FOUNDATIONS OF COMPUTING AND DECISION SCIENCES
Vol. 25 (2000) No. 3

AN EXTENDED CONIC FORMULATION FOR GEOMETRIC OPTIMIZATION

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Presented by J. Teghem

Abstract. The author has recently proposed a new way of formulating two classical classes of structured convex problems, geometric and l_p-norm optimization, using dedicated convex cones. This approach has some advantages over the traditional formulation: it simplifies the proofs of the well-known associated duality properties (i.e. weak and strong duality) and the design of a polynomial algorithm becomes straightforward.

In this article, we make a step towards the description of a common framework that includes these two classes of problems. Indeed, we present an extended variant of the cone for geometric optimization previously introduced by the author and show it is equally suitable to formulate this class of problems. This new cone has the additional advantage of being very similar to the cone used for l_p-norm optimization, which opens the way to a common generalization.

1 Introduction

1.1 Geometric optimization

Geometric optimization forms an important class of problems that enables practitioners to model a large variety of real-world applications, mostly in the field of engineering design [5]. This class of problems is usually known for historical reasons under the names of *geometric programming*. However, because of the strong connection of the term "programming" with computer science, we prefer to use the more natural word "optimization".

These problems can be expressed as follows: we first need to define the following two sets $R = \{0, 1, 2, \ldots, r\}$ and $I = \{1, 2, \ldots, n\}$ and let $\{I_k\}_{k \in R}$ be a partition of I into $r + 1$ classes, i.e. satisfying

$$\cup_{k \in R} I_k = I \text{ and } I_k \cap I_l = \emptyset \text{ for all } k \neq l \ .$$

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The primal geometric optimization problem is the following:

$$\inf G_0(t) \quad \text{s.t.} \quad t \in \mathbb{R}^m_{++} \text{ and } G_k(t) \leq 1 \text{ for all } k \in R \setminus \{0\} \ , \tag{1}$$

where t is the m-dimensional column vector we want to optimize and the functions G_k defining the objective and the constraints are so-called posynomials, defined by

$$G_k : \mathbb{R}^m_{++} \mapsto \mathbb{R}_{++} : t \mapsto \sum_{i \in I_k} C_i \prod_{j=1}^{m} t_j^{a_{ij}} \ ,$$

where exponents a_{ij} are arbitrary real numbers and coefficients C_i are required to be strictly positive (hence the name *pos*ynomial). These functions are very well suited for the formulation of constraints that come from the laws of physics or economics (either directly or using an empirical fit).

Although not convex itself (choose for example $G_0 : t \mapsto t^{1/2}$ as the objective, which is not a convex function), a geometric optimization problem can be easily transformed into a convex problem, for which a Lagrangean dual can be explicitly written. This transformation uses the following change of variables:

$$t_j = e^{y_j} \text{ for all } j \in \{1, 2, \ldots, m\} \ , \tag{2}$$

to become

$$\inf g_0(y) \quad \text{s.t.} \quad g_k(y) \leq 1 \text{ for all } k \in R \setminus \{0\} \ . \tag{3}$$

The functions g_k are defined to satisfy $g_k(y) = G_k(t)$ when (2) holds, which means

$$g_k : \mathbb{R}^m \mapsto \mathbb{R}_{++} : y \mapsto \sum_{i \in I_k} C_i \prod_{j=1}^{m} (e^{y_j})^{a_{ij}} = \sum_{i \in I_k} e^{-c_i + \sum_{j=1}^m y_j a_{ij}} = \sum_{i \in I_k} e^{a_i^T y - c_i} \ ,$$

where $a_i = (a_{i1}, a_{i2}, \ldots, a_{im})^T$ is an m-dimensional column vector and the coefficient vector $c \in \mathbb{R}^n$ is given by $c_i = -\log C_i$. Note that unlike the original variables t and coefficients C, variables y and coefficients c are not required to be strictly positive and can take any real value.

It is straightforward to check that functions g_k are now convex, hence that (3) is a convex optimization problem. However, we will not establish this property here but rather derive it from the fact that problem (3) can be cast as a conic optimization problem (see Subsection 3.1). Moreover, following others [3, 7], we will not use this formulation but instead work with a slight variation featuring a linear objective:

$$\sup b^T y \quad \text{s.t.} \quad g_k(y) \leq 1 \text{ for all } k \in R \ , \tag{PG}$$

where $b \in \mathbb{R}^m$ and 0 has been removed from set R (i.e. this set is now equal to $\{1, 2, \ldots, r\}$). Problems in the form (3) (and (1)) can be easily expressed in this format [1, Section 5.1] and all the results we are going to obtain about problem (PG) can be easily translated back to these more traditional settings, so that we can concentrate our attention on formulation (PG) without any loss of generality.

The dual problem for (PG) can be written as follows

$$\inf \ c^T x + \sum_{k \in R} \sum_{\substack{i \in I_k \\ x_i > 0}} x_i \log \frac{x_i}{\sum_{i \in I_k} x_i} \quad \text{s.t.} \quad Ax = b \text{ and } x \geq 0 \ , \tag{DG}$$

where $x \in \mathbb{R}^n_+$ is the vector of nonnegative dual variables we have to optimize.

Several duality results are known for the primal-dual pair (PG)–(DG), some being mere consequences of convexity (e.g. weak duality), others being specific to this particular class of problems (e.g. the absence of a duality gap). These properties were first studied in the late sixties, and can be found for example in the book of Duffin, Peterson and Zener [5].

1.2 Conic optimization

Conic optimization[1] deals with a class of problems that is essentially equivalent to the class of convex problems, i.e. minimization of a convex function over a convex set. However, formulating a convex problem in a conic way has the advantage of providing a very symmetric form for the dual problem and often gives a new insight about its structure.

The basic ingredient of conic optimization is a convex cone. Recall that a set C is a *cone* if and only if it is closed under nonnegative scalar multiplication, i.e. if $x \in C \Rightarrow \lambda x \in C$ for all $\lambda \in \mathbb{R}_+$. Moreover, a set is convex if and only if it contains the whole segment joining any two of its points. However, when dealing with cones, this assumption is equivalent to closedness under addition, i.e. cone C is convex if $x \in C$ and $y \in C \Rightarrow x + y \in C$.

In order to avoid some technical nuisances, the convex cones we are going to consider will be required to be closed, pointed and solid, according to the following additional definitions:

Definition 1.1 *A cone C is* solid *if and only if* $\operatorname{int} C \neq \emptyset$ *(where* $\operatorname{int} S$ *denotes the interior of set S).*

Definition 1.2 *A cone C is* pointed *if and only if* $C \cap -C = \{0\}$.

These two properties basically mean that C is a full-dimensional cone that does not contain any straight line passing through the origin.

We are now in position to define a conic optimization problem: let $C \subseteq \mathbb{R}^n$ a pointed, solid, closed convex cone. The (primal) conic optimization problem is defined as

$$\inf_x \ c^T x \quad \text{s.t.} \quad Ax = b \text{ and } x \in C \ , \tag{CP}$$

where $x \in \mathbb{R}^n$ is the column vector we are optimizing and the problem data is given by cone C, a $m \times n$ matrix A and two column vectors b and c belonging respectively

[1]Proofs for the theorems stated in this subsection can be found e.g. in [8, 9].

to $\mathbb{R}^m$ and $\mathbb{R}^n$. This problem can be viewed as the minimization of a linear function over the intersection of a convex cone and an affine subspace.

It is well-known that this class of problems is equivalent to the class of convex problems, see e.g. [4]. However, the usual Lagrangean dual of a conic problem can be also expressed very nicely in a conic form, using the notion of dual cone.

Definition 1.3 *The* dual *of a cone $C \subseteq \mathbb{R}^n$ is defined by*

$$C^* = \{x^* \in \mathbb{R}^n \mid x^T x^* \geq 0 \text{ for all } x \in C\} .$$

The following theorem stipulates that the dual of a cone is always a closed convex cone.

Theorem 1.1 *If C is a closed convex cone, its dual C^* is another closed convex cone. Moreover, the dual $(C^*)^*$ of C^* is equal to C.*

Closedness is essential for $(C^*)^* = C$ to hold (without the closedness assumption on C, we only have $(C^*)^* = \operatorname{cl} C$ where $\operatorname{cl} S$ denotes the closure of set S). The additional notions of solidness and pointedness also behave well when taking the dual of a convex cone (indeed, these two properties are dual to each other).

Theorem 1.2 *If C is a solid, pointed, closed convex cone, its dual C^* is another solid, pointed, closed convex cone.*

The dual of our primal conic problem (CP) can be stated as

$$\sup_{(y,s)} b^T y \quad \text{s.t.} \quad A^T y + s = c \text{ and } s \in C^* , \tag{CD}$$

where $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$ are the column vectors we are optimizing, the other quantities A, b and c being the same as in (CP). It is immediate to notice that this dual problem has the same kind of structure as the primal problem, i.e. it also involves optimizing a linear function over the intersection of a convex cone and an affine subspace. The only differences are the direction of the optimization (maximization instead of minimization) and the way the affine subspace is described (it is a translation of the range space of A^T, while primal involved a translation of the null space of A). It is also possible to show that the dual of this dual problem is equivalent to the primal problem, using the fact that $(C^*)^* = C$.

The two conic problems of this primal-dual pair are strongly related to each other, as demonstrated by several duality theorems. In order to keep this presentation short, we only cite the most basic of these theorems, called the *weak duality* property:

Theorem 1.3 (Weak duality) *Let x a feasible (i.e. satisfying the constraints) solution for* (CP), *and (y, s) a feasible solution for* (CD). *We have*

$$b^T y \leq c^T x ,$$

equality occurring if and only if the following orthogonality condition *is satisfied:*

$$x^T s = 0 .$$

This theorem shows that any primal (resp. dual) feasible solution provides an upper (resp. lower) bound for the dual (resp. primal) problem.

1.3 Aim of the article

In [1], the author starts by defining an appropriate convex cone that allows him to express geometric optimization problems as conic programs, his aim being to apply the general duality theory for conic optimization [8, 9] to these problems and prove in a seamless way the various well-known duality theorems of geometric optimization. The goal of this article is to introduce a variation of this convex cone that keeps its ability to model geometric optimization problems but bears more resemblance with the cone that was introduced for l_p-norm optimization in [2], hinting for a common generalization of these two families of cones.

This article is organized as follows: following this brief introduction to geometric and conic optimization, Section 2 introduces the convex cones needed to model geometric optimization and studies some of their properties. Section 3 constitutes the main part of this article and demonstrates how the above-mentioned cones enable us to model primal and dual geometric optimization problems in a seamless fashion. Modelling the primal problem with our first cone is rather straightforward and writing down its dual is immediate, but some work is needed to prove the equivalence with the traditional formulation of a dual geometric optimization problem. Finally, concluding remarks in Section 4 provide some insight about the relevance of our approach and hint at some possible ways to make use of it.

2 The geometric cone

Let us introduce the geometric cone $\mathcal{G}^n$, which will allow us to give a conic formulation of geometric optimization problems.

Definition 2.1 *Let $n \in \mathbb{N}$. The* geometric cone $\mathcal{G}^n$ *is defined by*

$$\mathcal{G}^n = \left\{(x, \theta, \kappa) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \mid \theta \sum_{i=1}^n e^{-\frac{x_i}{\theta}} \leq \kappa\right\}$$

using in the case of a zero denominator the following convention:

$$e^{-\frac{x_i}{0}} = 0\,.$$

We observe that this convention results in $(x, 0, \kappa) \in \mathcal{G}^n$ for all $x \in \mathbb{R}^n_+$ and $\kappa \in \mathbb{R}_+$. As a special case, we mention that $\mathcal{G}^0$ is the 2-dimensional nonnegative orthant $\mathbb{R}^2_+$.

In order to use the conic formulation described in the previous section, we first prove that $\mathcal{G}^n$ is a convex cone.

Theorem 2.1 *$\mathcal{G}^n$ is a convex cone.*

Proof. Let us first introduce the following function

$$f_n : \mathbb{R}^n_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+ : (x, \theta) \mapsto \sum_{i=1}^n \theta e^{-\frac{x_i}{\theta}}\,.$$

With the convention mentioned above, its effective domain is $\mathbb{R}_+^{n+1}$. It is straightforward to check that f_n is positively homogeneous, i.e. $f_n(\lambda x, \lambda\theta) = \lambda f_n(x, \theta)$ for $\lambda \geq 0$. Moreover, f_n is subadditive, i.e. $f_n(x + x', \theta + \theta') \leq f_n(x, \theta) + f_n(x', \theta')$. In order to show this property, we can work on each term of the sum separately, which means that we only need to prove the following inequality for all $x, x' \in \mathbb{R}$ and $\theta, \theta' \in \mathbb{R}_+$:

$$\theta e^{-\frac{x}{\theta}} + \theta' e^{-\frac{x'}{\theta'}} \geq (\theta + \theta') e^{-\frac{x+x'}{\theta+\theta'}} .$$

First observe that this inequality holds when $\theta = 0$ or $\theta' = 0$. For example, when $\theta = 0$, we have to check that $\theta' e^{-\frac{x'}{\theta'}} \geq \theta' e^{-\frac{x+x'}{\theta'}}$, which is a consequence of the fact that $x \mapsto e^{-x}$ is a decreasing function. When $\theta + \theta' > 0$, let us recall the well-known fact that $x \mapsto e^{-x}$ is a convex function on $\mathbb{R}_+$, implying that $\lambda e^{-a} + \lambda' e^{-a'} \geq e^{-(\lambda a + \lambda a')}$ for any nonnegative a, a', λ and λ' satisfying $\lambda + \lambda' = 1$. Choosing $a = \frac{x}{\theta}$, $a' = \frac{x'}{\theta'}$, $\lambda = \frac{\theta}{\theta+\theta'}$ and $\lambda' = \frac{\theta'}{\theta+\theta'}$, we find that

$$\frac{\theta}{\theta+\theta'} e^{-\frac{x}{\theta}} + \frac{\theta'}{\theta+\theta'} e^{-\frac{x'}{\theta'}} \geq e^{-\frac{\theta}{\theta+\theta'}\frac{x}{\theta} - \frac{\theta'}{\theta+\theta'}\frac{x'}{\theta'}} ,$$

which, after multiplying by $(\theta + \theta')$, lead to the desired inequality

$$\theta e^{-\frac{x}{\theta}} + \theta' e^{-\frac{x'}{\theta'}} \geq (\theta + \theta') e^{-\frac{x+x'}{\theta+\theta'}} .$$

Positive homogeneity and subadditivity imply that f_n is a convex function. Since $f_n(x, \theta) \geq 0$ for all $x \in \mathbb{R}_+^n$ and $\theta \in \mathbb{R}_+$, we notice that $\mathcal{G}^n$ is the epigraph of f_n, i.e.

$$\operatorname{epi} f_n = \left\{ (x, \theta, \kappa) \in \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R} \mid f_n(x, \theta) \leq \kappa \right\} = \mathcal{G}^n .$$

$\mathcal{G}^n$ is thus the epigraph of a convex positively homogeneous function, hence a convex cone [6]. □

We now proceed to prove some properties of the geometric cone $\mathcal{G}^n$.

Theorem 2.2 *$\mathcal{G}$ is closed.*

Proof. Let $\{(x^k, \theta^k, \kappa^k)\}$ a sequence of points in $\mathbb{R}_+^{n+2}$ such that $(x^k, \theta^k, \kappa^k) \in \mathcal{G}^n$ for all k and $\lim_{k\to\infty}(x^k, \theta^k, \kappa^k) = (x^\infty, \theta^\infty, \kappa^\infty)$. In order to prove that $\mathcal{G}^n$ is closed, it suffices to show that $(x^\infty, \theta^\infty, \kappa^\infty) \in \mathcal{G}^n$. Let us distinguish two cases:

- $\theta^\infty > 0$. Using the easily proven fact that functions $(x_i, \theta) \mapsto \theta e^{-\frac{x_i}{\theta}}$ are continuous on $\mathbb{R}_+ \times \mathbb{R}_{++}$, we have that

$$\theta^\infty \sum_{i=1}^n e^{-\frac{x_i^\infty}{\theta^\infty}} = \sum_{i=1}^n \lim_{k\to\infty} \theta^k e^{-\frac{x_i^k}{\theta^k}} = \lim_{k\to\infty} \sum_{i=1}^n \theta^k e^{-\frac{x_i^k}{\theta^k}} \leq \lim_{k\to\infty} \kappa^k = \kappa^\infty ,$$

  which implies $(x^\infty, \theta^\infty) \in \mathcal{G}^n$.

◇ $\theta^\infty = 0$. Since $(x^k, \theta^k, \kappa^k) \in \mathcal{G}^n$, we have $x^k \geq 0$ and $\kappa^k \geq 0$, which implies that $x^\infty \geq 0$ and $k^\infty \geq 0$. This shows that $(x^\infty, 0, \kappa^\infty) \in \mathcal{G}^n$..

In both cases, $(x^\infty, \theta^\infty, \kappa^\infty)$ is shown to belong to $\mathcal{G}^n$, which proves the claim. □

It is also interesting to identify the interior of this cone.

Theorem 2.3 *The interior of $\mathcal{G}^n$ is given by*

$$\operatorname{int} \mathcal{G}^n = \left\{(x, \theta, \kappa) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \mid \theta \sum_{i=1}^{n} e^{-\frac{x_i}{\theta}} < \kappa \right\} .$$

Proof. According to Lemma 7.3 in [6] we have

$$\operatorname{int} \mathcal{G}^n = \operatorname{int} \operatorname{epi} f_n = \{(x, \theta, \kappa) \mid (x, \theta) \in \operatorname{int} \operatorname{dom} f_n \text{ and } f_n(x, \theta) < \kappa\} .$$

The result then simply follows from the fact that $\operatorname{int} \operatorname{dom} f_n = \mathbb{R}^{n+1}_{++}$. □

Corollary 2.1 *The cone $\mathcal{G}^n$ is solid.*

Proof. It suffices to prove that there exists at least one point that belongs to $\operatorname{int} \mathcal{G}^n$ (Definition 1.1). Taking for example the point $(e, \frac{1}{n}, 1)$, where e stands for the n-dimensional all-one vector, we have

$$\sum_{i=1}^{n} \theta e^{-\frac{x_i}{\theta}} = e^{-n} < 1 = \kappa ,$$

and therefore $(e, \frac{1}{n}, 1) \in \operatorname{int} \mathcal{G}^n$. □

We also have the following fact:

Theorem 2.4 *$\mathcal{G}^n$ is pointed.*

Proof. The fact that $0 \in \mathcal{G}^n \subseteq \mathbb{R}^{n+2}_+$ implies that $\mathcal{G}^n \cap -\mathcal{G}^n = \{0\}$, i.e. $\mathcal{G}^n$ is pointed (Definition 1.2). □

To summarize, $\mathcal{G}^n$ is a solid pointed closed convex cone, hence suitable for conic optimization.

2.1 The dual geometric cone $(\mathcal{G})^*$

In order to express the dual of a conic problem involving the geometric cone $\mathcal{G}^n$, we need to find an explicit description of its dual.

Theorem 2.5 *The dual of $\mathcal{G}^n$ is given by*

$$(\mathcal{G}^n)^* = \left\{(x^*, \theta^*, \kappa^*) \in \mathbb{R}^n_+ \times \mathbb{R} \times \mathbb{R}_+ \mid \theta^* \geq \sum_{x_i^* < \kappa^*} \left(x_i^* \log \frac{x_i^*}{\kappa^*} - x_i^*\right) - \sum_{x_i^* \geq \kappa^*} \kappa^* \right\} .$$

Proof. Using Definition 1.3 for the dual cone, we have

$$(\mathcal{G}^n)^* = \left\{(x^*, \theta^*, \kappa^*) \in \mathbb{R}^n \times \mathbb{R}^2 \mid (x, \theta, \kappa)^T (x^*, \theta^*, \kappa^*) \geq 0 \text{ for all } (x, \theta, \kappa) \in \mathcal{G}^n\right\}$$

(the * superscript on variables x^* and θ^* is a reminder of their dual nature). We first note that in the case $\theta = 0$, we may choose any $x \in \mathbb{R}^n_+$ and $\kappa \in \mathbb{R}_+$ and have $(x, 0, \kappa) \in \mathcal{G}^n$, which means that the product

$$(x, \theta, \kappa)^T (x^*, \theta^*, \kappa^*) = x^T x^* + \theta\theta^* + \kappa\kappa^* = x^T x^* + \kappa\kappa^*$$

has to be nonnegative for all $(x, \kappa) \in \mathbb{R}^{n+1}_+$ and is easily seen to imply that x^* and κ^* are nonnegative. We may now suppose $\theta > 0$, $(x^*, \kappa^*) \geq 0$ and write

$$\begin{aligned}
& x^T x^* + \theta\theta^* + \kappa\kappa^* \geq 0 \quad \text{for all} \quad (x, \theta, \kappa) \in \mathcal{G}^n \\
\Leftrightarrow\ & x^T x^* + \theta\theta^* + \Big(\theta \sum_{i=1}^n e^{-\frac{x_i}{\theta}}\Big)\kappa^* \geq 0 \quad \text{for all} \quad (x, \theta) \in \mathbb{R}^n_+ \times \mathbb{R}_{++} \\
\Leftrightarrow\ & \theta^* \geq -\frac{x^T x^*}{\theta} - \kappa^* \sum_{i=1}^n e^{-\frac{x_i}{\theta}} \quad \text{for all} \quad (x, \theta) \in \mathbb{R}^n_+ \times \mathbb{R}_{++} \\
\Leftrightarrow\ & \theta^* \geq -t^T x^* - \kappa^* \sum_{i=1}^n e^{-t_i} \quad \text{for all} \quad t \in \mathbb{R}^n_+ \\
\Leftrightarrow\ & \theta^* \geq -\sum_{i=1}^n \big(t_i x_i^* + \kappa^* e^{-t_i}\big) \quad \text{for all} \quad t \in \mathbb{R}^n_+ \,,
\end{aligned}$$

where we have defined $t_i = \frac{x_i}{\theta}$ for convenience. We now proceed to seek the greatest possible lower bound on θ^*, examining each term of the sum separately: we have thus to seek the minimum of

$$t_i x_i^* + \kappa^* e^{-t_i} \,.$$

The derivative of this quantity with respect to t_i being equal to $x_i^* - \kappa^* e^{-t_i}$, we have a minimum when $t_i = -\log \frac{x_i^*}{\kappa^*}$, but we have to take into account the fact that t_i has to be nonnegative, which leads us to distinguish the following three cases

- $\kappa^* = 0$: in this case, the minimum is always equal to 0,
- $\kappa^* > 0$ and $x_i^* \leq \kappa^*$: in this case, the minimum is attained for a nonnegative t_i and is equal to $-x_i^* \log \frac{x_i^*}{\kappa^*} + x_i^*$, this quantity being taken as equal to zero in the case of $x_i^* = 0$,
- $\kappa^* > 0$ and $x_i^* > \kappa^*$: in this case, the minimum value for a nonnegative t is attained for $t = 0$ and is equal to κ^*.

These three cases can be summarized with

$$\inf_{t_i > 0} \big(t_i x_i^* + \kappa^* e^{-t_i}\big) = \begin{cases} -x_i^* \log \frac{x_i^*}{\kappa^*} + x_i^* & \text{when } x_i^* < \kappa^* \\ \kappa^* & \text{when } x_i^* \geq \kappa^* \end{cases} .$$

Since all of these lower bounds can be simultaneously attained with a suitable choice of t, we can state the final defining inequalities of our dual cone as

$$x^* \geq 0,\ \kappa^* \geq 0 \text{ and } \theta^* \geq \sum_{0<x_i^*<\kappa^*} \left(x_i^* \log \frac{x_i^*}{\kappa^*} - x_i^*\right) - \sum_{x_i^* \geq \kappa^*} \kappa^* .$$

□

As a special case, since $\mathcal{G}^0 = \mathbb{R}^2_+$, we check that $(\mathcal{G}^0)^* = (\mathbb{R}^2_+)^* = \mathbb{R}^2_+$, as expected.

Note 2.1 *It can be easily checked that the lower bound on θ^* appearing in the definition is always nonpositive, which means that $(x^*, \theta^*, \kappa^*) \in (\mathcal{G}^n)^*$ as soon as x^* and θ^* are nonnegative. This fact could have been guessed prior to any computation: noticing that $\mathcal{G}^n \subseteq \mathbb{R}^{n+2}_+$ and $(\mathbb{R}^{n+2}_+)^* = \mathbb{R}^{n+2}_+$, we immediately have that $(\mathcal{G}^n)^* \supseteq \mathbb{R}^{n+2}_+$, because taking the dual of a set inclusion reverses its direction.*

Finding the dual of $\mathcal{G}$ was a little involved, but establishing its properties is straightforward.

Theorem 2.6 *$(\mathcal{G})^*$ is a solid, pointed, closed convex cone. Moreover, $((\mathcal{G})^*)^* = \mathcal{G}$.*

The proof of this fact is immediate by Theorem 1.2 since $(\mathcal{G})^*$ is the dual of a solid, pointed, closed convex cone. □

The interior of $(\mathcal{G})^*$ is also rather easy to obtain:

Theorem 2.7 *The interior of $(\mathcal{G})^*$ is given by*

$$\operatorname{int}(\mathcal{G}^n)^* = \left\{ (x^*, \theta^*, \kappa^*) \in \mathbb{R}^n_{++} \times \mathbb{R} \times \mathbb{R}_{++} \mid \theta^* > \sum_{0<x_i^*<\kappa^*} \left(x_i^* \log \frac{x_i^*}{\kappa^*} - x_i^*\right) - \sum_{x_i^* \geq \kappa^*} \kappa^* \right\} .$$

Proof. We first note that $(\mathcal{G}^n)^*$, a convex set, is the epigraph of the following function

$$f_n : \mathbb{R}^n_+ \times \mathbb{R}_+ \mapsto \mathbb{R} : (x^*, \kappa^*) \mapsto \sum_{0<x_i^*<\kappa^*} \left(x_i^* \log \frac{x_i^*}{\kappa^*} - x_i^*\right) - \sum_{x_i^* \geq \kappa^*} \kappa^* ,$$

which implies that f_n is convex (by definition of a convex function). Hence we can apply Lemma 7.3 in [6] to get

$$\operatorname{int}(\mathcal{G}^n)^* = \operatorname{int} \operatorname{epi} f_n = \left\{ (x^*, \kappa^*, \theta^*) \in \operatorname{int} \operatorname{dom} f_n \times \mathbb{R} \mid \theta^* > f_n(x^*, \kappa^*) \right\} ,$$

which is exactly our claim since $\operatorname{int} \mathbb{R}^n_+ \times \mathbb{R}_+ = \mathbb{R}^n_{++} \times \mathbb{R}_{++}$. □

3 A conic formulation

This is the main section of this article, where we show how a primal-dual pair of geometric optimization problems can be modelled using the $\mathcal{G}^n$ and $(\mathcal{G}^n)^*$ cones.

3.1 Modelling geometric optimization

Let us restate here for convenience the definition of the standard primal geometric optimization problem:

$$\sup\ b^T y \quad \text{s.t.} \quad g_k(y) \leq 1 \text{ for all } k \in R\,, \tag{PG}$$

where functions g_k are defined by

$$g_k : \mathbb{R}^m \mapsto \mathbb{R}_{++} : y \mapsto \sum_{i \in I_k} e^{a_i^T y - c_i}\,.$$

In the rest of this article, we will use the following useful convention: v_S (resp. M_S) denotes the restriction of column vector v (resp. matrix M) to the components (resp. rows) whose indices belong to set S. We introduce a vector of auxiliary variables $s \in \mathbb{R}^n$ to represent the exponents used in functions g_k, more precisely we let

$$s_i = c_i - a_i^T y \text{ for all } i \in I \text{ or, in matrix form, } s = c - A^T y\,,$$

where A is a $m \times n$ matrix whose columns are a_i. Our problem becomes then

$$\sup\ b^T y \quad \text{s.t.} \quad s = c - A^T y \text{ and } \sum_{i \in I_k} e^{-s_i} \leq 1 \text{ for all } k \in R\,,$$

which is readily seen to be equivalent to the following, using the definition of $\mathcal{G}$,

$$\sup\ b^T y \quad \text{s.t.} \quad A^T y + s = c \text{ and } (s_{I_k}, 1, 1) \in \mathcal{G}^{\# I_k} \text{ for all } k \in R\,,$$

and finally to

$$\sup\ b^T y \quad \text{s.t.} \quad \begin{pmatrix} A^T \\ 0 \\ 0 \end{pmatrix} y + \begin{pmatrix} s \\ v \\ w \end{pmatrix} = \begin{pmatrix} c \\ e \\ e \end{pmatrix} \text{ and } (s_{I_k}, v_k, w_k) \in \mathcal{G}^{n_k} \text{ for all } k \in R\,, \tag{CPG}$$

where e is the all-one vector in $\mathbb{R}^r$, $n_k = \# I_k$ and two additional vectors of fictitious variables $v, w \in \mathbb{R}^r$ have been introduced, whose components are fixed to 1 by part of the linear constraints. This is exactly a conic optimization problem, in the dual form (CD), using variables $(\tilde{y}, \tilde{s})$, data $(\tilde{A}, \tilde{b}, \tilde{c})$ and a cone K^* such that

$$\tilde{y} = y,\ \tilde{s} = \begin{pmatrix} s \\ v \\ w \end{pmatrix},\ \tilde{A} = \begin{pmatrix} A & 0 & 0 \end{pmatrix},\ \tilde{b} = b,\ \tilde{c} = \begin{pmatrix} c \\ e \\ e \end{pmatrix},\ K^* = \mathcal{G}^{n_1} \times \mathcal{G}^{n_2} \times \cdots \times \mathcal{G}^{n_r}\,,$$

where K^* has been defined as the Cartesian product of several disjoint geometric cones, in order to deal with multiple conic constraints involving disjoint sets of variables. We also note that the fact that we have been able to model geometric optimization with a convex cone is a proof that these problems are convex.

3.2 Deriving the dual

Using properties of $\mathcal{G}$ and $(\mathcal{G})^*$ proved in the previous section, it is straightforward to show that K^* is a solid, pointed, closed convex cone whose dual is

$$(K^*)^* = K = (\mathcal{G}^{n_1})^* \times (\mathcal{G}^{n_2})^* \times \cdots \times (\mathcal{G}^{n_r})^* ,$$

another solid, pointed, closed convex cone, according to Theorem 1.2. This allows us to derive a dual problem to (CPG) in a completely mechanical way and find the following conic optimization problem, expressed in the primal form (CP):

$$\inf \begin{pmatrix} c \\ e \\ e \end{pmatrix}^T \begin{pmatrix} x \\ z \\ u \end{pmatrix} \quad \text{s.t.} \quad \begin{pmatrix} A & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ z \\ u \end{pmatrix} = b \text{ and } (x_{I_k}, z_k, u_k) \in (\mathcal{G}^{n_k})^* \ \forall k \in R , \tag{CDG}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^r$ and $u \in \mathbb{R}^r$ are the vectors we optimize. This problem can be simplified: making the conic constraints explicit, we find

$$\inf\ c^T x + e^T z + e^T u \text{ s.t.} \begin{cases} Ax = b,\ x_{I_k} \geq 0,\ u_k \geq 0 , \\ z_k \geq \sum_{\substack{i \in I_k \\ 0 < x_i < u_k}} (x_i \log \frac{x_i}{u_k} - x_i) - \sum_{\substack{i \in I_k \\ x_i \geq u_k}} u_k \ \forall k \in R , \end{cases}$$

which can be further reduced to

$$\inf\ c^T x + e^T u + \sum_{k \in R} \Big(\sum_{\substack{i \in I_k \\ 0 < x_i < u_k}} (x_i \log \frac{x_i}{u_k} - x_i) - \sum_{\substack{i \in I_k \\ x_i \geq u_k}} u_k \Big) \text{ s.t. } Ax = b,\ u \geq 0 \text{ and } x \geq 0 .$$

Indeed, since each variable z_k is free except for the inequality coming from the associated conic constraint, these inequalities must be satisfied with equality at each optimum solution and variables z can therefore be removed from the formulation. At this point, the formulation we have is simpler than the pure conic dual but is still different from the usual geometric optimization dual problem (DG) one can find in the literature. A little bit of calculus will help us to bridge the gap: let us fix k and consider the corresponding terms in the objective

$$c_{I_k}^T x_{I_k} + u_k + \sum_{\substack{i \in I_k \\ 0 < x_i < u_k}} (x_i \log \frac{x_i}{u_k} - x_i) - \sum_{\substack{i \in I_k \\ x_i \geq u_k}} u_k .$$

We would like to eliminate variable u_k, i.e. find for which value of u_k the previous quantity is minimum. It is first straightforward to check that such a value of u_k must satisfy $x_i < u_k$ for all $i \in I_k$, i.e. will only involve the first summation sign (since the value $-u_k$ in the second sum is attained as a limit case in the first sum when x_i tends to u_k from below). Taking the derivative with respect to u_k and equating it to zero we find

$$0 = 1 + \sum_{i \in I_k} x_i \frac{u_k}{x_i} \left(-\frac{x_i}{u_k^2}\right) = 1 - \frac{\sum_{i \in I_k} x_i}{u_k}, \text{ which implies } u_k = \sum_{i \in I_k} x_i .$$

Our objective terms become equal to

$$c_{I_k}^T x_{I_k} + \sum_{i \in I_k} x_i + \sum_{i \in I_k} \Big(x_i \log \frac{x_i}{\sum_{i \in I_k} x_i} - x_i\Big) = c_{I_k}^T x_{I_k} + \sum_{i \in I_k} x_i \log \frac{x_i}{\sum_{i \in I_k} x_i} ,$$

and leads to the following simplified dual problem

$$\inf \; c^T x + \sum_{k \in R} \sum_{\substack{i \in I_k \\ x_i > 0}} x_i \log \frac{x_i}{\sum_{i \in I_k} x_i} \quad \text{s.t.} \quad Ax = b \text{ and } x \geq 0 , \qquad \text{(DG)}$$

which is, as we expected, the traditional form of a dual geometric optimization problem that was presented in Section 1.1. This confirms the perfect relevance of our pair of primal-dual geometric cones as a tool to model the class of geometric optimization problems.

4 Concluding remarks

In this article, we have formulated geometric optimization problems in a conic way using some suitably defined convex cones $\mathcal{G}^n$ and $(\mathcal{G}^n)^*$. This approach has the following advantages:

- ◇ Classical results from the standard conic duality theory can be applied to derive the duality properties of a pair of geometric optimization problems, including weak and strong duality. This was done in [1, 2] and can be done here in a very similar fashion. This leads in our opinion to clearer proofs, the specificity of the class of problems under study being confined to the convex cone used in the formulation.

- ◇ Another advantage of our conic formulation is that it allows us to benefit with minimal work from the theory of polynomial interior-point methods for convex optimization developed in [4]. Indeed, finding a computable self-concordant barrier for our geometric cone $\mathcal{G}$ is all that is needed to build an algorithm able to solve a geometric optimization problem up to a given accuracy within a polynomial number of arithmetic operations.

- ◇ Unlike the cones described in [1], the pair of cones we have introduced in this chapter bears some strong similarities with the cones $\mathcal{L}^p$ and $\mathcal{L}_s^q$ used in [2] for l_p-norm optimization. We can indeed write the following definition of the cone $\mathcal{L}^p$

$$\mathcal{L}^p = \Big\{ (x, \theta, \kappa) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \mid \theta \sum_{i=1}^n \frac{1}{p_i} \Big| \frac{x_i}{\theta} \Big|^{p_i} \leq \kappa \Big\}$$

  and compare it to

$$\mathcal{G}^n = \Big\{ (x, \theta, \kappa) \in \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+ \mid \theta \sum_{i=1}^n e^{-\frac{x_i}{\theta}} \leq \kappa \Big\} .$$

The only difference between those two definitions is the function that is applied to the quantities $\frac{x_i}{\theta}$ for each term of the sum: the geometric cone $\mathcal{G}^n$ uses $x \mapsto e^{-x}$ while the l_p-norm cone $\mathcal{L}^p$ is based on $x \mapsto \frac{1}{p_i}|x|$. This observation is the first step towards the design of a common framework that would encompass geometric optimization, l_p-norm optimization and several other kinds of structured convex problems and that would allow the easy derivation of the associated duality properties.

Acknowledgements. The author is a research fellow supported by the F.N.R.S. (Belgian National Fund for Scientific Research). This research has been partly carried out during a visit to the SSOR department of the Delft University of Technology supported by a grant of the Communauté Française de Belgique, and the author wishes to thank Professors Kees Roos and Tamás Terlaky for their kind hospitality.

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Received November, 2000