# Quadratic approximation of some convex optimization problems using the arithmetic-geometric mean iteration

François Glineur
Center for Operations Research and Econometrics
Université catholique de Louvain
Voie du Roman Pays, 34
B-1348 Louvain-la-Neuve, Belgium
Email: francois.glineur@uclouvain.be

## 1 Introduction

Convex optimization includes many different classes of structured optimization problems, such as linear optimization, (convex) quadratic optimization, geometric optimization, semidefinite optimization, etc. Approximation of one class of problems $\mathscr{P}$ by another class $\mathscr{A}$ can be useful in several situations, e.g.

- when algorithms for class $\mathscr{A}$ are significantly faster than algorithms for class $\mathscr{P}$, so that one can hope to obtain an approximate solution to the original problem in much less time than is required to solve it exactly ;
- when the problem to be solved is discrete (i.e. integer or mixed integer programming) and its continuous relaxation belongs to problem class $\mathscr{P}$, while available branch-and-bound-type algorithms can only work with subproblems corresponding to relaxations of type $\mathscr{A}$ ;
- from a more practical point of view, when one requires to solve a problem of class $\mathscr{P}$ but only has access to a commercial solver for problems of class $\mathscr{A}$.

Previous work in this area includes the polyhedral approximation technique of the second-order cone proposed by Ben-Tal and Nemirovski [BTN01], which allows the efficient approximation of (integer) quadratic optimization problems by (integer) linear optimization.

## 2 Quadratic approximation

In this talk, we present recent results for a different situation: approximating some convex optimization problems by quadratic optimization.

Our main tool consists in the use of the *arithmetic-geometric mean (AGM) iteration* due to Gauss, Lagrange and Legendre, an algorithmic procedure with very fast convergence originally designed to compute some elliptic integrals (see e.g. [BB98]). More specifically, letting $a_0 = \alpha$, $b_0 = \beta$ with $\alpha > \beta > 0$ and defining the iteration

$$a_{n+1} = \frac{a_n + b_n}{2} \text{ and } b_{n+1} = \sqrt{a_n b_n}\ ,$$

we have that both sequences converge to a common limit

$$a_n \text{ and } b_n \to AG(\alpha,\beta) = \frac{\pi}{2I(\alpha,\beta)}$$

defined using the complete elliptic integral of the first kind:

$$I(\alpha,\beta) = \int_0^{\frac{\pi}{2}} \left(\alpha^2\cos^2\theta + \beta^2\sin^2\theta\right)^{-\frac{1}{2}} \mathrm{d}\theta\ .$$

Using the fact that several standard convex functions are also computable by the AGM [BB84], we show that the corresponding algorithmic procedure based on repeated AGM iterations can be converted into a short sequence of quadratic constraints, allowing the approximation of problems involving these convex functions by quadratic optimization, with arbitrary accuracy.

Examples of such functions include the exponential and the logarithm, as well as hyperbolic and trigonometric functions (restricted on well-chosen intervals to ensure convexity), allowing us to approximate for example geometric programming and $l_p$-norm optimization problems by quadratic optimization.

### References

[BTN01] A. Ben-Tal and A. Nemirovski, *On polyhedral approximations of the second-order cone*, Mathematics of Operations Research **26** (2001), no. 2, 193–205.

[BB84] J. M. Borwein and P. B. Borwein, *The arithmetic-geometric mean and fast computation of elementary functions*, SIAM Review, **26** (1984), pp. 351–366.

[BB98] J. M. Borwein and P. B. Borwein, *Pi and the AGM – A study in analytic number theory and computational complexity*, Canadian Mathematical Society Series of Monographs and Advanced Texts, 4, John Wiley & Sons Inc., New York, 1998. Reprint of the 1987 original, A Wiley-Interscience Publication.