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Virasoro symmetries for the Ablowitz-Ladik hierarchy and non-intersecting Brownian motion models

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Contents

Introduction

Random matrices, integrable systems and Virasoro constraints

In quantum mechanics, the state of a system is described by a wave function ψ which is solution of the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi,$$

where H is the Hamiltonian of the system. The energy levels in which the system can be found are given by the discrete part of the spectrum of the Hamiltonian. For certain systems, an exact knowledge of the Hamiltonian, and hence of its spectrum, is almost impossible. In the 1950's, Wigner [**69**] made the observation that the Hamiltonian of heavy nuclei could be modeled by a large size Hermitian random matrix. Indeed, the local statistical behavior of the eigenvalues of these random matrices very nicely model the local statistical behavior of the energy levels of heavy nuclei. The early mathematical theory of random matrices has been developped by Dyson [**32**], Gaudin [**38**], Mehta, ... We refer to the introduction of Mehta's book [**55**] for a historical overview of the field.

Central in random matrix theory is the concept of *matrix ensemble*, a set of matrices with a probability measure defined on it. A *random matrix* is a randomly chosen matrix in a given matrix ensemble, for the probability measure defined on it. Of particular interest is the distribution of the eigenvalues of a random matrix. The most famous example of a random matrix ensemble is the Gaussian Unitary Ensemble (GUE). It is the space $\mathcal{H}_n$ of complex $n \times n$ Hermitian matrices, with the probability measure

$$P(M)\mathrm{d}M = \frac{1}{Z_n}\, e^{-\mathrm{Tr}(M^2)}\, \mathrm{d}M, \tag{1}$$

where Z_n is a normalization constant, and

$$\mathrm{d}M = \prod_j \mathrm{d}M_{jj} \prod_{k<j} \mathrm{dRe}M_{kj}\, \mathrm{dIm}M_{kj}$$

is the Lebesgue measure on the independent variables of the matrix M. This probability measure is invariant under conjugation by a unitary matrix. Moreover the independent entries of a matrix M in this ensemble are statistically independent Gaussian random variables. The probability measure induces a joint probability density measure on the eigenvalues

$$P_n(x_1, \dots, x_n)\mathrm{d}x_1 \dots \mathrm{d}x_n = \frac{1}{Z'_n}\, \Delta_n(x)^2 \exp\Big(- \sum_{j=1}^n x_j^2 \Big)\, \mathrm{d}x_1 \dots \mathrm{d}x_n, \tag{2}$$

where $\Delta_n(x) = \det \big[x_j^{i-1}\big]_{1 \le i,j \le n}$ is the Vandermonde determinant, and Z'_n is a normalizing constant. The probability that a random GUE matrix has its spectrum in a set $E \subset \mathbb{R}$ is then simply the integral of this density over E^n. It can also be written as a Fredholm determinant

$$\mathbb{P}_n\big(\text{spectrum } M \text{ in } E^C\big) = \det\big(1 - K_n \chi_E\big),$$

with K_n an integral kernel that can be written in terms of Hermite polynomials, and χ_E is the indicator function of the set E. This formula is particularly interesting when studying large n assymptotics. Indeed, using well-known assymptotic formulas for the Hermite polynomials, we have

$$\lim_{n \to \infty} P\Big(\lambda_1, \dots, \lambda_n \notin \frac{\pi}{\sqrt{2n}}[-2a, 2a]\Big) = \det\big(I - K_{sin}\, \chi_{[-2a,2a]}\big),$$

with

$$K_{sin}(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$

the well-known Sine kernel.

An important development in the theory of random matrices in the 1980's was the discovery by Jimbo, Miwa, Môri and Sato [**46**] that the Fredholm determinant of the Sine kernel, appearing in the study of the distribution of the eigenvalues of large random matrices, as we have seen for the GUE ensemble, can be written in terms of a solution of the Painlevé V equation.

Theorem 1. (Jimbo-Miwa-Môri-Sato [**46**]) *We have*

$$\det\Big(1 - K_{\sin}\chi_{[-2a,2a]}\Big) = \exp\Big(\int_0^{\pi a} \frac{\sigma(x)}{x}dx\Big),$$

with σ the solution of the Painlevé V equation

$$(t\sigma'')^2 + 4(t\sigma' - \sigma)\big(t\sigma' - \sigma + (\sigma')^2\big) = 0,$$

so that

$$\sigma = -\frac{t}{\pi} - \frac{t^2}{\pi^2} - \frac{t^3}{\pi^3} + O(t^4), \quad \text{as } t \to 0.$$

This discovery makes the link between random matrix theory and the theory of integrable systems. The link with integrable systems has proven to be very fruitfull and has lead since the 1990's to a large number of new results.

The approach of Adler, Shiota and van Moerbeke [**6**] to random matrix theory lies within this perspective. The idea is to consider some deformations of the probabilities related to certain matrix ensembles by adding extra time variables. With respect to these new variables, the deformed probabilities are then special solutions of integrable hierarchies. An integrable hierarchy is a family of evolution equations

$$\frac{\partial u}{\partial t_j} = X_j(u), \quad j \in J, \tag{3}$$

on a manifold, with J a (finite or infinite) subset of $\mathbb{N}$, and the equations can be solved simultanously. This means that for all j, the vector field X_j is a symmetry for the other vector fields X_i, $i \neq j$, i.e. if $u(t)$ is a solution to the evolution equations (3), then

$$u(t) + \varepsilon X_j\big(u(t)\big)$$

is still a solution of the evolution equations (3), up to terms of order ε^2. Checking that X_j is a symmetry for the other vector fields X_i is rather simple. It suffices to check that

$$[X_j, X_i] = 0, \quad \forall i,$$

where $[\,\cdot\,,\,\cdot\,]$ is the Lie bracket of vector fields. The concept of symmetry can be generalized to time-dependent vector fields. A time-dependent vector field $Y(u,t)$ is a time-dependent symmetry of the evolution equations (3) if

$$u(t) \mapsto u(t) + \varepsilon Y\big(u(t), t\big),$$

maps a solution $u(t)$ of (3) on another solution of (3), up to terms of order ε^2. One easily checks that this is equivalent to

$$\frac{\partial Y}{\partial t_j} = [Y, X_j], \quad \forall j. \tag{4}$$

Following Fuchssteiner [**37**], we introduce the concept of master symmetries. These are time-independent vector fields V such that

$$[V, X_j] \neq 0, \qquad \big[[V, X_i], X_j\big] = 0, \qquad \forall i, j \in J. \tag{5}$$

Master symmetries are generators for time-dependent symmetries of (3) which are first order polynomial expressions in the time variables. Indeed, the vector field

$$Y = V + \sum_{j \in J} t_j [V, X_j]$$

satisfies (4). Master symmetries are related to the Virasoro algebra, and are usually connected with a bi-hamiltonian structure in the sense of Magri [**52**]. We leave aside the master symmetries for a while.

Typical hierarchies appearing in the Adler-Shiota-van Moerbeke approach are the KP hierarchy, the Toda lattice, ... As an example, let's have a look to the GUE ensemble. The probability measure (1) is deformed in the following way by introducing a family of time variables $t_1, t_2, t_3, \ldots$

$$P(M)\mathrm{d}M = \frac{1}{Z_n(t)} e^{-Tr(M^2) + \sum_{k=1}^{\infty} t_k Tr(M^k)} \mathrm{d}M.$$

Consequently, we have

$$\mathbb{P}_n\big(\text{spectrum } M \text{ in } E\big) = \frac{\tau_n(E;t)}{\tau_n(\mathbb{R};t)},$$

with

$$\tau_n(E;t) = \int_{\mathcal{H}_n(E)} P(M)\mathrm{d}M = \int_{E^n} \Delta_n(z)^2 \prod_{k=1}^{n} e^{\sum_{i=1}^{\infty} t_i z_k^i} e^{-z_k^2} \mathrm{d}z_k, \tag{6}$$

where $\mathcal{H}_n(E)$ is the set of $n \times n$ Hermitian matrices with spectrum in E. This function can be written in the form of a determinant of a finite moment matrix

$$\tau_n(E;t) = \det\big(\mu_{ij}(E;t)\big)_{0 \le i,j \le n-1},$$

where

$$\mu_{ij}(E;t) := \mu_{i+j}(E;t) = \big\langle z^i, z^j \big\rangle_{E,t} = \int_E z^{i+j} e^{-z^2 + \sum_{k=1}^{\infty} t_k z^k} dz,$$

are the moments associated to the scalar product $\langle \cdot\,, \cdot \rangle_{E,t}$ on the space of polynomials on the real line with complex coefficients. The link with integrable hierarchies goes through time-dependent orthogonal polynomials. Let $\{p_n(z;t)\}_{n \ge 0}$ be the sequence of time-dependent monic orthogonal polynomials associated with this scalar product. It can be proven that the polynomials $p_n(z;t)$ are given by the following expressions

$$p_n(z;t) = z^n \frac{\tau_n(E; t - [z^{-1}])}{\tau_n(E;t)}, \tag{7}$$

where $[z] = (z, \frac{z^2}{2}, \frac{z^3}{3}, \dots)$. It is a well-known fact that orthogonal polynomials on the real line satisfy three-term recurrence relations (see [**20**]):

$$zp_n(z;t) = p_{n+1}(z;t) + b_{n+1}(t)p_n(z;t) + a_n(t)p_{n-1}(z;t),$$

with initial condition $p_{-1}(z;t) = 0$ and $p_0(z;t) = 1$. These recurrence relations define a tri-diagonal semi-infinite matrix

$$L(t) = \begin{pmatrix} b_1(t) & 1 & & & \\ a_1(t) & b_2(t) & 1 & & \\ & a_2(t) & b_3(t) & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

This matrix satisfies the Toda lattice hierarchy

$$\frac{\partial L}{\partial t_i} = [L_+^i, L], \tag{8}$$

where $(\,\cdot\,)_+$ denotes the upper triangular part including the diagonal. Equations (7) and (8) imply that each of the functions $\tau_n(E;t)$, $n = 1, 2, \dots$, are tau-functions for the Toda lattice, in the sense of Sato theory. More details can be found in Chapter 5. The precise link between Sato's theory and the theory of orthogonal polynomials was first established in [**41**]. As a consequence, the functions $\tau_n(E;t)$ are solutions of the KP hierarchy, in particular, they satisfy the KP equation

$$\Big(\frac{\partial^4}{\partial t_1^4} + 3\frac{\partial^2}{\partial t_2^2} - 4\frac{\partial^2}{\partial t_1 \partial t_3}\Big)\log\tau_n + 6\Big(\frac{\partial^2}{\partial t_1^2}\log\tau_n\Big)^2 = 0. \tag{9}$$

The tau-functions $\tau_n(E;t)$ completely encode the Toda hierarchy. The entries of the matrix $L(t)$, the moments μ_k and the polynomials $p_n(z;t)$ can all be expressed in terms of these tau-functions. Based on Favard's theorem for orthogonal polynomials on the real line, we have the following correspondence

$$(\mu_k(t))_{k\geq 0} \leftrightarrow (\tau_n(t))_{n\geq 0} \leftrightarrow L(t). \tag{10}$$

A second important tool in the Adler-Shiota-van Moerbeke approach, besides the use of integrable hierarchies, is the application of Virasoro gauge transformations. As the functions $\tau_n(E;t)$ given in (6) are matrix integrals, we may change variables without changing the value of the integral (gauge invariance). Typically, for a set $E = \cup_{i=1}^r [c_{2i-1}, c_{2i}]$, this gauge invariance leads to linear constraints on the integrals

$$\mathcal{B}_m \tau_n(E;t) = \mathbb{V}_m^n \tau_n(E;t), \qquad m \geq -1, \tag{11}$$

with

$$\mathcal{B}_m = \sum_{j=1}^{2r} c_j^{m+1} \frac{\partial}{\partial c_j},$$

and $\mathbb{V}_m$ operators in the time variables, related to a representation of the Virasoro algebra in the space of formal power series in $t_1, t_2, \dots$. The Virasoro algebra is an infinite dimensional complex Lie algebra, obtained as a central extension of the complexification of the Lie algebra of vector fields on the unit circle S^1. The latter is a complex Lie algebra *d*, with basis

$$d_n = ie^{in\theta}\frac{d}{d\theta} = -z^{n+1}\frac{d}{dz},$$

where $n \in \mathbb{Z}$, $z = e^{i\theta}$, and commutation relations

$$[d_m, d_n] = (m-n)d_{m+n},$$

for $m, n \in \mathbb{Z}$. The *Virasoro algebra* is a central extension $d \oplus c\,\mathbb{C}$ of *d* by a one-dimensional center $c\,\mathbb{C}$, together with the commutation relations

$$[d_m, c] = 0,$$

$$[d_m, d_n] = (m-n)d_{m+n} + \delta_{m,-n}\,\frac{m^3-m}{12}\,c.$$

Combining (9) and the Virasoro constraints (11) evaluated on the locus $\{t_1 = t_2 = t_3 = \cdots = 0\}$, Adler-Shiota-van Moerbeke [**6**] have obtained a partial differential equation for $\tau_n(E;0)$, and thus for $\mathbb{P}_n\big(\text{spectrum } M \text{ in } E\big)\big|_{t=0}$, the variables being the endpoints of the set E. Defining $\mathbb{P}_n := \mathbb{P}_n\big(\text{spectrum } M \text{ in } E\big)\big|_{t=0}$, this PDE reads

$$\big(\mathcal{B}_{-1}^4 + 8n\mathcal{B}_{-1}^2 + 12\mathcal{B}_0^2 + 24\mathcal{B}_0 - 16\mathcal{B}_{-1}\mathcal{B}_1\big)\log\mathbb{P}_n + 6\big(\mathcal{B}_{-1}^2\log\mathbb{P}_n\big)^2 = 0.$$

In particular, when $E =]-\infty, x]$, then this PDE reduces to a 4th order ODE, which turns out to be a disguised form of the Painlevé IV equation.

In [**42**], Haine and Semengue propose an approach to this kind of problems 'at the level of the moments' (see also Faybusovich and Gekhtman [**35**] where the same idea appeared independently). This approach is based on the correspondence (10) given above. At the level of the moments, the Toda lattice equations (8) read

$$\frac{\partial\mu_k}{\partial t_i} = \mu_{k+i}, \quad i \geq 0, \quad k \geq 0,$$

and these equations define vector fields on the space of moments $T_i(\mu_k) = \mu_{k+i}$. The Toda vector fields T_i commute

$$[T_i, T_j] = 0, \quad \forall i, j \geq 0,$$

as can immediately be checked. Haine and Semengue then define the vector fields

$$V_j\,:\,\frac{\partial\mu_k}{\partial s_j} = (k+j+1)\mu_{k+j}, \quad j \geq -1.$$

These vector fields satisfy the commutation relations

$$[V_j, T_k] = kT_{k+j}, \quad [V_j, V_k] = (k-j)V_{j+k}. \tag{12}$$

So the vector fields V_j don't commute with the Toda vector fields. Comparing (12) with (5), one observes that the vector fields V_j, $j \geq -1$, form a Virasoro subalgebra of master symmetries for the Toda lattice. We repeat that master symmetries are not real symmetries of the hierarchy as they do not commute with the vector fields of the hierarchy. They are generators for time dependent symmetries of the hierarchy, which are first order polynomials in the time variables.

Translating the vector fields V_j to the level of the tau functions τ_n, the Virasoro constraints (11) can be recovered. So, in a certain sense, the Virasoro constraints are the expression of the master symmetries V_j at the level of the tau functions.

Random matrices, integrable systems and Virasoro constraints will play a central role in this thesis. We will be concerned with two different problems.

(1) In the first part, we will construct a Virasoro algebra of master symmetries for the Ablowitz-Ladik hierarchy. The first equation of the Ablowitz-Ladik hierarchy is a space discretization of the cubic nonlinear Schrödinger equation. Integrable deformations of the gap probabilities of the Circular Unitary random matrix ensemble are tau functions for this hierarchy. We will start with a study of this matrix ensemble, and construct Virasoro constraints for the deformed probabilities. These constraints will help us to obtain the master symmetries of the Ablowitz-Ladik hierarchy.

(2) In the second part we will study non-intersecting Brownian motion models. These models are closely related to Hermitian random matrix ensembles. Again, Virasoro constraints will play a crucial role in this part in the construction of PDE's satisfied by some probabilities.

The Ablowitz-Ladik hierarchy and the Circular Unitary Ensemble

In the first part of this thesis, we will construct an algebra of master symmetries for the Ablowitz-Ladik hierarchy. The Ablowitz-Ladik hierarchy is a hierarchy of compatible equations, and the first one is the Ablowitz-Ladik equation. This is a differential-difference equation which was introduced in 1975-1976 by Ablowitz and Ladik [**1**, **2**] in the form

$$\begin{cases} -i\frac{\partial q_k}{\partial t} = q_{k+1} - 2q_k + q_{k-1} - q_k r_k\big(q_{k+1} + q_{k-1}\big), \\ \\ -i\frac{\partial r_k}{\partial t} = -r_{k+1} + 2r_k - r_{k-1} + r_k q_k\big(r_{k+1} + r_{k-1}\big). \end{cases}$$

It is a space-discretization of the cubic nonlinear Schrödinger equation. Indeed, taking $r_k = \pm\overline{q_k}$, this equation reduces to

$$-i\frac{\partial q_k}{\partial t} = q_{k+1} - 2q_k + q_{k-1} \pm |q_k|^2(q_{k+1} + q_{k-1}).$$

After scaling

$$t \mapsto \varepsilon^{-2} t, \quad q_k \mapsto \varepsilon q_k,$$

the continuous limit $\varepsilon \to 0$ gives the cubic nonlinear Schrödinger equation

$$-iq_t = q_{xx} \mp 2q|q|^2.$$

The Ablowitz-Ladik equation is a discrete integrable Hamiltonian system. An infinite family of constants of motion can be found, which define the other vector fields of the hierarchy. The Ablowitz-Ladik equation received a lot of attention, as it is used in modeling several phenomena, such as wave propagation in optical fiber arrays. For a recent account of the huge literature on the Ablowitz-Ladik hierarchy, we refer the reader to Section 3.9 of [**39**]. The defocusing case ($y_k = \overline{x_k}$) has been studied in great detail by Nenciu [**58**]. In the same way as the Toda lattice is related to orthogonal polynomials on the real line, Nenciu establishes that the right tool to study the defocusing Ablowitz-Ladik hierarchy are the orthogonal Laurent polynomials on the unit circle in the complex plane. As proven by Cantero, Moral and Velazquez [**19**], orthogonal Laurent polynomials on the unit circle satisfy five-term recurrence relations. These recurrence relations define a penta-diagonal matrix, called a CMV-matrix. See also [**60**, **61**] for a discussion on CMV-matrices. Nenciu obtains Lax pairs for the defocusing Ablowitz-Ladik hierarchy using these CMV-matrices.

The Ablowitz-Ladik hierarchy has also been studied by Adler-van Moerbeke [**9**, **12**], under the name Toeplitz lattice. It appears in the context of random matrices and combinatorics, when dealing with integrals over the unitary group $U(n)$ for the Haar measure. Adler and van Moerbeke have obtained the Ablowitz-Ladik hierarchy as a reduction of the 2-Toda lattice. They consider the following time-dependent bilinear pairing on $\mathbb{C}[z] \times \mathbb{C}[z]$

$$\langle f, g \rangle = \oint_{S^1} f(z) g(z^{-1}) \, e^{\sum_{j=1}^{\infty} t_j z^j + \sum_{j=1}^{\infty} s_j z^{-j}} \rho(z) \, \frac{\mathrm{d}z}{2\pi i z}. \tag{13}$$

The moments $\mu_{i-j} := \mu_{i,j} = \langle z^i, z^j \rangle$ define a moment matrix $m_\infty = (\mu_{i,j})_{i,j \geq 0}$ which is Toeplitz, i.e. all elements on a same diagonal are equal. One sees immediately that as function of t, s, the moment matrix satisfies the following equations

$$\begin{cases} \frac{\partial m_\infty}{\partial t_n} = \Lambda^n \, m_\infty, \\ \\ \frac{\partial m_\infty}{\partial s_n} = m_\infty \left(\Lambda^T\right)^n, \end{cases} \qquad n \geq 1, \tag{14}$$

where $\Lambda = (\delta_{i,j-1})_{i,j \geq 0}$ is the usual shift matrix. At the level of the moments, these equations read

$$T_j \mu_k \equiv \frac{\partial \mu_k}{\partial t_j} = \mu_{k+j}, \qquad T_{-j} \mu_k \equiv \frac{\partial \mu_k}{\partial s_j} = \mu_{k-j}, \quad \forall j \geq 1. \tag{15}$$

Obviously $[T_i, T_j] = 0, \forall i, j \in \mathbb{Z}$, if we define $T_0\mu_k = \mu_k$. Let $\{p_n^{(1)}(\,\cdot\,; t, s), p_n^{(2)}(\,\cdot\,; t, s)\}_{n\geq 0}$ be the associated sequence of time-dependent monic bi-orthogonal polynomials. We have

$$\left\langle p_n^{(1)}, p_m^{(2)} \right\rangle = h_n \delta_{nm}, \quad h_n \neq 0,$$

and define $h = \operatorname{diag}(h_n)_{n\geq 0}$. We also define

$$x_n(t,s) = p_n^{(1)}(0; t, s), \qquad y_n(t,s) = p_n^{(2)}(0; t, s),$$

and $p^{(i)}(t, s; z) = \left(p_n^{(i)}(t, s; z)\right)_{n\geq 0}$, $i = 1, 2$. These polynomials satisfy recurrence relations

$$L_1 p^{(1)}(t, s; z) = z\, p^{(1)}(t, s; z), \quad \left(h^{-1} L_2 h\right)^T p^{(2)}(t, s; z) = z\, p^{(2)}(t, s; z),$$

with L_1 and L_2 matrices given by

$$L_1 = \begin{pmatrix} -x_1 y_0 & 1 & & & \\ -\frac{h_1}{h_0} x_2 y_0 & -x_2 y_1 & 1 & & O \\ -\frac{h_2}{h_0} x_3 y_0 & -\frac{h_2}{h_1} x_3 y_1 & -x_3 y_2 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\left(h^{-1} L_2 h\right)^T = \begin{pmatrix} -x_0 y_1 & 1 & & & \\ -\frac{h_1}{h_0} x_0 y_2 & -x_1 y_2 & 1 & & O \\ -\frac{h_2}{h_0} x_0 y_3 & -\frac{h_2}{h_1} x_1 y_3 & -x_2 y_3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Adler and van Moerbeke [**9**] prove that L_1 and L_2 are solutions of the 2-Toda lattice hierarchy described in [**67**]

$$\frac{\partial L_i}{\partial t_n} = [(L_1^n)_+, L_i], \quad \frac{\partial L_i}{\partial s_n} = [(L_2^n)_{--}, L_i], \qquad i = 1, 2, \; n = 1, 2, \ldots$$

where for a matrix A, we denote by A_+ (resp. A_{--}) the upper triangular part (resp. the strictly lower triangular part) of A. The particular form of the matrices L_1, L_2 is preserved by these evolution equations. The reduction of the 2-Toda lattice hierarchy to matrices with this particular form is called by Adler and van Moerbeke the Toeplitz lattice. The Toeplitz lattice equations on the variables x_n, y_n are exactly the evolution equations of the Ablowitz-Ladik hierarchy (see [**18**, **62**]). As proven by Ueno and Takasaki [**67**], the entries of the Lax operators L_1, L_2 of the 2-Toda lattice can ultimately be described in terms of a sequence of functions $(\tau_n(t,s))_{n\geq 0}$ called 2-Toda tau functions, satisfying some bilinear identities. When dealing with the Toeplitz reduction, these tau functions are given by finite determinants of the moment matrix:

$$\tau_n(t,s) = \det\left(\mu_{i,j}(t,s)\right)_{0\leq i,j\leq n-1}.$$

The starting point to obtain master symmetries for the Ablowitz-Ladik hierarchy is the Circular Unitary Ensemble from random matrix theory. The Circular Unitary Ensemble is the group $U(n)$ of unitary $n \times n$ matrices, with the normalized Haar measure as probability measure. The Weyl integral formula gives the induced density distribution on the eigenvalues of the matrices on the unit circle in the complex plane, and is given by

$$\frac{1}{n!}|\Delta_n(z)|^2 \prod_{k=1}^{n} \frac{dz_k}{2\pi i z_k}, \quad z_k = e^{i\theta_k}.$$

For $\eta, \theta \in]-\pi, \pi[$, with $\eta \leq \theta$, the probability that a random CUE matrix has no eigenvalues within an arc of circle $(\eta, \theta) = \{z \in S^1 \mid \eta < \arg(z) < \theta\}$ is given by

$$\tau_n(\eta, \theta) = \frac{1}{(2\pi)^n n!} \int_\theta^{2\pi+\eta} \cdots \int_\theta^{2\pi+\eta} \prod_{1 \leq k < l \leq n} \left| e^{i\theta_k} - e^{i\theta_l} \right|^2 d\theta_1 \ldots d\theta_n.$$

Following Adler-van Moerbeke [**9**], we introduce the 2-Toda time dependent tau functions

$$\tau_n(t, s; \eta, \theta) = \frac{1}{n!} \int_{[\theta, 2\pi+\eta]^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left(e^{\sum_{j=1}^{\infty} (t_j z_k^j + s_j z_k^{-j})} \frac{dz_k}{2\pi i z_k} \right), \tag{16}$$

with $z_k = e^{i\theta_k}$. These tau functions deform the probabilities $\tau_n(\eta, \theta) = \tau_n(0, 0; \eta, \theta)$. The main result of Chapter 1 is that these tau functions satisfy a set of Virasoro constraints indexed by *all* integers, decoupling into a boundary-part and a time-part. We have

Theorem 2. (Haine-Vanderstichelen [**43**])

(i) The tau functions $\tau_n(t, s; \eta, \theta), n \geq 1$, satisfy

$$\mathcal{B}_k(\eta, \theta)\tau_n(t, s; \eta, \theta) = L_k^{(n)} \tau_n(t, s; \eta, \theta), \quad k \in \mathbb{Z},$$

with $L_k^{(n)}, k \in \mathbb{Z}$, time-dependent differential operators, and

$$\mathcal{B}_k(\eta, \theta) = \frac{1}{i}\left(e^{ik\theta} \frac{\partial}{\partial \theta} + e^{ik\eta} \frac{\partial}{\partial \eta} \right); \quad i = \sqrt{-1}.$$

(ii) The operators $L_k^{(n)}, k \in \mathbb{Z}$, satisfy the commutation relations of the centerless Virasoro algebra, that is

$$\left[L_k^{(n)}, L_l^{(n)} \right] = (k - l) L_{k+l}^{(n)}, \quad k, l \in \mathbb{Z}.$$

The main surprise of this result is that the 2-Toda tau functions deforming the gap probabilities of the CUE ensemble satisfy a centerless *full* Virasoro algebra of constraints. This stands in contrast with the corresponding result for the deformed gap probabilities of the GUE ensemble and other Hermitian ensembles, which roughly satisfy only "half of" a Virasoro type algebra of constraints.

The integrals (16) can be expressed as Toeplitz determinants

$$\tau_n(t,s) = \det\big(\mu_{k-l}(t,s)\big)_{0\le k,l\le n-1}, \tag{17}$$

where

$$\mu_k(t,s) = \int_{S^1} z^k\, e^{\sum_{j=1}^{\infty}(t_j z^j + s_j z^{-j})} w(z)\, \frac{\mathrm{d}z}{2\pi i z}, \quad k \in \mathbb{Z}, \tag{18}$$

and $w(z)$ is some (complex-valued) weight function defined on the unit circle S^1. These moments satisfy the Ablowitz-Ladik hierarchy (15). Hence, the determinants (17) are very special instances of tau functions for the Ablowitz-Ladik hierarchy. This suggests, using the Virasoro constraints obtained in Chapter 1, following an idea introduced in [**42**] in the context of the 1-dimensional Toda lattices and explained earlier in this introduction, to define the following vector fields on the moments

$$V_j \mu_k = (k+j)\mu_{k+j}, \quad \forall j \in \mathbb{Z}. \tag{19}$$

These vector fields trivially satisfy the commutation relations

$$[V_i, V_j] = (j-i)V_{i+j}, \quad [V_i, T_j] = jT_{i+j}, \quad \forall i,j \in \mathbb{Z}, \tag{20}$$

from which it follows that

$$[[V_i, T_j], T_k] = j[T_{i+j}, T_k] = 0, \quad \forall i,j,k \in \mathbb{Z}. \tag{21}$$

Equations (20) and (21) mean that the vector fields $V_j, j \in \mathbb{Z}$, form a Virasoro algebra of master symmetries for the Ablowitz-Ladik hierarchy.

The Ablowitz-Ladik tau functions admit the following expansion

$$\tau_n(t,s) = \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} S_{i_{n-1}-(n-1),\ldots,i_0}(t) S_{j_{n-1}-(n-1),\ldots,j_0}(s),$$

where

$$p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} = \det\big(\mu_{i_k-j_l}(0,0)\big)_{0\le k,l\le n-1}, \tag{22}$$

are the so-called Plücker coordinates, and $S_{i_1,\ldots,i_k}(t)$ denote the Schur polynomials

$$S_{i_1,\ldots,i_k}(t) = \det\big(S_{i_r+s-r}(t)\big)_{1\le r,s\le k},$$

with $S_n(t)$ the elementary Schur polynomials. In Chapter 4, we shall establish the next result:

Theorem 3. (Haine-Vanderstichelen [**44**]) *For all $k \in \mathbb{Z}$, we have*

$$L_k^{(n)} \tau_n(t,s) = \sum_{\substack{0 \le i_0 < \cdots < i_{n-1} \\ 0 \le j_0 < \cdots < j_{n-1}}} V_k\left(p_{\substack{i_0,\ldots,i_{n-1} \\ j_0,\ldots,j_{n-1}}}\right) S_{i_{n-1}-(n-1),\ldots,i_0}(t) S_{j_{n-1}-(n-1),\ldots,j_0}(s),$$

with $L_k^{(n)}, k \in \mathbb{Z}$, defined as in Theorem 2, and $V_k\left(p_{\substack{i_0,\ldots,i_{n-1} \\ j_0,\ldots,j_{n-1}}}\right)$ the Lie derivative of the Plücker coordinates (22) *in the direction of the master symmetries V_k of the Ablowitz-Ladik hierarchy, as defined in* (19).

Thus the operators $L_k^{(n)}, k \in \mathbb{Z}$, precisely describe the master symmetries of the Ablowitz-Ladik hierarchy on the tau functions of this hierarchy.

To complete the picture, we give Lax equations for the master symmetries of the Ablowitz-Ladik hierarchy at the end of Chapter 4. The master symmetries of the Ablowitz-Ladik hierarchy form a full centerless Virasoro algebra. As we have seen, the centerless Virasoro algebra corresponds to the complexification of the Lie algebra of vector fields on the circle S^1, with basis

$$d_n = -z^{n+1} \frac{d}{dz}, \quad z = e^{i\theta},\ n \in \mathbb{Z}.$$

Define the matrices M_1, M_2 by

$$\frac{d}{dz} p^{(1)}(z;t,s) = M_1 p^{(1)}(z;t,s), \quad \frac{d}{dz} p^{(2)}(z;t,s) = M_2 p^{(2)}(z;t,s).$$

The operators $-z^{n+1}\frac{d}{dz}$ acting on $p^{(1)}(t,s;z)$ (or $p^{(2)}(t,s;z)$) can be expressed in terms of the matrices L_1, M_1, as long as $n+1 \ge 0$. Indeed, we have for $n+1 \ge 0$

$$-z^{n+1} \frac{d}{dz} p^{(1)}(z;t,s) = -M_1 L_1^{n+1} p^{(1)}(z;t,s).$$

When $n+1 < 0$, the operator $-z^{n+1}\frac{d}{dz}$ acting on $p^{(1)}(t,s;z)$ can not be expressed any more in terms of the matrices L_1, M_1 as L_1^{-1} is not defined. This, as well as the work of Nenciu [**58**], suggests to use bi-orthogonal Laurent polynomials for the study of the Ablowitz-Ladik hierarchy. Laurent polynomials are polynomials in z and z^{-1}. Let $\langle \cdot\,, \cdot \rangle$ be a time dependent bilinear pairing on the space of Laurent polynomials, for example the pairing (13). Applying a Gram-Schmidt bi-orthogonalization process to the basis $\{1, z, z^{-1}, z^2, z^{-2}, \ldots\}$, one obtains two sequences $\{f_n, g_n\}_{n \ge 0}$ of time-dependent monic right bi-orthogonal Laurent polynomials (See Chapter 2 for an exact definition), satisfying

$$\langle f_m, g_n \rangle = h_m \delta_{mn}.$$

Define the vectors $f = (f_n)_{n\geq 0}$ and $g = (g_n)_{n\geq 0}$. The multiplication by z in these bases is then given by two time-dependent matrices A_1, A_2

$$zf(z) = A_1 f(z), \qquad zg(z) = A_2 g(z),$$

and these matrices turn out to be penta-diagonal, generalizing the result obtained by Cantero, Moral and Velazquez [**19**], and also discussed by Simon [**60**,**61**], for orthogonal Laurent polynomials on the unit circle. The matrices A_1, A_2 are called (generalized) CMV-matrices. They are not independent, as A_2 is related to A_1^{-1} through

$$A_1^{-1} = hA_2^T h^{-1},$$

where $h = \mathrm{diag}(h_n)_{n\geq 0}$. We have

Theorem 4. *The Ablowitz-Ladik hierarchy is given by the Lax equations*

$$\frac{\partial A_1}{\partial t_n} = [A_1, (A_1^n)_{--}], \qquad \frac{\partial A_2}{\partial t_n} = [A_2, (A_2^{-n})_{--}], \qquad \forall n \in \mathbb{Z}, \tag{23}$$

where, for convenience, we put $s_n = t_{-n}$, $n \geq 0$.

We define the matrices D_1 and $(D_1^*)^T$ (respectively D_2 and $(D_2^*)^T$) representing the operator of derivation d/dz in the bases $\big(f_n(z)\big)_{n\geq 0}$ and $\big(h_n^{-1} g_n(z^{-1})\big)_{n\geq 0}$ (respectively $\big(g_n(z)\big)_{n\geq 0}$ and $\big(h_n^{-1} f_n(z^{-1})\big)_{n\geq 0}$):

$$\begin{aligned}
\frac{d}{dz} f(z) &= D_1\, f(z), &\qquad \frac{d}{dz}\big(h^{-1}\, g(z^{-1})\big) &= (D_1^*)^T\, \big(h^{-1}\, g(z^{-1})\big),\\
\frac{d}{dz} g(z) &= D_2\, g(z), &\qquad \frac{d}{dz}\big(h^{-1}\, f(z^{-1})\big) &= (D_2^*)^T\, \big(h^{-1}\, f(z^{-1})\big).
\end{aligned}$$

We obtain Lax equations for the Virasoro master symmetries of the Ablowitz-Ladik hierarchy in terms of the CMV matrices. We prove in Chapter 4

Theorem 5. (Haine-Vanderstichelen [**44**]) *For $k \in \mathbb{Z}$*

$$\begin{aligned}
V_k(A_1) &= \big[A_1, \big(D_1 A_1^{k+1}\big)_{--} + \big(A_1^{k+1} D_1^*\big)_{--} + k(A_1^k)_{--}\big],\\
V_k(A_2) &= \big[\big(D_2 A_2^{1-k}\big)_{--} + \big(A_2^{1-k} D_2^*\big)_{--} - k(A_2^{-k})_{--}, A_2\big],
\end{aligned}$$

where D_1 and $(D_1^*)^T$ (respectively D_2 and $(D_2^*)^T$) represent the operator of derivation d/dz in the bases $\big(f_n(z)\big)_{n\geq 0}$ and $\big(h_n^{-1} g_n(z^{-1})\big)_{n\geq 0}$ (respectively $\big(g_n(z)\big)_{n\geq 0}$ and $\big(h_n^{-1} f_n(z^{-1})\big)_{n\geq 0}$), with $f_n(z), g_n(z)$ the bi-orthogonal Laurent polynomials satisfying $\langle f_m, g_n\rangle = h_m \delta_{mn}$.

The theory of time-dependent bi-orthogonal Laurent polynomials is developped in Chapters 2 and 3. In Chapter 3 we construct the Lax equations (23) for the Ablowitz-Ladik hierarchy in terms of these CMV matrices A_1, A_2.

Non-intersecting Brownian motions

Random matrix theory is related with models of non-intersecting Brownian motions. Brownian motion was first described by the botanist Robert Brown in 1828. He was studying pollen particles in water under the microscope. He observed minute particles, ejected by the pollen grains, executing a jittery motion. He repeated the experiment with pollen coming from different plant species and also with particles of inorganic matter. In all these experiments he observed the same phenomenon, but he was unable to explain the origin of the motion. In 1877, Delsaux advanced the hypothesis that the changes in direction and speed of the particles were caused by the collisions of the particles with the water molecules. In 1905, Albert Einstein brought the solution of the problem to the attention of physicists. He determined the transition probability of the process using the heat equation. In 1906, Marian Smoluchowski obtained the Brownian motion as a scaling limit of random walks.

The rigourous mathematical theory to study Brownian motion was developed by Norbert Wiener in 1923. In mathematical terms, a real-valued stochastic process $\{B_t\}_{t\geq 0}$ is a standard Brownian motion starting at the origin if it is a Gaussian process such that

(1) $B_0 = 0$;

(2) for a sequence $0 \leq t_1 < t_2 < \cdots < t_k$ of times, the increments $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$ are independent random variables;

(3) for all $0 \leq s < t$, $B_t - B_s$ has a normal distribution $\mathcal{N}(0, t-s)$ with mean 0, and variance $t - s$;

(4) the path functions $t \mapsto B_t$ are almost surely continuous on $[0, +\infty[$.

In 1958, Karlin and McGregor [**49**] established a formula allowing one to compute the transition probability density $p_N(t, \vec{a}, \vec{b})$ to find N independent Brownian particles starting in $a_1 < \cdots < a_N$ at time $t = 0$ in positions $b_1, \ldots, b_N$ at a time $t > 0$ without any two of them ever having been coincident during the time interval $[0, t]$. It is given in terms of the transition probability density of one Brownian particle on the real line

$$p(t, x, y) = \frac{1}{\sqrt{\pi t}} e^{\frac{-(x-y)^2}{t}},$$

by the following determinant

$$p_N(t,\vec{\alpha},\vec{\beta}) := \det\begin{pmatrix} p(t,a_1,b_1) & \cdots & p(t,a_1,b_N) \\ \vdots & & \vdots \\ p(t,a_N,b_1) & \cdots & p(t,a_N,b_N) \end{pmatrix}.$$

In the second part of this thesis, we will be concerned with the study of N independent Brownian motions during a time-interval $[0,1]$, conditioned to start at positions $\alpha_1 < \cdots < \alpha_N$ at time $t=0$ and to end up in positions $\beta_1 < \cdots < \beta_N$, without two of them ever having been coincident during that time-interval. We will call this process non-intersecting Brownian motions. The probability density to find all the particles at time $0<t<1$ in positions $x_1,\dots,x_N$ is then, using the formula of Karlin-McGregor,

$$p_N\left(t;x_1,\dots,x_N \,\middle|\, \begin{matrix} \big(x_1(0),\dots,x_N(0)\big) = \big(\alpha_1,\dots,\alpha_N\big) \\ \big(x_1(1),\dots,x_N(1)\big) = \big(\beta_1,\dots,\beta_N\big) \end{matrix}\right)$$
$$= \frac{1}{Z_N} p_N(t,\vec{\alpha},\vec{x}) p_N(1-t,\vec{x},\vec{\beta}),$$

where Z_N is a normalizing constant. Interesting cases are the so-called confluent cases, where several particles start and/or end up in the same points. Of special interest are the two following cases:

(1) $\alpha_1 = \cdots = \alpha_N = 0$ and $\beta_1 = \cdots = \beta_N = 0$. After a simple rescaling, the distribution of the Brownian particles at time t then coincides with the distribution of the eigenvalues of a random GUE matrix.
(2) $\alpha_1 = \cdots = \alpha_N = 0$. After a simple rescaling, the distribution of the Brownian particles at time t coincides with the distribution of the eigenvalues of a random matrix from the Gaussian ensemble with external source $B = \text{diag}(\beta_1,\dots,\beta_N)$, as proven in [**15**].

The first confluent case, after rescaling, also describes the distribution of N Dyson Brownian motions on the real line. This process, discovered by Dyson [**31**] in 1962, describes the motion in time of the eigenvalues of a $N \times N$ Hermitian matrix whose real and imaginary parts of the entries perform independent Ornstein-Uhlenbeck-processes, with an initial distribution given by the invariant measure of the process. See Adler-Delépine-van Moerbeke [**3**] and Katori-Tanemura [**50**, **51**] for a detailed description of the relationship between Dyson Brownian motions, non-intersecting Brownian motions and Gaussian Hermitian matrix ensembles. In particular, in [**50**] both stochastic processes are obtained as scaling limits of the vicious walkers model.

In the two particular cases cited (i.e. non-intersecting Brownian motions with one starting position and one or several ending positions), the relationship between non-intersecting Brownian motions and Hermitian matrix models has led to a deeper comprehension of the diffusion problems. In both cases, partial differential equations

(PDE) for the finite N diffusions have been obtained (see [**6**, **8**, **14**]). For large N, upon taking appropriate scaling limits, different processes appear describing the transition probabilities of critical infinite dimensional diffusions, like the Airy process, the Sine process and the Dyson process (see [**11**, **51**, **63**]) for one starting and ending position, and for two or more ending positions the Pearcey process (see [**15**, **64**]), the Airy process with k outliers (see [**3**]), etc.

Consider now the following confluent case

$$\begin{aligned}(\alpha_1,\dots,\alpha_N) &= \big(\underbrace{a_1,a_1,\dots,a_1}_{m_1},\underbrace{a_2,a_2,\dots,a_2}_{m_2},\dots,\underbrace{a_q,a_q,\dots,a_q}_{m_q}\big),\\ (\beta_1,\dots,\beta_N) &= \big(\underbrace{b_1,b_1,\dots,b_1}_{n_1},\underbrace{b_2,b_2,\dots,b_2}_{n_2},\dots,\underbrace{b_p,b_p,\dots,b_p}_{n_p}\big),\end{aligned}$$

with $\sum_{i=1}^{q} m_i = \sum_{i=1}^{p} n_i = N$, for general p and q, $a_1 < a_2 < \cdots < a_q$ and $b_1 < b_2 < \cdots < b_p$. In this case, it is not known if the distribution of the positions of the non-intersecting Brownian particles at a given time $0 < t < 1$, is the same as the joint distribution of the eigenvalues of a matrix ensemble. For $p = q = 2$ this problem has first been studied by Daems-Kuijlaars [**22**] and Daems-Kuijlaars-Veys [**23**]. In these papers, the authors consider $N/2$ particles going from a to b, and $N/2$ particles going from $-a$ to $-b$. They show that the correlation functions of the positions of the non-intersecting Brownian motions have a determinantal form, with a kernel that can be expressed in terms of mixed multiple Hermite polynomials. They analyze the kernel in the large N limit, for a small separation of the starting and ending positions (i.e. when the product ab is sufficiently small), and find the limiting mean density of particles is supported by one or two intervals. Taking usual scaling limits of the kernel in the bulk and near the edges they find the Sine and the Airy kernel. For large separation of the starting and ending positions, those results have been extended by Delvaux-Kuijlaars [**26**]. In [**4**], Adler-Ferrari-van Moerbeke study a similar situation, but with an asymmetric number of paths in the left and right starting and ending positions. Recently, Adler-Ferrari-van Moerbeke [**5**] and also Delvaux-Kuijlaars-Zhang [**28**] (see also [**27**]) analyzed the large N-limit in a critical regime where the paths fill two tangent ellipses in the time-space plane. Using an appropriate double scaling limit, they prove the existence of a new process describing the diffusion of the particles near the point of tangency.

It is a hard problem to obtain concrete results about the processes describing the critical infinite dimensional diffusions, obtained as limiting situations of the problem of N non intersecting Brownian motions on the real line starting at q and ending at p prescribed positions, with $p, q \geq 2$. In the second part of this thesis we analyze the finite N diffusion for two or more starting and ending positions. We consider N non-intersecting Brownian motions $x_1(t),\dots,x_N(t)$ on $\mathbb{R}$, starting at time $t = 0$ in q

different points, and arriving in $t=1$ in p different points. If $\mathbb{P}^{a_1,\dots,a_q}_{b_1,\dots,b_p}\big(\text{all } x_i(t)\in E\big)$ denotes the probability to find all the particles in a set E at an intermediate time $0<t<1$, we prove the following theorem.

Theorem 6. (Adler-van Moerbeke-Vanderstichelen [**13**]) *For each value of the parameters $p\geq 1$ and $q\geq 1$, let K^* be the smallest positive integer such that*

$$(x^2-3x+4)(K^*)^2+(-x^2+3x+4)K^*-2x(x^2-2x-1)>0,$$

with $x=p+q$. Let E be a finite union of intervals. Under the assumptions $a_1+\cdots+a_q=0$ and $b_1+\cdots+b_p=0$, the function $\log\mathbb{P}^{a_1,\dots,a_q}_{b_1,\dots,b_p}\big(\text{all } x_i(t)\in E\big)$ satisfies a nonlinear PDE of order K^*+3 or less, the variables being the coordinates of the endpoints of the set E, and the coordinates of $a_1,\dots,a_q$ and $b_1,\dots,b_p$.

For example, for $4\leq x\leq 8$, the value of K^* in this theorem is given in the following table :

x	4	5	6	7	8
$\mathbf{K}^*$	3	4	5	5	5

The proof of this Theorem will be given in Chapter 6, and is based on the use of a particular integrable hierarchy, and Virasoro constraints. The use of these methods is suggested by the fact that the probability $\mathbb{P}^{a_1,\dots,a_q}_{b_1,\dots,b_p}\big(\text{all } x_i(t)\in E\big)$ has different descriptions:

(1) It can be written, after making a space and time transformation, as a block moment matrix

$$\mathbb{P}^{a_1,\dots,a_q}_{b_1,\dots,b_p}\big(\text{all } x_i(t)\in E\big) = \frac{1}{Z_N}\det\left[\left(\left\langle x^m\psi_i(x)\Big| y^n\varphi_j(y)\right\rangle\right)_{\substack{0\leq m\leq m_i-1\\ 0\leq n\leq n_j-1}}\right]_{\substack{1\leq i\leq q\\ 1\leq j\leq p}}, \tag{24}$$

where $\psi_i(x)=e^{\tilde a_i x}$, $\varphi_j(y)=e^{\tilde b_j y}$, and the following inner product

$$\left\langle x^m\psi_i(x)\Big| y^n\varphi_j(y)\right\rangle=\int_{\tilde E}x^{m+n}e^{(\tilde a_i+\tilde b_j)x}e^{-\frac{x^2}{2}}dx.$$

The $\sim$'s indicate that a space-time transformation has been performed.

(2) It can be written as a sum of multiple integrals

$$
\begin{aligned}
&\mathbb{P}^{a_1,\dots,a_q}_{b_1,\dots,b_p}\left(\text{all } x_i(t)\in E\right)\\
&= \frac{1}{Z_N}\sum_{\sigma\in S_N}(-1)^\sigma \int_{\tilde{E}^N}\Big(\prod_{i=1}^{N} e^{-\frac{x_i^2}{2}}\,dx_i\Big)\Big(\Delta_{m_1}(x_1,x_2,\dots,x_{m_1})\prod_{i=1}^{m_1}\psi_1(x_i)\Big)\\
&\times\dots\times\Big(\Delta_{m_q}(x_{m_1+\cdots+m_{q-1}+1},\dots,x_{m_1+\cdots+m_q})\prod_{i=1}^{m_q}\psi_q(x_{m_1+\cdots+m_{q-1}+i})\Big)\\
&\times\Big[\Big(\Delta_{n_1}(x_{\sigma(1)},\dots,x_{\sigma(n_1)})\prod_{i=1}^{n_1}\varphi_1(x_{\sigma(i)})\Big)\\
&\times\dots\\
&\times\Big(\Delta_{n_p}(x_{\sigma(n_1+\cdots+n_{p-1}+1)},\dots,x_{\sigma(n_1+\cdots+n_p)})\prod_{i=1}^{n_p}\varphi_p(x_{\sigma(n_1+\cdots+n_{p-1}+i)})\Big)\Big],
\end{aligned}
\tag{25}
$$

where Δ_n is the Vandermonde determinant, and S_N is the group of permutations of N elements.

As shown in Adler-van Moerbeke-Vanhaecke [**14**], the determinants of block moment matrices deformed in an appropriate way satisfy integrable hierarchies. Concretely, the determinant (24) is deformed by adding exponentials containing additional families of time variables, one family for each weight function φ_i and ψ_j, or equivalently, (25) is deformed by adding exponentials containing additional families of time variables, one family for each Vandermonde determinant. The determinants of the deformed block-moment matrices (24) are then tau functions for the multi-component KP hierarchy. The multi-component KP hierarchy is a very general hierarchy of integrable equations, describing the time-evolution of matrix-valued pseudo-differential operators, depending on several families of time variables. These operators can be expressed in terms of so-called tau-functions, which encode the whole hierarchy. As a consequence, the determinants of the deformed block moment matrices satisfy some nonlinear PDE's. The multi-component KP hierarchy is explained in Chapter 5.

As we have seen, matrix integrals deformed in an appropriate way satisfy Virasoro constraints (see [**6**]). Although we do not know if (25) for general p and q corresponds to (the reduction to polar coordinates of) a matrix integral, we show that each term in (25) separately satisfies Virasoro constraints. As a surprise, it appears that all the terms satisfy *the same* Virasoro constraints, and hence, by linearity, it follows that (25) satisfies Virasoro constraints.

Following the method developped by Adler-Shiota-van Moerbeke, Virasoro constraints with time and boundary parts can be used to eliminate all the partial derivatives with respect to the added time variables in the non-linear PDE's from the integrable hierarchy, and hence to obtain a non-linear PDE with respect to the variables of the unperturbed problem. The complexity of the problem studied does not enable one to perform concretely this elimination process and to obtain an explicit formula for arbitrary values $p, q > 2$. It is a priori not even obvious at all that it converges to a PDE after a finite number of steps! In Theorem 6 we prove, however, using a simple combinatorial argument, that it indeed does, and this for general p and q. We would like to emphasize that the existence of a PDE satisfied by $\mathbb{P}^{a_1,\dots,a_q}_{b_1,\dots,b_p}\bigl(\text{all } x_i(t) \in E\bigr)$ is not obvious at all. Our proof rests on two surprising facts, the first being that the perturbed problem satisfies Virasoro constraints, and the second that the elimination process converges after a finite number of steps.

Part 1

The Ablowitz-Ladik hierarchy and the Circular Unitary Ensemble

Chapter 1

The circular unitary ensemble

In this first chapter we introduce the concept of random matrix and random matrix ensemble. We will be mainly concerned with the study of one particular matrix ensemble: the circular unitary ensemble. The aim of this first chapter is to introduce in a self-contained way some methods and material that will be used later on.

1. Random matrices : Definition and Examples

A *matrix ensemble* is a set of matrices with a probability measure defined on it. A *random matrix* is a randomly chosen matrix in a given matrix ensemble, for the probability measure defined on it. Of particular interest is the distribution of the eigenvalues of a random matrix. We refer to Mehta's book [**55**] for a detailed discussion. We give some examples of matrix ensembles.

1.1. The Gaussian Unitary Ensemble (GUE). Let $\mathcal{H}_n$ be the space of complex $n \times n$ Hermitian matrices. A matrix $M \in \mathcal{H}_n$ has n^2 independent variables

$$M_{ii}^{(0)}, \quad M_{jk}^{(0)}, \quad M_{jk}^{(1)}, \qquad 1 \leq i \leq n,\ 1 \leq j < k \leq n,$$

respectively the real part of the diagonal elements (the imaginary part of the diagonal elements is zero), the real part of the elements above the diagonal, and the imaginary part of the elements above the diagonal. Consider the probability measure on $\mathcal{H}_n$

$$P(M)\mathrm{d}M = \frac{1}{Z_n}\, e^{-\mathrm{Tr}(M^2)}\, \mathrm{d}M, \tag{26}$$

where $Z_n = 2^{-n(n-1)/2}\pi^{n^2/2}$ is a normalization constant, and

$$\mathrm{d}M = \prod_{k\le j} \mathrm{d}M_{kj}^{(0)} \prod_{k<j} \mathrm{d}M_{kj}^{(1)}$$

is the Lebesgue measure on the independent variables of the matrix M. Two observations can be made concerning this probability measure:

(1) Developping the trace in the exponential, we get

$$P(M)\mathrm{d}M = \frac{1}{Z_n}\Big(\prod_{i=1}^{n} e^{-M_{ii}^{(0)2}}\mathrm{d}M_{ii}^{(0)}\Big) \times \Big(\prod_{1\le k<j\le n} e^{-2M_{kj}^{(0)2}}\mathrm{d}M_{kj}^{(0)}\Big)\Big(\prod_{1\le k<j\le n} e^{-2M_{kj}^{(1)2}}\mathrm{d}M_{kj}^{(1)}\Big)$$

and we observe that the independent variables of a matrix $M \in \mathcal{H}_n$ are also statistically independent random variables. They are distributed as Gaussian random variables, with zero mean.

(2) The probability measure (26) is invariant under the automorphism

$$h : \mathcal{H}_n \to \mathcal{H}_n,\ M \mapsto U^{-1}MU,$$

where $U \in U(n)$ is a $n \times n$ unitary matrix. Consequently, the measure is said to be unitary invariant.

These two observations explain the name of the matrix ensemble. Due to the unitary invariance of the probability measure (26), it induces a joint probability density measure on the eigenvalues

$$P_n(x_1,\dots,x_n)\mathrm{d}x_1\dots\mathrm{d}x_n = \frac{1}{Z_n'}\,\Delta_n(x)^2\exp\Big(-\sum_{j=1}^{n} x_j^2\Big)\,\mathrm{d}x_1\dots\mathrm{d}x_n, \tag{27}$$

where

$$\Delta_n(x) = \prod_{1\le i<j\le n}(x_j - x_i) = \det\big[x_j^{i-1}\big]_{1\le i,j\le n}, \tag{28}$$

is the Vandermonde determinant, and Z_n' is a normalizing constant. Consequently, the probability that a randomly chosen matrix in this ensemble has its spectrum in a set $E \subset \mathbb{R}$ is given by the integral

$$\begin{aligned}\mathbb{P}_n\big(\text{spectrum } M \text{ in } E\big) &= \frac{1}{Z_n}\int_{\mathcal{H}_n(E)} e^{-\mathrm{Tr}(M^2)}\mathrm{d}M\\ &= \frac{1}{Z_n'}\int_{E^n}\Delta_n(x)^2\exp\Big(-\sum_{j=1}^{n}x_j^2\Big)\,\mathrm{d}x_1\dots\mathrm{d}x_n,\end{aligned}$$

where $\mathcal{H}_n(E)$ is the set of Hermitian matrices with spectrum in E. We refer to [**55**,**25**] for a detailed discussion.

1.2. The Gaussian Ensemble with external source. Consider on $\mathcal{H}_n$, the space of complex $n \times n$ Hermitian matrices, the probability measure

$$P(M)\mathrm{d}M = \frac{1}{Z_n} e^{-\mathrm{Tr}\left(\frac{M^2}{2} - AM\right)} \mathrm{d}M, \tag{29}$$

where $\mathrm{d}M$ is, as in the example of the Gaussian Unitary Ensemble, the Lebesgue measure on the independent variables of $M \in \mathcal{H}_n$, and

$$A = \mathrm{diag}(\alpha_1, \dots, \alpha_n),$$

is a fixed diagonal matrix, $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. This measure is not invariant under conjugation by a unitary matrix any more. The probability that a randomly chosen matrix in this ensemble has its spectrum in a set $E \subset \mathbb{R}$ can be transformed in an integral over the eigenvalues, if all the α_i are distinct,

$$\begin{aligned}
\mathbb{P}_n\big(\text{spectrum } M \text{ in } E\big) &= \frac{1}{Z_n} \int_{\mathcal{H}_n(E)} e^{-\mathrm{Tr}\left(\frac{M^2}{2} - AM\right)} \mathrm{d}M \\
&= \frac{1}{Z_n'} \int_{E^n} \Delta_n(x)^2 \prod_{i=1}^n e^{-\frac{x_i^2}{2}} \mathrm{d}x_i \int_{U(n)} e^{\mathrm{Tr} AU \mathrm{diag}(x_1,\dots,x_n) U^{-1}} \mathrm{d}\mu_H(U) \\
&= \frac{1}{Z_n''} \int_{E^n} \Delta_n(x)^2 \prod_{i=1}^n e^{-\frac{x_i^2}{2}} \mathrm{d}x_i \, \frac{\det\left[e^{\alpha_i x_j}\right]_{1\le i,j\le n}}{\Delta_n(x)\Delta_n(\alpha)} \\
&= \frac{1}{Z_n'''} \int_{E^n} \Delta_n(x) \det\left[e^{-\frac{x_j^2}{2} + \alpha_i x_j}\right]_{1\le i,j\le n} \prod_{i=1}^n \mathrm{d}x_i,
\end{aligned}$$

where in the third step we have used the Harish-Chandra-Itzykson-Zuber formula to evaluate the integral over the unitary group $U(n)$, μ_H being the normalized Haar measure on $U(n)$. When several α_i coincide, the formula remains valid, upon taking appropriate limits. Suppose $\alpha_1, \dots, \alpha_{m_1} \to a_1$, $\dots$,$\alpha_{m_1+\cdots+m_{q-1}+1}, \dots, \alpha_{m_1+\cdots+m_q} \to a_q$, with $m_1 + \cdots + m_q = n$, then

$$\begin{aligned}
&\mathbb{P}_n^{a_1,\dots,a_q}\big(\text{spectrum } M \text{ in } E\big) \\
&\quad := \lim_{\substack{\alpha_1,\dots,\alpha_{m_1}\to a_1 \\ \cdots \\ \alpha_{m_1+\cdots+m_{q-1}+1},\dots,\alpha_{m_1+\cdots+m_q}\to a_q}} \mathbb{P}_n\big(\text{spectrum } M \text{ in } E\big) \\
&\quad = \frac{1}{Z_n''''} \int_{E^n} \Delta_n(x) \Big(\Delta_{m_1}(x^{(1)}) \prod_{i=1}^{m_1} e^{a_1 x_i}\Big) \times \dots \\
&\quad\quad \times \Big(\Delta_{m_q}(x^{(q)}) \prod_{i=1}^{m_q} e^{a_1 x_{m_1+\cdots+m_{q-1}+i}}\Big) \prod_{i=1}^n e^{-\frac{x_j^2}{2}} \mathrm{d}x_i,
\end{aligned} \tag{30}$$

where $x^{(1)} = (x_1, x_2, \dots, x_{m_1})$, $\dots$, $x^{(q)} = (x_{m_1+\cdots+m_{q-1}+1}, \dots, x_{m_1+\cdots+m_q})$. In [**15**] a non-intersecting Brownian motion interpretation is given of the Gaussian ensemble with external source. More details on non-intersecting Brownian motions can be found in chapter 6. We refer to [**15**] and references here-in for a detailed discussion of the Gaussian ensemble with external source.

1.3. The Circular Unitary Ensemble (CUE). The unitary group $U(n)$ is the set of $n \times n$ complex matrices $A \in \mathbb{C}^{n\times n}$ such that

$$AA^\dagger = I,$$

where I is the $n \times n$ identity matrix, together with matrix multiplication as group action. The eigenvalues of a unitary matrix all lie on the unit circle in the complex plane. The group $U(n)$ is a closed and bounded submanifold of $\mathbb{R}^{2n^2}$ of dimension n^2, after identification of $\mathbb{C}^{n\times n}$ with $\mathbb{R}^{2n^2}$, and thus it is a compact Lie group. The group acts on itself by left or right multiplication by an element of the group. As $U(n)$ is a compact group, it has a unique normalized measure μ_H that is both invariant under left and right multiplication:

$$\mu_H(hE) = \mu_H(E) = \mu_H(Eh),$$

for all $h \in U(n)$ and every measurable set E. This measure is called the normalized Haar measure. The Haar measure induces a joint eigenvalue measure density on the torus $(S^1)^n$, given by Weyl's formula

$$P_n(\theta_1, \dots, \theta_n)\mathrm{d}\theta_1 \dots \mathrm{d}\theta_n = \frac{1}{(2\pi)^n n!}|\Delta(e^{i\theta_1}, \dots, e^{i\theta_n})|^2 \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n. \tag{31}$$

The Circular Unitary Ensemble (CUE) is the set $U(n)$ of unitary $n \times n$ matrices, together with the normalized Haar measure μ_H on $U(n)$ as probability measure. Let $J \subset S^1$ be a subset of the unit circle. The probability that a randomly chosen matrix from $U(n)$ has all its spectrum in J is given by

$$\mathbb{P}_n(J) := \int_{U(n,J)} \mathrm{d}\mu_H(U),$$

where $U(n, J)$ is the set of unitary matrices with spectrum in J. Using Weyl's formula, this can be transformed in a multiple integral over the eigenvalues[1]

$$\mathbb{P}_n(J) = \int_{J^n} P_n(\theta_1, \dots, \theta_n)\mathrm{d}\theta_1 \dots \mathrm{d}\theta_n,$$

with $P_n(\theta_1, \dots, \theta_n)$ given in (31). Notice that $\Delta(e^{i\theta_1}, \dots, e^{i\theta_n}) = 0$ as soon as $\theta_l = \theta_k \mod 2\pi$ for any pair $l \neq k$. As a consequence, the eigenvalues of a randomly chosen matrix in $U(n)$ for the normalized measure are almost surely all distinct.

We will be mainly concerned with the circular unitary ensemble in this chapter.

[1]For simplicity, we denote by J a subset of S^1 and the set $\{\theta \in [0, 2\pi] | e^{i\theta} \in J\}$.

2. Joint eigenvalue probability density for the CUE : Fredholm determinants and Toeplitz moment matrices

2.1. Fredholm determinants. As shown in the preceding section, the joint eigenvalue probability density of the Circular Unitary Ensemble is

$$P_n(\theta_1, \dots, \theta_n) = \frac{1}{(2\pi)^n n!} \prod_{1 \le k < l \le n} |e^{i\theta_k} - e^{i\theta_l}|^2.$$

It is a symmetric function of its arguments. As we will see, it can be written as a determinant. Indeed, using the expression (28) of the Vandermonde determinant, we have

$$\begin{aligned}\prod_{1 \le k < l \le n} |e^{i\theta_k} - e^{i\theta_l}|^2 &= \det \left(e^{i(j-1)\theta_k}\right)_{1 \le j,k \le n} \det \left(e^{-i(j-1)\theta_k}\right)_{1 \le j,k \le n} \\ &= \det \left(\sum_{j=1}^{n} e^{i(j-1)(\theta_k - \theta_l)}\right)_{1 \le k,l \le n} \\ &= \det \left(\frac{1 - e^{in(\theta_k - \theta_l)}}{1 - e^{i(\theta_k - \theta_l)}}\right)_{1 \le k,l \le n} \\ &= \det \left(\frac{\sin \frac{n}{2}(\theta_k - \theta_l)}{\sin \frac{1}{2}(\theta_k - \theta_l)}\right)_{1 \le k,l \le n}.\end{aligned}$$

It follows that

$$P_n(\theta_1, \dots, \theta_n) = \frac{1}{n!} \det \left(\frac{1}{2\pi} \frac{\sin \frac{n}{2}(\theta_k - \theta_l)}{\sin \frac{1}{2}(\theta_k - \theta_l)}\right)_{1 \le k,l \le n}.$$

Defining the following integral kernel

$$K_n(\eta, \theta) := \frac{1}{2\pi} \frac{\sin \frac{n}{2}(\eta - \theta)}{\sin \frac{1}{2}(\eta - \theta)}, \tag{32}$$

this probability can be written

$$P_n(\theta_1, \dots, \theta_n) = \frac{1}{n!} \det \left(K(\theta_k, \theta_l)\right)_{1 \le k,l \le n}.$$

One checks that the kernel $K_n(\eta, \theta)$ has the reproducing kernel property, i.e. it satisfies

$$\int_0^{2\pi} K_n(\theta, \theta) \mathrm{d}\theta = n, \qquad \text{and} \qquad \int_0^{2\pi} K_n(\eta, \xi) K_n(\xi, \theta) d\xi = K_n(\eta, \theta).$$

This yields the important property

$$\begin{aligned}&\int_0^{2\pi} \det \left(K_n(\theta_k, \theta_l)\right)_{1 \le k,l \le m} \mathrm{d}\theta_m \\ &\qquad\qquad = (n - m + 1) \det \left(K_n(\theta_k, \theta_l)\right)_{1 \le k,l \le m-1},\end{aligned}$$

for all $1 \leq m \leq n$. The function $K_n(\theta, \theta)$ corresponds to the mean density of eigenvalues at θ. Indeed, if $\mathcal{I}$ is any subset of $[0, 2\pi]$, then we have

$$\int_{\mathcal{I}} K_n(\theta, \theta) \mathrm{d}\theta = \mathbb{E}\big(\text{number of eigenvalues in } \mathcal{I}\big).$$

One easily checks that the mean density of eigenvalues is

$$\rho := K_n(\theta, \theta) = \frac{n}{2\pi}.$$

We denote by $\mathcal{I}$ the union of p disjoint subintervals of $[0, 2\pi]$:

$$\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_p,$$

and by $\mathbb{P}_n(n_1, \ldots, n_p; \mathcal{I})$ the probability to find exactly n_1 eigenvalues in $\mathcal{I}_1, \ldots, n_p$ eigenvalues in $\mathcal{I}_p$, for a randomly chosen matrix of the Circular Unitary Ensemble, with $m = n_1 + \cdots + n_p \leq n$. We then have

$$\begin{aligned}
&\mathbb{P}_n(n_1, \ldots, n_p; \mathcal{I}) \\
&\quad = \binom{n}{n_1, \ldots, n_p, n-m} \int_0^{2\pi} \cdots \int_0^{2\pi} P_n(\theta_1, \ldots, \theta_n) \prod_{j_1=1}^{n_1} \chi_{\mathcal{I}_1}(\theta_{j_1}) \\
&\quad \times \prod_{j_2=n_1+1}^{n_1+n_2} \chi_{\mathcal{I}_2}(\theta_{j_2}) \cdots \prod_{j_p=n_1+\cdots+n_{p-1}+1}^{n_1+\cdots+n_p} \chi_{\mathcal{I}_p}(\theta_{j_p}) \\
&\quad \times \prod_{j=m+1}^{n} \big(1 - \chi_{\mathcal{I}}(\theta_j)\big) \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n.
\end{aligned}$$

It is possible to prove that (see [**25**, **55**])

$$\mathbb{P}_n(n_1, \ldots, n_p; \mathcal{I}) = \frac{(-1)^m}{n_1! \ldots n_p!} \frac{\partial^m D_n(\mathcal{I}; \lambda)}{\partial \lambda_1^{n_1} \ldots \partial \lambda_p^{n_p}} \Big|_{\lambda_1 = \cdots = \lambda_p = 1},$$

where

$$\begin{aligned}
&D_n(\mathcal{I}; \lambda) = \det\Big(1 - \sum_{j=1}^{p} \lambda_j K_n(\eta, \theta) \chi_{\mathcal{I}_j}(\theta)\Big) \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \\
&\qquad \times \int_{[0,2\pi]^m} \det\Big(\sum_{j=1}^{p} \lambda_j K_n(\theta_k, \theta_l) \chi_{\mathcal{I}_j}(\theta_l)\Big)_{1 \leq k, l \leq m} \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_m,
\end{aligned}$$

is the Fredholm determinant of the integral kernel K_n. As a consequence, we have the following formula for the gap probability

$$\mathbb{P}\big(\text{no eigenvalues in } \mathcal{I}\big) = \det\Big(1 - K_n(\eta, \theta) \chi_{\mathcal{I}}(\theta)\Big).$$

The expression of the above probabilities as Fredholm determinants of the kernel $K_n(\eta,\theta)$ is particularly useful when studying large n limits. As $n \to \infty$, the size of the matrices goes to infinity, and so does the density of eigenvalues. The mean spacing $\frac{1}{\rho}$ between successive eigenvalues tends to 0. Hence, we rescale the variables $\theta_j = \frac{x_j}{\rho}$ to normalize the mean spacing between successive eigenvalues. With this rescaling, we have

$$\lim_{n\to\infty} K_n\Big(\frac{2\pi x}{n}, \frac{2\pi y}{n}\Big) d\Big(\frac{2\pi y}{n}\Big) = \frac{\sin \pi(x-y)}{\pi(x-y)} dy =: K_{\sin}(x,y)dy.$$

This is the famous Sine-kernel, appearing also in the study of the GUE ensemble. We then have for $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_p \subset [0, 2\pi]$ a union of p disjoint subintervals, and for any fixed $n_1 + \cdots + n_p = m$:

$$\lim_{n\to\infty} \mathbb{P}_n(n_1, \ldots, n_m; \frac{2\pi}{n}\mathcal{I}) = \frac{(-1)^m}{n_1! \ldots n_p!} \frac{\partial^m D_{\sin}(\mathcal{I};\lambda)}{\partial\lambda_1^{n_1} \ldots \partial\lambda_p^{n_p}}\Big|_{\lambda_1=\cdots=\lambda_p=1},$$

where $D_{\sin}(\mathcal{I};\lambda)$ is the Fredholm determinant of the Sine-kernel. We have the following celebrated result of Jimbo, Miwa, Môri and Sato, linking Random Matrix Theory with integrable systems.

Theorem 1.1 (Jimbo-Miwa-Môri-Sato [**46**])**.** *We have*

$$\det\Big(1 - K_{sin}(\eta,\theta)\chi_{(-t/2,t/2)}(\theta)\Big) = 1 - F(t),$$

with

$$1 - F(t) = \exp\Big(\int_0^t \frac{\sigma(x)}{x} dx\Big), \qquad \textit{for } t \geq 0,$$

with σ the solution of the Painlevé V equation

$$(t\sigma'')^2 + 4(t\sigma' - \sigma)\big(t\sigma' - \sigma + (\sigma')^2\big) = 0,$$

so that

$$\sigma = -\frac{t}{\pi} - \frac{t^2}{\pi^2} - \frac{t^3}{\pi^3} + O(t^4), \quad \textit{as } t \to 0.$$

2.2. Joint eigenvalue probabilities as determinants of Toeplitz moment matrices. The probability that a $n \times n$ random CUE matrix has all its eigenvalues with argument within an arc of circle J is given by

$$\mathbb{P}_n(J) = \int_{J^n} P_n(\theta_1, \ldots, \theta_n) \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n,$$

where the joint eigenvalue density of the Circular Unitary Ensemble $P_n(\theta_1, \ldots, \theta_n)$ is given in (31). This probability can be written as the determinant of a Toeplitz matrix. A matrix A is a Toeplitz matrix if all the elements on the same diagonal are equal, i.e. if A can be written

$$A = \sum_{k=-n}^{n} a_k \Lambda^k,$$

with $\Lambda = (\delta_{i,j-1})_{1\le i,j\le n}$, and Λ^{-1} is interpreted as Λ^T. The probability $\mathbb{P}_n(J)$ can be written as the determinant of a finite Toeplitz moment matrix

$$\mathbb{P}_n(J) = \det m_n, \tag{33}$$

where

$$m_n = (\mu_{k-l})_{0\le j,k\le n-1}, \tag{34}$$

with μ_k the trigonometric moments defined by

$$\mu_k = \int_{S^1} z^k\,\rho(z)\,\frac{\mathrm{d}z}{2\pi i z}, \tag{35}$$

for $k \in \mathbb{Z}$, with the weight function $\rho(z) = \chi_J(z)$. Indeed, we have

$$\begin{aligned}\det m_n &= \int_{(S^1)^n} \det \begin{pmatrix} z_1^0 & \cdots & z_n^{-(n-1)} \\ \vdots & & \vdots \\ z_1^{n-1} & \cdots & z_n^0 \end{pmatrix} \prod_{j=1}^n \rho(z_j)\,\frac{\mathrm{d}z_j}{2\pi i z_j} \\ &= \int_{(S^1)^n} \det\left(z_l^{k-1}\right)_{1\le k,l\le n} z_1^0 z_2^{-1}\cdots z_n^{-(n-1)} \prod_{j=1}^n \rho(z_j)\,\frac{\mathrm{d}z_j}{2\pi i z_j}.\end{aligned}$$

Let S_n be the group of permutations of n elements and $\sigma \in S_n$. Relabeling the integration variables $(z_1,\dots,z_n) \to (z_{\sigma(1)},\dots,z_{\sigma(n)})$ in the multiple integral above gives

$$\begin{aligned}\det m_n = \int_{(S^1)^n} &(-1)^\sigma \det\left(z_l^{k-1}\right)_{1\le k,l\le n} \\ &\times\ z_{\sigma(1)}^0 z_{\sigma(2)}^{-1}\cdots z_{\sigma(n)}^{-(n-1)} \prod_{j=1}^n \rho(z_j)\,\frac{\mathrm{d}z_j}{2\pi i z_j},\end{aligned}$$

where $(-1)^\sigma$ is the signature of the permutation σ. The value of the integral is independent of the choice of $\sigma \in S_n$. Hence, summing over all $\sigma \in S_n$ and dividing by $n!$, the number of elements in S_n, we get

$$\begin{aligned}\det m_n = \frac{1}{n!}\sum_{\sigma\in S_n} \int_{(S^1)^n} &(-1)^\sigma \det\left(z_l^{k-1}\right)_{1\le k,l\le n} \\ &\times\ z_{\sigma(1)}^0 z_{\sigma(2)}^{-1}\cdots z_{\sigma(n)}^{-(n-1)} \prod_{j=1}^n \rho(z_j)\,\frac{\mathrm{d}z_j}{2\pi i z_j}.\end{aligned}$$

By definition of the Vandermonde determinant, this gives

$$
\begin{aligned}
&\det m_n \\
&= \frac{1}{n!}\int_{(S^1)^n} \det\left(z_l^{k-1}\right)_{1\le k,l\le n}\, \det\left(z_l^{-(k-1)}\right)_{1\le k,l\le n} \prod_{j=1}^n \rho(z_j)\,\frac{\mathrm{d}z_j}{2\pi i z_j} \\
&= \frac{1}{n!}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{j=1}^n \rho(z_j)\,\frac{\mathrm{d}z_j}{2\pi i z_j}.
\end{aligned}
$$

But this is $\mathbb{P}_n(J)$, establishing the identity (33).

Remark 1.2. *We observe that the above argument is independent of the nature of the weight function $\rho(z)$. Consequently, for an arbitrary weight function $\rho(z)$ we have*

$$
\det\Big[\oint_{S^1} z^{j-k}\,\rho(z)\,\frac{dz}{2\pi i z}\Big]_{0\le j,k<n} = \frac{1}{n!}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{j=1}^n \rho(z_j)\,\frac{dz_j}{2\pi i z_j}.
$$

Identity (33) is deeply connected with the theory of bi-orthogonal polynomials on the unit circle, as we will see. Good references for this material are [**9**, **12**, **36**], though this authors don't use the formalism of bi-moment functionals, which we will use in Chapter 2.

3. A differential equation due to Tracy and Widom

For $\eta, \theta \in]-\pi, \pi[$, with $\eta \le \theta$, the probability that a $n \times n$ random CUE matrix has no eigenvalues within an arc of circle $(\eta, \theta) = \{z \in S^1 | \eta < \arg(z) < \theta\}$ is given by

$$
\tau_n(\eta,\theta) = \frac{1}{(2\pi)^n n!}\int_\theta^{2\pi+\eta}\cdots\int_\theta^{2\pi+\eta} \prod_{1\le k<l\le n} |e^{i\varphi_k} - e^{i\varphi_l}|^2 \mathrm{d}\varphi_1\ldots\mathrm{d}\varphi_n. \tag{36}
$$

Obviously, this probability depends only on the length $\theta - \eta$. Consequently, without loss of generality, we can chose a symmetric arc of circle $(-\theta, \theta)$. We shall denote by

$$
R(\theta) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\log\tau_n(-\theta,\theta), \tag{37}
$$

the logarithmic derivative of the probability that an arc of circle of length 2θ contains no eigenvalues of a randomly chosen unitary matrix.

Using functional analytic techniques in the study of the kernel $K_n(\eta, \theta)$ in (32), Tracy and Widom prove in [**66**] that the function $R(\theta)$ satisfies the differential equation

$$
\begin{aligned}
&R(\theta)^2 + 2\sin\theta\,\cos\theta\,R(\theta)R'(\theta) + \sin^2\theta\,R'(\theta)^2 \\
&= \frac{1}{2}\Big(\frac{1}{4}\sin^2\theta\,\frac{R''(\theta)^2}{R'(\theta)} + \sin\theta\,\cos\theta\,R''(\theta) + \big(\cos^2\theta + n^2\sin^2\theta\big)\,R'(\theta)\Big).
\end{aligned} \tag{38}
$$

At the end of this chapter, we will give a new proof of this differential equation, and show it is a disguised form of the Painlevé VI equation. Our proof will be based on an integrable deformation of the multiple integral $\tau_n(\eta,\theta)$ defined in (36), and the construction of so-called Virasoro constraints for the deformed integral. This will be developped in the following section.

4. An integrable deformation of the joint probability distribution

The unitary matrix model was first discussed in [**54**] from the point of view of integrable deformations. Following [**9, 12**], let us consider the following deformation of the weight function $\rho(z) = \chi_J(z)$ defined on S^1, J being the arc of circle $[\eta,\theta]$,

$$\rho(z) \mapsto \rho(z) e^{\sum_{j=1}^{\infty} t_j z^j + \sum_{j=1}^{\infty} s_j z^{-j}},$$

in the trigonometric moments μ_k defined in (35), and the corresponding deformation of the moments

$$\mu_k(t,s) = \int_{S^1} z^k \, \rho(z) e^{\sum_{j=1}^{\infty} t_j z^j + \sum_{j=1}^{\infty} s_j z^{-j}} \, \frac{\mathrm{d}z}{2\pi i z}.$$

This deformation of the moments induces, through formula (33), a deformation of the probability $\tau_n(\eta,\theta)$

(39) $$\tau_n(t,s;\eta,\theta) = \det(\mu_{k-l}(t,s))_{0 \le j,k \le n-1},$$

or equivalently, working out the determinant

(40) $$\tau_n(t,s;\eta,\theta) = \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \Big(\rho(z_k) e^{\sum_{j=1}^{\infty} (t_j z_k^j + s_j z_k^{-j})} \, \frac{\mathrm{d}z_k}{2\pi i z_k} \Big),$$

with $z_k = e^{i\varphi_k}$, such that $\tau_n(0,0;\eta,\theta) = \tau_n(\eta,\theta)$. The deformed trigonometric moments satisfy the following simple equations

$$\frac{\partial \mu_k(t,s)}{\partial t_j} = \mu_{k+j}(t,s), \qquad \frac{\partial \mu_k(t,s)}{\partial s_j} = \mu_{k-j}(t,s), \qquad \forall j \ge 1.$$

These equations define the Ablowitz-Ladik hierarchy on the space of the trigonometric moments, as we will see in Chapter 3. The Ablowitz-Ladik hierarchy can be obtained as a reduction of the 2-Toda lattice described in [**67**]. The functions $\tau_n(t,s;\eta,\theta)$ in (40) are special instances of τ-functions in the sense of Sato theory for the 2-Toda lattice hierarchy, as we will see. Consequently, the sequence of τ-functions $(\tau_n(t,s;\eta,\theta))_{n\ge 0}$ satisfy the KP equation both in the t and the s variables :

(41) $$\Big(\frac{\partial^4}{\partial t_1^4} + 3 \frac{\partial^2}{\partial t_2^2} - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \Big) \log \tau_n + 6 \Big(\frac{\partial^2}{\partial t_1^2} \log \tau_n \Big)^2 = 0,$$

and

$$\Big(\frac{\partial^4}{\partial s_1^4} + 3 \frac{\partial^2}{\partial s_2^2} - 4 \frac{\partial^2}{\partial s_1 \partial s_3} \Big) \log \tau_n + 6 \Big(\frac{\partial^2}{\partial s_1^2} \log \tau_n \Big)^2 = 0.$$

In the first part of this section we give a proof of this statement based on orthogonality conditions of time-dependent polynomials and their Cauchy transforms. The proof is a particularization of a general proof given in [**14**]. In the second part of this section, we prove that the sequence of τ-functions $(\tau_n(t,s;\eta,\theta))_{n\geq 0}$ also satisfies linear PDE's with a boundary part (differentials with respect to η and θ) and a time part (differentials with respect to t and s), called Virasoro constraints.

From now on, we shall use the notation

$$\rho_{t,s}(z) = \rho(z)e^{\sum_{j=1}^{\infty}(t_j z^j + s_j z^{-j})}$$

for the deformed weight function.

4.1. The 2-Toda lattice and the KP equation. Let $\mathbb{C}[z]$ be the space of polynomials in the variable z with complex coefficients. We define on $\mathbb{C}[z]\times\mathbb{C}[z]$ the bilinear pairing

$$\langle f,g\rangle_{t,s} = \oint_{S^1} f(z)g(z^{-1})\rho_{t,s}(z)\frac{dz}{2\pi i z}.$$

Associated to this pairing, we define the trigonometric moments $\mu_{k,l}(t,s) = \left\langle z^k, z^l\right\rangle_{t,s}$ with $k,l\geq 0$. The bilinear pairing is completely determined by the sequence of its trigonometric moments. Obviously, the moments $\mu_{k,l}(t,s)$ only depend on the difference $k-l$. For simplicity, we shall omit the explicit dependence on the time variables (t,s) and we shall write $\mu_{k,l}(t,s)=\mu_{k,l}$. We define the semi-infinite moment matrix $m_\infty = (\mu_{k,l})_{k,l\geq 0}$. We have proven in section 2.2 that the multiple integral $\tau_n(t,s)$ can be represented as a Toeplitz determinant

$$\tau_n(t,s) = \det\left(\mu_{k,l}\right)_{0\leq k,l\leq n-1}.$$

Let $[z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \dots\right)$. Using the expansion $\ln(1-x) = -\sum_{j=1}^{\infty}\frac{x^j}{j}$, one has the following identities

$$\begin{cases} \mu_{k,l}(t-[z^{-1}],s) = \mu_{k,l} - \frac{1}{z}\mu_{k,l-1}, \\ \mu_{k,l}(t+[z^{-1}],s) = \sum_{j=0}^{\infty}\frac{1}{z^j}\mu_{k,l-j}, \end{cases}$$

and

$$\begin{cases} \mu_{k,l}(t,s-[z]) = \mu_{k,l} - z\mu_{k,l+1}, \\ \mu_{k,l}(t,s+[z]) = \sum_{j=0}^{\infty} z^j\mu_{k,l+j}. \end{cases} \tag{42}$$

It then follows that

$$
\begin{aligned}
\tau_n(t+[z^{-1}],s) &= \det\Big(\sum_{j=0}^{\infty}\frac{1}{z^j}\mu_{k,l-j}\Big)_{0\le k,l<n},\\
\tau_n(t-[z^{-1}],s) &= \det\Big(\mu_{k,l}-\frac{1}{z}\mu_{k,l-1}\Big)_{0\le k,l<n},\\
\tau_n(t,s+[z]) &= \det\Big(\sum_{j=0}^{\infty}z^j\mu_{k,l+j}\Big)_{0\le k,l<n},\\
\tau_n(t,s-[z]) &= \det\Big(\mu_{k,l}-z\mu_{k,l+1}\Big)_{0\le k,l<n}. \qquad (43)
\end{aligned}
$$

It is well-known that

$$
\begin{aligned}
p_n^{(1)}(t,s;z) &= \frac{1}{\tau_n(t,s)}\det\left(\begin{array}{ccc|c} & & & 1\\ & m_n & & z\\ & & & \vdots\\ \hline \mu_{n,0} & \cdots & \mu_{n,n-1} & z^n\end{array}\right),\\
p_n^{(2)}(t,s;z) &= \frac{1}{\tau_n(t,s)}\det\left(\begin{array}{ccc|c} & & & 1\\ & m_n^T & & z\\ & & & \vdots\\ \hline \mu_{0,n} & \cdots & \mu_{n-1,n} & z^n\end{array}\right), \qquad (44)
\end{aligned}
$$

define two sequences of monic bi-orthogonal polynomials with respect to the pairing $\langle\cdot\mid\cdot\rangle_{t,s}$, i.e. $p_n^{(i)}(t,s;z)$, $i=1,2$, is a polynomial of exact degree n, the coefficient of the highest order term being 1, and

$$
\Big\langle p_n^{(1)}(t,s,z)|p_m^{(2)}(t,s,z)\Big\rangle_{t,s} = h_n\delta_{nm},
$$

with $h_n=\frac{\tau_{n+1}}{\tau_n}$. We refer to Chapter 2 for a more detailed discussion on bi-orthogonal polynomials. As proven in [**9**], the polynomials can be written in terms of the functions $\tau_n(t,s)$:

$$
p_n^{(1)}(t,s,z) = z^n\frac{\tau_n(t-[z^{-1}],s)}{\tau_n(t,s)}, \qquad p_n^{(2)}(t,s,z) = z^n\frac{\tau_n(t,s-[z^{-1}])}{\tau_n(t,s)}. \qquad (45)
$$

As shown in [**14**], the Cauchy transforms of the polynomials $p_n^{(1)}(t,s,z)$ and $p_n^{(2)}(t,s,z)$ admit also expressions in terms of τ-function. We have

$$
\begin{aligned}
\Big\langle \frac{1}{z-u}\Big|p_n^{(2)}(t,s,u)\Big\rangle_{t,s} &= z^{-n-1}\frac{\tau_{n+1}(t+[z^{-1}],s)}{\tau_n(t,s)},\\
\Big\langle p_n^{(1)}(t,s,u)\Big|\frac{1}{z-u}\Big\rangle_{t,s} &= z^{-n-1}\frac{\tau_{n+1}(t,s+[z^{-1}])}{\tau_n(t,s)}. \qquad (46)
\end{aligned}
$$

Using expressions (45), (46), and the following simple, formal residue identities with $f(z)=\sum_{j=0}^{\infty} a_j z^j$

$$\oint_{z=\infty} f(z)\left\langle \frac{h(u)}{z-u} | g(u)\right\rangle_{t,s} \frac{dz}{2\pi i} = \langle f(u)h(u)|g(u)\rangle_{t,s}\,,$$

$$\oint_{z=0} f(z^{-1})\left\langle g(u)| \frac{h(u)}{z^{-1}-u}\right\rangle_{t,s} \frac{dz}{2\pi i z^2} = \langle g(u)|f(u)h(u)\rangle_{t,s}\,,$$

we obtain easily the following theorem.

Theorem 1.3. *The functions $\tau_n(t,s)$ satisfy the following bilinear identity*

$$\begin{aligned} &\oint_{z=\infty} \tau_n(t-[z^{-1}],s)\tau_{m+1}(t'+[z^{-1}],s')e^{\sum_{j=1}^{\infty}(t_j-t'_j)z^j} z^{n-m-1}dz \\ &= \oint_{z=0} \tau_{n+1}(t,s+[z])\tau_m(t',s'-[z])e^{-\sum_{j=1}^{\infty}(s_j-s'_j)z^{-j}} z^{n-m-1}dz, \end{aligned} \tag{47}$$

for all $n,m\geq 0$ and all t,t',s,s'.

The bilinear identities (47) are the bilinear identities of the 2-Toda lattice hierarchy, as described in [**67**]. These identities completely describe the 2-Toda lattice hierarchy. They admit an equivalent formulation in terms of the so-called Hirota symbol, defined by

$$p(\partial_t)f\circ g := p\Big(\frac{\partial}{\partial y}\Big)f(t+y)g(t-y)\Big|_{y=0},$$

for any polynomial p. Shifting the time variables $t\to t-a$, $t'\to t'+a$, $s\to s-b$, $s'\to s'+b$ in (47) and evaluating the residue in the left-hand side and the right-hand side, we have that the functions $\tau_n(t,s)$ satisfy the following bilinear identities in Hirota form

$$\begin{aligned} &\sum_{k=0}^{\infty} S_{k-n+m}(-2a)S_k(\tilde\partial_t)e^{\sum_{j=1}^{\infty}\left(a_j\frac{\partial}{\partial t_j}+b_j\frac{\partial}{\partial s_j}\right)}\tau_{m+1}\circ\tau_n \\ &= \sum_{k=0}^{\infty} S_{k+n-m}(2b)S_k(-\tilde\partial_s)e^{\sum_{j=1}^{\infty}\left(a_j\frac{\partial}{\partial t_j}+b_j\frac{\partial}{\partial s_j}\right)}\tau_{m}\circ\tau_{n+1}, \end{aligned} \tag{48}$$

for all $m,n\geq 0$, where S_k are the elementary Schur polynomials, and $\tilde\partial_t = \left(\frac{\partial}{\partial t_1},\frac{1}{2}\frac{\partial}{\partial t_2},\frac{1}{3}\frac{\partial}{\partial t_3},\dots\right)$. As a consequence of these bilinear identities, we can prove that the functions τ_n are solutions of the KP hierarchy, and in particular of the KP equation.

Corollary 1.4. *The functions $\tau_n(t,s)$ satisfy the KP hierarchy in the t and in the s variables, i.e. we have for all $k=0,1,2,\dots$ and all $n=1,2,\dots$*

$$\Big(S_{k+4}(\tilde\partial_t)-\frac{1}{2}\frac{\partial^2}{\partial t_1\partial t_{k+3}}\Big)\tau_n\circ\tau_n=0, \tag{49}$$

of which the first equation is the KP equation

$$\left(\Big(\frac{\partial}{\partial t_1}\Big)^4+3\Big(\frac{\partial}{\partial t_2}\Big)^2-4\frac{\partial^2}{\partial t_1\partial t_3}\right)\ln\tau_n+6\Big(\frac{\partial^2}{\partial t_1^2}\ln\tau_n\Big)^2=0, \tag{50}$$

and analoguous equations in the s *variables.*

PROOF. We shall only prove the statement for the t-variables. If we take $n=m+1$ and $b=0$ the right-hand of (48) is equal to 0 as $S_k(0)=0$ for $k>0$. Thus (48) gives

$$\begin{aligned}0&=e^{\sum_{j=1}^{\infty}a_j\frac{\partial}{\partial t_j}}\sum_{k=0}^{\infty}S_k(-2a)S_{k+1}(\tilde{\partial}_t)\tau_n\circ\tau_n\\&=\Big(1+\sum_{j=1}^{\infty}a_j\frac{\partial}{\partial t_j}+O(a^2)\Big)\\&\qquad\times\Big(\frac{\partial}{\partial t_1}+\sum_{k=1}^{\infty}\big(-2a_k+O(a^2)\big)S_{k+1}(\tilde{\partial}_t)\Big)\tau_n\circ\tau_n\\&=\left(\frac{\partial}{\partial t_1}+\sum_{k=1}^{\infty}a_k\Big(\frac{\partial^2}{\partial t_1\partial t_k}-2S_{k+1}(\tilde{\partial}_t)\Big)\right)\tau_n\circ\tau_n+O(a^2).\end{aligned}$$

We have $\frac{\partial}{\partial t_1}\tau_n\circ\tau_n=0$ for all n. This equation must be valid for all a_k. Consequently we have

$$0=\Big(\frac{\partial^2}{\partial t_1\partial t_k}-2S_{k+1}(\tilde{\partial}_t)\Big)\tau_n\circ\tau_n,\qquad\text{for } k=1,2,\ldots.$$

For $k=1$ and $k=2$ this equation is trivial. Consequently, we obtain (49) after relabeling. For $k=0$ an easy computation gives the KP equation (50). □

4.2. A centerless algebra of Virasoro constraints. In this subsection we prove that the sequence of τ-functions $(\tau_n(t,s;\eta,\theta))_{n\geq 0}$ satisfies linear PDE's with a boundary part (differentials with respect to η and θ) and a time part (differentials with respect to t and s), called Virasoro constraints. In their study of Painlevé equations satisfied (as functions of x) by integrals of Gessel's type $\mathbb{E}_{U(n)}\big[e^{x\,Tr(M+\overline{M})}\big]$, where the expectation $\mathbb{E}_{U(n)}$ refers to integration with respect to the Haar measure over the whole unitary group $U(n)$, Adler and van Moerbeke [**12**] consider the tau-functions $\big(\tau_n(t,s;-\pi,\pi)\big)_{n\geq 0}$, with τ_n defined in (40). Notice that

$$\mathbb{E}_{U(n)}\big[e^{x\,Tr(M+\overline{M})}\big]=\tau_n(t,s;-\pi,\pi)\big|_{t=s=(x,0,0,\ldots)}.$$

They prove that the tau-functions $\big(\tau_n(t,s;-\pi,\pi)\big)_{n\geq 0}$ satisfy the following constraints

$$L_k^n\tau_n(t,s;-\pi,\pi)=0,\qquad k=-1,0,1,$$

with

$$L^n_{-1} = -n\frac{\partial}{\partial s_1} - \sum_{j=1}^{\infty} js_j\frac{\partial}{\partial s_{j+1}} + \sum_{j=2}^{\infty} jt_j\frac{\partial}{\partial t_{j-1}} + nt_1,$$
$$L^n_0 = \sum_{j=1}^{\infty} jt_j\frac{\partial}{\partial t_j} - \sum_{j=1}^{\infty} js_j\frac{\partial}{\partial s_j},$$
$$L^n_1 = n\frac{\partial}{\partial t_1} + \sum_{j=1}^{\infty} jt_j\frac{\partial}{\partial t_{j+1}} - \sum_{j=2}^{\infty} js_j\frac{\partial}{\partial s_{j-1}} - ns_1.$$

For each $n \geq 0$, the operators L^n_k provide a representation of the $sl(2)$ algebra in the space of formal power series in t, s. In this subsection, we prove that the tau-functions $(\tau_n(t, s; \eta, \theta))_{n\geq 0}$, with arbitrary η, θ, satisfy a *full* Virasoro algebra of constraints, with a boundary part. The results in this section are based on [**43**]. Our proof is a non-trivial adaptation of the self-similarity argument exploited in the case of the Gaussian ensembles, based on the invariance of the integrals with respect to translations, see [**10**] and references therein. Here, we replace translations by appropriate rotations. More precisely, setting

$$\mathrm{d}I_n(t,s,z) = |\Delta_n(z)|^2 \prod_{\alpha=1}^{n} \Big(e^{\sum_{j=1}^{\infty}(t_j z_\alpha^j + s_j z_\alpha^{-j})} \ \frac{\mathrm{d}z_\alpha}{2\pi i z_\alpha}\Big), \tag{51}$$

with $z_\alpha = e^{i\varphi_\alpha}$ and $|\Delta_n(z)|^2 = \prod_{1\leq\alpha<\beta\leq n} |z_\alpha - z_\beta|^2$, we have the fundamental next proposition.

Proposition 1.5 (Haine-Vanderstichelen [**43**])**.** *The following variational formulas hold*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\, \mathrm{d}I_n\big(z_\alpha \mapsto z_\alpha e^{\varepsilon(z_\alpha^k - z_\alpha^{-k})}\big)\big|_{\varepsilon=0} = \big(L_k^{(n)} - L_{-k}^{(n)}\big)\, \mathrm{d}I_n, \tag{52}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\, \mathrm{d}I_n\big(z_\alpha \mapsto z_\alpha e^{i\varepsilon(z_\alpha^k + z_\alpha^{-k})}\big)\big|_{\varepsilon=0} = i\big(L_k^{(n)} + L_{-k}^{(n)}\big)\, \mathrm{d}I_n, \tag{53}$$

for all $k \geq 0$*, with*

$$L_k^{(n)} = \sum_{j=1}^{k-1} \frac{\partial^2}{\partial t_j \partial t_{k-j}} + n \frac{\partial}{\partial t_k} + \sum_{j=1}^{\infty} j t_j \frac{\partial}{\partial t_{j+k}}$$

$$- \sum_{j=k+1}^{\infty} j s_j \frac{\partial}{\partial s_{j-k}} - \sum_{j=1}^{k-1} j s_j \frac{\partial}{\partial t_{k-j}} - nks_k, \quad k \geq 1, \tag{54}$$

$$L_0^{(n)} = \sum_{j=1}^{\infty} j t_j \frac{\partial}{\partial t_j} - \sum_{j=1}^{\infty} j s_j \frac{\partial}{\partial s_j}, \tag{55}$$

$$L_{-k}^{(n)} = - \sum_{j=1}^{k-1} \frac{\partial^2}{\partial s_j \partial s_{k-j}} - n \frac{\partial}{\partial s_k} - \sum_{j=1}^{\infty} j s_j \frac{\partial}{\partial s_{j+k}}$$

$$+ \sum_{j=k+1}^{\infty} j t_j \frac{\partial}{\partial t_{j-k}} + \sum_{j=1}^{k-1} j t_j \frac{\partial}{\partial s_{k-j}} + nkt_k, \quad k \geq 1. \tag{56}$$

The proof of this proposition is based on the following elementary lemma, which we prove first.

Lemma 1.6. *Upon setting*

$$E = \prod_{\alpha=1}^{n} e^{\sum_{j=1}^{\infty} (t_j z_\alpha^j + s_j z_\alpha^{-j})},$$

the following four relations hold, for $k \geq 0$*,*

$$\Big(\frac{\partial}{\partial t_k} + n\delta_{k,0}\Big) E = \Big(\sum_{\alpha=1}^{n} z_\alpha^k\Big) E$$

$$\Big(\frac{\partial}{\partial s_k} + n\delta_{k,0}\Big) E = \Big(\sum_{\alpha=1}^{n} z_\alpha^{-k}\Big) E, \tag{57}$$

$$\left(\frac{1}{2} \sum_{\substack{i+j=k \\ i,j>0}} \frac{\partial^2}{\partial t_i \partial t_j} - \frac{n}{2}\delta_{k,0}\right) E = \left(\sum_{\substack{1 \leq \alpha < \beta \leq n \\ i+j=k \\ i,j>0}} z_\alpha^i z_\beta^j + \frac{k-1}{2} \sum_{\alpha=1}^{n} z_\alpha^k\right) E$$

$$\left(\frac{1}{2} \sum_{\substack{i+j=k \\ i,j>0}} \frac{\partial^2}{\partial s_i \partial s_j} - \frac{n}{2}\delta_{k,0}\right) E = \left(\sum_{\substack{1 \leq \alpha < \beta \leq n \\ i+j=k \\ i,j>0}} z_\alpha^{-i} z_\beta^{-j} + \frac{k-1}{2} \sum_{\alpha=1}^{n} z_\alpha^{-k}\right) E. \tag{58}$$

PROOF. The two relations (57) are trivial. We shall only give the proof of the first relation in (58). For $k > 0$, applying twice successively the first formula (57), we have

$$\left(\frac{1}{2}\sum_{\substack{i+j=k\\ i,j>0}}\frac{\partial^2}{\partial t_i\partial t_j}-\frac{n}{2}\,\delta_{k,0}\right)E=\frac{1}{2}\sum_{\substack{i+j=k\\ i,j>0}}\Big(\sum_{1\le\alpha\le n}z_\alpha^j\Big)\Big(\sum_{1\le\alpha\le n}z_\alpha^i\Big)\,E$$
$$=\frac{1}{2}\sum_{\substack{i+j=k\\ i,j>0}}\Big(\sum_{1\le\alpha,\beta\le n}z_\alpha^j\,z_\beta^i\Big)\,E$$
$$=\frac{1}{2}\sum_{\substack{i+j=k\\ i,j>0}}\Big(2\sum_{1\le\alpha<\beta\le n}z_\alpha^j\,z_\beta^i+\sum_{1\le\alpha\le n}z_\alpha^{i+j}\Big)\,E$$
$$=\left(\sum_{\substack{1\le\alpha<\beta\le n\\ i,j>0\\ i+j=k}}z_\alpha^i\,z_\beta^j+\frac{k-1}{2}\sum_{1\le\alpha\le n}z_\alpha^k\right)E.$$

This equation is also trivially satisfied for $k=0$. This proves the first equation in (58). The proof of the second equation in (58) is similar. □

We now turn to the proof of proposition 1.5.

PROOF. We shall first give the proof of (52). We split the computation into four contributions, corresponding to various factors in (51).

Contribution 1: For $k>0$, we have

$$\frac{\partial}{\partial\varepsilon}\left|\Delta_n\big(ze^{\varepsilon(z^k-z^{-k})}\big)\right|^2\Big|_{\varepsilon=0}$$
$$=|\Delta_n(z)|^2\sum_{1\le\alpha<\beta\le n}\left[\frac{\frac{\partial}{\partial\varepsilon}\Big(z_\alpha e^{\varepsilon\left(z_\alpha^k-z_\alpha^{-k}\right)}-z_\beta e^{\varepsilon\left(z_\beta^k-z_\beta^{-k}\right)}\Big)}{z_\alpha-z_\beta}\right.$$
$$\left.+\frac{\frac{\partial}{\partial\varepsilon}\Big(z_\alpha^{-1}e^{-\varepsilon\left(z_\alpha^k-z_\alpha^{-k}\right)}-z_\beta^{-1}e^{-\varepsilon\left(z_\beta^k-z_\beta^{-k}\right)}\Big)}{z_\alpha^{-1}-z_\beta^{-1}}\right]\Bigg|_{\varepsilon=0}$$
$$=|\Delta_n(z)|^2\sum_{1\le\alpha<\beta\le n}\frac{(z_\alpha+z_\beta)(z_\alpha^k-z_\beta^k-(z_\alpha^{-k}-z_\beta^{-k}))}{z_\alpha-z_\beta}.$$

As

$$z_\alpha^k-z_\beta^k=(z_\alpha-z_\beta)\sum_{i=0}^{k-1}z_\alpha^i\,z_\beta^{k-1-i},\tag{59}$$

and

$$z_\alpha^{-k}-z_\beta^{-k}=(z_\alpha^{-1}-z_\beta^{-1})\sum_{i=0}^{k-1}z_\alpha^{-i}\,z_\beta^{i-(k-1)}=-(z_\alpha-z_\beta)\sum_{i=0}^{k-1}z_\alpha^{-i-1}\,z_\beta^{i-k},\tag{60}$$

we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\varepsilon}\left|\Delta_n\left(ze^{\varepsilon(z^k-z^{-k})}\right)\right|^2\Big|_{\varepsilon=0}\\
&=|\Delta_n(z)|^2\sum_{1\le\alpha<\beta\le n}(z_\alpha+z_\beta)\Big(\sum_{i=0}^{k-1}z_\alpha^i z_\beta^{k-1-i}+\sum_{i=0}^{k-1}z_\alpha^{-i-1}z_\beta^{i-k}\Big).
\end{aligned}$$

Developing the product we get

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon}\left|\Delta_n\left(z\,e^{\epsilon\left(z^k-z^{-k}\right)}\right)\right|^2\Bigg|_{\epsilon=0}\\
&=\left|\Delta_n(z)\right|^2\sum_{1\le\alpha<\beta\le n}\left[2\sum_{\substack{i+j=k\\ i,j>0}}\left(z_\alpha^i z_\beta^j+z_\alpha^{-i}z_\beta^{-j}\right)+z_\alpha^k+z_\beta^k+z_\alpha^{-k}+z_\beta^{-k}\right]\\
&=\left|\Delta_n(z)\right|^2\left[2\sum_{\substack{1\le\alpha<\beta\le n\\ i+j=k\\ i,j>0}}\left(z_\alpha^i z_\beta^j+z_\alpha^{-i}z_\beta^{-j}\right)+\sum_{1\le\alpha\le n}(n-\alpha)\left(z_\alpha^k+z_\alpha^{-k}\right)\right.\\
&\qquad\left.+\sum_{1\le\beta\le n}(\beta-1)\left(z_\beta^k+z_\beta^{-k}\right)\right],
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon}\left|\Delta_n\left(z\,e^{\epsilon(z^k-z^{-k})}\right)\right|^2\Bigg|_{\epsilon=0}=\left|\Delta_n(z)\right|^2E^{-1}\times\\
&\quad\times\left[2\sum_{\substack{1\le\alpha<\beta\le n\\ i+j=k\\ i,j>0}}\left(z_\alpha^i z_\beta^j+z_\alpha^{-i}z_\beta^{-j}\right)+(n-1)\sum_{1\le\alpha\le n}\left(z_\alpha^k+z_\alpha^{-k}\right)\right]E.
\end{aligned}$$

Using the four relations (57) and (58), we obtain

$$\begin{aligned}
\text{(61)}\qquad &\frac{\partial}{\partial\varepsilon}\left|\Delta_n\left(ze^{\varepsilon(z^k-z^{-k})}\right)\right|^2\Big|_{\varepsilon=0}=2|\Delta_n(z)|^2E^{-1}\left[\frac12\sum_{\substack{i+j=k\\ i,j>0}}\frac{\partial^2}{\partial t_i\partial t_j}\right.\\
&\qquad\left.+\frac12\sum_{\substack{i+j=k\\ i,j>0}}\frac{\partial^2}{\partial s_i\partial s_j}+\frac{n-k}{2}\frac{\partial}{\partial t_k}+\frac{n-k}{2}\frac{\partial}{\partial s_k}\right]E,
\end{aligned}$$

which is also trivially satisfied for $k=0$.

Contribution 2: For $k \geq 0$, using the relations (57) in the last step, we have

$$
\begin{aligned}
&\frac{\partial}{\partial \epsilon} \prod_{\alpha=1}^{n} \mathrm{d}\Big(z_\alpha\, e^{\epsilon(z_\alpha^k - z_\alpha^{-k})}\Big)\Big|_{\epsilon=0} \\
&\qquad = \frac{\partial}{\partial \epsilon} \prod_{\alpha=1}^{n} \mathrm{d}\Big(z_\alpha\,\big(1+\epsilon(z_\alpha^k - z_\alpha^{-k}) + \mathscr{O}(\epsilon^2)\big)\Big)\Big|_{\epsilon=0} \\
&\qquad = \frac{\partial}{\partial \epsilon} \prod_{\alpha=1}^{n} \Big(\mathrm{d}z_\alpha + \epsilon(k+1) z_\alpha^k \mathrm{d}z_\alpha + \epsilon(k-1) z_\alpha^{-k}\mathrm{d}z_\alpha + \mathscr{O}(\epsilon^2)\big)\Big)\Big|_{\epsilon=0} \\
&\qquad = E^{-1} \sum_{\beta=1}^{n} \big((k+1) z_\beta^k + (k-1) z_\beta^{-k}\big)\, E \prod_{\alpha=1}^{n} \mathrm{d}z_\alpha \\
&\qquad = E^{-1}\Big[(k+1)\frac{\partial}{\partial t_k} + (k-1)\frac{\partial}{\partial s_k}\Big] E \prod_{\alpha=1}^{n} \mathrm{d}z_\alpha. && (62)
\end{aligned}
$$

Contribution 3: For $k \geq 0$, using the relations (57), we have

$$
\begin{aligned}
&\frac{\partial}{\partial \varepsilon} \prod_{\alpha=1}^{n} e^{\sum_{j=1}^{\infty}\Big(t_j\big(z_\alpha e^{\varepsilon(z_\alpha^k - z_\alpha^{-k})}\big)^j + s_j\big(z_\alpha e^{\varepsilon(z_\alpha^k - z_\alpha^{-k})}\big)^{-j}\Big)}\Big|_{\varepsilon=0} \\
&= \sum_{\alpha=1}^{n}\Big[\sum_{j=1}^{\infty} j t_j z_\alpha^j (z_\alpha^k - z_\alpha^{-k}) - \sum_{j=1}^{\infty} j s_j z_\alpha^{-j}(z_\alpha^k - z_\alpha^{-k})\Big] E \\
&= \Bigg[\sum_{j=1}^{\infty} j t_j \sum_{\alpha=1}^{n} z_\alpha^{j+k} - \sum_{j=1}^{k-1} j t_j \sum_{\alpha=1}^{n} z_\alpha^{j-k} - \sum_{j=k}^{\infty} j t_j \sum_{\alpha=1}^{n} z_\alpha^{j-k} \\
&\qquad - \sum_{j=1}^{k-1} j s_j \sum_{\alpha=1}^{n} z_\alpha^{k-j} - \sum_{j=k}^{\infty} j s_j \sum_{\alpha=1}^{n} z_\alpha^{k-j} + \sum_{j=1}^{\infty} j s_j \sum_{\alpha=1}^{n} z_\alpha^{-k-j}\Bigg] E \\
&= \Bigg[\sum_{j=1}^{\infty} j t_j \frac{\partial}{\partial t_{k+j}} - \sum_{j=1}^{k-1} j t_j \frac{\partial}{\partial s_{k-j}} - \sum_{j=k+1}^{\infty} j t_j \frac{\partial}{\partial t_{j-k}} - n k t_k \\
&\qquad - \sum_{j=1}^{k-1} j s_j \frac{\partial}{\partial t_{k-j}} - \sum_{j=k+1}^{\infty} j s_j \frac{\partial}{\partial s_{j-k}} - n k s_k + \sum_{j=1}^{\infty} j s_j \frac{\partial}{\partial s_{k+j}}\Bigg] E. && (63)
\end{aligned}
$$

Contribution 4: For $k \geq 0$, using the relations (57), we have

$$
\begin{aligned}
\frac{\partial}{\partial \varepsilon} \prod_{\alpha=1}^{n} \frac{1}{2\pi i z_\alpha e^{\varepsilon(z_\alpha^k - z_\alpha^{-k})}}\Big|_{\varepsilon=0} &= E^{-1}\Big[-\sum_{\alpha=1}^{n} z_\alpha^k + \sum_{\alpha=1}^{n} z_\alpha^{-k}\Big] E \prod_{\alpha=1}^{n} \frac{1}{2\pi i z_\alpha} \\
&= E^{-1}\Big[-\frac{\partial}{\partial t_k} + \frac{\partial}{\partial s_k}\Big] E \prod_{\alpha=1}^{n} \frac{1}{2\pi i z_\alpha}. && (64)
\end{aligned}
$$

Adding up (61), (62), (63) and (64) gives (52).

We now sketch briefly without comments the proof of (53). We split the computation in four contributions, corresponding to various factors in (51).

Contribution 1: For $k > 0$, we have

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon}\Big|\Delta_n\Big(z\, e^{i\epsilon(z^k+z^{-k})}\Big)\Big|^2\Big|_{\epsilon=0} \\
&= i\,\big|\Delta_n(z)\big|^2 \sum_{1\le\alpha<\beta\le n}\left[\frac{(z_\alpha+z_\beta)\big(z_\alpha^k-z_\beta^k+z_\alpha^{-k}-z_\beta^{-k}\big)}{z_\alpha-z_\beta}\right] \\
&= i\,\big|\Delta_n(z)\big|^2 E^{-1}\Bigg[2\sum_{\substack{1\le\alpha<\beta\le n\\ i+j=k\\ i,j>0}}\big(z_\alpha^i z_\beta^j - z_\alpha^{-i}z_\beta^{-j}\big) \\
&\qquad\qquad + (n-1)\sum_{1\le\alpha\le n}\big(z_\alpha^k - z_\alpha^{-k}\big)\Bigg]E \\
&= 2i\big|\Delta_n(z)\big|^2 E^{-1}\Bigg[\frac{1}{2}\sum_{\substack{i+j=k\\ i,j>0}}\frac{\partial^2}{\partial t_i\partial t_j} - \frac{1}{2}\sum_{\substack{i+j=k\\ i,j>0}}\frac{\partial^2}{\partial s_i\partial s_j} \\
&\qquad\qquad + \frac{n-k}{2}\,\frac{\partial}{\partial t_k} - \frac{n-k}{2}\,\frac{\partial}{\partial s_k}\Bigg]E,
\end{aligned} \tag{65}$$

which is also trivially satisfied for $k = 0$.

Contribution 2: For $k \ge 0$, using the relations (57) in the last step, we have

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon}\prod_{\alpha=1}^n \mathrm{d}\Big(z_\alpha\, e^{i\epsilon(z_\alpha^k+z_\alpha^{-k})}\Big)\Big|_{\epsilon=0} \\
&\qquad = i\,E^{-1}\sum_{\beta=1}^n\big((k+1)z_\beta^k-(k-1)z_\beta^{-k}\big)\,E\prod_{\alpha=1}^n \mathrm{d}z_\alpha \\
&\qquad = i\,E^{-1}\Big[(k+1)\frac{\partial}{\partial t_k}-(k-1)\frac{\partial}{\partial s_k}+2n\delta_{k,0}\Big]\,E\prod_{\alpha=1}^n \mathrm{d}z_\alpha.
\end{aligned} \tag{66}$$

Contribution 3: For $k \geq 0$, using the relations (57), we have

$$\frac{\partial}{\partial\epsilon}\prod_{\alpha=1}^{n} e^{\sum_{j=1}^{\infty}\left(t_j\left(z_\alpha\, e^{i\epsilon(z_\alpha^k+z_\alpha^{-k})}\right)^j + s_j\left(z_\alpha\, e^{i\epsilon(z_\alpha^k+z_\alpha^{-k})}\right)^{-j}\right)}\Bigg|_{\epsilon=0}$$

$$= i\left[\sum_{j=1}^{\infty} jt_j \sum_{\alpha=1}^{n} z_\alpha^{j+k} + \sum_{j=1}^{k-1} jt_j \sum_{\alpha=1}^{n} z_\alpha^{j-k} + \sum_{j=k}^{\infty} jt_j \sum_{\alpha=1}^{n} z_\alpha^{j-k}\right.$$
$$\left. - \sum_{j=1}^{k-1} js_j \sum_{\alpha=1}^{n} z_\alpha^{k-j} - \sum_{j=k}^{\infty} js_j \sum_{\alpha=1}^{n} z_\alpha^{k-j} - \sum_{j=1}^{\infty} js_j \sum_{\alpha=1}^{n} z_\alpha^{-k-j}\right] E$$

$$= i\left[\sum_{j=1}^{\infty} jt_j \frac{\partial}{\partial t_{j+k}} + \sum_{j=1}^{k-1} jt_j \frac{\partial}{\partial s_{k-j}} + \sum_{j=k+1}^{\infty} jt_j \frac{\partial}{\partial t_{j-k}} - \sum_{j=1}^{k-1} js_j \frac{\partial}{\partial t_{k-j}}\right.$$
$$\left. - \sum_{j=k+1}^{\infty} js_j \frac{\partial}{\partial s_{j-k}} - \sum_{j=1}^{\infty} js_j \frac{\partial}{\partial s_{j+k}} + nkt_k - nks_k\right] E. \tag{67}$$

Contribution 4: For $k \geq 0$, using the relations (57), we have

$$\frac{\partial}{\partial\epsilon}\prod_{\alpha=1}^{n}\frac{1}{2\pi i z_\alpha\, e^{i\epsilon(z_\alpha^k+z_\alpha^{-k})}}\Bigg|_{\epsilon=0}$$
$$= i\,E^{-1}\left[-\sum_{\alpha=1}^{n} z_\alpha^k - \sum_{\alpha=1}^{n} z_\alpha^{-k}\right] E \prod_{\alpha=1}^{n}\frac{1}{2\pi i z_\alpha}$$
$$= i\,E^{-1}\left[-\frac{\partial}{\partial t_k} - \frac{\partial}{\partial s_k} - 2n\delta_{k,0}\right] E \prod_{\alpha=1}^{n}\frac{1}{2\pi i z_\alpha}. \tag{68}$$

Adding up (65), (66), (67) and (68) gives (53). This concludes the proof of Proposition 1.5. □

Remark 1.7. *After [43] was completed, we noticed that Bowick et al. [17] have obtained the same result with a method closely related to ours, though full details are not given in their work.*

We are now able to state the main result of this section.

Theorem 1.8 (Haine-Vanderstichelen [**43**])**.** (i) *The tau functions* $\tau_n(t,s;\eta,\theta), n \geq 1$, *defined in* (40)*, satisfy*

$$\mathcal{B}_k(\eta,\theta)\tau_n(t,s;\eta,\theta) = L_k^{(n)}\tau_n(t,s;\eta,\theta), \quad k \in \mathbb{Z}, \tag{69}$$

with $L_k^{(n)}, k \in \mathbb{Z}$*, defined as in* (54), (55), (56)*, and*

$$\mathcal{B}_k(\eta,\theta) = \frac{1}{i}\left(e^{ik\theta}\frac{\partial}{\partial\theta} + e^{ik\eta}\frac{\partial}{\partial\eta}\right); \quad i = \sqrt{-1}. \tag{70}$$

(ii) *The operators* $L_k^{(n)}, k \in \mathbb{Z}$, *satisfy the commutation relations of the centerless Virasoro algebra, that is*

$$\big[L_k^{(n)}, L_l^{(n)}\big] = (k-l)L_{k+l}^{(n)}, \quad k, l \in \mathbb{Z}. \tag{71}$$

PROOF. (i) Denoting $z_\alpha = e^{i\varphi_\alpha}$, the change of variable $z_\alpha \mapsto z_\alpha e^{\varepsilon(z_\alpha^k - z_\alpha^{-k})}$ in the integral (40) gives the following transformation on the angle $\varphi_\alpha \mapsto \varphi_\alpha + 2\varepsilon \sin(k\varphi_\alpha)$, inducing a change in the limits of integration given by the inverse map

$$\varphi_\alpha \mapsto \varphi_\alpha - 2\varepsilon \sin(k\varphi_\alpha) + O(\varepsilon^2), \tag{72}$$

for ε small enough. Making the change of variable in the integral (40), with the corresponding change in the limits of integration, leaves it invariant. Thus, by differentiating the result with respect to ε and evaluating it at $\varepsilon = 0$, using the chain rule together with (52) and (72), we obtain

$$0 = \Big(-2\sin(k\theta)\frac{\partial}{\partial\theta} - 2\sin(k\eta)\frac{\partial}{\partial\eta} + L_k^{(n)} - L_{-k}^{(n)}\Big)\tau_n(t, s; \eta, \theta). \tag{73}$$

Similarly, the change of variable $z_\alpha \mapsto z_\alpha e^{i\varepsilon(z_\alpha^k + z_\alpha^{-k})}$ corresponds to the transformation $\varphi_\alpha \mapsto \varphi_\alpha + 2\varepsilon \cos(k\varphi_\alpha)$, with inverse

$$\varphi_\alpha \mapsto \varphi_\alpha - 2\varepsilon \cos(k\varphi_\alpha) + O(\varepsilon^2),$$

which, using (53), leads to

$$0 = \Big(-\frac{2}{i}\cos(k\theta)\frac{\partial}{\partial\theta} - \frac{2}{i}\cos(k\eta)\frac{\partial}{\partial\eta} + L_k^{(n)} + L_{-k}^{(n)}\Big)\tau_n(t, s; \eta, \theta). \tag{74}$$

Adding and subtracting (73) and (74) gives the constraints (69), with $\mathcal{B}_k(\eta, \theta)$ defined as in (70).

(ii) Consider the complex Lie algebra $\mathcal{A}$ given by the direct sum of two commuting copies of the Heisenberg algebra[2] with bases $\{\hbar_a, a_j | j \in \mathbb{Z}\}$ and $\{\hbar_b, b_j | j \in \mathbb{Z}\}$ and defining commutation relations

$$\begin{aligned}
&[\hbar_a, a_j] = 0 \quad , \quad [a_j, a_k] = j\delta_{j,-k}\hbar_a, \\
&[\hbar_b, b_j] = 0 \quad , \quad [b_j, b_k] = j\delta_{j,-k}\hbar_b, \\
&[\hbar_a, \hbar_b] = 0 \quad , \quad [a_j, b_k] = 0 \quad , \quad [\hbar_a, b_j] = 0 \quad , \quad [\hbar_b, a_j] = 0,
\end{aligned} \tag{75}$$

with $j, k \in \mathbb{Z}$. Let $\mathcal{B}$ be the space of formal power series in the variables $t_1, t_2, \ldots$ and $s_1, s_2, \ldots$, and consider the following representation of $\mathcal{A}$ in $\mathcal{B}$:

$$\begin{aligned}
&a_j = \frac{\partial}{\partial t_j} \quad , \quad a_{-j} = jt_j \quad , \quad b_j = \frac{\partial}{\partial s_j} \quad , \quad b_{-j} = js_j, \\
&a_0 = b_0 = \mu \quad , \quad \hbar_a = \hbar_b = 1,
\end{aligned} \tag{76}$$

[2]See Appendix A for a short introduction to the Heisenberg and the Virasoro algebra, and their oscillator representation.

for $j > 0$, and $\mu \in \mathbb{C}$. Define the operators

$$A_k^{(n)} = \frac{1}{2}\sum_{j\in\mathbb{Z}} : a_{-j}a_{j+k} : \quad , \quad B_k^{(n)} = \frac{1}{2}\sum_{j\in\mathbb{Z}} : b_{-j}b_{j+k} :, \tag{77}$$

where $k \in \mathbb{Z}$, a_j, b_j as in (76) with $\mu = n$, and where the colons indicate normal ordering, defined by

$$: a_j a_k := \begin{cases} a_j a_k & \text{if } j \leq k, \\ a_k a_j & \text{if } j > k, \end{cases}$$

and a similar definition for $: b_j b_k :$, obtained by changing the a's in b's in the former. Expanding the expressions in (77) we obtain for $k > 0$

$$\begin{aligned} A_0^{(n)} &= \sum_{j>0} j t_j \frac{\partial}{\partial t_j} + \frac{n^2}{2}, \\ A_k^{(n)} &= \frac{1}{2}\sum_{0<j<k} \frac{\partial^2}{\partial t_j \partial t_{k-j}} + \sum_{j>k}(j-k)t_{j-k}\frac{\partial}{\partial t_j} + n\frac{\partial}{\partial t_k}, \\ A_{-k}^{(n)} &= \frac{1}{2}\sum_{0<j<k} j(k-j)t_j t_{k-j} + \sum_{j>k} j t_j \frac{\partial}{\partial t_{j-k}} + nkt_k, \end{aligned} \tag{78}$$

and similar expressions for $B_k^{(n)}$, by changing the t-variables in s-variables. Using these notations, we can rewrite (54), (55) and (56) as follows

$$\begin{aligned} L_k^{(n)} &= A_k^{(n)} - B_{-k}^{(n)} + \frac{1}{2}\sum_{j=1}^{k-1}(a_j - b_{-j})(a_{k-j} - b_{j-k}), \quad k \geq 1 \\ L_0^{(n)} &= A_0^{(n)} - B_0^{(n)}, \\ L_{-k}^{(n)} &= A_{-k}^{(n)} - B_k^{(n)} - \frac{1}{2}\sum_{j=1}^{k-1}(a_{-j} - b_j)(a_{j-k} - b_{k-j}), \quad k \geq 1. \end{aligned} \tag{79}$$

As shown in [**48**] (see Lecture 2) the operators $A_k^{(n)}$, $k \in \mathbb{Z}$, provide a representation of the Virasoro algebra in $\mathcal{B}$ with central charge $c = 1$, that is

$$[A_k^{(n)}, A_l^{(n)}] = (k-l)A_{k+l}^{(n)} + \delta_{k,-l}\frac{k^3-k}{12}, \tag{80}$$

for $k, l \in \mathbb{Z}$. Similarly, the operators $B_k^{(n)}$ satisfy the commutation relations

$$[B_k^{(n)}, B_l^{(n)}] = (k-l)B_{k+l}^{(n)} + \delta_{k,-l}\frac{k^3-k}{12}, \tag{81}$$

for $k, l \in \mathbb{Z}$. Furthermore we have for $k, l \in \mathbb{Z}$

$$\begin{aligned} &[a_k, A_l^{(n)}] = k a_{k+l} \quad , \quad [b_k, B_l^{(n)}] = k b_{k+l}, \\ &[a_k, B_l^{(n)}] = 0 \quad , \quad [b_k, A_l^{(n)}] = 0. \end{aligned} \tag{82}$$

Let us now establish the commutation relations (71). We give the proof for $k, l \geq 0$, the other cases being similar. As $[A_i^{(n)}, B_j^{(n)}] = 0$, $i, j \in \mathbb{Z}$, we have using (75), (80), (81) and (82)

$$\begin{aligned}
[L_k^{(n)}, L_l^{(n)}] =&(k-l)\big(A_{k+l}^{(n)} - B_{-k-l}^{(n)}\big) \\
&- \frac{1}{2}\sum_{j=1}^{l-1} j(a_{j+k} - b_{-j-k})(a_{l-j} - b_{j-l}) \\
&- \frac{1}{2}\sum_{j=1}^{l-1}(l-j)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}) \\
&+ \frac{1}{2}\sum_{j=1}^{k-1} j(a_{j+l} - b_{-j-l})(a_{k-j} - b_{j-k}) \\
&+ \frac{1}{2}\sum_{j=1}^{k-1}(k-j)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}).
\end{aligned}$$

Relabeling the indices in the sums, we have

$$\begin{aligned}
[L_k^{(n)}, L_l^{(n)}] =&(k-l)\big(A_{k+l}^{(n)} - B_{-k-l}^{(n)}\big) \\
&- \frac{1}{2}\sum_{j=k+1}^{k+l-1} (j-k)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}) \\
&- \frac{1}{2}\sum_{j=1}^{l-1}(l-j)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}) \\
&+ \frac{1}{2}\sum_{j=l+1}^{k+l-1} (j-l)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}) \\
&+ \frac{1}{2}\sum_{j=1}^{k-1}(k-j)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}) \\
=&(k-l)L_{k+l}^{(n)}.
\end{aligned}$$

This concludes the proof. $\square$

5. The circular unitary ensemble and the Painlevé VI equation

In this section, using the method of [**6**], we establish the following result.

Theorem 1.9 (Tracy-Widom [**66**], Haine-Vanderstichelen [**43**]). *The function* $R(\theta)$ *defined in* (37) *satisfies* (38).

PROOF. Remembering the definition of $L_0^{(n)}$ in (55), the Virasoro constraint in (69) for $k=0$, evaluated along the locus $t=s=0$, gives

$$\left.\frac{\partial \log \tau_n(t,s;\eta,\theta)}{\partial \theta}\right|_{t=s=0} = -\left.\frac{\partial \log \tau_n(t,s;\eta,\theta)}{\partial \eta}\right|_{t=s=0}, \tag{83}$$

which is a reformulation of the fact that the gap probability $\tau_n(0,0;\eta,\theta)$ only depends on the length $\theta-\eta$.

Define the operator $\mathcal{D}=\frac{\partial}{\partial\theta}-\frac{\partial}{\partial\eta}$ and put for a fixed n

$$\begin{aligned} f(t,s;\eta,\theta) &= \log\tau_n(t,s;\eta,\theta), \\ g(\eta,\theta) &= -\frac{1}{2}\mathcal{D}\log\tau_n(t,s;\eta,\theta)\big|_{t=s=0}. \end{aligned} \tag{84}$$

Notice that for $k\geq 0$

$$\mathcal{D}^k\log\tau_n(t,s;\eta,\theta)\big|_{\substack{t=s=0\\ \eta=-\theta}} = \frac{\mathrm{d}^k}{\mathrm{d}\theta^k}\log\tau_n(t,s;-\theta,\theta)\big|_{t=s=0}.$$

Clearly, from the definition of $R(\theta)$ in (37), we have

$$R(\theta)=g(-\theta,\theta)=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\log\tau_n(t,s;-\theta,\theta)\big|_{t=s=0}.$$

Remembering the definition of $L_k^{(n)}$ in (54), the constraints in (69) for $k=1,2$, evaluated at $s=(s_1,s_2,s_3,\dots)=(0,0,0,\dots)$, can be written

$$\mathcal{B}_1(\eta,\theta)f\Big|_{s=0} = \sum_{j\geq1} jt_j\left.\frac{\partial f}{\partial t_{j+1}}\right|_{s=0} + n\left.\frac{\partial f}{\partial t_1}\right|_{s=0}, \tag{85}$$

$$\mathcal{B}_2(\eta,\theta)f\Big|_{s=0} = \sum_{j\geq1} jt_j\left.\frac{\partial f}{\partial t_{j+2}}\right|_{s=0} + \left.\frac{\partial^2 f}{\partial t_1^2}\right|_{s=0} + \left.\Big(\frac{\partial f}{\partial t_1}\Big)^2\right|_{s=0} + n\left.\frac{\partial f}{\partial t_2}\right|_{s=0}. \tag{86}$$

Using (83) and the definition of $g(\eta,\theta)$ (84), the constraint (85) evaluated along the locus $t=s=0$ gives

$$\left.\frac{\partial f}{\partial t_1}\right|_{t=s=0} = \frac{1}{in}\,(e^{i\eta}-e^{i\theta})g(\eta,\theta). \tag{87}$$

Consequently, along the locus $\eta=-\theta$, we have

$$\left.\frac{\partial f}{\partial t_1}\right|_{\substack{t=s=0\\ \eta=-\theta}} = -\frac{2}{n}\,\sin(\theta)R(\theta).$$

We then proceed by induction. We call

$$\frac{\partial^n f}{\partial t_{j_1}\partial t_{j_2}\dots\partial t_{j_n}},$$

a t derivative of weighted degree $|j| = j_1 + j_2 + \cdots + j_n$. Then, for $k \geq 1$, we compute the system formed by

$$\begin{cases} \text{all } t\text{-derivatives of weighted degree } k \text{ of (85)}, \\ \text{all } t\text{-derivatives of weighted degree } k-1 \text{ of (86)}, \end{cases} \tag{88}$$

evaluated at $t = s = 0$. For instance, for $k = 1$, (88) reduces to

$$\mathcal{B}_1(\eta,\theta)\Big(\frac{\partial f}{\partial t_1}\Big|_{t=s=0}\Big) = \frac{\partial f}{\partial t_2}\bigg|_{t=s=0} + n\,\frac{\partial^2 f}{\partial t_1^2}\bigg|_{t=s=0},$$
$$\mathcal{B}_2(\eta,\theta)f\Big|_{t=s=0} = \frac{\partial^2 f}{\partial t_1^2}\bigg|_{t=s=0} + n\,\frac{\partial f}{\partial t_2}\bigg|_{t=s=0} + \left(\frac{\partial f}{\partial t_1}\bigg|_{t=s=0}\right)^2.$$

After substitution of (87), this system of equations can be solved for $\frac{\partial^2 f}{\partial t_1^2}\Big|_{t=s=0}$ and $\frac{\partial f}{\partial t_2}\Big|_{t=s=0}$ in terms of η, θ, $g(\eta,\theta)$ and $\mathcal{D}g(\eta,\theta)$, whenever $n \neq 1$. Consequently, on the locus $\eta = -\theta$, the partials $\frac{\partial^2 f}{\partial t_1^2}\Big|_{\substack{t=s=0\\ \eta=-\theta}}$ and $\frac{\partial f}{\partial t_2}\Big|_{\substack{t=s=0\\ \eta=-\theta}}$ can be expressed in terms of θ, $R(\theta)$ and $R'(\theta)$.

For general $k \geq 1$, suppose all the t-derivatives of f of weighted degree k, evaluated at $t = s = 0$, have been expressed in terms of η, θ and $g(\eta,\theta), \ldots, \mathcal{D}^{k-1}g(\eta,\theta)$, whenever $n \neq 1, \ldots, k-1$. Then (88) is a system of linear equations where the unknowns are all the t-derivatives of f of weighted degree $k+1$, evaluated at $t = s = 0$, and the coefficients can be expressed in terms of η, θ and $g(\eta,\theta), \ldots, \mathcal{D}^{k-1}g(\eta,\theta)$. This is a system of $p(k)+p(k-1)$ linear equations in $p(k+1)$ unknowns, where $p(k)$ is the number of partitions of the natural number k. As $p(k+1) \leq p(k)+p(k-1)$, this system can be solved and all the t-derivatives of f of weighted degree $k+1$, evaluated at $t = s = 0$ can be expressed in terms of η, θ, and $g(\eta,\theta), \ldots, \mathcal{D}^k g(\eta,\theta)$, whenever $n \neq k$. Consequently, on the locus $\eta = -\theta$, the t-derivatives of f of weighted degree $k+1$, evaluated at $t = s = 0$ and on the locus $\eta = -\theta$, can be expressed in terms of θ, $R(\theta)$, $R'(\theta), \ldots, R^{(k)}(\theta)$.

Since the KP equation (41) contains t-derivatives of f of weighted degree less or equal to 4, by performing the above scheme up to $k = 3$, we can express all these derivatives, evaluated at $t = s = 0$ and $\eta = -\theta$, in terms of θ, $R(\theta)$ and its first three derivatives, whenever $n \geq 4$. This gives us a third order differential equation for $R(\theta)$:

$$0 = 4R(\theta)^2 - 2\big(n^2 + (1-n^2)\cos 2\theta\big)R'(\theta) + 8\sin 2\theta\, R(\theta)R'(\theta)$$
$$- 2\sin 2\theta\, R''(\theta) + \sin^2\theta\big(12R'(\theta)^2 - R'''(\theta)\big).$$

We refer to Appendix B for a detailed discussion of the above method. Multiplying the left-hand and the right-hand side of this equation with $\frac{1}{4}\sin\theta\,\Big(2\cos\theta\, R'(\theta) +$

$\sin\theta\, R''(\theta)\Big)$, we obtain

$$0=\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\sin^2\theta\, R'(\theta)W(\theta)\Big), \tag{89}$$

with

$$W(\theta)=R(\theta)^2+2\sin\theta\,\cos\theta\, R(\theta)R'(\theta)+\sin^2\theta\, R'(\theta)^2$$
$$-\frac{1}{2}\Big(\frac{1}{4}\sin^2\theta\,\frac{R''(\theta)^2}{R'(\theta)}+\sin\theta\,\cos\theta\, R''(\theta)+\big(\cos^2\theta+n^2\sin^2\theta\big)\,R'(\theta)\Big).$$

Equation (89) implies that $W(\theta)=0$, which is the equation (38), obtained by Tracy and Widom in [**66**]. This concludes the proof of Theorem 3.1. □

Remark 1.10. In the above proof, we had to assume that $n\geq 4$, where n is the size of the random unitary matrices. For $n=1,2,3$, the function $R(\theta)$ also satisfies (38), as can be shown by direct computation, using the representation (39) of the probability $\tau_n(\eta,\theta)$ as a Toeplitz determinant. It would be interesting to relate the proof with the original derivation in [**66**]. For the Gaussian ensembles, the relation between the two methods has been studied in [**59**].

Finally, similarly to the case of the Jacobi polynomial ensemble (see [**42**]), we observe that $R(\theta)$ in (37) is linked to the Painlevé VI equation. Precisely, we show that it is the restriction to the unit circle of a solution of (a special case of) the Painlevé VI equation, defined for $z\in\mathbb{C}$.

Corollary 1.11 (Haine-Vanderstichelen [**43**]). *Put $R(\theta)=r(e^{-2i\theta})$. Then, the function*

$$\sigma(z)=-i(z-1)r(z)-\frac{n^2}{4}z$$

satisfies the Okamoto-Jimbo-Miwa form of the Painlevé VI equation

$$[z(z-1)\sigma'']^2+4z(z-1)(\sigma')^3+4\sigma'\sigma^2+4(1-2z)\sigma(\sigma')^2$$
$$-c_1(\sigma')^2+[2(1-2z)c_4-c_2]\sigma'+4c_4\sigma-c_3=0, \tag{90}$$

with

$$c_1=n^2,\quad c_2=\frac{3n^4}{8},\quad c_3=\frac{n^6}{16},\quad c_4=-\frac{n^4}{16}. \tag{91}$$

PROOF. From (38), by a straightforward computation, putting $R(\theta)=r(e^{-2i\theta})$, we obtain that $r(z)$ satisfies

$$[z(z-1)r'']^2+4z^2(z-1)r'r''-4iz(z-1)^2(r')^3-4i(z^2-1)r(r')^2$$
$$+[4z^2-n^2(z-1)^2](r')^2-4ir^2r'=0. \tag{92}$$

Substituting in (92)

$$r(z) = i\frac{\sigma(z) + xz}{z - 1}$$

for some constant x, and multiplying the equation by $(z-1)^4$, we obtain

$$\begin{aligned}
0 =& n^2x^2(-1+z)^2 \\
&+ \Big[2n^2x(-1+z)^2 - 2n^2x(-1+z)^2z + 4x^2(-1+z)^2z\Big]\sigma' \\
&+ \Big[n^2(-1+z)^2 - 2n^2(-1+z)^2z + 8x(-1+z)^2z \\
&+ n^2(-1+z)^2z^2 - 4x(-1+z)^2z^2\Big](\sigma')^2 \\
&- 4(-1+z)^3z(\sigma')^3 + \Big(n^2(-1+z)^2 - 4x(-1+z)^2\Big)\sigma^2 \\
&- 4(-1+z)^2\sigma'\sigma^2 + \Big(2n^2x(-1+z)^2 - 4x^2(-1+z)^2\Big)\sigma \\
&+ \Big[2n^2(-1+z)^2 - 8x(-1+z)^2 - 2n^2(-1+z)^2z \\
&+ 8x(-1+z)^2z\Big]\sigma'\sigma \\
&+ \Big(-4(-1+z)^2 + 8(-1+z)^2z\Big)(\sigma')^2\sigma \\
&- (-1+z)^2\Big(z^2 - 2z^3 + z^4\Big)(\sigma'')^2.
\end{aligned}$$

Annihilating the coefficient of σ^2, one finds that $x = n^2/4$. With this choice of x, the new function $\sigma(z)$ satisfies

$$\begin{aligned}
0 = [z(z-1)\sigma'']^2 + 4z(z-1)(\sigma')^3 + 4\sigma'\sigma^2 + 4(1-2z)\sigma(\sigma')^2 \\
- n^2(\sigma')^2 + \frac{n^4}{4}(z-2)\sigma' - \frac{n^4}{4}\sigma - \frac{n^6}{16}.
\end{aligned}$$

This is the Painlevé VI equation (90) if we pick c_1, c_2, c_3 and c_4 as in (91), which establishes Corollary 3.3. $\square$

Chapter 2

Bi-orthogonal polynomials and bi-orthogonal Laurent polynomials on the unit circle

We introduce in this chapter the important concepts of bi-orthogonal polynomials and bi-orthogonal Laurent polynomials on the unit circle. We have already briefly seen in the first chapter the usefulness of bi-orthogonal polynomials in the theory of random matrices and integrable deformations. Bi-orthogonal polynomials and bi-orthogonal Laurent polynomials on the unit circle will play a crucial role in the following chapters.

1. Orthogonal polynomials on the real line

In this section we recall some well-known facts about orthogonal polynomials on the real line. We refer to [**20**] for more details and proofs.

Let $\mathbb{C}[x]$ be the complex vector space of polynomials in the variable $x \in \mathbb{R}$ with complex coefficients. For $n \geq 0$, we define $\mathbb{P}_n := \langle 1, x, \ldots, x^n \rangle$ the vector subspace of polynomials with degree less than or equal to n, and $\mathbb{P}_{-1} := \{0\}$ is the trivial subspace.

Definition 2.1. *A* moment functional *is a linear functional $\mathcal{L}$*

$$\mathscr{L} : \mathbb{C}[x] \to \mathbb{C}.$$

The moments *are defined by*

$$\mu_n = \mathscr{L}[x^n].$$

Due to the linearity of $\mathscr{L}$, it is completely determined by the sequence of moments $\{\mu_n\}_{n\geq 0}$.

Definition 2.2. *The moment functional $\mathscr{L}$ is* quasi-definite *if and only if $\Delta_n \neq 0$, for $n \geq 1$, where $\Delta_n := \det(\mu_{i+j})_{0\leq i,j\leq n-1}$. A sequence of moments $\{\mu_n\}_{n\geq 0}$ is quasi-definite if the moment functional defined by this sequence is quasi-definite.*

Given a moment functional $\mathscr{L}$, we define the concept of orthogonal polynomials with respect to $\mathscr{L}$.

Definition 2.3. *A sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ is* orthogonal *with respect to the moment functional $\mathscr{L}$ if*

(1) $P_n(x) \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$;
(2) $\mathscr{L}[P_m(x)P_n(x)] = 0$ *when* $m \neq n$;
(3) $\mathscr{L}[P_n(x)^2] \neq 0$.

A necessary and sufficient condition for the existence of a sequence of orthogonal polynomials with respect to $\mathscr{L}$ is that the moment functional $\mathscr{L}$ is quasi-definite.

A polynomial of degree n is said to be *monic* if the coefficient of the term of degree n is 1. We denote by $\{p_n(x)\}_{n\geq 0}$ the sequence of monic orthogonal polynomials associated to a quasi-definite moment functional $\mathscr{L}$. They are given by the following Heine-formula

$$p_n(x) = \frac{1}{\Delta_n}\det\begin{pmatrix} \mu_0 & \cdots & \mu_{n-1} & 1 \\ \mu_1 & \cdots & \mu_n & x \\ \vdots & & \vdots & \vdots \\ \mu_n & \cdots & \mu_{2n-1} & x^n \end{pmatrix}. \tag{93}$$

It is a well-known fact that orthogonal polynomials on the real line satisfy three-term recurrence relations.

Theorem 2.4. *Let $\mathscr{L}$ be a quasi-definite moment functional, and $\{p_n(x)\}_{n\geq 0}$ a sequence of monic orthogonal polynomials with respect to $\mathscr{L}$. Then there exist coefficients c_n and $\lambda_n \neq 0$ such that*

$$p_n(x) = (x - c_n)p_{n-1}(x) - \lambda_n p_{n-2}(x), \tag{94}$$

with $n = 1, 2, \ldots$ and initial conditions $p_{-1}(x) = 0$ and $p_0(x) = 1$.

Defining the semi-infinite vector $p(x) = \big(p_n(x)\big)_{n\geq 0}$ and the semi-infinite matrix

$$L = \begin{pmatrix} c_1 & 1 & & & \\ \lambda_2 & c_2 & 1 & & \\ & \lambda_3 & c_3 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

the recurrence relation (94) takes the simple form

$$x\,p(x) = L\,p(x).$$

The knowledge of the three-term recurrence relations is sufficient to reconstruct the quasi-definite moment functional. This result is known as Favard's Theorem.

Theorem 2.5 (Favard)**.** *Let $c_n, \lambda_n \in \mathbb{C}$ for $n \geq 0$, with $\lambda_n \neq 0$. Let $\{p_n(x)\}_{n\geq 0}$ be a sequence of monic polynomials satisfying the three-term recurrence relations* (94) *with initial conditions $p_{-1}(x) = 0$ and $p_0(x) = 1$. Then there is a unique quasi-definite moment functional $\mathscr{L}$ such that $\mathscr{L}[1] = \lambda_1$ and $\{p_n(x)\}_{n\geq 0}$ is a sequence of orthogonal polynomials for $\mathscr{L}$.*

To summarize this section, we have a correspondence between quasi-definite sequences of moments and three-band matrices :

$$\{\mu_n\}_{n\geq 0} \longleftrightarrow L = \begin{pmatrix} c_1 & 1 & & & \\ \lambda_2 & c_2 & 1 & & \\ & \lambda_3 & c_3 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

2. Bi-orhogonal polynomials

We give in this section a short introduction to bi-orthogonal polynomials. We refer to [**16**, **34**, **56**] for more details.

2.1. Definitions and existence theorem.

Let $\mathbb{C}[z]$ be the complex vector space of polynomials in the variable z with complex coefficients. For $n \geq 0$, we define $\mathbb{P}_n := \langle 1, z, \ldots, z^n \rangle$ the vector subspace of polynomials with degree less than or equal to n, and $\mathbb{P}_{-1} := \{0\}$ is the trivial subspace.

Definition 2.6. *A* bi-moment functional *is a bilinear functional $\mathcal{L}$*

$$\mathcal{L} \,:\, \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C}.$$

The bi-moments *are defined by*

$$\mu_{ij} = \mathcal{L}[z^i, z^j].$$

A bi-moment functional is uniquely determined by its bi-moments $\{\mu_{ij}\}_{i,j\geq 0}$. We define the semi-infinite bi-moment matrix

$$m_\infty = \left(\mu_{m,n}\right)_{m,n\geq 0}. \tag{95}$$

Definition 2.7. *The bi-moment functional $\mathcal{L}$ is* quasi-definite *if and only if $\Delta_n \neq 0$, for $n \geq 1$, where $\Delta_n := \det(\mu_{ij})_{0\leq i,j\leq n-1}$. The sequence of bi-moments $\{\mu_{ij}\}_{i,j\geq 0}$ (resp. the bi-moment matrix m_∞) is quasi-definite if the bi-moment functional defined by this sequence (resp. this bi-moment matrix) is quasi-definite.*

Given a bi-moment functional, we define the concept of bi-orthogonal polynomials.

Definition 2.8. *A sequence of polynomials $\{P_n^{(1)}(z), P_n^{(2)}(z)\}_{n\geq 0}$ is* bi-orthogonal *with respect to the bi-moment functional $\mathcal{L}$ if*

1. $P_n^{(1)}(z), P_n^{(2)}(z) \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$*;*
2. $\mathcal{L}[P_m^{(1)}(z), P_n^{(2)}(z)] = 0$ *when $m \neq n$;*
3. $\mathcal{L}[P_n^{(1)}(z), P_n^{(2)}(z)] \neq 0$.

For a given bi-moment functional, bi-orthogonal polynomials are uniquely determined, if they exist, if one fixes the leading coefficient of each polynomial. From now on, we will impose the leading coefficients to be equal to 1. The corresponding sequence of monic bi-orthogonal polynomials will be denoted by$\{p_n^{(1)}(z), p_n^{(2)}(z)\}_{n\geq 0}$, and we have

$$\mathcal{L}[p_m^{(1)}(z), p_n^{(2)}(z)] = h_n\delta_{mn}, \qquad h_n \neq 0,\ \forall n \in \mathbb{N}. \tag{96}$$

The following theorem is the analogue of a classical result for orthogonal polynomials. It guarantees the existence of a sequence of bi-orthogonal polynomials given a bi-moment functional $\mathcal{L}$, if and only if $\mathcal{L}$ is quasi-definite.

Theorem 2.9 (Bertola [**16**]). *Consider a bi-moment functional $\mathcal{L}$. There exist a sequence of bi-orthogonal polynomials with respect to $\mathcal{L}$ if and only if $\mathcal{L}$ is quasi-definite. Each polynomial in this sequence is uniquely determined up to an arbitrary non-zero factor. The monic sequence $\{p_n^{(1)}(z), p_n^{(2)}(z)\}_{n\geq 0}$ is given by the formulae*

$$p_n^{(1)}(z) = \frac{1}{\Delta_n}\det\begin{pmatrix} \mu_{0,0} & \cdots & \mu_{0,n-1} & 1 \\ \mu_{1,0} & \cdots & \mu_{1,n-1} & z \\ \vdots & & \vdots & \vdots \\ \mu_{n,0} & \cdots & \mu_{n,n-1} & z^n \end{pmatrix}, \tag{97}$$

$$p_n^{(2)}(z) = \frac{1}{\Delta_n}\det\begin{pmatrix} \mu_{0,0} & \mu_{0,1} & \cdots & \mu_{0,n} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1,0} & \mu_{n-1,1} & \cdots & \mu_{n-1,n} \\ 1 & z & \dots & z^n \end{pmatrix}. \tag{98}$$

We define the vectors $p^{(1)}(z) = \big(p_n^{(1)}(z)\big)_{n\geq 0}$ and $p^{(2)}(z) = \big(p_n^{(2)}(z)\big)_{n\geq 0}$. Defining the vector

$$\chi(z) = (1, z, z^2, \dots)^T,$$

these two vectors can be written

$$p^{(1)}(z) = S_1\chi(z), \qquad p^{(2)}(z) = h(S_2^T)^{-1}\chi(z),$$

where S_1 is a lower triangular matrix with all its diagonal elements equal to 1, S_2 is an upper triangular matrix such that the diagonal entries of $h^{-1}S_2$ are equal to 1, and $h = \mathrm{diag}(h_n)_{0\leq n\leq\infty}$ with h_n given in (96). It is easy to prove that the existence of a sequence of bi-orthogonal polynomials with respect to the bi-moment functional $\mathcal{L}$, is equivalent to the factorization of the bi-moment matrix m_∞

$$m_\infty = S_1^{-1}S_2,$$

with S_1, S_2 as above.

Both sequences $\{p_n^{(1)}(z)\}$ and $\{p_n^{(2)}(z)\}$ form a basis of $\mathbb{C}[z]$. We have the following theorem.

Theorem 2.10. *Let $\mathcal{L}$ be a quasi-definite bi-moment functional and $\{p_n^{(1)}(z), p_n^{(2)}(z)\}_{n\geq 0}$ a sequence of monic bi-orthogonal polynomials with respect to $\mathcal{L}$. Then for every polynomial $Q(z)$ of degree n,*

$$Q(z) = \sum_{k=0}^{n} c_k^{(1)}p_k^{(1)}(z), \qquad Q(z) = \sum_{k=0}^{n} c_k^{(2)}p_k^{(2)}(z),$$

where

$$c_k^{(1)} = \frac{\mathcal{L}[Q(z), p_k^{(2)}(z)]}{\mathcal{L}[p_k^{(1)}(z), p_k^{(2)}(z)]}, \qquad c_k^{(2)} = \frac{\mathcal{L}[p_k^{(1)}(z), Q(z)]}{\mathcal{L}[p_k^{(1)}(z), p_k^{(2)}(z)]},$$

for $0 \leq k \leq n$.

2.2. A Favard-like theorem for bi-orthogonal polynomials. Let $\{p_n^{(1)}(z), p_n^{(2)}(z)\}_{n\geq 0}$ be a sequence of monic bi-orthogonal polynomials for a quasi-definite bi-moment functional $\mathcal{L}$. In general these polynomials do not satisfy three-term recurrence relations, as for classical orthogonal polynomials. They however satisfy recurrence relations which are not given by finite band matrices :

$$zp_n^{(1)}(z) = \sum_{i=0}^{n+1} a_{in}p_{n+1-i}^{(1)}(z), \qquad zp_n^{(2)}(z) = \sum_{i=0}^{n+1} b_{in}p_{n+1-i}^{(2)}(z),$$

where $a_{ij}, b_{ij} \in \mathbb{C}$ and $a_{n0}, b_{n0} \neq 0$ for all $n \geq 0$. Those relations can be written in the following simpler form

$$zp^{(1)}(z) = L_1p^{(1)}(z), \qquad zp^{(2)}(z) = L_2^Tp^{(2)}(z), \tag{99}$$

where $p^{(1)}(z)$ and $p^{(2)}(z)$ are the vectors $\left(p_n^{(1)}(z)\right)_{n\geq 0}$ and $\left(p_n^{(2)}(z)\right)_{n\geq 0}$, and

$$L_1 = \sum_{l=0}^{\infty} \mathrm{diag}(a_{ln})_{n\in\mathbb{N}}\Lambda^{1-l} = \begin{pmatrix} a_{1,0} & a_{0,0} & & & \\ a_{2,1} & a_{1,1} & a_{0,1} & & O \\ a_{3,2} & a_{2,2} & a_{1,2} & a_{0,2} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{100}$$

$$L_2^T = \sum_{l=0}^{\infty} \mathrm{diag}(b_{ln})_{n\in\mathbb{N}}\Lambda^{1-l} = \begin{pmatrix} b_{1,0} & b_{0,0} & & & \\ b_{2,1} & b_{1,1} & b_{0,1} & & O \\ b_{3,2} & b_{2,2} & b_{1,2} & b_{0,2} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{101}$$

with $\Lambda = (\delta_{i,j-1})_{i,j\geq 0}$ and $a_{0n}, b_{0n} = 1$ for all $n \geq 0$. Favard's theorem for orthogonal polynomials on the real line can be extended to the case of bi-orthogonal polynomials. The proof can be found in [**16**].

Theorem 2.11 (Bertola [**16**]). *Let L_1 and L_2^T be semi-infinite matrices as in* (100) *and* (101)*, where $a_{0n}, b_{0n} = 1$ for all $n \geq 0$, and let $\{p_n^{(1)}(z)\}_{n\geq 0}$ and $\{p_n^{(2)}(z)\}_{n\geq 0}$ be sequences of monic polynomials defined by the recurrence relations* (99)*, with initial conditions $p_0^{(1)}(z) = 1$ and $p_0^{(2)}(z) = 1$. Let $\{h_n\}_{n\geq 0}$ be a sequence of complex numbers such that $h_n \neq 0$ for all $n \geq 0$. Then there exists a unique quasi-definite bi-moment functional $\mathcal{L}$ for which the sequence of monic polynomials $\{p_n^{(1)}(z), p_n^{(2)}(z)\}_{n\geq 0}$ is bi-orthogonal, and $\mathcal{L}[p_n^{(1)}, p_n^{(2)}] = h_n$.*

2.3. A Toeplitz bi-moment functional.

Let $\mathcal{L}$ be a bi-moment functional.

Definition 2.12. *The bi-moment functional $\mathcal{L}$ is a Toeplitz bi-moment functional if it satisfies the Toeplitz condition*

$$\mathcal{L}[z^{n+1}, z^{m+1}] = \mathcal{L}[z^n, z^m],$$

for all $m, n \geq 0$. The sequence of bi-moments $\{\mu_{ij}\}_{i,j\geq 0}$ is a sequence of Toeplitz bi-moments if the bi-moment functional defined by this sequence is a Toeplitz bi-moment functional.

A typical example of a Toeplitz bi-moment functional to have in mind is

$$\mathcal{L}[f, g] = \oint_{S^1} f(z)g(z^{-1})\rho(z)\frac{dz}{2\pi i z},$$

for some weight function $\rho(z)$.

We suppose from now on that $\mathcal{L}$ is a quasi-definite Toeplitz bi-moment functional. Let $\{p_n^{(1)}(z), p_n^{(2)}(z)\}_{n\geq 0}$ be the sequence of monic bi-orthogonal polynomials with respect to $\mathcal{L}$. The bi-orthogonality conditions give

$$\mathcal{L}[p_m^{(1)}(z), p_n^{(2)}(z)] = h_n\delta_{mn}, \qquad h_n \neq 0,\ \forall n \in \mathbb{N}.$$

We define

$$x_n = p_n^{(1)}(0), \qquad y_n = p_n^{(2)}(0).$$

In this section we prove that the sequence of monic bi-orthogonal polynomials $\{p_n^{(1)}(z), p_n^{(2)}(z)\}_{n\geq 0}$ satisfy two type of recurrence relations. First we prove that they satisfy Szegö-type recurrence relations, similar to the recursion relations obtained by Szegö for orthogonal polynomials on the unit circle. Secondly, they satisfy classical infinite band recurrence relations, as in the previous section. However, the semi-infinite matrices describing these recurrence relations have a particular form. A Favard-type Theorem exists for monic-bi-orthogonal polynomials associated with Toeplitz bi-moment functionals.

We start with the Szegö-type recurrence relations. We first need a small lemma.

Lemma 2.13. *Let $P(z)$ and $Q(z)$ be polynomials of degree n. If*

$$\mathcal{L}[P, z^j] = \mathcal{L}[Q, z^j] = 0, \qquad \forall 1 \leq j \leq n,$$

then $Q(z) = cP(z)$, with $c \in \mathbb{C}$.

PROOF. Let $P(z)$ be a polynomial of degree n such that $\mathcal{L}[P, z^j] = 0, \forall 1 \leq j \leq n$. Then $P(z)$ is a linear combination of the polynomials $p_0^{(1)}(z), \ldots, p_n^{(1)}(z)$

$$P(z) = \sum_{j=0}^{n} a_j p_j^{(1)}(z),$$

and the coefficients a_j satisfy

$$\begin{aligned} a_1 &= -a_0 \frac{\mathcal{L}[p_0^{(1)}, z]}{\mathcal{L}[p_1^{(1)}, z]}, \\ a_2 &= a_0 \frac{\mathcal{L}[p_0^{(1)}, z]\mathcal{L}[p_1^{(1)}, z^2] - \mathcal{L}[p_0^{(1)}, z^2]\mathcal{L}[p_1^{(1)}, z]}{\mathcal{L}[p_1^{(1)}, z]\mathcal{L}[p_2^{(1)}, z^2]}, \\ &etc. \end{aligned}$$

All the coefficients a_j, $1 \leq j \leq n$, are multiples of a_0 and are uniquely determined by a_0. It follows that if $Q(z)$ is another polynomial of degree n such that $\mathcal{L}[Q, z^j] = 0$, $\forall 1 \leq j \leq n$, then $Q(z) = cP(z)$, with $c \in \mathbb{C}$. $\square$

We then have a Szegö-type lemma for the monic bi-orthogonal polynomials.

Lemma 2.14 (Hisakado [**45**]). *We have for $n \geq 0$*

$$\begin{aligned} p_{n+1}^{(1)}(z) - zp_n^{(1)}(z) &= x_{n+1} z^n p_n^{(2)}(z^{-1}), \\ p_{n+1}^{(2)}(z) - zp_n^{(2)}(z) &= y_{n+1} z^n p_n^{(1)}(z^{-1}), \end{aligned}$$

where $x_n = p_n^{(1)}(0)$ and $y_n = p_n^{(2)}(0)$.

PROOF. The statement is trivial for $n = 0$. Suppose $n > 0$. For $1 \le j \le n$ we have on the one hand

$$\begin{aligned}\mathcal{L}\big[p_{n+1}^{(1)}(z) - zp_n^{(1)}(z), z^j\big] &= \mathcal{L}\big[p_{n+1}^{(1)}(z), z^j\big] - \mathcal{L}\big[p_n^{(1)}(z), z^{j-1}\big]\\ &= 0 - 0 = 0,\end{aligned}$$

and on the other hand

$$\mathcal{L}\big[z^n p_n^{(2)}(z^{-1}), z^j\big] = \mathcal{L}\big[z^{n-j}, p_n^{(2)}(z)\big] = 0.$$

As $z^n p_n^{(2)}(z^{-1})$ and $p_{n+1}^{(1)}(z) - zp_n^{(1)}(z)$ are both polynomials of degree n, the above equations imply by lemma 2.13 that

$$p_{n+1}^{(1)}(z) - zp_n^{(1)}(z) = cz^n p_n^{(2)}(z^{-1}),$$

for some $c \in \mathbb{C}$. We have $c = x_{n+1}$ as

$$p_{n+1}^{(1)}(z) - zp_n^{(1)}(z)\Big|_{z=0} = x_{n+1}, \qquad \text{and} \qquad z^n p_n^{(2)}(z^{-1})\Big|_{z=0} = 1.$$

This proves the first equation of the statement. The second equation is proven in a similar way. □

We have the following consequence of Hisakado's lemma.

Lemma 2.15. *We have for all* $n \ge 0$

$$x_{n+1}y_{n+1} = 1 - \frac{h_{n+1}}{h_n}, \tag{102}$$

where $h_n = \mathcal{L}[p_n^{(1)}, p_n^{(2)}]$.

PROOF. On the one hand we have

$$\begin{aligned}&\mathcal{L}\big[p_{n+1}^{(1)}(z) - zp_n^{(1)}(z), p_{n+1}^{(2)}(z) - zp_n^{(2)}(z)\big]\\ &\qquad = h_{n+1} - \mathcal{L}\big[p_{n+1}^{(1)}(z), zp_n^{(2)}(z)\big] - \mathcal{L}\big[zp_n^{(1)}(z), p_{n+1}^{(2)}(z)\big] + h_n\\ &\qquad = h_{n+1} - \mathcal{L}\big[p_{n+1}^{(1)}(z), z^{n+1}\big] - \mathcal{L}\big[z^{n+1}, p_{n+1}^{(2)}\big] + h_n\\ &\qquad = h_{n+1} - h_{n+1} - h_{n+1} + h_n\\ &\qquad = h_n - h_{n+1}.\end{aligned}$$

On the other hand we have using Hisakado's lemma

$$\begin{aligned}&\mathcal{L}\big[p_{n+1}^{(1)}(z) - zp_n^{(1)}(z), p_{n+1}^{(2)} - zp_n^{(2)}(z)\big]\\ &\qquad = \mathcal{L}\big[x_{n+1}z^n p_n^{(2)}(z^{-1}), y_{n+1}z^n p_n^{(1)}(z^{-1})\big]\\ &\qquad = x_{n+1}y_{n+1}\mathcal{L}\big[p_n^{(1)}(z), p_n^{(2)}(z)\big]\\ &\qquad = h_n x_{n+1}y_{n+1}.\end{aligned}$$

Comparing both identities gives the statement. □

We now turn to the classical infinite-band recurrence relations. It is proven in [**9**, **12**] that the recurrence relations for the monic bi-orthogonal polynomials $\{p_n^{(1)}(z), p_n^{(2)}(z)\}_{n\geq 0}$ take the particular form

$$zp^{(1)}(z) = l_1 p^{(1)}(z), \tag{103}$$

$$zp^{(2)}(z) = l_2^T p^{(2)}(z), \tag{104}$$

with

$$l_1 = \begin{pmatrix} -x_1y_0 & 1 & & & \\ -\frac{h_1}{h_0}x_2y_0 & -x_2y_1 & 1 & & O \\ -\frac{h_2}{h_0}x_3y_0 & -\frac{h_2}{h_1}x_3y_1 & -x_3y_2 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{105}$$

and

$$l_2^T = \begin{pmatrix} -x_0y_1 & 1 & & & \\ -\frac{h_1}{h_0}x_0y_2 & -x_1y_2 & 1 & & O \\ -\frac{h_2}{h_0}x_0y_3 & -\frac{h_2}{h_1}x_1y_3 & -x_2y_3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{106}$$

We have the following Favard-like theorem. The proof can be found in the appendix C.

Theorem 2.16. *Let $\{h_n; x_n, y_n\}_{n\geq 0}$ be sequences of complex numbers such that $x_0 = y_0 = 1$ and $\frac{h_n}{h_{n-1}} = 1 - x_ny_n \neq 0$ for all $n \geq 1$, and let $\{p_n^{(1)}(z)\}_{n\geq 0}$ and $\{p_n^{(2)}(z)\}_{n\geq 0}$ be sequences of monic polynomials defined by the recurrence relations (103) and (104), with initial conditions $p_0^{(1)}(z) = 1$ and $p_0^{(2)}(z) = 1$. Then there exists a unique quasi-definite Toeplitz bi-moment functional $\mathcal{L}$ such that $\{p_n^{(1)}(z), p_n^{(2)}(z)\}_{n\geq 0}$ is a sequence of bi-orthogonal polynomials with respect to $\mathcal{L}$ and*

$$\mathcal{L}[p_n^{(1)}(z), p_m^{(2)}(z)] = h_n\delta_{n,m}.$$

To summarize this section, we have a correspondence between sequences of quasi-definite Toeplitz moments, couples of semi-infinite matrices (l_1, l_2) as in (105) and (106) together with a non-zero constant h_0, and sequences $\{h_n; x_n, y_n\}_{n\geq 0}$ such that $x_0 = y_0 = 1$ and $\frac{h_n}{h_{n-1}} = 1 - x_ny_n \neq 0$ for all $n \geq 1$:

$$\{\mu_{i,j}\}_{i,j\geq 0} \longleftrightarrow \{(l_1, l_2); h_0\} \longleftrightarrow \{h_n; x_n, y_n\}_{n\geq 0}.$$

3. Bi-orhogonal Laurent polynomials

A Laurent polynomial, or simply a L-polynomial, in one variable z over a field $\mathbb{F}$ is a linear combination of positive and negative powers of z with coefficients in $\mathbb{F}$:

$$\sum_{k=m}^{n} c_k z^k, \qquad c_k \in \mathbb{F},\ m, n \in \mathbb{Z}.$$

Laurent polynomials in the variable z form a ring denoted $\mathbb{F}[z, z^{-1}]$.

In [**19**], Cantero-Moral-Velázquez consider the ring $\mathbb{C}[z, z^{-1}]$ of Laurent polynomials over $\mathbb{C}$, with z on the unit circle

$$S^1 = \{z \in \mathbb{C} |\, |z| = 1\}.$$

For a sesquilinear Hermitian[1] functional

$$\langle \cdot | \cdot \rangle \,:\, \mathbb{C}[z, z^{-1}] \times \mathbb{C}[z, z^{-1}] \to \mathbb{C},$$

satisfying the Toeplitz condition

$$\langle z^n | z^m \rangle = \left\langle z^{n-m} | 1 \right\rangle,$$

they define the concept of sequences of orthogonal Laurent polynomials on S^1 with respect to the functional $\langle \cdot | \cdot \rangle$ (in fact they distinguish between two types of sequences of orthogonal Laurent polynomials : *right* and *left* ones). They prove that there is a close relation between sequences of left orthogonal L-polynomials, sequences of right orthogonal L-polynomials and sequences of orthogonal polynomials on S^1. The main result in [**19**] is the existence of five-term recurrence relations for the sequences of left and right orthogonal L-polynomials on S^1. The main ingredient in the proof of these recurrence relations is the Toeplitz condition satisfied by the functional $\langle \cdot | \cdot \rangle$.

In this section, we translate the results obtained by Cantero-Moral-Velazquez to the case of bi-orthogonal L-polynomials on S^1 with respect to a quasi-definite bilinear Toeplitz bi-moment functional, as defined in the previous section. All their proofs are easily adapted to this case.

[1] A sesquilinear Hermitian functional satisfies

(1) $\forall f, g, h \in \mathbb{C}[z, z^{-1}], \forall \alpha, \beta \in \mathbb{C} \,:\, \langle f | \alpha g + \beta h \rangle = \overline{\alpha} \langle f | g \rangle + \overline{\beta} \langle f | h \rangle;$

(2) $\forall f, g \in \mathbb{C}[z, z^{-1}] \,:\, \langle f, g \rangle = \overline{\langle g, f \rangle}.$

3.1. Definitions - First properties. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle in the complex plane, and $\mathbb{C}[z, z^{-1}]$ the ring of L-polynomials over $\mathbb{C}$. A *bi-moment functional* on $\mathbb{C}[z, z^{-1}]$ is a bilinear form on $\mathbb{C}[z, z^{-1}] \times \mathbb{C}[z, z^{-1}]$

$$\mathcal{L} : \mathbb{C}[z, z^{-1}] \times \mathbb{C}[z, z^{-1}] \to \mathbb{C}, \quad (f, g) \mapsto \mathcal{L}[f, g].$$

If a bi-moment functional $\mathcal{L}$ satisfies the Toeplitz condition

$$\mathcal{L}[z^n, z^m] = \mathcal{L}[z^{n-m}, 1], \qquad \forall n, m \in \mathbb{Z}, \tag{107}$$

we shall simply call it a *Toeplitz bi-moment functional*. The bi-moments associated to $\mathcal{L}$ are

$$\mu_{mn} = \mathcal{L}[z^m, z^n], \qquad \forall m, n \in \mathbb{Z}.$$

Associated to $\mathcal{L}$ we also define the semi-infinite bi-moment matrix

$$\tilde{m}_\infty = \begin{pmatrix} \mu_{0,0} & \mu_{0,1} & \mu_{0,-1} & \cdots \\ \mu_{1,0} & \mu_{1,1} & \mu_{1,-1} & \cdots \\ \mu_{-1,0} & \mu_{-1,1} & \mu_{-1,-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{108}$$

A bi-moment functional is completely determined by the sequence of bi-moments $\{\mu_{ij}\}_{i,j\in\mathbb{Z}}$. We shall say that a sequence of bi-moments $\{\mu_{ij}\}_{i,j\in\mathbb{Z}}$ is a Toeplitz sequence of bi-moments if the bi-moment functional defined by this sequence is a Toeplitz bi-moment functional.

We define the vector subspaces

$$\mathbb{L}_{m,n} := \left\langle z^m, z^{m+1}, \ldots, z^{n-1}, z^n \right\rangle, \quad \forall m, n \in \mathbb{Z},\ m \le n,$$

and for $n \ge 0$

$$\mathbb{L}^+_{2n} := \mathbb{L}_{-n,n}, \qquad \mathbb{L}^+_{2n+1} := \mathbb{L}_{-n,n+1},$$
$$\mathbb{L}^-_{2n} := \mathbb{L}_{-n,n}, \qquad \mathbb{L}^-_{2n+1} := \mathbb{L}_{-n-1,n},$$

with the convention $\mathbb{L}^+_{-1} = \mathbb{L}^-_{-1} = \{0\}$.

Given a Toeplitz bi-moment functional $\mathcal{L}$, a Gram-Schmidt bi-orthogonalization process applied to the sequence $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$ gives a sequence of right bi-orthogonal Laurent polynomials in the sense of the following definition.

Definition 2.17 (Right bi-orthogonal L-polynomials)**.** *A sequence $\{f_n, g_n\}_{n\ge 0}$ is a sequence of right bi-othogonal L-polynomials with respect to $\mathcal{L}$ if*

(1) $f_n \in \mathbb{L}^+_n \setminus \mathbb{L}^+_{n-1}$*;*
(2) $g_n \in \mathbb{L}^+_n \setminus \mathbb{L}^+_{n-1}$*;*
(3) $\mathcal{L}[f_n, g_m] = h_n \delta_{n,m}$*, with* $h_n \neq 0$.

One can replace condition (3) in the definition by

$$
\text{(3')} \quad \left\{ \begin{array}{l} \left\{ \begin{array}{ll} \mathcal{L}[f_{2n}, z^k] = 0, & \text{if } -n+1 \leq k \leq n, \\ \mathcal{L}[f_{2n+1}, z^k] = 0, & \text{if } -n \leq k \leq n, \\ \mathcal{L}[f_{2n}, z^{-n}] \neq 0, & \\ \mathcal{L}[f_{2n+1}, z^{n+1}] \neq 0, & \end{array} \right. \\ \left\{ \begin{array}{ll} \mathcal{L}[z^k, g_{2n}] = 0, & \text{if } -n+1 \leq k \leq n, \\ \mathcal{L}[z^k, g_{2n+1}] = 0, & \text{if } -n \leq k \leq n, \\ \mathcal{L}[z^{-n}, g_{2n}] \neq 0, & \\ \mathcal{L}[z^{n+1}, g_{2n+1}] \neq 0. & \end{array} \right. \end{array} \right.
$$

We will say that the right bi-orthogonal L-polynomials $\{f_n, g_n\}_{n\geq 0}$ are monic if

$$
\begin{aligned}
&f_{2n}(z) - z^{-n} \text{ and } g_{2n}(z) - z^{-n} \in \left\langle z^{-n+1}, z^{-n+2}, \dots, z^{n} \right\rangle, \\
&f_{2n-1}(z) - z^{n} \text{ and } g_{2n-1}(z) - z^{n} \in \left\langle z^{-n+1}, z^{-n+2}, \dots, z^{n-1} \right\rangle.
\end{aligned}
$$

Similarly, one can apply a Gram-Schmidt bi-orthogonalization process to the sequence $\{1, z^{-1}, z, z^{-2}, z^2, \dots\}$. One gets a sequence of left bi-orthogonal Laurent polynomials in the sense of the following definition.

Definition 2.18 (Left bi-orthogonal L-polynomials)**.** *A sequence $\{f_n, g_n\}_{n\geq 0}$ is a sequence of left bi-othogonal L-polynomials with respect to $\mathcal{L}$ if*

(1) $f_n \in \mathbb{L}_n^- \setminus \mathbb{L}_{n-1}^-$*;*
(2) $g_n \in \mathbb{L}_n^- \setminus \mathbb{L}_{n-1}^-$*;*
(3) $\mathcal{L}[f_n, g_m] = h_n \delta_{n,m}$*, with* $h_n \neq 0$*.*

One can replace condition (3) in the definition by

$$
\text{(3')} \quad \left\{ \begin{array}{l} \left\{ \begin{array}{ll} \mathcal{L}[f_{2n}, z^k] = 0, & \text{if } -n \leq k \leq n-1, \\ \mathcal{L}[f_{2n+1}, z^k] = 0, & \text{if } -n \leq k \leq n, \\ \mathcal{L}[f_{2n}, z^{n}] \neq 0, & \\ \mathcal{L}[f_{2n+1}, z^{-n-1}] \neq 0, & \end{array} \right. \\ \left\{ \begin{array}{ll} \mathcal{L}[z^k, g_{2n}] = 0, & \text{if } -n \leq k \leq n-1, \\ \mathcal{L}[z^k, g_{2n+1}] = 0, & \text{if } -n \leq k \leq n, \\ \mathcal{L}[z^{n}, g_{2n}] \neq 0, & \\ \mathcal{L}[z^{-n-1}, g_{2n+1}] \neq 0. & \end{array} \right. \end{array} \right.
$$

We will say that the left bi-orthogonal L-polynomials $\{f_n, g_n\}_{n\geq 0}$ are monic if

$$
\begin{aligned}
&f_{2n}(z) - z^{n} \text{ and } g_{2n}(z) - z^{n} \in \left\langle z^{-n}, z^{-n+1}, \dots, z^{n-1} \right\rangle, \\
&f_{2n-1}(z) - z^{-n} \text{ and } g_{2n-1}(z) - z^{-n} \in \left\langle z^{-n+1}, z^{-n+2}, \dots, z^{n-1} \right\rangle.
\end{aligned}
$$

For a Toeplitz bi-moment functional $\mathcal{L}$, the existence of a sequence of right or left bi-orthogonal L-polynomials is guaranteed only under certain necessary and sufficient conditions on $\mathcal{L}$, as we will see. But we first need to establish the link that exists between bi-orthogonal Laurent polynomials and bi-orthogonal polynomials for a Toeplitz bi-moment functional. We start by proving that sequences of right and left bi-orthonal L-polynomials for a given Toeplitz bi-moment functional $\mathcal{L}$ are closely related to each other.

Theorem 2.19. *Let $f_n^*(z) = f_n(z^{-1})$ and $g_n^*(z) = g_n(z^{-1})$. Then $\{f_n, g_n\}_{n\geq 0}$ is a sequence of right bi-orthogonal L-polynomials with respect to $\mathcal{L}$ if and only if $\{g_n^*, f_n^*\}_{n\geq 0}$ is a sequence of left bi-orthogonal L-polynomials with respect to $\mathcal{L}$.*

PROOF. We have $f_n^*, g_n^* \in \mathbb{L}_n^- \setminus \mathbb{L}_{n-1}^-$ if and only if $f_n, g_n \in \mathbb{L}_n^+ \setminus \mathbb{L}_{n-1}^+$. The result then follows from

$$\mathcal{L}\big[g_m^*(z), f_n^*(z)\big] = \mathcal{L}\big[g_m(z^{-1}), f_n(z^{-1})\big] = \mathcal{L}\big[f_n(z), g_m(z)\big].$$

□

Sequences of right or left bi-orthogonal L-polynomials with respect to $\mathcal{L}$ are also very closely related to sequences of bi-orthogonal polynomials for $\mathcal{L}$, in the sense of section 2. This is proven in the next two theorems.

Theorem 2.20. *Let $\{f_n, g_n\}_{n\geq 0}$ be a sequence of L-polynomials and*

$$\begin{cases} p_{2n}^{(1)}(z) = z^n g_{2n}(z^{-1}), \\ p_{2n+1}^{(1)}(z) = z^n f_{2n+1}(z), \\ p_{2n}^{(2)}(z) = z^n f_{2n}(z^{-1}), \\ p_{2n+1}^{(2)}(z) = z^n g_{2n+1}(z), \end{cases} \tag{109}$$

The sequence $\{f_n, g_n\}_{n\geq 0}$ is a sequence of right bi-orthogonal L-polynomials with respect to $\mathcal{L}$ if and only if $\{p_n^{(1)}, p_n^{(2)}\}_{n\geq 0}$ is a sequence of bi-orthogonal polynomials with respect to $\mathcal{L}$. Furthermore we have

$$\mathcal{L}[f_n, g_n] = \mathcal{L}[p_n^{(1)}, p_n^{(2)}].$$

PROOF. For $\{p_n^{(1)}, p_n^{(2)}\}_{n\geq 0}$ defined as in (109) we have

$$\begin{aligned} p_{2n}^{(1)} \in \mathbb{P}_{2n} \setminus \mathbb{P}_{2n-1} \quad &\Leftrightarrow \quad g_{2n} \in \mathbb{L}_{2n}^+ \setminus \mathbb{L}_{2n-1}^+, \\ p_{2n+1}^{(1)} \in \mathbb{P}_{2n+1} \setminus \mathbb{P}_{2n} \quad &\Leftrightarrow \quad f_{2n+1} \in \mathbb{L}_{2n+1}^+ \setminus \mathbb{L}_{2n}^+, \\ p_{2n}^{(2)} \in \mathbb{P}_{2n} \setminus \mathbb{P}_{2n-1} \quad &\Leftrightarrow \quad f_{2n} \in \mathbb{L}_{2n}^+ \setminus \mathbb{L}_{2n-1}^+, \\ p_{2n+1}^{(2)} \in \mathbb{P}_{2n+1} \setminus \mathbb{P}_{2n} \quad &\Leftrightarrow \quad g_{2n+1} \in \mathbb{L}_{2n+1}^+ \setminus \mathbb{L}_{2n}^+. \end{aligned}$$

Furthermore we have using the Toeplitz condition (107)

$$\mathcal{L}\big[p_{2n+1}^{(1)}(z), z^k\big] = \mathcal{L}\big[z^n f_{2n+1}(z), z^k\big] = \mathcal{L}\big[f_{2n+1}(z), z^{k-n}\big],$$

and similarly

$$\mathcal{L}\big[p_{2n}^{(1)}(z), z^k\big] = \mathcal{L}\big[z^{n-k}, g_{2n}(z)\big],$$
$$\mathcal{L}\big[z^k, p_{2n+1}^{(2)}(z)\big] = \mathcal{L}\big[z^{k-n}, g_{2n+1}(z)\big],$$
$$\mathcal{L}\big[z^k, p_{2n}^{(2)}(z)\big] = \mathcal{L}\big[f_{2n}(z), z^{n-k}\big].$$

Consequently we have

$$\mathcal{L}\big[p_{2n+1}^{(1)}(z), z^k\big] = 0, \quad 0 \le k \le 2n \\ \Leftrightarrow \quad \mathcal{L}\big[f_{2n+1}(z), z^k\big] = 0, \quad -n \le k \le n,$$

$$\mathcal{L}\big[p_{2n}^{(1)}(z), z^k\big] = 0, \quad 0 \le k \le 2n-1 \\ \Leftrightarrow \quad \mathcal{L}\big[z^k, g_{2n}(z)\big] = 0, \quad -n+1 \le k \le n,$$

$$\mathcal{L}\big[z^k, p_{2n+1}^{(2)}(z)\big] = 0, \quad 0 \le k \le 2n \\ \Leftrightarrow \quad \mathcal{L}\big[z^k, g_{2n+1}(z)\big] = 0, \quad -n \le k \le n,$$

$$\mathcal{L}\big[z^k, p_{2n}^{(2)}(z)\big] = 0, \quad 0 \le k \le 2n-1 \\ \Leftrightarrow \quad \mathcal{L}\big[f_{2n}(z), z^k\big] = 0, \quad -n+1 \le k \le n,$$

and

$$\begin{aligned} \mathcal{L}\big[p_{2n+1}^{(1)}(z), z^{2n+1}\big] \neq 0 \quad &\Leftrightarrow \quad \mathcal{L}\big[f_{2n+1}(z), z^{n+1}\big] \neq 0, \\ \mathcal{L}\big[p_{2n}^{(1)}(z), z^{2n}\big] \neq 0 \quad &\Leftrightarrow \quad \mathcal{L}\big[z^{-n}, g_{2n}(z)\big] \neq 0, \\ \mathcal{L}\big[z^{2n+1}, p_{2n+1}^{(2)}(z)\big] \neq 0 \quad &\Leftrightarrow \quad \mathcal{L}\big[z^{n+1}, g_{2n+1}(z)\big] \neq 0, \\ \mathcal{L}\big[z^{2n}, p_{2n}^{(2)}(z)\big] \neq 0 \quad &\Leftrightarrow \quad \mathcal{L}\big[f_{2n}(z), z^{-n}\big] \neq 0. \end{aligned}$$

This concludes the proof. □

We have an analoguous result for left bi-orthogonal L-polynomials.

Theorem 2.21. *Let $\{f_n, g_n\}_{n\ge 0}$ be a sequence of L-polynomials and*

$$\begin{cases} \tilde{p}_{2n}^{(1)}(z) = z^n f_{2n}(z), \\ \tilde{p}_{2n+1}^{(1)}(z) = z^n g_{2n+1}(z^{-1}), \\ \tilde{p}_{2n}^{(2)}(z) = z^n g_{2n}(z), \\ \tilde{p}_{2n+1}^{(2)}(z) = z^n f_{2n+1}(z^{-1}), \end{cases}$$

The sequence $\{f_n, g_n\}_{n\ge 0}$ is a sequence of left bi-orthogonal L-polynomials with respect to $\mathcal{L}$ if and only if $\{\tilde{p}_n^{(1)}, \tilde{p}_n^{(2)}\}_{n\ge 0}$ is a sequence of bi-orthogonal polynomials with respect to $\mathcal{L}$.

PROOF. This theorem is a consequence of Theorems 2.19 and 2.20. □

We are now able to prove the existence and the unicity of bi-orthogonal L-polynomials with respect to $\mathcal{L}$. We first introduce the following definition.

Definition 2.22. *The Toeplitz bi-moment functional $\mathcal{L}$ is* quasi-definite *if and only if $\Delta_n \neq 0$, for $n \geq 1$, where $\Delta_n := \det(\mu_{ij})_{0\leq i,j\leq n-1}$. The sequence of Toeplitz bi-moments $\{\mu_{ij}\}_{i,j\in\mathbb{Z}}$ is quasi-definite if the Toeplitz bi-moment functional defined by this sequence is quasi-definite.*

We have the following theorem.

Theorem 2.23. *Consider a Toeplitz bi-moment functional $\mathcal{L}$. There exists a sequence of right bi-orthogonal L-polynomials with respect to $\mathcal{L}$ and a sequence of left bi-orthogonal L-polynomials with respect to $\mathcal{L}$ if and only if $\mathcal{L}$ is quasi-definite. Each L-polynomial in these sequences is uniquely determined up to an arbitrary non-zero factor.*

PROOF. By virtue of Theorems 2.20 and 2.21, the existence of a sequence of right or left bi-orthogonal L-polynomials with respect to $\mathcal{L}$ is equivalent to the existence of a sequence of bi-orthogonal polynomials with respect to $\mathcal{L}$. The proof then follows from Theorem 2.9. □

From now on we will always assume that $\mathcal{L}$ is a quasi-definite Toeplitz bi-moment functional on $\mathbb{C}[z,z^{-1}]$, and $\{f_n,g_n\}_{n\geq 0}$ is a sequence of monic right bi-orthogonal L-polynomials with respect to $\mathcal{L}$. As in Theorem 2.19, we define $f_n^*(z) = f_n(z^{-1})$ and $g_n^*(z) = g_n(z^{-1})$. The sequence $\{g_n^*, f_n^*\}_{n\geq 0}$ is a sequence of monic left bi-orthogonal L-polynomials for $\mathcal{L}$. We denote by $\{p_n^{(1)}, p_n^{(2)}\}_{n\geq 0}$ the associated sequence of monic bi-orthogonal polynomials with respect to $\mathcal{L}$. We define the vectors $f(z) = \big(f_n(z)\big)_{n\geq 0}$, $g(z) = \big(g_n(z)\big)_{n\geq 0}$, $f^*(z) = f(z^{-1}) = \big(f_n^*(z)\big)_{n\geq 0}$ and $g^*(z) = g(z^{-1}) = \big(g_n^*(z)\big)_{n\geq 0}$. These vectors can be written

$$f(z) = \tilde{S}_1\,\tilde{\chi}(z), \qquad g(z) = h\left(\tilde{S}_2^T\right)^{-1}\tilde{\chi}(z), \tag{110}$$

and

$$f^*(z) = \tilde{S}_1\,\tilde{\chi}(z^{-1}), \qquad g^*(z) = h\left(\tilde{S}_2^T\right)^{-1}\tilde{\chi}(z^{-1}), \tag{111}$$

where the vector $\tilde{\chi}(z)$ is defined by

$$\tilde{\chi}(z) = \left(1, z, z^{-1}, z^2, z^{-2}, \dots\right),$$

and $h = diag(h_n)_{0\leq n<\infty}$ with $h_n = \mathcal{L}[f_n, g_n]$, $\tilde{S}_1$ is a lower triangular matrix with all the diagonal elements equal to 1, and $\tilde{S}_2$ is an upper triangular matrix such that

$h^{-1}\,\tilde{S}_2$ has all diagonal elements equal to 1. The bi-moment matrix $\tilde{m}_\infty$ defined in (108) can be written in terms of the vector $\tilde{\chi}(z)$

$$\tilde{m}_\infty = \Big(\mathcal{L}\big[\big(\tilde{\chi}(z)\big)_m, \big(\tilde{\chi}(z)\big)_n\big]\Big)_{0\le m,n<\infty}$$

The existence of a sequence of right bi-orthogonal L-polynomials for $\mathcal{L}$ is equivalent to the existence of a factorization of the bi-moment matrix $\tilde{m}_\infty$ in a product of a lower triangular matrix and an upper triangular matrix with non-zero diagonal elements.

Proposition 2.24. *The bi-moment matrix $\tilde{m}_\infty$ factorizes in a product of a lower triangular matrix and an upper triangular matrix*

$$\tilde{m}_\infty = \tilde{S}_1^{-1}\,\tilde{S}_2.$$

PROOF. By bi-orthogonality of the sequence $\{f_n, g_n\}_{n\ge 0}$, we have

$$\mathcal{L}[f_m, g_n] = h_m \delta_{m,n}.$$

This can be written in matrix form

$$h = \big(\mathcal{L}[f_m, g_n]\big)_{0\le m,n<\infty}.$$

Using the expressions (135) we obtain

$$\begin{aligned} h &= \Big(\mathcal{L}\Big[\big(\tilde{S}_1\tilde{\chi}(z)\big)_m, \big(h(\tilde{S}_2^T)^{-1}\tilde{\chi}(z)\big)_n\Big]\Big)_{0\le m,n\le\infty} \\ &= \tilde{S}_1\,\tilde{m}_\infty\,\tilde{S}_2^{-1}\,h. \end{aligned}$$

Consequently we have

$$\tilde{m}_\infty = \tilde{S}_1^{-1}\tilde{S}_2.$$

□

Corollary 2.25. *The bi-moment matrix $\tilde{m}_\infty$ factorizes in a product of a lower triangular matrix with 1's on the diagonal, a diagonal matrix, and an upper triangular matrix with 1's on the diagonal*

$$\tilde{m}_\infty = \tilde{S}_1^{-1}\,h\,(h^{-1}\tilde{S}_2).$$

3.2. Operators $\tilde{\Lambda}$ and $\tilde{\Delta}$. In the previous section we defined the semi-infinite vector

$$\tilde{\chi}(z) = \big(1, z, z^{-1}, z^2, z^{-2}, \dots\big).$$

We define the semi-infinite matrix $\tilde{\Lambda}$ by

$$\tilde{\Lambda}\,\tilde{\chi}(z) = z\,\tilde{\chi}(z). \tag{112}$$

We have

$$\tilde{\Lambda} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and $\tilde{\Lambda}^{-1} = \tilde{\Lambda}^T$.

Similarly, we define the semi-infinite matrix $\tilde{\Delta}$ by

$$\tilde{\Delta}\,\tilde{\chi}(z) = \frac{d}{dz}\,\tilde{\chi}(z). \tag{113}$$

We have

$$\tilde{\Delta} = \tilde{D}\tilde{\Lambda}^T,$$

with $\tilde{D} = \text{diag}(0, 1, -1, 2, -2, \dots)$.

The operators $\tilde{\Lambda}$ and $\tilde{\Delta}$ satisfy the following commutation relation

$$[\tilde{\Lambda}, \tilde{\Delta}] = 1.$$

3.3. Five term recurrence relations. We have proven that bi-orthogonal L-polynomials and bi-orthogonal polynomials with respect to $\mathcal{L}$ are closely related. In section 2 we have seen that bi-orthogonal polynomials satisfy recurrence relations, but those recurrence relations in general do not have a fixed finite number of terms. In this section we prove that bi-orthogonal L-polynomials with respect to a quasi-definite Toeplitz bi-moment functional always satisfy five term recurrence relations. This result has first been obtained by Cantero-Moral-Velazquez [**19**] for orthogonal Laurent polynomials on the unit circle with respect to a sesquilinear Hermitian functional $\langle \cdot | \cdot \rangle$ satisfying the Toeplitz condition. The essential ingredient in their proof is the Toeplitz condition. Consequently, it can immediately be translated to the case of bi-orthogonal L-polynomials with respect to a bilinear functional.

Theorem 2.26. *Let $\{f_n, g_n\}_{n\geq 0}$ be a sequence of monic right bi-orthogononal L-polynomials with respect to $\mathcal{L}$. Then for $n \geq 0$ there exist five term recurrence relations*

$$zf_n(z) = \sum_{i=n-2}^{n+2} \alpha_{n,i} f_i(z), \quad zg_n(z) = \sum_{i=n-2}^{n+2} \beta_{n,i} g_i(z),$$
$$zf_n^*(z) = \sum_{i=n-2}^{n+2} \alpha_{n,i}^* f_i^*(z), \quad zg_n^*(z) = \sum_{i=n-2}^{n+2} \beta_{n,i}^* g_i^*(z),$$

where

$$\alpha_{n,i}^* = \frac{h_n}{h_i}\beta_{i,n}, \quad \beta_{n,i}^* = \frac{h_n}{h_i}\alpha_{i,n},$$

with $h_n = \mathcal{L}[f_n, g_n]$. Moreover, we have

$$\alpha_{2n-1,2n-3} = 0, \qquad \alpha_{2n,2n+2} = 0,$$
$$\beta_{2n-1,2n-3} = 0, \qquad \beta_{2n,2n+2} = 0.$$

PROOF. As $f_n \in \mathbb{L}_n^+ \setminus \mathbb{L}_{n-1}^+$, we have $zf_n(z) \in \mathbb{L}_{n+2}^+$. This implies that zf_n admits an expansion in terms of $f_0, \ldots, f_{n+2}$

$$zf_n(z) = \sum_{i=0}^{n+2} \alpha_{n,i} f_i(z),$$

with $\alpha_{n,i} \in \mathbb{C}$, $0 \leq i \leq n+2$. Consequently, by bi-orthogonality of the sequence $\{f_n, g_n\}_{n\geq 0}$ we have

$$\mathcal{L}[zf_n, g_m] = \sum_{i=0}^{n+2} h_i \alpha_{n,i} \delta_{i,m}.$$

But we also have

$$\mathcal{L}[zf_n, zg_k] = \mathcal{L}[f_n, g_k] = 0, \quad 0 \leq k \leq n-1,$$

and $\langle g_0, \ldots, g_{n-3}\rangle \subset \langle zg_0, \ldots, zg_{n-1}\rangle$. It follows that

$$\mathcal{L}[zf_n, g_k] = 0, \quad 0 \leq k \leq n-3.$$

Consequently we have $\alpha_{n,i} = 0$ if $i < n-2$, and thus

$$zf_n(z) = \sum_{i=n-2}^{n+2} \alpha_{n,i} f_i(z).$$

We prove that $\alpha_{2n,2n+2} = \alpha_{2n-1,2n-3} = 0$. We first prove that $\alpha_{2n,2n+2} = 0$. Indeed, we have $zf_{2n}(z) \in \langle z^{1-n}, \ldots, z^{1+n}\rangle$. Consequently, using condition (3') in the definition of right biorthogonal L-polynomials, we have $\mathcal{L}[zf_{2n}, g_{2n+2}] = 0$ and thus $\alpha_{2n,2n+2} = 0$. We also have $\alpha_{2n-1,2n-3} = 0$. Indeed, we have $\mathcal{L}[zf_{2n-1}, g_{2n-3}] = \mathcal{L}[f_{2n-1}, z^{-1}g_{2n-3}]$, and $z^{-1}g_{2n-3}(z) \in \langle z^{1-n}, \ldots, z^{n-2}\rangle$. From condition (3') in the definition of right biorthogonal L-polynomials it follows that $\mathcal{L}[f_{2n-1}, z^{-1}g_{2n-3}] = 0$ and thus $\mathcal{L}[zf_{2n-1}, g_{2n-3}] = 0$. A similar argument gives $\beta_{2n,2n+2} = \beta_{2n-1,2n-3} = 0$. The proof of the other recurrence relations is similar.

The coefficients in the recurrence relations satisfy

$$\alpha_{n,i} = \frac{\mathcal{L}[zf_n, g_i]}{\mathcal{L}[f_i, g_i]}, \quad \beta_{n,i} = \frac{\mathcal{L}[f_i, zg_n]}{\mathcal{L}[f_i, g_i]},$$
$$\alpha^*_{n,i} = \frac{\mathcal{L}[g^*_i, zf^*_n]}{\mathcal{L}[g^*_i, f^*_i]}, \quad \beta^*_{n,i} = \frac{\mathcal{L}[zg^*_n, f^*_i]}{\mathcal{L}[g^*_i, f^*_i]}.$$

It follows from the definition of $\{g^*_n, f^*_n\}_{n\geq 0}$ that

$$\alpha^*_{n,i} = \frac{\mathcal{L}[g^*_i, zf^*_n]}{\mathcal{L}[g^*_i, f^*_i]} = \frac{\mathcal{L}[f_n, zg_i]}{\mathcal{L}[f_i, g_i]} = \frac{\mathcal{L}[f_n, zg_i]}{\mathcal{L}[f_n, g_n]} \frac{\mathcal{L}[f_n, g_n]}{\mathcal{L}[f_i, g_i]} = \beta_{i,n} \frac{h_n}{h_i}.$$

Similarly we have

$$\beta^*_{n,i} = \frac{\mathcal{L}[zg^*_n, f^*_i]}{\mathcal{L}[g^*_i, f^*_i]} = \frac{\mathcal{L}[zf_i, g_n]}{\mathcal{L}[f_i, g_i]} = \frac{\mathcal{L}[zf_i, g_n]}{\mathcal{L}[f_n, g_n]} \frac{\mathcal{L}[f_n, g_n]}{\mathcal{L}[f_i, g_i]} = \alpha_{i,n} \frac{h_n}{h_i}.$$

This concludes the proof. □

Corollary 2.27. *With the same notations as in Theorem 2.26 we have*

$$z^{-1}f_n(z) = \sum_{i=n-2}^{n+2} \alpha^*_{n,i} f_i(z), \quad z^{-1}g_n(z) = \sum_{i=n-2}^{n+2} \beta^*_{n,i} g_i(z),$$
$$z^{-1}f^*_n(z) = \sum_{i=n-2}^{n+2} \alpha_{n,i} f^*_i(z), \quad z^{-1}g^*_n(z) = \sum_{i=n-2}^{n+2} \beta_{n,i} g^*_i(z),$$

PROOF. The corollary follows from Theorem 2.26 and the definition of the the L-polynomials $\{g^*_n, f^*_n\}_{n\geq 0}$. □

The five term recurrence relations obtained in Theorem 2.26 and Corollary 2.27 can be written in vector form

$$\begin{cases} zf(z) = A_1\, f(z), \\ zg(z) = A_2\, g(z), \\ z^{-1}f(z) = A^*_1\, f(z), \\ z^{-1}g(z) = A^*_2\, g(z), \end{cases} \qquad \begin{cases} zf^*(z) = A^*_1\, f^*(z), \\ zg^*(z) = A^*_2\, g^*(z), \\ z^{-1}f^*(z) = A_1\, f^*(z), \\ z^{-1}g^*(z) = A_2\, g^*(z), \end{cases} \tag{114}$$

with

$$A_1 = \left(\alpha_{i,j}\right)_{i,j\geq 0}, \qquad A_2 = \left(\beta_{i,j}\right)_{i,j\geq 0},$$

where $\alpha_{i,j} = \beta_{i,j} = 0$ if $|i-j| > 2$, and

$$A^*_1 = h\, A^T_2\, h^{-1}, \qquad A^*_2 = h\, A^T_1\, h^{-1}, \tag{115}$$

where $h = \mathrm{diag}(h_n)_{n\geq 0}$. We call the matrices A_1, A_2 the CMV-matrices. We have the following proposition.

Proposition 2.28.

$$A_1^* = A_1^{-1}, \qquad A_2^* = A_2^{-1}.$$

PROOF. We have

$$f(z) = z\, A_1^*\, f(z) = A_1^*\, A_1\, f(z),$$

and

$$f(z) = z^{-1}\, A_1\, f(z) = A_1\, A_1^*\, f(z).$$

Consequently we have $A_1^*\, A_1 = A_1\, A_1^* = 1$. The same argument applied to $g(z)$ gives $B_1^*\, B_1 = B_1\, B_1^* = 1$. □

The CMV-matrices admit the following factorizations.

Theorem 2.29. *We have*

$$A_1 = \tilde{S}_1\, \tilde{\Lambda}\, \tilde{S}_1^{-1}, \qquad A_2 = h\big(\tilde{S}_2^T\big)^{-1}\, \tilde{\Lambda}\, \tilde{S}_2^T\, h^{-1},$$

with $\tilde{S}_1$ and $\tilde{S}_2$ defined in (110).

PROOF. We have

$$A_1\, f(z) = z f(z) = z\, \tilde{S}_1\, \tilde{\chi}(z) = \tilde{S}_1\, \tilde{\Lambda}\, \tilde{\chi}(z) = \tilde{S}_1\, \tilde{\Lambda}\, \tilde{S}_1^{-1}\, f(z).$$

It follows that

$$A_1 = \tilde{S}_1\, \tilde{\Lambda}\, \tilde{S}_1^{-1}.$$

The proof of the second identity is similar. □

Corollary 2.30. *We have*

$$A_1^{-1} = \tilde{S}_2 \tilde{\Lambda}^T \tilde{S}_2^{-1}, \qquad A_2^{-1} = h(\tilde{S}_1^{-1})^T \tilde{\Lambda}^T \tilde{S}_1^T h^{-1}.$$

PROOF. This follows from (115), Proposition 2.28, and Theorem 2.29. □

Explicit expressions for the entries of the CMV-matrices can be found in terms of the variables x_n, y_n defined by

$$x_n = p_n^{(1)}(0), \qquad \text{and} \qquad y_n = p_n^{(2)}(0).$$

Theorem 2.31. *The non-zero entries of the CMV-matrices A_1 and A_2 are*

$$\begin{aligned} (A_1)_{2n-1,2n+1} &= 1, & (A_1)_{2n-1,2n-1} &= -x_{2n} y_{2n-1}, \\ (A_1)_{2n-1,2n} &= -x_{2n+1}, & (A_1)_{2n-1,2n-2} &= -x_{2n}(1 - x_{2n-1} y_{2n-1}), \end{aligned}$$

$$\begin{aligned} (A_1)_{2n,2n+1} &= y_{2n}, & (A_1)_{2n,2n-1} &= y_{2n-1}(1 - x_{2n} y_{2n}), \\ (A_1)_{2n,2n} &= -x_{2n+1} y_{2n}, & (A_1)_{2n,2n-2} &= (1 - x_{2n-1} y_{2n-1})(1 - x_{2n} y_{2n}), \end{aligned}$$

and

$$\begin{aligned} &(A_2)_{2n-1,2n+1} = 1, && (A_2)_{2n-1,2n-1} = -x_{2n-1}y_{2n}, \\ &(A_2)_{2n-1,2n} = -y_{2n+1}, && (A_2)_{2n-1,2n-2} = -y_{2n}(1 - x_{2n-1}y_{2n-1}), \end{aligned}$$

$$\begin{aligned} &(A_2)_{2n,2n+1} = x_{2n}, && (A_2)_{2n,2n-1} = x_{2n-1}(1 - x_{2n}y_{2n}), \\ &(A_2)_{2n,2n} = -x_{2n}y_{2n+1}, && (A_2)_{2n,2n-2} = (1 - x_{2n-1}y_{2n-1})(1 - x_{2n}y_{2n}). \end{aligned}$$

PROOF. (1) We have

$$(A_1)_{2n-1,2n+1} = \frac{1}{h_{2n+1}} \mathcal{L}\big[z f_{2n-1}(z), g_{2n+1}(z)\big].$$

By virtue of Theorem 2.20 we obtain

$$\begin{aligned} (A_1)_{2n-1,2n+1} &= \frac{1}{h_{2n+1}} \mathcal{L}\big[z^{2-n} p^{(1)}_{2n-1}(z), z^{-n} p^{(2)}_{2n+1}(z)\big] \\ &= \frac{1}{h_{2n+1}} \mathcal{L}\big[z^2 p^{(1)}_{2n-1}(z), p^{(2)}_{2n+1}(z)\big]. \end{aligned}$$

As $z^2 p^{(1)}_{2n-1}(z)$ is a monic polynomial of degree $2n+1$, using the bi-orthogonality of the polynomials, we have

$$(A_1)_{2n-1,2n+1} = \frac{1}{h_{2n+1}} \mathcal{L}\big[z^{2n+1}, p^{(2)}_{2n+1}(z)\big] = 1.$$

(2) We have

$$(A_1)_{2n-1,2n} = \frac{1}{h_{2n}} \mathcal{L}\big[z f_{2n-1}(z), g_{2n}(z)\big]$$

By virtue of Theorem 2.20 we obtain

$$\begin{aligned} (A_1)_{2n-1,2n} &= \frac{1}{h_{2n}} \mathcal{L}\big[z^{2-n} p^{(1)}_{2n-1}(z), z^{n} p^{(1)}_{2n}(z^{-1})\big] \\ &= \frac{1}{h_{2n}} \mathcal{L}\big[z^2 p^{(1)}_{2n-1}(z), z^{2n} p^{(1)}_{2n}(z^{-1})\big]. \end{aligned}$$

By using twice Lemma 2.14 we have

$$z^2 p^{(1)}_{2n-1}(z) = p^{(1)}_{2n+1}(z) - x_{2n+1} z^{2n} p^{(2)}_{2n}(z^{-1}) - x_{2n} z^{2n} p^{(2)}_{2n-1}(z^{-1}),$$

and hence

$$\begin{aligned} (A_1)_{2n-1,2n} = {} & \frac{1}{h_{2n}} \mathcal{L}\big[p^{(1)}_{2n+1}(z), z^{2n} p^{(1)}_{2n}(z^{-1})\big] \\ & - \frac{x_{2n+1}}{h_{2n}} \mathcal{L}\big[p^{(2)}_{2n}(z^{-1}), p^{(1)}_{2n}(z^{-1})\big] - \frac{x_{2n}}{h_{2n}} \mathcal{L}\big[p^{(2)}_{2n-1}(z^{-1}), p^{(1)}_{2n}(z^{-1})\big]. \end{aligned}$$

As $z^{2n}p_{2n}^{(1)}(z^{-1})$ is a polynomial of degree $2n$, the first term is equal to 0 by bi-orthogonality. The remaining terms give

$$\begin{aligned}(A_1)_{2n-1,2n} &= -\frac{x_{2n+1}}{h_{2n}}\mathcal{L}\big[p_{2n}^{(1)}(z), p_{2n}^{(2)}(z)\big] - \frac{x_{2n}}{h_{2n}}\mathcal{L}\big[p_{2n}^{(1)}(z), p_{2n-1}^{(2)}(z)\big] \\ &= -\frac{x_{2n+1}}{h_{2n}}\mathcal{L}\big[p_{2n}^{(1)}(z), p_{2n}^{(2)}(z)\big] \\ &= -x_{2n+1}.\end{aligned}$$

(3) We have

$$(A_1)_{2n-1,2n-1} = \frac{1}{h_{2n-1}}\mathcal{L}\big[zf_{2n-1}(z), g_{2n-1}(z)\big].$$

By virtue of Theorem 2.20 we obtain

$$\begin{aligned}(A_1)_{2n-1,2n-1} &= \frac{1}{h_{2n-1}}\mathcal{L}\big[z^{2-n}p_{2n-1}^{(1)}(z), z^{1-n}p_{2n-1}^{(2)}(z)\big] \\ &= \frac{1}{h_{2n-1}}\mathcal{L}\big[zp_{2n-1}^{(1)}(z), p_{2n-1}^{(2)}(z)\big].\end{aligned}$$

By using Lemma 2.14 we have

$$\begin{aligned}(A_1)_{2n-1,2n-1} &= \frac{1}{h_{2n-1}}\mathcal{L}\big[p_{2n}^{(1)}(z) - x_{2n}z^{2n-1}p_{2n-1}^{(2)}(z^{-1}), p_{2n-1}^{(2)}(z)\big] \\ &= -\frac{x_{2n}}{h_{2n-1}}\mathcal{L}\big[z^{2n-1}p_{2n-1}^{(2)}(z^{-1}), p_{2n-1}^{(2)}(z)\big] \\ &= -\frac{x_{2n}}{h_{2n-1}}\mathcal{L}\big[y_{2n-1}z^{2n-1}, p_{2n-1}^{(2)}(z)\big] \\ &= -x_{2n}y_{2n-1}.\end{aligned}$$

(4) We have

$$(A_1)_{2n-1,2n-2} = \frac{1}{h_{2n-2}}\mathcal{L}\big[zf_{2n-1}(z), g_{2n-2}(z)\big].$$

By virtue of Theorem 2.20 we obtain

$$\begin{aligned}(A_1)_{2n-1,2n-2} &= \frac{1}{h_{2n-2}}\mathcal{L}\big[z^{2-n}p_{2n-1}^{(1)}(z), z^{n-1}p_{2n-2}^{(1)}(z^{-1})\big] \\ &= \frac{1}{h_{2n-2}}\mathcal{L}\big[zp_{2n-1}^{(1)}(z), z^{2n-2}p_{2n-2}^{(1)}(z^{-1})\big].\end{aligned}$$

Using Lemma 2.14 we obtain

$$\begin{aligned}
&(A_1)_{2n-1,2n-2}\\
&\quad= \frac{1}{h_{2n-2}}\mathcal{L}\big[p_{2n}^{(1)}(z) - x_{2n}z^{2n-1}p_{2n-1}^{(2)}(z^{-1}), z^{2n-2}p_{2n-2}^{(1)}(z^{-1})\big]\\
&\quad= \frac{1}{h_{2n-2}}\mathcal{L}\big[p_{2n}^{(1)}(z), z^{2n-2}p_{2n-2}^{(1)}(z^{-1})\big]\\
&\qquad - \frac{x_{2n}}{h_{2n-2}}\mathcal{L}\big[zp_{2n-1}^{(2)}(z^{-1}), p_{2n-2}^{(1)}(z^{-1})\big].
\end{aligned}$$

The first term is equal to 0 as $z^{2n-2}p_{2n-2}^{(1)}(z^{-1})$ is a polynomial of degree $2n-2$. Consequently we have

$$\begin{aligned}
(A_1)_{2n-1,2n-2} &= -\frac{x_{2n}}{h_{2n-2}}\mathcal{L}\big[zp_{2n-2}^{(1)}(z), p_{2n-1}^{(2)}(z)\big]\\
&= -\frac{x_{2n}}{h_{2n-2}}\mathcal{L}\big[z^{2n-1}, p_{2n-1}^{(2)}(z)\big]\\
&= -\frac{h_{2n-1}}{h_{2n-2}}x_{2n}\\
&= -(1-x_{2n-1}y_{2n-1})x_{2n}.
\end{aligned}$$

(5) The other relations are proven in a similar way. □

We have an immediate corollary.

Corollary 2.32. *The non-zero entries of the modified CMV-matrices $\tilde{A}_1 = h^{-1}A_1h$ and $\tilde{A}_2 = h^{-1}A_2h$ are*

$$\begin{aligned}
&(\tilde{A}_1)_{2n-1,2n+1} = (1-x_{2n}y_{2n})(1-x_{2n+1}y_{2n+1}), && (\tilde{A}_1)_{2n-1,2n-1} = -x_{2n}y_{2n-1},\\
&(\tilde{A}_1)_{2n-1,2n} = -x_{2n+1}(1-x_{2n}y_{2n}), && (\tilde{A}_1)_{2n-1,2n-2} = -x_{2n},\\
&(\tilde{A}_1)_{2n,2n+1} = y_{2n}(1-x_{2n+1}y_{2n+1}), && (\tilde{A}_1)_{2n,2n-1} = y_{2n-1},\\
&(\tilde{A}_1)_{2n,2n} = -x_{2n+1}y_{2n}, && (\tilde{A}_1)_{2n,2n-2} = 1,
\end{aligned}$$

and

$$\begin{aligned}
&(\tilde{A}_2)_{2n-1,2n+1} = (1-x_{2n}y_{2n})(1-x_{2n+1}y_{2n+1}), && (\tilde{A}_2)_{2n-1,2n-1} = -x_{2n-1}y_{2n},\\
&(\tilde{A}_2)_{2n-1,2n} = -y_{2n+1}(1-x_{2n}y_{2n}), && (\tilde{A}_2)_{2n-1,2n-2} = -y_{2n},\\
&(\tilde{A}_2)_{2n,2n+1} = x_{2n}(1-x_{2n+1}y_{2n+1}), && (\tilde{A}_2)_{2n,2n-1} = x_{2n-1},\\
&(\tilde{A}_2)_{2n,2n} = -x_{2n}y_{2n+1}, && (\tilde{A}_2)_{2n,2n-2} = 1.
\end{aligned}$$

The next proposition is easily obtained from the preceding theorem and corollary.

Proposition 2.33. *The non-zero entries of $\frac{\partial A_1}{\partial y_{2n}}$ are*

$$\begin{aligned}
&\Big(\frac{\partial A_1}{\partial y_{2n}}\Big)_{2n,2n+1} = (A_1)_{2n-1,2n+1}, \quad \Big(\frac{\partial A_1}{\partial y_{2n}}\Big)_{2n,2n} = (A_1)_{2n-1,2n},\\
&\Big(\frac{\partial A_1}{\partial y_{2n}}\Big)_{2n,2n-1} = (A_1)_{2n-1,2n-1}, \quad \Big(\frac{\partial A_1}{\partial y_{2n}}\Big)_{2n,2n-2} = (A_1)_{2n-1,2n-2}.
\end{aligned} \tag{116}$$

The non-zero entries of $\frac{\partial A_1}{\partial x_{2n+1}}$ *are*

$$\Big(\frac{\partial A_1}{\partial x_{2n+1}}\Big)_{2n-1,2n} = -(A_1)_{2n-1,2n+1}, \quad \Big(\frac{\partial A_1}{\partial x_{2n+1}}\Big)_{2n,2n} = -(A_1)_{2n,2n+1},$$
$$\Big(\frac{\partial A_1}{\partial x_{2n+1}}\Big)_{2n+1,2n} = -(A_1)_{2n+1,2n+1}, \quad \Big(\frac{\partial A_1}{\partial x_{2n+1}}\Big)_{2n+2,2n} = -(A_1)_{2n+2,2n+1}.$$

The non-zero entries of $\frac{\partial \tilde{A}_1}{\partial y_{2n+1}}$ *are*

$$\Big(\frac{\partial \tilde{A}_1}{\partial y_{2n+1}}\Big)_{2n-1,2n+1} = (\tilde{A}_1)_{2n-1,2n}, \quad \Big(\frac{\partial \tilde{A}_1}{\partial y_{2n+1}}\Big)_{2n,2n+1} = (\tilde{A}_1)_{2n,2n},$$
$$\Big(\frac{\partial \tilde{A}_1}{\partial y_{2n+1}}\Big)_{2n+1,2n+1} = (\tilde{A}_1)_{2n+1,2n}, \quad \Big(\frac{\partial \tilde{A}_1}{\partial y_{2n+1}}\Big)_{2n+2,2n+1} = (\tilde{A}_1)_{2n+2,2n}. \tag{117}$$

The non-zero entries of $\frac{\partial \tilde{A}_1}{\partial x_{2n}}$ *are*

$$\Big(\frac{\partial \tilde{A}_1}{\partial x_{2n}}\Big)_{2n-1,2n-2} = -(\tilde{A}_1)_{2n,2n-2}, \quad \Big(\frac{\partial \tilde{A}_1}{\partial x_{2n}}\Big)_{2n-1,2n-1} = -(\tilde{A}_1)_{2n,2n-1},$$
$$\Big(\frac{\partial \tilde{A}_1}{\partial x_{2n}}\Big)_{2n-1,2n} = -(\tilde{A}_1)_{2n,2n}, \quad \Big(\frac{\partial \tilde{A}_1}{\partial x_{2n}}\Big)_{2n-1,2n+1} = -(\tilde{A}_1)_{2n,2n+1}.$$

The non-zero entries of $\frac{\partial A_2}{\partial y_{2n+1}}$ *are*

$$\Big(\frac{\partial A_2}{\partial y_{2n+1}}\Big)_{2n-1,2n} = -(A_2)_{2n-1,2n+1}, \quad \Big(\frac{\partial A_2}{\partial y_{2n+1}}\Big)_{2n,2n} = -(A_2)_{2n,2n+1},$$
$$\Big(\frac{\partial A_2}{\partial y_{2n+1}}\Big)_{2n+1,2n} = -(A_2)_{2n+1,2n+1}, \quad \Big(\frac{\partial A_2}{\partial y_{2n+1}}\Big)_{2n+2,2n} = -(A_2)_{2n+2,2n+1}.$$

The non-zero entries of $\frac{\partial A_2}{\partial x_{2n}}$ *are*

$$\Big(\frac{\partial A_2}{\partial x_{2n}}\Big)_{2n,2n-2} = (A_2)_{2n-1,2n-2}, \quad \Big(\frac{\partial A_2}{\partial x_{2n}}\Big)_{2n,2n-1} = (A_2)_{2n-1,2n-1},$$
$$\Big(\frac{\partial A_2}{\partial x_{2n}}\Big)_{2n,2n} = (A_2)_{2n-1,2n}, \quad \Big(\frac{\partial A_2}{\partial x_{2n}}\Big)_{2n,2n+1} = (A_2)_{2n-1,2n+1}.$$

The non-zero entries of $\frac{\partial \tilde{A}_2}{\partial y_{2n}}$ *are*

$$\Big(\frac{\partial \tilde{A}_2}{\partial y_{2n}}\Big)_{2n-1,2n-2} = -(\tilde{A}_2)_{2n,2n-2}, \quad \Big(\frac{\partial \tilde{A}_2}{\partial y_{2n}}\Big)_{2n-1,2n-1} = -(\tilde{A}_2)_{2n,2n-1},$$
$$\Big(\frac{\partial \tilde{A}_2}{\partial y_{2n}}\Big)_{2n-1,2n} = -(\tilde{A}_2)_{2n,2n}, \quad \Big(\frac{\partial \tilde{A}_2}{\partial y_{2n}}\Big)_{2n-1,2n+1} = -(\tilde{A}_2)_{2n,2n+1}.$$

The non-zero entries of $\frac{\partial \tilde{A}_2}{\partial x_{2n+1}}$ *are*

$$\Big(\frac{\partial \tilde{A}_2}{\partial x_{2n+1}}\Big)_{2n-1,2n+1} = (\tilde{A}_2)_{2n-1,2n}, \quad \Big(\frac{\partial \tilde{A}_2}{\partial x_{2n+1}}\Big)_{2n,2n+1} = (\tilde{A}_2)_{2n,2n},$$
$$\Big(\frac{\partial \tilde{A}_2}{\partial x_{2n+1}}\Big)_{2n+1,2n+1} = (\tilde{A}_2)_{2n+1,2n}, \quad \Big(\frac{\partial \tilde{A}_2}{\partial x_{2n+1}}\Big)_{2n+2,2n+1} = (\tilde{A}_2)_{2n+2,2n}.$$

In other words

(1) *the* $2n^{th}$ *line is the only non-zero line in* $\frac{\partial A_1}{\partial y_{2n}}$, *and coincides with the* $(2n-1)^{th}$ *line of* A_1*;*

(2) *the $2n^{th}$ column is the only non-zero column in $\frac{\partial A_1}{\partial x_{2n+1}}$, and coincides with the opposite of the $(2n+1)^{th}$ column of A_1;*
(3) *the $(2n+1)^{th}$ column is the only non-zero column in $\frac{\partial \tilde{A}_1}{\partial y_{2n+1}}$, and coincides with the $2n^{th}$ column of $\tilde{A}_1$;*
(4) *the $(2n-1)^{th}$ line is the only non-zero line in $\frac{\partial \tilde{A}_1}{\partial x_{2n}}$, and coincides with the opposite of the $2n^{th}$ line of $\tilde{A}_1$;*
(5) *the $2n^{th}$ column is the only non-zero column in $\frac{\partial A_2}{\partial y_{2n+1}}$, and coincides with the opposite of the $(2n+1)^{th}$ column of A_2;*
(6) *the $2n^{th}$ line is the only non-zero line in $\frac{\partial A_2}{\partial x_{2n}}$, and coincides with the $(2n-1)^{th}$ line of A_2;*
(7) *the $(2n-1)^{th}$ line is the only non-zero line in $\frac{\partial \tilde{A}_2}{\partial y_{2n}}$, and coincides with the opposite of the $2n^{th}$ line of $\tilde{A}_2$;*
(8) *the $(2n+1)^{th}$ column is the only non-zero column in $\frac{\partial \tilde{A}_2}{\partial x_{2n+1}}$, and coincides with the $2n^{th}$ column of $\tilde{A}_2$.*

3.4. The operators D_1, D_1^* and D_2, D_2^*. Remember from (110) that

$$f(z) = S_1\,\chi(z), \qquad g(z) = h\left(S_2^T\right)^{-1}\chi(z), \tag{118}$$

and, according to (114) and (115), these vectors satisfy

$$A_1\,f(z) = zf(z), \qquad A_1^T\big(h^{-1}\,g^*(z)\big) = z\big(h^{-1}\,g^*(z)\big), \tag{119}$$

$$A_2\,g(z) = zg(z), \qquad A_2^T\big(h^{-1}\,f^*(z)\big) = z\big(h^{-1}\,f^*(z)\big). \tag{120}$$

We define the semi-infinite matrices D_1, D_1^* and D_2, D_2^* by the relations

$$\frac{d}{dz}f(z) = D_1\,f(z), \qquad \frac{d}{dz}\big(h^{-1}\,g^*(z)\big) = (D_1^*)^T\left(h^{-1}\,g^*(z)\right), \tag{121}$$

$$\frac{d}{dz}g(z) = D_2\,g(z), \qquad \frac{d}{dz}\big(h^{-1}\,f^*(z)\big) = (D_2^*)^T\left(h^{-1}\,f^*(z)\right). \tag{122}$$

From (119), (121) and from (120), (122), we deduce that

$$[A_1, D_1] = 1, \qquad \text{and} \qquad [D_2^*, A_2] = 1. \tag{123}$$

The matrices D_1, D_1^* and D_2, D_2^* admit the following factorizations.

Lemma 2.34. *We have*

$$D_1 = \tilde{S}_1\,\tilde{D}\,\tilde{\Lambda}^T\,\tilde{S}_1^{-1}, \qquad D_2 = (\tilde{S}_2^T\,h^{-1})^{-1}\,\tilde{D}\,\tilde{\Lambda}^T\,(\tilde{S}_2^T\,h^{-1}), \tag{124}$$

$$D_1^* = -\tilde{S}_2\,\tilde{\Lambda}^T\,\tilde{D}\,\tilde{S}_2^{-1}, \qquad D_2^* = -(\tilde{S}_1^T\,h^{-1})^{-1}\,\tilde{\Lambda}^T\,\tilde{D}\,(\tilde{S}_1^T\,h^{-1}), \tag{125}$$

with $\tilde{D} = diag(0, 1, -1, 2, -2, \dots)$.

PROOF. Using (110) and (121), we have

$$D_1 f(z) = \frac{d}{dz}f(z) = \tilde{S}_1\frac{d}{dz}\tilde{\chi}(z).$$

By definition of the operator $\tilde{\Delta}$ in (113), we get

$$D_1 f(z) = \tilde{S}_1 \tilde{\Delta} \tilde{S}_1^{-1} f(z) = \tilde{S}_1 \tilde{D} \tilde{\Lambda}^T \tilde{S}_1^{-1} f(z).$$

This proves the first formula in (124).

We have using (111)

$$\frac{d}{dz} g^*(z) = h(\tilde{S}_2^T)^{-1} \frac{d}{dz} \tilde{\chi}(z^{-1}) = -h(\tilde{S}_2^T)^{-1} z^{-2} \Big(\frac{d}{du} \tilde{\chi}(u)\Big)\Big|_{u=z^{-1}},$$

which gives, by definition of $\tilde{\Delta}$,

$$\begin{aligned} \frac{d}{dz} g^*(z) &= -h(\tilde{S}_2^T)^{-1} \tilde{\Delta}\, z^{-2} \tilde{\chi}(z^{-1}) \\ &= -h(\tilde{S}_2^T)^{-1} \tilde{D} \tilde{\Lambda}^T \tilde{\Lambda}^2 \tilde{\chi}(z^{-1}) \\ &= -h(\tilde{S}_2^T)^{-1} \tilde{D} \tilde{\Lambda} \big(h(\tilde{S}_2^T)^{-1}\big)^{-1} g^*(z). \end{aligned}$$

Consequently, using the definition (121) of D_1^*

$$(D_1^*)^T h^{-1} g^*(z) = \frac{d}{dz}\big(h^{-1} g^*(z)\big) = -(\tilde{S}_2^T)^{-1} \tilde{D} \tilde{\Lambda} \tilde{S}_2^T \left(h^{-1} g^*(z)\right).$$

This proves the first formula in (125).

The proof of the two other formulas is identical, using the definitions of D_2 and D_2^* in (122). □

3.5. A Favard-like Theorem. In section 2 we mentioned the existence of a Favard-like Theorem for bi-orthogonal polynomials associated to a quasi-definite Toeplitz bi-moment functional (see Theorem 2.16). As bi-orthogonal L-polynomials with respect to a quasi-definite Toeplitz bi-moment functional are closely related to bi-orthogonal polynomials with respect to the same bi-moment functional, this Favard Theorem can readily be extended to the case of bi-orthogonal L-polynomials.

Theorem 2.35. *Let $\{h_n; x_n, y_n\}_{n\geq 0}$ be such that $x_0 = y_0 = 1$ and $\frac{h_n}{h_{n-1}} = 1 - x_n y_n \neq 0$ for all $n \geq 1$. Let A_1, A_2 be five-band matrices with entries as in Theorem 2.31. Let $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ be sequences of monic right L-polynomials defined by the recurrence relations* (114) *with initial conditions $f_0(z) = g_0(z) = 1$. Then there exist a unique quasi-definite Toeplitz bi-moment functional $\mathcal{L}$ such that the sequences of L-polynomials are right bi-orthogonal with respect to $\mathcal{L}$ and*

$$\mathcal{L}[f_n, g_m] = h_n \delta_{nm}.$$

This theorem generalizes a similar result for orthogonal Laurent polynomials on the unit circle obtained by Cruz-Barroso and Gonzalez-Vera [**21**]. But our proof rests completely on Theorems 2.16 and 2.20 and is independent of that given in [**21**].

To summarize this section on bi-orthogonal L-polynomials, we have a one-to-one correspondence between sequences of quasi-definite Toeplitz bi-moment functionals defined up to a multiplicative non-zero constant, couples of CMV-matrices (A_1, A_2), and sequences $\{x_n, y_n\}_{n\geq 0}$ with $x_0 = y_0 = 1$ and $1 - x_n y_n \neq 0$ for all $n \geq 1$:

$$\{\mu_{i,j}\}_{i,j\in\mathbb{Z}} \longleftrightarrow (A_1, A_2) \longleftrightarrow \{x_n, y_n\}_{n\geq 0}. \tag{126}$$

Chapter 3

The Ablowitz-Ladik hierarchy

In [**9**, **12**] Adler and van Moerbeke study time-dependent bi-orthogonal polynomials, in connexion with integrals over the unitary group $U(n)$. They consider the following time-dependent bilinear quasi-definite Toeplitz bi-moment functional on $\mathbb{C}[z] \times \mathbb{C}[z]$

$$\mathcal{L}[f,g] = \oint_{S^1} f(z)g(z^{-1})\, e^{\sum_{j=1}^{\infty} t_j z^j + \sum_{j=1}^{\infty} s_j z^{-j}}\, \rho(z)\, \frac{\mathrm{d}z}{2\pi i z}.$$

Let $\{p_n^{(1)}(\,\cdot\,;t,s), p_n^{(2)}(\,\cdot\,;t,s)\}_{n\geq 0}$ be the associated sequence of time-dependent monic bi-orthogonal polynomials, and define

$$x_n(t,s) = p_n^{(1)}(0;t,s), \qquad y_n(t,s) = p_n^{(2)}(0;t,s).$$

These polynomials satisfy recurrence relations defined by the time-dependent matrices l_1 and l_2 of the form (105) and (106). Adler and van Moerbeke [**9**] prove that $L_1 = l_1$ and $L_2 = h l_2 h^{-1}$ are solutions of the 2-Toda lattice hierarchy described in [**67**]

$$\frac{\partial L_i}{\partial t_n} = [(L_1^n)_+, L_i], \quad \frac{\partial L_i}{\partial s_n} = [(L_2^n)_{--}, L_i], \qquad i = 1,2,\ n = 1,2,\ldots$$

where for a matrix A, we denote by A_+ (resp. A_{--}) the upper triangular part (resp. the strictly lower triangular part) of A. The particular form of the matrices L_1, L_2 is preserved by these evolution equations. The reduction of the 2-Toda lattice hierarchy to matrices with this particular form is called by Adler and van Moerbeke the Toeplitz lattice. It is equivalent to the Ablowitz-Ladik hierarchy, see [**9**] and also [**62**, **18**]. In this chapter, we describe the Toeplitz lattice or the Ablowitz-Ladik hierarchy from the point of view of the bi-orthogonal L-polynomials on the unit circle. Using the correspondence (126) we will give three different descriptions of the Ablowitz-Ladik hierarchy.

1. The Ablowitz-Ladik vector fields on the space of bi-moments

Let $\mathcal{L} : \mathbb{C}[z, z^{-1}] \times \mathbb{C}[z, z^{-1}] \to \mathbb{C}$ be a quasi-definite Toeplitz bi-moment functional. It is completely determined by its bi-moments $\mu_{m,n} = \mathcal{L}[z^m, z^n]$. As with all Toeplitz bi-moment functionals, these bi-moments only depend on the difference $m - n$. We have

$$\mu_{m,n} := \mu_{m-n},$$

and we shall freely use both notations. The Ablowitz-Ladik hierarchy is defined on the space of quasi-definite Toeplitz bi-moments by the vector fields

$$\boxed{T_j \mu_k \equiv \frac{\partial \mu_k}{\partial t_j} = \mu_{k+j}, \qquad T_{-j}\mu_k \equiv \frac{\partial \mu_k}{\partial s_j} = \mu_{k-j}, \qquad \forall j \geq 1.} \tag{127}$$

Obviously, they satisfy the following commutation relations

$$[T_i, T_j] = 0, \qquad \forall i, j \in \mathbb{Z},$$

if we define $T_0 \mu_k = \mu_k$.

To the basis $\{1, z, z^2, \dots\}$ of the vector space $\mathbb{C}[z]$ we associate the semi-infinite Toeplitz bi-moment matrix

$$m_\infty = \left(\mu_{k-l}\right)_{k,l \geq 0}.$$

It follows immediately from (127) that

$$\begin{cases} \frac{\partial m_\infty}{\partial t_n} = \Lambda^n \, m_\infty, \\ \\ \frac{\partial m_\infty}{\partial s_n} = m_\infty \left(\Lambda^T\right)^n, \end{cases} \qquad n \geq 1, \tag{128}$$

where $\Lambda = (\delta_{i,j-1})_{i,j \geq 0}$ is the usual shift matrix.

To the basis $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$ of the vector space $\mathbb{C}[z, z^{-1}]$ we associate the semi-infinite bi-moment matrix $\tilde{m}_\infty$ defined by

$$\tilde{m}_\infty = \begin{pmatrix} \mu_{0,0} & \mu_{0,1} & \mu_{0,-1} & \cdots \\ \mu_{1,0} & \mu_{1,1} & \mu_{1,-1} & \cdots \\ \mu_{-1,0} & \mu_{-1,1} & \mu_{-1,-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Unlike m_∞, the bi-moment matrix $\tilde{m}_\infty$ is not a Toeplitz matrix. It follows from the definition of $\tilde{\Lambda}$ in (112) that the time evolution of the bi-moment matrix $\tilde{m}_\infty$ is given by the equations

$$\text{(129)} \qquad \begin{cases} \frac{\partial \tilde{m}_\infty}{\partial t_n} = \tilde{\Lambda}^n \, \tilde{m}_\infty, \\ \\ \frac{\partial \tilde{m}_\infty}{\partial s_n} = \tilde{\Lambda}^{-n} \, \tilde{m}_\infty, \end{cases} \qquad n \geq 1.$$

Notice that, because of the Toeplitz property satisfied by the bi-moments, we have the commutation relation

$$\text{(130)} \qquad [\tilde{\Lambda}, \tilde{m}_\infty].$$

Equations (127), (128) and (129) are three equivalent formulations of the Ablowitz-Ladik vector fields at the level of the bi-moments.

The expression of the Ablowitz-Ladik vector fields at the level of the bi-moment matrices m_∞ in (128) is particularly interesting in looking for explicit expressions for the flows on the bi-moments. These flows will be expressed in terms of elementary Schur polynomials $S_n(t)$, defined by the generating function

$$\text{(131)} \qquad \exp\big(\sum_{k=1}^{\infty} t_k x^k\big) = \sum_{n\in\mathbb{Z}} S_n(t_1, t_2, \ldots) x^n.$$

The first elementary Schur polynomials are easily found to be

$$\begin{aligned} &S_{-n}(t) = 0, \qquad \forall n \geq 1, \\ &S_0(t) = 1, \quad S_1(t) = t_1, \quad S_2(t) = \frac{t_1^2}{2} + t_2, \quad S_3(t) = \frac{t_1^3}{6} + t_1 t_2 + t_3, \\ &S_4(t) = \frac{1}{24}\left(t_1^4 + 12 t_1^2 t_2 + 12 t_2^2 + 24 t_1 t_3 + 24 t_4\right) \quad , \quad etc. \end{aligned}$$

The formal solution to the Cauchy problem (128) with given initial conditions $m_\infty(0,0) = M$, where M is a semi-infinite quasi-definite Toeplitz matrix, is given by

$$\text{(132)} \qquad m_\infty(t,s) = e^{\sum_{i=1}^{\infty} t_i \Lambda^i} \, M \, e^{\sum_{j=1}^{\infty} s_j (\Lambda^T)^j},$$

where

$$\begin{aligned} e^{\sum_{i=1}^{\infty} t_i \Lambda^i} = \sum_{i=0}^{\infty} S_i(t) \Lambda^i &= \begin{pmatrix} 1 & S_1(t) & S_2(t) & S_3(t) & S_4(t) & \cdots \\ 0 & 1 & S_1(t) & S_2(t) & S_3(t) & \cdots \\ 0 & 0 & 1 & S_1(t) & S_2(t) & \cdots \\ 0 & 0 & 0 & 1 & S_1(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \\ &= \big(S_{j-i}(t)\big)_{1 \leq i,j < \infty}. \end{aligned}$$

We then have

$$\begin{aligned}\mu_{k,l}(t,s) &= \left(e^{\sum_{i=1}^{\infty} t_i\Lambda^i}\, M\, e^{\sum_{j=1}^{\infty} s_j(\Lambda^T)^j}\right)_{k,l}\\ &= \left(\big(S_{j-i}(t)\big)_{1\le i,j<\infty}\, M\, \big(S_{j-i}(s)\big)^T_{1\le i,j<\infty}\right)_{k,l}\\ &= \sum_{m,n\ge 0} M_{m,n} S_{m-k}(t) S_{n-l}(s).\end{aligned}$$

Relabeling the indices $m \mapsto m+k$ and $l \mapsto n+l$, we obtain

$$\mu_{k,l}(t,s) = \sum_{m,n\ge 0} M_{m+k,n+l} S_m(t) S_n(s).$$

We thus have

$$\mu_{k,l}(t,s) = \mu_{k-l}(t,s) = \sum_{m,n=0}^{\infty} S_m(t)S_n(s)\mu_{k-l+m-n}(0,0). \tag{133}$$

This is the formal solution to the equations (127) with given initial conditions $\mu_{k-l}(0,0)$. Consequently, we observe that the Ablowitz-Ladik flows preserve the Toeplitz property of the bi-moments.

Example 3.1. *Let $\mathcal{L} : \mathbb{C}[z,z^{-1}] \times \mathbb{C}[z,z^{-1}] \to \mathbb{C}$ be a time-dependent Toeplitz bi-moment functional defined by*

$$\mathcal{L}[f,g] = \oint_{S^1} f(z)g(z^{-1})\, e^{\sum_{j=1}^{\infty} t_j z^j + \sum_{j=1}^{\infty} s_j z^{-j}}\, \rho(z)\, \frac{dz}{2\pi i z}.$$

We define for $k,l \in \mathbb{Z}$ the time-dependent bi-moments

$$\mu_{k,l}(t,s) = \mathcal{L}[z^k, z^l] = \oint_{S^1} z^{k-l}\, e^{\sum_{j=1}^{\infty} t_j z^j + \sum_{j=1}^{\infty} s_j z^{-j}}\, \rho(z)\, \frac{dz}{2\pi i z}. \tag{134}$$

The bi-moments clearly satisfy the equations (127)*. Expanding the exponential in terms of Schur polynomials, we get*

$$\mu_{k,l}(t,s) = \sum_{m,n=0}^{\infty} S_m(t)S_n(s) \oint_{S^1} z^{k+m-l-n}\, \rho(z)\, \frac{dz}{2\pi i z}.$$

This is an expansion of the form (133)*.*

Let $\{f_n, g_n\}_{n\ge 0}$ be the sequence of monic right bi-orthogonal L-polynomials with respect to $\mathcal{L}$, $\{g_n^*, f_n^*\}_{n\ge 0}$ the associated sequence of monic left bi-orthogonal L-polynomials, and $\{p_n^{(1)}, p_n^{(2)}\}_{n\ge 0}$ the associated sequence of monic bi-orthogonal polynomials with repsect to $\mathcal{L}$. All these polynomials depend on t and s. We will usually not indicate explicitly the dependence of all these polynomials on t and s. As in the preceding chapter, we define the semi-infinite vectors $f(z) = \big(f_n(z)\big)_{n\ge 0}$ and $g(z) = \big(g_n(z)\big)_{n\ge 0}$, and the matrices $\tilde{S}_1$ and $\tilde{S}_2$ such that

$$f(z) = \tilde{S}_1\, \tilde{\chi}(z), \qquad g(z) = h\left(\tilde{S}_2^T\right)^{-1} \tilde{\chi}(z), \tag{135}$$

where $\tilde{S}_1$ is a lower triangular matrix with all the diagonal elements equal to 1, and $h^{-1}\,\tilde{S}_2$ is an upper triangular matrix with all the diagonal elements equal to 1. We also define the vectors $p^{(1)}(z) = \big(p_n^{(1)}(z)\big)_{n\geq 0}$ and $p^{(2)}(z) = \big(p_n^{(2)}(z)\big)_{n\geq 0}$. These two vectors can be written

$$p^{(1)}(z) = S_1\chi(z), \qquad p^{(2)}(z) = h(S_2^T)^{-1}\chi(z),$$

where S_1 is a lower triangular matrix with all its diagonal elements equal to 1, S_2 is an upper triangular matrix such that the diagonal entries of $h^{-1}S_2$ are equal to 1. These matrices depend on t and s. We also define the functions

$$x_n(t,s) = p_n^{(1)}(0;t,s), \qquad y_n(t,s) = p_n^{(2)}(0;t,s). \tag{136}$$

2. The Ablowitz-Ladik hierarchy as a reduction of the 2-Toda lattice

In [**9**, **12**] Adler and van Moerbeke have obtained the Ablowitz-Ladik hierarchy as a reduction of the 2-Toda lattice. We briefly explain this here.

Let M be a semi-infinite quasi-definite Toeplitz matrix. As we have seen in Chapter 2, the quasi-definiteness of M implies that it factorizes

$$M = S_1(0,0)^{-1}S_2(0,0),$$

with $S_1(0,0)$ a lower triangular matrix with 1's on the principal diagonal, and $S_2(0,0)$ an invertible upper triangular matrix. For generic values of (t,s), the solution $m_\infty(t,s)$given by (132) to the Cauchy problem (128) with initial condition M, also factorizes as follows

$$m_\infty(t,s) = S_1(t,s)^{-1}S_2(t,s),$$

with $S_1(t,s)$ and $S_2(t,s)$ having the same properties as $S_1(0,0)$ and $S_2(0,0)$. We also define a diagonal matrix $h = \operatorname{diag}(h_n)_{n\geq 0}$, such that $h^{-1}S_2(t,s)$ is upper triangular with 1's on the principal diagonal. We then have

Theorem 3.2. *The vectors*

$$\Psi_1(z) := S_1\chi(z)e^{\sum_{j=1}^\infty t_j z^j},$$
$$\Psi_2^*(z) := (S_2^{-1})^T\chi(z^{-1})e^{-\sum_{j=1}^\infty s_j z^{-j}},$$

and the matrices $L_1 := S_1\Lambda S_1^{-1}$ and $L_2 := S_2\Lambda^{-1}S_2^{-1}$ satisfy the equations

$$\begin{cases} L_1\Psi_1 = z\Psi_1,\\ L_2^T\Psi_2^* = z^{-1}\Psi_2^*,\\ \frac{\partial\Psi_1}{\partial t_n} = (L_1^n)_+\Psi_1,\\ \frac{\partial\Psi_2^*}{\partial t_n} = -(L_1^n)_+^T\Psi_2^*,\\ \frac{\partial\Psi_1}{\partial s_n} = (L_2^n)_{--}\Psi_1,\\ \frac{\partial\Psi_2^*}{\partial s_n} = -(L_2^n)_{--}^T\Psi_2^*. \end{cases} \tag{137}$$

The compatibility conditions for (137) are given by the Lax equations

$$\text{(138)} \qquad \frac{\partial L_i}{\partial t_n} = \big[(L_1^n)_+, L_i\big], \quad \frac{\partial L_i}{\partial s_n} = \big[(L_2^n)_-, L_i\big],$$

with $i = 1, 2$. These are the Lax equations for the 2-Toda lattice. The vectors Ψ_1, Ψ_2^* are wave vectors for the 2-Toda lattice.

It is obvious that

$$\Psi_1(t, s; z) = p^{(1)}(t, s; z)e^{\sum_{j=1}^{\infty} t_j z^j},$$
$$\Psi_2^*(t, s; z) = h^{-1}p^{(2)}(t, s; z^{-1})e^{-\sum_{j=1}^{\infty} s_j z^{-j}},$$

where $\{p_n^{(1)}(t, s; z), p_n^{(2)}(t, s; z)\}_{n\geq 0}$ are bi-orthogonal polynomials in the sense of Chapter 2, for the bi-moment functional $\mathcal{L}$ associated to the bi-moment matrix $m_\infty(t, s)$. We have

$$\mathcal{L}[p_m^{(1)}(t, s; z), p_n^{(2)}(t, s; z)] = h_n\delta_{mn}.$$

The Lax matrices L_1, L_2 define reccurence relations on the bi-orthogonal polynomials:

$$L_1 p^{(1)}(t, s; z) = z\, p^{(1)}(t, s; z), \quad \big(h^{-1}L_2 h\big)^T p^{(2)}(t, s; z) = z\, p^{(2)}(t, s; z).$$

As we have seen in section 1, if $m_\infty(0, 0)$ is a Toeplitz matrix, then $m_\infty(t, s)$ is a Toeplitz matrix for all (t, s). By virtue of Theorem 2.16 the Lax matrices L_1, L_2 are then completely determined by the sequences $\{h_n; x_n, y_n\}_{n\geq 0}$, with $\frac{h_n}{h_{n-1}} = 1 - x_n y_n \neq 0$ for all $n \geq 1$, and

$$x_n(t, s) = p_n^{(1)}(t, s; 0), \qquad y_n(t, s) = p_n^{(2)}(t, s; 0).$$

We have

$$L_1 = l_1, \qquad L_2 = h\, l_2\, h^{-1},$$

with l_1, l_2 given in (105) and (106). Consequently, we see that the Ablowitz-Ladik hierarchy is a reduction of the 2-Toda lattice hierarchy.

In Section 4 of Chapter 1, we obtained in (45) the following expressions for the bi-orthogonal polynomials

$$p_n^{(1)}(t, s, z) = z^n \frac{\tau_n(t - [z^{-1}], s)}{\tau_n(t, s)}, \qquad p_n^{(2)}(t, s, z) = z^n \frac{\tau_n(t, s - [z^{-1}])}{\tau_n(t, s)},$$

with

$$\tau_n(t, s) = \det\big(\mu_{k,l}(t, s)\big)_{0\leq k,l\leq n-1},$$

with $\mu_{k,l}(t,s) = \big(m_\infty(t,s)\big)_{k,l}$. It follows that $\tau_n(t,s)$ is a tau-function for the 2-Toda lattice hierarchy in the sense of Sato. These tau-functions are completely determined by the sequence of bi-moments.

3. The Ablowitz-Ladik vector fields on the manifold of CMV-matrices

In this section we "dress up" the equations defining the Ablowitz-Ladik hierarchy (127) on the bi-moments. This leads to Lax pair representations for the hierarchy on the CMV matrices. In this section we shall denote the time variables $(t,s) = (t_1, t_2, \ldots, s_1, s_2, \ldots)$ of the AL hierarchy by $(t_k)_{k\in\mathbb{Z}}$, with $t_{-k} = s_k, k \geq 1$, and T_0 defined below (127). On the space of bi-moments, the Ablowitz-Ladik hierarchy equations (127) then take the particularly simple form

$$T_n \mu_j \equiv \frac{\partial \mu_j}{\partial t_n} = \mu_{j+n}, \qquad \forall n \in \mathbb{Z}, \tag{139}$$

or equivalently

$$\frac{\partial \tilde{m}_\infty}{\partial t_n} = \tilde{\Lambda}^n \, \tilde{m}_\infty, \qquad \forall n \in \mathbb{Z}. \tag{140}$$

For a square matrix A, we define

- A_0 the diagonal part of A;
- A_- (resp. A_+) the lower (resp. upper) triangular part of A;
- A_{--} (resp. A_{++}) the strictly lower (resp. strictly upper) triangular part of A.

We establish the following lemma, based on the factorization of the moment matrix $\tilde{m}_\infty$ in Proposition 2.24 in a product of a lower triangular and an upper triangular matrix.

Lemma 3.3. *We have for $n \in \mathbb{Z}$*

$$\frac{\partial \tilde{S}_1}{\partial t_n} \, \tilde{S}_1^{-1} = -(A_1^n)_{--}, \tag{141}$$

$$\big(\tilde{S}_2^T h^{-1}\big)^{-1} \, \frac{\partial \big(\tilde{S}_2^T h^{-1}\big)}{\partial t_n} = (A_2^{-n})_{--}. \tag{142}$$

PROOF. On the one hand, we have using Proposition 2.24

$$\frac{\partial \tilde{m}_\infty}{\partial t_n} = -\tilde{S}_1^{-1} \, \frac{\partial \tilde{S}_1}{\partial t_n} \, \tilde{S}_1^{-1} \, \tilde{S}_2 + \tilde{S}_1^{-1} \, \frac{\partial \tilde{S}_2}{\partial t_n}.$$

On the other hand, by virtue of equation (140) we have

$$\frac{\partial \tilde{m}_\infty}{\partial t_n} = \tilde{\Lambda}^n \, \tilde{m}_\infty = \tilde{\Lambda}^n \, \tilde{S}_1^{-1} \, \tilde{S}_2.$$

Consequently, as $A_1 = \tilde{S}_1 \tilde{\Lambda} \tilde{S}_1^{-1}$, we obtain

$$A_1^n = -\frac{\partial \tilde{S}_1}{\partial t_n} \tilde{S}_1^{-1} + \frac{\partial \tilde{S}_2}{\partial t_n} \tilde{S}_2^{-1}.$$

Since $\frac{\partial \tilde{S}_1}{\partial t_n}$ is strictly lower triangular, the first term in this equation is strictly lower triangular. The second term in this equation is upper triangular. Consequently, taking the strictly lower triangular part of both sides of the equation yields

$$\frac{\partial \tilde{S}_1}{\partial t_n} \tilde{S}_1^{-1} = -(A_1^n)_{--},$$

which establishes (141).

To establish the other formula, we write $\tilde{m}_\infty = (\tilde{S}_1^{-1}\, h)\,(h^{-1}\, \tilde{S}_2)$ which gives

$$\frac{\partial \tilde{m}_\infty}{\partial t_n} = \frac{\partial(\tilde{S}_1^{-1}\, h)}{\partial t_n} (h^{-1}\tilde{S}_2) + (\tilde{S}_1^{-1}\, h)\, \frac{\partial(h^{-1}\, \tilde{S}_2)}{\partial t_n}.$$

Using the commutation relation (130) and (140), we also have

$$\frac{\partial \tilde{m}_\infty}{\partial t_n} = \tilde{m}_\infty\, \tilde{\Lambda}^n = (\tilde{S}_1^{-1}\, h)\,(h^{-1}\, \tilde{S}_2)\, \tilde{\Lambda}^n.$$

As $A_2 = (\tilde{S}_2^T\, h^{-1})^{-1}\, \tilde{\Lambda}\, (\tilde{S}_2^T\, h^{-1})$, we obtain after some algebra

$$A_2^{-n} = \frac{\partial(\tilde{S}_1^{-1}\, h)^T}{\partial t_n} \left((\tilde{S}_1^{-1}\, h)^T\right)^{-1} + (\tilde{S}_2^T\, h^{-1})^{-1}\, \frac{\partial(\tilde{S}_2^T\, h^{-1})}{\partial t_n}.$$

Since $(\tilde{S}_1^{-1}\, h)^T$ is upper triangular, the first term in the right hand side of this equation is upper triangular. As $\tilde{S}_2^T\, h^{-1}$ is lower triangular with all diagonal entries equal to 1, the second term is strictly lower triangular. Consequently, taking the strictly lower triangular part of both sides of the equation yields

$$(\tilde{S}_2^T\, h^{-1})^{-1}\, \frac{\partial(\tilde{S}_2^T\, h^{-1})}{\partial t_n} = (A_2^{-n})_{--},$$

which establishes (142), completing the proof. □

We are now able to obtain a Lax pair representation for the Ablowitz-Ladik hierarchy.

Theorem 3.4 (Haine-Vanderstichelen [**44**])**.** *The "dressed up" form of the moment equation* (129) *gives the following Lax pair representation for the Ablowitz-Ladik hierarchy on the semi-infinite CMV matrices* (A_1, A_2)

$$\boxed{\frac{\partial A_1}{\partial t_n} = [A_1, (A_1^n)_{--}], \qquad \frac{\partial A_2}{\partial t_n} = [A_2, (A_2^{-n})_{--}], \qquad \forall n \in \mathbb{Z}.} \tag{143}$$

In the particular case of the defocusing Ablowitz-Ladik hierarchy (see introduction), this result had already been obtained by Nenciu [**58**].

PROOF. As $A_1 = \tilde{S}_1 \, \Lambda \, \tilde{S}_1^{-1}$ and $A_2 = (\tilde{S}_2^T \, h^{-1})^{-1} \, \Lambda \, (\tilde{S}_2^T \, h^{-1})$, we have

$$\frac{\partial A_1}{\partial t_n} = \Big[\frac{\partial \tilde{S}_1}{\partial t_n} \, \tilde{S}_1^{-1} \, , \, A_1 \Big],$$

and

$$\frac{\partial A_2}{\partial t_n} = \Big[A_2 \, , \, (\tilde{S}_2^T \, h^{-1})^{-1} \, \frac{\partial (\tilde{S}_2^T \, h^{-1})}{\partial t_n} \Big].$$

By Lemma 3.3 we obtain

$$\frac{\partial A_1}{\partial t_n} = [-(A_1^n)_{--}, A_1] \qquad \text{and} \qquad \frac{\partial A_2}{\partial t_n} = [A_2, (A_2^{-n})_{--}],$$

which establishes (143), concluding the proof. □

The following theorem states that the system of equations in (143) is consistent, i.e. that the flows induced by this system mutually commute. The proof is inspired on that given in [**67**] for the 2-Toda lattice.

Theorem 3.5. *The Ablowitz-Ladik hierarchy* (143) *is equivalent to the system of equations of Zakharov-Shabat type*

(144) $\quad \partial_{t_n} B_m - \partial_{t_m} B_n + [B_m, B_n] = 0,$

(145) $\quad \partial_{t_n} C_m - \partial_{t_m} C_n + [C_m, C_n] = 0,$

for $m, n \in \mathbb{Z}$, with $B_n := (A_1^n)_+$ and $C_n := (A_2^{-n})_+$.

PROOF. 1. We first prove that the first formula in (143) implies (144). We have $A_1^n = B_n + (A_1^n)_{--}$ for all $n \in \mathbb{Z}$. Consequently

$$[B_n, A_1^m] = [A_1^m, (A_1^n)_{--}],$$

as $[A_1^n, A_1^m] = 0$ for all $n, m \in \mathbb{Z}$. It follows that

(146) $\quad \big([B_n, A_1^m]\big)_+ = \big([B_m, (A_1^n)_{--}]\big)_+,$

as $[(A_1^m)_{--}, (A_1^n)_{--}]$ is strictly lower triangular.

From (143) we get for all $n, m \in \mathbb{Z}$

$$\frac{\partial A_1^m}{\partial t_n} = [B_n, A_1^m], \qquad \text{and} \qquad \frac{\partial A_1^n}{\partial t_m} = [B_m, A_1^n].$$

The upper triangular part of the difference of these equations gives

$$\begin{aligned} \frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} &= \big([B_n, A_1^m]\big)_+ - \big([B_m, A_1^n]\big)_+ \\ &= \big([B_n, A_1^m]\big)_+ - \big([B_m, B_n]\big)_+ - \big([B_m, (A_1^n)_{--}]\big)_+. \end{aligned}$$

Using (146), one gets (144).

2. We now prove that (144) gives the first formula of (143). For a matrix

$$A = \sum_{i=-\infty}^{n} a_i \Lambda^i,$$

with a_i diagonal matrices such that $a_n \neq 0$, we call n the order of A. From (144) we get

$$\frac{\partial A_1^m}{\partial t_n} - [B_n, A_1^m] = \frac{\partial (A_1^m)_{--}}{\partial t_n} + \frac{\partial B_n}{\partial t_m} - [B_n, (A_1^m)_{--}],$$

for all $n, m \in \mathbb{Z}$. In the right-hand side we have three terms whose order is $\leq |2n|$, for all $m \in \mathbb{Z}$. Consequently, the left-hand side is also of finite order $\leq |2n|$ for all $m \in \mathbb{Z}:$

$$\text{order}\Big(\frac{\partial A_1^m}{\partial t_n} - [B_n, A_1^m]\Big) \leq |2n|, \qquad \forall m \in \mathbb{Z}.$$

Suppose now

$$\frac{\partial A_1^m}{\partial t_n} - [B_n, A_1^m] \neq 0. \tag{147}$$

Then we have

$$\lim_{m\to\pm\infty} \text{order}\Big(\frac{\partial A_1^m}{\partial t_n} - [B_n, A_1^m]\Big) = +\infty,$$

and we obtain a contradiction with (147). We thus have

$$\frac{\partial A_1^m}{\partial t_n} - [B_n, A_1^m] = 0.$$

3. A similar argument proves that the second formula in (143) and (145) are equivalent. □

Corollary 3.6. *The vector fields defined by* (143) *commute.*

PROOF. We have by virtue of (143)

$$\begin{aligned}\partial_{t_m}(\partial_{t_n} A_1) - \partial_{t_n}(\partial_{t_m} A_1) &= \partial_{t_m}[B_n, A_1] - \partial_{t_n}[B_m, A_1] \\ &= [\partial_{t_m} B_n - \partial_{t_n} B_m, A_1] + \big[B_n, [B_m, A_1]\big] - \big[B_m, [B_n, A_1]\big].\end{aligned}$$

Using (144) we obtain

$$\begin{aligned}\partial_{t_m}(\partial_{t_n} A_1) - \partial_{t_n}(\partial_{t_m} A_1) &= \partial_{t_m}[B_n, A_1] - \partial_{t_n}[B_m, A_1] \\ &= \big[[B_m, B_n], A_1\big] + \big[B_n, [B_m, A_1]\big] - \big[B_m, [B_n, A_1]\big],\end{aligned}$$

which is equal to zero by virtue of the Jacobi identity. Similarly we get

$$\partial_{t_m}(\partial_{t_n} A_2) - \partial_{t_n}(\partial_{t_m} A_2) = 0.$$

□

4. Hamiltonian formalism

We prove in this section that the Ablowitz-Ladik hierarchy, as defined in (143), is a Hamiltonian system. We first prove the following lemma.

Lemma 3.7. *We have for $k \geq 1$*

$$\begin{cases} \frac{\partial \tilde{S}_1}{\partial t_k}\, \tilde{S}_1^{-1} = -(A_1^k)_{--}, \\ \frac{\partial h}{\partial t_k}\, h^{-1} = (A_1^k)_0, \\ \frac{\partial (h^{-1}\tilde{S}_2)}{\partial t_k}\, (h^{-1}\tilde{S}_2)^{-1} = (\tilde{A}_1^k)_{++}, \end{cases}$$

and

$$\begin{cases} \frac{\partial \tilde{S}_1}{\partial s_k}\, \tilde{S}_1^{-1} = -(\tilde{A}_2^k)^T_{--}, \\ \frac{\partial h}{\partial s_k}\, h^{-1} = (A_2^k)_0, \\ \frac{\partial (h^{-1}\tilde{S}_2)}{\partial s_k}\, (h^{-1}\tilde{S}_2)^{-1} = (A_2^k)^T_{--}. \end{cases}$$

PROOF. We have using Corollary 2.25

$$\frac{\partial \tilde{m}_\infty}{\partial t_k} = -\tilde{S}_1^{-1}\, \frac{\partial \tilde{S}_1}{\partial t_k}\, \tilde{S}_1^{-1}\, h\, (h^{-1}\tilde{S}_2) + \tilde{S}_1^{-1}\, \frac{\partial h}{\partial t_k} (h^{-1}\tilde{S}_2) + \tilde{S}_1^{-1}\, h\, \frac{\partial (h^{-1}\tilde{S}_2)}{\partial t_k}.$$

We also have by equation (129)

$$\frac{\partial \tilde{m}_\infty}{\partial t_k} = \tilde{\Lambda}^k\, \tilde{m}_\infty = \tilde{\Lambda}^k\, \tilde{S}_1^{-1}\, h\, (h^{-1}\tilde{S}_2).$$

Consequently, as $A_1 = \tilde{S}_1 \tilde{\Lambda} \tilde{S}_1^{-1}$, we obtain

$$A_1^k = -\frac{\partial \tilde{S}_1}{\partial t_k}\, \tilde{S}_1^{-1} + \frac{\partial h}{\partial t_k} h^{-1} + h \frac{\partial (h^{-1}\tilde{S}_2)}{\partial t_n}\, (h^{-1}\tilde{S}_2)^{-1} h^{-1}.$$

We notice that $\frac{\partial \tilde{S}_1}{\partial t_k}\, \tilde{S}_1^{-1}$ is strictly lower triangular, $\frac{\partial h}{\partial t_k} h^{-1}$ is diagonal, and $h \frac{\partial (h^{-1}\tilde{S}_2)}{\partial t_n}\, (h^{-1}\tilde{S}_2)^{-1} h^{-1}$ is strictly upper triangular. Consequently, taking the strictly lower triangular part, the diagonal part, and the strictly upper triangular part of this equation, we obtain the first part of the lemma. The second part of the lemma is obtained in a similar way. □

We deduce the following lemma.

Lemma 3.8.

$$\frac{\partial f(z)}{\partial t_k} = -(A_1^k)_{--} f(z), \qquad \frac{\partial f(z)}{\partial s_k} = -(\tilde{A}_2^k)^T_{++} f(z),$$
$$\frac{\partial g(z)}{\partial t_k} = -(\tilde{A}_1^k)^T_{++} g(z), \qquad \frac{\partial g(z)}{\partial s_k} = -(A_2^k)_{--} g(z).$$

PROOF. We have using Lemma 3.7

$$\frac{\partial f(z)}{\partial t_k} = \frac{\partial}{\partial t_k}\big(\tilde{S}_1\tilde{\chi}(z)\big) = \Big(\frac{\partial \tilde{S}_1}{\partial t_k}\tilde{S}_1^{-1}\Big)f(z) = -(A_1^k)_{--}f(z),$$

and

$$\begin{aligned}\frac{\partial g(z)}{\partial t_k} &= \frac{\partial}{\partial t_k}\Big(\big((h^{-1}\tilde{S}_2)^{-1}\big)^T\tilde{\chi}(z)\Big)\\ &= -\Big((h^{-1}\tilde{S}_2)^{-1}\frac{\partial(h^{-1}\tilde{S}_2)}{\partial t_k}(h^{-1}\tilde{S}_2)^{-1}\Big)^T\tilde{\chi}(z)\\ &= -\Big((h^{-1}\tilde{S}_2)^{-1}(\tilde{A}_1^k)_{++}\Big)^T\tilde{\chi}(z)\\ &= -(\tilde{A}_1^k)_{++}^T g(z).\end{aligned}$$

The proof of the two other equations is similar. □

On the space $\{(x_1, y_1, \dots) \,|\, 1 - x_k y_k \neq 0,\ x_k, y_k \in \mathbb{C},\ \forall k \geq 1\}$, consider the symplectic form

$$\omega := -\sum_{k=1}^{\infty}\frac{dx_k \wedge dy_k}{1 - x_k y_k}. \tag{148}$$

The following theorem, due to Adler-van Moerbeke [**9**], proves that the Ablowitz-Ladik hierarchy is a Hamiltonian system. We give a new proof of this theorem, based on bi-orthogonal L-polynomials.

Theorem 3.9 (Adler-van Moerbeke [**9**])**.** *The functions $x_n(t,s), y_n(t,s)$ defined in* (136) *satisfy the following integrable Hamiltonian system*

$$\boxed{\begin{cases}\frac{\partial x_n}{\partial t_k} = (1 - x_n y_n)\frac{\partial H_k^{(1)}}{\partial y_n},\\ \frac{\partial x_n}{\partial s_k} = (1 - x_n y_n)\frac{\partial H_k^{(2)}}{\partial y_n},\\ \frac{\partial y_n}{\partial t_k} = -(1 - x_n y_n)\frac{\partial H_k^{(1)}}{\partial x_n},\\ \frac{\partial y_n}{\partial s_k} = -(1 - x_n y_n)\frac{\partial H_k^{(2)}}{\partial x_n},\end{cases} \quad , \qquad n \geq 1,\ k \geq 1,} \tag{149}$$

with boundary condition $x_0 = y_0 = 1$, where $H_k^{(1)} = -\frac{1}{k}TrA_1^k$ and $H_k^{(2)} = \frac{1}{k}TrA_2^k = \frac{1}{k}TrA_1^{-k}$, $k = 1, 2, \dots$ are integrals in involution with respect to the symplectic form ω.

PROOF. We shall only prove the first equation. The proof of the three remaining equations is similar.

We first consider $\frac{\partial x_{2n+1}}{\partial t_k}$. We have $x_{2n+1} = p^{(1)}_{2n+1}(0) = z^n f_{2n+1}(z)|_{z=0}$. Consequently, using Lemma 3.8 we get

$$\begin{aligned}\frac{\partial x_{2n+1}}{\partial t_k} &= -\Big((A_1^k)_{--} z^n f(z)\Big)_{2n+1}\Big|_{z=0}\\ &= -\sum_{l<2n+1} (A_1^k)_{2n+1,l} z^n f_l(z)\Big|_{z=0}.\end{aligned}$$

But $z^n f_l(z)\big|_{z=0} = 0$ for $l < 2n$, and as f_l are monic L-polynomials, we have $z^n f_{2n}(z)\big|_{z=0} = 1$. Consequently we obtain

$$\frac{\partial x_{2n+1}}{\partial t_k} = -(A_1^k)_{2n+1,2n} = -\frac{h_{2n+1}}{h_{2n}}(\tilde{A}_1^k)_{2n+1,2n},$$

with $\tilde{A}_1 = h^{-1}A_1 h$. We have

$$\begin{aligned}\frac{\partial x_{2n+1}}{\partial t_k} &= -\frac{h_{2n+1}}{h_{2n}} \sum_{l=2n-1}^{2n+2} (\tilde{A}_1^{k-1})_{2n+1,l}(\tilde{A}_1)_{l,2n}\\ &= -\frac{h_{2n+1}}{h_{2n}} \sum_{l=2n-1}^{2n+2} (\tilde{A}_1^{k-1})_{2n+1,l}\Big(\frac{\partial \tilde{A}_1}{\partial y_{2n+1}}\Big)_{l,2n+1}, \qquad (150)\end{aligned}$$

where we have applied (117). Defining on the space of square matrices the scalar product

$$\langle A, B\rangle = \mathrm{Tr}(AB),$$

one easily checks that

$$\frac{\partial}{\partial x}\mathrm{Tr}A^n = \Big\langle nA^{n-1}, \frac{\partial A}{\partial x}\Big\rangle.$$

Using this identity, equation (150) gives

$$\begin{aligned}\frac{\partial x_{2n+1}}{\partial t_k} &= -\frac{h_{2n+1}}{h_{2n}}\Big\langle \tilde{A}_1^{k-1}, \frac{\partial \tilde{A}_1}{\partial y_{2n+1}}\Big\rangle,\\ &= -\frac{h_{2n+1}}{h_{2n}}\frac{\partial}{\partial y_{2n+1}}\frac{1}{k}\mathrm{Tr}(\tilde{A}_1^k).\end{aligned}$$

By the invariance of a trace under cyclic permutation of its arguments, we get

$$\frac{\partial x_{2n+1}}{\partial t_k} = -\frac{h_{2n+1}}{h_{2n}}\frac{\partial}{\partial y_{2n+1}}\frac{1}{k}\mathrm{Tr}(A_1^k).$$

Defining $H_k^{(1)} = -\frac{1}{k}\mathrm{Tr}A_1^k$, this expression can be written

$$\frac{\partial x_{2n+1}}{\partial t_k} = (1 - x_{2n+1}y_{2n+1})\frac{\partial H_k^{(1)}}{\partial y_{2n+1}},$$

as $\frac{h_{2n+1}}{h_{2n}} = (1 - x_{2n+1}y_{2n+1})$ by virtue of Proposition 2.15.

We now consider $\frac{\partial x_{2n}}{\partial t_k}$. We have $x_{2n} = p_{2n}^{(1)}(0) = z^n g_{2n}(z^{-1})\big|_{z=0}$. Using Lemma 3.8 we get

$$\begin{aligned}\frac{\partial x_{2n}}{\partial t_k} &= -\Big((\tilde{A}_1^k)_{++}^T z^n g(z^{-1})\Big)_{2n}\Big|_{z=0}\\ &= -\sum_{l<2n}(\tilde{A}_1^k)_{l,2n} z^n g_l(z^{-1})\Big|_{z=0}\\ &= -(\tilde{A}_1^k)_{2n-1,2n},\end{aligned}$$

as $z^n g_l(z^{-1})\big|_{z=0} = 0$ for $l < 2n-1$, and $z^n g_{2n-1}(z^{-1})\big|_{z=0} = 1$ by monicity. As $\tilde{A}_1 = h^{-1}A_1 h$, we get

$$\begin{aligned}\frac{\partial x_{2n}}{\partial t_k} &= -\frac{h_{2n}}{h_{2n-1}}(A_1^k)_{2n-1,2n}\\ &= -\frac{h_{2n}}{h_{2n-1}}\sum_{l=2n-2}^{2n+1}(A_1)_{2n-1,l}(A_1^{k-1})_{l,2n}\\ &= -\frac{h_{2n}}{h_{2n-1}}\sum_{l=2n-2}^{2n+1}\Big(\frac{\partial A_1}{\partial y_{2n}}\Big)_{2n,l}(A_1^{k-1})_{l,2n},\end{aligned}$$

by virtue of (116). This gives

$$\begin{aligned}\frac{\partial x_{2n}}{\partial t_k} &= -\frac{h_{2n}}{h_{2n-1}}\left\langle\frac{\partial A_1}{\partial y_{2n}}, A_1^{k-1}\right\rangle\\ &= -\frac{h_{2n}}{h_{2n-1}}\frac{\partial}{\partial y_{2n}}\frac{1}{k}\mathrm{Tr}(A_1^k)\\ &= (1 - x_{2n}y_{2n})\frac{\partial H_k^{(1)}}{\partial y_{2n}}.\end{aligned}$$

We now prove that the Hamiltonians $H_k^{(1)}$ and $H_k^{(2)}$ are integrals in involution with respect to the symplectic form ω. Indeed, let $\{\,\cdot\,,\,\cdot\,\}$ be the Poisson bracket associated to ω. We have for $n > 0$ and $m \in \mathbb{Z}$

$$\frac{\partial \mathrm{Tr}A_1^m}{\partial t_n} = \{H_n^{(1)}, \mathrm{Tr}A_1^m\}, \qquad \frac{\partial \mathrm{Tr}A_1^m}{\partial s_n} = \{H_n^{(2)}, \mathrm{Tr}A_1^m\}.$$

The Lax form (143) of the Ablowitz-Ladik hierarchy then implies

$$\{H_n^{(1)}, \mathrm{Tr}A_1^m\} = \mathrm{Tr}\frac{\partial A_1^m}{\partial t_n} = \mathrm{Tr}[B_n, A_1^m] = 0,$$

and similarly

$$\{H_n^{(2)}, \mathrm{Tr}A_1^m\} = 0.$$

□

5. The first flows of the Ablowitz-Ladik hierarchy : the Ablowitz-Ladik equation

Consider the space $\{(x_1, y_1, \dots)|1 - x_k y_k \neq 0,\ x_k, y_k \in \mathbb{C},\ \forall k \geq 1\}$ going with the symplectic form ω defined in (148). We first compute the first flows of the Ablowitz-Ladik hierarchy. The first equations in (149) are

$$\frac{\partial x_n}{\partial t_1} = (1 - x_n y_n)x_{n+1}, \qquad \frac{\partial x_n}{\partial s_1} = -(1 - x_n y_n)x_{n-1},$$
$$\frac{\partial y_n}{\partial t_1} = -(1 - x_n y_n)y_{n-1}, \qquad \frac{\partial y_n}{\partial s_1} = (1 - x_n y_n)y_{n+1}.$$

Taking the following linear combination $\frac{\partial}{\partial t} := \frac{\partial}{\partial t_1} - \frac{\partial}{\partial s_1}$ of the vector fields gives

$$\frac{\partial x_k}{\partial t} = x_{k+1} + x_{k-1} - x_k y_k \big(x_{k+1} + x_{k-1}\big),$$

$$\frac{\partial y_k}{\partial t} = -y_{k+1} - y_{k-1} + y_k x_k \big(y_{k+1} + y_{k-1}\big).$$

Define new variables $\tilde{x}_n, \tilde{y}_n$ such that

$$x_n = \tilde{x}_n e^{2t}, \qquad y_n = \tilde{y}_n e^{-2t}.$$

These new variables satisfy

$$\frac{\partial \tilde{x}_k}{\partial t} = \tilde{x}_{k+1} - 2\tilde{x}_k + \tilde{x}_{k-1} - \tilde{x}_k \tilde{y}_k \big(\tilde{x}_{k+1} + \tilde{x}_{k-1}\big),$$

(151)

$$\frac{\partial \tilde{y}_k}{\partial t} = -\tilde{y}_{k+1} + 2\tilde{y}_k - \tilde{y}_{k-1} + \tilde{y}_k \tilde{x}_k \big(\tilde{y}_{k+1} + \tilde{y}_{k-1}\big).$$

These equations are the Ablowitz-Ladik equations, obtained by Ablowitz and Ladik [**1**, **2**] as a discretization of the nonlinear cubic Schrödinger equation. Indeed, after rescaling

$$t \mapsto \epsilon^{-2} t, \quad x_k \mapsto \epsilon x_k, \quad y_k \mapsto \epsilon y_k,$$

the continuous limit $\epsilon \to 0$ in (151) gives the system of partial differential equations

(152) $$x_t = x_{qq} - 2x^2 y, \qquad y_t = -y_{qq} + 2xy^2,$$

where q is the space variable. Upon making the change of variables $t \mapsto it$, with $i = \sqrt{-1}$, and making the reduction $y = \pm\overline{x}$, where $\overline{x}$ is the complex conjugate of x, the system (152) reduces to the cubic Schrödinger equation

(153) $$-ix_t = x_{qq} \mp 2x|x|^2.$$

Chapter 4

Master Symmetries of the Ablowitz-Ladik hierarchy

In Chapter 1 we constructed in (54), (55) and (56) a family of operators $L_k^{(n)}$, $k \in \mathbb{Z}$, satisfying the commutation relations of the centerless Virasoro algebra. Following an idea introduced in [**42**] in the context of 1-dimensional Toda lattices, we prove that these operators precisely describe the master symmetries of the Ablowitz-Ladik hierarchy on the tau functions of this hierarchy. We also construct Lax pairs for the master symmetries, by translating their action on the manifold of CMV-matrices. The results presented in this chapter are joined work with L. Haine [**44**].

1. Time-dependent symmetries

Let M be a differential manifold of class C^∞, and X a vector field on M. An integral curve of X is a smooth curve $c : I \to M$ defined on an open interval $I \subset \mathbb{R}$ such that

$$\frac{d}{dt}c(t) = X\big(c(t)\big), \quad \forall t \in I. \tag{154}$$

We denote by $t \mapsto \phi_X^t(p)$ the unique maximal integral curve of X with initial condition $\phi^0(p) = p$. The map $p \mapsto \phi^t(p)$ (for fixed t) is called the flow. A time-independent vector field X is called an autonomous vector field.

We first recall the concept of a symmetry of X. A **symmetry** of X is a vector field Y on M such that

$$[X, Y] = 0, \tag{155}$$

where $[X, Y]$ is the Lie bracket of the vector fields X, Y. This is equivalent to saying that the flows ϕ_X^t and ϕ_Y^s commute

$$\phi_X^t \circ \phi_Y^s(p) = \phi_Y^s \circ \phi_X^t(p), \qquad \forall p \in M,$$

for all values t, s for which the expressions in this equation make sense. Consequently,

$$t \mapsto \phi_Y^s\big(\phi_X^t(p)\big)$$

is also an integral curve of X for fixed s sufficiently small.

We now turn to the more general case of time-dependent symmetries. Let Y be a time-dependent vector field on M depending smoothly on time, i.e.

$$Y\,:\,\mathbb{R}\times M \to TM,\ (t,p) \mapsto Y(t,p) \in T_pM,$$

is a smooth mapping, with TM the tangent bundle of M, and T_pM the tangent space to M at $p \in M$. An integral curve of Y is a smooth curve $c\,:\,I \to M$ defined on an open interval $I \subset \mathbb{R}$ such that

$$\frac{d}{dt}c(t) = Y\big(t, c(t)\big), \quad \forall t \in I. \tag{156}$$

We denote $t \mapsto c_Y(t; t_0, p)$ the unique maximal integral curve of Y on M with initial condition p at time $t = t_0$. To the time-dependent vector field Y on M, we associate the autonomous vector field $\tilde{Y}$ on $\mathbb{R}\times M$

$$\tilde{Y}(t,p) = \frac{\partial}{\partial t} + Y(t,p) \in \mathbb{R}\times T_pM.$$

In a similar way, we extend the vector field X to a vector field $\tilde{X}$ on $\mathbb{R}\times M$ by

$$\tilde{X}(t,p) = \frac{\partial}{\partial t} + X(p) \in \mathbb{R}\times T_pM.$$

The flows $\tilde{\phi}_X^t(t_0,p)$ and $\tilde{\phi}_Y^t(t_0,p)$ of $\tilde{X}$ and $\tilde{Y}$ on $\mathbb{R}\times M$ are then given by

$$\tilde{\phi}_X^t(t_0,p) = \big(t+t_0; \phi_X^{t+t_0}(p)\big), \qquad \tilde{\phi}_Y^t(t_0,p) = \big(t+t_0; c_Y(t+t_0; t_0, p)\big).$$

We call Y a **time-dependent symmetry** of X if $\tilde{Y}$ is a symmetry of $\tilde{X}$ on $\mathbb{R}\times M$. A necessary and sufficient condition for this to happen is

$$[\tilde{X}, \tilde{Y}] = 0,$$

or equivalently

$$\frac{\partial Y}{\partial t} + [X, Y] = 0. \tag{157}$$

Consequently, the flows $\tilde{\phi}^t_X(t_0, p)$ and $\tilde{\phi}^s_Y(t_0, p)$ of $\tilde{X}$ and $\tilde{Y}$ on $\mathbb{R} \times M$ will commute. Hence,

$$t \mapsto c_Y\big(s + t + t_0; t + t_0, \phi_X^{t+t_0}(p)\big),$$

is an integral curve of X, for s sufficiently small.

2. Master symmetries

Following Fuchsteiner [**37**] we introduce the concept of master symmetries. Let M be a smooth differential manifold and $\mathscr{X}(M)$ the algebra of vector fields on M. Consider a vector field X on M and the evolution equation (154). A vector field Z on M is a X-generator of degree n if

$$\Big[[\dots[Z, \underbrace{X], \dots, X], X\Big]}_{n+1} = 0.$$

It then immediately follows from (157) that the time-dependent vector field

$$Y_Z = \sum_{k=0}^{n} \frac{t^k}{k!} \Big[[\dots[Z, \underbrace{X], \dots, X], X\Big]}_{k},$$

is a time-dependent symmetry of X. One easily checks that every time-dependent symmetry of X which is polynomial in t is of this form. Consequently, we have a one-to-one correspondence between the set of X-generators and the set of polynomial time-dependent symmetries of X. The symmetries of X are generators of degree 0.

A master symmetry Z of X is a X-generator of degree 1 which is not a symmetry:

$$\big[[Z, X], X\big] = 0, \qquad \text{and} \qquad [Z, X] \neq 0.$$

Given Z, one constructs the time-dependent symmetry

$$Y_Z = Z + t[Z, X]. \tag{158}$$

We can generalize this definition and define the concept of master symmetries of a hierarchy $\{X_j\}_{j \in J}$, $J \subset \mathbb{N}$, of commuting vector fields on M. A master symmetry of the hierarchy $\{X_j\}_{j \in J}$ is a vector field Z on M such that

$$\big[[Z, X_i], X_j\big] = 0, \quad \text{and} \quad [Z, X_i] \neq 0,$$

for all $i, j \in J$. If Z is a master symmetry of the hierarchy, then

$$Y_z = Z + \sum_{j \in J} t_j [Z, X_j],$$

is a time-dependent symmetry of all the vector fields X_j.

3. Plücker coordinates for the tau-functions of the Ablowitz-Ladik hierarchy

The Plücker coordinates for the tau-functions of the Ablowitz-Ladik hierarchy are defined in terms of the elementary Schur polynomials. The elementary Schur polynomials $S_n(t)$ are defined in (131). As a direct consequence of the definition of the elementary Schur polynomials, we notice that

$$nS_n(t) = \sum_{1\le k\le n} kt_k S_{n-k}(t), \quad \text{and} \quad \frac{\partial}{\partial t_j} S_i(t) = S_{i-j}(t). \tag{159}$$

As we have seen in Chapter 3, the tau functions of the Ablowitz-Ladik hierarchy are given by

$$\tau_n(t,s) = \det\bigl(\mu_{k-l}(t,s)\bigr)_{0\le k,l<n}. \tag{160}$$

It immediately follows from the generating function (131) that the moments μ_k admit the following expansion in terms of elementary Schur polynomials

$$\mu_k(s,t) = \sum_{n,m=0}^{\infty} S_m(t) S_n(s) \mu_{k+m-n}(0,0). \tag{161}$$

It follows that the tau functions (160) admit the expansion

$$\tau_n(t,s) = \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} S_{i_{n-1}-(n-1),\ldots,i_0}(t) S_{j_{n-1}-(n-1),\ldots,j_0}(s),$$

where

$$p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} = \det\bigl(\mu_{i_k-j_l}(0,0)\bigr)_{0\le k,l<n}, \tag{162}$$

are the so-called Plücker coordinates, and $S_{i_1,\ldots,i_k}(t)$ denote the Schur polynomials

$$S_{i_1,\ldots,i_k}(t) = \det\bigl(S_{i_r+s-r}(t)\bigr)_{1\le r,s\le k}.$$

Indeed, using the expansion (161) we have

$$\tau_n(t,s) = \sum_{\substack{0\le i_0,i_1,\ldots,i_{n-1}\\ 0\le j_0,j_1,\ldots,j_{n-1}}} \det\bigl[\mu_{k-l+i_k-j_l}(0,0)\bigr]_{0\le k,l<n} \times S_{i_0}(t)\ldots S_{i_{n-1}}(t) S_{j_0}(s)\ldots S_{j_{n-1}}(s).$$

Relabeling the indices as follows

$$i_k \mapsto i_k - k, \qquad j_l \mapsto j_l - l,$$

we get

$$
\begin{aligned}
&\tau_n(t,s)\\
&= \sum_{\substack{0\le i_0,\dots,i_{n-1}\\ 0\le j_0,\dots,j_{n-1}}} \det\left[\mu_{i_k-j_l}(0,0)\right]_{0\le k,l<n} S_{i_0}(t)S_{i_1-1}(t)\dots S_{i_{n-1}-(n-1)}(t)\\
&\qquad\qquad \times S_{j_0}(s)S_{j_1-1}(s)\dots S_{j_{n-1}-(n-1)}(s)\\
&= \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} \sum_{\sigma_1,\sigma_2\in S_n} (-1)^{\sigma_1}(-1)^{\sigma_2} \det\left[\mu_{i_k-j_l}(0,0)\right]_{0\le k,l<n}\\
&\quad \times\ S_{i_{\sigma_1(0)}}(t)\, S_{i_{\sigma_1(1)}-1}(t)\ \dots\ S_{i_{\sigma_1(n-1)}-(n-1)}(t)\\
&\quad \times\ S_{j_{\sigma_2(0)}}(s)\, S_{j_{\sigma_2(1)}-1}(s)\ \dots\ S_{j_{\sigma_2(n-1)}-(n-1)}(s)\\
&= \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}}\, S_{i_{n-1}-(n-1),\dots,i_0}(t)\, S_{j_{n-1}-(n-1),\dots,j_0}(s).
\end{aligned}
$$

4. Master symmetries of the Ablowitz-Ladik hierarchy

Following an idea introduced in [**42**] in the context of the 1-dimensional Toda lattices, we define the following vector fields on the Toeplitz bi-moments

$$\boxed{V_j\mu_k = (k+j)\mu_{k+j}, \quad \forall j\in\mathbb{Z}.} \tag{163}$$

These vector fields trivially satisfy the commutation relations

$$[V_i,V_j] = (j-i)V_{i+j} \tag{164}$$

$$[V_i,T_j] = jT_{i+j}, \quad \forall i,j\in\mathbb{Z}, \tag{165}$$

from which it follows that

$$[[V_i,T_j],T_j] = j[T_{i+j},T_j] = 0, \quad \forall i,j\in\mathbb{Z}. \tag{166}$$

Equations (164), (165) and (166) mean that the vector fields V_j, $j\in\mathbb{Z}$, form a Virasoro algebra of master symmetries, in the sense of Fuchssteiner [**37**], for the Ablowitz-Ladik hierarchy.

We shall establish the next result:

Theorem 4.1 (Haine-Vanderstichelen [**44**]). *For all $k\in\mathbb{Z}$, we have*

$$
\begin{aligned}
&L_k^{(n)}\tau_n(t,s) =\\
&\sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} V_k\Big(p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}}\Big) S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s),
\end{aligned}
$$

with $L_k^{(n)}, k \in \mathbb{Z}$, *defined as in* (54), (55), (56), *and* $V_k\left(p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}}\right)$ *the Lie derivative of the Plücker coordinates* (162) *in the direction of the master symmetries* V_k *of the Ablowitz-Ladik hierarchy, as defined in* (163).

Thus the operators $L_k^{(n)}, k \in \mathbb{Z}$, precisely describe the master symmetries of the Ablowitz-Ladik hierarchy on the tau functions of this hierarchy. Since master symmetries are usually connected with a bi-hamiltonian structure in the sense of Magri [**52**] (see [**24**] for an overview, and also [**53**] and [**40**] for connexions with bispectral problems), it suggests investigating the existence of a bi-Hamiltonian structure for the Ablowitz-Ladik hierarchy, and in particular the link with the recursion operator for this hierarchy that was found in [**33**].

The plan of the rest of this section is as follows. First we shall translate the master symmetries on the Plücker coordinates $p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}}$. Next we shall compute the action of the Virasoro operators on the products of Schur polynomials $S_{i_{n-1}-(n-1),\ldots,i_0}(t)S_{j_{n-1}-(n-1),\ldots,j_0}(s)$. Finally we shall end with the proof of Theorem 4.1.

4.1. Some algebraic lemmas. We shall need the following lemmas. In order to formulate them, we introduce some notations. Given n vectors $x_1, \ldots, x_n \in \mathbb{R}^n$, we shall denote by $|x_1x_2\ldots x_n|$ the determinant of the $n \times n$ matrix formed with the columns x_i. Also, given two vectors x and y, $x \wedge y$ denotes the usual wedge product, with components $(x \wedge y)_{rs} = x_ry_s - x_sy_r$. Finally, for an $n \times n$ matrix A, A_r will denote the rth column of A, and A_r^T the rth column of the transposed matrix, and $tr(A)$ will mean the trace of A. With these conventions, we have the following lemma.

Lemma 4.2 (Haine-Semengue [**42**])**.** *Let* A *and* B *be* $n\times n$ *matrices, with* A *invertible. Then*

$$(i)\ \sum_{r=1}^{n} |A_1 \ldots A_{r-1}B_rA_{r+1}\ldots A_n| = (\det A)tr(BA^{-1}),$$

$$(ii)\ \sum_{1\le r<s\le n} |A_1 \ldots A_{r-1}B_rA_{r+1}\ldots A_{s-1}B_sA_{s+1}\ldots A_n|$$

$$= (\det A)\sum_{1\le r<s\le n}\big((BA^{-1})_r \wedge (BA^{-1})_s\big)_{rs}.$$

PROOF. (i) Let A, B be $n \times n$ matrices, with A invertible. As A is invertible, its colums form a basis of $\mathbb{C}^n$ and thus we have

$$B_r = Ac^{(r)} = \sum_j c_j^{(r)} A_j, \tag{167}$$

for a certain $c^{(r)} \in \mathbb{C}^n$, whose components are $c_j^{(r)} = (A^{-1}B)_{jr}$. It then follows that

$$\begin{aligned}
\sum_{r=1}^{n} &|A_1 \dots A_{r-1} B_r A_{r+1} \dots A_n| \\
&= \sum_{r=1}^{n} |A_1 \dots A_{r-1} \big(\sum_j c_j^{(r)} A_j\big) A_{r+1} \dots A_n| \\
&= \det A \sum_{r=1}^{n} c_r^{(r)} \\
&= (\det A) tr(BA^{-1}).
\end{aligned}$$

(ii) Using (167), we have

$$\begin{aligned}
&\sum_{1 \le r < s \le n} |A_1 \dots A_{r-1} B_r A_{r+1} \dots A_{s-1} B_s A_{s+1} \dots A_n| \\
&= \sum_{1 \le r < s \le n} |A_1 \dots A_{r-1} \big(\sum_j c_j^{(r)} A_j\big) A_{r+1} \dots \\
&\qquad\qquad A_{s-1} \big(\sum_j c_j^{(s)} A_j\big) A_{s+1} \dots A_n| \\
&= \sum_{1 \le r < s \le n} |A_1 \dots A_{r-1} \big(c_r^{(r)} A_r + c_s^{(r)} A_s\big) A_{r+1} \dots \\
&\qquad\qquad A_{s-1} \big(c_r^{(s)} A_r + c_s^{(s)} A_s\big) A_{s+1} \dots A_n| \\
&= \det A \sum_{1 \le r < s \le n} (c_r^{(r)} c_s^{(s)} - c_s^{(r)} c_r^{(s)}) \\
&= \det A \sum_{1 \le r < s \le n} \big((A^{-1}B)_r \wedge (A^{-1}B)_s\big)_{rs}.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\sum_{1 \le r < s \le n} &|A_1 \dots A_{r-1} B_r A_{r+1} \dots A_{s-1} B_s A_{s+1} \dots A_n| \\
&= \det A \sum_{1 \le r < s \le n} \big((BA^{-1})_r \wedge (BA^{-1})_s\big)_{rs},
\end{aligned}$$

where we have used the fact that for X, Y two $n \times n$ matrices, we have

$$\sum_{1 \le r < s \le n} \big((XY)_r \wedge (XY)_s\big)_{rs} = \sum_{1 \le r < s \le n} \big((YX)_r \wedge (YX)_s\big)_{rs}. \tag{168}$$

This concludes the proof of the lemma. $\square$

We will also need a transposed version of this lemma.

Lemma 4.3. *With the same conditions as in Lemma 4.2, we have*

$$(i)\ \sum_{r=1}^{n} |A_1^T \dots A_{r-1}^T (B)_r^T A_{r+1}^T \dots A_n^T| = \sum_{r=1}^{n} |A_1 \dots A_{r-1} B_r A_{r+1} \dots A_n|,$$

$$(ii)\ \sum_{1 \le r < s \le n} |A_1^T \dots A_{r-1}^T B_r^T A_{r+1}^T \dots A_{s-1}^T B_s^T A_{s+1}^T \dots A_n^T| = \sum_{1 \le r < s \le n} |A_1 \dots A_{r-1} B_r A_{r+1} \dots A_{s-1} B_s A_{s+1} \dots A_n|.$$

PROOF. Both formulas are direct consequences of Lemma 4.2, by observing that for X, Y two $n \times n$ matrices, we have (168) and

$$(X_r^T \wedge X_s^T)_{rs} = (X_r \wedge X_s)_{rs}.$$

$\square$

We give two consequences of this lemma. First we particularize the preceding lemma to the Plücker coordinates, and then we particularize it to the Schur polynomials.

Lemma 4.4. *For $m \in \mathbb{Z}$ we have*

$$(i)\ \sum_{l=1}^{n} p_{\substack{i_0,\dots,i_{n-1} \\ j_0,\dots,j_{n-l}-m,\dots,j_{n-1}}} = \sum_{l=1}^{n} p_{\substack{i_0,\dots,i_{n-l}+m,\dots,i_{n-1} \\ j_0,\dots,j_{n-1}}},$$

$$(ii)\ \sum_{1 \le r < s \le n} p_{\substack{i_0,\dots,i_{n-1} \\ j_0,\dots,j_{n-s}-m,\dots,j_{n-r}-m,\dots,j_{n-1}}} = \sum_{1 \le r < s \le n} p_{\substack{i_0,\dots,i_{n-s}+m,\dots,i_{n-r}+m,\dots,i_{n-1} \\ j_0,\dots,j_{n-1}}}.$$

PROOF. Define the $n \times n$ matrices

$$A = (\mu_{i_k - j_l})_{0 \le k,l \le n-1}, \quad \text{and} \quad B(m) = (\mu_{i_k - j_l + m})_{0 \le k,l \le n-1}.$$

We then have

$$
\begin{aligned}
&\sum_{l=1}^{n} p^{\,i_0,\ldots,i_{n-1}}_{\,j_0,\ldots,j_{n-l}-m,\ldots,j_{n-1}} \\
&\quad = \sum_{l=1}^{n} \left| A_1 \ldots A_{n-l-1} \big(B(m)\big)_{n-l} A_{n-l+1} \ldots A_{n-1} \right| \\
&\quad = \sum_{l=1}^{n} \left| A_1^T \ldots A_{n-l-1}^T \big(B(m)\big)_{n-l}^T A_{n-l+1}^T \ldots A_{n-1}^T \right|, \\
&\quad = \sum_{l=1}^{n} p_{\substack{i_0,\ldots,i_{n-l}+m,\ldots,i_{n-1}, \\ j_0,\ldots,j_{n-1}}}
\end{aligned}
$$

where we have used Lemma 4.3(i) in the second equality. This proves (i). The proof of (ii) is similar. □

Lemma 4.5. *The following holds*

$$
(i)\ \sum_{l=1}^{n} S_{i_{n-1}-(n-1),\ldots,i_{n-l}-(n-l)-1,\ldots,i_0}(t)
= \det \begin{pmatrix}
S_{i_{n-1}-n}(t) & S_{i_{n-1}-(n-2)}(t) & \cdots & S_{i_{n-1}}(t) \\
S_{i_{n-2}-n}(t) & S_{i_{n-2}-(n-2)}(t) & \cdots & S_{i_{n-2}}(t) \\
\vdots & \vdots & & \vdots \\
S_{i_0-n}(t) & S_{i_0-(n-2)}(t) & \cdots & S_{i_0}(t)
\end{pmatrix},
$$

$$
(ii)\ \sum_{l=1}^{n-1} S_{i_{n-1}-(n-1),\ldots,i_{n-l}-(n-l)+1,\ldots,i_1-1}(t)
= \det \begin{pmatrix}
S_{i_{n-1}-(n-1)}(t) & \cdots & S_{i_{n-1}-2}(t) & S_{i_{n-1}}(t) \\
S_{i_{n-2}-(n-1)}(t) & \cdots & S_{i_{n-2}-2}(t) & S_{i_{n-2}}(t) \\
\vdots & & \vdots & \vdots \\
S_{i_1-(n-1)}(t) & \cdots & S_{i_1-2}(t) & S_{i_1}(t)
\end{pmatrix}.
$$

PROOF. We prove (i). Define the $n \times n$ matrices

$$
\begin{aligned}
A &= \big(S_{i_{n-k}-(n-k)+l-k}(t)\big)_{1\le k,l\le n}, \\
B(m) &= \big(S_{i_{n-k}-(n-k)+l-k+m}(t)\big)_{1\le k,l\le n}.
\end{aligned}
$$

We have $S_{i_{n-1}-(n-1),\dots,i_0}(t) = \det A$. It then follows that

$$\sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t) = \sum_{l=1}^{n} \left| A_1^T \dots A_{n-l-1}^T \big(B(-1)\big)_{n-l}^T A_{n-l+1}^T \dots A_{n-1}^T \right|.$$

Using Lemma 4.3(i) we get

$$\sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t) = \sum_{l=1}^{n} \left| A_1 \dots A_{n-l-1} \big(B(-1)\big)_{n-l} A_{n-l+1} \dots A_{n-1} \right|.$$

In the right-hand side, in the l^{th} term, the l^{th} and $(l-1)^{th}$ colums coincide in the determinant, provided that $l \neq 1$. Consequently, only the first term of the right-hand side gives a non zero contribution. This proves (i). The proof of (ii) is similar. $\square$

4.2. Expression of the master symmetries on the Plücker coordinates. We now translate the master symmetries on Plücker coordinates.

Lemma 4.6. *Let $V_k p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}}$ denote the Lie derivative of the Plücker coordinates in the direction of the vector fields V_k. Then for $k \in \mathbb{Z}$,*

$$\begin{aligned} &V_k p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \\ &= \sum_{l=0}^{n-1} (i_l + k) p_{\substack{i_0,\dots,i_{l-1},i_l+k,i_{l+1},\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} - \sum_{l=0}^{n-1} j_l p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{l-1},j_l-k,j_{l+1},\dots,j_{n-1}}} \\ &= \sum_{l=0}^{n-1} i_l p_{\substack{i_0,\dots,i_{l-1},i_l+k,i_{l+1},\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} - \sum_{l=0}^{n-1} (j_l - k) p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{l-1},j_l-k,j_{l+1},\dots,j_{n-1}}} . \end{aligned} \tag{169}$$

PROOF. Fix $0 \leq i_0 < i_1 < \dots < i_{n-1}$ and $0 \leq j_0 < j_1 < \dots < j_{n-1}$. We introduce the $n \times n$ matrices

$$A = \big(\mu_{i_k - j_l}(0,0)\big)_{0 \leq k,l \leq n-1}, \quad B(m) = \big(\mu_{i_k - j_l + m}(0,0)\big)_{0 \leq k,l \leq n-1},$$

as well as the diagonal matrix $D = \mathrm{diag}(j_0, \dots, j_{n-1})$. We notice that $p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} = \det A$, by definition of the Plücker coordinates. From the definition of V_k and using

Leibniz's rule we find for $k \in \mathbb{Z}$

$$V_k p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} = \sum_{l=0}^{n-1} \begin{vmatrix} \mu_{i_0-j_0} & \mu_{i_0-j_1} & \cdots & \mu_{i_0-j_{n-1}} \\ \vdots & \vdots & & \vdots \\ \mu_{i_{l-1}-j_0} & \mu_{i_{l-1}-j_1} & \cdots & \mu_{i_{l-1}-j_{n-1}} \\ (i_l-j_0+k)\mu_{i_l-j_0+k} & (i_l-j_1+k)\mu_{i_l-j_1+k} & \cdots & (i_l-j_{n-1}+k)\mu_{i_l-j_{n-1}+k} \\ \mu_{i_{l-1}-j_0} & \mu_{i_{l-1}-j_1} & \cdots & \mu_{i_{l-1}-j_{n-1}} \\ \vdots & \vdots & & \vdots \\ \mu_{i_{n-1}-j_0} & \mu_{i_{n-1}-j_1} & \cdots & \mu_{i_{n-1}-j_{n-1}} \end{vmatrix},$$

or equivalently,

$$\begin{aligned} V_k p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} &= \sum_{l=0}^{n-1} (i_l+k) p_{\substack{i_0,\dots,i_{l-1},i_l+k,i_{l+1},\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \\ &\qquad - \sum_{l=1}^{n} \left| A_1^T \dots A_{l-1}^T \big(B(k)D\big)_l^T A_{l+1}^T \dots A_n^T \right|. \end{aligned}$$

Using Lemma 4.3 we obtain

$$\begin{aligned} V_k p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} &= \sum_{l=0}^{n-1} (i_l+k) p_{\substack{i_0,\dots,i_{l-1},i_l+k,i_{l+1},\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \\ &\quad - \sum_{l=1}^{n} \left| A_1 \dots A_{l-1} \big(B(k)D\big)_l A_{l+1} \dots A_n \right| \\ &= \sum_{l=0}^{n-1} (i_l+k) p_{\substack{i_0,\dots,i_{l-1},i_l+k,i_{l+1},\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \\ &\quad - \sum_{l=1}^{n} j_{l-1} \left| A_1 \dots A_{l-1} \big(B(k)\big)_l A_{l+1} \dots A_n \right|. \end{aligned}$$

This gives the first equality in (169). The second equality in (169) can be derived from the first one by using Lemma 4.4(i). □

4.3. Action of the Virasoro operators $L_k^{(n)}$ on the Schur polynomials. Next we shall compute the action of the Virasoro operators on the products of Schur polynomials $S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s)$. In (79) we found the following expressions for the Virasoro operators

$$\begin{aligned} L_k^{(n)} &= A_k^{(n)} - B_{-k}^{(n)} + \frac{1}{2}\sum_{j=1}^{k-1} (a_j - b_{-j})(a_{k-j} - b_{j-k}), \quad k \geq 1 \\ L_0^{(n)} &= A_0^{(n)} - B_0^{(n)}, \\ L_{-k}^{(n)} &= A_{-k}^{(n)} - B_k^{(n)} - \frac{1}{2}\sum_{j=1}^{k-1} (a_{-j} - b_j)(a_{j-k} - b_{k-j}), \quad k \geq 1, \end{aligned} \tag{170}$$

with $A_k^{(n)}$ and $B_k^{(n)}$ defined in (77), or (78). We have the following lemma.

Lemma 4.7.

$$(i)\ L_0^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s) = \sum_{l=0}^{n-1} (i_l - j_l) S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s),$$

$$\begin{aligned}(ii)\ L_1^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s) &= \sum_{l=1}^{n} i_{n-l} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s) \\ &\quad - \sum_{l=1}^{n} (j_{n-l}+1) S_{i_{n-1}-(n-1),\dots,i_0}(t) \\ &\qquad \times S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s),\end{aligned}$$

$$\begin{aligned}(iii)\ L_2^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s) &= \sum_{l=1}^{n} i_{n-l} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-2,\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s) \\ &\quad + \sum_{1\le k<l\le n} S_{i_{n-1}-(n-1),\dots,i_{n-k}-(n-k)-1,\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t) \\ &\qquad \times S_{j_{n-1}-(n-1),\dots,j_0}(s) \\ &\quad - \sum_{l=1}^{n} (j_{n-l}+2) S_{i_{n-1}-(n-1),\dots,i_0}(t) \\ &\qquad \times S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+2,\dots,j_0}(s) \\ &\quad - \sum_{1\le k<l\le n} S_{i_{n-1}-(n-1),\dots,i_0}(t) \\ &\qquad \times S_{j_{n-1}-(n-1),\dots,j_{n-k}-(n-k)+1,\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s) \\ &\quad + s_1 \sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s) \\ &\quad - s_1 \sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s).\end{aligned}$$

PROOF. By using Leibniz's rule we have for $j \geq 1$,

$$\frac{\partial}{\partial t_j} S_{i_{n-1}-(n-1),\dots,i_0}(t) = \sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-j,\dots,i_0}(t), \tag{171}$$

and

$$\frac{\partial^2}{\partial t_1^2} S_{i_{n-1}-(n-1),\dots,i_0}(t) = \sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-2,\dots,i_0}(t) + 2 \sum_{1\leq r<s\leq n} S_{i_{n-1}-(n-1),\dots,i_{n-r}-(n-r)-1,\dots,i_{n-s}-(n-s)-1,\dots,i_0}(t). \tag{172}$$

Define the following $n \times n$ matrices

$$A(t) := \begin{pmatrix} S_{i_{n-1}-(n-1)}(t) & \dots & S_{i_{n-1}}(t) \\ \vdots & & \vdots \\ S_{i_0-(n-1)}(t) & \dots & S_{i_0}(t) \end{pmatrix},$$

$$B(j;t) := \begin{pmatrix} S_{i_{n-1}-(n-1)-j}(t) & \dots & S_{i_{n-1}-j}(t) \\ \vdots & & \vdots \\ S_{i_0-(n-1)-j}(t) & \dots & S_{i_0-j}(t) \end{pmatrix},$$

and $D = diag(n-1, n-2, \dots, 0)$. We shall denote $\hat{A}(s)$ and $\hat{B}(j;s)$ the same matrices with $t \to s$ and $(i_0, \dots, i_{n-1}) \to (j_0, \dots, j_{n-1})$. From the definition of the elementary Schur polynomials we have for $j \geq 0$,

$$\sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+j}} S_i(t) = (i-j) S_{i-j}(t),$$

$$\sum_{k=j+1}^{\infty} kt_k \frac{\partial}{\partial t_{k-j}} S_i(t) = (i+j) S_{i+j}(t) - \sum_{1\leq l\leq j} lt_l S_{i+j-l}(t).$$

Consequently, by first using Leibniz's rule and then Lemma 4.2(i) we have for $j \geq 0$

$$\sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+j}} S_{i_{n-1}-(n-1),\dots,i_0}(t) = \sum_{l=1}^{n} (i_{n-l}-j) S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-j,\dots,i_0}(t) - \big(\det A(t)\big) tr\big(A(t)^{-1} B(j,t) D\big), \tag{173}$$

$$
\begin{aligned}
\sum_{k=j+1}^{\infty} kt_k \frac{\partial}{\partial t_{k-j}} S_{i_{n-1}-(n-1),\ldots,i_0}(t)& \\
&= \sum_{l=1}^{n} (i_{n-l}+j) S_{i_{n-1}-(n-1),\ldots,i_{n-l}-(n-l)+j,\ldots,i_0}(t) \\
&- \big(\det A(t)\big) tr\big(A(t)^{-1} B(-j,t) D\big) \\
&- \sum_{m=1}^{j} mt_m \big(\det A(t)\big) tr\big(A(t)^{-1} B(m-j,t)\big). \qquad (174)
\end{aligned}
$$

We are now ready to prove the lemma.

(i) From (170), we have $L_0^{(n)} = A_0^{(n)} - B_0^{(n)}$. Using (173) with $j=0$, we obtain

$$
\begin{aligned}
&A_0^{(n)} S_{i_{n-1}-(n-1),\ldots,i_0}(t) \\
&\quad = \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_k} S_{i_{n-1}-(n-1),\ldots,i_0}(t) + \frac{n^2}{2} S_{i_{n-1}-(n-1),\ldots,i_0}(t) \\
&\quad = \sum_{l=1}^{n} i_{n-l} S_{i_{n-1}-(n-1),\ldots,i_0}(t) - \big(\det A(t)\big) tr\big(A(t)^{-1} B(0,t) D\big) \\
&\qquad + \frac{n^2}{2} S_{i_{n-1}-(n-1),\ldots,i_0}(t).
\end{aligned}
$$

We have $B(0,t) = A(t)$, and thus

$$
\begin{aligned}
\big(\det A(t)\big) tr\big(A(t)^{-1} B(0,t) D\big) &= \big(\det A(t)\big) tr(D) \\
&= \frac{n(n-1)}{2} S_{i_{n-1}-(n-1),\ldots,i_0}(t).
\end{aligned}
$$

Consequently, we get

$$
A_0^{(n)} S_{i_{n-1}-(n-1),\ldots,i_0}(t) = \Big[\sum_{l=1}^{n} i_{n-l} + \frac{n}{2} \Big] S_{i_{n-1}-(n-1),\ldots,i_0}(t).
$$

Similarly, we get

$$
B_0^{(n)} S_{j_{n-1}-(n-1),\ldots,j_0}(s) = \Big[\sum_{l=1}^{n} j_{n-l} + \frac{n}{2} \Big] S_{j_{n-1}-(n-1),\ldots,j_0}(s).
$$

Combining both equations, we obtain (i).

(ii) We have $L_1^{(n)} = A_1^{(n)} - B_{-1}^{(n)}$. We compute, using (171) and (173)

$$\begin{aligned} A_1^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) &= \Big[\sum_{j=1}^{\infty} j t_j \frac{\partial}{\partial t_{j+1}} + n\frac{\partial}{\partial t_1}\Big] S_{i_{n-1}-(n-1),\dots,i_0}(t) \\ &= \sum_{l=1}^{n} \big(i_{n-l}+n-1\big) S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t) \\ &\quad - \big(\det A(t)\big) tr\big(A(t)^{-1}B(1,t)D\big) \end{aligned}$$

By virtue of Lemma 4.2(i), we have

$$\big(\det A(t)\big) tr\big(A(t)^{-1}B(1,t)D\big) = (n-1)\big|\big(B(1,t)\big)_1 A_2(t)\dots A_n(t)\big|.$$

But by virtue of Lemma 4.5(i), this is equal to

$$\begin{aligned} &\big(\det A(t)\big) tr\big(A(t)^{-1}B(1,t)D\big) \\ &\qquad = (n-1)\sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t). \end{aligned}$$

Hence, we obtain

$$A_1^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) = \sum_{l=1}^{n} i_{n-l} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t). \tag{175}$$

Similarly, we have using (174)

$$\begin{aligned} B_{-1}^{(n)} S_{j_{n-1}-(n-1),\dots,j_0}(s) &= \Big[\sum_{j=2}^{\infty} j s_j \frac{\partial}{\partial s_{j-1}} + n s_1\Big] S_{j_{n-1}-(n-1),\dots,j_0}(s) \\ &= \sum_{l=1}^{n} (j_{n-l}+1) S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s) \\ &\quad - \big(\det \hat{A}(s)\big) tr\big(\hat{A}(s)^{-1}\hat{B}(-1,s)D\big) \\ &\quad - s_1\big(\det \hat{A}(s)\big) tr\big(\hat{A}(s)^{-1}\hat{B}(0,s)\big) + n s_1 S_{j_{n-1}-(n-1),\dots,j_0}(s). \end{aligned}$$

We have using Lemma 4.2(i)

$$\big(\det \hat{A}(s)\big) tr\big(\hat{A}(s)^{-1}\hat{B}(-1,s)D\big) = 0,$$

and, obviously, we also have

$$\big(\det \hat{A}(s)\big) tr\big(\hat{A}(s)^{-1}\hat{B}(0,s)\big) = n S_{j_{n-1}-(n-1),\dots,j_0}(s).$$

Consequently we obtain

$$\begin{aligned} &B_{-1}^{(n)} S_{j_{n-1}-(n-1),\dots,j_0}(s) \\ &\qquad = \sum_{l=1}^{n} (j_{n-l}+1) S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s). \end{aligned} \tag{176}$$

Substracting (175) and (176) gives (ii).

(iii) From (170), we have

$$L_2^{(n)} = A_2^{(n)} - B_{-2}^{(n)} + \frac{1}{2}\Big(\frac{\partial}{\partial t_1} - s_1\Big)^2.$$

We study separately the contributions of the three terms in the operator $L_2^{(n)}$ on the product of Schur functions. We start with the contribtion of $A_2^{(n)}$. We compute, using (171), (172) and (173)

$$\begin{aligned}
&A_2^{(n)} S_{i_{n-1}-(n-1),\ldots,i_0}(t)\\
&\quad = \Big[\frac{1}{2}\frac{\partial^2}{\partial t_1^2} + \sum_{j=1}^{\infty} j t_j \frac{\partial}{\partial t_{j+2}} + n\frac{\partial}{\partial t_2}\Big] S_{i_{n-1}-(n-1),\ldots,i_0}(t)\\
&\quad = \sum_{l=1}^{n} \big(i_{n-l} + n - \frac{3}{2}\big) S_{i_{n-1}-(n-1),\ldots,i_{n-l}-(n-l)-2,\ldots,i_0}(t)\\
&\qquad + \sum_{1\le k<l\le n} S_{i_{n-1}-(n-1),\ldots,i_{n-k}-(n-k)-1,\ldots,i_{n-l}-(n-l)-1,\ldots,i_0}(t)\\
&\qquad - \big(\det A(t)\big) tr\big(A(t)^{-1}B(2,t)D\big).
\end{aligned}$$

The last term in this equation gives by developing the trace

$$\begin{aligned}
&\big(\det A(t)\big) tr\big(A(t)^{-1}B(2,t)D\big)\\
&\quad = \big(\det A(t)\big)\Big[(n-1)\big(A(t)^{-1}B(2,t)\big)_{11} + (n-2)\big(A(t)^{-1}B(2,t)\big)_{22}\Big].
\end{aligned}$$

We have

$$\begin{aligned}
&\big(\det A(t)\big) tr\big(A(t)^{-1}B(2,t)\big)\\
&\qquad\qquad = \big(\det A(t)\big)\Big[\big(A(t)^{-1}B(2,t)\big)_{11} + \big(A(t)^{-1}B(2,t)\big)_{22}\Big],
\end{aligned}$$

and by a short computation

$$\begin{aligned}
&\big(A(t)^{-1}B(2,t)\big)_{22}\\
&\qquad\qquad = -\sum_{1\le k<l\le n}\Big(\big(A(t)^{-1}B(1,t)\big)_k \wedge \big(A(t)^{-1}B(1,t)\big)_l\Big)_{kl}.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
\big(\det A(t)\big) tr\big(A(t)^{-1}B(2,t)D\big) &= (n-1)\big(\det A(t)\big) tr\big(A(t)^{-1}B(2,t)\big)\\
&\quad + \big(\det A(t)\big)\sum_{1\le k<l\le n}\Big(\big(A(t)^{-1}B(1,t)\big)_k \wedge \big(A(t)^{-1}B(1,t)\big)_l\Big)_{kl}.
\end{aligned}$$

Using Lemma 4.2, we obtain

$$
\begin{aligned}
&\big(\det A(t)\big) tr\big(A(t)^{-1}B(2,t)D\big)\\
&=(n-1)\sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-2,\dots,i_0}(t)\\
&\quad+\sum_{1\le k<l\le n} S_{i_{n-1}-(n-1),\dots,i_{n-k}-(n-k)-1,\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
(177)\qquad & A_2^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t)\\
&= \sum_{l=1}^{n}\big(i_{n-l}-\frac{1}{2}\big) S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-2,\dots,i_0}(t).
\end{aligned}
$$

We now turn to the contribution of $B_{-2}^{(n)}$. We have using (174)

$$
\begin{aligned}
&B_{-2}^{(n)} S_{j_{n-1}-(n-1),\dots,j_0}(s)\\
&= \Big[\frac{1}{2}s_1^2+\sum_{j=3}^{\infty} jt_j\frac{\partial}{\partial s_{j-2}}+2ns_2\Big] S_{j_{n-1}-(n-1),\dots,j_0}(s)\\
&= \Big[\frac{1}{2}s_1^2+2ns_2\Big] S_{j_{n-1}-(n-1),\dots,j_0}(s)\\
&\quad+\sum_{l=1}^{n}\big(j_{n-l}+2\big) S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+2,\dots,j_0}(s)\\
&\quad-\big(\det \hat{A}(s)\big) tr\big(\hat{A}(s)^{-1}\hat{B}(-2,s)D\big)\\
&\quad-\sum_{m=1}^{2} ms_m\big(\det \hat{A}(s)\big) tr\big(\hat{A}(s)^{-1}\hat{B}(m-2,s)\big).
\end{aligned}
$$

By a similar argument as above, we have

$$
\begin{aligned}
&\big(\det \hat{A}(s)\big) tr\big(\hat{A}(s)^{-1}\hat{B}(-2,s)D\big)\\
&= -\big(\det \hat{A}(s)\big)\sum_{1\le k<l\le n}\Big(\big(\hat{A}(s)^{-1}\hat{B}(-1,s)\big)_k\wedge\big(\hat{A}(s)^{-1}\hat{B}(-1,s)\big)_l\Big)_{kl},
\end{aligned}
$$

and thus using Lemma 4.2(ii), we obtain

$$
\begin{aligned}
&\big(\det \hat{A}(s)\big) tr\big(\hat{A}(s)^{-1}\hat{B}(-2,s)D\big)\\
&= -\sum_{1\le k<l\le n} S_{j_{n-1}-(n-1),\dots,j_{n-k}-(n-k)+1,\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s).
\end{aligned}
$$

We also have, using Lemma 4.2(i),

$$\sum_{m=1}^{2} ms_m\big(\det \hat{A}(s)\big)tr\big(\hat{A}(s)^{-1}\hat{B}(m-2,s)\big)$$
$$= s_1\sum_{l=1}^{n} S_{j_{n-1}-(n-1),\ldots,j_{n-l}-(n-l)+1,\ldots,j_0}(s)+2ns_2S_{j_{n-1}-(n-1),\ldots,j_0}(s).$$

Consequently, we have

$$\begin{aligned}
&B_{-2}^{(n)}S_{j_{n-1}-(n-1),\ldots,j_0}(s)\\
&\quad=\sum_{l=1}^{n}\big(j_{n-l}+2\big)S_{j_{n-1}-(n-1),\ldots,j_{n-l}-(n-l)+2,\ldots,j_0}(s)\\
&\qquad+\sum_{1\leq k<l\leq n} S_{j_{n-1}-(n-1),\ldots,j_{n-k}-(n-k)+1,\ldots,j_{n-l}-(n-l)+1,\ldots,j_0}(s)\\
&\qquad-s_1\sum_{l=1}^{n}S_{j_{n-1}-(n-1),\ldots,j_{n-l}-(n-l)+1,\ldots,j_0}(s)\\
&\qquad+\frac{1}{2}s_1^2S_{j_{n-1}-(n-1),\ldots,j_0}(s).
\end{aligned}\tag{178}$$

Finally, we turn to the contribution of the term $\frac{1}{2}\Big(\frac{\partial}{\partial t_1}-s_1\Big)^2$. We have using (171) and (172)

$$\begin{aligned}
&\frac{1}{2}\Big[\frac{\partial}{\partial t_1}-s_1\Big]^2S_{i_{n-1}-(n-1),\ldots,i_0}(t)\\
&=\frac{1}{2}\Big[\frac{\partial^2}{\partial t_1^2}-2s_1\frac{\partial}{\partial t_1}+s_1^2\Big]S_{i_{n-1}-(n-1),\ldots,i_0}(t)\\
&=\frac{1}{2}\sum_{l=1}^{n}S_{i_{n-1}-(n-1),\ldots,i_{n-l}-(n-l)-2,\ldots,i_0}(t)\\
&\quad+\sum_{1\leq k<l\leq n}S_{i_{n-1}-(n-1),\ldots,i_{n-k}-(n-k)-1,\ldots,i_{n-l}-(n-l)-1,\ldots,i_0}(t)\\
&\quad-s_1\sum_{l=1}^{n}S_{i_{n-1}-(n-1),\ldots,i_{n-l}-(n-l)-1,\ldots,i_0}(t)+\frac{1}{2}s_1^2S_{i_{n-1}-(n-1),\ldots,i_0}(t).
\end{aligned}\tag{179}$$

Combining (177), (178) and (179), we obtain (iii). □

Remark 4.8. *We observe that by definition of the operators $L_k^{(n)}$ we have*

$$L_{-k}^{(n)}S_{i_{n-1}-(n-1),\ldots,i_0}(t)S_{j_{n-1}-(n-1),\ldots,j_0}(s)$$
$$=-L_k^{(n)}S_{i_{n-1}-(n-1),\ldots,i_0}(t)S_{j_{n-1}-(n-1),\ldots,j_0}(s)\Big|_{\substack{t\leftrightarrow s\\(i_0,\ldots,i_{n-1})\leftrightarrow(j_0,\ldots,j_{n-1})}}.$$

4.4. Expression of the master symmetries on the manifold of tau functions: proof of the main theorem. We now turn to the last part of this section. We will prove Theorem 4.1. We first prove the following lemma.

Lemma 4.9.

$$\sum_{l=1}^{n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} S_{i_{n-1}-(n-1),\ldots,i_0}(t) \times S_{j_{n-1}-(n-1),\ldots,j_{n-l}-(n-l)+1,\ldots,j_0}(s)$$

$$+ \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ -1,j_1,\ldots,j_{n-1}}} S_{i_{n-1}-(n-1),\ldots,i_0}(t)\, S_{j_{n-1}-(n-1),\ldots,j_1-1,0}(s)$$

$$= \sum_{l=1}^{n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} S_{i_{n-1}-(n-1),\ldots,i_{n-l}-(n-l)-1,\ldots,i_0}(t) \times S_{j_{n-1}-(n-1),\ldots,j_0}(s). \tag{180}$$

PROOF. For simplicity, we will use the notations

$$\mathcal{S}_i(t) = S_{i_{n-1}-(n-1),\ldots,i_0}(t), \quad \mathcal{S}_j(s) = S_{j_{n-1}-(n-1),\ldots,j_0}(s), \tag{181}$$

when no 'special' shift on the indices of the Schur functions occur. Relabeling each term in the first sum of the left-hand side of (180) in the following way $j_{n-l} \mapsto j_{n-l}-1$ gives

$$\sum_{l=1}^{n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\ldots,j_{n-l}-(n-l)+1,\ldots,j_0}(s)$$

$$= \sum_{l=1}^{n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-l}-1<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-l}-1,\ldots,j_{n-1}}} \mathcal{S}_i(t)\, \mathcal{S}_j(s).$$

On the one hand, for a fixed $1 \le l \le n-1$, if $j_{n-l} = j_{n-l-1}+1$, then $p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-l}-1,\ldots,j_{n-1}}} = 0$. On the other hand, for a fixed $2 \le l \le n$, if $j_{n-l} = j_{n-l+1}$, then $S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l),\dots,j_0}(s) = 0$. Therefore

$$\sum_{l=1}^{n} \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s)$$

$$= \sum_{l=1}^{n} \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-l}-1,\dots,j_{n-1}}} \mathcal{S}_i(t)\, \mathcal{S}_j(s)$$

$$- \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0< j_1<\dots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s).$$

Consequently, the left-hand side of (180) is equal to

$$\sum_{l=1}^{n} \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s)$$

$$+ \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0< j_1<\dots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s)$$

$$= \sum_{l=1}^{n} \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-l}-1,\dots,j_{n-1}}} \mathcal{S}_i(t)\, \mathcal{S}_j(s). \tag{182}$$

Similarly, one can show that the right-hand side of (180) is equal to

$$\sum_{l=1}^{n} \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t)\, \mathcal{S}_j(s)$$

$$= \sum_{l=1}^{n} \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-l}+1,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\, \mathcal{S}_j(s). \tag{183}$$

By virtue of Lemma 4.4, (182) and (183) are equal. □

Proof of Theorem 4.1: We will prove the theorem for $k \ge 0$. The case $k < 0$ is similar. Using the Plücker expansion of $\tau_n(t)$, and Lemmas 4.6 and 4.7 we have for $k = 0, 1$, using the notations (181),

$$V_k\tau_n(s,t) = \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} V_k p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\, \mathcal{S}_j(s)$$

$$= \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} L_k^{(n)} \mathcal{S}_i(t)\, \mathcal{S}_j(s)$$

$$= L_k^{(n)}\tau_n(s,t),$$

where, in the second equality, we have performed some relabeling of the indices as in the proof of Lemma 4.9. We will finish the proof with the case $k = 2$, for which we provide some more details, but first we prove the theorem for general $k \geq 3$. We proceed by induction. Assume the theorem holds for some $k \geq 2$. We will establish it for $k+1$. The argument follows from the commutation relations (71) and (164). We have

$$\begin{aligned}(k-1)V_{k+1}\tau_n(s,t) &= \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} [V_1, V_k] p_{\substack{i_0,\dots,i_{n-1}\\j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\mathcal{S}_j(s)\\ &= \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\j_0,\dots,j_{n-1}}} [L_k^{(n)}, L_1^{(n)}]\mathcal{S}_i(t)\,\mathcal{S}_j(s)\\ &= (k-1)L_{k+1}^{(n)}\tau_n(s,t),\end{aligned}$$

where in the second equality we have used the induction hypothesis.

We now provide some details for the case $k = 2$. Using Lemmas 4.7 and 4.9 we have

$$\begin{aligned}&L_2^{(n)}\tau_n(s,t) = T_1 + T_2 + T_3 + T_4\\ &\quad - s_1 \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\-1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s),\end{aligned} \tag{184}$$

with

$$\begin{aligned}T_1 &:= \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\j_0,\dots,j_{n-1}}}\\ &\qquad \times \sum_{l=1}^{n} i_{n-l} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-2,\dots,i_0}(t)\,\mathcal{S}_j(s),\\ T_2 &:= -\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\j_0,\dots,j_{n-1}}}\\ &\qquad \times \sum_{l=1}^{n}(j_{n-l}+2)\mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+2,\dots,j_0}(s),\\ T_3 &:= \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\j_0,\dots,j_{n-1}}}\\ &\quad \times \sum_{1\le k<l\le n} S_{i_{n-1}-(n-1),\dots,i_{n-k}-(n-k)-1,\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t)\,\mathcal{S}_j(s),\end{aligned}$$

$$T_4 := -\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} \times \sum_{1\le k<l\le n} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\ldots,j_{n-k}-(n-k)+1,\ldots,j_{n-l}-(n-l)+1,\ldots,j_0}(s).$$

We will consider separately the four terms T_1, T_2, T_3, T_4. By arguments similar to that used in the proof of Lemma 4.9, and using the fact that $S_{i_{n-1}-(n-1),\ldots,i_0}(t) = 0$ if $i_k < 0$ for some $0 \le k \le n-1$, we get for T_1

$$T_1 = \sum_{l=1}^{n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2) p_{\substack{i_0,\ldots,i_{n-l}+2,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} \mathcal{S}_i(t)\,\mathcal{S}_j(s)$$
$$+ \sum_{l=1}^{n-1} \sum_{\substack{-1\le i_0-1<\cdots<i_{n-l-1}-1\\ =i_{n-l}<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2) p_{\substack{i_0,\ldots,i_{n-l}+2,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} \mathcal{S}_i(t)\,\mathcal{S}_j(s)$$
$$- \sum_{l=2}^{n} \sum_{\substack{-1\le i_0-1<\cdots<i_{n-l-1}-1\\ <i_{n-l}+1=i_{n-l+1}<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2) p_{\substack{i_0,\ldots,i_{n-l}+2,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} \times\, \mathcal{S}_i(t)\,\mathcal{S}_j(s).$$

The two last terms in this expression annihilate, i.e.

$$0 = \sum_{l=1}^{n-1} \sum_{\substack{-1\le i_0-1<\cdots<i_{n-l-1}-1\\ =i_{n-l}<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2) p_{\substack{i_0,\ldots,i_{n-l}+2,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} \times\, \mathcal{S}_i(t)\,\mathcal{S}_j(s)$$
$$- \sum_{l=2}^{n} \sum_{\substack{-1\le i_0-1<\cdots<i_{n-l-1}-1\\ <i_{n-l}+1=i_{n-l+1}<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2) p_{\substack{i_0,\ldots,i_{n-l}+2,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} \times\, \mathcal{S}_i(t)\,\mathcal{S}_j(s). \tag{185}$$

Indeed, we have for $1 \leq l \leq n-1$

$$\sum_{\substack{-1\leq i_0-1<\cdots<i_{n-l-1}-1\\ =i_{n-l}<\cdots<i_{n-1}\\ 0\leq j_0<\cdots<j_{n-1}}} (i_{n-l}+2)p_{\substack{i_0,\ldots,i_{n-l}+2,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}}\mathcal{S}_i(t)\,\mathcal{S}_j(s) =$$

$$\sum_{\substack{-1\leq k_0-1<\cdots<k_{n-l-1}\\ =k_{n-l}-1<\cdots<k_{n-1}\\ 0\leq j_0<\cdots<j_{n-1}}} (k_{n-l-1}+2)p_{\substack{k_0,\ldots,k_{n-l-2},k_{n-l},k_{n-l-1}+2,k_{n-l+1},\ldots,k_{n-1}\\ j_0,\ldots,j_{n-1}}}\times$$

$$S_{k_{n-1}-(n-1),\ldots,k_{n-l+1}-(n-l+1),k_{n-l-1}-(n-l),k_{n-l}-(n-l-1),k_{n-l-2}-(n-l-2),\ldots,k_0}(t) \times \mathcal{S}_j(s),$$

where we have made the relabeling $i_{n-l-1} \mapsto k_{n-l}$, $i_{n-l} \mapsto k_{n-l-1}$, and $i_m \mapsto k_m$ if $m \neq n-l-1, n-l$. As the Plücker coordinates and the Schur functions are determinants, we have, permuting lines in the determinants,

$$p_{\substack{k_0,\ldots,k_{n-l-2},k_{n-l},k_{n-l-1}+2,k_{n-l+1},\ldots,k_{n-1}\\ j_0,\ldots,j_{n-1}}} = -p_{\substack{k_0,\ldots,k_{n-l-1}+2,\ldots,k_{n-1}\\ j_0,\ldots,j_{n-1}}},$$

and

$$S_{k_{n-1}-(n-1),\ldots,k_{n-l+1}-(n-l+1),k_{n-l-1}-(n-l),k_{n-l}-(n-l-1),k_{n-l-2}-(n-l-2),\ldots,k_0}(t) = -\mathcal{S}_k(t),$$

and hence

$$\sum_{\substack{-1\leq i_0-1<\cdots<i_{n-l-1}-1=i_{n-l}<\cdots<i_{n-1}\\ 0\leq j_0<\cdots<j_{n-1}}} (i_{n-l}+2)p_{\substack{i_0,\ldots,i_{n-l}+2,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} \times \mathcal{S}_i(t)\,\mathcal{S}_j(s)$$

$$= \sum_{\substack{-1\leq k_0-1<\cdots<k_{n-l-1}=k_{n-l}-1<\cdots<k_{n-1}\\ 0\leq j_0<\cdots<j_{n-1}}} (k_{n-l-1}+2)p_{\substack{k_0,\ldots,k_{n-l-1}+2,\ldots,k_{n-1}\\ j_0,\ldots,j_{n-1}}} \times \mathcal{S}_k(t)\,\mathcal{S}_j(s).$$

Summing this expression for $1 \leq l \leq n-1$, and relabeling $l \mapsto l-1$ we get (185). Consequently we obtain

$$T_1 = \sum_{l=1}^{n} \sum_{\substack{0\leq i_0<\cdots<i_{n-1}\\ 0\leq j_0<\cdots<j_{n-1}}} (i_{n-l}+2)p_{\substack{i_0,\ldots,i_{n-l}+2,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}}\mathcal{S}_i(t)\,\mathcal{S}_j(s). \tag{186}$$

By similar arguments, we have

$$T_2 = -\sum_{l=1}^{n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} j_{n-l} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-l}-2,\dots,j_{n-1}}} \mathcal{S}_i(t)\,\mathcal{S}_j(s)$$

$$+ \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t)\,S_{j_{n-1}-(n-1),\dots,j_1-1,1}(s), \tag{187}$$

$$T_3 = \sum_{1\le k<l\le n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-l}+1,\dots,i_{n-k}+1,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\,\mathcal{S}_j(s), \tag{188}$$

$$T_4 = -\sum_{1\le k<l\le n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-l}-1,\dots,j_{n-k}-1,\dots,j_{n-1}}} \mathcal{S}_i(t)\mathcal{S}_j(s)$$

$$+ \sum_{1\le k\le n-1} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-k}-1,\dots,j_{n-1}}}$$

$$\times\ \mathcal{S}_i(t)\,S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s). \tag{189}$$

Substituting (186), (187), (188) and (189) in (184), using Lemmas 4.4 and 4.6 we obtain

$$L_2^{(n)}\tau_n(s,t) = \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} V_2 p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\,\mathcal{S}_j(s)$$

$$+ \sum_{k=1}^{n-1} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-k}-1,\dots,j_{n-1}}} \mathcal{S}_i(t)\,S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s)$$

$$- s_1 \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t)\,S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s)$$

$$+ \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t)\,S_{j_{n-1}-(n-1),\dots,j_1-1,1}(s).$$

We prove that the last three terms in this expression annihilate

$$
\begin{aligned}
0 = &\sum_{k=1}^{n-1} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ -1,j_1,\ldots,j_{n-k}-1,\ldots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\ldots,j_1-1,0}(s)\\
&- s_1 \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ -1,j_1,\ldots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\ldots,j_1-1,0}(s)\\
&+ \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ -1,j_1,\ldots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\ldots,j_1-1,1}(s), \qquad (190)
\end{aligned}
$$

and hence

$$
L_2^{(n)}\tau_n(s,t) = \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} V_2 p_{\substack{i_0,\ldots,i_{n-1}\\ j_0,\ldots,j_{n-1}}} \mathcal{S}_i(t)\, \mathcal{S}_j(s). \qquad (191)
$$

Indeed, developping the determinant $S_{j_{n-1}-(n-1),\ldots,j_1-1,1}(s)$ with respect to the last line, using the fact that the first elementary Schur polynomials are $S_0(s) = 1$ and $S_1(s) = s_1$, and Lemma 4.5(ii), we have

$$
\begin{aligned}
&\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ -1,j_1,\ldots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\ldots,j_1-1,1}(s)\\
&= s_1 \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ -1,j_1,\ldots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\ldots,j_1-1}(s)\\
&\quad - \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ -1,j_1,\ldots,j_{n-1}}} \mathcal{S}_i(t)\\
&\qquad\qquad \times \sum_{l=1}^{n-1} S_{j_{n-1}-(n-1),\ldots,j_{n-l}-(n-l)+1,\ldots,j_1-1}(s).
\end{aligned}
$$

By an argument similar to that of the proof of Lemma 4.9, we get

$$
\begin{aligned}
&\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ -1,j_1,\ldots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\ldots,j_1-1,1}(s)\\
&= s_1 \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ -1,j_1,\ldots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\ldots,j_1-1}(s)\\
&\quad - \sum_{l=1}^{n-1} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1}\\ -1,j_1,\ldots,j_{n-l}-1,\ldots,j_{n-1}}} \mathcal{S}_i(t)\, S_{j_{n-1}-(n-1),\ldots,j_1-1}(s).
\end{aligned}
$$

Noticing that $S_{j_{n-1}-(n-1),\ldots,j_1-1}(s) = 0$ when $j_1 = 0$, and

$$
S_{j_{n-1}-(n-1),\ldots,j_1-1}(s) = S_{j_{n-1}-(n-1),\ldots,j_1-1,0}(s),
$$

when $j_1 > 0$, we get (190), and hence (191). This proves the case $k = 2$ and finishes the proof.

$\square$

5. A Lax pair for the master symmetries

In the preceding sections, we have defined on the bi-moments vector fields V_k, $k \in \mathbb{Z}$, defining the master symmetries of the Ablowitz-Ladik hierarchy. We have obtained the expression of these vector fields on the manifold of τ-functions. In this section we translate the action of the vector fields V_k, $k \in \mathbb{Z}$, on the manifold of CMV-matrices A_1.

We first decompose the vector fields V_k defined in (163)

$$V_k = kT_k + \mathscr{V}_k,$$

where T_k are the Ablowitz-Ladik vector fields. At the level of the bi-moments, the vector fields $\mathscr{V}_k$ are given by

$$\boxed{\mathscr{V}_k \;:\; \frac{d}{du_k}\mu_j = j\mu_{j+k}, \qquad j,k \in \mathbb{Z},} \tag{192}$$

or equivalently, at the level of the bi-moment matrix $\tilde{m}_\infty$

$$\boxed{\frac{d\tilde{m}_\infty}{du_k} = \tilde{D}\tilde{\Lambda}^k\tilde{m}_\infty - \tilde{\Lambda}^k\tilde{m}_\infty\tilde{D} = [\tilde{D}, \tilde{\Lambda}^k\tilde{m}_\infty].} \tag{193}$$

These vector fields satisfy the following commutation relations

$$[\mathscr{V}_i, \mathscr{V}_j] = (j-i)\mathscr{V}_{i+j},$$
$$[\mathscr{V}_i, T_j] = jT_{i+j}.$$

It follows that

$$[[\mathscr{V}_i, T_j], T_k] = 0, \quad \forall i,j,k \in \mathbb{Z}.$$

Consequently, like the vector fields V_k, the vector fields $\mathscr{V}_j$, $j \in \mathbb{Z}$, form a Virasoro algebra of master symmetries for the Ablowitz-Ladik hierarchy.

Lemma 4.10. *We have for $k \in \mathbb{Z}$*

$$\frac{d\tilde{S}_1}{du_k}\,\tilde{S}_1^{-1} = -\big(D_1\,A_1^{k+1}\big)_{--} - \big(A_1^{k+1}\,D_1^{*}\big)_{--}, \tag{194}$$

$$(\tilde{S}_2^T\,h^{-1})^{-1}\,\frac{d(\tilde{S}_2^T\,h^{-1})}{du_k} = -\big(D_2\,A_2^{1-k}\big)_{--} - \big(A_2^{1-k}\,D_2^{*}\big)_{--}. \tag{195}$$

PROOF. By substituting the factorisation $\tilde{m}_\infty = \tilde{S}_1^{-1}\,\tilde{S}_2$ of the moment matrix into (193), we obtain

$$-\tilde{S}_1^{-1}\,\frac{d\tilde{S}_1}{du_k}\,\tilde{S}_1^{-1}\,\tilde{S}_2 + \tilde{S}_1^{-1}\,\frac{d\tilde{S}_2}{du_k} = \tilde{D}\,\tilde{\Lambda}^k\,\tilde{S}_1^{-1}\,\tilde{S}_2 - \tilde{\Lambda}^k\,\tilde{S}_1^{-1}\,\tilde{S}_2\,\tilde{D}.$$

Multiplying this equation on the left by $\tilde{S}_1$ and on the right by $\tilde{S}_2^{-1}$, we get

$$-\frac{d\tilde{S}_1}{du_k}\,\tilde{S}_1^{-1} + \frac{d\tilde{S}_2}{du_k}\,\tilde{S}_2^{-1} = \underbrace{\tilde{S}_1\,\tilde{D}\,\tilde{\Lambda}^k\,\tilde{S}_1^{-1}}_{\text{Term 1}} - \underbrace{\tilde{S}_1\,\tilde{\Lambda}^k\,\tilde{S}_1^{-1}\,\tilde{S}_2\,\tilde{D}\,\tilde{S}_2^{-1}}_{\text{Term 2}}. \tag{196}$$

Using the factorisation of A_1 given in Theorem 2.29 and the factorisation of D_1 in (124), Term 1 gives

$$\begin{aligned}\text{Term 1} &= \tilde{S}_1\,\tilde{D}\,\tilde{\Lambda}^T\,\tilde{\Lambda}^{k+1}\,\tilde{S}_1^{-1}\\ &= \big(\tilde{S}_1\,\tilde{D}\,\tilde{\Lambda}^T\,\tilde{S}_1^{-1}\big)\big(\tilde{S}_1\,\tilde{\Lambda}^{k+1}\,\tilde{S}_1^{-1}\,\big) = D_1 A_1^{k+1}.\end{aligned}$$

Similarly, the second term gives

$$\text{Term 2} = A_1^k\,\tilde{S}_2\,\tilde{D}\,\tilde{S}_2^{-1} = A_1^{k+1}\,A_1^{-1}\,\tilde{S}_2\,\tilde{D}\,\tilde{S}_2^{-1}.$$

Using the factorisation of A_1^{-1} in Corollary 2.30 we get

$$\begin{aligned}\text{Term 2} &= A_1^{k+1}\,\big(\tilde{S}_2\,\tilde{\Lambda}^T\,\tilde{S}_2^{-1}\big)\,\tilde{S}_2\,\tilde{D}\,\tilde{S}_2^{-1}\\ &= A_1^{k+1}\,\big(\tilde{S}_2\,\tilde{\Lambda}^T\,\tilde{D}\,\tilde{S}_2^{-1}\big) = -A_1^{k+1}\,D_1^{*},\end{aligned}$$

where we have used the expression of D_1^* in Lemma 2.34. Substituting these results in (196), we obtain

$$-\frac{d\tilde{S}_1}{du_k}\,\tilde{S}_1^{-1} + \frac{d\tilde{S}_2}{du_k}\,\tilde{S}_2^{-1} = D_1\,A_1^{k+1} + A_1^{k+1}\,D_1^{*}.$$

The first term in the left-hand side is strictly lower triangular, while the second term in the left-hand side is upper triangular. Consequently, taking the strictly lower triangular part in both sides, we obtain

$$\frac{d\tilde{S}_1}{du_k}\,\tilde{S}_1^{-1} = -\big(D_1\,A_1^{k+1}\big)_{--} - \big(A_1^{k+1}\,D_1^{*}\big)_{--},$$

which establishes (194).

To establish the other formula, we substitute the factorisation $\tilde{m}_\infty = (\tilde{S}_1^{-1}\, h)\,(h^{-1}\,\tilde{S}_2)$ into equation (193) rewritten as

$$\frac{d\tilde{m}_\infty}{du_k} = [\tilde{D}, \tilde{m}_\infty\, \tilde{\Lambda}^k],$$

which follows from the commutation relation (130). This gives

$$\begin{aligned}\frac{d(\tilde{S}_1^{-1}\, h)}{du_k}\,(h^{-1}\,\tilde{S}_2) + (\tilde{S}_1^{-1}\, h)\,\frac{d(h^{-1}\,\tilde{S}_2)}{du_k}& \\ = \tilde{D}\,(\tilde{S}_1^{-1}\, h)\,(h^{-1}\,\tilde{S}_2)\,\tilde{\Lambda}^k - (\tilde{S}_1^{-1}\, h)\,(h^{-1}\,\tilde{S}_2)\,\tilde{\Lambda}^k\,\tilde{D}&.\end{aligned}$$

Multiplying this equation on the left by $(\tilde{S}_1^{-1}\, h)^{-1}$ and on the right by $(h^{-1}\,\tilde{S}_2)^{-1}$, we get

$$\begin{aligned}(\tilde{S}_1^{-1}\, h)^{-1}\,\frac{d(\tilde{S}_1^{-1}\, h)}{du_k} + \frac{d(h^{-1}\,\tilde{S}_2)}{du_k}\,(h^{-1}\,\tilde{S}_2)^{-1} \\ = \underbrace{(\tilde{S}_1^{-1}\, h)^{-1}\,\tilde{D}\,(\tilde{S}_1^{-1}\, h)\,(h^{-1}\,\tilde{S}_2)\,\tilde{\Lambda}^k\,(h^{-1}\,\tilde{S}_2)^{-1}}_{\text{Term 1}} \\ - \underbrace{(h^{-1}\,\tilde{S}_2)\,\tilde{\Lambda}^k\,\tilde{D}\,(h^{-1}\,\tilde{S}_2)^{-1}}_{\text{Term 2}}.\end{aligned} \tag{197}$$

Using the factorisation of A_2 in Theorem 2.29 and the factorisation of D_2 in (125), Term 2 gives

$$\begin{aligned}\text{Term 2} &= (h^{-1}\,\tilde{S}_2)\,\tilde{\Lambda}^{k-1}\,\tilde{\Lambda}\,\tilde{D}\,(h^{-1}\,\tilde{S}_2)^{-1} \\ &= (h^{-1}\,\tilde{S}_2)\,\tilde{\Lambda}^{k-1}\,(h^{-1}\,\tilde{S}_2)^{-1}\,(h^{-1}\,\tilde{S}_2)\,\tilde{\Lambda}\,\tilde{D}\,(h^{-1}\,\tilde{S}_2)^{-1} \\ &= (A_2^T)^{1-k}\,D_2^T.\end{aligned}$$

Similarly, using the factorisation of A_2 in Theorem 2.29, Term 1 gives

$$\begin{aligned}\text{Term 1} &= (\tilde{S}_1^{-1}\, h)^{-1}\,\tilde{D}\,(\tilde{S}_1^{-1}\, h)\,(A_2^T)^{-k} \\ &= (\tilde{S}_1^{-1}\, h)^{-1}\,\tilde{D}\,(\tilde{S}_1^{-1}\, h)\,(A_2^T)^{-1}(A_2^T)^{1-k}.\end{aligned}$$

Using the factorisation of A_2^{-1} in Corollary 2.30 and the factorisation of D_2^* in (125), we get

$$\text{Term 1} = (h^{-1}\,\tilde{S}_1)\,\tilde{D}\,\tilde{\Lambda}\,(h^{-1}\,\tilde{S}_1)^{-1}\,(A_2^T)^{1-k} = -(D_2^*)^T\,(A_2^T)^{1-k}.$$

Substituting these results in the transpose of (197), we obtain

$$\begin{aligned}\frac{d(\tilde{S}_1^{-1}\, h)^T}{du_k}\big((\tilde{S}_1^{-1}\, h)^T\big)^{-1} + (\tilde{S}_2^T\, h^{-1})^{-1}\,\frac{d(\tilde{S}_2^T\, h^{-1})}{du_k}& \\ = -D_2\,A_2^{1-k} - A_2^{1-k}\,D_2^*&.\end{aligned}$$

Since $(\tilde{S}_1^{-1}\, h)^T$ is upper triangular and $\tilde{S}_2^T\, h^{-1}$ is lower triangular with diagonal elements equal to 1, by taking the strictly lower part of both sides of this equation, we obtain (195). This concludes the proof of the lemma. □

We are now able to obtain a Lax pair representation for the master symmetries vector fields $\mathcal{V}_k, k \in \mathbb{Z}$.

Theorem 4.11 (Haine-Vanderstichelen [**44**]). *The "dressed up" form of the moment equation* (193) *gives the following Lax pair representation for the master symmetries vector fields $\mathcal{V}_k$ on the semi-infinite CMV matrices (A_1, A_2)*

$$\boxed{\begin{aligned} \tfrac{d}{du_k} A_1 &= \Big[A_1\,,\, \big(D_1\, A_1^{k+1}\big)_{--} + \big(A_1^{k+1}\, D_1^*\big)_{--}\Big], &\quad \forall k \in \mathbb{Z},\\ \tfrac{d}{du_k} A_2 &= \Big[\big(D_2\, A_2^{1-k}\big)_{--} + \big(A_2^{1-k}\, D_2^*\big)_{--}\,,\, A_2\Big], &\quad \forall k \in \mathbb{Z}, \end{aligned}} \tag{198}$$

or equivalently

$$\boxed{\begin{aligned} \tfrac{d}{du_k} A_1 &= A_1^{k+1} + \Big[\big(D_1\, A_1^{k+1}\big)_{+} - \big(A_1^{k+1}\, D_1^*\big)_{--}\,,\, A_1\Big], &\quad \forall k \in \mathbb{Z},\\ \tfrac{d}{du_k} A_2 &= A_2^{1-k} + \Big[A_2\,,\, \big(A_2^{1-k}\, D_2^*\big)_{+} - \big(D_2\, A_2^{1-k}\big)_{--}\Big], &\quad \forall k \in \mathbb{Z}. \end{aligned}}$$

PROOF. As $A_1 = \tilde{S}_1\, \tilde{\Lambda}\, \tilde{S}_1^{-1}$ and $A_2 = (\tilde{S}_2^T\, h^{-1})^{-1}\, \tilde{\Lambda}\, (\tilde{S}_2^T\, h^{-1})$, we have

$$\begin{aligned} \frac{dA_1}{du_k} &= \Big[\frac{d\tilde{S}_1}{du_k}\, \tilde{S}_1^{-1}\,,\, A_1\Big]\\ \frac{dA_2}{du_k} &= \Big[A_2\,,\, (\tilde{S}_2^T\, h^{-1})^{-1}\, \frac{d(\tilde{S}_2^T\, h^{-1})}{du_k}\Big]. \end{aligned}$$

Using (194) and (195) in Lemma 4.10, we obtain (198).

From the commutation relations (123), one readily obtains that

$$\begin{aligned} &\big[A_1\,,\, \big(D_1\, A_1^{k+1}\big)_{+}\big] + \big[A_1\,,\, \big(D_1\, A_1^{k+1}\big)_{--}\big] = A_1^{k+1},\\ &\big[(A_2^{1-k}\, D_2^*)_{+}\,,\, A_2\big] + \big[\big(A_2^{1-k}\, D_2^*\big)_{--}\,,\, A_2\big] = A_2^{1-k}, \end{aligned}$$

which gives the equivalent formulation for the representation of the master symmetries on the CMV-matrices (A_1, A_2). This concludes the proof. $\square$

As a consequence, it is easy to obtain the Lax pair representation of the master symmetries V_k on the manifold of CMV-matrices.

Corollary 4.12 (Haine-Vanderstichelen [**44**]). *On the CMV-matrices (A_1, A_2), the master symmetries V_k, $\forall k \in \mathbb{Z}$, admit the Lax pair representation*

$$\begin{aligned} V_k(A_1) &= \Big[A_1\,,\, \big(D_1\, A_1^{k+1}\big)_{--} + \big(A_1^{k+1}\, D_1^*\big)_{--} + k\big(A_1^{k}\big)_{--}\Big],\\ V_k(A_2) &= \Big[\big(D_2\, A_2^{1-k}\big)_{--} + \big(A_2^{1-k}\, D_2^*\big)_{--} - k\big(A_2^{-k}\big)_{--}\,,\, A_2\Big]. \end{aligned}$$

Using the explicit form of the CMV matrices (A_1, A_2) in Theorem 2.31 and Theorem 4.11, one can compute the first few master symmetry vector fields $\mathscr{V}_2, \mathscr{V}_{-1}, \mathscr{V}_0, \mathscr{V}_1$ in terms of the variables x_n, y_n.

$$
\begin{aligned}
\mathscr{V}_{-2}(x_n) = {} & (n-4)x_{n-2}(1-x_{n-1}y_{n-1})(1-x_ny_n) \\
& - x_{n-1}(1-x_ny_n)\big((n-4)x_{n-1}y_n + (n-1)x_ny_{n+1}\big) \\
& - 2x_{n-1}(1-x_ny_n)\sum_{k=1}^{n} y_kx_{k-1} + x_n\sum_{k=1}^{n} y_k^2x_{k-1}^2 \\
& - 2x_n\sum_{k=2}^{n} y_kx_{k-2} + 2x_n\sum_{k=2}^{n} y_ky_{k-1}x_{k-1}x_{k-2}, \\
\mathscr{V}_{-2}(y_n) = {} & -ny_{n+2}(1-x_ny_n)(1-x_{n+1}y_{n+1}) \\
& + y_{n+1}(1-x_ny_n)\big(nx_ny_{n+1} + (n-1)x_{n-1}y_n\big) \\
& + 2y_{n+1}(1-x_ny_n)\sum_{k=1}^{n} y_kx_{k-1} - y_n\sum_{k=1}^{n} y_k^2x_{k-1}^2 \\
& + 2y_n\sum_{k=2}^{n} y_kx_{k-2} - 2y_n\sum_{k=2}^{n} y_ky_{k-1}x_{k-1}x_{k-2},
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{V}_{-1}(x_n) &= (n-2)x_{n-1}(1-x_ny_n) - x_n\sum_{k=1}^{n} y_kx_{k-1}, \\
\mathscr{V}_{-1}(y_n) &= -ny_{n+1}(1-x_ny_n) + y_n\sum_{k=1}^{n} y_kx_{k-1},
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{V}_{0}(x_n) &= nx_n, \\
\mathscr{V}_{0}(y_n) &= -ny_n,
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{V}_{1}(x_n) &= nx_{n+1}(1-x_ny_n) - x_n\sum_{k=1}^{n} x_ky_{k-1}, \\
\mathscr{V}_{1}(y_n) &= -(n-2)y_{n-1}(1-x_ny_n) + y_n\sum_{k=1}^{n} x_ky_{k-1}.
\end{aligned}
$$

Part 2

Non-intersecting Brownian motions

Non-intersecting Brownian motion models are closely related to Hermitian random matrix ensembles, as was shown by Aptekarev-Bleher-Kuijlaars [**15**], using a result by Karlin-McGregor [**49**]. Karlin and McGregor established a formula allowing one to compute the transition probability density $p_N(t,\vec{a},\vec{b})$ to find N independent Brownian particles starting in $a_1 < \cdots < a_N$ at time $t=0$ in positions $b_1,\ldots,b_N$ at a time $t>0$ without any two of them ever having been coincident during the time interval $[0,t]$. It is given in terms of the transition probability density of one Brownian particle on the real line

$$p(t,x,y)=\frac{1}{\sqrt{\pi t}}e^{\frac{-(x-y)^2}{t}},$$

by the following determinant

$$p_N(t,\vec{\alpha},\vec{\beta}):=\det\begin{pmatrix} p(t,a_1,b_1) & \cdots & p(t,a_1,b_N)\\ \vdots & & \vdots\\ p(t,a_N,b_1) & \cdots & p(t,a_N,b_N)\end{pmatrix}.$$

In the second part of this thesis, we will consider N independent Brownian motions during a time-interval $[0,1]$, conditioned to start at positions

$$(\alpha_1,\ldots,\alpha_N) \;=\; \big(\underbrace{a_1,a_1,\ldots,a_1}_{m_1},\underbrace{a_2,a_2,\ldots,a_2}_{m_2},\ldots,\underbrace{a_q,a_q,\ldots,a_q}_{m_q}\big),$$

at time $t=0$ and to end up in positions

$$(\beta_1,\ldots,\beta_N) \;=\; \big(\underbrace{b_1,b_1,\ldots,b_1}_{n_1},\underbrace{b_2,b_2,\ldots,b_2}_{n_2},\ldots,\underbrace{b_p,b_p,\ldots,b_p}_{n_p}\big),$$

at time $t=1$, with $\sum_{i=1}^q m_i = \sum_{i=1}^p n_i = N$, for general p and q, $a_1<a_2<\cdots<a_q$ and $b_1<b_2<\cdots<b_p$, without two of them ever having been coincident during that time-interval. For arbitrary p and q, it is not known if the distribution of the positions of the non-intersecting Brownian particles at a given time $0<t<1$, is the same as the joint distribution of the eigenvalues of a random matrix. Using the formula of Karlin-McGregor, the probability to find all the Brownian particles in a set $E=\cup_{i=1}^r[c_{2i-1},c_{2i}]\subset\mathbb{R}$ at an intermediate time $0<t<1$ is given by a block moment matrix

$$\begin{aligned}&\mathbb{P}^{a_1,\ldots,a_q}_{b_1,\ldots,b_p}\big(\text{all } x_i(t)\in E\big)\\ &\qquad\qquad =\frac{1}{Z_N}\det\left[\Big(\Big\langle x^m\psi_i(x)\Big|y^n\varphi_j(y)\Big\rangle\Big)_{\substack{0\le m\le m_i-1\\0\le n\le n_j-1}}\right]_{\substack{1\le i\le q\\1\le j\le p}},\end{aligned}$$

where $\psi_i(x)=e^{\tilde{a}_i x}$, $\varphi_j(y)=e^{\tilde{b}_j y}$, and the following inner product

$$\Big\langle x^m\psi_i(x)\Big|y^n\varphi_j(y)\Big\rangle=\int_{\tilde{E}}x^{m+n}e^{(\tilde{a}_i+\tilde{b}_j)x}e^{-\frac{x^2}{2}}dx.$$

The $\sim$'s in the former formula indicate that a space-time transformation has been performed. We prove in Chapter 6 the existence, for general p and q, of a partial differential equation (PDE) satisfied by the function $\log \mathbb{P}^{a_1,\dots,a_q}_{b_1,\dots,b_p}\big(\text{all } x_i(t) \in E\big)$. The variables are the coordinates of the starting and ending points of the particles, and the boundary points of the set E. The proof of the existence of such a PDE, using Virasoro constraints and the multicomponent KP hierarchy, is based on the method of elimination of the unwanted partials. We start in Chapter 5 with a definition of the KP and multi-component KP hierarchies.

Chapter 5

KP and multi-component KP hierarchy

The method of integrable deformations in random matrix theory and in the theory of non-intersecting Brownian motions on the real line, reduces certain problems to the study of moment matrices and block moment matrices with regard to one or several weights, deformed by adding one set of times for each weight function. It is proven in [**14**] that the determinants of these time-dependent (block) moment matrices are tau-functions for the KP and multi-component KP hierarchy, as they satisfy the bilinear identities which completely encode the hierarchies. The proof is based on the orthogonality conditions of the (multiple) orthogonal polynomials for the deformed weight functions.

1. KP hierarchy

This section is based on [**29**] for the introduction of the KP hierarchy using pseudo-differential operators, and on [**7**] for the link with time-dependent orthogonal polynomials.

1.1. Sato Theory.

1.1.1. *Notations.* A formal pseudo-differential operator of order m is a formal series

$$R = \sum_{j=-\infty}^{m} u_j(x)\partial^j,$$

where $\partial = \frac{\partial}{\partial x}$ and ∂^{-1} is the operator of formal integration, such that

$$\partial \cdot \partial^{-1} = \partial^{-1} \cdot \partial = 1,$$

and $u_j(x)$ are some functions of the variable x. Let $\mathcal{R}$ be the set of pseudo-differential operators of finite order. Two elements of $\mathcal{R}$ can be added together very naturally. Let $R = \sum_{j=-\infty}^{m} u_j(x)\partial^j$ and $\tilde{R} = \sum_{j=-\infty}^{n} v_j(x)\partial^j$ be pseudo-differential operators of order m and n respectively, with $m \geq n$, then their sum is a pseudo-differential operator of order not higher then m defined by

$$R + \tilde{R} = \sum_{j=-\infty}^{m} \big(u_j(x) + v_j(x)\big)\partial^j,$$

with the convention $v_j = 0$ if $j > n$. Multiplication of two pseudo-differential operators is defined by the following extension of Leibniz's rule

$$\partial^n \cdot f(x) = \sum_{j\geq 0} \binom{n}{j} \frac{\partial^j f(x)}{\partial x^j} \partial^{n-j}, \qquad \forall n \in \mathbb{Z},$$

where for $n \in \mathbb{Z}$, $j \geq 0$,

$$\binom{n}{j} = \frac{n(n-1)\cdots(n-j+1)}{j!}.$$

This makes the set $\mathcal{R}$ of pseudo-differential operators into an associative ring.

Consider a pseudo-differential operator of order m

$$R = \sum_{j=-\infty}^{m} u_j(x)\partial^j,$$

with $u_m(x) = 1$. Then there exists unique pseudo-differential operators

$$R^{-1} = \partial^{-m} + \sum_{j<-m} v_j(x)\partial^j, \qquad R^{1/m} = \partial + \sum_{j\leq 0} w_j(x)\partial^j,$$

such that $\left(R^{1/m}\right)^m = R$ and $R \cdot R^{-1} = R^{-1} \cdot R = 1$. These pseudo-differential operators commute with R, i.e. $[R, R^{-1}] = [R, R^{1/m}] = 0$. We also define the formal adjoint R^* of R, by

$$R^* = \sum_{j=-\infty}^{m} (-\partial)^j \cdot u_j(x).$$

For two pseudo-differential operators R_1 and R_2, one has $(R_1R_2)^* = R_2^*R_1^*$.

Finally, for a pseudo-differential operator $R = \sum_{j=-\infty}^{m} u_j(x)\partial^j$ we define

$$R_+ = \sum_{j=0}^{m} u_j(x)\partial^j$$

the differential part, and

$$R_- = \sum_{j<0} u_j(x)\partial^j$$

the integral (or Volterra) part of R.

1.1.2. *Lax form of the KP hierarchy.* Let L be a pseudo-differential operator of order 1,

$$L = \partial + u_1(x,t)\partial^{-1} + u_2(x,t)\partial^{-2} + \dots,$$

where u_j are functions of x and of a family of time parameters $t = (t_1, t_2, \dots)$.

Definition 5.1. *The Kadomtsev-Petviashvili hierarchy, or simply the KP hierarchy, is the set of differential equations in Lax form*

$$\frac{\partial L}{\partial t_n} = [B_n, L], \qquad n = 1, 2, \dots \tag{199}$$

where $B_n = (L^n)_+$.

At first, we observe that we can (and will) identify $x = t_1$. Indeed, the equation in the KP hierarchy corresponding to $n = 1$ is

$$\frac{\partial L}{\partial t_1} = [\partial, L],$$

or equivalently

$$\forall j \geq 1 \,:\, \frac{\partial u_j}{\partial t_1} = \frac{\partial u_j}{\partial x}.$$

The KP hierarchy is a compatible system, i.e. two or more equations can be solved simultanously. One can find functions u_i of two or more variables t_n satisfying the corresponding equations in the KP hierarchy with respect to each variable. This follows from the fact that the vector fields defined in (199) commute.

Proposition 5.2. *Equations* (199) *of the KP hierarchy are equivalent to the system of equations of Zakharov-Shabat type*

$$\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + [B_n, B_m] = 0, \qquad \forall n, m \geq 1. \tag{200}$$

Consequently the vector fields defined in (199) *commute.*

PROOF. 1. We first deduce (200) from (199). Equations (199) of the KP hierarchy imply for all $n, m \geq 1$

$$\frac{\partial L^n}{\partial t_m} = [B_m, L^n].$$

Hence, as $[L^n, L^m] = 0$ we have

$$\begin{aligned}\frac{\partial L^n}{\partial t_m} - \frac{\partial L^m}{\partial t_n} &= [B_n - L^n, B_m - L^m] - [B_n, B_m] \\ &= [(L^n)_-, (L^m)_-] - [B_n, B_m],\end{aligned}$$

where we have used the equality $L^n = B_n + (L^n)_-$. Taking the differential part of this equation gives the Zakharov-Shabat equations (200).

2. We now prove that (200) implies (199). From (200), using the decomposition $L^n = B_n + (L^n)_-$, we deduce

$$\frac{\partial L^n}{\partial t_m} - [B_m, L^n] = \frac{\partial}{\partial t_m}(L^n)_- + \frac{\partial B_m}{\partial t_n} - [B_m, (L^n)_-]. \tag{201}$$

In the right-hand side of this equation, we have three terms of which the orders satisfy

(1) $\operatorname{ord}\left(\frac{\partial}{\partial t_m}(L^n)_-\right) < 0$;
(2) $\operatorname{ord}\left(\frac{\partial B_m}{\partial t_n}\right) < m - 1$;
(3) $\operatorname{ord}\left([B_m, (L^n)_-]\right) < m - 1$.

Hence, for fixed m, the order of the left-hand side of (201) is bounded for all $n \geq 0$

$$\operatorname{ord}\left(\frac{\partial L^n}{\partial t_m} - [B_m, L^n]\right) \leq m - 1, \tag{202}$$

where $\operatorname{ord}(\,\cdot\,)$ is the order of the pseudo-differential operator in the brackets. Suppose now that

$$\frac{\partial L^n}{\partial t_m} - [B_m, L^n] \neq 0.$$

But it then follows that

$$\lim_{n \to +\infty} \operatorname{ord}\left(\frac{\partial L^n}{\partial t_m} - [B_m, L^n]\right) = +\infty,$$

which is in contradiction with (202).

3. We finally prove that the flows of the vector fields defined by (199) commute. We have, by virtue of (199)

$$\begin{aligned}\frac{\partial}{\partial t_m}\left(\frac{\partial L}{\partial t_n}\right) - \frac{\partial}{\partial t_n}\left(\frac{\partial L}{\partial t_m}\right) &= \frac{\partial}{\partial t_m}[B_n, L] - \frac{\partial}{\partial t_n}[B_m, L] \\ &= \left[\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n}, L\right] + \big[B_n, [B_m, L]\big] - \big[B_m, [B_n, L]\big].\end{aligned}$$

Using (200) we obtain

$$\frac{\partial}{\partial t_m}\Big(\frac{\partial L}{\partial t_n}\Big) - \frac{\partial}{\partial t_n}\Big(\frac{\partial L}{\partial t_m}\Big)$$
$$= \big[[B_m, B_n], L\big] + \big[B_n, [B_m, L]\big] - \big[B_m, [B_n, L]\big].$$

This is equal to zero due to the Jacobi identity. □

A consequence of the commutation of the flows of the vector fields defined in (199) is that each equation in the KP hierarchy generates a symmetry for the other equations.

Example 5.3. *We have*

$$B_2 = \partial^2 + 2u_1, \qquad B_3 = \partial^3 + 3u_1\partial + 3(u_1' + u_2),$$

where the prime stands for derivation with respect to the x-variable. Consequently taking $n = 3$ and $m = 2$ in the Zakharov-Shabat equations (200) *we obtain the system*

$$\begin{cases} 2\frac{\partial u_1}{\partial t_3} - 3\frac{\partial^2 u_1}{\partial t_1 \partial t_2} - 3\frac{\partial u_2}{\partial t_2} + \frac{\partial^3 u_1}{\partial t_1^3} + 3\frac{\partial^2 u_2}{\partial t_1^2} - 6u_1\frac{\partial u_1}{\partial t_1} = 0, \\ -3\frac{\partial u_1}{\partial t_2} + 3\frac{\partial^2 u_1}{\partial t_1^2} + 6\frac{\partial u_2}{\partial t_1} = 0. \end{cases}$$

Eliminating u_2 in these equations, and remembering that $t_1 = x$, we get

$$0 = 3\frac{\partial^2 u_1}{\partial t_2^2} - \frac{\partial}{\partial x}\Big(4\frac{\partial u_1}{\partial t_3} - \frac{\partial^3 u_1}{\partial x^3} - 12u_1\frac{\partial u_1}{\partial x}\Big).$$

We define $v = 2u_1$. This function then satisfies

$$0 = 3\frac{\partial^2 v}{\partial t_2^2} - \frac{\partial}{\partial x}\Big(4\frac{\partial v}{\partial t_3} - \frac{\partial^3 v}{\partial x^3} - 6v\frac{\partial v}{\partial x}\Big).$$

This is the classical form of the KP equation. It is related to the form (41) *of the KP equation through the identity*

$$v = 2\frac{\partial^2 \log \tau_n}{\partial t_1^2}.$$

1.1.3. *Wave function.* The KP hierarchy (199), and its equivalent description in the form of Zakharov-Shabat equations (200), appear as compatibility conditions of the linear problem for $n = 1, 2, \ldots$

$$L\psi = z\psi,$$
$$\frac{\partial \psi}{\partial t_n} = B_n\psi,$$

where ψ is an eigenfunction of the pseudo-differential operator L, and z is a spectral parameter. We are interested in the time evolution of the eigenfunctions ψ. To study this time evolution, we are going to compare the eigenfunctions ψ with the eigenfunctions of the constant operator ∂.

It is convenient to represent the pseudo-differential operator L by

$$L = S\,\partial\,S^{-1},$$

where S is a pseudo-differential operator of order 0

$$S = \sum_{j=0}^{\infty} w_j \partial^{-j},$$

with $w_0 = 1$. This representation is called the formal dressing of L, and S is called the wave operator. The coefficients of S are determined inductively, using the relation

$$LS = S\,\partial.$$

One gets

$$u_i = -w_i' + Q_i(w_1, \ldots, w_{i-1}), \qquad i = 1, 2, \ldots$$

where Q_i are differential polynomials. Notice that S is not unique. Indeed, the dressing operator S is determined up to multiplication on the right by a pseudo-differential operator of order 0 with constant coefficients $1 + \sum_{j=1}^{\infty} c_i \partial^{-i}$. We have the following proposition giving the time evolution of the wave operator.

Proposition 5.4. *The KP vector fields on S are given by*

$$\frac{\partial S}{\partial t_m} = -(L^m)_- S. \tag{203}$$

The flows commute.

PROOF. We check that the vector fields (203) induce the KP Lax equations on L. We have

$$\begin{aligned}
\frac{\partial L}{\partial t_m} &= \frac{\partial}{\partial t_m}(S\,\partial\,S^{-1}) \\
&= \Big(\frac{\partial S}{\partial t_m}\Big)\,\partial\,S^{-1} - S\,\partial\,S^{-1}\Big(\frac{\partial S}{\partial t_m}\Big)\,S^{-1} \\
&= -(L^m)_- S\,\partial\,S^{-1} + S\,\partial\,S^{-1}\,(L^m)_-\,SS^{-1} \\
&= [L, (L^m)_-] \\
&= [(L^m)_+, L].
\end{aligned}$$

These flows commute. Indeed, we have

$$\begin{aligned}
\frac{\partial}{\partial t_m}\Big(\frac{\partial S}{\partial t_n}\Big) &= -\Big(\frac{\partial L^n}{\partial t_m}\Big)_- S - (L^n)_- \frac{\partial S}{\partial t_m} \\
&= -[L^m_+, L^n]_- S + (L^n)_-(L^m)_- S \\
&= [L^m_-, L^n]_- S + (L^n)_-(L^m)_- S \\
&= [L^m_-, L^n_-] S + [L^m_-, L^n_+]_- S + (L^n)_-(L^m)_- S \\
&= L^m_- L^n_- S - [L^n_+, L^m]_- S \\
&= \frac{\partial}{\partial t_n}\Big(\frac{\partial S}{\partial t_m}\Big).
\end{aligned}$$

□

Pseudo-differential operators are not real operators. Indeed, their action on functions is not defined unless they are purely differential. We define the action of the pseudo-differential operators on the function $\exp\xi(t,z) = \exp\big(\sum_{k=1}^{\infty} t_k z^k\big)$, by

$$\begin{aligned}
&\partial^m \xi(t,z) = z^m, \\
&\partial^m \exp\xi(t,z) = z^m \exp\xi(t,z),
\end{aligned}$$

for all $m \in \mathbb{Z}$. The wave function is then defined as

$$\Psi(t,z) = S\exp\xi(t,z) = \Big[1 + \sum_{j=1}^{\infty} w_j(t) z^{-j}\Big] \exp\xi(t,z).$$

We define $\hat{\Psi}(t,z) = 1 + \sum_{j=1}^{\infty} w_j(t) z^{-j}$, such that

$$\Psi(t,z) = \hat{\Psi}(t,z)\exp\xi(t,z).$$

We also define the adjoint wave funtion

$$\begin{aligned}
\Psi^*(t,z) &= (S^*)^{-1}\exp\big(-\xi(t,z)\big) \\
&= \Big[1 + \sum_{j=1}^{\infty} w^*_j(t) z^{-j}\Big] \exp\big(-\xi(t,z)\big).
\end{aligned}$$

We define $\hat{\Psi}^*(t,z) = 1 + \sum_{j=1}^{\infty} w^*_j(t) z^{-j}$, such that

$$\Psi^*(t,z) = \hat{\Psi}^*(t,z)\exp\big(-\xi(t,z)\big).$$

It is then easy to prove that the wave and adjoint wave functions satisfy the following equations

$$\begin{aligned}
&L\Psi = z\Psi, \qquad \frac{\partial}{\partial t_m}\Psi = B_m\Psi, \\
&L^*\Psi^* = z\Psi^*, \qquad \frac{\partial}{\partial t_m}\Psi^* = -(L^m)^*_+\Psi^*.
\end{aligned}$$

We consider two types of residues :

$$\text{Res}_\partial \sum a_j \partial^j = a_{-1}, \qquad \text{Res}_z \sum a_j z^j = a_{-1}.$$

It is then easy to check that if P, Q are two pseudo-differential operators, then

$$\text{Res}_z \Big[\big(Pe^{xz}\big) \cdot \big(Qe^{-xz}\big) \Big] = \text{Res}_\partial \big(PQ^*\big). \tag{204}$$

Theorem 5.5. *The wave and adjoint wave functions Ψ and Ψ^* satisfy the following bilinear identity*

$$Res_z \Bigg[\Bigg(\Big(\frac{\partial}{\partial t_1}\Big)^{j_1} \Big(\frac{\partial}{\partial t_2}\Big)^{j_2} \cdots \Big(\frac{\partial}{\partial t_m}\Big)^{j_m} \Psi(t,z) \Bigg) \Psi^*(t,z) \Bigg] = 0,$$

for any $m \geq 1$ and multi-index $(j_1, j_2, \ldots, j_m)$, or equivalently

$$Res_z \Big[\Psi(t', z) \Psi^*(t,z) \Big] = 0,$$

for any t, t', where $\Psi(t', z)$ should be understood as a formal Taylor series expansion around t.

Proof. As $\frac{\partial}{\partial t_m} \Psi = B_m \Psi$, we have

$$\begin{aligned} \text{Res}_z \Bigg[\Bigg(\Big(\frac{\partial}{\partial t_1}\Big)^{j_1} \Big(\frac{\partial}{\partial t_2}\Big)^{j_2} \cdots \Big(\frac{\partial}{\partial t_m}\Big)^{j_m} \Psi(t,z) \Bigg) \Psi^*(t,z) \Bigg] \\ = \text{Res}_z \Big[\Big(R\Psi(t,z) \Big) \Psi^*(t,z) \Big], \end{aligned}$$

where R is some purely differential operator, i.e. $R_- = 0$. We then have

$$\begin{aligned} \text{Res}_z \Bigg[\Bigg(\Big(\frac{\partial}{\partial t_1}\Big)^{j_1} \Big(\frac{\partial}{\partial t_2}\Big)^{j_2} \cdots \Big(\frac{\partial}{\partial t_m}\Big)^{j_m} \Psi(t,z) \Bigg) \Psi^*(t,z) \Bigg] \\ = \text{Res}_z \Big[\Big(RS \exp \xi(t,z) \Big) (S^*)^{-1} \exp\big(-\xi(t,z) \big) \Big] \\ = \text{Res}_z \Big[\Big(RSe^{xz} \Big) (S^*)^{-1} e^{-xz} \Big]. \end{aligned}$$

By virtue of (204) this gives

$$\begin{aligned} \text{Res}_z \Bigg[\Bigg(\Big(\frac{\partial}{\partial t_1}\Big)^{j_1} \Big(\frac{\partial}{\partial t_2}\Big)^{j_2} \cdots \Big(\frac{\partial}{\partial t_m}\Big)^{j_m} \Psi(t,z) \Bigg) \Psi^*(t,z) \Bigg] \\ = \text{Res}_\partial \big[RSS^{-1} \big] = \text{Res}_\partial [R] = 0. \end{aligned}$$

□

The inverse also holds true, i.e. the equations of the hierarchy are contained in these bilinear identities. The proof can be found in [**29**].

Theorem 5.6. *Let*

$$\Psi(t,z) = \sum_{j=0}^{\infty} w_j(t) z^{-j} \exp \xi(t,z),$$
$$\Psi^*(t,z) = \sum_{j=0}^{\infty} w_j^*(t) z^{-j} \exp\big(-\xi(t,z)\big),$$

be formal power series, with $w_0(t) = w_0^*(t) = 1$, such that

$$\mathit{Res}_z \Big[\Psi(t',z)\Psi^*(t,z)\Big] = 0,$$

holds for any t, t'. Then $\Psi(t,z)$ and $\Psi^*(t,z)$ are wave and adjoint wave functions for the KP hierarchy, i.e. there exists a pseudo-differential operator S of order 0, such that

$$\Psi(t,z) = S \exp \xi(t,z), \qquad \Psi^*(t,z) = (S^*)^{-1} \exp\big(-\xi(t,z)\big),$$

and $L = S\,\partial\,S^{-1}$ is a solution of the KP hierarchy.

A consequence of the bilinear identities is that the wave and adjoint wave functions $\Psi(t,z)$ and $\Psi^*(t,z)$ of the KP hierarchy can be expressed in terms of one single function $\tau(t)$, called tau function. A proof can be found in [**29**].

Theorem 5.7. *If $\Psi(t,z)$ and $\hat{\Psi}^*(t,z)$ are wave functions for the KP hierarchy, then there exists a function $\tau(t) = \tau(t_1, t_2, \dots)$ such that*

$$\Psi(t,z) = \frac{\tau(t - [z^{-1}])}{\tau(t)} \exp \xi(t,z),$$
$$\Psi^*(t,z) = \frac{\tau(t + [z^{-1}])}{\tau(t)} \exp\big(-\xi(t,z)\big),$$

where $[z] = \big(z, \frac{z^2}{2}, \frac{z^3}{3}, \dots\big)$. The tau function is determined up to multiplication by $ce^{\sum_{j=1}^{\infty} c_j t_j}$ with $c, c_1, c_2, \dots$ arbitrary constants.

It follows that all the coefficients of the wave operators, and of the solution L to the KP hierarchy, can be expressed in terms of the tau function. The bilinear identities can also be expressed in terms of the tau function:

$$\text{(205)} \qquad \mathrm{Res}_z \Big[\tau(t' - [z^{-1}])\tau(t + [z^{-1}]) \exp \xi(t' - t, z)\Big] = 0.$$

1.2. Time-dependent orthogonal polynomials. Let $\rho(x)$ be a weight function on $\mathbb{R}$, decaying fast enough when $x \to \pm\infty$, and consider the following formal deformation of the weight

$$\rho_t(x) := \rho(x) e^{\sum_{j=1}^{\infty} t_j x^j}.$$

We define the symmetric inner product of two functions f, g by

$$\langle f, g\rangle_t = \int_{\mathbb{R}} f(x)g(x)\rho_t(x)dx.$$

Associated with this inner product, we define the moments $\mu_{k,l}(t) = \left\langle x^k, x^l\right\rangle_t$ with $k, l \geq 0$. Obviously, the moments $\mu_{k,l}(t)$ only depend on $k+l$. For simplicity, we shall omit the explicit dependence on the time variables t and we shall write $\mu_{k,l}(t) = \mu_{k,l}$. We also define the Hankel moment matrices

$$m_n(t) = (\mu_{k,l})_{0\leq k,l\leq n-1}.$$

Using an argument similar to that of section 2.2 of Chapter 1, we get the following identity, expressing the determinants of these matrices as multiple integrals

$$\tau_n(t) = \det m_n(t) = \frac{1}{n!}\int_{\mathbb{R}^n} \Delta_n^2(x)\prod_{k=1}^{n}\rho_t(x_k)dx_k.$$

The aim of this section is to prove that these determinants are tau functions for the KP hierarchy. We will show that these determinants satisfy the bilinear identities (205). The proof of this fact is a consequence of the orthogonality conditions of the orthogonal polynomials for the weight $\rho_t(x)$. This subsection is based on [**7**], and also [**68**]. The connexion between Sato's theory of tau functions and the theory of orthogonal polynomials was first established in [**41**].

Let $p_n(x) := p_n(t; x)$, $n \geq 0$, be the monic orthogonal polynomials on the real line with respect to the weight $\rho_t(x)$. By virtue of (93), they admit the following Heine representation

$$p_n(t;x) = \frac{1}{\tau_n(t,s)}\det\left(\begin{array}{ccc|c} & & & 1\\ & m_n(t,s) & & x\\ & & & \vdots\\ \hline \mu_{n,0}(t,s) & \ldots & \mu_{n,n-1}(t,s) & x^n\end{array}\right), \tag{206}$$

These polynomials can also be written using the determinants $\tau_n(t)$.

Proposition 5.8.

$$p_n(t;x) = x^n\frac{\tau_n(t-[x^{-1}])}{\tau_n(t)}, \tag{207}$$

with $[x] = \left(x, \frac{x^2}{2}, \frac{x^3}{3}, \ldots\right)$.

PROOF. Using the expansion $1 - x = \exp\left(-\sum_{j=1}^{\infty}\frac{x^j}{j}\right)$, one can prove that

$$\mu_{k,l}(t-[x^{-1}]) = \mu_{k,l} - \frac{1}{x}\mu_{k+1,l},$$

and hence

$$\tau_n(t-[x^{-1}]) = \det\Big(\mu_{k,l} - \frac{1}{x}\mu_{k+1,l}\Big)_{0\le k,l\le n-1}. \tag{208}$$

We then have

$$\begin{aligned} x^n\tau_n(t-[x^{-1}]) &= \det\Big(x\mu_{k,l} - \mu_{k+1,l}\Big)_{0\le k,l\le n-1} \\ &= \det\Big(x\vec{\mu}_0 - \vec{\mu}_1, x\vec{\mu}_1 - \vec{\mu}_2, \ldots, x\vec{\mu}_{n-1} - \vec{\mu}_n\Big), \end{aligned}$$

where $\vec{\mu}_j = (\mu_{j,0}, \ldots, \mu_{j,n-1}) \in \mathbb{R}^n$. By row operations, we get

$$\begin{aligned} &x^n\tau_n(t-[x^{-1}]) \\ &\qquad = \det\Big(\sum_{j=0}^{n-1}\frac{x\vec{\mu}_j - \vec{\mu}_{j+1}}{x^j}, \sum_{j=0}^{n-2}\frac{x\vec{\mu}_{j+1} - \vec{\mu}_{j+2}}{x^j}, \ldots, x\vec{\mu}_{n-1} - \vec{\mu}_n\Big), \end{aligned}$$

which, after simplification of the expressions gives

$$x^n\tau_n(t-[x^{-1}]) = \det\Big(x\vec{\mu}_0 - \frac{\vec{\mu}_n}{x^{n-1}}, x\vec{\mu}_1 - \frac{\vec{\mu}_n}{x^{n-2}}, \ldots, x\vec{\mu}_{n-1} - \vec{\mu}_n\Big).$$

Adding one row and one column to this determinant gives

$$x^n\tau_n(t-[x^{-1}]) = \frac{1}{x^n}\det\left(\begin{array}{c|c} x\vec{\mu}_0 - \frac{\vec{\mu}_n}{x^{n-1}} & 0 \\ x\vec{\mu}_1 - \frac{\vec{\mu}_n}{x^{n-2}} & 0 \\ \vdots & \vdots \\ x\vec{\mu}_{n-1} - \vec{\mu}_n & 0 \\ \vec{\mu}_n & x^n \end{array}\right),$$

which, making row operations, gives

$$x^n\tau_n(t-[x^{-1}]) = \frac{1}{x^n}\det\left(\begin{array}{c|c} x\vec{\mu}_0 & x \\ x\vec{\mu}_1 & x^2 \\ \vdots & \vdots \\ x\vec{\mu}_{n-1} & x^n \\ \vec{\mu}_n & x^n \end{array}\right) = \tau_n(t)p_n(x).$$

□

Let $q_n(x)$ be the Cauchy-transforms of the polynomials $p_n(x)$:

$$q_n(x) = x\int_{\mathbb{R}}\frac{p_n(y)\rho_t(y)}{x-y}dy.$$

These functions have also expressions in terms of the determinants $\tau_n(t)$.

Proposition 5.9.

$$q_n(x) = x^{-n}\frac{\tau_{n+1}(t+[x^{-1}])}{\tau_n(t)}. \tag{209}$$

PROOF. Using the expansion $1 - x = \exp\big(-\sum_{j=1}^{\infty} \frac{x^j}{j}\big)$, one can prove that

$$\mu_{k,l}(t + [x^{-1}]) = \sum_{j=0}^{\infty} \frac{1}{x^j} \mu_{k,l+j},$$

and hence

$$\tau_{n+1}(t + [x^{-1}]) = \det\Big(\sum_{j=0}^{\infty} \frac{1}{x^j} \mu_{k,l+j}\Big)_{0 \le k,l \le n}. \tag{210}$$

We then have

$$x^{-n}\tau_{n+1}(t + [x^{-1}]) = x^{-n} \det\Big(\sum_{j=0}^{\infty} \frac{\vec{\mu}_j^T}{x^j}, \sum_{j=0}^{\infty} \frac{\vec{\mu}_{j+1}^T}{x^j}, \ldots, \sum_{j=0}^{\infty} \frac{\vec{\mu}_{j+n}^T}{x^j}\Big),$$

where $\vec{\mu}_j = (\mu_{0,j}, \ldots, \mu_{n,j}) \in \mathbb{R}^{n+1}$. By column operations, we get

$$\begin{aligned} x^{-n}\tau_{n+1}(t + [x^{-1}]) &= x^{-n} \det\Big(\vec{\mu}_0^T, \vec{\mu}_1^T, \ldots, \vec{\mu}_{n-1}^T, \sum_{j=0}^{\infty} \frac{\vec{\mu}_{j+n}^T}{x^j}\Big) \\ &= \det\Big(\vec{\mu}_0^T, \vec{\mu}_1^T, \ldots, \vec{\mu}_{n-1}^T, \sum_{j=0}^{\infty} \frac{\vec{\mu}_j^T}{x^j}\Big). \end{aligned}$$

After substitution of the integral expression of the moments in the last column, we get

$$\begin{aligned} &x^{-n}\tau_{n+1}(t + [x^{-1}]) \\ &= \det\left(\begin{array}{c|c} & \int_{\mathbb{R}} \sum_{j=0}^{\infty} \left(\frac{y}{x}\right)^j \rho_t(y)dy \\ & \int_{\mathbb{R}} \sum_{j=0}^{\infty} \left(\frac{y}{x}\right)^j y\rho_t(y)dy \\ \vec{\mu}_0^T, \vec{\mu}_1^T, \ldots, \vec{\mu}_{n-1}^T & \vdots \\ & \int_{\mathbb{R}} \sum_{j=0}^{\infty} \left(\frac{y}{x}\right)^j y^{n-1}\rho_t(y)dy \\ & \int_{\mathbb{R}} \sum_{j=0}^{\infty} \left(\frac{y}{x}\right)^j y^n\rho_t(y)dy \end{array}\right) \\ &= \int_{\mathbb{R}} \sum_{j=0}^{\infty} \left(\frac{y}{x}\right)^j \rho_t(y)dy \det\left(\begin{array}{c|c} & 1 \\ & y \\ \vec{\mu}_0^T, \vec{\mu}_1^T, \ldots, \vec{\mu}_{n-1}^T & \vdots \\ & y^{n-1} \\ & y^n \end{array}\right). \end{aligned}$$

Using the Heine representation of the monic orthogonal polynomials $p_n(x)$ we obtain

$$\begin{aligned} x^{-n}\tau_{n+1}(t + [x^{-1}]) &= \tau_n(t) \int_{\mathbb{R}} \sum_{j=0}^{\infty} \left(\frac{y}{x}\right)^j p_n(y)\rho_t(y)dy \\ &= x\tau_n(t) \int_{\mathbb{R}} \frac{p_n(y)\rho_t(y)}{x - y} dy. \end{aligned}$$

□

Before proving that the determinants $\tau_n(t)$ satisfy the bilinear identities of the KP hierarchy, we need the following simple, formal residue identity.

Lemma 5.10. *For holomorphic functions $f(z) = \sum_{j=0}^{\infty} a_j z^j$ and $g(z)$, we have*

$$\int_{\mathbb{R}} f(y)g(y)dy = \frac{1}{2\pi i}\oint_{z=\infty} dz f(z)\int_{\mathbb{R}} \frac{g(y)}{z-y}dy.$$

PROOF. We have

$$\begin{aligned}\frac{1}{2\pi i}\oint_{z=\infty} dz f(z)\int_{\mathbb{R}} \frac{g(y)}{z-y}dy\\ &= \operatorname{Res}_{z=\infty}\Big[\Big(\sum_{j=0}^{\infty} a_j z^j\Big)\Big(\frac{1}{z}\sum_{j\geq 0} z^{-j}\int_{\mathbb{R}} g(y)y^j dy\Big)\Big]\\ &= \sum_{j\geq 0} a_j \int_{\mathbb{R}} g(y)y^j dy\\ &= \int_{\mathbb{R}} g(y)\sum_{j\geq 0} a_j y^j dy\\ &= \int_{\mathbb{R}} g(y)f(y)dy.\end{aligned}$$

□

We are now able to state the next theorem.

Theorem 5.11. *The functions $\tau_n(t,s)$ satisfy the following bilinear identity*

$$\oint_{z=\infty} \tau_n(t-[z^{-1}])\tau_{m+1}(t'+[z^{-1}])e^{\sum_{j=1}^{\infty}(t_j-t'_j)z^j} z^{n-m-1}dz = 0, \tag{211}$$

for all $n \geq m+1$ and all $t, t' \in \mathbb{C}^\infty$.

PROOF. Using the representations (207) and (209) of p_n and q_{m+1} in terms of tau functions, we have

$$\begin{aligned}\frac{1}{\tau_n(t)\tau_m(t')}\oint_{z=\infty} \tau_n(t-[z^{-1}])\tau_{m+1}(t'+[z^{-1}])e^{\sum_{j=1}^{\infty}(t_j-t'_j)z^j} z^{n-m-1}dz\\ = \oint_{z=\infty} p_n(t;z)q_m(t';z)e^{\sum_{j=1}^{\infty}(t_j-t'_j)z^j}\frac{dz}{z}.\end{aligned}$$

But q_m is the Cauchy-transform of the polynomial p_m. Hence, we have by virtue of Lemma 5.10

$$\begin{aligned}&\frac{1}{\tau_n(t)\tau_m(t')}\oint_{z=\infty}\tau_n(t-[z^{-1}])\tau_{m+1}(t'+[z^{-1}])e^{\sum_{j=1}^{\infty}(t_j-t'_j)z^j}z^{n-m-1}dz\\ &=\oint_{z=\infty}p_n(t;z)e^{\sum_{j=1}^{\infty}(t_j-t'_j)z^j}\frac{dz}{z}\int_{\mathbb{R}}\frac{p_m(t';y)}{z-y}e^{\sum_{k=1}^{\infty}t'_k y^k}\rho(y)dy\\ &=2\pi i\int_{\mathbb{R}}p_n(t;z)p_m(t';z)e^{\sum_{j=1}^{\infty}t_j z^j}\rho(z)dz.\end{aligned}$$

This is equal to 0 when $n \geq m+1$. □

This bilinear identity includes the bilinear identity of the KP hierarchy, for $m+1=n$.

2. Multi-component KP hierarchy

The multi-component KP hierarchy is a matrix hierarchy. It is a generalization of the KP hierarchy explained in the former section. This section is based on [**67**] and [**30**] for the Sato theory of the multi-component KP hierarchy, and on [**14**] for the link with time-dependent multiple orthogonal polynomials.

2.1. Sato theory.

2.1.1. *Notations.* A formal matrix pseudo-differential operator of order m is a formal series

$$R=\sum_{j=-\infty}^{m}u_j\partial^j,$$

where ∂ is the derivation operator defined by

$$\partial=\sum_{\alpha=1}^{r}\partial_{x^{(\alpha)}},$$

and ∂^{-1} is its formal inverse, such that

$$\partial\cdot\partial^{-1}=\partial^{-1}\cdot\partial=1,$$

and $u_j(x)\in\mathbb{C}^{r\times r}$ are some functions of the variables $x=(x^{(1)},\dots,x^{(r)})$. Let $\mathcal{R}^{(r)}$ be the set of $r\times r$-matrix pseudo-differential operators of finite order. Addition and multiplication of two elements of $\mathcal{R}^{(r)}$ are defined as in the scalar case. This makes the set $\mathcal{R}^{(r)}$ of pseudo-differential operators into an associative ring.

We define the formal adjoint R^* of R, by

$$R^*=\sum_{j=-\infty}^{m}(-\partial)^j\cdot u_j(x)^T,$$

where $u_j(x)^T$ is the transposed matrix of $u_j(x)$. For two pseudo-differential operators R_1 and R_2, one has $(R_1R_2)^* = R_2^*R_1^*$.

Finally, for a pseudo-differential operator $R = \sum_{j=-\infty}^{m} u_j(x)\partial^j$ we define

$$R_+ = \sum_{j=0}^{m} u_j(x)\partial^j$$

the differential part, and

$$R_- = \sum_{j<0} u_j(x)\partial^j$$

the integral (or Volterra) part of R.

2.1.2. *Lax form of the multi-component KP hierarchy.* Consider r families of time variables $t = (t^{(1)}, t^{(2)}, \dots, t^{(r)})$, with

$$t^{(\alpha)} = (t_1^{(\alpha)}, t_2^{(\alpha)}, \dots), \qquad \alpha = 1, \dots, r.$$

We identify $x^{(\alpha)} = t_1^{(\alpha)}$, $\alpha = 1, \dots, r$. Let L and U_α, $\alpha = 1, \dots, r$, be matrix pseudo-differential operators given by

$$L = \sum_{j=-\infty}^{1} u_j\partial^j, \qquad U_\alpha = \sum_{j=-\infty}^{0} u_{j,\alpha}\partial^j,$$

where $u_j, u_{j,\alpha}$ are $r \times r$ matrix-valued functions of the variables $x = (x^{(1)}, \dots, x^{(r)})$ and of the families of t-time variables given above, such that $u_1 = \mathbf{1}_r$ is the $r \times r$ identity matrix, $u_0 = 0$, and $u_{0,\alpha} = E_\alpha$, with

$$E_\alpha = (\delta_{k,\alpha}\delta_{l,\alpha})_{1\le k,l\le r} = \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}.$$

We suppose the following conditions for $\alpha, \beta = 1, \dots, r$

$$[L, U_\alpha] = 0, \qquad [U_\alpha, U_\beta] = 0,$$

and

$$\sum_{\alpha=1}^{r} U_\alpha = \mathbf{1}_r, \qquad U_\alpha U_\beta = \delta_{\alpha,\beta}U_\beta.$$

Definition 5.12. *The r-component KP hierarchy is defined by the equations in the Lax form*

$$\frac{\partial L}{\partial t_n^{(\alpha)}} = [B_n^{(\alpha)}, L], \qquad \frac{\partial U_\beta}{\partial t_n^{(\alpha)}} = [B_n^{(\alpha)}, U_\beta], \tag{212}$$

where $B_n^{(\alpha)} = \big(L^n U_\alpha\big)_+$, with $\alpha, \beta = 1, \dots, r$ and $n = 1, 2, \dots$.

The system of equations (212) is a compatible system. Indeed, we have the following theorem, whose proof is similar to that of Proposition 5.2.

Proposition 5.13. *The system of equations* (212) *is equivalent to the system of equations of Zakharov-Shabat type*

$$\frac{\partial B_m^{(\alpha)}}{\partial t_n^{(\beta)}} - \frac{\partial B_n^{(\beta)}}{\partial t_m^{(\alpha)}} + [B_m^{(\alpha)}, B_n^{(\beta)}] = 0, \tag{213}$$

with $\alpha, \beta = 1, \dots, r$, and $m, n = 1, 2, \dots$. Consequently, the vector fields defined in (212) *commute.*

2.1.3. *Wave function.* Consider the formal dressing

$$L = \hat{W} \partial \hat{W}^{-1},$$
$$U_\alpha = \hat{W} E_\alpha \hat{W}^{-1},$$
$$\frac{\partial \hat{W}}{\partial t_n^{(\alpha)}} = B_n^{(\alpha)} \hat{W} - \hat{W} E_\alpha \partial^n,$$

by a $r \times r$ matrix pseudo-differential operator

$$\hat{W} = \sum_{j=0}^{\infty} w_j(t) \partial^{-j},$$

with $w_0 = \mathbf{1}_r$. As for the scalar case, the matrices $w_j(t)$ are determined recursively, using the relation

$$L\hat{W} = \hat{W}\partial.$$

We define the action of the matrix pseudo-differential operators on the function $\exp\big(\sum_{\alpha=1}^r \xi\big(t^{(\alpha)}, z\big) E_\alpha\big)$, by

$$\partial^m \exp \sum_{\alpha=1}^{r} \xi\big(t^{(\alpha)}, z\big) E_\alpha = z^m \exp \sum_{\alpha=1}^{r} \xi\big(t^{(\alpha)}, z\big) E_\alpha,$$

for all $m \in \mathbb{Z}$. The wave function is then defined as

$$W(t, z) = \hat{W} \exp \sum_{\alpha=1}^{r} \xi\big(t^{(\alpha)}, z\big) E_\alpha = \sum_{j=0}^{\infty} w_j(t) z^{-j} \exp \sum_{\alpha=1}^{r} \xi\big(t^{(\alpha)}, z\big) E_\alpha.$$

We also define the adjoint wave function

$$W^*(t,z) = (\hat{W}^*)^{-1} \exp\Big(-\sum_{\alpha=1}^{r} \xi\big(t^{(\alpha)}, z\big) E_\alpha \Big).$$

The wave and adjoint wave functions satisfy the following equations

$$LW = zW, \qquad U_\alpha W = W E_\alpha, \qquad \frac{\partial W}{\partial t_n^{(\alpha)}} = B_n^{(\alpha)} W,$$

$$L^* W^* = zW^*, \qquad U_\alpha^* W^* = W^* E_\alpha, \qquad \frac{\partial W^*}{\partial t_n^{(\alpha)}} = -\big(B_n^{(\alpha)}\big)^* W^*.$$

The r-component KP hierarchy (212), and its equivalent formulation (213), arise as compatibility conditions of this linear system.

Remark 5.14. *Notice that the pseudo-differential operator $\hat{W}$ is defined up to multiplication on the right by a series $\sum_{j=0}^{\infty} c_j \partial^{-j}$ with constant diagonal matrices c_j and $c_0 = \mathbf{1}_r$. Correspondingly, the wave function is defined up to a multiplication by $\sum_{j=0}^{\infty} c_j z^{-j}$.*

As for the scalar case, the wave and adjoint wave function satisfy bilinear identities. These identities completely determine the multi-component KP hierarchy. We first prove an elementary lemma on residue pairing.

Lemma 5.15. *If P, Q are $r \times r$ matrix pseudo-differential operators, then*

$$\mathit{Res}_z \Big[\Big(P e^{\sum_{\alpha=1}^{r} z t_1^{(\alpha)} E_\alpha} \Big) \Big(Q e^{-\sum_{\alpha=1}^{r} z t_1^{(\alpha)} E_\alpha} \Big)^T \Big] = \mathit{Res}_\partial \big[P Q^* \big].$$

PROOF. Let $P = \sum p_j \partial^j$ and $Q = \sum q_j \partial^j$ be $r \times r$ matrix pseudo-differential operators. On the one hand we have

$$\begin{aligned} \mathrm{Res}_z \Big[\Big(P e^{\sum_{\alpha=1}^{r} z t_1^{(\alpha)} E_\alpha} \Big) \Big(Q e^{-\sum_{\alpha=1}^{r} z t_1^{(\alpha)} E_\alpha} \Big)^T \Big] & \\ = \mathrm{Res}_z \Big[\Big(\sum p_j z^j \Big) \Big(\sum q_k (-z)^k \Big)^T \Big] & \\ = \sum_{j+k=-1} (-1)^k p_j q_k^T. & \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \mathrm{Res}_\partial \big[PQ^* \big] &= \mathrm{Res}_\partial \Big[\Big(\sum p_j \partial^j \Big) \Big(\sum (-\partial)^k q_k^T \Big) \Big] \\ &= \mathrm{Res}_\partial \Big[\sum_{j,k} p_j \partial^j (-\partial)^k q_k^T \Big) \Big] \\ &= \sum_{j+k=-1} (-1)^k p_j q_k^T. \end{aligned}$$

□

We then have the following theorem, whose proof is similar to the scalar case.

Theorem 5.16. *The wave and adjoint wave functions W and W^* satisfy the following bilinear identity*

$$Res_z\Big[\big(\partial_{k_1,\alpha_1}\dots\partial_{k_s,\alpha_s}W(t,z)\big)\big(W^*(t,z)\big)^T\Big]=0,$$

for any $s\geq 0$, $1\leq\alpha_i\leq r$, $k_i=1,2,\dots$, and where ∂_{k_i,α_i} stands for the derivative $\frac{\partial}{\partial t^{(\alpha_i)}_{k_i}}$. Equivalently, we have

$$Res_z\Big[W(z,t')\big(W^*(z,t)\big)^T\Big]=0, \tag{214}$$

for any t,t', where $W(z,t')$ should be understood as a formal Taylor series expansion around t. Conversely, if there are two expressions of the form $W(t,z)=\sum_{j=0}^{\infty}w_j(t)z^{-j}\exp\sum_{\alpha=1}^{r}\xi\big(t^{(\alpha)},z\big)E_\alpha$ and $W^*(t,z)=\sum_{j=0}^{\infty}v_j(t)z^{-j}\exp\Big(-\sum_{\alpha=1}^{r}\xi\big(t^{(\alpha)},z\big)E_\alpha\Big)$ with $w_0=v_0=\mathbf{1}_r$ such that (214) *holds for them, then they are wave and adjoint wave functions for the multi-component KP hierarchy.*

These bilinear identities completely characterize the wave functions. A consequence of these identities is that the wave and adjoint wave functions can be expressed in terms of several tau-functions. The proof can be found in [**30**].

Theorem 5.17. *If $W(z,t)$ and $W^*(z,t)$ are wave and adjoint wave functions for the r-component KP hierarchy, then there exists functions $\tau(t)$ and $\tau_{\alpha,\beta}(t)$, $1\leq\alpha,\beta\leq r$, $\alpha\neq\beta$, such that*

$$W(z,t)_{\alpha,\beta}=\begin{cases} c_\alpha(z)\frac{\tau(t^{(\alpha)}-[z^{-1}])\exp\xi(t^{(\alpha)},z)}{\tau(t)}, & \text{if } \alpha=\beta,\\[2ex] z^{-1}c_\beta(z)\frac{\tau_{\alpha,\beta}(t^{(\beta)}-[z^{-1}])\exp\xi(t^{(\beta)},z)}{\tau(t)}, & \text{if } \alpha\neq\beta,\end{cases} \tag{215}$$

and

$$W^*(z,t)_{\alpha,\beta}=\begin{cases} \frac{1}{c_\alpha(z)}\frac{\tau(t^{(\alpha)}+[z^{-1}])\exp\left(-\xi(t^{(\alpha)},z)\right)}{\tau(t)}, & \text{if } \alpha=\beta,\\[2ex] -z^{-1}\frac{1}{c_\beta(z)}\frac{\tau_{\beta,\alpha}(t^{(\beta)}+[z^{-1}])\exp\left(-\xi(t^{(\beta)},z)\right)}{\tau(t)}, & \text{if } \alpha\neq\beta,\end{cases} \tag{216}$$

where $c_\beta(z)=\sum_{j=0}^{\infty}c_{j\beta}z^{-j}$ is a constant series (i.e. independent of t) with $c_{0\beta}=1$.

Remark 5.18. *As a consequence of Remark 5.14, one can chose $c_\beta(z)=1$, $\forall\beta$, in the expressions* (215) *and* (216).

2.2. Time-dependent multiple orthogonal polynomials. Define two sets of weights on $\mathbb{R}$

$$\psi_1(x), \dots, \psi_q(x), \quad \text{and} \quad \varphi_1(y), \dots, \varphi_p(y), \qquad \text{with } x, y \in \mathbb{R},$$

and two multi-indices

$$\vec{m} = (m_1, \dots, m_q), \quad \vec{n} = (n_1, \dots, n_p), \quad \text{with } |\vec{m}| = |\vec{n}|,$$

with $|\vec{m}| = \sum_{i=1}^{q} m_i$, $|\vec{n}| = \sum_{j=1}^{p} n_j$, and consider a (not necessarily symmetric) inner product $\langle \cdot \,|\, \cdot \rangle$. Following [**22**] we introduce the notion of mixed multiple orthogonal polynomials (MOPS). Let $A_1(y), \dots, A_p(y)$ be polynomials in the variable y, and set

$$Q(y) = \sum_{j=1}^{p} A_j(y)\varphi_j(y).$$

There are two possible normalizations :

- Fix $\alpha \in \{1, \dots, q\}$. The polynomials $A_1, \dots, A_p$ are said to be **Type I normalized** mixed multiple orthogonal polynomials with respect to ψ_α, if $deg(A_\beta) \leq n_\beta - 1$, with $\beta = 1, \dots, p$, and we have the following orthogonality conditions
$$\left\langle x^i \psi_{\alpha'}(x) \,\middle|\, Q(y) \right\rangle = \delta_{\alpha,\alpha'}\delta_{i,m_\alpha - 1}, \qquad i = 0, \dots, m_{\alpha'} - 1, \quad 1 \leq \alpha' \leq q.$$
- Fix $\beta \in \{1, \dots, p\}$. The polynomials $A_1, \dots, A_p$ are said to be **Type II normalized** mixed multiple orthogonal polynomials with respect to φ_β, if A_β is monic of degree n_β and $deg(A_{\beta'}) \leq n_{\beta'} - 1$ for $1 \leq \beta' \leq p$ with $\beta' \neq \beta$, and we have the following orthogonality conditions
$$\left\langle x^i \psi_\alpha(x) \,\middle|\, Q(y) \right\rangle = 0, \qquad i = 0, \dots, m_\alpha - 1, \quad 1 \leq \alpha \leq q.$$

Similarly, let $B_1(x), \dots, B_q(x)$ be polynomials in the variable x, and set

$$P(x) = \sum_{j=1}^{q} B_j(x)\psi_j(x).$$

As in the former case, there are two possible normalizations : **Type I normalization** with respect to φ_β, for a fixed $1 \leq \beta \leq p$, and **Type II normalization** with respect to ψ_α, for a fixed $1 \leq \alpha \leq q$. These normalizations are obtained by interchanging the role of $\varphi \leftrightarrow \psi$ and $\vec{m} \leftrightarrow \vec{n}$, $p \leftrightarrow q$, $x \leftrightarrow y$ in the above normalizations.

Consider now the deformed weights depending on time parameters $s^{(\alpha)} = \big(s_1^{(\alpha)}, s_2^{(\alpha)}, \dots\big)$, $1 \leq \alpha \leq q$, and $t^{(\beta)} = \big(t_1^{(\beta)}, t_2^{(\beta)}, \dots\big)$, $1 \leq \beta \leq p$, denoted by

$$\psi_\alpha^{-s}(x) := \psi_\alpha(x) e^{-\sum_{k=1}^{\infty} s_k^{(\alpha)} x^k}, \quad \text{and} \quad \varphi_\beta^{t}(y) := \varphi_\beta(y) e^{\sum_{k=1}^{\infty} t_k^{(\beta)} y^k}.$$

For each set of integers

$$\vec{m} = (m_1, \dots, m_q), \quad \vec{n} = (n_1, \dots, n_p), \quad \text{with } |\vec{m}| = |\vec{n}|,$$

consider the moment matrix $T_{\vec{m}\vec{n}}$ of size $|\vec{m}| = |\vec{n}|$, composed of pq blocks of sizes $m_i n_j$

$$T_{\vec{m}\vec{n}} = \begin{pmatrix} \left(\left\langle x^i \psi_1^{-s}(x) \,|\, y^j \varphi_1^t(y) \right\rangle \right)_{\substack{0 \le i < m_1 \\ 0 \le j < n_1}} & \cdots & \left(\left\langle x^i \psi_1^{-s}(x) \,|\, y^j \varphi_p^t(y) \right\rangle \right)_{\substack{0 \le i < m_1 \\ 0 \le j < n_p}} \\ \vdots & & \vdots \\ \left(\left\langle x^i \psi_q^{-s}(x) \,|\, y^j \varphi_1^t(y) \right\rangle \right)_{\substack{0 \le i < m_q \\ 0 \le j < n_1}} & \cdots & \left(\left\langle x^i \psi_q^{-s}(x) \,|\, y^j \varphi_p^t(y) \right\rangle \right)_{\substack{0 \le i < m_q \\ 0 \le j < n_p}} \end{pmatrix},$$

where the moments are taken with regard to the inner product $\langle \cdot \,|\, \cdot \rangle$. We define the determinants

$$\tau_{\vec{m}\vec{n}}(s^{(1)}, \dots, s^{(q)}; t^{(1)}, \dots, t^{(p)}) := \det T_{\vec{m}\vec{n}}. \tag{217}$$

In the following proposition, it is shown that one can define mixed multiple orthogonal polynomials and their Cauchy transforms using these determinants.

Proposition 5.19 (Adler, van Moerbeke, Vanhaecke [**14**]). *Let* $\vec{e}_1 = (1, 0, 0 \dots)$, $\vec{e}_2 = (0, 1, 0, 0, \dots)$, ... *and put*

$$\varepsilon_{\alpha,\alpha'}(\vec{n}) = \begin{cases} (-1)^{n_{\alpha'+1} + n_{\alpha'+2} + \cdots + n_\alpha + 1}, & \text{if } \alpha > \alpha', \\ (-1)^{n_{\alpha+1} + n_{\alpha+2} + \cdots + n_{\alpha'}}, & \text{if } \alpha < \alpha', \end{cases}$$

and

$$\epsilon_{\alpha,\beta}(\vec{m}, \vec{n}) = (-1)^{m_1 + \cdots + m_\alpha} (-1)^{n_1 + \cdots + n_\beta}.$$

Writing explicitly only the shifted time variables, we have the following four statements.

(1) *For* $1 \le \beta, \beta' \le p$, *let*

$$\begin{aligned} Q_{\vec{m}\vec{n}}^{(\beta,\beta')}(z) &= \varepsilon_{\beta,\beta'}(\vec{n}) z^{n_{\beta'}-1} \frac{\tau_{\vec{m},\vec{n}+\vec{e}_\beta-\vec{e}_{\beta'}}(t^{(\beta')} - [z^{-1}])}{\tau_{\vec{m}\vec{n}}}, \qquad \beta' \neq \beta, \\ Q_{\vec{m}\vec{n}}^{(\beta,\beta)}(z) &= z^{n_\beta} \frac{\tau_{\vec{m}\vec{n}}(t^{(\beta)} - [z^{-1}])}{\tau_{\vec{m}\vec{n}}}. \end{aligned} \tag{218}$$

Then $Q_{\vec{m}\vec{n}}^{(\beta,1)}(y), \dots, Q_{\vec{m}\vec{n}}^{(\beta,p)}(y)$ *are Type II normalized mixed multiple orthogonal polynomials with respect to* φ_β^t.

(2) *For* $1 \le \alpha \le q$ *and* $1 \le \beta \le p$, *let*

$$P_{\vec{m}\vec{n}}^{(\alpha,\beta)}(z) = \epsilon_{\alpha,\beta}(\vec{m}, \vec{n}) z^{n_\beta - 1} \frac{\tau_{\vec{m}-\vec{e}_\alpha, \vec{n}-\vec{e}_\beta}(t^{(\beta)} - [z^{-1}])}{\tau_{\vec{m}\vec{n}}}. \tag{219}$$

Then $P_{\vec{m}\vec{n}}^{(\alpha,1)}(y), \dots, P_{\vec{m}\vec{n}}^{(\alpha,p)}(y)$ *are Type I normalized mixed multiple orthogonal polynomials with respect to* ψ_α^{-s}.

(3) *For* $1 \leq \alpha \leq q$ *and* $1 \leq \beta \leq p$*, the Cauchy transforms of*

$$Q^{(\beta)}_{\vec{m}\vec{n}}(y) = \sum_{\beta'=1}^{p} Q^{(\beta,\beta')}_{\vec{m}\vec{n}}(y)\varphi^t_{\beta'}(y),$$

with respect to ψ^{-s}_α*, with* $Q^{(\beta,\beta')}_{\vec{m}\vec{n}}(y)$ *defined in* (218)*, can be expressed in terms of the determinants* $\tau_{\vec{m}\vec{n}}$ *as follows*

$$\left\langle \frac{\psi^{-s}_\alpha(x)}{z-x} \,\middle|\, Q^{(\beta)}_{\vec{m}\vec{n}}(y) \right\rangle = \epsilon_{\alpha,\beta}(\vec{m},\vec{n}) z^{-m_\alpha - 1} \frac{\tau_{\vec{m}+\vec{e}_\alpha,\vec{n}+\vec{e}_\beta}(s^{(\alpha)} - [z^{-1}])}{\tau_{\vec{m}\vec{n}}}. \tag{220}$$

(4) *For* $1 \leq \alpha, \alpha' \leq q$*, the Cauchy transforms of*

$$P^{(\alpha')}_{\vec{m}\vec{n}}(y) = \sum_{\beta=1}^{p} P^{(\alpha',\beta)}_{\vec{m}\vec{n}}(y)\varphi^t_{\beta}(y),$$

with respect to ψ^{-s}_α*, with* $P^{(\alpha,\beta)}_{\vec{m}\vec{n}}(y)$ *defined in* (219)*, can be expressed in terms of the determinants* $\tau_{\vec{m}\vec{n}}$ *as follows*

$$\begin{aligned} \left\langle \tfrac{\psi^{-s}_\alpha(x)}{z-x} \,\middle|\, P^{(\alpha')}_{\vec{m}\vec{n}}(y) \right\rangle &= \varepsilon_{\alpha',\alpha}(\vec{m}) z^{-m_\alpha - 1} \tfrac{\tau_{\vec{m}+\vec{e}_\alpha-\vec{e}_{\alpha'},\vec{n}}(s^{(\alpha)}-[z^{-1}])}{\tau_{\vec{m}\vec{n}}}, \quad \alpha' \neq \alpha \\ \left\langle \tfrac{\psi^{-s}_\alpha(x)}{z-x} \,\middle|\, P^{(\alpha')}_{\vec{m}\vec{n}}(y) \right\rangle &= z^{-m_\alpha} \tfrac{\tau_{\vec{m},\vec{n}}(s^{(\alpha)}-[z^{-1}])}{\tau_{\vec{m}\vec{n}}}. \end{aligned} \tag{221}$$

And similarly.

Proposition 5.20 (Adler, van Moerbeke, Vanhaecke [**14**])**.** *Writing explicitly only the shifted time variables, we have the following four statements.*

(1) *For* $1 \leq \alpha, \alpha' \leq q$*, let*

$$\begin{aligned} Q^{*(\alpha,\alpha')}_{\vec{n}\vec{m}}(z) &= \varepsilon_{\alpha,\alpha'}(\vec{m}) z^{m_{\alpha'}-1} \tfrac{\tau_{\vec{m}+\vec{e}_\alpha-\vec{e}_{\alpha'},\vec{n}}(s^{(\alpha')}+[z^{-1}])}{\tau_{\vec{m}\vec{n}}}, \qquad \alpha' \neq \alpha, \\ Q^{*(\alpha,\alpha)}_{\vec{n}\vec{m}}(z) &= z^{m_\alpha} \tfrac{\tau_{\vec{m}\vec{n}}(s^{(\alpha)}+[z^{-1}])}{\tau_{\vec{m}\vec{n}}}. \end{aligned} \tag{222}$$

Then $Q^{*(\alpha,1)}_{\vec{n}\vec{m}}(x), \dots, Q^{*(\alpha,q)}_{\vec{n}\vec{m}}(x)$ *are Type II normalized mixed multiple orthogonal polynomials with respect to* ψ^{-s}_α*.*

(2) *For* $1 \leq \alpha \leq q$ *and* $1 \leq \beta \leq p$*, let*

$$P^{*(\beta,\alpha)}_{\vec{n}\vec{m}}(z) = \epsilon_{\beta,\alpha}(\vec{n},\vec{m}) z^{m_\alpha - 1} \frac{\tau_{\vec{m}-\vec{e}_\alpha,\vec{n}-\vec{e}_\beta}(s^{(\alpha)} + [z^{-1}])}{\tau_{\vec{m}\vec{n}}}. \tag{223}$$

Then $P^{*(\beta,1)}_{\vec{m}\vec{n}}(x), \dots, P^{*(\beta,p)}_{\vec{m}\vec{n}}(x)$ *are Type I normalized mixed multiple orthogonal polynomials with respect to* φ^t_β*.*

(3) *For* $1 \leq \alpha \leq q$ *and* $1 \leq \beta \leq p$*, the Cauchy transforms of*

$$Q^{*(\alpha)}_{\vec{n}\vec{m}}(x) = \sum_{\alpha'=1}^{q} Q^{*(\alpha,\alpha')}_{\vec{n}\vec{m}}(x)\psi^{-s}_{\alpha'}(x),$$

with respect to φ^t_β, *with* $Q^{*(\alpha,\alpha')}_{\vec{n}\vec{m}}(y)$ *defined in* (222), *can be expressed in terms of the determinants* $\tau_{\vec{m}\vec{n}}$ *as follows*

$$\left\langle Q^{*(\alpha)}_{\vec{n}\vec{m}}(x) \,\middle|\, \frac{\varphi^t_\beta(y)}{z-y} \right\rangle = \epsilon_{\beta,\alpha}(\vec{n},\vec{m}) z^{-n_\beta - 1} \frac{\tau_{\vec{m}+\vec{e}_\alpha, \vec{n}+\vec{e}_\beta}(t^{(\beta)} + [z^{-1}])}{\tau_{\vec{m}\vec{n}}}. \tag{224}$$

(4) *For* $1 \le \beta, \beta' \le p$, *the Cauchy transforms of*

$$P^{*(\beta)}_{\vec{n}\vec{m}}(x) = \sum_{\alpha=1}^{q} P^{*(\beta,\alpha)}_{\vec{n}\vec{m}}(x) \psi^{-s}_\alpha(x),$$

with respect to φ^t_β, *with* $P^{*(\beta,\alpha)}_{\vec{n}\vec{m}}(y)$ *defined in* (223), *can be expressed in terms of the determinants* $\tau_{\vec{m}\vec{n}}$ *as follows*

$$\begin{aligned} &\left\langle P^{*(\beta)}_{\vec{n}\vec{m}}(x) \,\middle|\, \tfrac{\varphi^t_{\beta'}(y)}{z-y} \right\rangle = \varepsilon_{\beta,\beta'}(\vec{n}) z^{-n_{\beta'}-1} \tfrac{\tau_{\vec{m},\vec{n}+\vec{e}_{\beta'}-\vec{e}_\beta}(t^{(\beta')}+[z^{-1}])}{\tau_{\vec{m}\vec{n}}}, \\ &\left\langle P^{*(\beta)}_{\vec{n}\vec{m}}(x) \,\middle|\, \tfrac{\varphi^t_{\beta}(y)}{z-y} \right\rangle = z^{-n_\beta} \tfrac{\tau_{\vec{m},\vec{n}}(t^{(\beta)}+[z^{-1}])}{\tau_{\vec{m}\vec{n}}}. \end{aligned} \tag{225}$$

Define the $(p+q) \times (p+q)$ matrix

$$W_{\vec{m}\vec{n}}(z;s,t) := \begin{pmatrix} \left(Q^{(\beta,\beta')}_{\vec{m}\vec{n}}\right)_{\substack{1\le\beta\le p\\1\le\beta'\le p}} & \left(\left\langle \frac{\psi^{-s}_\alpha(x)}{z-x} \,\middle|\, Q^{(\beta)}_{\vec{m}\vec{n}}(y) \right\rangle\right)_{\substack{1\le\beta\le p\\1\le\alpha\le q}} \\ \left(P^{(\alpha,\beta)}_{\vec{m}\vec{n}}\right)_{\substack{1\le\alpha\le q\\1\le\beta\le p}} & \left(\left\langle \frac{\psi^{-s}_\alpha(x)}{z-x} \,\middle|\, P^{(\alpha')}_{\vec{m}\vec{n}}(y) \right\rangle\right)_{\substack{1\le\alpha'\le q\\1\le\alpha\le q}} \end{pmatrix} \times \Delta(z),$$

where

$$\Delta(z) = \operatorname{diag}\left(e^{\xi(t^{(1)},z)}, \dots, e^{\xi(t^{(p)},z)}, e^{\xi(s^{(1)},z)}, \dots, e^{\xi(s^{(q)},z)}\right),$$

and the adjoint matrix

$$W^*_{\vec{m}\vec{n}}(z;s,t) := \begin{pmatrix} \left(\left\langle P^{*(\beta')}_{\vec{n}\vec{m}}(x) \,\middle|\, \frac{\varphi^t_\beta(y)}{z-y} \right\rangle\right)_{\substack{1\le\beta'\le p\\1\le\beta\le p}} & \left(-P^{*(\beta,\alpha)}_{\vec{n}\vec{m}}\right)_{\substack{1\le\beta\le p\\1\le\alpha\le q}} \\ \left(-\left\langle Q^{*(\alpha)}_{\vec{n}\vec{m}}(x) \,\middle|\, \frac{\varphi^t_\beta(y)}{z-y} \right\rangle\right)_{\substack{1\le\alpha\le q\\1\le\beta\le p}} & \left(Q^{*(\alpha,\alpha')}_{\vec{n}\vec{m}}\right)_{\substack{1\le\alpha\le q\\1\le\alpha'\le q}} \end{pmatrix} \times \Delta(z)^{-1}.$$

In [**14**] it is proven that $W_{\vec{m}\vec{n}}, W^*_{\vec{m}\vec{n}}$ satisfy bilinear identities. We state the result without proof. It is essentially based on the orthogonality relations between the polynomials, and follows the same ideas as used in Section 4.1 of Chapter 1.

Theorem 5.21 (Adler, van Moerbeke, Vanhaecke [**14**]). *The wave matrix $W_{\vec{m}\vec{n}}$ and the adjoint wave matrix $W^*_{\vec{m}\vec{n}}$ satisfy the following bilinear identities*

$$\oint_{z=\infty} W_{\vec{m}\vec{n}}(z;s,t)W^*_{\vec{m}'\vec{n}'}(z;\tilde{s},\tilde{t})^T dz = 0, \tag{226}$$

for any $t,\tilde{t},s,\tilde{s} \in \mathbb{C}^\infty$ and any multi-indices $\vec{m} = (m_1,\dots,m_q)$, $\vec{m}' = (m'_1,\dots,m'_q)$, $\vec{n} = (n_1,\dots,n_p)$ and $\vec{n}' = (n'_1,\dots,n'_p)$, with $|\vec{m}| = |\vec{n}|$ and $|\vec{m}'| = |\vec{n}'|$. This bilinear identity is equivalent to the following identity satisfied by the determinants of the block matrices $\tau_{\vec{m}\vec{n}}$

$$\sum_{\beta=1}^{p}\oint_{\infty}(-1)^{\sigma_\beta(\vec{n})}\tau_{\vec{m},\vec{n}-\vec{e}_\beta}(t^{(\beta)}-[z^{-1}])\tau_{\vec{m}',\vec{n}'+\vec{e}_\beta}(\tilde{t}^{(\beta)}+[z^{-1}])$$
$$e^{\sum_{k=1}^{\infty}(t_k^{(\beta)}-\tilde{t}_k^{(\beta)})z^k}z^{n_\beta-n'_\beta-2}dz$$

$$= \sum_{\alpha=1}^{q}\oint_{\infty}(-1)^{\sigma_\alpha(\vec{m})}\tau_{\vec{m}+\vec{e}_\alpha,\vec{n}}(s^{(\alpha)}-[z^{-1}])\tau_{\vec{m}'-\vec{e}_\alpha,\vec{n}'}(\tilde{s}^{(\alpha)}+[z^{-1}])$$
$$e^{\sum_{k=1}^{\infty}(s_k^{(\alpha)}-\tilde{s}_k^{(\alpha)})z^k}z^{m'_\alpha-m_\alpha-2}dz, \tag{227}$$

for all $\vec{m},\vec{n},\vec{m}',\vec{n}'$ such that $|\vec{m}'| = |\vec{n}'|+1$ and $|\vec{m}| = |\vec{n}|-1$, and all $s,t,\tilde{s},\tilde{t} \in \mathbb{C}^\infty$, and where

$$\sigma_\alpha(\vec{m}) = \sum_{\alpha'=1}^{\alpha}(m_{\alpha'}-m'_{\alpha'}), \quad \textit{and} \quad \sigma_\beta(\vec{n}) = \sum_{\beta'=1}^{\beta}(n_{\beta'}-n'_{\beta'}).$$

Define the matrices

$$\tilde{W}_{\vec{m}\vec{n}}(z;s,t) := W_{\vec{m}\vec{n}}(z;s,t) \times \Xi_{\vec{m}\vec{n}}(z),$$
$$\tilde{W}^*_{\vec{m}\vec{n}}(z;s,t) := W^*_{\vec{m}\vec{n}}(z;s,t) \times \Xi_{\vec{m}\vec{n}}(z)^{-1},$$

where

$$\Xi_{\vec{m}\vec{n}}(z) := \operatorname{diag}\big(z^{-n_1},\dots,z^{-n_p},z^{m_1},\dots,z^{m_q}\big).$$

The matrices $\tilde{W}_{\vec{m}\vec{n}}(z;s,t), \tilde{W}^*_{\vec{m}\vec{n}}(z;s,t)$ have exactly the same form as the wave and adjoint wave functions of the multi-component KP hierarchy in (215) and (216). Furthermore, they satisfy the bilinear identity (226) with $\vec{m} = \vec{m}'$ and $\vec{n} = \vec{n}'$, characterizing the $(p+q)$-component KP hierarchy. Consequently, by virtue of Theorem 5.16, they are wave and adjoint wave functions for the $(p+q)$-component KP hierarchy.

Define the Hirota symbol between functions $f = f(t_1, t_2, \dots)$ and $g = g(t_1, t_2, \dots)$, given a polynomial $p(t_1, t_2, \dots)$, namely

$$p\Big(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \dots\Big) f \circ g := p\Big(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots\Big) f(t+y) g(t-y)\Big|_{y=0}.$$

This operation extends readily to the case where $p(t_1, t_2, \dots)$ is a Taylor series in $t_1, t_2, \dots$. Remember the elementary Schur polynomials S_l are defined by $e^{\sum_{k=1}^{\infty} t_k z^k} := \sum_{k=0}^{\infty} S_k(t) z^k$, for $l \geq 0$, and $S_l(t) = 0$ for $l < 0$. Moreover, set

$$S_l(\tilde{\partial}_t) := S_l\Big(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots\Big).$$

With these notations, computing the residues about $z = \infty$ in the contour integrals (227), the functions $\tau_{\vec{m}\vec{n}}$, with $|\vec{m}| = |\vec{n}|$, are found to satisfy the following PDE's :

$$\tau_{\vec{m}\vec{n}}^2 \frac{\partial^2}{\partial t_{j+1}^{(\beta)} \partial t_1^{(\beta')}} \log \tau_{\vec{m}\vec{n}} = S_{j+2\delta_{\beta\beta'}}(\tilde{\partial}_{t^{(\beta)}}) \tau_{\vec{m},\vec{n}+\vec{e}_\beta-\vec{e}_{\beta'}} \circ \tau_{\vec{m},\vec{n}-\vec{e}_\beta+\vec{e}_{\beta'}},$$

$$\tau_{\vec{m}\vec{n}}^2 \frac{\partial^2}{\partial s_{j+1}^{(\alpha)} \partial s_1^{(\alpha')}} \log \tau_{\vec{m}\vec{n}} = S_{j+2\delta_{\alpha\alpha'}}(\tilde{\partial}_{s^{(\alpha)}}) \tau_{\vec{m}-\vec{e}_\alpha+\vec{e}_{\alpha'},\vec{n}} \circ \tau_{\vec{m}+\vec{e}_\alpha-\vec{e}_{\alpha'},\vec{n}},$$

$$\tau_{\vec{m}\vec{n}}^2 \frac{\partial^2}{\partial s_1^{(\alpha)} \partial t_{j+1}^{(\beta)}} \log \tau_{\vec{m}\vec{n}} = -S_j(\tilde{\partial}_{t^{(\beta)}}) \tau_{\vec{m}+\vec{e}_\alpha,\vec{n}+\vec{e}_\beta} \circ \tau_{\vec{m}-\vec{e}_\alpha,\vec{n}-\vec{e}_\beta},$$

(228) $$\tau_{\vec{m}\vec{n}}^2 \frac{\partial^2}{\partial t_1^{(\beta)} \partial s_{j+1}^{(\alpha)}} \log \tau_{\vec{m}\vec{n}} = -S_j(\tilde{\partial}_{s^{(\alpha)}}) \tau_{\vec{m}-\vec{e}_\alpha,\vec{n}-\vec{e}_\beta} \circ \tau_{\vec{m}+\vec{e}_\alpha,\vec{n}+\vec{e}_\beta}.$$

As a consequence, we have the following corollary.

Corollary 5.22. *The function $\tau_{\vec{m},\vec{n}}(s,t)$ satisfies the following identities for $|\vec{m}| = |\vec{n}|$ and $1 \leq k, k' \leq q$, $1 \leq l, l' \leq p$, $k \neq k'$, $l \neq l'$,*

$$\frac{\partial}{\partial t_1^{(l)}} \ln \frac{\tau_{\vec{m},\vec{n}+\vec{e}_l-\vec{e}_{l'}}}{\tau_{\vec{m},\vec{n}-\vec{e}_l+\vec{e}_{l'}}} = \frac{\frac{\partial^2}{\partial t_2^{(l)} \partial t_1^{(l')}} \ln \tau_{\vec{m},\vec{n}}}{\frac{\partial^2}{\partial t_1^{(l)} \partial t_1^{(l')}} \ln \tau_{\vec{m},\vec{n}}},$$

$$\frac{\partial}{\partial t_1^{(l)}} \ln \frac{\tau_{\vec{m}+\vec{e}_k,\vec{n}+\vec{e}_l}}{\tau_{\vec{m}-\vec{e}_k,\vec{n}-\vec{e}_l}} = \frac{\frac{\partial^2}{\partial t_2^{(l)} \partial s_1^{(k)}} \ln \tau_{\vec{m},\vec{n}}}{\frac{\partial^2}{\partial t_1^{(l)} \partial s_1^{(k)}} \ln \tau_{\vec{m},\vec{n}}},$$

$$\frac{\partial}{\partial s_1^{(k)}} \ln \frac{\tau_{\vec{m}-\vec{e}_k+\vec{e}_{k'},\vec{n}}}{\tau_{\vec{m}+\vec{e}_k-\vec{e}_{k'},\vec{n}}} = \frac{\frac{\partial^2}{\partial s_2^{(k)} \partial s_1^{(k')}} \ln \tau_{\vec{m},\vec{n}}}{\frac{\partial^2}{\partial s_1^{(k)} \partial s_1^{(k')}} \ln \tau_{\vec{m},\vec{n}}},$$

$$\frac{\partial}{\partial s_1^{(k)}} \ln \frac{\tau_{\vec{m}-\vec{e}_k,\vec{n}-\vec{e}_l}}{\tau_{\vec{m}+\vec{e}_k,\vec{n}+\vec{e}_l}} = \frac{\frac{\partial^2}{\partial s_2^{(k)} \partial t_1^{(l)}} \ln \tau_{\vec{m},\vec{n}}}{\frac{\partial^2}{\partial s_1^{(k)} \partial t_1^{(l)}} \ln \tau_{\vec{m},\vec{n}}}.$$

PROOF. We shall only give the proof of the first identity. The two first elementary Schur polynomials are given by

$$S_0(x_1, x_2, \dots) = 1, \qquad S_1(x_1, x_2, \dots) = x_1.$$

Consequently, the first equation in (228) with $j = 0$ and $l \neq l'$ gives

$$\begin{aligned} \tau_{\vec{m}\vec{n}}^2 \frac{\partial^2}{\partial t_1^{(l)} \partial t_1^{(l')}} \log \tau_{\vec{m}\vec{n}} &= S_0(\tilde{\partial}_{t^{(l)}}) \tau_{\vec{m},\vec{n}+\vec{e}_l-\vec{e}_{l'}} \circ \tau_{\vec{m},\vec{n}-\vec{e}_l+\vec{e}_{l'}} \\ &= \tau_{\vec{m},\vec{n}+\vec{e}_l-\vec{e}_{l'}} \tau_{\vec{m},\vec{n}-\vec{e}_l+\vec{e}_{l'}}, \end{aligned} \tag{229}$$

while for $j = 1$ and $l \neq l'$ it gives

$$\begin{aligned} \tau_{\vec{m}\vec{n}}^2 \frac{\partial^2}{\partial t_2^{(l)} \partial t_1^{(l')}} \log \tau_{\vec{m}\vec{n}} &= S_1(\tilde{\partial}_{t^{(l)}}) \tau_{\vec{m},\vec{n}+\vec{e}_l-\vec{e}_{l'}} \circ \tau_{\vec{m},\vec{n}-\vec{e}_l+\vec{e}_{l'}} \\ &= \tau_{\vec{m},\vec{n}-\vec{e}_l+\vec{e}_{l'}} \frac{\partial}{\partial t_1^{(l)}} \tau_{\vec{m},\vec{n}+\vec{e}_l-\vec{e}_{l'}} - \tau_{\vec{m},\vec{n}+\vec{e}_l-\vec{e}_{l'}} \frac{\partial}{\partial t_1^{(l)}} \tau_{\vec{m},\vec{n}-\vec{e}_l+\vec{e}_{l'}}. \end{aligned} \tag{230}$$

Taking the ratio of (230) and (229) yields the first formula of Corollary 6.3. The other identities are obtained in a similar way. □

Chapter 6

Non-intersecting Brownian motions leaving from and going to several points

This Chapter is mainly based on [**13**]

1. Non-intersecting Brownian motions : the Karlin-McGregor formula

Let $x(t)$ be the path of a Brownian particle on $\mathbb{R}$, starting at time $t = 0$ in α, and with Gaussian transition probability density given by

$$p(t, x, y) = \frac{1}{\sqrt{\pi t}} e^{\frac{-(x-y)^2}{t}}. \tag{231}$$

It is a stationary Markov process. We will use the notation

$$p(t, x, E) = \int_E p(t, x, y) dy,$$

for the probability to find the Brownian particle in $E \subset \mathbb{R}$ at time $t > 0$, knowing it was in x at time $t = 0$. Consider now N Brownian particles $x_1(t), x_2(t), \ldots, x_N(t)$ in $\mathbb{R}$, leaving from distinct points $\alpha_1 < \alpha_2 < \cdots < \alpha_N$, executing simultaneously and independently the process described above. As the N processes are independent, we have

$$\begin{aligned}\mathbb{P}^{\vec{\alpha}}\big(\vec{x}(t) \in \vec{E}\big) &:= P\big(\vec{x}(t) \in E_1 \times \cdots \times E_N \,\big|\, \vec{x}(0) = \vec{\alpha}\big)\\ &= \prod_{i=1}^{N} p(t, \alpha_i, E_i),\end{aligned}$$

where $E_i \subset \mathbb{R}$, $1 \leq i \leq N$, $\vec{x}(t) = \big(x_1(t), \ldots, x_N(t)\big)$ and $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$. We are interested in the probability density that N Brownian particles starting at prescribed positions at time $t = 0$, end up at prescribed positions at a time $t > 0$, without two of them ever having been coincident during the time interval $[0, t]$. It is a well-known result due to Karlin and McGregor [**49**] that this probability is given by a determinant expressed in terms of the transition probability density $p(t, x, y)$. In this section, we give the proof of this formula.

We define the set $F \subset \mathbb{R}^N$ of coincident states

$$F = \big\{(x_1, \ldots, x_N) \in \mathbb{R}^N \,\big|\, \exists i < j \,:\, x_i = x_j\big\}.$$

A permutation $\lambda \in S_N$ is called a transposition if there exist $1 \leq i < j \leq N$ such that

$$\lambda(i) = j, \qquad \lambda(j) = i, \qquad \lambda(k) = k \quad \text{if } k \neq i, j.$$

We use the notation $\lambda = (i, j)$. A coincident state $\vec{x} = (x_1, \ldots, x_N) \in F$ is said to belong to the transposition (i, j) with $i < j$, if $x_1, \ldots, x_{j-1}$ are all different but $x_i = x_j$. Consequently, every coincident state belongs to a unique transposition. For a transposition λ, the set of all coincident states belonging to λ will be denoted $F(\lambda)$. The set of all transpositions of $1, \ldots, N$ will be denoted Λ. We have of course $F = \cup_{\lambda \in \Lambda} F(\lambda)$, and the sets $F(\lambda)$ are disjoint. Let $T(\vec{x})$ be the time of first coincidence (the time of first hitting F), i.e.

$$T(\vec{x}) = \inf\{t > 0 \,|\, \exists 1 \leq i < j \leq N \,:\, x_i(t) = x_j(t)\}.$$

As the Brownian particles have continuous path functions $x_i(t)$, and transition probabilities $p(t; x, y)$ continuous in t and x, we have the following identity

$$\begin{aligned}P^{\vec{\alpha}}\big(\vec{x}(t) \in \vec{E}\big) &= P^{\vec{\alpha}}\big(\vec{x}(t) \in \vec{E},\, T(\vec{x}) > t\big)\\ &\quad + \int_0^t ds \int_F d\vec{y}\, P^{\vec{\alpha}}\big(\vec{x}(s) = \vec{y},\, T(\vec{x}) = s\big)\, P^{\vec{y}}\big(\vec{x}(t-s) \in \vec{E}\big). \qquad (232)\end{aligned}$$

See for instance [**49**] for a proof.

Theorem 6.1 (Karlin-McGregor [**49**]). *The transition probability density to find the Brownian particles in $\gamma_1 < \gamma_2 < \cdots < \gamma_N$ at a time $0 < t$, without two of them ever having been coincident during the time interval $[0, t]$, is given by the determinant*

$$P^{\vec{\alpha}}\big(\vec{x}(t) = \vec{\gamma},\, T(\vec{x}) > t\big) = \det\big[p(t, \alpha_i, \gamma_j)\big]_{1 \leq i,j \leq N}.$$

PROOF. On the one hand we have

$$\sum_{\sigma\in S_N}(-1)^\sigma P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_\sigma\big) = \sum_{\sigma\in S_N}(-1)^\sigma\prod_{i=1}^N p(t,\alpha_i,\gamma_{\sigma(i)})$$

$$= \det\big[p(t,\alpha_i,\gamma_j)\big]_{1\le i,j\le N}, \tag{233}$$

where $\vec\gamma_\sigma = (\gamma_{\sigma(1)},\dots,\gamma_{\sigma(N)})$. On the other hand, by virtue of (232) we have for $\sigma\in S_N$

$$P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_\sigma\big) = P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_\sigma,\ T(\vec x)>t\big) + \int_0^t ds\int_F d\vec y\, P^{\vec\alpha}\big(\vec x(s)=\vec y,\ T(\vec x)=s\big)\, P^{\vec y}\big(\vec x(t-s)=\vec\gamma_\sigma\big).$$

Consequently

$$\sum_{\sigma\in S_N}(-1)^\sigma\Big(P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_\sigma\big) - P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_\sigma,\ T(\vec x)>t\big)\Big)$$

$$= \sum_{\sigma\in S_N}(-1)^\sigma\int_0^t ds\int_F d\vec y\, P^{\vec\alpha}\big(\vec x(s)=\vec y,\ T(\vec x)=s\big)\, P^{\vec y}\big(\vec x(t-s)=\vec\gamma_\sigma\big)$$

$$= \sum_{\sigma\in S_N}\sum_{\lambda\in\Lambda}(-1)^\sigma\int_0^t ds\int_{F(\lambda)} d\vec y\, P^{\vec\alpha}\big(\vec x(s)=\vec y,\ T(\vec x)=s\big) \times P^{\vec y}\big(\vec x(t-s)=\vec\gamma_\sigma\big)$$

$$= -\sum_{\sigma\in S_N}\sum_{\lambda\in\Lambda}(-1)^{\sigma\circ\lambda}\int_0^t ds\int_{F(\lambda)} d\vec y\, P^{\vec\alpha}\big(\vec x(s)=\vec y,\ T(\vec x)=s\big) \times P^{\vec y}\big(\vec x(t-s)=\vec\gamma_{\sigma\circ\lambda}\big).$$

For a fixed $\lambda\in\Lambda$ define $\tilde\sigma=\sigma\circ\lambda$, $\sigma\in S_N$. When σ runs over S_N, then so does $\tilde\sigma=\sigma\circ\lambda$. So we get

$$\sum_{\sigma\in S_N}(-1)^\sigma\Big(P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_\sigma\big) - P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_\sigma,\ T(\vec x)>t\big)\Big)$$

$$= -\sum_{\lambda\in\Lambda}\sum_{\tilde\sigma\in S_N}(-1)^{\tilde\sigma}\int_0^t ds\int_{F(\lambda)} d\vec y\, P^{\vec\alpha}\big(\vec x(s)=\vec y,\ T(\vec x)=s\big) \times P^{\vec y}\big(\vec x(t-s)=\vec\gamma_{\tilde\sigma}\big)$$

$$= -\sum_{\tilde\sigma\in S_N}(-1)^{\tilde\sigma}\Big(P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_{\tilde\sigma}\big) - P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_{\tilde\sigma},\ T(\vec x)>t\big)\Big).$$

Consequently we have

$$\sum_{\sigma\in S_N}(-1)^\sigma\Big(P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_\sigma\big)-P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_\sigma,\ T(\vec x)>t\big)\Big)=0,$$

and thus

$$\sum_{\sigma\in S_N}(-1)^\sigma P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_\sigma\big)=\sum_{\sigma\in S_N}(-1)^\sigma P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_\sigma,\ T(\vec x)>t\big).$$

As $\gamma_1<\gamma_2<\cdots<\gamma_N$, all the terms except one are equal to zero in the right-hand side of this relation. We get

$$\sum_{\sigma\in S_N}(-1)^\sigma P^{\vec\alpha}\big(\vec x(t)=\vec\gamma_\sigma\big)=P^{\vec\alpha}\big(\vec x(t)=\vec\gamma,\ T(\vec x)>t\big).$$

Comparing this with (233) concludes the proof. □

This Theorem enables us to compute the probability density to find the Brownian particles in positions $\beta_1<\beta_2<\cdots<\beta_N$ at time $t=1$ and in positions $\gamma_1<\gamma_2<\cdots<\gamma_N$ at an intermediate time $0<t<1$, without two of them ever having been coincident during the time interval $[0,1]$. We have

$$\begin{aligned}&P^{\vec\alpha}\big(\vec x(t)=\vec\gamma,\ \vec x(1)=\vec\beta,\ T(\vec x)>1\big)\\&\qquad=\det\big[p(t,\alpha_i,\gamma_j)\big]_{1\le i,j\le N}\det\big[p(1-t,\gamma_i,\beta_j)\big]_{1\le i,j\le N},\end{aligned}$$

and we have

$$\begin{aligned}&P^{\vec\alpha}\big(\vec x(1)=\vec\beta,\ T(\vec x)>1\big)\\&\quad=\int_{y_1<y_2<\cdots<y_N}\det\big[p(t,\alpha_i,y_j)\big]_{1\le i,j\le N}\\&\qquad\qquad\times\ \det\big[p(1-t,y_i,\beta_j)\big]_{1\le i,j\le N}d\vec y\\&\quad=\frac{1}{N!}\int_{\mathbb R^N}\det\big[p(t,\alpha_i,y_j)\big]_{1\le i,j\le N}\det\big[p(1-t,y_i,\beta_j)\big]_{1\le i,j\le N}d\vec y.\end{aligned}$$

Consequently, the conditional probability to find the Brownian particles in a set E at an intermediate time $0<t<1$, provided the particles are in β at time $t=1$, without two of them ever having been coincident during the time interval $[0,1]$, is given by

$$\begin{aligned}&P^{\vec\alpha}\big(\vec x(t)\in E^N\,|\,\vec x(1)=\vec\beta,\ T(\vec x)>1\big)\\&\quad=\frac{P^{\vec\alpha}\big(\vec x(t)\in E^N,\ \vec x(1)=\vec\beta,\ T(\vec x)>1\big)}{P^{\vec\alpha}\big(\vec x(1)=\vec\beta,\ T(\vec x)>1\big)}\\&\quad=\frac{\int_{E^N}\det\big[p(t,\alpha_i,y_j)\big]_{1\le i,j\le N}\det\big[p(1-t,y_i,\beta_j)\big]_{1\le i,j\le N}d\vec y}{\int_{\mathbb R^N}\det\big[p(t,\alpha_i,y_j)\big]_{1\le i,j\le N}\det\big[p(1-t,y_i,\beta_j)\big]_{1\le i,j\le N}d\vec y}\\&\quad=:\mathbb P^{\vec\alpha}_{\vec\beta}\big(\vec x(t)\in E^N\big).\end{aligned}\tag{234}$$

2. Non-intersecting Brownian motions with q starting points and p ending points

Consider N Brownian motions $x_1(t)$, $x_2(t)$, $\ldots$, $x_N(t)$ in $\mathbb{R}$, conditioned to leave from distinct points $\alpha_1 < \alpha_2 < \cdots < \alpha_N$ at time $t = 0$ and to end up at distinct points $\beta_1 < \beta_2 < \cdots < \beta_N$ at time $t = 1$, without two of them ever having been coincident during the time interval $[0, 1]$. By virtue of (234), the conditional probability that all $x_i(t)$ belong to a set $E \subset \mathbb{R}$ at a given time $0 < t < 1$ is

$$\begin{aligned}
&\mathbb{P}^{\vec{\alpha}}_{\vec{\beta}}\big(\vec{x}(t) \in E^N\big)\\
&= \frac{1}{Z_N} \int_{E^N} \det\big[p(t, \alpha_i, x_j)\big]_{1\le i,j\le N} \det\big[p(1-t, x_i, \beta_j)\big]_{1\le i,j\le N} \prod_{i=1}^{N} dx_i\\
&= \frac{1}{\tilde{Z}_N} \int_{E^N} \det\big[e^{\frac{2\alpha_i x_j}{t}}\big]_{1\le i,j\le N} \det\big[e^{\frac{2\beta_i x_j}{1-t}}\big]_{1\le i,j\le N} \prod_{i=1}^{N} e^{\frac{-x_i^2}{t(1-t)}}\, dx_i,
\end{aligned}$$

where Z_N and $\tilde{Z}_N$ are normalizing factors. In particular, if

$$\begin{aligned}
(\alpha_1, \ldots, \alpha_N) &= \big(\underbrace{a_1, a_1, \ldots, a_1}_{m_1}, \underbrace{a_2, a_2, \ldots, a_2}_{m_2}, \ldots, \underbrace{a_q, a_q, \ldots, a_q}_{m_q}\big),\\
&\qquad\qquad a_1 < a_2 < \cdots < a_q,\\
(\beta_1, \ldots, \beta_N) &= \big(\underbrace{b_1, b_1, \ldots, b_1}_{n_1}, \underbrace{b_2, b_2, \ldots, b_2}_{n_2}, \ldots, \underbrace{b_p, b_p, \ldots, b_p}_{n_p}\big),\\
&\qquad\qquad b_1 < b_2 < \cdots < b_p,
\end{aligned}$$

with $\sum_{i=1}^{q} a_i = \sum_{i=1}^{p} b_i = 0$ and $\sum_{i=1}^{q} m_i = \sum_{i=1}^{p} n_i = N$, then we have

$$\begin{aligned}
&\mathbb{P}^{a_1,\ldots,a_q}_{b_1,\ldots,b_p}\big(\vec{x}(t) \in E^N\big)\\
&:= \mathbb{P}\left(\vec{x}(t) \in E^N \,\middle|\, \begin{array}{l} \big(x_1(0), \ldots, x_N(0)\big) = \big(\underbrace{a_1, \ldots, a_1}_{m_1}, \ldots, \underbrace{a_q, \ldots, a_q}_{m_q}\big)\\ \big(x_1(1), \ldots, x_N(1)\big) = \big(\underbrace{b_1, \ldots, b_1}_{n_1}, \ldots, \underbrace{b_p, \ldots, b_p}_{n_p}\big)\end{array}\right)\\
&= \lim_{\substack{\alpha_1,\ldots,\alpha_{m_1}\to a_1\\ \cdots\\ \alpha_{m_1+\cdots+m_{q-1}+1},\ldots,\alpha_{m_1+\cdots+m_q}\to a_q\\ \beta_1,\ldots,\beta_{n_1}\to b_1\\ \cdots\\ \beta_{n_1+\cdots+n_{p-1}+1},\ldots,\beta_{n_1+\cdots+n_p}\to b_p}} \mathbb{P}^{\vec{\alpha}}_{\vec{\beta}}\big(\vec{x}(t) \in E^N\big).
\end{aligned}$$

Consequently, we obtain

$$\mathbb{P}^{a_1,\dots,a_q}_{b_1,\dots,b_p}\big(\vec{x}(t)\in E^N\big)=\frac{1}{\hat{Z}_N}\int_{E^N}\prod_{i=1}^N e^{\frac{-x_i^2}{t(1-t)}}\,dx_i$$

$$\times\ \det\begin{pmatrix}\left(x_j^i e^{\frac{2a_1x_j}{t}}\right)_{\substack{0\le i\le m_1-1\\1\le j\le N}}\\ \vdots\\ \left(x_j^i e^{\frac{2a_qx_j}{t}}\right)_{\substack{0\le i\le m_q-1\\1\le j\le N}}\end{pmatrix}.\det\begin{pmatrix}\left(x_j^i e^{\frac{2b_1x_j}{1-t}}\right)_{\substack{0\le i\le n_1-1\\1\le j\le N}}\\ \vdots\\ \left(x_j^i e^{\frac{2b_px_j}{1-t}}\right)_{\substack{0\le i\le n_p-1\\1\le j\le N}}\end{pmatrix}.$$

We have, using the change of variables $x_i=\sqrt{\frac{t(1-t)}{2}}\,y_i$, $1\le i\le N$,

$$\mathbb{P}^{a_1,\dots,a_q}_{b_1,\dots,b_p}\big(\vec{x}(t)\in E^N\big)=P_{p,q}\Big(\sqrt{\frac{2}{t(1-t)}}\,E;\sqrt{\frac{2(1-t)}{t}}\,a,\sqrt{\frac{2t}{1-t}}\,b\Big),\tag{235}$$

with the normalized problem being

$$P_{p,q}(E;a,b):=\frac{1}{Z_{p,q}}\int_{E^N}\Big(\prod_{i=1}^N e^{\frac{-x_i^2}{2}}\,dx_i\Big)$$
$$\det\big[\tilde{\psi}_i(x_j)\big]_{1\le i,j\le N}\ \det\big[\tilde{\varphi}_i(x_j)\big]_{1\le i,j\le N},\tag{236}$$

where $Z_{p,q}$ is a normalizing factor, and where we have introduced the following notation

$$\begin{aligned}\big(\tilde{\psi}_1(x),\dots,\tilde{\psi}_N(x)\big):=&\Big(e^{a_1x},xe^{a_1x},\dots,x^{m_1-1}e^{a_1x},e^{a_2x},xe^{a_2x},\dots,\\&x^{m_2-1}e^{a_2x},\dots,e^{a_qx},xe^{a_qx},\dots,x^{m_q-1}e^{a_qx}\Big),\\ \big(\tilde{\varphi}_1(x),\dots,\tilde{\varphi}_N(x)\big):=&\Big(e^{b_1x},xe^{b_1x},\dots,x^{n_1-1}e^{b_1x},e^{b_2x},xe^{b_2x},\dots,\\&x^{n_2-1}e^{b_2x},\dots,e^{b_px},xe^{b_px},\dots,x^{n_p-1}e^{b_px}\Big).\end{aligned}$$

In the following proposition, we consider a general situation, of which (236) is a special case by setting $V(x)=\frac{x^2}{2}$, $\psi_i(x)=e^{a_ix},\ 1\le i\le q$, and $\varphi_i(x)=e^{b_ix},\ 1\le i\le p$.

Proposition 6.2 (Adler-van Moerbeke-Vanderstichelen [**13**]). *Given an arbitrary potential $V(x)$ and arbitrary functions $\psi_1(x),\dots,\psi_q(x)$ and $\varphi_1(x),\dots,\varphi_p(x)$, define*

($N = m_1 + \cdots + m_q = n_1 + \ldots n_p$)

$$\begin{aligned}\big(\tilde\psi_1(x),\ldots,\tilde\psi_N(x)\big) &:= \Big(\psi_1(x), x\psi_1(x),\ldots,x^{m_1-1}\psi_1(x),\psi_2(x),\\ &\qquad x\psi_2(x),\ldots,x^{m_2-1}\psi_2(x),\ldots,\psi_q(x),x\psi_q(x),\ldots,x^{m_q-1}\psi_q(x)\Big),\\ \big(\tilde\varphi_1(x),\ldots,\tilde\varphi_N(x)\big) &:= \Big(\varphi_1(x), x\varphi_1(x),\ldots,x^{n_1-1}\varphi_1(x),\varphi_2(x),\\ &\qquad x\varphi_2(x),\ldots,x^{n_2-1}\varphi_2(x),\ldots,\varphi_p(x),x\varphi_p(x),\ldots,x^{n_p-1}\varphi_p(x)\Big).\end{aligned}$$

We have

$$\begin{aligned}&N!\int_{E^N}\Big(\prod_{i=1}^N e^{-V(x_i)}\,dx_i\Big)\,\det\big[\tilde\psi_i(x_j)\big]_{1\le i,j\le N}\,\det\big[\tilde\varphi_i(x_j)\big]_{1\le i,j\le N}\\ &=(N!)^2\det\left[\int_E\tilde\psi_i(x)\tilde\varphi_j(x)e^{-V(x)}dx\right]_{1\le i,j\le N}\\ &=\binom{N}{m_1,m_2,\ldots,m_q}\binom{N}{n_1,n_2,\ldots,n_p}\int_{E^N}\Big(\prod_{i=1}^N e^{-V(x_i)}\,dx_i\Big)\\ &\quad\times\Big(\Delta_{m_1}(x^{(1)})\prod_{i=1}^{m_1}\psi_1(x_i)\Big)\times\ldots\times\Big(\Delta_{m_q}(x^{(q)})\prod_{i=1}^{m_q}\psi_q(x_{m_1+\cdots+m_{q-1}+i})\Big)\\ &\quad\times\sum_{\sigma\in S_N}(-1)^\sigma\Big[\Big(\Delta_{n_1}(x_{\sigma(1)},\ldots,x_{\sigma(n_1)})\prod_{i=1}^{n_1}\varphi_1(x_{\sigma(i)})\Big)\\ &\quad\times\cdots\times\Big(\Delta_{n_p}(x_{\sigma(n_1+\cdots+n_{p-1}+1)},\ldots,x_{\sigma(n_1+\cdots+n_p)})\\ &\quad\times\prod_{i=1}^{n_p}\varphi_p(x_{\sigma(n_1+\cdots+n_{p-1}+i)})\Big)\Big],\end{aligned}\tag{237}$$

where $x^{(1)} = (x_1, x_2, \ldots, x_{m_1})$, $\ldots$, $x^{(q)} = (x_{m_1+\cdots+m_{q-1}+1}, \ldots, x_{m_1+\cdots+m_q})$, *and* $\Delta_n(x_1,\ldots,x_n) = \det[x_j^{i-1}]_{1\le i,j\le n}$ *is the Vandermonde determinant.*

PROOF. Let $\tilde P(E;p,q)$ be left hand side in (237)

$$\tilde P(E;p,q) := N!\int_{E^N}\Big(\prod_{i=1}^N e^{-V(x_i)}\,dx_i\Big)\times\det\big[\tilde\psi_i(x_j)\big]_{1\le i,j\le N}\,\det\big[\tilde\varphi_i(x_j)\big]_{1\le i,j\le N}.$$

The first identity in (237) is a consequence of applying the following standard identity

$$\det[a_{ij}]_{1\le i,j\le N}\det[b_{ij}]_{1\le i,j\le N} = \sum_{\sigma\in S_N}\det\big[a_{i,\sigma(j)}b_{j,\sigma(j)}\big]_{1\le i,j\le N},$$

and distributing the integration over the different columns.

We prove now the second identity in (237). Working out the determinant $\det\left[\tilde{\psi}_i(x_j)\right]_{1\le i,j\le N}$ we obtain

$$\tilde{P}(E;p,q) = N! \sum_{\sigma\in S_N} (-1)^\sigma \int_{E^N} \Big(\prod_{i=1}^N e^{-V(x_i)}\,dx_i\Big) \times \Big(\prod_{i=1}^N \tilde{\psi}_i(x_{\sigma(i)})\Big) \det\left[\tilde{\varphi}_i(x_j)\right]_{1\le i,j\le N}.$$

In each term of this summation, we make the change of variables $x_i = y_{\sigma^{-1}(i)}$, $1\le i\le N$. We have

$$\tilde{P}(E;p,q) = (N!)^2 \int_{E^N} \Big(\prod_{i=1}^N e^{-V(y_i)}\,dy_i\Big) \times \Big(\prod_{i=1}^N \tilde{\psi}_i(y_i)\Big) \det\left[\tilde{\varphi}_i(y_j)\right]_{1\le i,j\le N},$$

since $\det\left[\tilde{\varphi}_i(y_j)\right]_{1\le i,j\le N} = (-1)^\sigma \det\left[\tilde{\varphi}_i(x_j)\right]_{1\le i,j\le N}$. Now take $\sigma_1\in S_{m_1}, \sigma_2\in S_{m_2},\dots,\sigma_q\in S_{m_q}$ arbitrarily and define the permutation $\sigma := \sigma_1\times\sigma_2\times\cdots\times\sigma_q \in S_N$. Consider the following change of variables $y\to z$ defined by σ:

$$\begin{aligned}(y_1,\dots,y_{m_1}) &= (z_{\sigma_1(1)},\dots,z_{\sigma_1(m_1)}),\\ (y_{m_1+1},\dots,y_{m_1+m_2}) &= (z_{m_1+\sigma_2(1)},\dots,z_{m_1+\sigma_2(m_2)}),\\ &\vdots\\ (y_{m_1+\cdots+m_{q-1}+1},\dots,y_{m_1+\cdots+m_q}) &= (z_{m_1+\cdots+m_{q-1}+\sigma_q(1)},\dots,z_{m_1+\cdots+m_{q-1}+\sigma_q(m_q)}).\end{aligned}$$

This change of variables leaves the integral unchanged. Consequently, if we sum over all the permutations $\sigma_1\times\sigma_2\times\cdots\times\sigma_q \in S_{m_1}\times\cdots\times S_{m_q}$ and divide by $m_1!m_2!\dots m_q!$, we have by the definition of $\tilde{\psi}_i$ and $\Delta_n(z)$

$$\begin{aligned}\tilde{P}(E;p,q) &= N!\binom{N}{m_1,m_2,\dots,m_q}\int_{E^N}\Big(\prod_{i=1}^N e^{-V(z_i)}\,dz_i\Big)\Big(\Delta_{m_1}(z^{(1)})\prod_{i=1}^{m_1}\psi_1(z_i)\Big)\\ &\quad\times\Big(\Delta_{m_2}(z^{(2)})\prod_{i=1}^{m_2}\psi_2(z_{m_1+i})\Big)\times\cdots\times\\ &\quad\times\Big(\Delta_{m_q}(z^{(q)})\prod_{i=1}^{m_q}\psi_q(z_{m_1+\cdots+m_{q-1}+i})\Big)\det\left[\tilde{\varphi}_i(z_j)\right]_{1\le i,j\le N}.\end{aligned}\tag{238}$$

We develop the determinant $\det\left[\tilde{\varphi}_i(z_j)\right]_{1\le i,j\le N}$ in this expression

$$\begin{aligned}\det\left[\tilde{\varphi}_i(z_j)\right]_{1\le i,j\le N} &= \sum_{\sigma\in S_N}(-1)^\sigma\prod_{i=1}^N\tilde{\varphi}_i\big(z_{\sigma(i)}\big)\\ &= \sum_{\sigma\in S_N}(-1)^\sigma\Big(\prod_{i=1}^{n_1}\varphi_1\big(z_{\sigma(i)}\big)z_{\sigma(i)}^{i-1}\Big)\Big(\prod_{i=1}^{n_2}\varphi_2\big(z_{\sigma(n_1+i)}\big)z_{\sigma(n_1+i)}^{i-1}\Big)\\ &\qquad\times\cdots\times\Big(\prod_{i=1}^{n_p}\varphi_p\big(z_{\sigma(n_1+\cdots+n_{p-1}+i)}\big)z_{\sigma(n_1+\cdots+n_{p-1}+i)}^{i-1}\Big).\end{aligned}$$

Fix $\tilde{\sigma}_1\in S_{n_1},\ldots,\tilde{\sigma}_p\in S_{n_p}$ and, for a given permutation $\sigma\in S_N$, let $\tilde{\sigma}\in S_N$ be such that $\sigma=\tilde{\sigma}\circ(\tilde{\sigma}_1\times\tilde{\sigma}_2\times\cdots\times\tilde{\sigma}_p)$, but when $\tilde{\sigma}$ runs over S_N, then so does σ. Consequently we have

$$\begin{aligned}&\det\left[\tilde{\varphi}_i(z_j)\right]_{1\le i,j\le N}\\ &\quad= \sum_{\tilde{\sigma}\in S_N}(-1)^{\tilde{\sigma}}(-1)^{\tilde{\sigma}_1}\ldots(-1)^{\tilde{\sigma}_p}\Big(\prod_{i=1}^{n_1}\varphi_1\big(z_{\tilde{\sigma}\circ\tilde{\sigma}_1(i)}\big)z_{\tilde{\sigma}\circ\tilde{\sigma}_1(i)}^{i-1}\Big)\times\cdots\\ &\qquad\times\Big(\prod_{i=1}^{n_p}\varphi_p\big(z_{\tilde{\sigma}(n_1+\cdots+n_{p-1}+\tilde{\sigma}_p(i))}\big)z_{\tilde{\sigma}(n_1+\cdots+n_{p-1}+\tilde{\sigma}_p(i))}^{i-1}\Big).\end{aligned}$$

We substitute this expression in equation (238). For each $\tilde{\sigma}\in S_N$ we further sum over $\tilde{\sigma}_1\in S_{n_1},\tilde{\sigma}_2\in S_{n_2},\ldots,\tilde{\sigma}_p\in S_{n_p}$ and so must divide by $n_1!n_2!\ldots n_p!$ as we have overcounted. We then obtain the second equality in (237). This ends the proof. □

As a consequence, setting $V(x)=\frac{x^2}{2}$, $\psi_i(x)=e^{a_ix}$, $1\le i\le q$, and $\varphi_i(x)=e^{b_ix}$, $1\le i\le p$, in (237), we have for the normalized problem

$$\begin{aligned}P_{p,q}(E;a,b) &= \frac{1}{Z}\int_{E^N}\Big(\prod_{i=1}^N e^{-\frac{x_i}{2}}\,dx_i\Big)\Big(\Delta_{m_1}(x^{(1)})\prod_{i=1}^{m_1}e^{a_1x_i}\Big)\times\cdots\\ &\times\Big(\Delta_{m_q}(x^{(q)})\prod_{i=1}^{m_q}e^{a_qx_{m_1+\cdots+m_{q-1}+i}}\Big)\\ &\times\sum_{\sigma\in S_N}(-1)^\sigma\Big[\Big(\Delta_{n_1}(x_{\sigma(1)},\ldots,x_{\sigma(n_1)})\prod_{i=1}^{n_1}e^{b_1x_{\sigma(i)}}\Big)\times\cdots\\ &\times\Big(\Delta_{n_p}(x_{\sigma(n_1+\cdots+n_{p-1}+1)},\ldots,x_{\sigma(n_1+\cdots+n_p)})\prod_{i=1}^{n_p}e^{b_px_{\sigma(n_1+\cdots+n_{p-1}+i)}}\Big)\Big],\end{aligned}\tag{239}$$

where Z is a normalizing constant.

Suppose $p=q=1$. Then we have $\alpha_i=\beta_i=0$, $1\le i\le N$, and all the Brownian particles start at $t=0$ in $x=0$, and end up at $t=1$ in $x=0$. Then the sum in the normalized problem (239) reduces to

$$P_{1,1}(E;0,0) = \frac{1}{Z}\int_{E^N} \Delta_N(x)^2 \Big(\prod_{i=1}^{N} e^{-\frac{x_i}{2}}\, dx_i\Big).$$

Remembering (27), we notice that this is nothing but the probability that a randomly chosen $N \times N$ Hermitian matrix from the GUE ensemble has all its eigenvalues in E. Through transformation (235), we have thus an interpretation of the GUE ensemble in terms of non-intersecting Brownian motions leaving at time $t = 0$ from and ending up at time $t = 1$ in the origin.

Suppose now $q = 1$. Then we have $\alpha_i = 0$, $1 \le i \le N$, and all the Brownian particles start at $t = 0$ in $x = 0$, and n_1 end up in b_1, n_2 end up in b_2, ..., n_p end up in b_p at time $t = 1$. All the terms in the normalized problem (239) are equal, and we get

$$\begin{aligned} P_{p,1}(E;0,b) = \frac{1}{Z}\int_{E^N} \Delta_N(x) \Big(\Delta_{n_1}(x^{(1)}) \prod_{i=1}^{n_1} e^{b_1 x_i}\Big) \\ \times \cdots \times \Big(\Delta_{n_p}(x^{(p)}) \prod_{i=1}^{n_p} e^{b_p x_{n_1+\cdots+n_{p-1}+i}}\Big)\Big(\prod_{i=1}^{N} e^{-\frac{x_i}{2}}\, dx_i\Big). \end{aligned}$$

Comparing this relation with (30), we notice that this is nothing but the probability that a $N \times N$ Hermitian matrix from the Gaussian ensemble with external source

$$A = \mathrm{diag}\big(\underbrace{b_1, \dots, b_1}_{n_1}, \dots, \underbrace{b_p, \dots, b_p}_{n_p}\big),$$

has all its eigenvalues in E. Through transformation (235), we have thus an interpretation of the Gaussian ensemble with external source in terms of non-intersecting Brownian motions leaving at time $t = 0$ from the origin and ending up at time $t = 1$ in p different points $b_1 < \cdots < b_p$.

The two special cases we have treated here ($p = 1$ and/or $q = 1$) and their interpretation in terms of joint eigenvalue probabilities of matrix ensembles have been studied and are very well understood. We refer to [**8, 11, 22, 23, 63, 64**] and references herein, and to [**15**] for the interpretation in terms of matrix ensembles. The case when both $p \neq 1$ and $q \neq 1$ differs from these two particular cases as no interpretation of the Brownian motion model in terms of matrix ensembles is known.

3. An integrable deformation of the joint probability density function

The connection of the problem of non-intersecting brownian motions on $\mathbb{R}$ with the multi-component KP hierarchy is explained in [**14**]. The main ideas are being sketched in this section. We will deform $P_{p,q}(E;a,b)$ defined in (236) by adding extra time variables

$$t^{(1)} = \big(t_1^{(1)}, t_2^{(1)}, \dots\big), \quad \dots \quad , \quad t^{(p)} = \big(t_1^{(p)}, t_2^{(p)}, \dots\big),$$
$$s^{(1)} = \big(s_1^{(1)}, s_2^{(1)}, \dots\big), \quad \dots \quad , \quad s^{(q)} = \big(s_1^{(q)}, s_2^{(q)}, \dots\big),$$

and auxiliary variables

$$(\alpha_1, \dots, \alpha_q), \quad (\beta_1, \dots, \beta_p),$$

such that $\sum_{i=1}^{q} \alpha_i = \sum_{j=1}^{p} \beta_j = 0$. First set

$$\psi_i^{-s}(x) := e^{a_i x + \alpha_i x^2 - \sum_{j=1}^{\infty} s_j^{(i)} x^j}, \quad 1 \le i \le q,$$
$$\varphi_i^{t}(x) := e^{b_i x + \beta_i x^2 + \sum_{j=1}^{\infty} t_j^{(i)} x^j}, \quad 1 \le i \le p.$$

We define

$$P_{p,q}\big(E;a,b;(t,s),(\alpha,\beta)\big) = \frac{\tau^{E}_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b)}{\tau^{\mathbb{R}}_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b)}, \tag{240}$$

with

$$\begin{aligned}\tau^{E}_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b) :=& \frac{1}{N!}\int_{E^N} \Big(\prod_{i=1}^{N} e^{\frac{-x_i^2}{2}}\, dx_i\Big) \\ & \times \det\big[\tilde{\psi}_i^{-s}(x_j)\big]_{1\le i,j\le N} \det\big[\tilde{\varphi}_i^{t}(x_j)\big]_{1\le i,j\le N},\end{aligned} \tag{241}$$

where

$$(t,s) = \big(t^{(1)}, \dots, t^{(p)}; s^{(1)}, \dots, s^{(q)}\big),$$
$$(\alpha,\beta) = (\alpha_1, \dots, \alpha_{q-1}; \beta_1, \dots, \beta_{p-1}),$$
$$(a,b) = (a_1, \dots, a_{q-1}; b_1, \dots, b_{p-1}),$$

and

$$\begin{aligned}\big(\tilde{\psi}_1^{-s}(x), \dots, \tilde{\psi}_N^{-s}(x)\big) :=& \Big(\psi_1^{-s}(x), x\psi_1^{-s}(x), \dots, x^{m_1-1}\psi_1^{-s}(x), \dots, \\ & \psi_q^{-s}(x), x\psi_q^{-s}(x), \dots, x^{m_q-1}\psi_q^{-s}(x)\Big),\\ \big(\tilde{\varphi}_1^{t}(x), \dots, \tilde{\varphi}_N^{t}(x)\big) :=& \Big(\varphi_1^{t}(x), x\varphi_1^{t}(x), \dots, x^{n_1-1}\varphi_1^{t}(x), \dots, \\ & \varphi_p^{t}(x), x\varphi_p^{t}(x), \dots, x^{n_p-1}\varphi_p^{t}(x)\Big).\end{aligned}$$

Observe that $P_{p,q}(E;a,b) = P_{p,q}\big(E;a,b;(t,s),(\alpha,\beta)\big)\big|_{\mathcal{L}}$, where $\mathcal{L} = \{(t,s) = (0,0), \alpha=\beta=0\}$. By virtue of proposition 6.2 we have

$$\tau^E_{\vec m,\vec n}(t,s;\alpha,\beta;a,b) = \det\left[\int_E \tilde\psi_i^{-s}(x)\tilde\varphi_j^t(x)e^{\frac{-x^2}{2}}\,dx\right]_{1\le i,j\le N}$$

$$= \det\begin{pmatrix} \left(\int_E x^{i+j}\psi_1^{-s}\varphi_1^t e^{-\frac{x^2}{2}}\,dx\right)_{\substack{0\le i<m_1\\0\le j<n_1}} & \dots & \left(\int_E x^{i+j}\psi_1^{-s}\varphi_p^t e^{-\frac{x^2}{2}}\,dx\right)_{\substack{0\le i<m_1\\0\le j<n_p}} \\ \vdots & & \vdots \\ \left(\int_E x^{i+j}\psi_q^{-s}\varphi_1^t e^{-\frac{x^2}{2}}\,dx\right)_{\substack{0\le i<m_q\\0\le j<n_1}} & \dots & \left(\int_E x^{i+j}\psi_q^{-s}\varphi_p^t e^{-\frac{x^2}{2}}\,dx\right)_{\substack{0\le i<m_q\\0\le j<n_p}} \end{pmatrix}, \tag{242}$$

where for simplicity we have left out the dependence of φ_i^{-s} and ψ_j^t on x. We have also

$$\begin{aligned}&\tau^E_{\vec m,\vec n}(t,s;\alpha,\beta;a,b)\\ &= \frac{1}{\prod_{i=1}^q m_i!\prod_{j=1}^p n_j!}\int_{E^N}\Big(\Delta_{m_1}(x^{(1)})\prod_{i=1}^{m_1}\psi_1^{-s}(x_i)e^{\frac{-x_i^2}{2}}\,dx_i\Big)\times\dots\times\\ &\Big(\Delta_{m_q}(x^{(q)})\prod_{i=m_1+\dots+m_{q-1}+1}^{m_1+\dots+m_q}\psi_q^{-s}(x_i)e^{\frac{-x_i^2}{2}}\,dx_i\Big)\\ &\times\sum_{\sigma\in S_N}(-1)^\sigma\Big[\Big(\Delta_{n_1}(x_{\sigma(1)},\dots,x_{\sigma(n_1)})\prod_{i=1}^{n_1}\varphi_1^t(x_{\sigma(i)})\Big)\times\dots\times\\ &\Big(\Delta_{n_p}(x_{\sigma(n_1+\dots+n_{p-1}+1)},\dots,x_{\sigma(n_1+\dots+n_p)})\prod_{i=n_1+\dots+n_{p-1}+1}^{n_1+\dots+n_p}\varphi_p^t(x_{\sigma(i)})\Big)\Big].\end{aligned}\tag{243}$$

The determinant of the moment matrix (217) with regard to the inner product $\langle f,g\rangle = \int_E f(z)g(z)e^{-z^2/2}dz$, with

$$\psi_i(x) := e^{a_ix+\alpha_ix^2},\quad 1\le i\le q,\qquad \varphi_j(x) := e^{b_jx+\beta_jx^2},\quad 1\le j\le p,$$

is the same as the determinant (242). Therefore, by virtue of Theorem 5.21, $\tau^E_{\vec m,\vec n}(t,s;\alpha,\beta;a,b)$ satisfies the $(p+q)$-component KP hierarchy. A direct consequence of Corollary 5.22 is :

Corollary 6.3. *The function $\tau^E_{\vec m,\vec n}(t,s;\alpha,\beta;a,b)$ satisfies the following identities, $1\le k,k'\le q$, $1\le l,l'\le p$, $k\ne k'$, $l\ne l'$,*

$$\frac{\partial}{\partial t_1^{(l)}}\ln\frac{\tau^E_{\vec m,\vec n+\vec e_l-\vec e_{l'}}}{\tau^E_{\vec m,\vec n-\vec e_l+\vec e_{l'}}} = \frac{\frac{\partial^2}{\partial t_2^{(l)}\partial t_1^{(l')}}\ln\tau^E_{\vec m,\vec n}}{\frac{\partial^2}{\partial t_1^{(l)}\partial t_1^{(l')}}\ln\tau^E_{\vec m,\vec n}},$$

$$\frac{\partial}{\partial t_1^{(l)}}\ln\frac{\tau^E_{\vec m+\vec e_k,\vec n+\vec e_l}}{\tau^E_{\vec m-\vec e_k,\vec n-\vec e_l}} = \frac{\frac{\partial^2}{\partial t_2^{(l)}\partial s_1^{(k)}}\ln\tau^E_{\vec m,\vec n}}{\frac{\partial^2}{\partial t_1^{(l)}\partial s_1^{(k)}}\ln\tau^E_{\vec m,\vec n}},$$

$$\frac{\partial}{\partial s_1^{(k)}}\ln\frac{\tau^E_{\vec m-\vec e_k+\vec e_{k'},\vec n}}{\tau^E_{\vec m+\vec e_k-\vec e_{k'},\vec n}}=\frac{\frac{\partial^2}{\partial s_2^{(k)}\partial s_1^{(k')}}\ln\tau^E_{\vec m,\vec n}}{\frac{\partial^2}{\partial s_1^{(k)}\partial s_1^{(k')}}\ln\tau^E_{\vec m,\vec n}},$$

$$\frac{\partial}{\partial s_1^{(k)}}\ln\frac{\tau^E_{\vec m-\vec e_k,\vec n-\vec e_l}}{\tau^E_{\vec m+\vec e_k,\vec n+\vec e_l}}=\frac{\frac{\partial^2}{\partial s_2^{(k)}\partial t_1^{(l)}}\ln\tau^E_{\vec m,\vec n}}{\frac{\partial^2}{\partial s_1^{(k)}\partial t_1^{(l)}}\ln\tau^E_{\vec m,\vec n}}. \tag{244}$$

4. Virasoro constraints

Let us introduce the following differential operators

$$\begin{aligned}\mathbb{J}^{(1)}_{m,k}(t)=&\frac{\partial}{\partial t_m}+(-m)t_{-m}+k\,\delta_{0,m},\\ \mathbb{J}^{(2)}_{m,k}(t)=&\frac{1}{2}\Big(\sum_{i+j=m}\frac{\partial^2}{\partial t_i\partial t_j}+2\sum_{i\geq 1}it_i\frac{\partial}{\partial t_{i+m}}+\sum_{i+j=-m}it_ijt_j\Big)\\ &+\Big(k+\frac{m+1}{2}\Big)\Big(\frac{\partial}{\partial t_m}+(-m)t_{-m}\Big)+\frac{k(k+1)}{2}\,\delta_{m,0}.\end{aligned}$$

Those operators satisfy the Heisenberg and Virasoro algebra respectively

$$\begin{aligned}\big[\mathbb{J}^{(1)}_{k,n}(t),\mathbb{J}^{(1)}_{l,n}(t)\big]&=k\,\delta_{k,-l},\\ \big[\mathbb{J}^{(2)}_{k,n}(t),\mathbb{J}^{(2)}_{l,n}(t)\big]&=(k-l)\mathbb{J}^{(2)}_{k+l,n}-\Big(\frac{k^3-k}{6}\Big)\,\delta_{k,-l},\end{aligned}$$

and interact as follows

$$\big[\mathbb{J}^{(2)}_{k,n}(t),\mathbb{J}^{(1)}_{l,n}(t)\big]=-l\,\mathbb{J}^{(1)}_{k+l,n}(t)+\frac{k(k+1)}{2}\delta_{k,-l}.$$

We have the following lemma, proven by Adler and van Moerbeke [**10**].

Lemma 6.4 (Adler-van Moerbeke [**10**])**.** *Given $\rho(z)=e^{-V(z)}$, with*

$$-\frac{\rho'(z)}{\rho(z)}=V'(z)=\frac{g(z)}{f(z)}=\frac{\sum_{i=0}^\infty\nu_iz^i}{\sum_{i=0}^\infty\mu_iz^i},$$

the integrand

$$dI_N(z;t):=\Delta_N(z)\prod_{k=1}^N\Big(e^{\sum_{i=1}^\infty t_iz_k^i}\rho(z_k)dz_k\Big)$$

satisfies the variational formula

$$\begin{aligned}&\frac{d}{d\epsilon}dI_N\big(z_i\mapsto z_i+\epsilon f(z_i)z_i^{k+1};t\big)\Big|_{\epsilon=0}\\ &\qquad=\sum_{l=0}^\infty\Big(\mu_l\,\mathbb{J}^{(2)}_{k+l,N}(t)-\nu_l\,\mathbb{J}^{(1)}_{k+l+1,N}(t)\Big)dI_N(z;t),\end{aligned}$$

for each $k \geq -1$. The contribution of the factor $\prod_{i=1}^{N} dz_i$ in this equation is

$$\sum_{l=0}^{\infty} \mu_l \, (l+k+1) \, \mathbb{J}^{(1)}_{k+l,N} \, dI_N(z;t).$$

We define, for a given permutation $\sigma \in S_n$, the integrands

$$dI^{\sigma}_{\vec{m},\vec{n}}\big(x;(t,s)\big) = \Big(\Delta_{m_1}(x^{(1)}) \prod_{i=1}^{m_1} \psi_1^{-s^{(1)}}(x_i) e^{\frac{-x_i^2}{2}} \, dx_i\Big) \times \ldots \times$$

$$\Big(\Delta_{m_q}(x^{(q)}) \prod_{i=m_1+\cdots+m_{q-1}+1}^{m_1+\cdots+m_q} \psi_q^{-s^{(q)}}(x_i) e^{\frac{-x_i^2}{2}} \, dx_i\Big)$$

$$\times \Big(\Delta_{n_1}(x_{\sigma(1)}, \ldots, x_{\sigma(n_1)}) \prod_{i=1}^{n_1} \varphi_1^{t^{(1)}}(x_{\sigma(i)})\Big) \times \cdots \times$$

$$\Big(\Delta_{n_p}(x_{\sigma(n_1+\cdots+n_{p-1}+1)}, \ldots, x_{\sigma(n_1+\cdots+n_p)}) \prod_{i=n_1+\cdots+n_{p-1}+1}^{n_1+\cdots+n_p} \varphi_p^{t^{(p)}}(x_{\sigma(i)})\Big). \tag{245}$$

We are looking for a variational equation for

$$dI^{\sigma}_{\vec{m},\vec{n}}\Big(x_i \mapsto x_i + \epsilon x_i^{k+1}; (t,s)\Big).$$

We have the following lemma.

Lemma 6.5 (Adler-van Moerbeke-Vanderstichelen [**13**])**.** *The integrand $dI^{\sigma}_{\vec{m},\vec{n}}\big(x;(t,s)\big)$ as defined in (245), satisfies the following variational equation for each $\sigma \in S_N$ and $k \geq -1$*

$$\frac{d}{d\epsilon} dI^{\sigma}_{\vec{m},\vec{n}}\Big(x_i \mapsto x_i + \epsilon x_i^{k+1}; (t,s)\Big)\Big|_{\epsilon=0} = \mathbb{V}_k^{\vec{m},\vec{n}}\big(dI^{\sigma}_{\vec{m},\vec{n}}\big),$$

with

$$\begin{aligned}
&\mathbb{V}_k^{\vec{m},\vec{n}} \\
&:= \sum_{i=1}^{q} \Big[\mathbb{J}^{(2)}_{k,m_i}(-s^{(i)}) + a_i \, \mathbb{J}^{(1)}_{k+1,m_i}(-s^{(i)}) - \big(1-2\alpha_i\big) \, \mathbb{J}^{(1)}_{k+2,m_i}(-s^{(i)})\Big] \\
&+ \sum_{i=1}^{p} \Big[\mathbb{J}^{(2)}_{k,n_i}(t^{(i)}) + b_i \, \mathbb{J}^{(1)}_{k+1,n_i}(t^{(i)}) + 2\beta_i \, \mathbb{J}^{(1)}_{k+2,n_i}(t^{(i)}) \\
&\qquad - (k+1) \, \mathbb{J}^{(1)}_{k,n_i}(t^{(i)})\Big].
\end{aligned} \tag{246}$$

PROOF. By the Leibniz rule, applying Lemma 6.4 to each factor in (245) and adding all these contributions yields (246). $\square$

As the variational formula in Lemma 6.4 is independent of the labeling of the variables in the integrand, it is a trivial but very important fact that the operator $\mathbb{V}_k^{\vec{m},\vec{n}}$ as defined in (246) is independent of the choice of $\sigma \in S_n$. As a consequence, we have the following theorem.

Theorem 6.6 (Adler-van Moerbeke-Vanderstichelen [**13**]). *The function $\tau^E_{\vec{m},\vec{n}}(t,s)$ as defined in (241) satisfies the following Virasoro constraints*

$$\mathcal{B}_k\, \tau^E_{\vec{m},\vec{n}} = \mathbb{V}_k^{\vec{m},\vec{n}}\, \tau^E_{\vec{m},\vec{n}}, \qquad k \geq -1, \tag{247}$$

with

$$\mathcal{B}_k = \sum_{i=1}^{2r} c_i^{k+1} \frac{\partial}{\partial c_i},$$

for $E = \cup_{i=1}^r [c_{2i-1}, c_{2i}] \subset \mathbb{R}$.

Proof. By virtue of formula (243) expressing $\tau^E_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b)$ as a N-uple integral over E, we have

$$\tau^E_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b) = \frac{1}{\prod_{i=1}^q m_i! \prod_{j=1}^p n_j!} \sum_{\sigma \in S_N} (-1)^\sigma \tau^{E,\sigma}_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b),$$

where $\tau^{E,\sigma}_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b)$ is defined by

$$\tau^{E,\sigma}_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b) := \int_{E^N} dI^\sigma_{\vec{m},\vec{n}}(x;(t,s)),$$

with $dI^\sigma_{\vec{m},\vec{n}}(x;(t,s))$ as in (245). For a fixed permutation $\sigma \in S_N$ and $k \geq -1$, we apply the change of variables $x_i \mapsto x_i + \epsilon x_i^{k+1}$, $1 \leq i \leq N$, given in lemma 6.5, in the integral defining $\tau^{E,\sigma}_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b)$. This change of variables leaves the integral invariant, but induces a change of limits of integration, given by the inverse map

$$c_i \mapsto c_i - \epsilon c_i^{k+1} + O(\epsilon^2), \quad 1 \leq i \leq 2r,$$

for ϵ small enough. Consequently, differentiating the result with respect to ϵ and evaluating it at $\epsilon = 0$, using the fundamental theorem of integral calculus together with Lemma 6.5, we obtain

$$\mathcal{B}_k\, \tau^{E,\sigma}_{\vec{m},\vec{n}} = \mathbb{V}_k^{\vec{m},\vec{n}}\, \tau^{E,\sigma}_{\vec{m},\vec{n}}, \qquad k \geq -1, \tag{248}$$

with

$$\mathcal{B}_k = \sum_{i=1}^{2r} c_i^{k+1} \frac{\partial}{\partial c_i}.$$

As noticed earlier, the operator $\mathbb{V}_k^{\vec{m},\vec{n}}$ does not depend on $\sigma \in S_N$. Consequently, summing (248) over $\sigma \in S_N$ and dividing by $\prod_{i=1}^q m_i! \prod_{j=1}^p n_j!$, we obtain

$$\mathcal{B}_k\, \tau^E_{\vec{m},\vec{n}} = \mathbb{V}_k^{\vec{m},\vec{n}}\, \tau^E_{\vec{m},\vec{n}}, \qquad k \geq -1.$$

This concludes the proof. ∎

When specializing the differential equations (247) to $k=-1$ and $k=0$, we find that the tau-function $\tau^E_{\vec{m},\vec{n}}$ satisfies respectively

$$\begin{aligned}\mathcal{B}_{-1}\,\tau &= \sum_{i\geq 2}\left(\sum_{l=1}^{q} i s_i^{(l)}\frac{\partial}{\partial s_{i-1}^{(l)}} + \sum_{l=1}^{p} i t_i^{(l)}\frac{\partial}{\partial t_{i-1}^{(l)}}\right)\tau + \sum_{l=1}^{q}(1-2\alpha_l)\frac{\partial\tau}{\partial s_1^{(l)}}\\ &\quad + 2\sum_{l=1}^{p}\beta_l\frac{\partial\tau}{\partial t_1^{(l)}} + \Big(\sum_{l=1}^{p} n_l t_1^{(l)} - \sum_{l=1}^{q} m_l s_1^{(l)}\Big)\tau\\ &\quad + \Big(\sum_{l=1}^{q} a_l m_l + \sum_{l=1}^{p} b_l n_l\Big)\tau,\\ \mathcal{B}_0\,\tau &= \sum_{i\geq 1}\left(\sum_{l=1}^{q} i s_i^{(l)}\frac{\partial}{\partial s_i^{(l)}} + \sum_{l=1}^{p} i t_i^{(l)}\frac{\partial}{\partial t_i^{(l)}}\right)\tau - \sum_{l=1}^{q} a_l\frac{\partial\tau}{\partial s_1^{(l)}}\\ &\quad + \sum_{l=1}^{p} b_l\frac{\partial\tau}{\partial t_1^{(l)}} + \sum_{l=1}^{q}(1-2\alpha_l)\frac{\partial\tau}{\partial s_2^{(l)}} + 2\sum_{l=1}^{p}\beta_l\frac{\partial\tau}{\partial t_2^{(l)}}\\ &\quad + \frac{1}{2}\Big(\sum_{l=1}^{q} m_l^2 + \sum_{l=1}^{p} n_l^2\Big)\tau.\end{aligned}\tag{249}$$

The Virasoro constraints (247) play a very crucial role in finding a PDE for the function $\log\mathbb{P}^{a_1,\dots,a_q}_{b_1,\dots,b_p}\big(all\ x_i(t)\in E\big)$ in the variables $a_1,\dots,a_q$, $b_1,\dots,b_p$ and the endpoints of the set E. In the next section, we will prove the existence of a PDE for the logarithm of the normalized problem $P_{p,q}(E;a,b)$ defined in (236). The normalized problem is related to the function $\tau^E_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b)$ on the locus $\mathcal{L}=\{(t,s)=0,(\alpha,\beta)=0\}$ through formula (240). As we have seen, this function is a tau-function of the $(p+q)$-component KP hierarchy and thus satisfies the PDE's (230). As the Virasoro constraints involve derivatives with respect to the endpoints of the set E, as well as derivatives with respect to the time variables (t,s), we will prove that they can be used to eliminate all the derivatives with respect to the time variables in (230) on the locus $\mathcal{L}$. The proof proceeds in two main steps. In the first step, the Virasoro constraints, together with the linear conditions imposed on a_i,α_i,b_j,β_j, $1\leq i\leq q$ and $1\leq j\leq p$

$$\sum_{i=1}^{q} a_i = \sum_{i=1}^{p} b_i = \sum_{i=1}^{q}\alpha_i = \sum_{i=1}^{p}\beta_i = 0,\tag{250}$$

are used to express on the locus $\mathcal{K}=\{(t,s)=0\}$ all the derivatives with respect to the time variables in (230) in terms of derivatives with respect to the auxiliary variables $\alpha_1,\dots,\alpha_q$ and $\beta_1,\dots,\beta_p$. In the second step, using a combinatorial argument, it will be shown that, on the locus $\mathcal{L}$, all these derivatives with respect to $\alpha_1,\dots,\alpha_q$ and $\beta_1,\dots,\beta_p$ can be eliminated. Both steps will be performed in the next section. We end this section with some consequences of Theorem 6.6.

From the linear conditions (250) it follows that the function $\tau^E_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b)$ as defined in (241) satisfies the following equations

$$\sum_{l=1}^{q}\frac{\partial\tau}{\partial s_i^{(l)}}+\sum_{l=1}^{p}\frac{\partial\tau}{\partial t_i^{(l)}}=0,\quad i\geq 1, \tag{251}$$

and

$$\frac{\partial\tau}{\partial a_i}=-\frac{\partial\tau}{\partial s_1^{(i)}}+\frac{\partial\tau}{\partial s_1^{(q)}},\quad 1\leq i\leq q-1, \tag{252}$$

$$\frac{\partial\tau}{\partial b_i}=\frac{\partial\tau}{\partial t_1^{(i)}}-\frac{\partial\tau}{\partial t_1^{(p)}},\quad 1\leq i\leq p-1, \tag{253}$$

$$\frac{\partial\tau}{\partial \alpha_i}=-\frac{\partial\tau}{\partial s_2^{(i)}}+\frac{\partial\tau}{\partial s_2^{(q)}},\quad 1\leq i\leq q-1, \tag{254}$$

$$\frac{\partial\tau}{\partial \beta_i}=\frac{\partial\tau}{\partial t_2^{(i)}}-\frac{\partial\tau}{\partial t_2^{(p)}},\quad 1\leq i\leq p-1. \tag{255}$$

From these equations, we deduce two families of identities. Let $f:=\log\tau^E_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b)$. Firstly, using equation (252) we have

$$\sum_{l=1}^{q}a_l\frac{\partial f}{\partial s_1^{(l)}}=-\sum_{l=1}^{q-1}a_l\frac{\partial f}{\partial a_l}, \tag{256}$$

since $\sum_{l=1}^{q}a_l=0$, and similarly

$$\begin{aligned}
&\sum_{l=1}^{q}\alpha_l\frac{\partial f}{\partial s_1^{(l)}}=-\sum_{l=1}^{q-1}\alpha_l\frac{\partial f}{\partial a_l},\\
&\sum_{l=1}^{p}b_l\frac{\partial f}{\partial t_1^{(l)}}=\sum_{l=1}^{p-1}b_l\frac{\partial f}{\partial b_l},\qquad \sum_{l=1}^{p}\beta_l\frac{\partial f}{\partial t_1^{(l)}}=\sum_{l=1}^{p-1}\beta_l\frac{\partial f}{\partial b_l},\\
&\sum_{l=1}^{q}\alpha_l\frac{\partial f}{\partial s_2^{(l)}}=-\sum_{l=1}^{q-1}\alpha_l\frac{\partial f}{\partial \alpha_l},\qquad \sum_{l=1}^{p}\beta_l\frac{\partial f}{\partial t_2^{(l)}}=\sum_{l=1}^{p-1}\beta_l\frac{\partial f}{\partial \beta_l}.
\end{aligned} \tag{257}$$

Secondly, using equation (252) we have

$$\sum_{i=1}^{q}\frac{\partial f}{\partial s_1^{(i)}}=-\sum_{i=1}^{q-1}\frac{\partial f}{\partial a_l}+q\frac{\partial f}{\partial s_1^{(q)}},$$

and thus

$$\frac{\partial f}{\partial s_1^{(q)}}=\frac{1}{q}\sum_{i=1}^{q}\frac{\partial f}{\partial s_1^{(i)}}+\frac{1}{q}\sum_{i=1}^{q-1}\frac{\partial f}{\partial a_l}. \tag{258}$$

Using again equation (252), we obtain

$$\frac{\partial f}{\partial s_1^{(j)}}=\frac{1}{q}\sum_{i=1}^{q}\frac{\partial f}{\partial s_1^{(i)}}+\frac{1}{q}\sum_{i=1}^{q-1}\frac{\partial f}{\partial a_l}-\frac{\partial f}{\partial a_j},\quad 1\leq j\leq q-1. \tag{259}$$

Equations (258) and (259) can be summarized as follows

$$\frac{\partial f}{\partial s_1^{(j)}} = \frac{1}{q}\sum_{i=1}^{q}\frac{\partial f}{\partial s_1^{(i)}} + \frac{1}{q}\sum_{i=1}^{q-1}\frac{\partial f}{\partial a_l} - (1-\delta_{jq})\frac{\partial f}{\partial a_j}, \quad 1 \le j \le q. \tag{260}$$

Similarly, we have

$$\begin{aligned}
\frac{\partial f}{\partial t_1^{(j)}} &= \frac{1}{p}\sum_{i=1}^{p}\frac{\partial f}{\partial t_1^{(i)}} - \frac{1}{p}\sum_{i=1}^{p-1}\frac{\partial f}{\partial b_l} + (1-\delta_{jp})\frac{\partial f}{\partial b_j}, \quad 1 \le j \le p,\\
\frac{\partial f}{\partial s_2^{(j)}} &= \frac{1}{q}\sum_{i=1}^{q}\frac{\partial f}{\partial s_2^{(i)}} + \frac{1}{q}\sum_{i=1}^{q-1}\frac{\partial f}{\partial \alpha_l} - (1-\delta_{jq})\frac{\partial f}{\partial \alpha_j}, \quad 1 \le j \le q,\\
\frac{\partial f}{\partial t_2^{(j)}} &= \frac{1}{p}\sum_{i=1}^{p}\frac{\partial f}{\partial t_2^{(i)}} - \frac{1}{p}\sum_{i=1}^{p-1}\frac{\partial f}{\partial \beta_l} + (1-\delta_{jp})\frac{\partial f}{\partial \beta_j}, \quad 1 \le j \le p.
\end{aligned} \tag{261}$$

Substituting relations (256),(257), (260), (261) in the Virasoro constraints (249), we get

$$\begin{aligned}
A_j f &= \frac{\partial f}{\partial s_1^{(j)}} + \frac{1}{q}\sum_{i\ge 2}\left(\sum_{l=1}^{q} i s_i^{(l)}\frac{\partial}{\partial s_{i-1}^{(l)}} + \sum_{l=1}^{p} i t_i^{(l)}\frac{\partial}{\partial t_{i-1}^{(l)}}\right) f\\
&\quad + \frac{1}{q}\Big(\sum_{l=1}^{p} n_l t_1^{(l)} - \sum_{l=1}^{q} m_l s_1^{(l)}\Big) + \frac{1}{q}\Big(\sum_{l=1}^{q} a_l m_l + \sum_{l=1}^{p} b_l n_l\Big),\\
&\qquad\qquad 1 \le j \le q,\\
B_j f &= -\frac{\partial f}{\partial t_1^{(j)}} + \frac{1}{p}\sum_{i\ge 2}\left(\sum_{l=1}^{q} i s_i^{(l)}\frac{\partial}{\partial s_{i-1}^{(l)}} + \sum_{l=1}^{p} i t_i^{(l)}\frac{\partial}{\partial t_{i-1}^{(l)}}\right) f\\
&\quad + \frac{1}{p}\Big(\sum_{l=1}^{p} n_l t_1^{(l)} - \sum_{l=1}^{q} m_l s_1^{(l)}\Big) + \frac{1}{p}\Big(\sum_{l=1}^{q} a_l m_l + \sum_{l=1}^{p} b_l n_l\Big),\\
&\qquad\qquad 1 \le j \le p,
\end{aligned} \tag{262}$$

and

$$\hat{A}_j f = \frac{\partial f}{\partial s_2^{(j)}} + \frac{1}{q}\sum_{i\geq 1}\left(\sum_{l=1}^{q} i s_i^{(l)} \frac{\partial}{\partial s_i^{(l)}} + \sum_{l=1}^{p} i t_i^{(l)} \frac{\partial}{\partial t_i^{(l)}}\right) f + \frac{K}{q},$$

$$1 \leq j \leq q,$$

$$\hat{B}_j f = -\frac{\partial f}{\partial t_2^{(j)}} + \frac{1}{p}\sum_{i\geq 1}\left(\sum_{l=1}^{q} i s_i^{(l)} \frac{\partial}{\partial s_i^{(l)}} + \sum_{l=1}^{p} i t_i^{(l)} \frac{\partial}{\partial t_i^{(l)}}\right) f + \frac{K}{p},$$

$$1 \leq j \leq p, \tag{263}$$

where

$$K := \frac{1}{2}\Big(\sum_{l=1}^{q} m_l^2 + \sum_{l=1}^{p} n_l^2\Big),$$

and

$$A_j = -(1-\delta_{jq})\frac{\partial}{\partial a_j} + \frac{1}{q}\left(\mathcal{B}_{-1} + \sum_{l=1}^{q-1}\frac{\partial}{\partial a_l} - 2\Big(\sum_{l=1}^{q-1}\alpha_l\frac{\partial}{\partial a_l} + \sum_{l=1}^{p-1}\beta_l\frac{\partial}{\partial b_l}\Big)\right),$$
$$1 \leq j \leq q,$$

$$B_j = -(1-\delta_{jp})\frac{\partial}{\partial b_j} + \frac{1}{p}\left(\mathcal{B}_{-1} + \sum_{l=1}^{p-1}\frac{\partial}{\partial b_l} - 2\Big(\sum_{l=1}^{q-1}\alpha_l\frac{\partial}{\partial a_l} + \sum_{l=1}^{p-1}\beta_l\frac{\partial}{\partial b_l}\Big)\right),$$
$$1 \leq j \leq p,$$

$$\begin{aligned}\hat{A}_j = -(1-\delta_{jq})\frac{\partial}{\partial \alpha_j} + \frac{1}{q}\Bigg(&\mathcal{B}_0 - \Big(\sum_{l=1}^{q-1} a_l\frac{\partial}{\partial a_l} + \sum_{l=1}^{p-1} b_l \frac{\partial}{\partial b_l}\Big)\\ &+ \sum_{l=1}^{q-1}\frac{\partial}{\partial \alpha_l} - 2\Big(\sum_{l=1}^{q-1}\alpha_l\frac{\partial}{\partial \alpha_l} + \sum_{l=1}^{p-1}\beta_l\frac{\partial}{\partial \beta_l}\Big)\Bigg), \quad 1 \leq j \leq q,\end{aligned}$$

$$\begin{aligned}\hat{B}_j = -(1-\delta_{jp})\frac{\partial}{\partial \beta_j} + \frac{1}{p}\Bigg(&\mathcal{B}_0 - \Big(\sum_{l=1}^{q-1} a_l\frac{\partial}{\partial a_l} + \sum_{l=1}^{p-1} b_l \frac{\partial}{\partial b_l}\Big)\\ &+ \sum_{l=1}^{p-1}\frac{\partial}{\partial \beta_l} - 2\Big(\sum_{l=1}^{q-1}\alpha_l\frac{\partial}{\partial \alpha_l} + \sum_{l=1}^{p-1}\beta_l\frac{\partial}{\partial \beta_l}\Big)\Bigg), \quad 1 \leq j \leq p.\end{aligned}$$

Observe that the operators A_j, $1 \leq j \leq q$, and B_j, $1 \leq j \leq p$, all commute.

Lemma 6.7 (Adler-van Moerbeke-Vanderstichelen [**13**]). *On the locus* $\mathcal{K} = \{(t,s) = 0\}$*, the function* $f := \log \tau^{E}_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b)$ *satisfies the Virasoro constraints*

$$\begin{aligned}
\frac{\partial^2 f}{\partial s_1^{(j)} \partial s_1^{(k)}} &= A_j A_k f + \frac{m_j}{q} + \frac{m_k}{q} - \frac{N}{q^2} + \frac{2}{q^2}\big(\langle \alpha, m\rangle + \langle \beta, n\rangle\big),\\
\frac{\partial^2 f}{\partial t_1^{(j)} \partial t_1^{(k)}} &= B_j B_k f + \frac{n_j}{p} + \frac{n_k}{p} - \frac{N}{p^2} + \frac{2}{p^2}\big(\langle \alpha, m\rangle + \langle \beta, n\rangle\big),\\
\frac{\partial^2 f}{\partial s_1^{(j)} \partial t_1^{(k)}} &= -A_j B_k f - \frac{m_j}{p} - \frac{n_k}{q} + \frac{N}{pq} - \frac{2}{pq}\big(\langle \alpha, m\rangle + \langle \beta, n\rangle\big),\\
\frac{\partial^2 f}{\partial s_1^{(k)} \partial s_2^{(j)}} &= \Big(\hat{A}_j - \frac{1}{q}\Big) A_k f + \frac{2}{q^2}\big(\langle a, m\rangle + \langle b, n\rangle\big),\\
\frac{\partial^2 f}{\partial t_1^{(k)} \partial t_2^{(j)}} &= \Big(\hat{B}_j - \frac{1}{p}\Big) B_k f + \frac{2}{p^2}\big(\langle a, m\rangle + \langle b, n\rangle\big),\\
\frac{\partial^2 f}{\partial s_1^{(k)} \partial t_2^{(j)}} &= -\Big(\hat{B}_j - \frac{1}{p}\Big) A_k f - \frac{2}{pq}\big(\langle a, m\rangle + \langle b, n\rangle\big),\\
\frac{\partial^2 f}{\partial s_2^{(j)} \partial t_1^{(k)}} &= -\Big(\hat{A}_j - \frac{1}{q}\Big) B_k f - \frac{2}{pq}\big(\langle a, m\rangle + \langle b, n\rangle\big),
\end{aligned}$$

where $\langle \alpha, m\rangle = \sum_{i=1}^{q} \alpha_i m_i$ *and* $\langle \beta, n\rangle = \sum_{i=1}^{p} \beta_i n_i$*.*

PROOF. We compute on the locus $\mathcal{K}$, using (262) and (263) that

$$\begin{aligned}
&A_j A_k f\big|_{\mathcal{K}}\\
&= A_j \Bigg[\frac{\partial f}{\partial s_1^{(k)}} + \frac{1}{q}\sum_{i\geq 2}\Bigg(\sum_{l=1}^{q} i s_i^{(l)} \frac{\partial}{\partial s_{i-1}^{(l)}} + \sum_{l=1}^{p} i t_i^{(l)} \frac{\partial}{\partial t_{i-1}^{(l)}}\Bigg) f\\
&\qquad + \frac{1}{q}\Big(\sum_{l=1}^{p} n_l t_1^{(l)} - \sum_{l=1}^{q} m_l s_1^{(l)}\Big) + \frac{1}{q}\Big(\sum_{l=1}^{q} a_l m_l + \sum_{l=1}^{p} b_l n_l\Big)\Bigg]\Bigg|_{\mathcal{K}}\\
&= \Bigg[\frac{\partial}{\partial s_1^{(k)}} + \frac{1}{q}\sum_{i\geq 2}\Bigg(\sum_{l=1}^{q} i s_i^{(l)} \frac{\partial}{\partial s_{i-1}^{(l)}} + \sum_{l=1}^{p} i t_i^{(l)} \frac{\partial}{\partial t_{i-1}^{(l)}}\Bigg)\Bigg] A_j f\big|_{\mathcal{K}}\\
&\qquad + \frac{1}{q} A_j \Big(\sum_{l=1}^{q} a_l m_l + \sum_{l=1}^{p} b_l n_l\Big)\Big|_{\mathcal{K}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial s_1^{(k)}}\Bigg[\frac{\partial f}{\partial s_1^{(j)}} + \frac{1}{q}\sum_{i\geq 2}\left(\sum_{l=1}^{q} i s_i^{(l)}\frac{\partial}{\partial s_{i-1}^{(l)}} + \sum_{l=1}^{p} i t_i^{(l)}\frac{\partial}{\partial t_{i-1}^{(l)}}\right) f \\
&\qquad + \frac{1}{q}\Big(\sum_{l=1}^{p} n_l t_1^{(l)} - \sum_{l=1}^{q} m_l s_1^{(l)}\Big) + \frac{1}{q}\Big(\sum_{l=1}^{q} a_l m_l + \sum_{l=1}^{p} b_l n_l\Big)\Bigg]\Bigg|_{\mathcal{K}} \\
&\quad + \frac{1}{q}\Bigg[- (1-\delta_{j,q})\frac{\partial}{\partial a_j} \\
&\qquad + \frac{1}{q}\left(\mathcal{B}_{-1} + \sum_{l=1}^{q-1}\frac{\partial}{\partial a_l} - 2\Big(\sum_{l=1}^{q-1}\alpha_l\frac{\partial}{\partial a_l} + \sum_{l=1}^{p-1}\beta_l\frac{\partial}{\partial b_l}\Big)\right)\Bigg] \\
&\qquad \times \Big(\sum_{l=1}^{q} a_l m_l + \sum_{l=1}^{p} b_l n_l\Big)\Big|_{\mathcal{K}} \\
&= \frac{\partial^2 f}{\partial s_1^{(j)}\partial s_1^{(k)}} - \frac{m_k}{q} + \frac{1}{q^2}\Big(- q(m_j - m_q)(1-\delta_{j,q}) + \sum_{l=1}^{q-1}(m_l - m_q) \\
&\qquad\qquad - 2\sum_{l=1}^{q-1}\alpha_l(m_l - m_q) - 2\sum_{l=1}^{p-1}\beta_l(n_l - n_p)\Big).
\end{aligned}
$$

Since $\sum_{l=1}^{q-1}(m_l - m_q) = N - qm_q$, $\sum_{l=1}^{q-1}\alpha_l(m_l - m_q) = \langle\alpha, m\rangle$ and $\sum_{l=1}^{p-1}\beta_l(n_l - n_p) = \langle\beta, n\rangle$, we obtain

$$
A_j A_k f\big|_{\mathcal{K}} = \frac{\partial^2 f}{\partial s_1^{(j)}\partial s_1^{(k)}} - \frac{m_j}{q} - \frac{m_k}{q} + \frac{N}{q^2} - \frac{2}{q^2}\big(\langle\alpha, m\rangle + \langle\beta, n\rangle\big).
$$

The proof of the other relations is analogous. □

5. Existence of a PDE for $\log P_{p,q}(E;a,b)$

In this section we prove that, under the assumptions $a_1 + \cdots + a_q = 0$ and $b_1 + \cdots + b_p = 0$, the function $\log P_{p,q}(E;a,b)$, with $P_{p,q}(E;a,b)$ as defined in (236), satisfies a nonlinear PDE, the variables being $a_1,\ldots,a_{q-1}$, $b_1,\ldots,b_{p-1}$ and the coordinates of the endpoints of the set E, i.e. $c_1,\ldots,c_{2r}$. To perform this, we first show that the function $f := \log \tau^{E}_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b)$ satisfies a system of $\frac{1}{2}(p+q)(p+q-1)$ equations on the locus $\mathcal{L}$, containing partial derivatives with respect to $a_1,\ldots,a_{q-1}$, $b_1,\ldots,b_{p-1}$, $\alpha_1,\ldots,\alpha_{q-1}$, $\beta_1,\ldots,\beta_{p-1}$ and the coordinates of the endpoints of the set E.

We define the operators

$$A_j^{\mathcal{L}} = -(1-\delta_{jq})\frac{\partial}{\partial a_j} + \frac{1}{q}\Big(\mathcal{B}_{-1} + \sum_{l=1}^{q-1}\frac{\partial}{\partial a_l}\Big), \quad 1 \le j \le q,$$

$$B_j^{\mathcal{L}} = -(1-\delta_{jp})\frac{\partial}{\partial b_j} + \frac{1}{p}\Big(\mathcal{B}_{-1} + \sum_{l=1}^{p-1}\frac{\partial}{\partial b_l}\Big), \quad 1 \le j \le p, \tag{264}$$

$$\hat{\mathcal{B}}_0 = \mathcal{B}_0 - \Big(\sum_{l=1}^{q-1} a_l\frac{\partial}{\partial a_l} + \sum_{l=1}^{p-1} b_l\frac{\partial}{\partial b_l}\Big).$$

We then have

$$A_j = A_j^{\mathcal{L}} - \frac{2}{q}\Big(\sum_{l=1}^{q-1}\alpha_l\frac{\partial}{\partial a_l} + \sum_{l=1}^{p-1}\beta_l\frac{\partial}{\partial b_l}\Big), \quad 1 \le j \le q,$$

$$B_j = B_j^{\mathcal{L}} - \frac{2}{p}\Big(\sum_{l=1}^{q-1}\alpha_l\frac{\partial}{\partial a_l} + \sum_{l=1}^{p-1}\beta_l\frac{\partial}{\partial b_l}\Big), \quad 1 \le j \le p,$$

$$\hat{A}_j = \frac{1}{q}\hat{\mathcal{B}}_0 - (1-\delta_{jq})\frac{\partial}{\partial \alpha_j} - \frac{2}{q}\Big(\sum_{l=1}^{q-1}\alpha_l\frac{\partial}{\partial \alpha_l} + \sum_{l=1}^{p-1}\beta_l\frac{\partial}{\partial \beta_l}\Big) + \frac{1}{q}\sum_{l=1}^{q-1}\frac{\partial}{\partial \alpha_l}, \quad 1 \le j \le q, \tag{265}$$

$$\hat{B}_j = \frac{1}{p}\hat{\mathcal{B}}_0 - (1-\delta_{jp})\frac{\partial}{\partial \beta_j} - \frac{2}{p}\Big(\sum_{l=1}^{q-1}\alpha_l\frac{\partial}{\partial \alpha_l} + \sum_{l=1}^{p-1}\beta_l\frac{\partial}{\partial \beta_l}\Big) + \frac{1}{p}\sum_{l=1}^{p-1}\frac{\partial}{\partial \beta_l}, \quad 1 \le j \le p.$$

We also introduce the following notation

$$-\partial_{\beta_j} + \frac{1}{p}\partial_\beta = -(1-\delta_{jp})\frac{\partial}{\partial \beta_j} + \frac{1}{p}\sum_{l=1}^{p-1}\frac{\partial}{\partial \beta_l},$$

$$-\partial_{b_j} + \frac{1}{p}\partial_b = -(1-\delta_{jp})\frac{\partial}{\partial b_j} + \frac{1}{p}\sum_{l=1}^{p-1}\frac{\partial}{\partial b_l},$$

$$-\partial_{\alpha_k} + \frac{1}{q}\partial_\alpha = -(1-\delta_{kq})\frac{\partial}{\partial \alpha_k} + \frac{1}{q}\sum_{l=1}^{q-1}\frac{\partial}{\partial \alpha_l},$$

$$-\partial_{a_k} + \frac{1}{q}\partial_a = -(1-\delta_{kq})\frac{\partial}{\partial a_k} + \frac{1}{q}\sum_{l=1}^{q-1}\frac{\partial}{\partial a_l}, \tag{266}$$

with $1 \le j \le p$ and $1 \le k \le q$. Note the two sets of operators on the first two rows respectively sum to zero as we sum j from 1 to p, and similarly for the operators on the last two rows, as we sum k from 1 to q. With these notations we have the following.

Theorem 6.8 (Adler-van Moerbeke-Vanderstichelen [**13**]). *The function* $f := \log \tau^E_{\vec{m},\vec{n}}(t,s;\alpha,\beta;a,b)$ *satisfies the following* $\frac{1}{2}(p+q)(p+q-1)$ *equations*[1] *on the locus* $\mathcal{L}$

$$\begin{aligned}
&\Big\{A_j^{\mathcal{L}}(\frac{1}{p}\partial_\beta - \partial_{\beta_k})f, A_j^{\mathcal{L}}B_k^{\mathcal{L}}f + \frac{m_j}{p} + \frac{n_k}{q} - \frac{N}{pq}\Big\}_{A_j^{\mathcal{L}}} \\
&- \Big\{B_k^{\mathcal{L}}(\frac{1}{q}\partial_\alpha - \partial_{\alpha_j})f, A_j^{\mathcal{L}}B_k^{\mathcal{L}}f + \frac{m_j}{p} + \frac{n_k}{q} - \frac{N}{pq}\Big\}_{B_k^{\mathcal{L}}} = G_{jk}^{AB}, \\
&\hspace{20em} 1 \le j \le q, 1 \le k \le p,
\end{aligned}$$

$$\begin{aligned}
&\Big\{A_k^{\mathcal{L}}(\frac{1}{q}\partial_\alpha - \partial_{\alpha_j})f, A_j^{\mathcal{L}}A_k^{\mathcal{L}}f + \frac{m_j}{q} + \frac{m_k}{q} - \frac{N}{q^2}\Big\}_{A_k^{\mathcal{L}}} \\
&+ \Big\{A_j^{\mathcal{L}}(\frac{1}{q}\partial_\alpha - \partial_{\alpha_k})f, A_j^{\mathcal{L}}A_k^{\mathcal{L}}f + \frac{m_j}{q} + \frac{m_k}{q} - \frac{N}{q^2}\Big\}_{A_j^{\mathcal{L}}} = G_{jk}^{A}, \\
&\hspace{20em} 1 \le j < k \le q,
\end{aligned} \tag{267}$$

$$\begin{aligned}
&\Big\{B_k^{\mathcal{L}}(\frac{1}{p}\partial_\beta - \partial_{\beta_j})f, B_j^{\mathcal{L}}B_k^{\mathcal{L}}f + \frac{n_j}{p} + \frac{n_k}{p} - \frac{N}{p^2}\Big\}_{B_k^{\mathcal{L}}} \\
&+ \Big\{B_j^{\mathcal{L}}(\frac{1}{p}\partial_\beta - \partial_{\beta_k})f, B_j^{\mathcal{L}}B_k^{\mathcal{L}}f + \frac{n_j}{p} + \frac{n_k}{p} - \frac{N}{p^2}\Big\}_{B_j^{\mathcal{L}}} = G_{jk}^{B}, \\
&\hspace{20em} 1 \le j < k \le p,
\end{aligned}$$

where G_{jk}^A, G_{jk}^B *and* G_{jk}^{AB} *only depend on* f, *its derivatives with respect to* $a_1,\dots,a_{q-1}$, $b_1,\dots,b_{p-1}$, *and its differentials up to the third order with respect to the operators* $A_j^{\mathcal{L}}$, $B_j^{\mathcal{L}}$ *and* $\hat{\mathcal{B}}_0$, *evaluated on the locus* $\mathcal{L}$.

PROOF. Using equations (262) and (263) we obtain on the locus $\mathcal{K}$

$$\begin{aligned}
\frac{\partial}{\partial s_1^{(j)}} \log \frac{\tau^E_{\vec{m}-\vec{e_j}+\vec{e_k},\vec{n}}}{\tau^E_{\vec{m}+\vec{e_j}-\vec{e_k},\vec{n}}}\Bigg|_{\mathcal{K}} &= A_j \log \frac{\tau^E_{\vec{m}-\vec{e_j}+\vec{e_k},\vec{n}}}{\tau^E_{\vec{m}+\vec{e_j}-\vec{e_k},\vec{n}}} + \frac{2}{q}(a_j - a_k), \\
\frac{\partial}{\partial s_1^{(j)}} \log \frac{\tau^E_{\vec{m}-\vec{e_j},\vec{n}-\vec{e_k}}}{\tau^E_{\vec{m}+\vec{e_j},\vec{n}+\vec{e_k}}}\Bigg|_{\mathcal{K}} &= A_j \log \frac{\tau^E_{\vec{m}-\vec{e_j},\vec{n}-\vec{e_k}}}{\tau^E_{\vec{m}+\vec{e_j},\vec{n}+\vec{e_k}}} + \frac{2}{q}(a_j + b_k), \\
\frac{\partial}{\partial t_1^{(j)}} \log \frac{\tau^E_{\vec{m},\vec{n}+\vec{e_j}-\vec{e_k}}}{\tau^E_{\vec{m},\vec{n}-\vec{e_j}+\vec{e_k}}}\Bigg|_{\mathcal{K}} &= -B_j \log \frac{\tau^E_{\vec{m},\vec{n}+\vec{e_j}-\vec{e_k}}}{\tau^E_{\vec{m},\vec{n}-\vec{e_j}+\vec{e_k}}} + \frac{2}{p}(b_j - b_k), \\
\frac{\partial}{\partial t_1^{(j)}} \log \frac{\tau^E_{\vec{m}+\vec{e_k},\vec{n}+\vec{e_j}}}{\tau^E_{\vec{m}-\vec{e_k},\vec{n}-\vec{e_j}}}\Bigg|_{\mathcal{K}} &= -B_j \log \frac{\tau^E_{\vec{m}+\vec{e_k},\vec{n}+\vec{e_j}}}{\tau^E_{\vec{m}-\vec{e_k},\vec{n}-\vec{e_j}}} + \frac{2}{p}(a_k + b_j).
\end{aligned} \tag{268}$$

[1] $\{f,g\}_X = g\,X(f) - f\,X(g)$

Substituting the first equation in (268) into the third equation in (244) and using lemma 6.7, we have on the locus $\mathcal{K}$

$$
\begin{aligned}
(269) \qquad & A_j \log \frac{\tau^E_{\vec{m}-\vec{e_j}+\vec{e_k},\vec{n}}}{\tau^E_{\vec{m}+\vec{e_j}-\vec{e_k},\vec{n}}} \\
&= \frac{\left(\hat{A}_j - \frac{1}{q}\right) A_k f + \frac{2}{q^2}\big(\langle a, m\rangle + \langle b, n\rangle\big)}{A_j A_k f + \frac{m_j}{q} + \frac{m_k}{q} - \frac{N}{q^2} + \frac{2}{q^2}\big(\langle \alpha, m\rangle + \langle \beta, n\rangle\big)} - \frac{2}{q}(a_j - a_k).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(270) \qquad & B_j \ \ln \frac{\tau^E_{\vec{m},\vec{n}-\vec{e_j}+\vec{e_k}}}{\tau^E_{\vec{m},\vec{n}+\vec{e_j}-\vec{e_k}}} \\
&= \frac{\left(\hat{B}_j - \frac{1}{p}\right) B_k f + \frac{2}{p^2}\big(\langle a, m\rangle + \langle b, n\rangle\big)}{B_j B_k f + \frac{n_j}{p} + \frac{n_k}{p} - \frac{N}{p^2} + \frac{2}{p^2}\big(\langle \alpha, m\rangle + \langle \beta, n\rangle\big)} - \frac{2}{p}(b_j - b_k),
\end{aligned}
$$

$$
\begin{aligned}
(271) \qquad & B_j \ \ln \frac{\tau^E_{\vec{m}-\vec{e_k},\vec{n}-\vec{e_j}}}{\tau^E_{\vec{m}+\vec{e_k},\vec{n}+\vec{e_j}}} \\
&= \frac{-\left(\hat{B}_j - \frac{1}{p}\right) A_k f - \frac{2}{pq}\big(\langle a, m\rangle + \langle b, n\rangle\big)}{-A_k B_j f - \frac{m_k}{p} - \frac{n_j}{q} + \frac{N}{pq} - \frac{2}{pq}\big(\langle \alpha, m\rangle + \langle \beta, n\rangle\big)} - \frac{2}{p}(a_k + b_j),
\end{aligned}
$$

$$
\begin{aligned}
(272) \qquad & A_j \ \ln \frac{\tau^E_{\vec{m}-\vec{e_j},\vec{n}-\vec{e_k}}}{\tau^E_{\vec{m}+\vec{e_j},\vec{n}+\vec{e_k}}} \\
&= \frac{-\left(\hat{A}_j - \frac{1}{q}\right) B_k f - \frac{2}{pq}\big(\langle a, m\rangle + \langle b, n\rangle\big)}{-A_j B_k f - \frac{m_j}{p} - \frac{n_k}{q} + \frac{N}{pq} - \frac{2}{pq}\big(\langle \alpha, m\rangle + \langle \beta, n\rangle\big)} - \frac{2}{q}(a_j + b_k).
\end{aligned}
$$

Let us denote equations (269)-(272), with indices chosen as above, by $(269)_{jk}$, $(270)_{jk}$, $(271)_{jk}$ and $(272)_{jk}$. We compute $A_j(271)_{kj} - B_k(272)_{jk}$, and we obtain

$$
\begin{aligned}
0 = A_j &\left(\frac{-\left(\hat{B}_k - \frac{1}{p}\right) A_j f - \frac{2}{pq}\big(\langle a, m\rangle + \langle b, n\rangle\big)}{-A_j B_k f - \frac{m_j}{p} - \frac{n_k}{q} + \frac{N}{pq} - \frac{2}{pq}\big(\langle \alpha, m\rangle + \langle \beta, n\rangle\big)} - \frac{2}{p}(a_j + b_k) \right) \\
-B_k &\left(\frac{-\left(\hat{A}_j - \frac{1}{q}\right) B_k f - \frac{2}{pq}\big(\langle a, m\rangle + \langle b, n\rangle\big)}{-A_j B_k f - \frac{m_j}{p} - \frac{n_k}{q} + \frac{N}{pq} - \frac{2}{pq}\big(\langle \alpha, m\rangle + \langle \beta, n\rangle\big)} - \frac{2}{q}(a_j + b_k) \right),
\end{aligned}
$$

since $[A_j,B_k]=0$. Define $\sigma(a,b)=\langle a,m\rangle+\langle b,n\rangle$. As A_j and B_k are first order differential operators, we have

$$0=\frac{\Big\{\Big(\hat{B}_k-\frac{1}{p}\Big)A_jf+\frac{2\sigma(a,b)}{pq},A_jB_kf+\frac{m_j}{p}+\frac{n_k}{q}-\frac{N}{pq}+\frac{2\sigma(\alpha,\beta)}{pq}\Big\}_{A_j}}{\Big(-A_jB_kf-\frac{m_j}{p}-\frac{n_k}{q}+\frac{N}{pq}-\frac{2\sigma(\alpha,\beta)}{pq}\Big)^2}-\frac{\Big\{\Big(\hat{A}_j-\frac{1}{q}\Big)B_kf+\frac{2\sigma(a,b)}{pq},A_jB_kf+\frac{m_j}{p}+\frac{n_k}{q}-\frac{N}{pq}+\frac{2\sigma(\alpha,\beta)}{pq}\Big\}_{B_k}}{\Big(-A_jB_kf-\frac{m_j}{p}-\frac{n_k}{q}+\frac{N}{pq}-\frac{2\sigma(\alpha,\beta)}{pq}\Big)^2}-\frac{2}{p}A_j(a_j+b_k)+\frac{2}{q}B_k(a_j+b_k). \tag{273}$$

Similarly, we compute, for $j\neq k$, $A_k(269)_{jk}+A_j(269)_{kj}$ and $B_k(270)_{jk}+B_j(270)_{kj}$, and we obtain

$$0=\frac{\Big\{\Big(\hat{A}_j-\frac{1}{q}\Big)A_kf+\frac{2\sigma(a,b)}{q^2},A_jA_kf+\frac{m_j}{q}+\frac{m_k}{q}-\frac{N}{q^2}+\frac{2\sigma(\alpha,\beta)}{q^2}\Big\}_{A_k}}{\Big(A_jA_kf+\frac{m_j}{q}+\frac{m_k}{q}-\frac{N}{q^2}+\frac{2\sigma(\alpha,\beta)}{q^2}\Big)^2}+\frac{\Big\{\Big(\hat{A}_k-\frac{1}{q}\Big)A_jf+\frac{2\sigma(a,b)}{q^2},A_kA_jf+\frac{m_k}{q}+\frac{m_j}{q}-\frac{N}{q^2}+\frac{2\sigma(\alpha,\beta)}{q^2}\Big\}_{A_j}}{\Big(A_jA_kf+\frac{m_j}{q}+\frac{m_k}{q}-\frac{N}{q^2}+\frac{2\sigma(\alpha,\beta)}{q^2}\Big)^2}-\frac{4}{q}, \tag{274}$$

and

$$0=\frac{\Big\{\Big(\hat{B}_j-\frac{1}{p}\Big)B_kf+\frac{2\sigma(a,b)}{p^2},B_jB_kf+\frac{n_j}{p}+\frac{n_k}{p}-\frac{N}{p^2}+\frac{2\sigma(\alpha,\beta)}{p^2}\Big\}_{B_k}}{\Big(B_jB_kf+\frac{n_j}{p}+\frac{n_k}{p}-\frac{N}{p^2}+\frac{2\sigma(\alpha,\beta)}{p^2}\Big)^2}+\frac{\Big\{\Big(\hat{B}_k-\frac{1}{p}\Big)B_jf+\frac{2\sigma(a,b)}{p^2},B_kB_jf+\frac{n_k}{p}+\frac{n_j}{p}-\frac{N}{p^2}+\frac{2\sigma(\alpha,\beta)}{p^2}\Big\}_{B_j}}{\Big(B_kB_jf+\frac{n_k}{p}+\frac{n_j}{p}-\frac{N}{p^2}+\frac{2\sigma(\alpha,\beta)}{p^2}\Big)^2}-\frac{4}{p}. \tag{275}$$

Using (265) we compute

$$\begin{aligned}A_jA_kf&=\left(A_j^{\mathcal{L}}-\frac{2}{q}\Big(\sum_{l=1}^{q-1}\alpha_l\frac{\partial}{\partial a_l}+\sum_{l=1}^{p-1}\beta_l\frac{\partial}{\partial b_l}\Big)\right)\\&\qquad\times\left(A_k^{\mathcal{L}}-\frac{2}{q}\Big(\sum_{l=1}^{q-1}\alpha_l\frac{\partial}{\partial a_l}+\sum_{l=1}^{p-1}\beta_l\frac{\partial}{\partial b_l}\Big)\right)f\\&=A_j^{\mathcal{L}}A_k^{\mathcal{L}}f+\mathcal{O}(\alpha,\beta).\end{aligned}$$

Similarly we have

$$A_jB_kf=A_j^{\mathcal{L}}B_k^{\mathcal{L}}f+\mathcal{O}(\alpha,\beta),\qquad B_jB_kf=B_j^{\mathcal{L}}B_k^{\mathcal{L}}f+\mathcal{O}(\alpha,\beta),$$

and

$$
\begin{aligned}
\Big(\hat{A}_j - \frac{1}{q}\Big) A_k f = \Big(\frac{1}{q}\hat{\mathcal{B}}_0 - (\partial_{\alpha_j} - \frac{1}{q}\partial_\alpha) - \frac{1}{q}\Big) A_k^{\mathcal{L}} f \\
+ \frac{2}{q}(\partial_{a_j} - \frac{1}{q}\partial_a) f + \mathcal{O}(\alpha,\beta), \\
\Big(\hat{A}_j - \frac{1}{q}\Big) B_k f = \Big(\frac{1}{q}\hat{\mathcal{B}}_0 - (\partial_{\alpha_j} - \frac{1}{q}\partial_\alpha) - \frac{1}{q}\Big) B_k^{\mathcal{L}} f \\
+ \frac{2}{p}(\partial_{a_j} - \frac{1}{q}\partial_a) f + \mathcal{O}(\alpha,\beta), \\
\Big(\hat{B}_j - \frac{1}{p}\Big) A_k f = \Big(\frac{1}{p}\hat{\mathcal{B}}_0 - (\partial_{\beta_j} - \frac{1}{p}\partial_\beta) - \frac{1}{p}\Big) A_k^{\mathcal{L}} f \\
+ \frac{2}{q}(\partial_{b_j} - \frac{1}{p}\partial_b) f + \mathcal{O}(\alpha,\beta), \\
\Big(\hat{B}_j - \frac{1}{p}\Big) B_k f = \Big(\frac{1}{p}\hat{\mathcal{B}}_0 - (\partial_{\beta_j} - \frac{1}{p}\partial_\beta) - \frac{1}{p}\Big) B_k^{\mathcal{L}} f \\
+ \frac{2}{p}(\partial_{b_j} - \frac{1}{p}\partial_b) f + \mathcal{O}(\alpha,\beta).
\end{aligned}
$$

Consequently, on the locus $\mathcal{L}$ the equation (273) can be written

$$
\begin{aligned}
0 = \frac{\Big\{\Big(\frac{1}{p}\hat{\mathcal{B}}_0 - (\partial_{\beta_k} - \frac{1}{p}\partial_\beta) - \frac{1}{p}\Big) A_j^{\mathcal{L}} f + \frac{2}{q}(\partial_{b_k} - \frac{1}{p}\partial_b) f + \frac{2\sigma(a,b)}{pq}, A_j^{\mathcal{L}} B_k^{\mathcal{L}} f + \frac{m_j}{p} + \frac{n_k}{q} - \frac{N}{pq}\Big\}_{A_j^{\mathcal{L}}}}{\Big(- A_j^{\mathcal{L}} B_k^{\mathcal{L}} f - \frac{m_j}{p} - \frac{n_k}{q} + \frac{N}{pq}\Big)^2} \\
- \frac{\Big\{\Big(\frac{1}{q}\hat{\mathcal{B}}_0 - (\partial_{\alpha_j} - \frac{1}{q}\partial_\alpha) - \frac{1}{q}\Big) B_k^{\mathcal{L}} f + \frac{2}{p}(\partial_{a_j} - \frac{1}{q}\partial_a) f + \frac{2\sigma(a,b)}{pq}, A_j^{\mathcal{L}} B_k^{\mathcal{L}} f + \frac{m_j}{p} + \frac{n_k}{q} - \frac{N}{pq}\Big\}_{B_k^{\mathcal{L}}}}{\Big(- A_j^{\mathcal{L}} B_k^{\mathcal{L}} f - \frac{m_j}{p} - \frac{n_k}{q} + \frac{N}{pq}\Big)^2} \\
+ \frac{2}{p} - \frac{2}{q}.
\end{aligned}
$$

Putting all the terms which do not contain derivatives of f with respect to α_i's or β_j's in the left hand side, we obtain

$$
\begin{aligned}
G_{jk}^{AB} = \Big\{(\frac{1}{p}\partial_\beta - \partial_{\beta_k}) A_j^{\mathcal{L}} f, A_j^{\mathcal{L}} B_k^{\mathcal{L}} f + \frac{m_j}{p} + \frac{n_k}{q} - \frac{N}{pq}\Big\}_{A_j^{\mathcal{L}}} \\
- \Big\{(\frac{1}{q}\partial_\alpha - \partial_{\alpha_j}) B_k^{\mathcal{L}} f, A_j^{\mathcal{L}} B_k^{\mathcal{L}} f + \frac{m_j}{p} + \frac{n_k}{q} - \frac{N}{pq}\Big\}_{B_k^{\mathcal{L}}},
\end{aligned}
$$

where

$$
\begin{aligned}
G_{jk}^{AB} &:= \Big(\frac{2}{q} - \frac{2}{p}\Big)\Big(- A_j^{\mathcal{L}} B_k^{\mathcal{L}} f - \frac{m_j}{p} - \frac{n_k}{q} + \frac{N}{pq}\Big)^2 \\
&- \Big\{\Big(\frac{1}{p}\hat{\mathcal{B}}_0 - \frac{1}{p}\Big) A_j^{\mathcal{L}} f + \frac{2}{q}(\partial_{b_k} - \frac{1}{p}\partial_b) f + \frac{2\sigma(a,b)}{pq}, A_j^{\mathcal{L}} B_k^{\mathcal{L}} f + \frac{m_j}{p} + \frac{n_k}{q} - \frac{N}{pq}\Big\}_{A_j^{\mathcal{L}}} \\
&+ \Big\{\Big(\frac{1}{q}\hat{\mathcal{B}}_0 - \frac{1}{q}\Big) B_k^{\mathcal{L}} f + \frac{2}{p}(\partial_{a_j} - \frac{1}{q}\partial_a) f + \frac{2\sigma(a,b)}{pq}, A_j^{\mathcal{L}} B_k^{\mathcal{L}} f + \frac{m_j}{p} + \frac{n_k}{q} - \frac{N}{pq}\Big\}_{B_k^{\mathcal{L}}}.
\end{aligned}
$$

Similarly, on the locus $\mathcal{L}$, the equations (274) and (275) can be written

$$G^A_{jk} = \Big\{(\frac{1}{q}\partial_\alpha - \partial_{\alpha_j})A^{\mathcal{L}}_k f, A^{\mathcal{L}}_j A^{\mathcal{L}}_k f + \frac{m_j}{q} + \frac{m_k}{q} - \frac{N}{q^2}\Big\}_{A^{\mathcal{L}}_k} + \Big\{(\frac{1}{q}\partial_\alpha - \partial_{\alpha_k})A^{\mathcal{L}}_j f, A^{\mathcal{L}}_j A^{\mathcal{L}}_k f + \frac{m_j}{q} + \frac{m_k}{q} - \frac{N}{q^2}\Big\}_{A^{\mathcal{L}}_j},$$

$$G^B_{jk} = \Big\{(\frac{1}{p}\partial_\beta - \partial_{\beta_j})B^{\mathcal{L}}_k f, B^{\mathcal{L}}_j B^{\mathcal{L}}_k f + \frac{n_j}{p} + \frac{n_k}{p} - \frac{N}{p^2}\Big\}_{B^{\mathcal{L}}_k} + \Big\{(\frac{1}{p}\partial_\beta - \partial_{\beta_k})B^{\mathcal{L}}_j f, B^{\mathcal{L}}_j B^{\mathcal{L}}_k f + \frac{n_j}{p} + \frac{n_k}{p} - \frac{N}{p^2}\Big\}_{B^{\mathcal{L}}_j},$$

where

$$\begin{aligned} G^A_{jk} &:= \frac{4}{q}\Big(A^{\mathcal{L}}_j A^{\mathcal{L}}_k f + \frac{m_j}{q} + \frac{m_k}{q} - \frac{N}{q^2}\Big)^2 \\ &- \Big\{\Big(\frac{1}{q}\hat{\mathcal{B}}_0 - \frac{1}{q}\Big)A^{\mathcal{L}}_k f + \frac{2}{q}(\partial_{a_j} - \frac{1}{q}\partial_a)f + \frac{2\sigma(a,b)}{q^2}, A^{\mathcal{L}}_j A^{\mathcal{L}}_k f + \frac{m_j}{q} + \frac{m_k}{q} - \frac{N}{q^2}\Big\}_{A^{\mathcal{L}}_k} \\ &- \Big\{\Big(\frac{1}{q}\hat{\mathcal{B}}_0 - \frac{1}{q}\Big)A^{\mathcal{L}}_j f + \frac{2}{q}(\partial_{a_k} - \frac{1}{q}\partial_a)f + \frac{2\sigma(a,b)}{q^2}, A^{\mathcal{L}}_j A^{\mathcal{L}}_k f + \frac{m_j}{q} + \frac{m_k}{q} - \frac{N}{q^2}\Big\}_{A^{\mathcal{L}}_j}, \end{aligned}$$

$$\begin{aligned} G^B_{jk} &:= \frac{4}{p}\Big(B^{\mathcal{L}}_j B^{\mathcal{L}}_k f + \frac{n_j}{p} + \frac{n_k}{p} - \frac{N}{p^2}\Big)^2 \\ &- \Big\{\Big(\frac{1}{p}\hat{\mathcal{B}}_0 - \frac{1}{p}\Big)B^{\mathcal{L}}_k f + \frac{2}{p}(\partial_{b_j} - \frac{1}{p}\partial_b)f + \frac{2\sigma(a,b)}{p^2}, B^{\mathcal{L}}_j B^{\mathcal{L}}_k f + \frac{n_j}{p} + \frac{n_k}{p} - \frac{N}{p^2}\Big\}_{B^{\mathcal{L}}_k} \\ &- \Big\{\Big(\frac{1}{p}\hat{\mathcal{B}}_0 - \frac{1}{p}\Big)B^{\mathcal{L}}_j f + \frac{2}{p}(\partial_{b_k} - \frac{1}{p}\partial_b)f + \frac{2\sigma(a,b)}{p^2}, B^{\mathcal{L}}_j B^{\mathcal{L}}_k f + \frac{n_j}{p} + \frac{n_k}{p} - \frac{N}{p^2}\Big\}_{B^{\mathcal{L}}_j}. \end{aligned}$$

□

In order to obtain a PDE for $f = \log \tau^E_{\vec{m},\vec{n}}(0;a,b)$ or for $\log P_{p,q}(E,a,b)$, we need to eliminate the partial derivatives of f with respect to $\alpha_1,\dots,\alpha_{q-1},\beta_1,\dots,\beta_{p-1}$ from the equations (267) in Theorem 6.8. Define

$$\begin{aligned} X_i &= (\frac{1}{q}\partial_\alpha - \partial_{\alpha_i})f\big|_{\mathcal{L}}, \quad 1 \le i \le q, \\ Y_i &= (\frac{1}{p}\partial_\beta - \partial_{\beta_i})f\big|_{\mathcal{L}}, \quad 1 \le i \le p. \end{aligned}$$

Note that we have $\sum_{i=1}^q X_i = \sum_{i=1}^p Y_i = 0$, and $\sum_{i=1}^q A^{\mathcal{L}}_i = \sum_{i=1}^p B^{\mathcal{L}}_i = \mathcal{B}_{-1}$. Consequently, there are among $A^{\mathcal{L}}_i$, $1 \le i \le q$, and $B^{\mathcal{L}}_j$, $1 \le j \le p$, only $p+q-1$ linearly independent differential operators. Set $A^{\mathcal{L}}_q = \mathcal{B}_{-1} - \sum_{i=1}^{q-1} A^{\mathcal{L}}_i$ and $B^{\mathcal{L}}_p = \mathcal{B}_{-1} - \sum_{i=1}^{p-1} B^{\mathcal{L}}_i$.

With these notations, the equations (267) can be written

$$\begin{aligned}
&\Big\{A_j^{\mathcal{L}}Y_k, A_j^{\mathcal{L}}B_k^{\mathcal{L}}f+\frac{m_j}{p}+\frac{n_k}{q}-\frac{N}{pq}\Big\}_{A_j^{\mathcal{L}}}\\
&-\Big\{B_k^{\mathcal{L}}X_j, A_j^{\mathcal{L}}B_k^{\mathcal{L}}f+\frac{m_j}{p}+\frac{n_k}{q}-\frac{N}{pq}\Big\}_{B_k^{\mathcal{L}}}=G_{jk}^{AB}, \quad 1\le j\le q, 1\le k\le p,\\
&\Big\{A_k^{\mathcal{L}}X_j, A_j^{\mathcal{L}}A_k^{\mathcal{L}}f+\frac{m_j}{q}+\frac{m_k}{q}-\frac{N}{q^2}\Big\}_{A_k^{\mathcal{L}}}\\
&\quad+\Big\{A_j^{\mathcal{L}}X_k, A_j^{\mathcal{L}}A_k^{\mathcal{L}}f+\frac{m_j}{q}+\frac{m_k}{q}-\frac{N}{q^2}\Big\}_{A_j^{\mathcal{L}}}=G_{jk}^{A}, \quad 1\le j<k\le q,\\
&\Big\{B_k^{\mathcal{L}}Y_j, B_j^{\mathcal{L}}B_k^{\mathcal{L}}f+\frac{n_j}{p}+\frac{n_k}{p}-\frac{N}{p^2}\Big\}_{B_k^{\mathcal{L}}}\\
&\quad+\Big\{B_j^{\mathcal{L}}Y_k, B_j^{\mathcal{L}}B_k^{\mathcal{L}}f+\frac{n_j}{p}+\frac{n_k}{p}-\frac{N}{p^2}\Big\}_{B_j^{\mathcal{L}}}=G_{jk}^{B}, \quad 1\le j<k\le p,
\end{aligned}$$

or

$$\begin{aligned}
&\big(A_j^{\mathcal{L}}\big)^2Y_k-\big(B_k^{\mathcal{L}}\big)^2X_j-\Big(\frac{1}{c_{jk}^{AB}}A_j^{\mathcal{L}}c_{jk}^{AB}\Big)A_j^{\mathcal{L}}Y_k\\
&\qquad+\Big(\frac{1}{c_{jk}^{AB}}B_k^{\mathcal{L}}c_{jk}^{AB}\Big)B_k^{\mathcal{L}}X_j=g_{jk}^{AB}, \quad 1\le j\le q, 1\le k\le p,
\end{aligned}$$

$$\begin{aligned}
&\big(A_j^{\mathcal{L}}\big)^2X_k+\big(A_k^{\mathcal{L}}\big)^2X_j-\Big(\frac{1}{c_{jk}^{A}}A_j^{\mathcal{L}}c_{jk}^{A}\Big)A_j^{\mathcal{L}}X_k\\
&\qquad-\Big(\frac{1}{c_{jk}^{A}}A_k^{\mathcal{L}}c_{jk}^{A}\Big)A_k^{\mathcal{L}}X_j=g_{jk}^{A}, \quad 1\le j<k\le q,\\
&\big(B_j^{\mathcal{L}}\big)^2Y_k+\big(B_k^{\mathcal{L}}\big)^2Y_j-\Big(\frac{1}{c_{jk}^{B}}B_j^{\mathcal{L}}c_{jk}^{B}\Big)B_j^{\mathcal{L}}Y_k\\
&\qquad-\Big(\frac{1}{c_{jk}^{B}}B_k^{\mathcal{L}}c_{jk}^{B}\Big)B_k^{\mathcal{L}}Y_j=g_{jk}^{B}, \quad 1\le j<k\le p,
\end{aligned}\tag{276}$$

where

$$\begin{aligned}
c_{jk}^{AB}&:=A_j^{\mathcal{L}}B_k^{\mathcal{L}}f+\frac{m_j}{p}+\frac{n_k}{q}-\frac{N}{pq},\\
c_{jk}^{A}&:=A_j^{\mathcal{L}}A_k^{\mathcal{L}}f+\frac{m_j}{q}+\frac{m_k}{q}-\frac{N}{q^2},\\
c_{jk}^{B}&:=B_j^{\mathcal{L}}B_k^{\mathcal{L}}f+\frac{n_j}{p}+\frac{n_k}{p}-\frac{N}{p^2},
\end{aligned}\tag{277}$$

and

$$g_{jk}^{AB} := \frac{G_{jk}^{AB}}{c_{jk}^{AB}}, \qquad g_{jk}^{A} := \frac{G_{jk}^{A}}{c_{jk}^{A}}, \qquad g_{jk}^{B} := \frac{G_{jk}^{B}}{c_{jk}^{B}}. \tag{278}$$

We have thus a system of $M = \frac{1}{2}(p+q)(p+q-1)$ linear equations in the $p+q-2$ unknown functions $X_1,\dots,X_{q-1}$ and $Y_1,\dots,Y_{p-1}$ and at most all their first and second order derivatives with respect to the independent commuting differential operators $A_i^{\mathcal{L}}$, $1 \le i \le q-1$, $B_j^{\mathcal{L}}$, $1 \le j \le p-1$, and $\mathcal{B}_{-1}$. We think at all these quantities as unknowns. At this point, we have a system with a smaller number of linear equations then unknowns. The general strategy is to keep differentiating the equations and show that at some point we must reach a balance between the number of equations and the number of unknowns, leading to the vanishing of a determinant *at the first point this occurs*, which must yield a nontrivial relation. Let Z_M be the set of linear equations (276), and define

$$Z_M^K := [1 + \mathcal{B}_{-1} + A_1^{\mathcal{L}} + \cdots + A_{q-1}^{\mathcal{L}} + B_1^{\mathcal{L}} + \cdots + B_{p-1}^{\mathcal{L}}]^K Z_M,$$

the set of equations obtained by taking the equations of Z_M and all their derivatives up to the Kth order with respect to the differential operators $\mathcal{B}_{-1}, A_1^{\mathcal{L}},\dots,A_{q-1}^{\mathcal{L}}, B_1^{\mathcal{L}},\dots,B_{p-1}^{\mathcal{L}}$. The number of equations in Z_M^K is simply M times the number of monomials of degree K chosen from a set of $p+q$ variables, i.e.

$$\begin{aligned} &M\binom{p+q+K-1}{K} \\ &\qquad = \frac{1}{2}(p+q)\frac{(K+p+q-1)(K+p+q-2)\dots(K+1)}{(p+q-2)!}. \end{aligned}$$

The set of equations Z_M^K is a set of linear equations in the $p+q-2$ unknown functions $X_1,\dots,X_{q-1}$ and $Y_1,\dots,Y_{p-1}$ and at most all their first, second, …, $(K+2)^{\text{th}}$ order derivatives with respect to the differential operators $A_i^{\mathcal{L}}$, $1 \le i \le q-1$, $B_j^{\mathcal{L}}$, $1 \le j \le p-1$, and $\mathcal{B}_{-1}$. Let L be the number of unknowns in these equations. Then

$$\begin{aligned} L &\le (p+q-2)\binom{p+q+K+2-1}{K+2} \\ &\qquad = (p+q-2)\frac{(K+p+q+1)(K+p+q)\dots(K+3)}{(p+q-1)!}. \end{aligned}$$

From these considerations, it is clear that a sufficient condition to have $Card(Z_M^K) > L$ is

$$\frac{1}{2}(p+q)\frac{(K+p+q-1)!}{K!(p+q-2)!} > (p+q-2)\frac{(K+p+q+1)!}{(K+2)!(p+q-1)!},$$

or, simplifying this expression,

$$(x^2-3x+4)K^2 + (-x^2+3x+4)K - 2x(x^2-2x-1) > 0,$$

where we have noted $x = p+q$. We observe that, with p and q fixed, for K sufficiently large, this inequality is satisfied, since $x^2 - 3x + 4 > 0$. Let K^* be the smallest value of K such that this inequality is satisfied, and note k^* the number of equations in $Z_M^{K^*}$, i.e.

$$k^* = \frac{1}{2}(p+q)(p+q-1)\binom{p+q+K^*-1}{K^*},$$

and L^* the number of unknowns in the set of linear equations $Z_M^{K^*}$. Let us note these unknowns $x_1, \ldots, x_{L^*}$. Then the system of k^* linear equations that we have obtained can be written

$$\big[a_{ij}(f)\big]_{\substack{1\le i\le k^*\\ 1\le j\le L^*+1}} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_{L^*} \end{pmatrix} = 0.$$

As $k^* > L^*$, we can select the L^*+1 first equations in this system and construct the following system

$$\big[a_{ij}(f)\big]_{\substack{1\le i\le L^*+1\\ 1\le j\le L^*+1}} \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_{L^*} \end{pmatrix} = 0.$$

But then, necessarily, we have

$$\det \big[a_{ij}(f)\big]_{\substack{1\le i\le L^*+1\\ 1\le j\le L^*+1}} = 0.$$

This is a PDE of order (K^*+3) for the function f, with variables $a_1, \ldots, a_{q-1}$, $b_1, \ldots, b_{p-1}$ and $c_1, \ldots, c_{2r}$. Since $f = \log \tau^E_{\vec{m},\vec{n}}(0;a,b) = \log P_{p,q}(E;a,b) + \log \tau^{\mathbb{R}}_{\vec{m},\vec{n}}(0;a,b)$, this yields a nonlinear PDE of order K^*+3 in the variables $a_1, \ldots, a_{q-1}$, $b_1, \ldots, b_{p-1}$ and the endpoints of E for $\log P_{p,q}(E;a,b)$, and thus throuhg formula (235) for $\log \mathbb{P}^{a_1,\ldots,a_q}_{b_1,\ldots,b_p}\big(all\ \ x_i(t)\ \in\ E\big)$, in terms of the input $\log \tau^{\mathbb{R}}_{\vec{m},\vec{n}}(0;a,b)$, which we think of as known[2]. We thus have proven the following theorem.

Theorem 6.9 (Adler-van Moerbeke-Vanderstichelen [**13**]). *For each value of the parameters $p \ge 1$ and $q \ge 1$, let K^* be the smallest positive integer such that*

$$(x^2-3x+4)(K^*)^2 + (-x^2+3x+4)K^* - 2x(x^2-2x-1) > 0,$$

with $x = p+q$. Let E be a finite union of intervals. Under the assumptions $a_1 + \cdots + a_q = 0$ and $b_1 + \cdots + b_p = 0$, the function $\log \mathbb{P}^{a_1,\ldots,a_q}_{b_1,\ldots,b_p}\big(all\ x_i(t) \in E\big)$ satisfies a nonlinear PDE of order K^*+3 or less, the variables being the coordinates of the endpoints of the set E, and the coordinates of $a_1, \ldots, a_q$ and $b_1, \ldots, b_p$.

[2]See Appendix D for a discussion of this problem.

For example, for $4 \leq x \leq 8$, the value of K^* in this theorem is given in the following table :

$\mathbf{x}$	4	5	6	7	8
$\mathbf{K}^*$	3	4	5	5	5

6. Non-intersecting Brownian motions with two starting points and two ending points

In this section we consider a particular situation of the problem studied in the preceding sections. Consider N non-intersecting Brownian motions $x_1(t), x_2(t), \dots, x_N(t)$ in $\mathbb{R}$, conditionned to start at time $t = 0$ at two different points and to end up at $t = 1$ in two different points, with the coordinates of the starting points and the coordinates of the ending points both satisfying a linear condition. In this particular case, all the results of the preceding sections hold with $p = q = 2$. Consequently, by virtue of Theorem 6.9, we know that the probability to find all the particles in a certain set $E = \bigcup_{i=1}^r [c_{2i-1}, c_{2i}] \subset \mathbb{R}$ at a given time $0 < t < 1$, satisfies a nonlinear PDE of order 6, the variables being the coordinates of the starting and ending points, and the endpoints of the set E. The aim of this section is to improve the result of Theorem 6.9 in the particular case when $p = q = 2$ and to describe this PDE more precisely.

For the sake of clarity, we first recall some notations. Consider N non-intersecting Brownian motions $x_1(t), x_2(t), \dots, x_N(t)$ in $\mathbb{R}$, with m_1 particles leaving from a and m_2 particles leaving from $-a$, and n_1 particles ending in b and n_2 particles ending in $-b$. We denote

$$\begin{aligned}
&\mathbb{P}^{+a,-a}_{+b,-b}\big(\text{all } x_i(t) \in E\big) \\
&:= \mathbb{P}\left(\text{all } x_i(t) \in E \;\middle|\; \begin{array}{l} \big(x_1(0), \dots, x_N(0)\big) = \big(\underbrace{a, \dots, a}_{m_1}, \underbrace{-a, \dots, -a}_{m_2}\big) \\ \big(x_1(1), \dots, x_N(1)\big) = \big(\underbrace{b, \dots, b}_{n_1}, \underbrace{-b, \dots, -b}_{n_2}\big) \end{array}\right),
\end{aligned}$$

the probability to find all the particles in a set $E \subset \mathbb{R}$, at a given time $0 < t < 1$. We have

$$\mathbb{P}^{a,-a}_{b,-b}\big(\text{all } x_i(t) \in E\big) = P_{2,2}\Big(\sqrt{\frac{2}{t(1-t)}}\, E; \sqrt{\frac{2(1-t)}{t}}\, a, \sqrt{\frac{2t}{1-t}}\, b\Big),$$

with the normalized problem defined in (236). We deform $P_{2,2}(E; a, b)$ by adding four families of extra time variables

$$\begin{aligned}
t^{(1)} &= \big(t^{(1)}_1, t^{(1)}_2, \dots\big), \quad t^{(2)} = \big(t^{(2)}_1, t^{(2)}_2, \dots\big), \\
s^{(1)} &= \big(s^{(1)}_1, s^{(1)}_2, \dots\big), \quad s^{(2)} = \big(s^{(2)}_1, s^{(2)}_2, \dots\big),
\end{aligned}$$

and the two parameters $\alpha, \beta \in \mathbb{C}$. We have

$$P_{2,2}\big(E; a, b; (t,s), (\alpha,\beta)\big) = \frac{\tau^{E}_{m_1 m_2; n_1, n_2}(t, s; \alpha, \beta; a, b)}{\tau^{\mathbb{R}}_{m_1 m_2; n_1, n_2}(t, s; \alpha, \beta; a, b)}, \tag{279}$$

with $\tau^{E}_{m_1 m_2; n_1, n_2}(t, s; \alpha, \beta; a, b)$ defined in (241), and where $(t,s) = \big(t^{(1)}, t^{(2)}; s^{(1)}, s^{(2)}\big)$. The function $P_{2,2}(E; a, b)$ is then simply given by $P_{2,2}\big(E; a, b; (t,s), (\alpha,\beta)\big)$ evaluated along the locus $\mathcal{L} = \{(t,s) = (0,0), \alpha = \beta = 0\}$. We define the function $f := \log \tau^{E}_{m_1, m_2; n_1, n_2}(t, s; \alpha, \beta; a, b)$. The following particular version of Theorem 6.8 holds.

Theorem 6.10 (Adler-van Moerbeke-Vanderstichelen [**13**]). *Put $X = -\frac{1}{2}\frac{\partial f}{\partial \alpha}\big|_{\mathcal{L}}$ and $Y = -\frac{1}{2}\frac{\partial f}{\partial \beta}\big|_{\mathcal{L}}$. The function $f = \log \tau^{E}_{m_1, m_2; n_1, n_2}(t, s; \alpha, \beta; a, b)$ satisfies the following* 6 *equations on the locus $\mathcal{L}$*

$$\begin{aligned}&\Big\{A_1^{\mathcal{L}} Y, A_1^{\mathcal{L}} B_1^{\mathcal{L}} f + \frac{m_1}{2} + \frac{n_1}{2} - \frac{N}{4}\Big\}_{A_1^{\mathcal{L}}} \\ &\qquad - \Big\{B_1^{\mathcal{L}} X, A_1^{\mathcal{L}} B_1^{\mathcal{L}} f + \frac{m_1}{2} + \frac{n_1}{2} - \frac{N}{4}\Big\}_{B_1^{\mathcal{L}}} = G_{11}^{AB},\end{aligned} \tag{280}$$

$$\begin{aligned}&-\Big\{A_1^{\mathcal{L}} Y, A_1^{\mathcal{L}} B_2^{\mathcal{L}} f + \frac{m_1}{2} + \frac{n_2}{2} - \frac{N}{4}\Big\}_{A_1^{\mathcal{L}}} \\ &\qquad - \Big\{B_2^{\mathcal{L}} X, A_1^{\mathcal{L}} B_2^{\mathcal{L}} f + \frac{m_1}{2} + \frac{n_2}{2} - \frac{N}{4}\Big\}_{B_2^{\mathcal{L}}} = G_{12}^{AB},\end{aligned} \tag{281}$$

$$\begin{aligned}&\Big\{A_2^{\mathcal{L}} Y, A_2^{\mathcal{L}} B_1^{\mathcal{L}} f + \frac{m_2}{2} + \frac{n_1}{2} - \frac{N}{4}\Big\}_{A_2^{\mathcal{L}}} \\ &\qquad + \Big\{B_1^{\mathcal{L}} X, A_2^{\mathcal{L}} B_1^{\mathcal{L}} f + \frac{m_2}{2} + \frac{n_1}{2} - \frac{N}{4}\Big\}_{B_1^{\mathcal{L}}} = G_{21}^{AB},\end{aligned} \tag{282}$$

$$\begin{aligned}&-\Big\{A_2^{\mathcal{L}} Y, A_2^{\mathcal{L}} B_2^{\mathcal{L}} f + \frac{m_2}{2} + \frac{n_2}{2} - \frac{N}{4}\Big\}_{A_2^{\mathcal{L}}} \\ &\qquad + \Big\{B_2^{\mathcal{L}} X, A_2^{\mathcal{L}} B_2^{\mathcal{L}} f + \frac{m_2}{2} + \frac{n_2}{2} - \frac{N}{4}\Big\}_{B_2^{\mathcal{L}}} = G_{22}^{AB},\end{aligned} \tag{283}$$

$$\Big\{A_2^{\mathcal{L}} X, A_1^{\mathcal{L}} A_2^{\mathcal{L}} f + \frac{N}{4}\Big\}_{A_2^{\mathcal{L}}} - \Big\{A_1^{\mathcal{L}} X, A_1^{\mathcal{L}} A_2^{\mathcal{L}} f + \frac{N}{4}\Big\}_{A_1^{\mathcal{L}}} = G_{12}^{A}, \tag{284}$$

$$\Big\{B_2^{\mathcal{L}} Y, B_1^{\mathcal{L}} B_2^{\mathcal{L}} f + \frac{N}{4}\Big\}_{B_2^{\mathcal{L}}} - \Big\{B_1^{\mathcal{L}} Y, B_1^{\mathcal{L}} B_2^{\mathcal{L}} f + \frac{N}{4}\Big\}_{B_1^{\mathcal{L}}} = G_{12}^{B}, \tag{285}$$

where

$$A_j = A_j^{\mathcal{L}} - \Big(\alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b}\Big), \quad B_j = B_j^{\mathcal{L}} - \Big(\alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial b}\Big), \quad 1 \le j \le 2,$$

$$A_j^{\mathcal{L}} = \frac{1}{2}\Big(\mathcal{B}_{-1} + (-1)^j \frac{\partial}{\partial a}\Big), \quad B_j^{\mathcal{L}} = \frac{1}{2}\Big(\mathcal{B}_{-1} + (-1)^j \frac{\partial}{\partial b}\Big), \quad 1 \le j \le 2,$$

and where G_{jk}^A, G_{jk}^B and G_{jk}^{AB} only depend on f, its derivatives with respect to $a_1, \dots, a_{q-1}$, $b_1, \dots, b_{p-1}$, and its differentials up to the third order with respect to the operators $A_j^{\mathcal{L}}$, $B_j^{\mathcal{L}}$ and $\hat{\mathcal{B}}_0$, evaluated on the locus $\mathcal{L}$.

The equations in Theorem 6.10 can be written

$$\begin{aligned}
&\big(A_2^{\mathcal{L}}\big)^2 X - \big(A_1^{\mathcal{L}}\big)^2 X - \Big(A_2^{\mathcal{L}} \log c_{12}^A\Big) A_2^{\mathcal{L}} X + \Big(A_1^{\mathcal{L}} \log c_{12}^A\Big) A_1^{\mathcal{L}} X = g_{12}^A,\\
&\big(B_2^{\mathcal{L}}\big)^2 Y - \big(B_1^{\mathcal{L}}\big)^2 Y - \Big(B_2^{\mathcal{L}} \log c_{12}^B\Big) B_2^{\mathcal{L}} Y + \Big(B_1^{\mathcal{L}} \log c_{12}^B\Big) B_1^{\mathcal{L}} Y = g_{12}^B,\\
&\big(A_1^{\mathcal{L}}\big)^2 Y - \big(B_1^{\mathcal{L}}\big)^2 X - \Big(A_1^{\mathcal{L}} \log c_{11}^{AB}\Big) A_1^{\mathcal{L}} Y + \Big(B_1^{\mathcal{L}} \log c_{11}^{AB}\Big) B_1^{\mathcal{L}} X = g_{11}^{AB},\\
&\big(A_2^{\mathcal{L}}\big)^2 Y + \big(B_1^{\mathcal{L}}\big)^2 X - \Big(A_2^{\mathcal{L}} \log c_{21}^{AB}\Big) A_2^{\mathcal{L}} Y - \Big(B_1^{\mathcal{L}} \log c_{21}^{AB}\Big) B_1^{\mathcal{L}} X = g_{21}^{AB},\\
&-\big(A_1^{\mathcal{L}}\big)^2 Y - \big(B_2^{\mathcal{L}}\big)^2 X + \Big(A_1^{\mathcal{L}} \log c_{12}^{AB}\Big) A_1^{\mathcal{L}} Y + \Big(B_2^{\mathcal{L}} \log c_{12}^{AB}\Big) B_2^{\mathcal{L}} X = g_{12}^{AB},\\
&-\big(A_2^{\mathcal{L}}\big)^2 Y + \big(B_2^{\mathcal{L}}\big)^2 X + \Big(A_2^{\mathcal{L}} \log c_{22}^{AB}\Big) A_2^{\mathcal{L}} Y - \Big(B_2^{\mathcal{L}} \log c_{22}^{AB}\Big) B_2^{\mathcal{L}} X = g_{22}^{AB},
\end{aligned} \tag{286}$$

where the c_{ij}'s and the g_{ij}'s are defined in (277) and (278) with $p = q = 2$.

The four differential operators $A_1^{\mathcal{L}}, A_2^{\mathcal{L}}, B_1^{\mathcal{L}}, B_2^{\mathcal{L}}$ are not linearly independent. Indeed, we have

$$A_1^{\mathcal{L}} + A_2^{\mathcal{L}} = B_1^{\mathcal{L}} + B_2^{\mathcal{L}} = \mathcal{B}_{-1}, \qquad A_2^{\mathcal{L}} - A_1^{\mathcal{L}} = \frac{\partial}{\partial a}, \qquad B_2^{\mathcal{L}} - B_1^{\mathcal{L}} = \frac{\partial}{\partial b}.$$

Consequently, we will rewrite the six equations (286) in terms of the three independent, commuting differential operators $\frac{\partial}{\partial a}$, $\frac{\partial}{\partial b}$ and $\mathcal{B}_{-1}$. Before performing this, let us introduce some notations. First, we will write

$$\frac{\partial F}{\partial a} = F_a, \quad \frac{\partial F}{\partial b} = F_b, \quad \mathcal{B}_{-1} = F_c,$$

for a function F. Next, we put

$$\begin{aligned}
&G_1 := g_{12}^A, \quad G_2 := g_{12}^B, \quad G_3 := -g_{11}^{AB} + g_{21}^{AB} - g_{12}^{AB} + g_{22}^{AB},\\
&G_4 := g_{11}^{AB} + g_{21}^{AB} - g_{12}^{AB} - g_{22}^{AB}, \quad G_5 := \frac{1}{2}\big(g_{11}^{AB} - g_{21}^{AB} - g_{12}^{AB} + g_{22}^{AB}\big),\\
&G_6 := -\frac{1}{2}\big(g_{11}^{AB} + g_{21}^{AB} + g_{12}^{AB} + g_{22}^{AB}\big),
\end{aligned}$$

and

$$\begin{aligned}
&\Delta_1 := \log c_{12}^A, \quad \Delta_2 := \log c_{12}^B, \quad \Delta_3 := \log \frac{c_{11}^{AB} c_{21}^{AB}}{c_{12}^{AB} c_{22}^{AB}},\\
&\Delta_4 := \log \frac{c_{11}^{AB} c_{12}^{AB}}{c_{21}^{AB} c_{22}^{AB}}, \quad \Delta_5 := \log \frac{c_{21}^{AB} c_{12}^{AB}}{c_{11}^{AB} c_{22}^{AB}},\\
&\Delta_6 := \log\big(c_{11}^{AB} c_{21}^{AB} c_{12}^{AB} c_{22}^{AB}\big).
\end{aligned}$$

We then define

$$\alpha := \frac{1}{4}(-\Delta_{3c} + \Delta_{6b}), \qquad \beta := \frac{1}{4}(\Delta_{6c} - \Delta_{3b}),$$
$$\gamma := \frac{1}{4}(\Delta_{4c} + \Delta_{5b}), \qquad \delta := \frac{1}{4}(\Delta_{5c} + \Delta_{4b}),$$

and

$$\hat{\alpha} := \frac{1}{4}(-\Delta_{4c} + \Delta_{6a}), \qquad \hat{\beta} := \frac{1}{4}(\Delta_{6c} - \Delta_{4a}),$$
$$\hat{\gamma} := \frac{1}{4}(\Delta_{3c} + \Delta_{5a}), \qquad \hat{\delta} := \frac{1}{4}(\Delta_{5c} + \Delta_{3a}).$$

Taking adequate linear combinations of the equations (286), and using all these notations, we obtain the following equivalent system

$$\begin{aligned}
&X_{ac} = G_1 + \frac{1}{2}\Delta_{1a}X_c + \frac{1}{2}\Delta_{1c}X_a,\\
&Y_{bc} = G_2 + \frac{1}{2}\Delta_{2b}Y_c + \frac{1}{2}\Delta_{2c}Y_b,\\
&X_{cc} + X_{bb} = G_3 + \beta X_c + \alpha X_b + \hat{\delta}Y_c + \hat{\gamma}Y_a,\\
&Y_{cc} + Y_{aa} = G_4 + \delta X_c + \gamma X_b + \hat{\beta}Y_c + \hat{\alpha}Y_a,\\
&X_{bc} - Y_{ac} = G_5 + \frac{\alpha}{2}X_c + \frac{\beta}{2}X_b - \frac{\hat{\alpha}}{2}Y_c - \frac{\hat{\beta}}{2}Y_a,\\
&0 = G_6 + \frac{\gamma}{2}X_c + \frac{\delta}{2}X_b - \frac{\hat{\gamma}}{2}Y_c - \frac{\hat{\delta}}{2}Y_a.
\end{aligned} \tag{287}$$

This is a system of linear equations in the variables

$$X_a, X_b, X_c, X_{ac}, X_{bc}, X_{bb}, X_{cc}, Y_a, Y_b, Y_c, Y_{ac}, Y_{bc}, Y_{aa}, Y_{cc}.$$

The coefficients of this linear system depend on $\log \tau^E_{m_1,m_2;n_1,n_2}(0; a, b)$ and its partial derivatives up to the third order with respect to a, b and the endpoints of E. They are highly related. Indeed, there are only twelve non zero independent coefficients $\alpha, \beta, \gamma, \delta, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, \Delta_{1b}, \Delta_{1c}, \Delta_{2a}, \Delta_{2c}$. We will now generate new linear equations by applying successively the derivatives $\frac{\partial}{\partial a}$, $\frac{\partial}{\partial b}$ and $\mathcal{B}_{-1} = \frac{\partial}{\partial c}$ to this system. Consider the following system of 37 equations

$$
\begin{aligned}
&(287.1), (287.2), (287.3), (287.4), (287.5), (287.6), (287.6)_a, (287.6)_b, (287.6)_c,\\
&(287.1)_a, (287.1)_b, (287.1)_c, (287.2)_a, (287.2)_b, (287.2)_c,\\
&(287.3)_a, (287.3)_b, (287.3)_c, (287.4)_a, (287.4)_b, (287.4)_c,\\
&(287.5)_a, (287.5)_b, (287.5)_c, (287.6)_{aa}, (287.6)_{ab}, (287.6)_{ac}, (287.6)_{bb}, (287.6)_{bc}, (287.6)_{cc},\\
&(287.1)_{bb} - (287.2)_{aa} - (287.5)_{ab},\\
&(287.1)_{bb} + (287.1)_{cc} - (287.3)_{ac},\\
&(287.2)_{aa} + (287.2)_{cc} - (287.4)_{bc},\\
&(287.5)_{aa} + (287.5)_{bb} + (287.5)_{cc} + (287.2)_{ab} - (287.1)_{ab} + (287.4)_{ac} - (287.3)_{bc},\\
&(287.6)_{abc} - \frac{\delta}{2} \times (287.1)_{bb} - \frac{\gamma}{2} \times (287.1)_{bc} + \frac{\hat{\delta}}{2} \times (287.2)_{aa} + \frac{\hat{\gamma}}{2} \times (287.2)_{ac},\\
&(287.6)_{aac} - \frac{\delta}{2} \times (287.1)_{ab} - \frac{\gamma}{2} \times (287.1)_{ac} + \frac{\hat{\delta}}{2} \times ((287.1)_{ab} - (287.5)_{aa})\\
&\qquad\qquad + \frac{\hat{\gamma}}{2} \times ((287.1)_{bc} - (287.5)_{ac}),\\
&(287.6)_{bbc} - \frac{\delta}{2} \times ((287.5)_{bb} + (287.2)_{ab}) - \frac{\gamma}{2} \times ((287.5)_{bc} + (287.2)_{ac}) + \frac{\hat{\delta}}{2} \times (287.2)_{ab}\\
&\qquad\qquad + \frac{\hat{\gamma}}{2} \times (287.2)_{bc}.
\end{aligned}
$$

These are linear equations, the variables being all the first, second and third order derivatives in a, b, c of X and Y, except X_{aaa} and Y_{bbb}. Consequently, there are 36 variables. Constructing the vector $\vec{x} := (1, X_a, X_b, X_c, Y_a, Y_b, Y_c, \dots)^T \in \mathbb{C}^{37}$ (the first component being one, followed by the 36 variables), this system of linear equations can be written

$$\left[a_{ij}(f)\right]_{1\le i,j\le 37}.\vec{x} = 0,$$

where a_{ij} are differential operators of order less or equal then 6. But then we have necessarily that

$$\det \left[a_{ij}(f)\right]_{1\le i,j\le 37} = 0.$$

This is a PDE for $f = \log P_{2,2}(E; a, b) + \log \tau^{\mathbb{R}}_{m_1,m_2;n_1,n_2}(0; 0; a, b)$. Thus using the structure of the 6 equations in Theorem 6.10, we have obtained a much better result than the one in Theorem 6.9. Indeed, performing in detail the general method described in the proof of Theorem 6.9, one obtains in the case when $p = q = 2$ a PDE given by a determinant of a 107×107 matrix that is equal to zero. Thus in any particular case, one can do much better than the general case in Theorem 6.9.

Appendix A

The Virasoro algebra and the oscillator representation

We briefly discuss the Heisenberg and the Virasoro algebras, and also their oscillator representation. Our discussion is based on [**48**].

1. The Heisenberg algebra

The Heisenberg algebra is the complex Lie algebra $\mathcal{A}$, with basis $\{\hbar, a_j | j \in \mathbb{Z}\}$, and defining commutation relations

$$[\hbar, a_j] = 0, \qquad [a_j, a_k] = j\delta_{j,-k}\hbar.$$

The central elements of $\mathcal{A}$ are $\hbar$ and a_0. This algebra admits a representation in the Fock space $\mathcal{B}$ of formal power series in infinitely many variables $t_1, t_2, \ldots$, the so-called oscillator representation of the Heisenberg algebra. This representation of $\mathcal{A}$ in $\mathcal{B}$ is defined in the following way

$$a_j = \frac{\partial}{\partial t_j}, \quad a_{-j} = j\hbar t_j, \quad a_0 = \mu 1, \quad \hbar = \hbar 1,$$

where $j > 0$, and $\mu, \hbar \in \mathbb{C}$. If $\hbar \neq 0$, then this representation of $\mathcal{A}$ in $\mathcal{B}$ is irreducible.

2. The Virasoro algebra

Let us consider the Lie algebra $Vect\, S^1$ of real vector fields on the unit circle S^1. As a manifold, S^1 is diffeomorphic to $\mathbb{R}/2\pi\mathbb{Z}$, and thus, any real vector field on S^1 is of the form

$$f(\theta)\frac{d}{d\theta},$$

where $f(\theta)$ is a smooth real-valued, 2π-periodic function. The Lie bracket of vector fields is

$$[f(\theta)\frac{d}{d\theta}, g(\theta)\frac{d}{d\theta}] = \big(f(\theta)g'(\theta) - f'(\theta)g(\theta)\big)\frac{d}{d\theta}.$$

A basis for $Vect\, S^1$ is given by the vector fields

$$\frac{d}{d\theta}, \quad \cos(n\theta)\frac{d}{d\theta}, \qquad \sin(n\theta)\frac{d}{d\theta}, \quad n = 1, 2, \dots$$

The complexification of $Vect\, S^1$ defines a complex Lie algebra *d*, with basis

$$d_n = i\, e^{in\theta}\frac{d}{d\theta} = -z^{n+1}\frac{d}{dz},$$

where $n \in \mathbb{Z}$, $z = e^{i\theta}$. The commutation relations are

$$[d_m, d_n] = (m-n)d_{m+n},$$

for $m, n \in \mathbb{Z}$.

Let us now consider the central extension $\hat{d}$ of *d* by a one-dimensional center $c\,\mathbb{C}$

$$\hat{d} = d \oplus c\mathbb{C},$$

together with the commutation relations

$$[d_m, d_n] = (m-n)d_{m+n} + a(m,n)c,$$
$$[d_m, c] = 0,$$

where $m, n \in \mathbb{Z}$, and $a(m,n) \in \mathbb{C}$. The anticommutativity of the Lie bracket and the Jacobi identity imposes the function $a(m,n)$ to be of the form

$$a(m,n) = (\alpha m + \beta m^3)\delta_{m,-n},$$

with $\alpha, \beta \in \mathbb{C}$. For $\beta = 0$, the Lie algebra $\hat{d}$ is a direct sum of Lie algebras *d* and $c\mathbb{C}$. A non-trivial central extension of the Lie algebra *d* is given by the Virasoro algebra with central charge *c*, corresponding to the choice $\alpha = -\beta = -1/12$. It is the complex Lie algebra with basis $\{d_m,\ m \in \mathbb{Z};\ c\}$, and the defining commutation relations

$$[d_m, c] = 0,$$
$$[d_m, d_n] = (m-n)d_{m+n} + \delta_{m,-n}\,\frac{m^3 - m}{12}\,c.$$

Every non-trivial central extension of the Lie algebra *d* by a one-dimensional center is isomorphic to the Virasoro algebra.

3. The oscillator representation of the Virasoro algebra

The oscillator representation of the Virasoro algebra is a representation of this algebra in the Fock space $\mathcal{B}$. We introduce the operators

$$A_k^{(n)} = \frac{1}{2} \sum_{j \in \mathbb{Z}} : a_{-j} a_{j+k} :,$$

where $k \in \mathbb{Z}$, a_j as in (76) with $\mu = n$, and where the colons indicate normal ordering, defined by

$$: a_j a_k := \begin{cases} a_j a_k & \text{if } j \leq k, \\ a_k a_j & \text{if } j > k. \end{cases}$$

As shown in [**48**], these operators satisfy the following commutation relations

$$[a_k, A_l^{(n)}] = k a_{k+l}, \tag{288}$$

and

$$[A_k^{(n)}, A_l^{(n)}] = (k-l) A_{k+l}^{(n)} + \delta_{k,-l} \frac{k^3 - k}{12}, \tag{289}$$

for $k, l \in \mathbb{Z}$. As a consequence, we have obtained a representation of the Virasoro algebra with central charge $c = 1$ in the Fock space $\mathcal{B}$.

Appendix B

Saturation of the Virasoro constraints for the CUE

In this appendix, we give the details of the proof of Theorem 1.9. Put for a fixed n

$$f(t,s;\eta,\theta)=\log\tau_n(t,s;\eta,\theta).$$

Remembering the definition of $L_0^{(n)}$ in (55), the Virasoro constraint in (69) for $k=0$, evaluated along the locus $t=s=0$, gives

$$\left.\frac{\partial f}{\partial\theta}\right|_{t=s=0}+\left.\frac{\partial f}{\partial\eta}\right|_{t=s=0}=0.$$

We also have

$$\left.\frac{\partial f}{\partial\theta}\right|_{\substack{t=s=0\\ \eta=-\theta}}-\left.\frac{\partial f}{\partial\eta}\right|_{\substack{t=s=0\\ \eta=-\theta}}=\left.\frac{\mathrm{d}}{\mathrm{d}\theta}f(t,s;-\theta,\theta)\right|_{t=s=0}.$$

Consequently we have

$$\left.\frac{\partial f}{\partial\theta}\right|_{\substack{t=s=0\\ \eta=-\theta}}=\left.\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}f(t,s;-\theta,\theta)\right|_{t=s=0},$$

$$\left.\frac{\partial f}{\partial\eta}\right|_{\substack{t=s=0\\ \eta=-\theta}}=\left.-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}f(t,s;-\theta,\theta)\right|_{t=s=0},$$

and more generally

$$\frac{\partial^{n_1+n_2} f}{\partial \eta^{n_1} \partial \theta^{n_2}}\bigg|_{\substack{t=s=0\\ \eta=-\theta}} = (-1)^{n_1}\Big(\frac{1}{2}\Big)^{n_1+n_2} \frac{d^{n_1+n_2}}{d\theta^{n_1+n_2}} f(t,s;-\theta,\theta)\bigg|_{t=s=0}. \tag{290}$$

We define the function

$$R(\theta) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta} f(t,s;-\theta,\theta)\bigg|_{t=s=0}. \tag{291}$$

Remembering the definition of $L_k^{(n)}$ in (54), the constraints in (69) for $k = 1, 2$, evaluated at $s = (s_1, s_2, s_3, \dots) = (0, 0, 0, \dots)$, can be written

$$\mathcal{B}_1(\eta,\theta) f\Big|_{s=0} = \sum_{j\geq 1} j t_j \frac{\partial f}{\partial t_{j+1}}\bigg|_{s=0} + n\,\frac{\partial f}{\partial t_1}\bigg|_{s=0}, \tag{292}$$

$$\begin{aligned}\mathcal{B}_2(\eta,\theta) f\Big|_{s=0} &= \sum_{j\geq 1} j t_j \frac{\partial f}{\partial t_{j+2}}\bigg|_{s=0} + \frac{\partial^2 f}{\partial t_1^2}\bigg|_{s=0}\\ &\quad + \Big(\frac{\partial f}{\partial t_1}\Big)^2\bigg|_{s=0} + n\,\frac{\partial f}{\partial t_2}\bigg|_{s=0}.\end{aligned} \tag{293}$$

The constraint (292) evaluated along the locus $t = s = 0$ gives

$$\frac{\partial f}{\partial t_1}\bigg|_{t=s=0} = \frac{1}{n}\mathscr{B}_1 f\big|_{t=s=0}, \tag{294}$$

for $n \neq 0$. Next, we call

$$\frac{\partial^n f}{\partial t_{j_1}\partial t_{j_2}\dots\partial t_{j_n}},$$

a t derivative of weighted degree $|j| = j_1 + j_2 + \cdots + j_n$. Then, for $k \geq 1$, we compute the system formed by

$$\begin{cases}\text{all } t\text{-derivatives of weighted degree } k \text{ of (292)},\\ \text{all } t\text{-derivatives of weighted degree } k-1 \text{ of (293)},\end{cases} \tag{295}$$

evaluated at $t = s = 0$. For $k = 1$, (295) reduces to

$$\mathcal{B}_1(\eta,\theta)\Big(\frac{\partial f}{\partial t_1}\Big|_{t=s=0}\Big) = \frac{\partial f}{\partial t_2}\bigg|_{t=s=0} + n\,\frac{\partial^2 f}{\partial t_1^2}\bigg|_{t=s=0},$$

$$\mathcal{B}_2(\eta,\theta) f\Big|_{t=s=0} = \frac{\partial^2 f}{\partial t_1^2}\bigg|_{t=s=0} + n\,\frac{\partial f}{\partial t_2}\bigg|_{t=s=0} + \left(\frac{\partial f}{\partial t_1}\bigg|_{t=s=0}\right)^2.$$

This is, for $n \neq 1$, a rank two linear system for the "unknowns" $\frac{\partial^2 f}{\partial t_1^2}\Big|_{t=s=0}$ and $\frac{\partial f}{\partial t_2}\Big|_{t=s=0}$. After substitution of (294), this system of equations can be solved for $\frac{\partial^2 f}{\partial t_1^2}\Big|_{t=s=0}$ and $\frac{\partial f}{\partial t_2}\Big|_{t=s=0}$ in terms of $\mathcal{B}_2 f$ and $\mathcal{B}_1^2 f$, whenever $n \geq 2$. We obtain

$$\frac{\partial f}{\partial t_2}\bigg|_{t=s=0} = \frac{1}{n(1-n^2)}\Big[\mathcal{B}_1^2 f - n^2\mathcal{B}_2 f + \big(\mathcal{B}_1 f\big)^2\Big]\Big|_{t=s=0}, \tag{296}$$

$$\frac{\partial^2 f}{\partial t_1^2}\bigg|_{t=s=0} = \frac{1}{1-n^2}\Big[-\mathcal{B}_1^2 f + \mathcal{B}_2 f - \frac{1}{n^2}\big(\mathcal{B}_1 f\big)^2\Big]\Big|_{t=s=0}. \tag{297}$$

For $k=2$, (295) reduces to

$$\begin{aligned}
\mathcal{B}_1\frac{\partial^2 f}{\partial t_1^2}\bigg|_{t=s=0} &= 2\frac{\partial^2 f}{\partial t_1 \partial t_2}\bigg|_{t=s=0} + n\,\frac{\partial^3 f}{\partial t_1^3}\bigg|_{t=s=0},\\
\mathcal{B}_1\frac{\partial f}{\partial t_2}\bigg|_{t=s=0} &= 2\frac{\partial f}{\partial t_3}\bigg|_{t=s=0} + n\,\frac{\partial^2 f}{\partial t_1 \partial t_2}\bigg|_{t=s=0},\\
\mathcal{B}_2\frac{\partial f}{\partial t_1}\bigg|_{t=s=0} &= \frac{\partial f}{\partial t_3}\bigg|_{t=s=0} + \frac{\partial^3 f}{\partial t_1^3}\bigg|_{t=s=0} + 2\frac{\partial f}{\partial t_1}\bigg|_{t=s=0}\frac{\partial^2 f}{\partial t_1^2}\bigg|_{t=s=0}\\
&\quad + n\,\frac{\partial^2 f}{\partial t_1 \partial t_2}\bigg|_{t=s=0}.
\end{aligned}$$

This is, for $n \neq 2$, a rank three linear system for the "unknowns"

$$\frac{\partial^2 f}{\partial t_1 \partial t_2}\bigg|_{t=s=0}, \quad \frac{\partial f}{\partial t_3}\bigg|_{t=s=0}, \quad \frac{\partial^3 f}{\partial t_1^3}\bigg|_{t=s=0},$$

and solving it we obtain

$$\begin{aligned}
&\frac{\partial^2 f}{\partial t_1 \partial t_2}\bigg|_{t=s=0}\\
&\qquad = \frac{1}{4-n^2}\bigg[2\mathcal{B}_1\frac{\partial^2 f}{\partial t_1^2} + n\mathcal{B}_1\frac{\partial f}{\partial t_2} - 2n\mathcal{B}_2\frac{\partial f}{\partial t_1} + 4n\frac{\partial f}{\partial t_1}\frac{\partial^2 f}{\partial t_1^2}\bigg]\bigg|_{t=s=0},
\end{aligned} \tag{298}$$

$$\begin{aligned}
&\frac{\partial^3 f}{\partial t_1^3}\bigg|_{t=s=0}\\
&\qquad = \frac{1}{4-n^2}\bigg[-n\mathcal{B}_1\frac{\partial^2 f}{\partial t_1^2} - 2\mathcal{B}_1\frac{\partial f}{\partial t_2} + 4\mathcal{B}_2\frac{\partial f}{\partial t_1} - 8\frac{\partial f}{\partial t_1}\frac{\partial^2 f}{\partial t_1^2}\bigg]\bigg|_{t=s=0},
\end{aligned} \tag{299}$$

(300)
$$\left.\frac{\partial f}{\partial t_3}\right|_{t=s=0} = \frac{1}{4-n^2}\left[-n\mathcal{B}_1\frac{\partial^2 f}{\partial t_1^2}+\bigl(2-n^2\bigr)\mathcal{B}_1\frac{\partial f}{\partial t_2}+n^2\mathcal{B}_2\frac{\partial f}{\partial t_1}-2n^2\frac{\partial f}{\partial t_1}\frac{\partial^2 f}{\partial t_1^2}\right]\Bigg|_{t=s=0}.$$

For $k=3$, (295) reduces to

$$\begin{aligned}
\mathcal{B}_1\left.\frac{\partial^3 f}{\partial t_1^3}\right|_{t=s=0} &= 3\left.\frac{\partial^3 f}{\partial t_1^2\partial t_2}\right|_{t=s=0} + n\left.\frac{\partial^4 f}{\partial t_1^4}\right|_{t=s=0},\\
\mathcal{B}_1\left.\frac{\partial^2 f}{\partial t_1\partial t_2}\right|_{t=s=0} &= 2\left.\frac{\partial^2 f}{\partial t_1\partial t_3}\right|_{t=s=0} + \left.\frac{\partial^2 f}{\partial t_2^2}\right|_{t=s=0} + n\left.\frac{\partial^3 f}{\partial t_1^2\partial t_2}\right|_{t=s=0},\\
\mathcal{B}_1\left.\frac{\partial f}{\partial t_3}\right|_{t=s=0} &= 3\left.\frac{\partial f}{\partial t_4}\right|_{t=s=0} + n\left.\frac{\partial^2 f}{\partial t_1\partial t_3}\right|_{t=s=0},\\
\mathcal{B}_2\left.\frac{\partial^2 f}{\partial t_1^2}\right|_{t=s=0} &= 2\left.\frac{\partial^2 f}{\partial t_1\partial t_3}\right|_{t=s=0} + \left.\frac{\partial^4 f}{\partial t_1^4}\right|_{t=s=0} + 2\left.\Bigl(\frac{\partial^2 f}{\partial t_1^2}\Bigr)^2\right|_{t=s=0}\\
&\quad + 2\left.\frac{\partial f}{\partial t_1}\frac{\partial^3 f}{\partial t_1^3}\right|_{t=s=0} + n\left.\frac{\partial^3 f}{\partial t_1^2\partial t_2}\right|_{t=s=0},\\
\mathcal{B}_2\left.\frac{\partial f}{\partial t_2}\right|_{t=s=0} &= 2\left.\frac{\partial f}{\partial t_4}\right|_{t=s=0} + \left.\frac{\partial^3 f}{\partial t_1^2\partial t_2}\right|_{t=s=0} + 2\left.\frac{\partial f}{\partial t_1}\frac{\partial^2 f}{\partial t_1\partial t_2}\right|_{t=s=0}\\
&\quad + n\left.\frac{\partial^2 f}{\partial t_2^2}\right|_{t=s=0}.
\end{aligned}$$

This is, for $n \neq 3$, a rank four linear system for the "unknowns"

(301)
$$\left.\frac{\partial^3 f}{\partial t_1^2\partial t_2}\right|_{t=s=0},\quad \left.\frac{\partial^4 f}{\partial t_1^4}\right|_{t=s=0},\quad \left.\frac{\partial^2 f}{\partial t_1\partial t_3}\right|_{t=s=0},\quad \left.\frac{\partial^2 f}{\partial t_2^2}\right|_{t=s=0},\quad \left.\frac{\partial f}{\partial t_4}\right|_{t=s=0},$$

and solving it we obtain

(302)
$$\left.\frac{\partial^3 f}{\partial t_1^2\partial t_2}\right|_{t=s=0} = \frac{1}{9-n^2}\left[4\mathcal{B}_1\frac{\partial^3 f}{\partial t_1^3}+3n\mathcal{B}_1\frac{\partial^2 f}{\partial t_1\partial t_2}+2\mathcal{B}_1\frac{\partial f}{\partial t_3}-4n\mathcal{B}_2\frac{\partial^2 f}{\partial t_1^2}+8n\Bigl(\frac{\partial^2 f}{\partial t_1^2}\Bigr)^2 + 8n\frac{\partial f}{\partial t_1}\frac{\partial^3 f}{\partial t_1^3}-3\mathcal{B}_2\frac{\partial f}{\partial t_2}+6\frac{\partial f}{\partial t_1}\frac{\partial^2 f}{\partial t_1\partial t_2}\right]\Bigg|_{t=s=0},$$

(303)
$$\frac{\partial^4 f}{\partial t_1^4}\bigg|_{t=s=0} =$$
$$\frac{1}{n(9-n^2)}\bigg[-(3+n^2)\mathcal{B}_1\frac{\partial^3 f}{\partial t_1^3} - 9n\mathcal{B}_1\frac{\partial^2 f}{\partial t_1\partial t_2} - 6\mathcal{B}_1\frac{\partial f}{\partial t_3} + 12n\mathcal{B}_2\frac{\partial^2 f}{\partial t_1^2}$$
$$- 24n\Big(\frac{\partial^2 f}{\partial t_1^2}\Big)^2 - 24n\frac{\partial f}{\partial t_1}\frac{\partial^3 f}{\partial t_1^3} + 9\mathcal{B}_2\frac{\partial f}{\partial t_2} - 18\frac{\partial f}{\partial t_1}\frac{\partial^2 f}{\partial t_1\partial t_2}\bigg]\bigg|_{t=s=0},$$

(304)
$$\frac{\partial^2 f}{\partial t_1\partial t_3}\bigg|_{t=s=0} =$$
$$\frac{1}{2n(9-n^2)}\bigg[3(1-n^2)\mathcal{B}_1\frac{\partial^3 f}{\partial t_1^3} + 3n(3-n^2)\mathcal{B}_1\frac{\partial^2 f}{\partial t_1\partial t_2} + 2(3-n^2)\mathcal{B}_1\frac{\partial f}{\partial t_3}$$
$$- 3n(1-n^2)\mathcal{B}_2\frac{\partial^2 f}{\partial t_1^2} + 6n(1-n^2)\Big(\frac{\partial^2 f}{\partial t_1^2}\Big)^2 + 6n(1-n^2)\frac{\partial f}{\partial t_1}\frac{\partial^3 f}{\partial t_1^3}$$
$$- 3(3-n^2)\mathcal{B}_2\frac{\partial f}{\partial t_2} + 6(3-n^2)\frac{\partial f}{\partial t_1}\frac{\partial^2 f}{\partial t_1\partial t_2}\bigg]\bigg|_{t=s=0},$$

(305)
$$\frac{\partial^2 f}{\partial t_2^2}\bigg|_{t=s=0} =$$
$$\frac{1}{n(9-n^2)}\bigg[-(3+n^2)\mathcal{B}_1\frac{\partial^3 f}{\partial t_1^3} - n^3\mathcal{B}_1\frac{\partial^2 f}{\partial t_1\partial t_2} - 6\mathcal{B}_1\frac{\partial f}{\partial t_3} + n(3+n^2)\mathcal{B}_2\frac{\partial^2 f}{\partial t_1^2}$$
$$-2n(3+n^2)\Big(\frac{\partial^2 f}{\partial t_1^2}\Big)^2 - 2n(3+n^2)\frac{\partial f}{\partial t_1}\frac{\partial^3 f}{\partial t_1^3} + 9\mathcal{B}_2\frac{\partial f}{\partial t_2} - 18\frac{\partial f}{\partial t_1}\frac{\partial^2 f}{\partial t_1\partial t_2}\bigg]\bigg|_{t=s=0},$$

(306)
$$\frac{\partial f}{\partial t_4}\bigg|_{t=s=0} =$$
$$\frac{1}{2(9-n^2)}\bigg[-(1-n^2)\mathcal{B}_1\frac{\partial^3 f}{\partial t_1^3} - n(3-n^2)\mathcal{B}_1\frac{\partial^2 f}{\partial t_1\partial t_2} + 4\mathcal{B}_1\frac{\partial f}{\partial t_3}$$
$$+ n(1-n^2)\mathcal{B}_2\frac{\partial^2 f}{\partial t_1^2} - 2n(1-n^2)\Big(\frac{\partial^2 f}{\partial t_1^2}\Big)^2 - 2n(1-n^2)\frac{\partial f}{\partial t_1}\frac{\partial^3 f}{\partial t_1^3}$$
$$+ (3-n^2)\mathcal{B}_2\frac{\partial f}{\partial t_2} - 2(3-n^2)\frac{\partial f}{\partial t_1}\frac{\partial^2 f}{\partial t_1\partial t_2}\bigg]\bigg|_{t=s=0}.$$

Suppose now $n \geq 4$. Substituting (303), (304) and (305) into the KP equation (41), which contains t-derivatives of f of weighted degree less or equal to 4, we obtain

$$0 = \mathcal{B}_1\left[2\frac{\partial^3 f}{\partial t_1^3} + 3n\frac{\partial^2 f}{\partial t_1 \partial t_2} + 4\frac{\partial f}{\partial t_3}\right] + \mathcal{B}_2\left[-3n\frac{\partial^2 f}{\partial t_1^2} - 6\frac{\partial f}{\partial t_2}\right] + \frac{\partial f}{\partial t_1}\left[6n\frac{\partial^3 f}{\partial t_1^3} + 12\frac{\partial^2 f}{\partial t_1 \partial t_2}\right]\Bigg|_{t=s=0}.$$

After substitution of (298), (299) and (300) into this equation we obtain

$$0 = \mathcal{B}_1\left[\mathcal{B}_1\frac{\partial f}{\partial t_2} + 2\mathcal{B}_2\frac{\partial f}{\partial t_1} - 4\frac{\partial f}{\partial t_1}\frac{\partial^2 f}{\partial t_1^2}\right] + \mathcal{B}_2\left[-3n\frac{\partial^2 f}{\partial t_1^2} - 6\frac{\partial f}{\partial t_2}\right] + 6\frac{\partial f}{\partial t_1}\mathcal{B}_1\frac{\partial^2 f}{\partial t_1^2}\Bigg|_{t=s=0},$$

which, after substitution of (294), (296) and (297), multiplying by $n(1-n^2)$ and using the commutation relation $[\mathcal{B}_k, \mathcal{B}_l] = (l-k)\mathcal{B}_{k+l}$ in the second equality, gives

$$\begin{aligned}
0 &= \mathcal{B}_1^2\Big[\mathcal{B}_1^2 f - n^2\mathcal{B}_2 f + (\mathcal{B}_1 f)^2\Big] + 2(1-n^2)\mathcal{B}_1\mathcal{B}_2\mathcal{B}_1 f \\
&\quad - 4(\mathcal{B}_1^2 f)\Big[\mathcal{B}_2 f - \mathcal{B}_1^2 f - \frac{1}{n^2}(\mathcal{B}_1 f)^2\Big] \\
&\quad + 3\mathcal{B}_2\Big[(n^2-2)\mathcal{B}_1^2 f + n^2\mathcal{B}_2 f - (\mathcal{B}_1 f)^2\Big] \\
&\quad + 2(\mathcal{B}_1 f)\mathcal{B}_1\Big[\mathcal{B}_2 f - \mathcal{B}_1^2 f - \frac{1}{n^2}(\mathcal{B}_1 f)^2\Big]\Big|_{t=s=0} \\
&= \mathcal{B}_1^4 f - 4\mathcal{B}_2\mathcal{B}_1^2 f + 2(1-2n^2)\mathcal{B}_3\mathcal{B}_1 f - 2n^2\mathcal{B}_4 f + 6\big(\mathcal{B}_1^2 f\big)^2 \\
&\qquad - 4\big(\mathcal{B}_1^2 f\big)\big(\mathcal{B}_2 f\big) + 3n^2\mathcal{B}_2^2 f - 4\big(\mathcal{B}_1 f\big)\big(\mathcal{B}_2\mathcal{B}_1 f\big) + 2\big(\mathcal{B}_1 f\big)\big(\mathcal{B}_3 f\big).
\end{aligned}$$

Evaluating this equation on the locus $\eta = -\theta$ and using relation (290), we obtain

$$\begin{aligned}
0 =& 2\sin^2\theta\Big[4R(\theta)^2 - 2\big(n^2 + (1-n^2)\cos 2\theta\big)R'(\theta) + 8\sin 2\theta R(\theta)R'(\theta) \\
&- 2\sin 2\theta R''(\theta) + \sin^2\theta\big(12R'(\theta)^2 - R'''(\theta)\big)\Big],
\end{aligned}$$

or equivalently

$$\begin{aligned}
0 =& 4R(\theta)^2 - 2\big(n^2 + (1-n^2)\cos 2\theta\big)R'(\theta) + 8\sin 2\theta R(\theta)R'(\theta) \\
&- 2\sin 2\theta R''(\theta) + \sin^2\theta\big(12R'(\theta)^2 - R'''(\theta)\big).
\end{aligned}$$

Appendix C

Bi-orhogonal polynomials : Proof of Theorem 2.16

Working out the recurrence relations (103) and (104), we find

$$p_0^{(1)}(z) = 1, \qquad p_{n+1}^{(1)}(z) = zp_n^{(1)}(z) + \sum_{k=0}^{n} \frac{h_n}{h_k} x_{n+1} y_k p_k^{(1)}(z), \quad n \geq 0$$

and

$$p_0^{(2)}(z) = 1, \qquad p_{n+1}^{(2)}(z) = zp_n^{(2)}(z) + \sum_{k=0}^{n} \frac{h_n}{h_k} x_k y_{n+1} p_k^{(2)}(z), \quad n \geq 0.$$

By virtue of theorem 2.11, there exist a unique quasi-definite bi-moment functional $\mathcal{L}$ for which the two sequences of polynomials $\{p_n^{(1)}(z)\}_{n\geq 0}$ and $\{p_n^{(2)}(z)\}_{n\geq 0}$ satisfy the bi-orthogonality conditions

$$\mathcal{L}[p_n^{(1)}(z), p_m^{(2)}(z)] = h_n \delta_{n,m}.$$

It remains to show that the bi-moment matrix is a Toeplitz matrix, i.e. $\mu_{i,j} = \mu_{k,l}$ whenever $i - j = k - l$.

We first show that

$$\mathcal{L}[zp_m^{(1)}(z), zp_n^{(2)}(z)] = \mathcal{L}[p_m^{(1)}(z), p_n^{(2)}(z)]. \tag{307}$$

On the one hand, using the recurrence relations we have

$$\begin{aligned}
h_{n+1}\delta_{m,n} &= \mathcal{L}[p^{(1)}_{m+1}(z), p^{(2)}_{n+1}(z)] \\
&= \mathcal{L}[zp^{(1)}_m(z) + \sum_{k=0}^{m} \frac{h_m}{h_k} x_{m+1} y_k p^{(1)}_k(z), p^{(2)}_{n+1}(z)] \\
&= \mathcal{L}[zp^{(1)}_m(z), p^{(2)}_{n+1}(z)] + \sum_{k=0}^{m} \frac{h_m}{h_k} x_{m+1} y_k \mathcal{L}[p^{(1)}_k(z), p^{(2)}_{n+1}(z)] \\
&= \mathcal{L}[zp^{(1)}_m(z), p^{(2)}_{n+1}(z)] + \sum_{k=0}^{m} h_m x_{m+1} y_k \delta_{k,n+1} \\
&= \mathcal{L}[zp^{(1)}_m(z), zp^{(2)}_n(z) + \sum_{j=0}^{n} \frac{h_n}{h_j} x_j y_{n+1} p^{(2)}_j(z)] \\
&\quad + \sum_{k=0}^{m} h_m x_{m+1} y_k \delta_{k,n+1} \\
&= \mathcal{L}[zp^{(1)}_m(z), zp^{(2)}_n(z)] + \sum_{j=0}^{n} \frac{h_n}{h_j} x_j y_{n+1} \mathcal{L}[zp^{(1)}_m(z), p^{(2)}_j(z)] \\
&\quad + \sum_{k=0}^{m} h_m x_{m+1} y_k \delta_{k,n+1}.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
\mathcal{L}[zp^{(1)}_m(z), zp^{(2)}_n(z)] &= h_{n+1}\delta_{m,n} \\
&\quad - \sum_{j=0}^{n} \frac{h_n}{h_j} x_j y_{n+1} \mathcal{L}[zp^{(1)}_m(z), p^{(2)}_j(z)] - \sum_{k=0}^{m} h_m x_{m+1} y_k \delta_{k,n+1}.
\end{aligned} \tag{308}$$

On the other hand, we have

$$\begin{aligned}
h_j\delta_{m+1,j} &= \mathcal{L}[p^{(1)}_{m+1}(z), p^{(2)}_j(z)] \\
&= \mathcal{L}[zp^{(1)}_m(z) + \sum_{k=0}^{m} \frac{h_m}{h_k} x_{m+1} y_k p^{(1)}_k(z), p^{(2)}_j(z)] \\
&= \mathcal{L}[zp^{(1)}_m(z), p^{(2)}_j(z)] + \sum_{k=0}^{m} h_m x_{m+1} y_j \delta_{k,j},
\end{aligned}$$

and thus

$$\mathcal{L}[zp^{(1)}_m(z), p^{(2)}_j(z)] = h_j\delta_{m+1,j} - \sum_{k=0}^{m} h_m x_{m+1} y_j \delta_{k,j}. \tag{309}$$

Combining equations (308) and (309) we obtain

$$
\begin{aligned}
&\mathcal{L}[zp_m^{(1)}(z), zp_n^{(2)}(z)] \\
&= h_{n+1}\delta_{m,n} + \sum_{j=0}^{n} \frac{h_n}{h_j} x_j y_{n+1} \sum_{k=0}^{m} h_m x_{m+1} y_j \delta_{k,j} \\
&\quad - \sum_{j=0}^{n} h_n x_j y_{n+1} \delta_{m+1,j} - \sum_{k=0}^{m} h_m x_{m+1} y_k \delta_{k,n+1} \\
&= h_{n+1}\delta_{m,n} + h_n h_m x_{m+1} y_{n+1} \sum_{j=0}^{n} \sum_{k=0}^{m} \frac{1}{h_j} x_j y_j \delta_{k,j} \\
&\quad - \sum_{j=0}^{n} h_n x_j y_{n+1} \delta_{m+1,j} - \sum_{k=0}^{m} h_m x_{m+1} y_k \delta_{k,n+1} \\
&= h_{n+1}\delta_{m,n} + h_n h_m x_{m+1} y_{n+1} \sum_{j=1}^{n} \sum_{k=1}^{m} \frac{1}{h_j} x_j y_j \delta_{k,j} \\
&\quad + \frac{h_n h_m}{h_0} x_{m+1} y_{n+1} - \sum_{j=0}^{n} h_n x_j y_{n+1} \delta_{m+1,j} - \sum_{k=0}^{m} h_m x_{m+1} y_k \delta_{k,n+1} \\
&= h_{n+1}\delta_{m,n} + h_n h_m x_{m+1} y_{n+1} \sum_{j=1}^{n} \sum_{k=1}^{m} \frac{1}{h_j} \Big(1 - \frac{h_j}{h_{j-1}}\Big) \delta_{k,j} \\
&\quad + \frac{h_n h_m}{h_0} x_{m+1} y_{n+1} - \sum_{j=0}^{n} h_n x_j y_{n+1} \delta_{m+1,j} - \sum_{k=0}^{m} h_m x_{m+1} y_k \delta_{k,n+1} \\
&= h_{n+1}\delta_{m,n} + h_n h_m x_{m+1} y_{n+1} \sum_{j=1}^{n} \sum_{k=1}^{m} \Big(\frac{1}{h_j} - \frac{1}{h_{j-1}}\Big) \delta_{k,j} \\
&\quad + \frac{h_n h_m}{h_0} x_{m+1} y_{n+1} - \sum_{j=0}^{n} h_n x_j y_{n+1} \delta_{m+1,j} - \sum_{k=0}^{m} h_m x_{m+1} y_k \delta_{k,n+1}.
\end{aligned}
$$

(a) We first consider the case $m > n$. Then we have

$$
\begin{aligned}
&\mathcal{L}[xp_m^{(1)}(x), yp_n^{(2)}(y)] \\
&= h_n h_m x_{m+1} y_{n+1} \sum_{j=1}^{n} \Big(\frac{1}{h_j} - \frac{1}{h_{j-1}}\Big) + \frac{h_n h_m}{h_0} x_{m+1} y_{n+1} \\
&\quad - h_m x_{m+1} y_{n+1} \\
&= h_n h_m x_{m+1} y_{n+1} \Big(\frac{1}{h_n} - \frac{1}{h_0}\Big) + \frac{h_n h_m}{h_0} x_{m+1} y_{n+1} - h_m x_{m+1} y_{n+1} \\
&= 0,
\end{aligned}
$$

and as $\mathcal{L}[p_m^{(1)}(x), p_n^{(2)}(y)] = 0$ for $m > n$, we get

$$\mathcal{L}[xp_m^{(1)}(x), yp_n^{(2)}(y)] = \mathcal{L}[p_m^{(1)}(x), p_n^{(2)}(y)].$$

(b) We now consider the case $m < n$. We have

$$\begin{aligned}
&\mathcal{L}[xp_m^{(1)}(x), yp_n^{(2)}(y)] \\
&\quad = h_n h_m x_{m+1} y_{n+1} \sum_{k=1}^{m} \Big(\frac{1}{h_k} - \frac{1}{h_{k-1}} \Big) + \frac{h_n h_m}{h_0} x_{m+1} y_{n+1} \\
&\qquad - h_n x_{m+1} y_{n+1} \\
&\quad = h_n h_m x_{m+1} y_{n+1} \Big(\frac{1}{h_m} - \frac{1}{h_0} \Big) + \frac{h_n h_m}{h_0} x_{m+1} y_{n+1} - h_n x_{m+1} y_{n+1} \\
&\quad = 0.
\end{aligned}$$

As $\mathcal{L}[p_m^{(1)}(x), p_n^{(2)}(y)] = 0$ when $m < n$, we get

$$\mathcal{L}[xp_m^{(1)}(x), yp_n^{(2)}(y)] = \mathcal{L}[p_m^{(1)}(x), p_n^{(2)}(y)].$$

(c) Finally, if $m = n$, then we have

$$\begin{aligned}
&\mathcal{L}[xp_n^{(1)}(x), yp_n^{(2)}(y)] \\
&\quad = h_{n+1} + h_n^2 x_{n+1} y_{n+1} \sum_{j=1}^{n} \Big(\frac{1}{h_j} - \frac{1}{h_{j-1}} \Big) + \frac{h_n^2}{h_0} x_{n+1} y_{n+1} \\
&\quad = h_{n+1} + h_n^2 x_{n+1} y_{n+1} \Big(\frac{1}{h_n} - \frac{1}{h_0} \Big) + \frac{h_n^2}{h_0} x_{n+1} y_{n+1} \\
&\quad = h_{n+1} + h_n^2 x_{n+1} y_{n+1} \frac{1}{h_n} \\
&\quad = h_n \Big(\frac{h_{n+1}}{h_n} + x_{n+1} y_{n+1} \Big) \\
&\quad = h_n.
\end{aligned}$$

We also have

$$\mathcal{L}[p_n^{(1)}(x), p_n^{(2)}(y)] = h_n.$$

Consequently, we have

$$\mathcal{L}[xp_n^{(1)}(x), yp_n^{(2)}(y)] = \mathcal{L}[p_n^{(1)}(x), p_n^{(2)}(y)].$$

This proves (307) for all $m, n \geq 0$.

Consider now the bi-moment $\mu_{m+1,n+1}$. The monomial z^m (resp. z^n) can be written as a linear combination of the polynomials $p_0^{(1)}(z), \dots, p_m^{(1)}(z)$ (resp.

$p_0^{(2)}(z), \ldots, p_n^{(2)}(z))$:

$$z^m = p_m^{(1)}(z) + \sum_{i=0}^{m-1} a_i p_i^{(1)}(z),$$

$$z^n = p_n^{(2)}(z) + \sum_{j=0}^{n-1} b_j p_j^{(2)}(z).$$

We thus have

$$\begin{aligned}
\mu_{m+1,n+1} &= \mathcal{L}[z^{m+1}, z^{n+1}] \\
&= \mathcal{L}\Big[z\Big(p_m^{(1)}(z) + \sum_{i=0}^{m-1} a_i p_i^{(1)}(z)\Big), z\Big(p_n^{(2)}(z) + \sum_{j=0}^{n-1} b_j p_j^{(2)}(z)\Big)\Big] \\
&= \mathcal{L}[zp_m^{(1)}(z), zp_n^{(2)}(z)] + \sum_{i=0}^{m-1} a_i \mathcal{L}[zp_i^{(1)}(z), zp_n^{(2)}(z)] \\
&\quad + \sum_{j=0}^{n-1} b_j \mathcal{L}[zp_m^{(1)}(z), zp_j^{(2)}(z)] + \sum_{\substack{0\le i\le m-1 \\ 0\le j\le n-1}} a_i b_j \mathcal{L}[zp_i^{(1)}(z), zp_j^{(2)}(z)].
\end{aligned}$$

Using (307), we obtain

$$\begin{aligned}
&\mu_{m+1,n+1} \\
&= \mathcal{L}[p_m^{(1)}(z), p_n^{(2)}(z)] + \sum_{i=0}^{m-1} a_i \mathcal{L}[p_i^{(1)}(z), p_n^{(2)}(z)] \\
&\quad + \sum_{j=0}^{n-1} b_j \mathcal{L}[p_m^{(1)}(z), p_j^{(2)}(z)] + \sum_{\substack{0\le i\le m-1 \\ 0\le j\le n-1}} a_i b_j \mathcal{L}[p_i^{(1)}(z), p_j^{(2)}(z)] \\
&= \mathcal{L}\Big[p_m^{(1)}(z) + \sum_{i=0}^{m-1} a_i p_i^{(1)}(z), p_n^{(2)}(z) + \sum_{j=0}^{n-1} b_j p_j^{(2)}(z)\Big] \\
&= \mathcal{L}[z^m, z^n].
\end{aligned}$$

Consequently, we have $\mu_{m+1,n+1} = \mu_{m,n}$, for arbitrary m, n. This implies $\mu_{i,j} = \mu_{k,l}$ whenever $i - j = k - l$, and thus the bi-moment matrix is Toeplitz.

Appendix D

Brownian motions : The integral over the full range

In section 5 of Chapter 6 we have shown that the function $f = \log \tau^{E}_{\vec{m},\vec{n}}(0;a,b) = \log P_{p,q}(E;a,b) + \log \tau^{\mathbb{R}}_{\vec{m},\vec{n}}(0;a,b)$ satisfies a nonlinear PDE, and in section 6 of the same chapter we have described this PDE when $p = q = 2$. To obtain a PDE for $\log P_{p,q}(E;a,b)$ it is necessary to evaluate $\log \tau^{\mathbb{R}}_{\vec{m},\vec{n}}(0;a,b)$. This has been done in the particular case when $p = 1$ (see the appendix in [**8**]), but it seems harder to evaluate this function when $p, q \geq 2$. In this appendix, we conjecture some results about the evaluation of $\log \tau^{\mathbb{R}}_{\vec{m},\vec{n}}(0;a,b)$ when $p = q = 2$.

Consider N non-intersecting Brownian motions $x_1(t), x_2(t), \ldots, x_N(t)$ in $\mathbb{R}$, with m_1 particles leaving from a and m_2 particles leaving from $-a$, and n_1 particles ending in b and n_2 particles ending in $-b$. We suppose $m_1 = n_1$ and $m_2 = n_2$. We denote

$$\begin{aligned}&\mathbb{P}^{+a,-a}_{+b,-b}\big(\text{all } x_i(t) \in E\big)\\ &:= \mathbb{P}\left(\text{all } x_i(t) \in E \left| \begin{array}{l} \big(x_1(0), \ldots, x_N(0)\big) = \big(\underbrace{a, \ldots, a}_{m_1}, \underbrace{-a, \ldots, -a}_{m_2}\big) \\ \big(x_1(1), \ldots, x_N(1)\big) = \big(\underbrace{b, \ldots, b}_{m_1}, \underbrace{-b, \ldots, -b}_{m_2}\big) \end{array} \right.\right),\end{aligned}$$

the probability to find all the particles in a set $E \subset \mathbb{R}$, at a given time $0 < t < 1$. We have

$$\mathbb{P}^{a,-a}_{b,-b}\big(\text{all } x_i(t) \in E\big) = P_{2,2}\big(\tilde{E};\tilde{a},\tilde{b}\big),$$

with the normalized problem defined in (236), and where

$$\tilde{a} := \sqrt{\frac{2(1-t)}{t}}\,a, \qquad \tilde{b} := \sqrt{\frac{2t}{1-t}}\,b, \qquad \tilde{E} := \sqrt{\frac{2}{t(1-t)}}\,E.$$

Notice that $\tilde{a}\tilde{b} = 2ab$. As shown in (240), we have

$$P_{2,2}\big(\tilde{E};\tilde{a},\tilde{b}\big) = \frac{\tau^{E}_{\vec{m},\vec{n}}(0;\tilde{a},\tilde{b})}{\tau^{\mathbb{R}}_{\vec{m},\vec{n}}(0;\tilde{a},\tilde{b})},$$

with $\tau^{E}_{\vec{m},\vec{n}}(0;\tilde{a},\tilde{b})$ defined in (241). We try to evaluate the function

$$\tau^{\mathbb{R}}_{\vec{m},\vec{n}}(0;\tilde{a},\tilde{b}) = \det\begin{pmatrix} \left(\int_{\mathbb{R}} x^{i+j} e^{(\tilde{a}+\tilde{b})x} e^{-\frac{x^2}{2}}\,dx\right)_{\substack{0\le i<m_1\\0\le j<m_1}} & \left(\int_{\mathbb{R}} x^{i+j} e^{(\tilde{a}-\tilde{b})x} e^{-\frac{x^2}{2}}\,dx\right)_{\substack{0\le i<m_1\\0\le j<m_2}} \\ \left(\int_{\mathbb{R}} x^{i+j} e^{(-\tilde{a}+\tilde{b})x} e^{-\frac{x^2}{2}}\,dx\right)_{\substack{0\le i<m_2\\0\le j<m_1}} & \left(\int_{\mathbb{R}} x^{i+j} e^{(-\tilde{a}-\tilde{b})x} e^{-\frac{x^2}{2}}\,dx\right)_{\substack{0\le i<m_2\\0\le j<m_2}} \end{pmatrix}.$$

Define

$$\mu_{i+j}(c) = \int_{\mathbb{R}} x^{i+j} e^{-\frac{x^2}{2}+cx}dx.$$

We have

$$\mu_0(c) = \sqrt{2\pi}e^{\frac{c^2}{2}}, \quad \text{and} \quad \mu_{i+j}(c) = \left(\frac{d}{dc}\right)^{i+j}\mu_0(c).$$

Define also the polynomials (Hermite polynomials up to a slight change of variables)

$$p_j(x) = e^{-\frac{x^2}{2}}\left(\frac{d}{dx}\right)^{j} e^{\frac{x^2}{2}}.$$

We then have

$$\begin{aligned}
&\tau^{\mathbb{R}}_{\vec{m},\vec{n}}(0;\tilde{a},\tilde{b})\\
&= \det\begin{pmatrix} \big(\mu_{i+j}(\tilde{a}+\tilde{b})\big)_{\substack{0\le i\le m_1-1\\0\le j\le m_1-1}} & \big(\mu_{i+j}(\tilde{a}-\tilde{b})\big)_{\substack{0\le i\le m_1-1\\0\le j\le m_2-1}} \\ \big(\mu_{i+j}(-\tilde{a}+\tilde{b})\big)_{\substack{0\le i\le m_2-1\\0\le j\le m_1-1}} & \big(\mu_{i+j}(-\tilde{a}-\tilde{b})\big)_{\substack{0\le i\le m_2-1\\0\le j\le m_2-1}} \end{pmatrix}\\
&= (2\pi)^{\frac{m_1+m_2}{2}} e^{\frac{m_1+m_2}{2}(\tilde{a}+\tilde{b})^2} e^{-4m_1\tilde{a}\tilde{b}}\\
&\quad\times\det\begin{pmatrix} \big(e^{4\tilde{a}\tilde{b}}p_{i+j}(\tilde{a}+\tilde{b})\big)_{\substack{0\le i\le m_1-1\\0\le j\le m_1-1}} & \big(p_{i+j}(\tilde{a}-\tilde{b})\big)_{\substack{0\le i\le m_1-1\\0\le j\le m_2-1}} \\ \big(p_{i+j}(-\tilde{a}+\tilde{b})\big)_{\substack{0\le i\le m_2-1\\0\le j\le m_1-1}} & \big(p_{i+j}(-\tilde{a}-\tilde{b})\big)_{\substack{0\le i\le m_2-1\\0\le j\le m_2-1}} \end{pmatrix}.
\end{aligned}$$

We will use the following lemma.

Lemma D.1. *Consider the block matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with A, D square matrices, and D invertible. Then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det D \times \det\bigl(A - BD^{-1}C\bigr).$$

PROOF. Doing row and column operations, we have

$$\begin{aligned}\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \times \det\begin{pmatrix} A & B \\ D^{-1}C & I \end{pmatrix} \\ &= \det D \times \det\begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{pmatrix} \\ &= \det D \times \det\bigl(A - BD^{-1}C\bigr).\end{aligned}$$

□

Using this lemma, we have

$$\begin{aligned}\tau^{\mathbb{R}}_{\vec{m},\vec{n}}(0;\tilde{a},\tilde{b}) = (2\pi)^{\frac{m_1+m_2}{2}} e^{\frac{m_1+m_2}{2}(\tilde{a}+\tilde{b})^2} e^{-4m_1\tilde{a}\tilde{b}} \det D \\ \times \det\bigl(A - BD^{-1}C\bigr),\end{aligned}$$

where

$$\begin{aligned} A &= \Bigl(e^{4\tilde{a}\tilde{b}} p_{i+j}(\tilde{a}+\tilde{b})\Bigr)_{\substack{0\le i\le m_1-1\\ 0\le j\le m_1-1}}, & B &= \Bigl(p_{i+j}(\tilde{a}-\tilde{b})\Bigr)_{\substack{0\le i\le m_1-1\\ 0\le j\le m_2-1}}, \\ C &= \Bigl(p_{i+j}(-\tilde{a}+\tilde{b})\Bigr)_{\substack{0\le i\le m_2-1\\ 0\le j\le m_1-1}}, & D &= \Bigl(p_{i+j}(-\tilde{a}-\tilde{b})\Bigr)_{\substack{0\le i\le m_2-1\\ 0\le j\le m_2-1}}, \end{aligned} \tag{310}$$

and it is well-known that $\det D = \prod_{i=0}^{m_2-1} i\,!$. Let us note

$$\begin{aligned} X = e^{4\tilde{a}\tilde{b}} - \sum_{j=0}^{m_2-1} \frac{(4\tilde{a}\tilde{b})^j}{j\,!} &= (4\tilde{a}\tilde{b})^{m_2} \frac{1}{2\pi i} \oint_{\Gamma_{0,4\tilde{a}\tilde{b}}} \frac{e^z\,dz}{z^{m_2}(z-4\tilde{a}\tilde{b})} \\ &= (8ab)^{m_2} \frac{1}{2\pi i} \oint_{\Gamma_{0,8ab}} \frac{e^z\,dz}{z^{m_2}(z-8ab)}, \end{aligned} \tag{311}$$

where $\Gamma_{0,4\tilde{a}\tilde{b}}$ denotes a closed contour containing 0 and $4\tilde{a}\tilde{b}$ in the complex plane. Computer observations point out that for A, B, C, D as in (310) we have

$$\det\bigl(A - BD^{-1}C\bigr) = \det\Bigl(p_{i+j}(\tilde{a}+\tilde{b})X + P_{i,j}(\tilde{a},\tilde{b})\Bigr)_{\substack{0\le i\le m_1-1\\ 0\le j\le m_1-1}}, \tag{312}$$

where $P_{i,j}(\tilde{a},\tilde{b})$ is a polynomial in $\tilde{a},\tilde{b}$ such that $P_{i,j}(\tilde{a},\tilde{b}) = P_{j,i}(\tilde{b},\tilde{a})$, $P_{0,0}(\tilde{a},\tilde{b}) = 0$, and the order of $P_{i,j}(\tilde{a},\tilde{b})$ is $2(m_1-1)+i+j$ when $i+j>0$. We will develop (312) in the large m_2-limit.

Let us consider the large m_2-limit, keeping m_1 fixed, see Figure 1. If $m_1 = 0$, for large m_2 the mean density of brownian particles at any time $0 < t < 1$ is supported

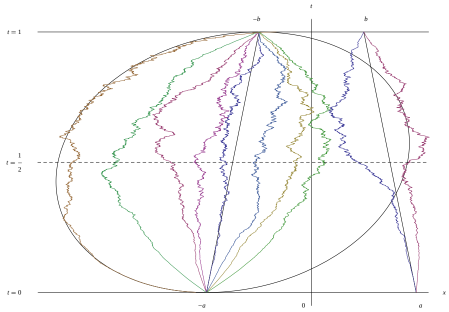

FIGURE 1. Non-intersecting Brownian motions in the large m_2-limit, with m_1 fixed

by an interval with endpoints given by $\pm\sqrt{2m_2 t(1-t)} - a(1-t) - bt$. When m_1 is fixed but not necessarily zero, the non-intersecting nature of the cloud of m_2 particles implies that the largest one will again reach a height of about $\sqrt{\frac{m_2}{2}} - \frac{a+b}{2}$ at $t = \frac{1}{2}$. We will consider the following scaling given in [**4**] for the starting and the target points

$$a = \frac{1}{2}\sqrt{\frac{m_2}{2}}\Big(1 + \frac{A}{m_2^{1/3}}\Big), \quad \text{and} \quad b = \frac{1}{2}\sqrt{\frac{m_2}{2}}\Big(1 - \frac{B}{m_2^{1/3}}\Big). \tag{313}$$

With this scaling, the m_1 wanderers will interact with the bulk of m_2 particles (m_2 very large), upon considering regions close to $x = \sqrt{\frac{m_2}{2}} - \frac{a+b}{2}$ and $t = \frac{1}{2}$, namely at space-time positions (x, t) which scale like

$$t = \frac{1}{2} + \frac{1}{2}\frac{\tau}{m_2^{1/3}},$$
$$x = \frac{1}{2}\sqrt{\frac{m_2}{2}} + \frac{\xi - \tau^2}{2\sqrt{2}m_2^{1/6}} + \frac{1}{4\sqrt{2}}m_2^{1/6}(B - A) + \frac{1}{4\sqrt{2}}\frac{(A+B)\tau}{m_2^{1/6}}.$$

We suppose $A < B$. Under this scaling, the quantity

$$8ab = m_2\Big(1 + \frac{A - B}{m_2^{1/3}} - \frac{AB}{m_2^{2/3}}\Big),$$

is strictly less than m_2, for m_2 large enough. Consequently, by Cauchy's theorem, the contour $\Gamma_{0,8ab}$ in (311) can be taken to be a circle centered at the origin and of radius m_2. We will follow [**4**] to obtain an assymptotic expansion for X. Making the change of variable $z = um_2$ in the integral defining X we have

$$X = (8ab)^{m_2}\frac{m_2^{-m_2}}{2\pi i}\oint_{|u|=1}\frac{e^{m_2 F(u)}}{u - 1 - (A-B)m_2^{-1/3} + ABm_2^{-2/3}}du, \tag{314}$$

where

$$F(u) := u - \ln u = 1 + \frac{1}{2}(u-1)^2 + O\big((u-1)^3\big),$$

with

$$\Re\big(F(u)\big) = \Re(u) - \ln(|u|). \tag{315}$$

The stationary points of the function $F(u)$ are solution of the equation $F'(u) = 0$, and thus there is one stationary point at $u = 1$. We can deform the path $|u| = 1$ into $\gamma_\delta = \{1+iy| -\delta \leq y \leq \delta\}$ plus a circle segment γ' centered at the origin and joining the extremities of γ_δ. It follows from (315) that γ_δ is tangent to the steep descent path through $u = 1$. We choose $\delta = m_2^{-2/5}$. Then the contribution to the integral coming from γ_δ is given by

$$\int_{\gamma_\delta} \frac{e^{m_2 F(u)}}{u - 1 - (A-B)m_2^{-1/3} + ABm_2^{-2/3}} du = \frac{-m_2^{1/3} e^{m_2}}{(A-B)} \int_{1-im_2^{-2/5}}^{1+im_2^{-2/5}} e^{\frac{m_2}{2}(u-1)^2} du \big(1 + \mathcal{O}(m_2^{-1/5})\big).$$

Making the change of variable $\omega = -i\sqrt{\frac{m_2}{2}}(u-1)$, we obtain

$$\int_{\gamma_\delta} \frac{e^{m_2 F(u)}}{u - 1 - (A-B)m_2^{-1/3} + ABm_2^{-2/3}} du = \frac{-i\sqrt{2}m_2^{-1/6} e^{m_2}}{(A-B)} \int_{-\frac{1}{\sqrt{2}}m_2^{1/10}}^{\frac{1}{\sqrt{2}}m_2^{1/10}} e^{-\omega^2} d\omega \big(1 + \mathcal{O}(m_2^{-1/5})\big).$$

As

$$\int_{\frac{1}{\sqrt{2}}m_2^{1/10}}^{+\infty} e^{-\omega^2} d\omega = o\big(m_2^{-1/5}\big),$$

we have

$$\int_{\gamma_\delta} \frac{e^{m_2 F(u)}}{u - 1 - (A-B)m_2^{-1/3} + ABm_2^{-2/3}} du = \frac{-i\sqrt{2\pi}m_2^{-1/6} e^{m_2}}{(A-B)} \big(1 + \mathcal{O}(m_2^{-1/5})\big).$$

Let us now evaluate the contribution to the integral coming from γ'. Along γ', we have $u = \sqrt{1+\delta^2}e^{i\theta}$ with $\cos\theta \leq \frac{1}{\sqrt{1+\delta^2}}$ and $\delta = m_2^{-2/5}$, and thus

$$\begin{aligned} \left|e^{m_2 F(u)}\right| &= e^{m_2 \Re(F(u))} = e^{m_2\left(\sqrt{1+\delta^2}\cos\theta - \frac{1}{2}\ln(1+\delta^2)\right)} \\ &\leq e^{m_2} e^{-\frac{m_2}{2}\ln(1+\delta^2)}. \end{aligned}$$

It follows that

$$e^{-m_2}\left|e^{m_2 F(u)}\right| = \mathcal{O}\Big(e^{-\frac{1}{2}m_2^{1/5}}\Big).$$

Along γ', we also have

$$\Big|\frac{1}{u-1-(A-B)m_2^{-1/3}+ABm_2^{-2/3}}\Big|$$
$$\leq \frac{1}{\big|\sqrt{1+\delta^2}-1-|A-B|m_2^{-1/3}-|AB|m_2^{-2/3}\big|},$$

and thus

$$\Big|\frac{1}{u-1-(A-B)m_2^{-1/3}+ABm_2^{-2/3}}\Big| \leq \frac{-m_2^{1/3}}{A-B}\big(1+\mathcal{O}(m_2^{-1/3})\big).$$

It follows that the contribution to the integral in (314) from γ' is of order $O(e^{-cm_2^{1/5}})$ smaller then the main contribution coming from γ_δ, for some $0 < c < \frac{1}{2}$. Consequently we have

$$\oint_{\Gamma_{0,4\tilde{a}\tilde{b}}} \frac{e^z\,dz}{z^{m_2}(z-4\tilde{a}\tilde{b})}$$
$$= -\sqrt{2\pi} i m_2^{-1/6}\Big(\frac{e}{m_2}\Big)^{m_2}\frac{1}{(A-B)}\big(1+\mathcal{O}(m_2^{-1/5})\big).$$

It follows that in the large m_2-limit

$$X = \frac{-m_2^{-1/6}}{\sqrt{2\pi}(A-B)}$$
$$\times \exp\Big(m_2+(A-B)m_2^{2/3}-\frac{1}{2}(A^2+B^2)m_2^{1/3}+\frac{1}{3}(A^3-B^3)\Big)$$
$$\times \big(1+\mathcal{O}(m_2^{-1/5})\big).$$

Consequently, in the large m_2-limit, for m_1 fixed, $\tau^{\mathbb{R}}_{\vec{m},\vec{n}}(\tilde{a},\tilde{b})$ is expected to behave like

$$\tau^{\mathbb{R}}_{\vec{m},\vec{n}}(\tilde{a},\tilde{b})$$
$$\approx (2\pi)^{\frac{m_1+m_2}{2}} e^{\frac{m_1+m_2}{2}(\tilde{a}+\tilde{b})^2} e^{-4m_2\tilde{a}\tilde{b}}\Big(\prod_{i=0}^{m_1-1} i\,!\Big)\Big(\prod_{j=0}^{m_2-1} j\,!\Big)X^{m_2}.$$

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