A time-varying extremum-seeking control approach

M. Guay, S. Dhaliwal and D. Dochain

Abstract— This paper considers the solution of a real-time optimization problem using adaptive extremum seeking control. It is assumed that the equations describing the dynamics of the nonlinear system and the cost function to be minimized are unknown and that the objective function is measured. The main contribution of the paper is to formulate the extremumseeking problem as a time-varying estimation problem. The proposed approach is shown to avoid the need for averaging results which minimizes the impact of the choice of dither signal on the performance of the extremum seeking control system. A simulation is used to illustrate the effectiveness of the proposed technique.

I. INTRODUCTION

Extremum-seeking control (ESC) has been the subject of considerable research effort over the last decade. This approach, which dates back to the 1920s [1], is an ingenious mechanism by which a system can be driven the optimum of a measured variable of interest [2]. The revived interest in the field was primarily sparked by Krstic and co-workers who provided an elegant proof of the convergence of a standard perturbation based extremum seeking scheme for a general class of nonlinear systems. The main drawback of ESC is the lack of transient performance guarantees. As highlighted in the proof of Krstic and Wang [3], the stability analysis relies on two components: 1) an averaging analysis of the persistently perturbed ESC loop and 2) a time-scale separation of ESC closed-loop dynamics between the fast transients of the system dynamics and the slow quasi steadystate extremum-seeking task. This analysis demonstrates that the three key parameters for ESC, which include the amplitude and frequency of the dither signal and the gain of the gradient algorithm, must be chosen very carefully to guarantee convergence to a neighbourhood of the unknown optimum. The amplitude of the dither signal must be large enough to provide sufficient excitation but cannot be too large to ensure that a sufficiently small neighbourhood of the optimum is reached. The frequency of the dither signal, taken as the singular perturbation parameter, must be small enough to ensure that the stability properties of the slow and fast time scales are not altered when they are interconnected. Finally, the choice of the gain of the gradient update used in the ESC cannot be adjusted freely since the convergence of the ESC depends largely on the magnitude of the unknown Hessian of the steady-state measured output.

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Over the last few years, many researchers have considered various approaches to overcome the limitations of ESC. In [4], the performance limitations associated with ESC were considered in detail. Tighter bounds on the tuning parameters as well as more precise statements on the guarantees of convergence were proposed. The non-local properties on ESC was studied in [5]. This work extends the work in [3] by considering the case where the fast dynamics can be assumed to be uniformly global asymptotically stable along the equilbrium manifold. More precise statements concerning the dependence of the stability properties on the tuning parameters are provided. In [6], [7] and [8], an alternative ESC algorithm is considered where an adaptive control and estimation approach is used. The key aspect of this approach is that the equilibrium map is parameterized and the parameters are estimated with the help of a tailored adaptive estimation technique. The results in [9] unify the approaches based on singular perturbation and parameter estimation by considering the case where the objective function is parameterized in a known fashion. Recent work reported in [10] have proposed a Newton-based extremum-seeking technique that provides an estimate of the inverse of the Hessian of the cost function. This technique can effectively alleviate the convergence problems associated with the increase of the gain of the Newton update.

In this paper, we provide an alternative extremum-seeking technique which is based on the estimation of the gradient as a time-varying parameter. We consider a parameter estimation routine for nonlinear systems with the time-varying parameters proposed in [11] and apply it in the context of an extremum seeking controller design approach. The timevarying parameter estimation technique is used to remove the need for averaging system to establish the convergence of the extremum seeking controller to the unknown steadystate optimum of a measured output function. It also avoids the need to use the frequency of the dither signal as a singular perturbation parameter. The proposed ESC algorithm provides more freedom in the tuning of the ESC loop to achieve improvements in transient performance.

The paper is organized as follows. A brief problem description is given in section II. In section III, the proposed ESC controller is presented for the case of process described by a static map. The application to an unknown dynamical system is presented in section IV. A brief simulation study is presented in V followed by brief conclusions in VI.

II. PROBLEM DESCRIPTION

Consider a nonlinear system

$$\dot{x} = f(x, u) \tag{1}$$

$$y = h(x) \tag{2}$$

where $x \in \mathbb{R}^n$ is the vector of state variables, u is the vector of input variables taking values in $\mathcal{U} \subset \mathbb{R}^p$ and $y \in \mathbb{R}$ is the variable to be minimized. It is assumed that f(x, u) is a smooth vector valued functions of x and u and that h(x) is a smooth function of x.

The objective is to steer the system to the equilibrium x^* and u^* that achieves the minimum value of $y(=h(x^*))$. The equilibrium (or steady-state) map is the *n* dimensional vector $\pi(u)$ which is such that:

$$f(\pi(u), u) = 0.$$

The equilibirum cost function is given by:

$$y = h(\pi(u)) = \ell(u) \tag{3}$$

Thus, at equilibrium, the problem is reduced to finding the minimizer u^* of $y = \alpha(u^*)$.

Some basic assumption are required to ensure that this problem is well-posed.

Assumption 1: The equilibrium cost (3) is such that

1)

2)

$$\frac{\partial^2 \ell}{\partial u \partial u^T} > \alpha I, \ \forall u \in \mathcal{U}$$

 $\frac{\partial \ell(u^*)}{\partial u} = 0$

In this section, we consider the extremum-seeking problem for a static map:

 $y = \ell(u)$

that satisfies Assumption 1.

In addition, the following assumptions are required. Assumption 2: The static-map ℓ is such that

1)

$$\|y\| \le Y$$

2)

$$\left\|\frac{\partial\ell}{\partial u}\right\| \le L_1$$

3)

$$\left\|\frac{\partial^2 \ell}{\partial u \partial u^T}\right\| \le L_2$$

 $\forall u \in \mathcal{U}$ with positive constants Y > 0, $L_1 > 0$ and $L_2 > 0$

In the development below, the minimization of y is performed in real-time. The input u is taken as a time-varying signal. That is,

$$y(t) = \ell(u(t)) \tag{4}$$

If one differentiates (4) with respect to time, the following dynamics are obtained:

$$\dot{y} = \frac{\partial \ell}{\partial u} \dot{u}.$$

Defining z = y, $\theta(t) = \frac{\partial \ell}{\partial u}$, one can therefore write the following dynamical system:

$$\dot{z} = \dot{u}\theta(t). \tag{5}$$

We make the following assumption concerning the input dynamics.

Assumption 3: The input signal u(t) is such that $\forall t \geq t_0 \geq 0$

1)
$$u(t) \in \mathcal{U}$$

2) $\|\dot{u}\| \le c_1$

with positive constant $c_1 > 0$

By construction, we have that:

$$\dot{\theta}(t) = \frac{\partial^2 \ell}{\partial u \partial u^T} \dot{u}$$

and Assumptions 2 and 3, it follows that

$$\|\theta(t)\| \le L_2 c_1 = L_\theta. \tag{6}$$

The design of the extremum seeking routine is based on the dynamics (5). The first step consists in the estimation of the time-varying parameters $\theta(t)$. In the second step, we define a suitable controller that achieves the extremumseeking task.

A. Parameter estimation

Let the estimator model for (5) be chosen as

$$\dot{\hat{z}} = \dot{u}\hat{\theta}(t) + Ke + c(t)^T\hat{\theta}(t), \qquad K > 0, \tag{7}$$

where $e = z - \hat{z}$ is the estimation error. The time varying parameter c(t) is the solution of the differential equation:

$$\dot{c}(t)^T = -Kc(t)^T + \dot{u}, \qquad c(t_0) = 0.$$
 (8)

The prediction error dynamics are given by:

$$\hat{e} = \dot{u}\tilde{\theta}(t) - Ke - c(t)^T\hat{\theta}(t)$$
(9)

where $e(t_0) = x(t_0) - \hat{x}(t_0)$. We define the auxiliary variable $\eta = e - c(t)^T \tilde{\theta}(t)$. The dynamics of η are as follows:

$$\dot{\eta} = -K\eta - c(t)^T \dot{\theta}(t), \qquad \eta(t_0) = e(t_0)$$
 (10)

An estimate of η is generated from

ė

$$\dot{\hat{\eta}} = -K\hat{\eta}.\tag{11}$$

As a result, the dynamics of the estimation error $\tilde{\eta}=\eta-\hat{\eta}$ are

$$\dot{\tilde{\eta}} = -K\tilde{\eta} - c(t)^T\dot{\theta}(t), \qquad \tilde{\eta}(t_0) = 0.$$
(12)

Following [12], the preferred parameter estimation update approach is given as follows.

Let $\Sigma \in \mathbb{R}^{n_{\theta} \times n_{\theta}}$ be generated from

$$\dot{\Sigma} = c(t)c(t)^T - k_T \Sigma, \qquad \Sigma(t_0) = \alpha I \succ 0,$$
 (13)

where α and k_T are strictly positive constant to be assigned.

Based on (7),(8) and (11), one considers the parameter update law as proposed in [13] is given by

$$\dot{\Sigma}^{-1} = -\Sigma^{-1} c(t) c(t)^T \Sigma^{-1} + k_T \Sigma^{-1}, \Sigma^{-1}(t_0) = \frac{1}{\alpha} I, \quad (14)$$

$$\dot{\hat{\theta}}(t) = \operatorname{proj}\left\{\gamma \Sigma^{-1} c(t)(e - \hat{\eta}), \hat{\theta}(t)\right\}, \hat{\theta}(t_0) = \theta^0 \in \Theta^0, \quad (15)$$

where $\operatorname{Proj}\{\phi, \hat{\theta}(t)\}$ denotes a Lipschitz projection operator [14] such that

$$-\operatorname{Proj}\{\phi, \hat{\theta}(t)\}^T \tilde{\theta}(t) \le -\phi^T \tilde{\theta}(t), \tag{16}$$

$$\hat{\theta}(t_0) \in \Theta^0 \implies \hat{\theta}(t) \in \Theta, \forall t \ge t_0$$
 (17)

where $\Theta \triangleq B(\hat{\theta}(t), z_{\theta})$, where $\hat{\theta}(t)$ and z_{θ} are the parameter estimate and and uncertainty set radius. Note that by Assumption 2, the uncertainty set radius z_{θ} can be set to L_1 .

Assumption 4: There exists constants $\alpha_1 > 0$ and T > 0 such that

$$\int_{t}^{t+T} c(\tau)c(\tau)^{T} d\tau \ge \alpha_{1} I \tag{18}$$

 $\forall t > 0.$

B. Controller design

The simplest extremum-seeking controller possible in this case is given by:

$$\dot{u} = -k\hat{\theta} + d(t) \tag{19}$$

where d(t) is a bounded differ signal with $||d(t)|| \le D$ and k > 0.

Note that the controller is such that $\|\dot{u}\| \leq kL_1 + D$.

Theorem 1: Let Assumptions 2 to 4 hold. The extremumseeking controller (19), (14) and (15) is such that the system converges exponentially to a neighbourhood of the minimizer u^* of the static cost y. The size of this neighbourhood is adjustable by increasing the gains K, k_T and k.

Proof: (Sketch) We consider the Lyapunov function:

$$W = \frac{1}{2}\tilde{\eta}^T\tilde{\eta} + \frac{1}{2}\tilde{\theta}^T\Sigma\tilde{\theta} + \frac{1}{2}\theta^T\theta.$$

Upon differentiating and exploiting the properties of the projection algorithm, we have that:

$$\begin{split} \dot{W} &\leq -\tilde{\eta}^{T} K \tilde{\eta} + \tilde{\eta}^{T} c(t)^{T} \dot{\theta} \\ &+ \tilde{\theta}(t)^{T} \Sigma(t) \dot{\theta}(t) - k_{T} \frac{1}{2} \tilde{\theta}(t)^{T} \Sigma \tilde{\theta}(t) \\ &- \frac{1}{2} (e - \hat{\eta})^{T} (e - \hat{\eta}) + \frac{1}{2} \tilde{\eta}^{T} \tilde{\eta} = \theta^{T} \dot{\theta} \end{split}$$
(20)

where k_1 and k_T are positive constants to be assigned. Recall that $\theta(t) = \frac{\partial \ell}{\partial u}$ and that $\theta = \hat{\theta} + \tilde{\theta}$, Then

$$\dot{\theta} = \frac{\partial^2 \alpha}{\partial u \partial u^T} \dot{u} = -k\Gamma \theta + k\Gamma \tilde{\theta} + \Gamma d(t)$$

where $\Gamma = \frac{\partial^2 \alpha}{\partial u \partial u^T}$. Upon substitution, one obtains: $\dot{W} \leq -\tilde{\eta}^T K \tilde{\eta} - k \tilde{\eta}^T c(t)^T \Gamma \theta + k \tilde{\eta}^T c(t)^T \Gamma \tilde{\theta}$ $+ \tilde{\eta}^T c(t)^T \Gamma d(t) - k_T \tilde{\theta}(t)^T \Sigma \tilde{\theta}(t) - \frac{1}{2} (e - \hat{\eta})^T (e - \hat{\eta})$ $+ \frac{1}{2} \tilde{\eta}^T \tilde{\eta} - k_T \tilde{\theta}(t)^T \Sigma \tilde{\theta}(t) - \frac{1}{2} (e - \hat{\eta})^T (e - \hat{\eta})$ $+ \frac{1}{2} \tilde{\eta}^T \tilde{\eta} - k \tilde{\theta}(t)^T \Sigma \Gamma \theta + k \tilde{\theta}(t)^T \Sigma \Gamma \tilde{\theta}$ $+ \tilde{\theta}(t)^T \Sigma \Gamma d(t) - k \theta^T \Gamma \theta + k \theta^T \Gamma \tilde{\theta} + \theta^T \Gamma d(t)$

The boundedness of the matrix $\Sigma(t)$ can be shown as follows. By integration, one gets:

$$\Sigma(t) = e^{-k_T t} \Sigma(0) + \int_0^t e^{-k_T (t-\tau)} c(\tau) c(\tau)^T d\tau$$
$$\geq \int_{t-T}^t e^{-k_T (t-\tau)} c(\tau) c(\tau)^T d\tau \geq e^{-k_T T} \alpha_1 I = \gamma_1 I$$

By the boundedness of c(t), one can also write,

$$\Sigma(t) \le \Sigma(0) + \beta_2 \int_0^t e^{-k_T(t-\tau)} d\tau I \le \alpha I + \beta_2 I = \gamma_2 I.$$

As a result, we get that:

$$\gamma_1 I \leq \Sigma(t) \leq \gamma_2 I$$
 and $\gamma_2^{-1} I \leq \Sigma(t)^{-1} \leq \gamma_1^{-1} I$.

Completing the squares, exploiting the boundedness of $\Sigma(t)$ and Γ and rearranging, we obtain the following inequality:

$$\begin{split} \dot{W} &\leq -\tilde{\eta}^{T} \left(K - \frac{1}{2}I - \frac{L_{2}k(k_{1} + k_{2}) + k_{3}}{2}c(t)^{T}c(t) \right) \tilde{\eta} \\ &- \left(k_{T} - \frac{L_{2}}{2k_{2}}\gamma_{1} - \gamma_{1} \left(L_{2}k\frac{(k_{4} + k_{5})}{2} + L_{2}k \right) \right) \\ &- \frac{L_{2}kk_{6}}{2} \right) \tilde{\theta}^{T} \tilde{\theta} - \frac{1}{2}(e - \hat{\eta})^{T}(e - \hat{\eta}) \\ &- \left(k - \frac{k\gamma_{2}}{2k_{4}} - \frac{k}{2k_{1}} - \left(\frac{k}{2k_{6}} + \frac{k_{7}}{2} \right) \right) \theta^{T} \Gamma \theta \\ &+ \left(\frac{L_{2}}{2k_{3}} + \frac{\gamma_{2}L_{2}}{2k_{5}} + \frac{1}{2k_{7}} \right) d(t)^{T} d(t) \end{split}$$

We let:

$$K = k_{\eta_1} I + k_{\eta_2} c(t)^T c(t)$$

Based on the last inequality, it follows that there exist constants k, k_T , k_{η_1} and k_{η_2} such that:

$$\dot{W} \leq -k_{\eta}\tilde{\eta}^{T}\tilde{\eta} - k_{\theta}\tilde{\theta}^{T}\tilde{\theta} - \frac{1}{2}(e-\hat{\eta})^{T}(e-\hat{\eta}) -k_{y}\theta^{T}\Gamma\theta + k_{d}\|d(t)\|^{2} \leq -2k_{\eta}V_{\tilde{\eta}} - 2\frac{k_{\theta}}{\gamma_{2}}V_{\tilde{\theta}} - \frac{1}{2}(e-\hat{\eta})^{T}(e-\hat{\eta}) -\frac{k_{y}}{\alpha}\theta^{T}\theta + k_{d}\|d(t)\|^{2} < -k_{W}W + k_{d}\|d(t)\|^{2}$$
(21)

where k_{η} , k_{θ} , k_{y} and k_{d} are strictly positive constants.

It follows that $\tilde{\eta}$, $\tilde{\theta}$ and θ convergence exponentially to a neighbourhood of the origin. The size of this neighbourhood depends on the choice of gains k, k_T and K and the magnitude of the dither signal.

IV. OPTIMIZATION IN DYNAMICAL SYSTEMS

In this section, we consider the initial extremum-control system which consists in steering the unknown dynamical system (1) to the equilibrium that minimizes the measure cost function (2).

The closed-loop extremum seeking control system is given by:

$$\epsilon \dot{x} = f(x, u)$$

$$\dot{u} = -k\hat{\theta}_{1}(t) + d(t)$$

$$\dot{\hat{\theta}} = \operatorname{proj} \left\{ \Sigma^{-1}c(e - \hat{\eta}) \right\}$$

$$\dot{\hat{\eta}} = -K\hat{\eta}$$

$$\dot{c} = -Kc + \dot{u}\dot{u}^{T}$$

$$\dot{\hat{y}} = \dot{u}(t)\hat{\theta} + Ke + c^{T}\dot{\hat{\theta}}$$

$$\dot{\Sigma} = cc^{T} - k_{T}\Sigma$$
(22)

where $e = h(x) - \hat{y}$.

As in other works on extremum-seeking control, the closed-loop dynamics of the system are written in error form in terms of a two time-scale system where t is the slow time-scale and the system's dynamics are assumed to evolve over a fast time-scale $\frac{t}{\epsilon}$. The parameter $\epsilon > 0$ is a small strictly positive parameter to be assigned.

Let us define the deviation variables $\tilde{x} = x - \pi(u)$ and $\tilde{u} = u - u^*$ where u^* is the local minimizer of the steady-state map $y = \ell(u)$. The auxiliary variable is defined as above. However, one must take into account the measurement of the cost function over the fast time-scale. That is:

$$\eta = e - c^T \tilde{\theta}$$

where $e = h(\tilde{x} + \pi(\tilde{u} + u^*)) - \hat{y}$ represents the error dynamics in the fast and the slow time scale. As a result, one obtains the following dynamics:

$$\dot{\eta} = \frac{1}{\epsilon} \frac{\partial h(x)}{\partial x} f(x, u) + \frac{\partial h(x)}{\partial x} \frac{\partial \pi(u)}{\partial u} \dot{u} - \dot{\dot{y}} \\ - \dot{c}^T \ddot{\theta} + c^T \dot{\dot{\theta}} - c^T \dot{\theta}$$

Substituting for $\dot{\hat{y}}$ and \dot{c} , one obtains:

$$\dot{\eta} = \frac{1}{\epsilon} \frac{\partial h(x)}{\partial x} f(x, u) + \frac{\partial h(x)}{\partial x} \frac{\partial \pi(u)}{\partial u} \dot{u} - \frac{\partial h(\pi(u))}{\partial x} \frac{\partial \pi(u)}{\partial u} \dot{u} \\ - K\eta - c^T \dot{\theta}$$

As a result, the η dynamics is affected by the fast and slow dynamics. In the following, we will assume that the gain matrix K is such that

$$K = K_s + \frac{1}{\epsilon}K_f$$

where K_s and K_f are such that $K_i + K_i^T > \beta_i I$ with $\beta_i > 0$ for i = s, f. The estimation error dynamics $\tilde{\eta}$ are given by:

$$\begin{split} \dot{\tilde{\eta}} &= -\frac{1}{\epsilon} K_f \tilde{\eta} - K_s \tilde{\eta} - c^T \dot{\theta} \frac{1}{\epsilon} \frac{\partial h(x)}{\partial x} f(x, u) \\ &+ \left(\frac{\partial h(x)}{\partial x} \frac{\partial \pi(u)}{\partial u} \dot{u} - \frac{\partial h(\pi(u))}{\partial x} \frac{\partial \pi(u)}{\partial u} \dot{u} \right) \end{split}$$

Similarly, one must also rewrite the parameter estimation error as:

$$\begin{split} \tilde{\theta} &= -\text{proj}\left\{\Sigma^{-1}c(e-\hat{\eta})\right\} + \dot{\theta} \\ &= -\text{proj}\left\{\gamma\Sigma^{-1}c(t)c(t)\tilde{\theta} + \Sigma^{-1}c(t)\tilde{\eta}\right\} + \dot{\theta} \end{split}$$

The gradient algorithm can be written in deviation form as:

$$\dot{\tilde{u}} = -k\hat{\theta} + d(t) = -k\theta + k\tilde{\theta} + d(t)$$

Finally, the gradient of the cost function is given by:

$$\theta = -k\Gamma\theta + k\Gamma\theta + \Gamma d(t).$$

One can write the system (22) in deviation form as follows:

$$\begin{aligned} \epsilon \dot{\tilde{x}} &= f(\tilde{x} + \pi(\tilde{u} + u^*), \tilde{u} + u^*) - \epsilon \frac{\partial \pi}{\partial u} \dot{\tilde{u}} \\ \epsilon \dot{\tilde{\eta}} &= \frac{\partial h(x)}{\partial x} f(x, u) - K_f \tilde{\eta} - \epsilon K_s \tilde{\eta} - \epsilon c^T \dot{\theta} \\ &+ \epsilon \left(\frac{\partial h(x)}{\partial x} \frac{\partial \pi(u)}{\partial u} \dot{u} - \frac{\partial h(\pi(u))}{\partial x} \frac{\partial \pi(u)}{\partial u} \dot{u} \right) \quad (23) \\ \dot{\tilde{u}} &= -k\theta + k\tilde{\theta} + d(t) \\ \dot{\tilde{\theta}} &= -\operatorname{proj} \left\{ \gamma \Sigma^{-1} c(t) c(t) \tilde{\theta} + \Sigma^{-1} c(t) \tilde{\eta} \right\} + \dot{\theta} \\ \dot{\theta} &= -k\Gamma\theta + k\Gamma\tilde{\theta} + k\Gamma\tilde{\theta} + \Gamma d(t). \end{aligned}$$

This system assumes that standard singular perturbation form. Following the standard nomenclature, the *reduced* system $(\tilde{x} = \pi(u))$ is given by:

$$\begin{aligned} \dot{\tilde{u}} &= -k\theta + k\tilde{\theta} + d(t) \\ \dot{\tilde{\theta}} &= -\text{proj}\left\{\gamma\Sigma^{-1}c(t)c(t)\tilde{\theta} + \Sigma^{-1}c(t)\tilde{\eta}\right\} + \dot{\theta} \end{aligned} (24) \\ \dot{\tilde{\eta}} &= -K\tilde{\eta} - c^{T}\dot{\theta} \\ \dot{\theta} &- k\Gamma\theta + k\Gamma\tilde{\theta} + \Gamma d(t) \end{aligned}$$

The boundary layer system is given by

$$\frac{d\tilde{x}}{d\tau} = f(\tilde{x} + \pi(\tilde{u} + u^*), \tilde{u} + u^*) \\
\frac{d\tilde{\eta}}{d\tau} = \frac{\partial h(\tilde{x} + \pi(\tilde{u} + u^*))}{\partial x} f(\tilde{x} + \pi(\tilde{u} + u^*), \tilde{u} + u^*) - K_f \tilde{\eta}$$
(25)

Assumption 5: The origin of the nonlinear system (1) is locally exponentially stable $\forall u \in \mathcal{U}$.

Let $\mathcal{X}_r = \{\tilde{x} \in \mathbb{R}^n \mid ||\tilde{x}|| \le r\}$ for r > 0, a positive constant. Similarly, let $\mathcal{E}_r = \{\tilde{\eta} \in \mathbb{R}^m \mid ||\tilde{\eta}|| \le r\}$.

Assumption 6: The vector field $f(\tilde{x} + \pi(\tilde{u} + u^*), \tilde{u} + u^*)$ is such that:

$$||f(\tilde{x} + \pi(\tilde{u} + u^*), \tilde{u} + u^*)|| \le L_f ||\tilde{x}|$$

 $\forall \tilde{x} \in \mathcal{X} \text{ and } \forall u \in \mathcal{U} \text{ where } L_f > 0 \text{ is a positive constant.}$ Similarly we assume the following:

Assumption 7: The output map h(x) is such that:

$$\left\|\frac{\partial h(\tilde{x} + \pi(\tilde{u} + u^*))}{\partial x}\right\| \le L_h$$

2)

1)

$$\left\|\frac{\partial h(\tilde{x}+\pi(u))}{\partial x} - \frac{\partial h(\tilde{y}+\pi(u))}{\partial x}\right\| \le L_h \|\tilde{x}-\tilde{y}\|$$

 $\forall \tilde{x} \text{ and } \tilde{y} \in \mathcal{X} \text{ and } \forall u \in \mathcal{U} \text{ where } L_h > 0 \text{ is a positive constant.}$

Finally, we make the following assumption concerning the steady-state map, $\pi(u)$

Assumption 8: The steady-state map $\pi(u)$ is such that:

$$\left\|\frac{\partial \pi(u)}{\partial u}\right\| \le L_{\pi}$$

 $\forall u \in \mathcal{U}$ where $L_{\pi} > 0$ is a positive constant.

By assumptions 6, 7 and 8, it follows that there exists a K_f such that the origin of the boundary layer (25) is locally exponentially stable.

It then follows that there exists a Lyapunov function $V(\tilde{x}, \tilde{\eta})$ and positive constants, α_1 , α_2 , α_3 , α_4 , α_5 and α_6 such that:

1)

$$\alpha_1(\|\tilde{x}\|^2 + \|\tilde{\eta}\|^2) \le V(\tilde{x}, \tilde{\eta}) \le \alpha_2(\|\tilde{x}\|^2 + \|\tilde{\eta}\|^2)$$

2)

$$\frac{dV}{d\tau} = \frac{\partial V}{\partial \tilde{x}} f(\tilde{x} + \pi(u), u)
+ \frac{\partial V}{\partial \tilde{\eta}} \left(\frac{\partial h(\tilde{x} + \pi(u))}{\partial x} f(\tilde{x} + \pi(u), u) - K_f \tilde{\eta} \right)
\leq -\alpha_3 \|\tilde{x}\|^2 - \alpha_4 \|\tilde{\eta}\|^2$$

3)

$$\left\| \left[\frac{\partial V}{\partial \tilde{x}}, \frac{\partial V}{\partial \tilde{\eta}} \right] \right\| \le \alpha_5 \|\tilde{x}\| + \alpha_6 \|\tilde{\eta}\|$$

 $\forall u \in \mathcal{U}, \ \tilde{x} \in \mathcal{X}_r.$

Theorem 2: Consider nonlinear system (1) and the cost function (2). Let Assumptions 1 to 8 be fulfilled then the time-varying parameter estimation scheme and the extremum-seeking controller (19) is such that for every $\epsilon \in$ $(0, \epsilon^*)$, the closed-loop system converges exponentially to a neighbourhood of the unknown local minimum of the cost function (2). The size of the neighbourhood depends on the choice of gains K, k_T and k and the magnitude of the dither signal d(t).

Proof: To study the stability of the two time-scale closed-loop system (23), we propose the Lyapunov function:

$$\mathcal{V} = \delta W + (1 - \delta) V(\tilde{x}, \tilde{\eta})$$

where $\delta \in (0, 1)$.

Differentiating with respect to t, one obtains:

$$\begin{split} \dot{\mathcal{V}} &\leq -\delta k_W W + \delta k_d \|d(t)\|^2 \\ &+ \delta \tilde{\eta}^T \left(\frac{\partial h(x)}{\partial x} \frac{\partial \pi(u)}{\partial u} \dot{u} - \frac{\partial h(\pi(u))}{\partial x} \frac{\partial \pi(u)}{\partial u} \dot{u} \right) \\ &- \frac{(1-\delta)}{\epsilon} \alpha_3 \|\tilde{x}\|^2 - \frac{(1-\delta)}{\epsilon} \alpha_4 \|\tilde{\eta}\|^2. \end{split}$$

By Assumptions 6, 7 and 8, the inequality becomes:

$$\dot{\mathcal{V}} \leq -\delta k_W W + \delta k_d \|d(t)\|^2 + \delta L_\pi L_h (kL_1 + D) \|\tilde{x}\| \|\tilde{\eta}\| - \frac{(1-\delta)}{\epsilon} \alpha_3 \|\tilde{x}\|^2 - \frac{(1-\delta)}{\epsilon} \alpha_4 \|\tilde{\eta}\|^2.$$

It then follows that there exists an ϵ^* such that $\forall \epsilon \in (0, \epsilon^*)$,

$$\dot{\mathcal{V}} \le -\delta k_W W - k_e \|\tilde{x}\|^2 - k_e \|\tilde{\eta}\|^2 + \delta k_d \|d(t)\|^2$$

where k_e is a strictly positive constant.

V. SIMULATION EXAMPLE

A. Single input static map problem

Let us consider the simple quadratic cost given by:

$$y = 0.5 + 0.1u + 0.2u^2$$

We consider the application of the algorithm with the following tuning parameters, $d(t) = 0.1 \sin(t)$, k = 100 and $k_1 = k_2 = k_3 = k_4 = 100$. In contrast to standard perturbation based ESC, the choice of dither is completely arbitrary.

Simulation results are shown in Figure 1, 2, 3 and 4. Figure 1 shows that progress of the cost function and the corresponding input value as a function of time. For the value of gain k = 100, the long term affect of the dither signal is completely removed. The parameter estimates and the true time-varying parameters are shown on Figure 2. The parameter estimation is shown to converge quickly to the true parameters. In addition, the gradient is shown to approach zero as required.

The results demonstrate that the observed performance of the extremum seeking controller differs from existing perturbation ESC techniques. The proposed technique is such that the injected perturbation is completely removed as the gradient of the cost function is reduced. The asymptotic convergence is therefore independent of the amplitude and the frequency of the dither signal. Figure 3 shows that result for a dither signal $d(t) = \sin(10t)$ with an optimization gain of k = 10. If one increases the gain to k = 100, the trajectories of the extremum seeking controller shown 4 are obtained. In addition, the speed of convergence is dramatically improved when the optimization gain is increased.

VI. CONCLUSION

In this paper, an alternative ESC technique was proposed. The technique is based on the time-varying estimation of the unknown gradient. The ESC algorithm is shown to provide local exponential convergence of the closed-loop system to the unknown optimum. The technique simplifies the tuning of such schemes by avoiding the limitations associated with choice of dither.

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Fig. 2. Plot the parameter estimates as a function of time for the single-input example.



Fig. 3. Plot of the cost function and the corresponding control as a function of time for a smaller gain and high dither amplitude.



Fig. 1. Plot of the cost function and the corresponding control as a function of time for the single-input example.



Fig. 4. Plot of the cost function and the corresponding control as a function of time for a larger and high dither amplitude.