ_1^0 generated by the collection $\mathcal{A}_1^0 = \left\{G_{s_i}^0\right\}_{s_i\in S_i}$ and by the set intersection $\cap$ as binary operation.

$$\Theta_1^0 = \begin{cases} \mathcal{A}_1^0 = \begin{cases} A_{i,1}^0 = G_{[a_1\cdot]}^0 = \left\{\frac{1}{4} \le \alpha_2 \le \frac{3}{4}\right\} \\ A_{i,2}^0 = G_{[b_1x_1]}^0 = \left\{\frac{3}{4} \le \alpha_2 \le 1\right\} \\ A_{i,3}^0 = G_{[b_1y_1]}^0 = \left\{0 \le \alpha_2 \le \frac{1}{4}\right\} \end{cases} \\ A_{i,4}^0 = G_{[a_1\cdot]}^0 \cap G_{[b_1x_1]}^0 = \left\{\alpha_2 = \frac{3}{4}\right\} \\ A_{i,5}^0 = G_{[a_1\cdot]}^0 \cap G_{[b_1y_1]}^0 = \left\{\alpha_2 = \frac{1}{4}\right\} \\ A_{i,6}^0 = \varnothing \end{cases}$$

Consider the randomized strategy $[b_1z_1] = \frac{1}{2}[a_1\cdot] + \frac{1}{2}[b_1y_1]$: the corresponding compatible set $G_{[b_1z_1]}^0$ is $G_{[a_1\cdot]}^0 \cap G_{[b_1y_1]}^0$. More generally the introduction of any mixed strategy does not affect Θ_1^0.

Given Θ_1^0 derive the collection of all admissible orderings:

$$\Psi_1^0 = \begin{cases} \psi_1 = A_{i,1}^0 \succ A_{i,2}^0 \succ A_{i,3}^0 \succ A_{i,4}^0 \succ A_{i,5}^0 \succ A_{i,6}^0 \\ \psi_2 = A_{i,2}^0 \succ A_{i,1}^0 \succ A_{i,3}^0 \succ A_{i,4}^0 \succ A_{i,5}^0 \succ A_{i,6}^0 \\ \psi_3 = A_{i,3}^0 \succ A_{i,2}^0 \succ A_{i,1}^0 \succ A_{i,4}^0 \succ A_{i,5}^0 \succ A_{i,6}^0 \\ \vdots \end{cases}$$

Finally define the strategy orderings $F_1^0\left(\Psi_1^0\right)$ corresponding to each ψ_i:

$$F_1^0\left(\Psi_1^0\right) = \begin{cases} F_1^0\left(\psi_1\right) = \begin{cases} a_1. \succ b_1x_1 \succ b_1y_1 \succ b_1x_1 \succ a_1. \succ b_1x_1 \\ a_1. \succ b_1x_1 \succ b_1y_1 \succ a_1. \succ b_1y_1 \succ b_1y_1 \\ a_1. \succ b_1x_1 \succ b_1y_1 \succ b_1x_1 \succ b_1y_1 \succ b_1x_1 \\ \vdots \end{cases} \\ F_1^0\left(\psi_2\right) = \begin{cases} b_1x_1 \succ a_1. \succ b_1x_1 \succ a_1. \succ b_1x_1 \\ b_1x_1 \succ a_1. \succ b_1y_1 \succ a_1. \succ b_1y_1 \succ b_1y_1 \\ b_1x_1 \succ a_1. \succ b_1y_1 \succ b_1x_1 \succ b_1y_1 \succ b_1x_1 \\ \vdots \end{cases} \\ \vdots \end{cases}$$

For every set $A_{i,h}^0$ the corresponding set $A_{i,h}^1$ has to be defined. First consider the set $A_{i,1}^0 = G_{[a_1\cdot]}^0 = \left\{\frac{1}{4} \le \alpha_2 \le \frac{3}{4}\right\}$.

Then $BR_2^{-1}\left(A_{i,1}^0\right) = \{(1-\beta_1-\beta_2)[a_1\cdot] + \beta_1[b_1x_1] + \beta_2[b_1y_1] | \beta_2 = \beta_1\}$

and $A^1_{i,1} = BR_1^{-1}\left(BR_2^{-1}\left(A^0_{i,1}\right)\right) = A^0_{i,1}$.

Consider, on the other hand, $A^0_{i,2} = G^0_{[b_1 x_1]} = \left\{\frac{3}{4} \leq \alpha_2 \leq 1\right\}$. The set $BR_2^{-1}\left(A^0_{i,2}\right)$ is given by $\{(1-\beta_1-\beta_2)\,[a_1\cdot] + \beta_1[b_1x_1] + \beta_2[b_1y_1] | \beta_2 \geq \beta_1\}$ and $BR_1^{-1}\left(BR_2^{-1}\left(A^0_{i,2}\right)\right) = \left\{\alpha_2 = \frac{3}{4}\right\}$. Thus the mixed strategies corresponding to the orderings in $F^0_1(\psi_2)$ are not part of the strategy set $\bar{S}_1$.

1.2.2 Properties

Proposition 1 (Existence). *Every normal form game Γ has a refined equilibrium.*

Proof. Existence comes immediately given the constructive definition of ϵ-refined equilibria and lemma 1.2.1. $\square$

Proposition 2 (Admissibility). *In every refined equilibrium the players' strategies are undominated.*

Proof. If an equilibrium strategy s_i is weakly dominated by $\tilde{s}_i$ then $G_{s_i} \subset G_{\tilde{s}_i}$ and consequently will never be ranked above it. $\square$

Proposition 3 (Invariance). *Every refined equilibrium is invariant with respect to the introduction of a randomly redundant strategy.*

Proof. Given the initial game Γ create a new game $\hat{\Gamma}$ by introducing a randomly redundant strategy $\hat{s}_i$ for some player i with $\hat{G}^0_{\hat{s}_i} \neq \varnothing$[5]. If a randomized strategy σ_i is a best reply to a strategy profile σ_{-i} then every pure strategy s_i in the support of σ_i is a best reply to σ_{-i} as well. Conversely, if every s_i in the support of σ_i is a best reply to σ_{-i} also σ_i is a best response. Thus $\hat{G}^0_{\hat{s}_i} \in \Theta^0_i$ and $\Theta^0_i = \hat{\Theta}^0_i$ since $\hat{G}^0_{\hat{s}_i} = \bigcap_{s_i \in S_{\hat{s}_i}} G^0_{s_i}$ where $S_{\hat{s}_i}$ is the set of pure strategies in the support of the mixed strategy $\sigma_i \in \Delta S_i$ payoff equivalent to $\hat{s}_i$. As a consequence, the set Ψ^0_i

[5]With a slight abuse of notation all the sets and correspondences relative to the new game $\hat{\Gamma}$ are identified by hat as superscript.

remains unchanged i.e. $\hat{\Psi}_i^0 = \Psi_i^0$ and any new pure ordering in $\hat{\Omega}_i^0$ is a linear combination of some pure orderings in the corresponding set Ω_i^0 of the original game.
Now assume that $\hat{G}_{\hat{s}_i}^0 = \varnothing$. If $\varnothing \in \Theta_i$ then the proof just outlined applies. Suppose, on the other hand, that $\varnothing \notin \Theta_i$. This in turn implies that $\bigcap_{s_i \in S_i} G_{s_i}^0 \neq \varnothing$. Therefore for any randomly redundant strategy $\tilde{s}_i$ it must be $G_{\tilde{s}_i}^0 \neq \varnothing$, hence a contradiction.
The introduction of the randomly redundant strategy $\hat{s}_i$ also modifies the set of possible beliefs of any other player $j \neq i$. However, any set $G_{s_j}^0$ in the original game Γ is just the projection on Σ_{-j} of the corresponding set $\hat{G}_{s_j}^0$ with projection function $p : \hat{\Sigma}_i \times \Sigma_{-i} \rightarrow \Sigma$ such that $p(\hat{\sigma}_i, \sigma{-}i) = \sigma$ if and only if $(\hat{\sigma}_i, \sigma{-}i)$ and σ are payoff equivalent for every player i. $\square$

Proposition 4 (Properness)**.** *The set of refined equilibria is not a subset of the set of proper equilibria.*

	L	C	R
T	$3,2$	$2,0$	$1,1$
M	$3,1$	$1,2$	$2,0$
B	$2,1$	$1,1$	$0,0$

Figure 1.8: Game with refined not proper equilibrium

Game $A_{1.1}$ presented in Figure 1.3 proves that a proper equilibrium is not necessarily a refined one. Consider now the normal form game represented in Figure 1.8: the equilibrium $\{[T],[L]\}$, while refined, is not proper. However, this equilibrium seems to be reasonable since it becomes a proper equilibrium once player 2's dominated strategy $[R]$ is deleted. Note that Proposition 4 implies that the concept of refined equilibria is not a refinement of proper equilibria. This is not surprising since otherwise invariance would be violated.

Proposition 5 (Backwards Induction)**.** *The intersection of the sets of refined and sequential equilibria might be empty.*

Proof. The sets of sequential equilibria of two games with the same reduced normal form that differ just for a randomly redundant strategy might have empty intersection. Thus, given Propositions 3 and 4, a game can be constructed with a unique refined equilibrium that is not sequential. □

If a refined equilibrium were always in a stable set of equilibria as defined in Mertens [1989] including, by definition, a proper hence sequential equilibrium, Proposition 5 would be of minor importance. Unfortunately, the result presented in Proposition 6 excludes this case.

Proposition 6 (Elimination of dominated strategies)**.** *A refined equilibrium of a game $\bar{\Gamma}$ obtained from the original game Γ by deleting a dominated strategy doesn't always belong to the same connected component of Nash equilibria a refined equilibrium of Γ belongs to.*

First note that a player i's dominated strategy s_i can be concealed, without changing the strategic structure of the game, by the introduction of a zero sum subgame with a unique equilibrium strategy payoff equivalent to s_i. The game presented in Figure 1.9 is a modified version of the game proposed in Figure 1.3: player 1's dominated strategy $[B]$ has been replaced by a zero sum game; the unique equilibrium of the zero sum game is $\left\{\frac{1}{2}[B_1]+\frac{1}{2}[B_2];\frac{1}{2}[L]+\frac{1}{2}[C]\right\}$ where $\frac{1}{2}[B_1]+\frac{1}{2}[B_2]$ is payoff equivalent to strategy $[B]$. Therefore there is no reason to expect different sets of refined equilibria in the two games. However since strategies $[B_1]$ and $[B_2]$ are undominated the sets $G^0_{B_1}$ and $G^0_{B_2}$ are not empty. The sets $G^1_{B_1}$ and $G^1_{B_2}$ are to be considered to exclude $\sigma^*=\{[T],[R]\}$ as a refined equilibrium. However the role of dominated strategies is much more involved.

	L	C	R
T	$2, 2$	$2, 2$	$2, 2$
M	$3, 1$	$3, 1$	$0, 0$
B_1	$-4, 4$	$4, -4$	$1, 1$
B_2	$4, -4$	$-4, 4$	$1, 1$

Figure 1.9: Game with concealed dominated strategy

Note that concealing a dominated strategy implies that it still remains, indirectly, part of the game. Intuitively one should also expect that the elimination of a dominated strategy should have no impact on a Nash equilibrium refinement. The consequences of the elimination of dominated strategies has been widely discussed in Kohlberg and Mertens [1986] recognizing that this intuitive idea is wrong. Consider the game represented in Figure 1.10: the equilibrium (a_1, x_2) is proper and refined. However, if the dominated strategy c_1 were dropped the only refined equilibrium would be (b_1, z_2). This is a negative result since this equilibrium is not even in the connected component including (a_1, x_2).

A slight variation of the model, obtained by adopting the lower inverse of a correspondence in the definition of the sets $\{A_i^n\}$ for every player i and every n, would solve the example. However this modification could lead to violate the property of existence.

1.2.3 Conclusions

The evaluation of refined equilibria is controversial. As a main result the invariance property is satisfied improving on the concept of proper equilibrium. This result stems from a setting that is closer to the forward induction than the backwards induction approach: a sin-

	x_2	y_2	z_2
a_1	$4, 2$	$2, 2$	$0, 0$
b_1	$1, 0$	$3, 0$	$1, 1$
c_1	$1, 1$	$0, 0$	$0, 0$
d_1	$0, 0$	$1, 1$	$0, 0$

Figure 1.10: Game $E_{1.1}$

gle valued solution concept that always conforms with backwards induction cannot verify the property of invariance. However not even refined equilibria satisfy any reasonable formulation of the property of iterated deletion of dominated strategies. It remains a cue for future research whether a slight modification of the model could lead to more promising results.

Chapter 2

Strategic stability of equilibria: the missing paragraph

2.1 Introduction

As recalled in the previous chapter, Kohlberg and Mertens [1986] defined the concept of strategic stability of Nash equilibria by introducing a set of desirable properties that an equilibrium should satisfy:

Existence: every game has at least one solution;

Connectedness: every solution is connected;

Backwards induction: a solution of a tree contains a backwards induction equilibrium of the tree;

Invariance: a solution of a game is also a solution of any game with the same reduced normal form;

Admissibility: the players' strategies are undominated at any point in a solution;

Iterated dominance and forward induction: a solution of a game Γ contains the solution of any game $\tilde{\Gamma}$ obtained from Γ by deleting a strategy that is either dominated or that is always an inferior response in every equilibrium of the set.

Among others, admissibility and iterated dominance, jointly considered, lead naturally to conclude that just a set of Nash equilibria could fit strategic stability. Given game $A_{2.1}$, represented in Figure 2.1, if strategy M were deleted, then, by admissibility, just $(2,2)$ would be strategically stable. Conversely, if B were deleted, the unique admissible equilibrium would be $(3,2)$. Thus, a strategically stable set of equilibria should include both $(2,2)$ and $(3,2)$.

	L	R
T	$3,2$	$2,2$
M	$1,1$	$0,0$
B	$0,0$	$1,1$

Figure 2.1: Game $A_{2.1}$

Kohlberg and Mertens [1986] proposed three different definitions of stable sets trying to satisfy all the desirable properties.

For the sake of convenience recall the definitions of hyperstable, fully stable and stable sets of equilibria and their flaws:

Definition 3. *Q is a hyperstable set of equilibria of a game Γ if it is minimal with respect to the following property:*
Q is a closed set of Nash equilibria of Γ such that, for any equivalent game, and for any perturbation of the normal form of that game, there is a Nash equilibrium close to Q.

Existence of hyperstable sets proves that, while an essential single valued equilibrium might not exist, an essential set of equilibria does exist for any normal form game.

However an hyperstable set might violate admissibility. Fully stable sets are then introduced to improve on hyperstability by restricting the collection of allowed perturbations to those which arise from strategy perturbations:

Definition 4. *Q is a fully stable set of equilibria of a game Γ if it is minimal with respect to the following property:*
Q is a closed set of Nash equilibria of the game Γ satisfying: for any $\epsilon > 0$ there exists some $\delta > 0$ such that, whenever each player's strategy set is restricted to some compact convex polyhedron in the interior of the simplex at an (Hausdorff) distance less than δ from the simplex, then the resulting game has an equilibrium point ϵ-close to Q.

While every fully stable set of equilibria of a normal form game always contains a proper[1] (hence perfect and sequential in every extensive form game with that normal form) equilibrium, still admissibility might be violated.

This flaw is due to the fact that each player's strategic choice is allowed to be affected by perturbations of his own strategies.

Then the authors allowed just for perturbations in which every pure strategy s_i of each player i is perturbed in the same amount towards the same completely mixed strategy. This led to the definition of stable sets of equilibria:

Definition 5. *Q is a stable set of equilibria of a game Γ if it is minimal with respect to the following property:*
Q is a closed set of Nash equilibria of the game Γ satisfying: for any $\epsilon > 0$ there exists some $\delta_0 > 0$ such that for any completely mixed strategy vector $\sigma_1 \ldots \sigma_n$ (n players) and for any $\delta_1 \ldots \delta_n$ $(0 < \delta_i < \delta_0)$, the perturbed game where every strategy $s_i \in S_i$ of player i is replaced by $(1 - \delta_i)\, s_i + \delta_i \sigma_i$ has an equilibrium ϵ-close to Q.

However stable sets might not satisfy the backwards induction requirement.

The first contribution of this paper is the proof that a natural generalization of the definition of stable sets of equilibria, named F- stable

[1]See Myerson [1978].

sets, satisfies all the desirable properties proposed by Kohlberg and Mertens [1986]: the backwards induction requirement is easily satisfied, without violating admissibility, by allowing every pure strategy to be replaced by a set of its perturbations instead of a unique one.

However F - stable sets might violate the player splitting property as introduced by Mertens [1989]:

Player splitting: given a partition of the information set of some player, such that no play intersects two different partition elements, the new game, where this player is replaced by a set of agents each managing one of the partition elements, has the same stable sets as the old game.

This flaw is not surprising since every agent of a splitted player will choose his perturbed strategy independently just to maximize his individual payoff. Conversely, a single player would correlate his agents mistakes in order to maximize his overall payoff.

Then the tension between the properties of backwards induction and player splitting seems the real knot to be untied. This is confirmed also by the discussion in Hillas [1990].

This conflict is solved by the definition of G - stable sets where the collection of games exploited to identify a strategically stable set of equilibria of a normal form game Γ includes perturbations of a class of new games each obtained from Γ by introducing inessential transformations.

2.2 *F* - stable equilibria

Let $\Gamma = \left\{I, \{\Sigma_i\}_{i\in I}, \{u_i\}_{i\in I}\right\}$ be a finite n player game, where I is the finite set of players indexed by i, Σ_i is player i's compact, convex strategy-polyhedron (in Euclidean space) being S_i his pure strategy set and u_i his multi linear payoff function defined on $\Sigma = \prod\limits_{i\in I}\Sigma_i$.

Define the set P_ϵ of perturbations η as $P_\epsilon = \left\{\epsilon \cdot \vec{\sigma} |\, 0 < \epsilon < 1, \vec{\sigma} \in \Sigma^k \backslash \partial\Sigma^k, k \in \mathbb{Z}\right\}$ where Σ^k is the k-fold Cartesian product of Σ with k integer number.

Let $\eta_i = \epsilon \cdot \vec{\sigma_i}$ be the k-dimensional vector that represents the projection onto $\Sigma_i^k \backslash \partial\Sigma_i^k$ of the perturbation $\eta = \epsilon \cdot \vec{\sigma}$.

Given $\eta \in P_\epsilon$ let $\tau_\eta\left(s_i\right) = \left(1-\epsilon\right)\vec{s_i} + \eta_i$ be the k-dimensional strategy vector where $\vec{s_i}$ is the k-dimensional vector with each entry equal to s_i.

Define $\Gamma_\eta = \left\{I, \left\{\bar{\Sigma}_i\right\}_{i\in I}, \left\{u_i\right\}_{i\in I}\right\}$ as the game obtained from Γ by replacing each pure strategy s_i of each player i by the vector $\tau_\eta\left(s_i\right)$.

The definition of stable set of equilibria is then modified accordingly:

Definition 6. *Q is an F - stable set of equilibria of a game Γ if it is a set of equilibria minimal with respect to the following property F:*

Property (F). *Q is a closed set of Nash equilibria of Γ satisfying: for any $\delta > 0$ there exists some $\epsilon_0 > 0$ such that any perturbed game Γ_η with $\eta \in P_\epsilon$ and $\epsilon_0 > \epsilon > 0$ has an equilibrium δ-close to Q.*

Note that the proposed definition is halfway the definitions of fully stable and stable set of equilibria. In particular, the new collection of strategy polyhedral includes all those allowed by the definition of stable equilibria as special cases in which $k = 1$, and is a proper subset of the ones defining a fully stable set.

2.2.1 Properties

Given the definition of F - stable equilibria, it has to be proved that it is consistent with the properties defining a strategically stable set of equilibria:

Proposition 7 (Existence). *Every normal form game Γ has an F - stable set of equilibria.*

Proof. Existence of F - stable sets comes from existence of a fully stable set of equilibria for any normal form game Γ as proved in Kohlberg and Mertens [1986]. The definition of F - stable equilibria is less demanding than the definition of fully stable equilibria since a narrower set of perturbations is allowed. $\square$

Proposition 8 (Connectedness)**.** *Every game has an F - stable set contained in a single connected component of the set of Nash equilibria.*

Proof. Kohlberg and Mertens [1986] proved that every game has an hyperstable set of equilibria contained in a single connected component of the set of Nash equilibria. Thus, since every hyperstable set includes an F - stable set of equilibria, every normal form game has an F - stable set which is contained in a single connected component of the set of Nash equilibria. $\square$

Proposition 9 (Backwards induction)**.** *An F - stable set of any finite normal form game Γ always includes a proper equilibrium of Γ.*

Proof. Given Γ construct a perturbed game $\tilde{\Gamma}$ as follows: first for each player $i \in I$ define the set $E_i = \left\{ e_j^i \in S_i^n | j = 1, \dots n! \right\}$ of all permutations e_j^i of his pure strategies, where S_i^n is the n-fold Cartesian product of S_i and $n = |S_i|$.

Then, for every player i, construct, from each ordering e_j^i, a completely mixed strategy $\sigma_i\left(e_j^i\right)$ such that, when $\sigma_i\left(e_j^i\right)$ is chosen, each strategy $s_i \in S_i$ is played with probability $\epsilon^{k-1}\left(1-\epsilon\right)/\left(1-\epsilon^n\right)$, where k corresponds to its position in the ordering e_j^i.

Given the set $\tilde{E}_i = \left\{ \sigma_i\left(e_j^i\right) | e_j^i \in E_i \right\}$ of the $n!$ totally mixed strategies for every player i, define a new game $\tilde{\Gamma}$ in which each strategy s_i of each player i is replaced by the following set of perturbed strategies:

$$\left\{ \left(1-\epsilon\right) s_i + \epsilon \sigma_i\left(e_j^i\right) \right\}_{\sigma_i\left(e_j^i\right) \in \tilde{E}_i} \tag{2.1}$$

In the new game $\tilde{\Gamma}$, when a player chooses a pure strategy $\left(1-\epsilon\right) s_i + \epsilon \sigma_i\left(e_j^i\right)$ he actually selects with probability $\left(1-\epsilon\right)$ the strategy s_i and, with probability ϵ a lottery over a given ordering of his pure strategies in S_i.

Pick an equilibrium point of the new game in the neighborhood of its set of equilibria. As proved in Kohlberg and Mertens [1986] it is an ϵ-proper equilibrium of the initial game. $\square$

Proposition 10 (Invariance). *Every F - stable set is also an F - stable set of any equivalent game (i.e. having the same reduced normal form).*

Proof. From a geometrical point of view, the definition of F - stable equilibria considers just polyhedra $\bar{\Sigma}_i$ within the strategy simplex of each player i that are the convex hull of any finite collection of polyhedra allowed by the definition of stable sets of equilibria in Kohlberg and Mertens [1986].

If a new strategy $\hat{s}_i$, linear combination of a finite number of pure strategies in S_i, were explicitly introduced as an additional pure strategy in S_i it would be perturbed as any other pure strategy.

Then, given any convex strategy polyhedron $\bar{\Sigma}_i$ allowed by the definition of F - stable sets in the original game Γ, each strategy in $\tau_\eta\left(\hat{s}_i\right)$ would be represented by a point on a side of $\bar{\Sigma}_i$. $\square$

Proposition 11 (Admissibility). *Given any equilibrium in an F - stable set, every equilibrium strategy for every player i is undominated.*

Proof. It is well known that a stable set of equilibria satisfies admissibility. Despite a larger set of perturbations is now allowed, any F - stable set still satisfies this property since each $\vec{\sigma}$ in P_ϵ is in the interior of Σ^k and all strategies in S_i for every player i are identically perturbed. $\square$

Given these last two properties, one can verify immediately the (i, α)-ordinality of F - stable sets, using Theorem 2 in Mertens [2004].

Proposition 12 (Iterated dominance and forward induction). *(A) An F - stable set of a game Γ contains the F - stable set of any game $\bar{\Gamma}$ obtained from Γ by deleting a dominated strategy and (B) an F - stable set of a game*

Γ *contains the* F*-stable set of any game* $\tilde{\Gamma}$ *obtained from* Γ *by deleting a strategy that is an inferior response in all the equilibria of the set (Forward induction).*

Proof. Given a perturbation $\bar{\Gamma}_\eta$ of the game $\bar{\Gamma}$ without the dominated strategy $\bar{s}_i$, construct a close-by perturbation in two steps: first introduce $\bar{s}_i$ in the strategy set $\bar{S}_i$ of player i and perturb it like any other strategy of the game. Then construct the perturbed game $\Gamma_\eta(z)$ by slightly perturbing any of player i's strategies towards $\bar{s}_i$ by z. The game $\Gamma_\eta(z)$ is thus a perturbation of the initial game Γ. Obviously in no equilibrium of $\Gamma_\eta(z)$ the eliminated strategy will be played and taking the limit for $z \to 0$ of these equilibria will give an equilibrium of $\bar{\Gamma}_\eta$ close to the F-stable set.

The proof of the part (B) of Proposition (12) is identical and thus omitted. □

In Mertens [1989] two additional properties were added to the ones proposed in Kohlberg and Mertens [1986]:

Small worlds and decomposition: a strategically stable set of equilibria should be immune to the introduction of irrelevant players.

A subset $J \subset I$ of players represents a small world if their payoffs do not depend on the strategies of the outside players in $I \backslash J$.

Let $\Gamma_J = \left\{J, \{\Sigma_i\}_{i\in J}, \{u_i\}_{i\in J}\right\}$ be the game played by the insiders. For a small world the outside players should be considered irrelevant and should have no impact on the stable sets of equilibria of Γ_J.

Similarly, a game that consists of N small worlds should decompose. If each of N disjoint sets of players constitutes a small world and plays a different game in a separated room, then it should not matter if the N games were analyzed jointly or separately.

Proposition 13 (Small worlds)**.** *If* J *is a small world in* Γ*, then a set of equilibria is an* F*-stable set of the game* Γ_J *if and only if it is the projection of an* F*-stable set of the game* Γ*.*

Proposition 14 (Decomposition). *If for a game Γ both J and $I\backslash J$ constitute small worlds in Γ, then a set Q of equilibria is an F - stable set of the game Γ if and only if $Q = Q_J \times Q_{I\backslash J}$ with Q_J an F - stable set of the game Γ_J and $Q_{I\backslash J}$ an F - stable set of the game $\Gamma_{I\backslash J}$.*

Both these properties[2] are self evident since, given any small world J in I, when every single strategy of each player $i \in J$ is replaced by a set of its perturbations the set J of players remains a small world in Γ_η.

Player Splitting: given a partition of the information set of some player, such that no play intersects two different partition elements, consider the new game obtained by letting a different agent of this player manage each of these partition elements, and receive the same payoff as this player for those play that intersect his own information sets while he receives an arbitrary payoff on the other plays. This new game, where this player is replaced by these agents, should have, according to Mertens [1989], the same stable sets as the old game.

Unfortunately, this property is not satisfied for F - stable sets since the agents of a single player maximize their payoffs independently while a single player correlates his agents strategies and mistakes.

The sender - receiver game represented in Figure 2.2 proposed by Hillas [1990] clarifies the point. First consider the game as a five player game and look just at the equilibria in which all the types of the sender play strategy L. The simplex in Figure 2.3 represents the receiver's strategy space and the corresponding choices by the sender.

The collection of stable sets of equilibria, according to the original definition in Kohlberg and Mertens [1986], consist of four pairs $\{W, Y\}, \{W, Z\}, \{X, Y\}, \{X, Z\}$[3]. Since every agent's strategy set consists of just two strategies the KM-stable sets and the F-stable sets coincide.

[2]For a complete discussion of both properties see Mertens [1992].

[3]The letters $\{X, Y, W, Z\}$ refers to the points labeled in Figure 2.3.

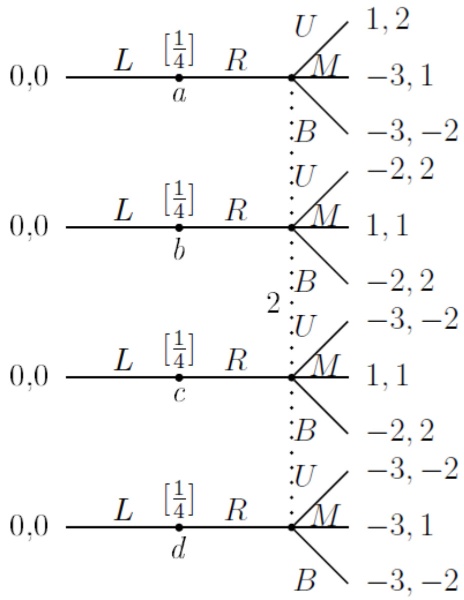

Figure 2.2: Hillas signaling game

This is no longer true if the sender is regarded as a single player. Consider, as an example, the pair of strategies $(RLLL; LLLR)$. For every strategy of the receiver in XW, $RLLL$ is strictly preferred, by player i, to $LLLR$.

In other terms, for player i, a mistake by his agent a is less costly than a mistake by his agent d. Thus when choosing a perturbed strategy he would play the one with $RLLL$ receiving a higher probability than $LLLR$. On the other hand, agents aim just at maximizing their own utility and have no chance to coordinate their strategies. Since, given a strategy in XW, the sender will never willingly choose R at either c or d the receiver, at his information set, should put all weight on the types a and b which upsets the equilibrium.

Therefore, while the set of perturbed games has to be enlarged to satisfy the property of backward induction, when this happens in a natural way by replacing every pure strategy with a set of its perturbations, the player splitting property is violated. This approach and re-

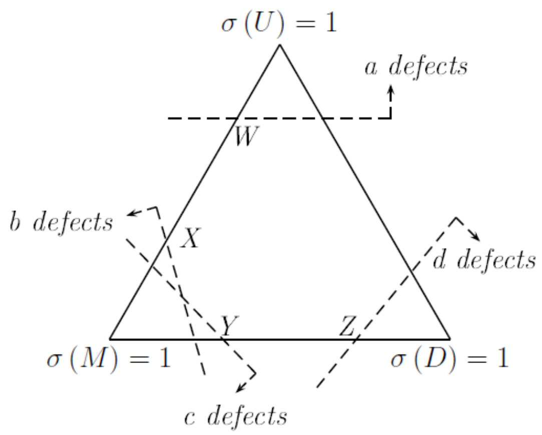

Figure 2.3: Hillas signaling game: sender's strategic choices

sults, while reached independently, are analogue to the ones proposed in an unpublished paper by Reny[4].

2.3 On the introduction of dominated strategies and irrelevant players

The effect of the introduction of a new dominated strategy within the strategy set of a player has been widely discussed by Kohlberg and Mertens [1986]:

(...) *One might think that the iterated dominance requirement should apply not just for deletion but also for addition of dominated strategies, i.e. that a strategically stable set of equilibria in a game G must be contained in a strategically stable set of equilibria in any game G' obtained from G by addition of a dominated strategy. However we disagree* (...).

To support their thesis, Kohlberg and Mertens [1986] proposed the game represented in Figure 2.4. Every single point on the interval $(3, 2)$ to $(2, 2)$ is a strategically stable equilibrium.

[4]Prof. P. Reny private communication.

	L	R
T	$3,2$	$2,2$

Figure 2.4: Game $A_{2.2}$

However when the dominated strategy $[M]$ is added (Figure 2.5) by admissibility only $(3,2)$ is strategically stable.

	L	R
T	$3,2$	$2,2$
M	$1,1$	$0,0$

Figure 2.5: Game $A_{2.3}$

More interestingly, when a second dominated strategy $[B]$ is introduced (Figure 2.6) then any strategically stable set must include both $(3,2)$ and $(2,2)$. Thus the introduction of dominated strategies might shrink the number of stable sets not their width.

The property of Iterated dominance and Forward Induction as defined in Kohlberg and Mertens [1986] states this formally.

	L	R
T	$3,2$	$2,2$
M	$1,1$	$0,0$
B	$0,0$	$1,1$

Figure 2.6: Game $A_{2.4}$

Consider, as a second example, the three player game proposed by Gul where Player 1 starts by either taking an outside option $[s_1^1]$ which yields payoffs $(2,0,0)$ or moving into a simultaneous move subgame represented by Figure 2.7 where each of the three players has two choices.

It is well known that while this game admits a unique sequential equilibrium $\sigma^* = \left\{\frac{1}{2}\left[s_1^2\right] + \frac{1}{2}\left[s_1^3\right]; \frac{1}{2}\left[s_2^1\right] + \frac{1}{2}\left[s_2^2\right]; \frac{1}{2}\left[s_3^1\right] + \frac{1}{2}\left[s_3^2\right]\right\}$ there exists a stable set of equilibria $\{[s_1^1]; [s_2^1]; [s_3^1]\} \cup \{[s_1^1]; [s_2^2]; [s_3^2]\}$ that doesn't contain it.

	s_3^1	s_3^2
s_2^1	$0, 3, 3$	$5, 1, 1$
s_2^2	$5, 1, 1$	$1, 0, 0$

s_1^2

	s_3^1	s_3^2
s_2^1	$1, 0, 0$	$5, 1, 1$
s_2^2	$5, 1, 1$	$0, 3, 3$

s_1^3

Figure 2.7: Gul's game: simultaneous move sub-game

The sequential equilibrium σ^* is preferred by player 1 to any equilibrium within the stable set S. However, although player 1 moves first, there is no chance for him to induce σ^*. Consider the set of perturbed strategies $\left\{\hat{s}_1^j\right\}_{j=1}^3$ with $\hat{s}_1^j = (1-\epsilon)\left[s_1^j\right] + \epsilon\ \left(\frac{1}{2}\left[s_1^2\right] + \frac{1}{2}\left[s_1^3\right]\right)$ for every $j = \{1, 2, 3\}$.

Given the choice of $\hat{s}_1^1$, even if the equilibrium strategy $\sigma_i^* = \frac{1}{2}\left[s_1^2\right] + \frac{1}{2}\left[s_1^3\right]$ is played within the subgame reached with vanishing probability ϵ, there exists a Nash equilibrium in which players 2 and 3 play, respectively, s_2^1 and s_3^1. Given $[s_2^1, s_3^1]$, player 1 would confirm the choice of $\hat{s}_1^1$ since the perturbation of his pure strategies is unique and cannot be modified.

Consider now the introduction of a dominated strategy s_1^4 for player 1. First, equation (2.2) implies that strategy s_1^4 is strictly dominated by the outside option given a value of ϵ positive and arbitrarily close to zero. Second, equation (2.3) defines the new payoffs given the dominated strategy s_1^4 and implies, for instance, that the strategy profile (s_1^4, s_3^1) corresponds, for player 2, to the strategy profile (s_1^2, s_3^1) in the original game.

	s_3^1	s_3^2
s_2^1	$2-\epsilon, 3, 0$	$2-\epsilon, 1, 1$
s_2^2	$2-\epsilon, 1, 1$	$2-\epsilon, 3, 0$

s_1^4

Figure 2.8: Gul's game: new dominated strategy s_1^4

$$u_1\left(s_1^4, s_{-1}\right) = u_1\left(s_1^1, s_{-1}\right) - \epsilon \quad \text{for} \quad \forall s_{-1} \in S_{-1} \quad \epsilon > 0 \tag{2.2}$$

$$u_2\left(s_1^4, \cdot, s_3^1\right) = u_2\left(s_1^2, \cdot, s_3^1\right); \; u_2\left(s_1^4, \cdot, s_3^2\right) = u_2\left(s_1^3, \cdot, s_3^2\right);$$
$$u_3\left(s_1^4, s_2^1, \cdot\right) = u_3\left(s_1^3, s_2^1, \cdot\right); \; u_3\left(s_1^4, s_2^2, \cdot\right) = u_3\left(s_1^2, s_2^2, \cdot\right) \tag{2.3}$$

Loosely speaking the game represented in Figure 2.8 is strategically equivalent, for players 2 and 3, to the subgame in Figure 2.7: given s_1^4, if player 3 played s_3^1 then player 2 would play $s_2^1 = BR_2\left(s_1^4, s_3^1\right)$ with associated payoff 3.

Given s_2^1 player 3 would play s_3^2 with associated payoff 1 that is what he would get in the original subgame if, once given the strategy profile $\left(s_3^1, s_2^1\right)$ as initial point, player 1 could freely choose between s_1^2 and s_1^3. Since $BR_1\left(s_3^1, s_2^1\right) = s_1^3$ then player 3 would deviate to s_3^2 with $u_3\left(s_1^3, s_2^1, s_3^2\right) = 1$.

In other terms the dominated strategy mimics what would happen in the original game when player 1's outside strategy s_1^1 is replaced by an n-tuple of differently perturbed strategies as in the fictitious game devised to prove the backwards induction property for F - stable equilibria.

Thus if each player i's strategy were perturbed towards s_1^4 with the highest vanishing probability the resulting perturbed game would have σ^* as its unique equilibrium.

Finally, given the initial set of players I, define any additional player j as irrelevant if his strategic choices don't affect player i's utility for every $i \in I$.

It is obvious that if a finite collection J of $M \geq 0$ irrelevant players were added to I, a strategy profile σ_I is an equilibrium of the initial game if and only if it is the projection on $\Sigma_I = \prod\limits_{i \in I} \Sigma_i$ of an equilibrium of the new game.

2.4 *G* - stable equilibria

Given $\Gamma = \left\{I, \{S_i\}_{i\in I}, \{u_i\}_{i\in I}\right\}$, define a new collection of games in two steps: first add to I a collection J of $M \geq 0$ irrelevant players with $H = I \cup J$. Then define, for every player $i \in I$ and every strategy $s_i \in S_i$, a finite set Φ_{s_i} including both s_i and N_i new strategies $\{\bar{s}_{i,n}\}_{n=1}^{N_i}$ strictly dominated by s_i with $N_i \geq 1$.

Let $\tilde{S}_i$ be player i's new strategy set obtained by adding to S_i the finite collection $\bar{S}_i$ of $|S_i| \times N_i$ dominated strategies and define the projection function $\pi_i : \tilde{S}_i \to S_i$ from player i's new strategy set to S_i with, for every $\tilde{s}_i \in \tilde{S}_i$, $\pi_i(\tilde{s}_i) = s_i$ if and only if $\tilde{s}_i \in \Phi_{s_i}$.

Given ϵ_i strictly positive and arbitrarily close to zero, and any real valued multilinear functions $\bar{u}_{i,n}$ defined on $\prod\limits_{i \in H} \Sigma_i$, determine for every $\{s_j\}_{j\neq i} \in \tilde{S}_{-i}$, the payoffs of each n^{th} strictly dominated strategy $\bar{s}_{i,n}$ in every Φ_{s_i} as:

$$u_{i,n}(\bar{s}_{i,n}, \{s_j\}_{j\neq i}) = \epsilon_i(\bar{u}_{i,n}(\pi_i(\bar{s}_{i,n}), \{f_{j,n}(s_j)\}_{j\neq i}) \tag{2.4}$$

with $f_{j,n} : \tilde{\Sigma}_j \to \Sigma_j$ linear function independent of i. For any other player j we have:

$$u_j(\bar{s}_{i,n}, \{s_z\}_{z\notin\{i,j\}}, s_j) = u_j(\pi_i(\bar{s}_{i,n}), \{\pi_z(s_z)\}_{z\notin\{i,j\}}, s_j)$$
$$\text{for } \forall \{s_z\}_{z\notin\{i,j\}} \in \tilde{S}_{-ij},\ \forall s_j \in S_j \text{ and } \forall j \neq i \quad (2.5)$$

Condition (2.4) implies that, given values of every ϵ_i sufficiently close to zero, each new strategy in $\{\bar{s}_{i,n}\}_{n=1}^{N_i}$ is strictly dominated for every player i.

Condition (2.5) excludes that any (weakly) dominated strategy $s_j \in S_j$ for any player $j \neq i$ could become undominated given the introduction of new dominated strategies for his opponents.

Let ζ the collection of all games $\tilde{\Gamma} = \left\{H, \left\{\tilde{S}_i\right\}_{i\in H}, \{u_i\}_{i\in H}, \{\epsilon_i, \bar{u}_{i,n}, \{f_{j,n}\}_{j\neq i}\right\}_{n=1}^{N_i}\Big\}_{i\in I}\Big\}$ each obtained from Γ by adding any finite set J of M irrelevant players and including in every strategy set $\{S_i\}_{i\in I}$ a finite collection $\bar{S}_i$ of $|S_i| \times N_i$ dominated strategies verifying conditions (2.4) and (2.5).

For each game $\tilde{\Gamma}$ in ζ redefine the set P_ϵ of perturbations η as $P_\epsilon = \left\{\epsilon \cdot \vec{\sigma} \mid 0 < \epsilon < 1,\ \vec{\sigma} \in \bar{\Sigma}_I^k \backslash \partial \bar{\Sigma}_I^k,\ k \in \mathbb{Z}\right\}$ where $\bar{\Sigma}_I = \Delta \prod_{i\in I} \bar{S}_i$ and $\bar{\Sigma}_I^k$ is the k-fold Cartesian product of $\bar{\Sigma}_I$ with k integer number.

Let $\eta_i = \epsilon \cdot \vec{\sigma}_i$ be the k-dimensional vector that represents the projection onto $\bar{\Sigma}_i^k \backslash \partial \bar{\Sigma}_i^k$ of the perturbation $\eta = \epsilon \cdot \vec{\sigma}$.

Given $\eta \in P_\epsilon$ let $\tau_\eta(s_i) = (1-\epsilon)\ \vec{s}_i + \eta_i$ be the k-dimensional vector of strategies replacing strategy s_i in player i's strategy set $\tilde{S}_i$ where $\vec{s}_i$ is the k-dimensional vector with each entry equal to s_i.

Let $\tilde{\Gamma}_\eta$ be the game obtained from $\tilde{\Gamma}$ by replacing each pure strategy $s_i \in \tilde{S}_i$ of each player i in I by the vector $\tau_\eta(s_i)$.

The definition of stable set of equilibria is then modified accordingly:

Definition 7. *Q is a G - stable set of equilibria of a game Γ if it is a set of equilibria, minimal with respect to the following property G:*

Property (G). *Q is a closed set of Nash equilibria of Γ satisfying: for any $\delta > 0$ there exists some $\epsilon_0 > 0$ such that any perturbed game $\tilde{\Gamma}_\eta$ with $\eta \in P_\epsilon$ and $\epsilon_0 > \epsilon > 0$ has an equilibrium whose projection onto $\Sigma_I = \prod_{i \in I} \Sigma_i$ is δ-close to Q.*

2.4.1 Properties

Given the definition of G - stable equilibria its properties are now analyzed:

Proposition 15 (Existence). *Every normal form game Γ has a G - stable set of equilibria.*

Proof. Since the projection onto $\Sigma_I = \prod_{i \in I} \Sigma_i$ of the set of equilibria of each game in ζ coincides with the set of equilibria of the initial game Γ, existence of G - stable sets comes easily from existence of an hyperstable set of equilibria for any normal form game as proved in Kohlberg and Mertens [1986]. □

Proposition 16 (Connectedness). *Every game has a G - stable set contained in a single connected component of the set of Nash equilibria.*

Proof. Kohlberg and Mertens [1986] proved that every game has an hyperstable set of equilibria contained in a single connected component of the set of Nash equilibria. Thus, since every hyperstable set includes a G - stable set of equilibria, every normal form game has a G - stable set which is contained in a single connected component of the set of Nash equilibria. □

Proposition 17 (Backwards induction). *A G - stable set of any finite normal form game Γ always includes a proper equilibrium of Γ.*

Proof. Given the initial game $\Gamma = \left\{I, \{S_i\}_{i \in I}, \{u_i\}_{i \in I}\right\}$ let $M = 0$ and introduce for every player i and every strategy $s_i \in S_i$ a unique new dominated strategy $\bar{s}_{i,1}$ with $\bar{S}_i$ the strategy set including just the new dominated strategies.

Assume that $f_{j,1}(s_j) = \pi_j(s_j)$ for every $s_j \in \tilde{S}_j$, $\bar{u}_{i,n} = u_i$ for every player $i \in I$.

Thus player i's payoff from any dominated strategy $\bar{s}_{i,1}$ in $\bar{S}_i$ is obtained by rescaling the payoff corresponding to $\pi_1(\bar{s}_{i,1})$ in Γ by an identical factor ϵ_i.

Besides, by condition (2.5), every strategy $\bar{s}_{i,n}$ is payoff equivalent to $\pi_i(\bar{s}_{i,n})$ for any strategy $s_j \in S_j$ and any player $j \neq i$ in I.

Define a new game $\tilde{\Gamma} = \left\{I, \left\{\tilde{S}_i\right\}_{i\in I}, \{u_i\}_{i\in I}\right\}$ with $\tilde{S}_i = S_i \bigcup \bar{S}_i$ and consider, for each player i, the set $\bar{E}_i = \left\{\bar{e}_i^j \in \bar{S}_i^n\right\}_{j=1}^{n!}$ of all permutations $\bar{e}_i^j$ of his new dominated strategies, where $\bar{S}_i^n$ is the n-fold Cartesian product of $\bar{S}_i$ and $n = |\bar{S}_i|$.

Then, for every player i, construct, from each ordering $\bar{e}_i^j$, a mixed strategy $\sigma_i\left(\bar{e}_i^j\right)$ such that when $\sigma_i\left(\bar{e}_i^j\right)$ is chosen, each strategy in the ordering $\bar{e}_i^j$ is played with probability $\frac{(1-\epsilon)\epsilon^{h-1}}{(1-\epsilon^n)}$ being h its position in $\bar{e}_i^j$.

Given $\tilde{E}_i = \left\{\sigma\left(\bar{e}_i^j\right) | \bar{e}_i^j \in \bar{E}_i\right\}$, define $\hat{\Gamma}$ as the game derived from $\tilde{\Gamma}$ in which each strategy $\tilde{s}_i \in \tilde{S}_i$ of each player i is replaced by the following set of perturbed strategies:

$$\left\{(1-\epsilon_i)\,\tilde{s}_i + \epsilon_i \sigma_i\left(\bar{e}_i^j\right)\right\}_{\sigma_i\left(\bar{e}_i^j\right)\in\tilde{E}_i} \tag{2.6}$$

Therefore each player when choosing a pure strategy in the new game $\hat{\Gamma}$ actually chooses with probability $(1-\epsilon_i)$ a strategy in $\tilde{S}_i$ and, with probability ϵ_i a lottery over a given ordering of his new dominated strategies in $\bar{S}_i$.

The fictitious game $\hat{\Gamma}$ is strategically equivalent to the one presented in the proof of the property of backwards induction proposed for F-stable equilibria even if we now consider orderings among strategies in $\bar{S}_i$ instead of S_i.

Pick an equilibrium point of the new game $\hat{\Gamma}$ in the neighbourhood of its set of equilibria. It is an ϵ-proper equilibrium of the initial game. □

Proposition 18 (Invariance)**.** *Every G - stable set is also a G - stable set of any equivalent game (i.e. having the same reduced normal form).*

Proof. The proof of invariance of G - stable sets is almost identical to the one proposed for stable sets and F - stable sets since, in each game $\tilde{\Gamma}_\eta$, all strategies are identically perturbed.

However, for G - stable sets, it has to be considered the effect of the introduction of a randomly redundant strategy on the set ζ.

Given the initial game Γ and any new game $\tilde{\Gamma}$ in the corresponding set ζ, if a randomly redundant strategy $\dot{s}_i$ were added to S_i the collection $\Phi_{\dot{s}_i}$ of new strategies should be considered. Each strategy in $\Phi_{\dot{s}_i}$, however, would correspond to a randomization of a finite set of dominated strategies within the set $\tilde{S}_i$ defining $\tilde{\Gamma}$.

Besides, if a randomly redundant strategy $\dot{s}_j$ were added to player j's strategy set with $j \neq i$, there would be no effect on the payoffs of player i's strategies in $\bar{S}_i$ provided that, by construction, $f_{j,n}$ is a linear function for every n. □

Proposition 19 (Admissibility)**.** *Given any equilibrium in a G - stable set, every equilibrium strategy for every player i is undominated.*

Proof. For this property to be verified, condition (2.5) is crucial since, as already pointed out, it ensures that a (weakly) dominated strategy for player j could not be made undominated by the introduction of a new dominated strategy for some player $i \neq j$.

Once this possibility has been excluded, the property comes easily since, given $N_i \geq 1$ for every player $i \in I$, de facto it is as if every strategy s_i in S_i were to be played with strictly positive probability in every perturbed game. □

Proposition 20 (Iterated dominance and forward induction)**.** (A) *A G - stable set of a game Γ contains the G - stable set of any game obtained from Γ*

by deleting a dominated strategy and (B) *a G - stable set of a game* Γ *contains the G - stable set of any game obtained from* Γ *by deleting a strategy that is an inferior response in all the equilibria of the set (Forward induction).*

Proof. Given a perturbed game $\tilde{\hat{\Gamma}}_\eta$ derived from the initial game $\hat{\Gamma}$ without the dominated strategy $\hat{s}_i$, construct a close-by perturbation $\Gamma_\eta(z)$ in two steps: first introduce the dominated strategy $\hat{s}_i$ in the strategy set $\hat{S}_i$ of player i. Perturb all strategies in $\Phi_{\hat{s}_i}$ like any other strategy in $\tilde{\hat{\Gamma}}_\eta$. The introduction of $\hat{s}_i$ implies $N_i \geq 1$ new dominated strategies. Let $\dot{S}_i$ the collection of all the dominated strategies $\dot{s}_{i,n}$ with $\pi_i(\dot{s}_{i,n}) = \hat{s}_i$.

Then construct the perturbed game $\Gamma_\eta(z)$ by slightly perturbing any player i's strategy towards a completely mixed strategy in $\dot{S}_i$ by z. The game $\Gamma_\eta(z)$ is a perturbation of the initial game. Obviously in no equilibrium the dominated strategy $\hat{s}_i$ will be played and taking the limit for $z \to 0$ of these equilibria will give an equilibrium of $\hat{\Gamma}_\eta$ close to the G - stable set.

The proof of the part (B) of Proposition (12) is identical and thus omitted. □

Since a G-stable set of a normal form game Γ is defined by considering, among others, games obtained from Γ by adding new dominated strategies one might expect that a stronger version of the property of iterated dominance holds. However for the property of admissibility to be verified, just a subset of all possible dominated strategies is considered. Hence iterated dominance can be verified just in its original formulation.

Proposition 21 (Small worlds and Decomposition)**.** *A G - stable set of any finite game* Γ *satisfies small worlds and decomposition axioms.*

Proof. This property is self evident given that the introduction of a set of irrelevant players is one of the allowed modifications of the original game Γ. □

Proposition 22 (Player splitting). *Given a partition of the information set of some player, such that no play intersects two different partition elements, consider the new game obtained by letting a different agent of this player man each of these partition elements, and receive the same payoff as this player for those play that intersect his own information sets while he receives an arbitrary payoff on the other plays. This new game, where this player is replaced by these agents, has the same G - stable sets as the old game.*

Given the initial game Γ create a new game Γ^a by splitting player i into K agents indexed by k each managing one partition element of his information set.

The main difference between the two games is that a single player trembles in a completely correlated way while, once splitted, his agents tremble independently. However, since only one partition of the game is going to occur, only the marginal probabilities matter.

Consider player i's set of new dominated strategies $\bar{S}_i$ and, for each n^{th} new dominated strategy $\bar{s}_{i,n} \in \bar{S}_i$ with payoffs defined by $\left\{\epsilon_i, \bar{u}_{i,n}, \{f_{j,n}\}_{j\neq i}\right\}$, let $\pi_i\left(\bar{s}_{i,n}\right)$ its projection on S_i.

With a slight abuse of notation indicate the behavioral strategy s_{i_k} for agent k implied by $s_i = \pi_i\left(\bar{s}_{i,n}\right)$ as $s_{i_k} = \pi_{i_k}\left(\bar{s}_{i,n}\right)$. For every strategy $\bar{s}_{i,n} \in \bar{S}_i$, introduce for every agent k of player i a dominated strategy $\bar{s}_{i_k,n}$ with $\pi_{i_k}\left(\bar{s}_{i_k,n}\right) = \pi_{i_k}\left(\bar{s}_{i,n}\right)$ and payoffs defined by $\left\{\epsilon_{i_k}, \tilde{u}_{i_k,n}, \left\{\tilde{f}_{j,n}\right\}_{j\neq i_k}\right\}$ as follows:

$$u_{i_k,n}\left(\bar{s}_{i_k,n}, \{s_j\}_{j\neq i_k}\right) = \epsilon_{i_k}\left(\tilde{u}_{i_k,n}\left(\pi_{i_k}(\bar{s}_{i_k,n}), \left\{\tilde{f}_{j,n}\left(s_j\right)\right\}_{j\neq i_k}\right)\right) \tag{2.7}$$

with for every agent i_k of player i:

$$\begin{array}{l} \tilde{u}_{i_k,n} = \bar{u}_{i,n} \\ \epsilon_{i_k} = \epsilon_i \quad \text{for } \forall i_k \\ \tilde{f}_{j,n}\left(s_j\right) = \left\{ \begin{array}{l} f_{j,n}\left(s_j\right) \ \forall j \neq i \\ \pi_{i_{\hat{k}}}\left(\bar{s}_{i,n}\right) \ \forall i_{\hat{k}} \neq i_k \end{array} \right. \end{array} \tag{2.8}$$

Thus the definition of k^{th} agent's payoff strategy associated to $\bar{s}_{i_k,n}$ induces a correlation among player i's agents mimicking player i's strategic choices. Similarly, given N_{i_k} dominated strategies for every agent of player i, $\prod_{k \in K} |N_{i_k}|$ dominated strategies should be introduced in the original game and player i's corresponding payoffs can be appropriately defined to replicate his agents' strategic choices.

2.5 Conclusions

This chapter introduces, as major contribution, a new class of perturbed games that stem from games obtained by introducing irrelevant players and new dominated strategies in the original game.

Both variations have no effect on the set of equilibria but allow to widen the resulting stable sets of equilibria up to satisfy all properties proposed in Kohlberg and Mertens [1986] and in Mertens [1989].

It remains unclear, and a topic for further research, if the introduction of new properties might lead to prefer G - stable equilibria or M - stable equilibria as proposed in Mertens [1989]. A first insight could be offered by determining the geometric relation between the different sets following the contribution by Govindan [1995].

Chapter 3

Stable correlated equilibria

3.1 Introduction

A game with communication arises when players have the opportunity to communicate with each other prior to the choice of their actions in the actual game.

In this setting, the presence of a trustful mediator is a particularly powerful device since it allows players to implement correlated strategies. The mediator privately recommends actions to each player according to the realization of an agreed upon correlation device and each player decides whether to obey the recommendation.

A correlated equilibrium, as defined in Aumann [1974], is a self-enforcing correlated strategy: no player has an incentive to deviate from the received recommendation given the information at her disposal.

Formally, given a normal form game $\Gamma = \left\{I, \{S_i\}_{i\in I}, \{u_i\}_{i\in I}\right\}$, a correlated equilibrium is any strategy profile σ^* in ΔS with $S = \prod\limits_{i\in I} S_i$ such that for every player the following collection of incentive constraints is satisfied:

$$\sum_{s_{-i} \in S_{-i}} \sigma^* (s_i, s_{-i}) (u_i (s_i, s_{-i}) - u_i (e_i, s_{-i})) \geq 0 \quad \text{for } \forall s_i \in S_i \text{ and } \forall e_i \in S_i \tag{3.1}$$

If at least one constraint were binding, the corresponding correlated equilibrium might disappear if, for each player i, every strategy had a strictly positive vanishing probability.

Thus Myerson [1986] introduced the concept of acceptable correlated equilibria that is correlated equilibria in which the obedient behaviour by every player could still be rational when each player i always has a strictly positive vanishing probability ϵ_i of trembling so deviating with respect to the mediator's recommendation.

Acceptable correlated equilibria hinge on the related concept of acceptable strategies: a pure strategy $s_i \in S_i$ is acceptable if and only if it can be rationally used by an obedient player i when the trembling probabilities are arbitrarily close to zero.

Given the initial game Γ, an acceptable correlated equilibrium is any correlated equilibrium of the game obtained from Γ by deleting unacceptable strategies. Thus the set of acceptable correlated equilibria is closed and convex.

Myerson [1986] proved that the set of weakly dominated strategies is a subset of the set of unacceptable strategies. Therefore no weakly dominated strategy is played with strictly positive probability in an acceptable correlated equilibrium.

	w_2	x_2	y_2	z_2
w_1	$2,2$	$1,1$	$0,0$	$0,0$
x_1	$1,1$	$1,1$	$2,0$	$2,0$
y_1	$0,0$	$0,2$	$3,0$	$0,3$
z_1	$0,0$	$0,2$	$0,3$	$3,0$

Figure 3.1: Game $A_{3.1}$

More interestingly, the set of unacceptable strategies might include also undominated strategies. As an example, in the game represented in Figure 3.1, there are no (weakly) dominated strategies but y_1, z_1, y_2 and z_2 are all unacceptable.

Unfortunately an equilibrium that is neither perfect nor a linear combination of perfect equilibria could be an acceptable correlated equilibrium according to the definition proposed by Myerson [1986]. This seems a consequence of the fact that in a three player game an equilibrium might be not perfect even if no weakly dominated strategy is played.

	x_2	y_2
x_1	$1, 1, 1$	$4, 4, 0$
y_1	$3, 2, 2$	$3, 2, 2$

x_3

	x_2	y_2
x_1	$1, 1, 1$	$0, 0, 4$
y_1	$0, 0, 0$	$0, 0, 0$

y_3

Figure 3.2: Game $A_{3.2}$ - Three player game in strategic form

Consider the game represented in Figure 3.2. Both strategy profiles (x_1, x_2, y_3) and (y_1, x_2, x_3) are Nash equilibria of the game but (y_1, x_2, x_3) is not perfect. However, all strategies are acceptable so both equilibria are acceptable correlated equilibria.

Mertens and Dhillon [1996] overcame this flaw by introducing the concept of perfect correlated equilibria (PCE) that applies the analogue of Selten perfection (Selten [1975]) to correlated equilibria.

In more details Mertens and Dhillon [1996] defined a perfect correlated equilibrium (PCE) of a normal form game Γ as a perfect equilibrium of an extended game obtained from Γ by introducing a correlation device.

A perfect correlated equilibrium distribution (PCED) is then a probability distribution over the set of pure strategy profiles S determined by a perfect equilibrium of some extended game.

The authors show that in a simple two-player game the revelation principle fails to hold for PCEDs i.e. certain PCEDs are not obtainable by using each player i's strategy set as the message space. The reason for this failure is that players need more 'coordinates' in their messages to encode information about how they are to tremble. Just for the two player case the two-fold product of each strategy space suffices as the message space, while Mertens and Dhillon [1996] were unable to extend their results to N-player games.

A new concept of stable correlated equilibria is here introduced to overcome the limits of both approaches and to try to extend the idea of stable sets, as proposed by Kohlberg and Mertens [1986], to correlated equilibria. It is interesting to note that every desirable property defining strategically stable sets in non cooperative games remains a natural requirement in a cooperative setting:

Existence: every normal form game Γ has a stable set of correlated equilibria;

Convexity: since the set of correlated equilibria as defined by Aumann [1974] is convex, the property of connectedness should be enhanced by requiring that convexity is preserved for any stable set of correlated equilibria;

Backwards induction: since backwards induction implies that at every point during any play of the game each player believes that his prescribed strategy will maximize his expected payoff in the remainder of the game, there is no reason to assume that a refinement of correlated equilibria should exclude all of these Nash equilibria;

Invariance: a solution of a game is also a solution of any game with the same reduced normal form. As in a non cooperative setting also in a cooperative one there is no reason to assume that the introduction, among the pure strategies, of a randomly redundant one for a player i, should modify his strategic choices.

Admissibility: players' strategies are undominated at any point in the stable set of correlated equilibria;

Iterated dominance and forward induction: the stable set of correlated equilibria of a game Γ contains the stable set of correlated equilibria of any game $\tilde{\Gamma}$ obtained from Γ by deleting a strategy that is either dominated or that is always an inferior response in every equilibrium of the set;

Small worlds: if J is a small world in Γ, then a set of strategy profiles is a stable set of correlated equilibria of the game Γ_J if and only if it is the projection of a stable set of correlated equilibria of the game Γ;

Decomposition: if for a game Γ both J and $I\backslash J$ constitute small worlds in Γ, then a set Q of equilibria is a stable set of correlated equilibria of the game Γ if and only if $Q = Q_J \times Q_{I\backslash J}$ with Q_J an a stable set of correlated equilibria of the game Γ_J and $Q_{I\backslash J}$ a stable set of correlated equilibria of the game $\Gamma_{I\backslash J}$;

Player splitting: given a partition of the information set of some player, such that no play intersects two different partition elements, the new game, where this player is replaced by a set of agents each managing one of the partition elements, has the same stable set of correlated equilibria as the initial game;

Unfortunately this last property doesn't seem to be satisfied. This result is similar to the one obtained when the same setting is applied to non cooperative games.

3.2 Stable correlated equilibria

Given the initial game $\Gamma = \left\{I, \{S_i\}_{i\in I}, \{u_i\}_{i\in I}\right\}$ define the set P^ϵ of perturbations η as $P^\epsilon = \{\sigma \cdot \epsilon \,|\, 0 < \epsilon < 1 \,\text{and}\, \sigma \in \Sigma\backslash\partial\Sigma\}$. Let S^ϵ the resulting set of perturbed pure strategies profiles with $S^\epsilon = \{(1-\epsilon)\, s + \eta|\, s \in S \,\text{and}\, \eta \in P^\epsilon\}$ and S_i^ϵ the projection of S^ϵ onto S_i.

The definition of stable correlated equilibria relies on the assumption that each player i's strategic behavior is independent of his strategies' perturbations. This result would come easily if all strategies of every player were identically perturbed or, not equivalently, if each player i's payoff function were redefined as:

$$\bar{u}_i(\sigma_i^\epsilon, \sigma_{-i}^\epsilon) = u_i(\lim_{\epsilon \to 0} \sigma_i^\epsilon, \sigma_{-i}^\epsilon) \quad \text{for} \quad \forall \sigma^\epsilon \in \Sigma^\epsilon = \Delta S^\epsilon \tag{3.2}$$

Condition (3.2) implies that each player i's utility is independent of his own strategies' perturbations.

The two approaches might lead to different results since if we considered, for each pure strategy profile $s \in S$ with $S = \prod_{i \in I} S_i$, a set of perturbations P^ϵ with $|P^\epsilon| > 1$, the choice of the perturbation would be part of player i's strategic behavior if and only if condition (3.2) were not imposed on player i's utility function.

Thus, given a perturbed strategy σ_{-i}^ϵ, the set of player i's best replies might be different in the two settings with $BR_i\left(\sigma_{-i}^\epsilon\right) \subseteq \overline{BR}_i\left(\sigma_{-i}^\epsilon\right)$ where $\overline{BR}_i\left(\sigma_{-i}^\epsilon\right)$ is the set of best replies by player i when condition (3.2) holds.

Condition (3.2) might be disputable, mainly in a non cooperative setting. It seems less critical in the realm of correlated equilibria since, while each player receives a clear recommendation from the mediator, his strategic decision will depend only on the unknown recommendations σ_{-i}^* privately sent to his opponents and their strategy perturbations might be regarded as part of the uncertainty about σ_{-i}^*.

Given $\Gamma = \left\{I, \{S_i\}_{i \in I}, \{u_i\}_{i \in I}\right\}$, define, for every player i, the collection $\mathcal{A}_i$ of all convex and compact sets $G_{s_i} = BR_i^{-1}(s_i) := \{\sigma_{-i} \in \Sigma_{-i} | BR_i (\sigma_{-i}) \cap s_i \neq \varnothing\}$ with $G_{s_i} \subseteq \Sigma_{-i}$ and $s_i \in S_i$.

Given the minimal set Θ_i generated by $\mathcal{A}_i$ with set difference and set intersection as binary operations, construct a new strategy set as follows: first define the collection Ψ_i of all orderings over the sets $A_i \in$

Θ_i such that, given any pair of non empty sets $(A_i, \hat{A}_i)$, if $A_i \subseteq \hat{A}_i$ then $\hat{A}_i$ must be ranked above A_i. Since not all orderings of the sets A_i in Θ_i might be considered, the set Ψ_i has cardinality $|\Psi_i| \leq K_i!$ with $|\Theta_i| = K_i$.

Then introduce a correspondence $F_i : \Psi_i \to \Omega_i$ where Ω_i is the collection of all orderings ω_i of any set of K_i, non necessarily distinct, pure strategys in S_i. For each ordering $\psi_i \in \Psi_i$, define $F_i(\psi_i)$ as the set of all orderings in Ω_i such that A_i in ψ_i and s_i in ω_i are identically ranked only if $A_i \subseteq G_{s_i}$.

Let $w_{s_i} = \frac{(1-\epsilon)\epsilon^{k-1}}{1-\epsilon^{K_i}}$ be the weight assigned to the strategy s_i when ranked in k^{th} position in the ordering ω_i with $\epsilon > 0$ and arbitrarily close to zero.

Then, for every ordering ω_i in $F_i(\psi_i)$, a (mixed) strategy $\sigma_{\omega_i} \in \Sigma_i$ can be defined in which each strategy s_i is played with a probability equal to its weight.

Consider a new game $\bar{\Gamma} = \left\{ I, \left\{ \bar{S}_i \right\}_{i \in I}, \left\{ \bar{u}_i \right\}_{i \in I} \right\}$ with $\bar{u}_i$ defined by condition (3.2) and $\bar{S}_i = \{ \sigma_{\omega_i} | \omega_i \in F_i(\psi_i) \text{ with } \psi_i \in \Psi_i \}$ for every $i \in I$.

If payoff functions $\{u_i\}_{i \in I}$ were considered, we'd eventually confine ourselves to perturbations that exclude dominated strategies since every ordering including a dominated strategy would be strictly dominated.

The mediator will select a point in $\bar{\Sigma}$ and will recommend privately to each player a strategy in $\bar{\Sigma}_i$ to be played. Note that any recommended strategy will be regarded by every player i as payoff equivalent to the limit of the corresponding perturbed strategy in ΔS_i and, by construction, any weakly dominated equilibrium strategy won't be part of any mediator's recommendation.

Definition 8 (ϵ-stable correlated equilibrium)**.** *An ϵ-stable correlated equilibrium of a game Γ is any correlated equilibrium of the perturbed game $\bar{\Gamma}$.*

Definition 9 (Stable correlated equilibrium)**.** *A stable correlated equilibrium of a game Γ is the limit of an ϵ-fully stable correlated equilibrium for $\epsilon \to 0$.*

3.2.1 Properties

Given the definition of stable correlated equilibria its properties are now analyzed:

Proposition 23 (Convexity)**.** *The set of stable correlated equilibria is convex.*

Proof. The limit of the set of ϵ-correlated equilibria of the game $\bar{\Gamma}$ is the limit of the set of solutions of a system of linear inequalities. □

Proposition 24. *The set of stable correlated equilibria does always contain a proper equilibrium.*

Proof. In order to ensure that a stable set of equilibria of a non cooperative game does contain a proper equilibrium Kohlberg and Mertens [1986] had to create a fictitious game $\tilde{\Gamma}$ whose equilibria are all ϵ-proper equilibria of the original game since a stable set has to include at least one equilibrium of each allowed perturbed game.

Conversely, since the set of ϵ-correlated equilibria includes every Nash equilibrium of the perturbed game $\bar{\Gamma}$, it is enough to prove that at least one equilibrium of $\bar{\Gamma}$ is an ϵ-proper equilibrium of the initial game Γ.

Recall that the fictitious game $\tilde{\Gamma}$ proposed by Kohlberg and Mertens [1986] to prove existence of proper equilibria is derived from the initial game Γ by redefining each player i's strategy set S_i as follows: given the collection Ψ_i of all orderings ψ_i of player i's pure strategies, let $w_{s_i} = \frac{(1-\epsilon)\epsilon^{k-1}}{1-\epsilon^K}$ be the weight assigned to the pure strategy s_i when ranked in k^{th} position in a given ordering ψ_i with ϵ strictly positive and arbitrarily close to zero. For each ordering ψ_i determine a completely mixed strategy σ_{ψ_i} such that, given σ_{ψ_i}, each pure strategy $s_i \in S_i$ is played with a probability equal to its weight in ψ_i. Denote player i's new strategy set as $\tilde{S}_i = \{\sigma_{\psi_i}\}_{\psi_i \in \Psi_i}$. Kohlberg and Mertens [1986] proved that any equilibrium of the resulting game $\tilde{\Gamma} = \left\{I, \left\{\tilde{S}_i\right\}_{i \in I}, \{u_i\}_{i \in I}\right\}$ is an ϵ-proper equilibrium of the original game Γ.

With respect to the model proposed in Kohlberg and Mertens [1986], the first restriction imposed on the set of admissible orderings by the definition of ϵ-correlated equilibria, is that a (weakly) dominated strategy $s_i \in S_i$ has to receive a weight w_{s_i} lower, by a factor of ϵ, than the corresponding dominant strategy. Besides, for every player i, each ordering might not include all the pure strategies in S_i and a single strategy might appear many times in the same ordering.

However, given condition (3.2), every ϵ-proper equilibrium strategy $\hat{\sigma}_i^{\epsilon}$ could represent a mediator's recommendation $\hat{\sigma}_i^{\epsilon}$ that an obedient player will follow provided that $\lim_{\epsilon \to 0} \hat{\sigma}_i^{\epsilon}$ is a best reply in S_i to $\hat{\sigma}_{-i}^{\epsilon}$ for every player i. $\square$

Proposition 25 (Admissibility)**.** *The set of stable correlated equilibria satisfies admissibility.*

Proof. By construction for each player i in every admissible ordering ω_i, a weakly dominated strategy $s_i \in S_i$ has to receive a weight lower, by a factor of ϵ, than the weight assigned to the corresponding dominant strategy. Therefore no weakly dominated strategy belongs to $\lim_{\epsilon \to 0} \bar{S}_i$. $\square$

Proposition 26 (Iterated dominance)**.** *The stable set of correlated equilibria of a game Γ contains the stable set of correlated equilibria of any game obtained from Γ by deleting either a dominated strategy or a strategy that is an inferior response in all the equilibria of the set.*

Proof. First note that the set $\varnothing$ is always included in Θ_i: if not, then $G = \bigcap_{s_i \in S_i} G_{s_i} \neq \varnothing$ and the intersection of all sets $\{G_{s_i} \backslash G\}_{s_i \in S_i}$ would be empty. Therefore the deletion of a dominated strategy wouldn't affect the collection of sets Θ_i for every player i.

Consider the game $\tilde{\Gamma} = \left\{I, \left\{\tilde{S}_i\right\}_{i\in I}, \{u_i\}_{i\in I}\right\}$ obtained from Γ by deleting the dominated strategy $\tilde{s}_i$.

Define a new correspondence $\tilde{F}_i : \Psi_i \to \tilde{\Omega}_i$ where $\tilde{\Omega}_i$ is the set of all orderings ω_i of any set of $\tilde{K}_i$, non necessarily distinct, pure strategies in $\tilde{S}_i$ with $|\tilde{S}_i| = \tilde{K}_i$.

For each ordering $\psi_i \in \Psi_i$ we would have $\tilde{F}_i(\psi_i) \subseteq F_i(\psi_i)$ since $\tilde{F}_i(\psi_i)$ will not include strategy orderings with $\tilde{s}_i$ equally ranked as $\varnothing$. If any mixed strategies σ_{ω_i} corresponding to an ordering ω_i in $F_i(\psi_i) \backslash \tilde{F}_i(\psi_i)$ were added to the perturbed game, no ϵ - correlated equilibrium of $\tilde{\Gamma}$ would be eliminated: the mediator could always select a point in $\prod_{i\in I} \tilde{S}_i$ and, given condition (3.2), player i will have no incentive to deviate to any new strategy σ_{ω_i} with $\omega_i \in F_i(\psi_i) \backslash \tilde{F}_i(\psi_i)$. $\square$

Proposition 27 (Invariance)**.** *The set of stable correlated equilibria satisfies invariance.*

Proof. Given the initial game Γ create a new game $\hat{\Gamma}$ by introducing a randomly redundant strategy[1] $\hat{s}_i$ for some player i.

Assume that $\hat{G}_{\hat{s}_i} \neq \varnothing$ and recall that if a randomized strategy σ_i is a best reply to a strategy σ_{-i} then every pure strategy s_i in the support of σ_i is a best reply to σ_{-i} as well. Therefore $\hat{G}_{\hat{s}_i} \in \Theta_i$ and $\Theta_i = \hat{\Theta}_i$ since $\hat{G}_{\hat{s}_i} = \bigcap_{s_i \in F_{\hat{s}_i}} G_{s_i}$ where $F_{\hat{s}_i}$ is the support of the mixed strategy $\sigma \in \Delta S_i$ payoff equivalent to $\hat{s}_i$. As a consequence, the set Ψ_i remains unchanged i.e. $\hat{\Psi}_i = \Psi_i$ and any new pure ordering in $\hat{\Omega}_i$ is strategically equivalent to a linear combination of some pure orderings in the corresponding set Ω_i of the original game.

Now assume that $\hat{G}_{\hat{s}_i} = \varnothing$. If $\varnothing \in \Theta$ then the proof just outlined applies. Suppose, on the other hand, that $\varnothing \notin \Theta$. This in turn implies

[1] A pure strategy $\hat{s}_i$ is randomly redundant if and only if it is payoff equivalent to a mixed strategy $\sigma_i \in \Sigma$.

that $\bigcap\limits_{s_i \in S_i} G_{s_i} \neq \varnothing$. Therefore for any randomly redundant strategy $\hat{s}_i$ it must be $G_{\hat{s}_i} \neq \varnothing$, hence a contradiction.

The introduction of the randomly redundant strategy modifies also the set of possible beliefs of any other player $j \neq i$. However, any set G_{s_j} in the original game Γ is just the projection on Σ_{-j} of the corresponding set $\hat{G}_{s_j}$. □

Proposition 28 (Small worlds). *If J is a small world in Γ, then a set of equilibria is a stable set of correlated equilibria of the game Γ_J if and only if it is the projection of a stable set of correlated equilibria of the game Γ.*

Proposition 29 (Decomposition). *If for a game Γ both J and $I \backslash J$ constitute small worlds in Γ, then a set Q of equilibria is a stable set of correlated equilibria of the game Γ if and only if $Q = Q_J \times Q_{I \backslash J}$ with Q_J a stable set of correlated equilibria of the game Γ_J and $Q_{I \backslash J}$ a stable set of correlated equilibria of the game $\Gamma_{I \backslash J}$.*

Both these properties are self evident since, given any small world J in I, when every single strategy of each player $i \in J$ is replaced by a set of its perturbations the set J of players remains a small world in Γ_η.

The last property to be analyzed is player splitting. Unfortunately this natural requirement doesn't seem to be satisfied.

3.3 Conclusions

It is worth noting that the proposed model is far from the one proposed in Kohlberg and Mertens [1986]; the connection with Mertens' stability is confined to the attempt of satisfying the properties defining strategic stability as defined in Mertens [1989]. It is disappointing to observe that the missing property is the property of player splitting since one could expect that, in a cooperative setting, this requirement would be naturally satisfied.

General conclusions

This thesis introduces, as major contribution, a new definition of strategically stable set of equilibria by considering a new class of perturbed games obtained by adding a set of irrelevant players and a collection of dominated strategies to the initial game. Both variations have no effect on the set of equilibria of the original game but allow to widen the resulting stable sets of equilibria up to satisfy all the properties proposed in Kohlberg and Mertens [1986] and in Mertens [1989]. Besides a new fictitious game is proposed to define a new single valued Nash equilibrium refinement, namely *refined equilibria*, and to extend strategic stability to correlated equilibria. In both cases the desired goals are not reached even if there are significant positive results: a refined equilibrium satisfies the invariance property improving on the concept of proper equilibrium. This result stems from a setting that is closer to the forward induction than the backwards induction approach: a single valued solution concept that always conforms with backwards induction cannot verify the property of invariance. Similarly the concept of stable correlated equilibria improve on both perfect and acceptable correlated equilibria since it can be applied to n-player games and it excludes equilibria that are not perfect.

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