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A tentative based on smooth fit.

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# Optimal time to invest when the price processes are geometric Brownian motions

A tentative based on smooth fit.

Joachim GAHUNGU [1] and Yves SMEERS[2]

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**Abstract**

This paper considers the problem of the optimal timing of the exchange of the sum of *n* geometric Brownian motions for the sum of *m* others. We propose a closed form determinable stopping time based on the heuristic principle of smooth fit. We cannot prove that this stopping time is optimal. However, we show numerically on examples that it is a potentially useful candidate: letting $S^{\diamond}$ denote the stopping region induced by our stopping time we show that (i) $S^{-} \subset S^{\diamond} \subset S^{+}$ where $S^{-}$ and $S^{+}$ are well-known subset and superset of the optimal stopping region; (ii) stopping at the first entry time of $S^{\diamond}$ offers a better payoff than stopping at the first entry time of $S^{-}$ or $S^{+}$, especially when assets are correlated.

**Keywords**: optimal stopping, geometric Brownian motion, smooth fit.

**JEL Classification**: D81, G11

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# 1 Introduction

This paper considers the problem of determining the optimal stopping time for the exchange of the sum of $m$ geometric Brownian motions for the sum of $n$ others, hereafter referred to as the $(n, m)$ exchange. This problem can model optimal timing of investments when revenues and costs evolve over time as geometric Brownian motions.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\{\mathcal{F}_t \subset \mathcal{F}, t \in [0, \infty[\}$ a family of $\sigma$-algebras increasing in $t$, right-continuous and completed by sets of $\Omega$ having $\mathbb{P}$-measure zero. We note $X_t^x(\omega) : \Re_+ \times \Omega \to \Re_+^{n+m}$ with $n, m \geq 1$ a $(n+m)$-dimensional geometric Brownian motion (GBM) starting at $x \in \Re_+^{n+m}$ at time $t = 0$. In other words, denoting by $B(t, \omega)$ the $(n + m)$-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, one has $X_t^x(\omega) = X^x(t, \omega)$ satisfies the stochastic differential equations (SDEs)

$$X_0 = x; \quad \mathrm{d}X_i(t, \omega) = \mu_i X_i(t)\mathrm{d}t + \sigma_i X_i(t)\mathrm{d}B_i(t, \omega) \tag{1}$$

for $i = 1, \ldots, n + m$ and some vectors $\mu$, $\sigma \in \Re_+^{n+m}$. We note $\rho_{ij}\mathrm{d}t \triangleq \mathbb{E}\left[\mathrm{d}B_i \cdot \mathrm{d}B_j\right] = \mathrm{Cov}\left[\mathrm{d}B_i \cdot \mathrm{d}B_j\right]$.

Define the linear *reward function* $g : \Re^{n+m} \to \Re$ by

$$g(x) \triangleq \sum_{i=1}^{n} x_i - \sum_{j=n+1}^{n+m} x_j = \sum_{i=1}^{n+m} c_i x_i \tag{2}$$

where $c \in \Re^{n+m}$, defined by

$$c_i \triangleq \begin{cases} 1 & \text{for} \quad i = 1, \ldots, n \\ -1 & \text{for} \quad i = n+1, \ldots, n+m \end{cases} \tag{3}$$

indicates whether $X_i$ is an income or a cost.

Let $\mathcal{S}$ denote the set of stopping times, containing $\tau = \infty$ and note $\mathbb{E}^x$ the expectation w.r.t. the probability law $\mathbb{P}^x$ generated by the stochastic process $X^x(t, \omega)$ since its departure from $x$. For a given stopping time $\tau \in \mathcal{S}$ and discount rate $r > 0$, let $J(\tau, x)$ be the *performance* associated to $\tau$:

$$J(\tau, x) \triangleq \mathbb{E}^x\left[e^{-r\tau} g\left(X_\tau\right)\right]. \tag{4}$$

We want to solve the optimal stopping problem

$$\tau^\star(x, \omega) = \arg\sup_{\tau \in \mathcal{S}} J(\tau, x) \tag{5}$$

i.e. we are looking for a random time $\tau^\star(x, \omega)$ that maximizes $J$, for all $x$. The optimal performance $f(x) \triangleq J(\tau^\star(x), x)$ is called the *value function.* The *stopping region* of the problem is the set $S_{n,m} \in \Re_+^{n+m}$ such that

$$x \in S_{n,m} \Leftrightarrow \tau^\star(x, \omega) = 0 \quad \text{a.s.} \quad \mathbb{P}^x.$$

The components of the optimal stopping problem in hand are simple mathematical objects: (a) all assets are geometric Brownian motions and (b) the payoff function is linear. Notwithstanding this simplicity there is no characterization of the optimal stopping rule so far for arbitrary $n$ and $m$. Partial results exist that we summarize below.

**1.** McDonald and Siegel (1986) solve the problem for $n = m = 1$. They determine the optimal investment rule for the exchange of one geometric Brownian motion for another:

$$\begin{aligned}\tau^\star(x,\omega) &= \arg\sup_{\tau\in\mathcal{S}} \mathbb{E}^x\left[e^{-r\tau}\left(X_1(\tau) - X_2(\tau)\right)\right] \\ &= \inf\left\{t : X_1^x(t,\omega) \geq C_{12}X_2^x(t,\omega)\right\}\end{aligned}$$

with

$$C_{12} \triangleq \frac{\lambda_1}{\lambda_1 - 1} \tag{6}$$

where $\lambda_1$ is the positive root of

$$Q_{12}(\lambda) \triangleq \frac{1}{2}\left(\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2\right)\lambda\left(\lambda - 1\right) + \left(\mu_1 - \mu_2\right)\lambda - (r - \mu_2). \tag{7}$$

In other words,

$$S_{1,1} = \left\{x \in \Re_+^2 : x_1 \geq C_{12}x_2\right\}. \tag{8}$$

This first result is fundamental for our class of problems: advanced results on $(n,m)$ exchanges are expressed via this solution of the two dimensional case. Note that optimal exchange of the GBM $X_j$ for the GBM $X_i$ leads to a coefficient $C_{ij}$ in a similar way: $C_{ij} = \lambda_1/(\lambda_1 - 1)$ where $\lambda_1$ is the positive root of a quadratic $Q_{ij}(\lambda)$ defined as in Eq. (7).

**2.** Olsen and Stensland (1992) prove that the value function of the $(n,m)$ exchange is homogeneous of degree one. Importantly, they provide a *sufficient* condition for optimal stopping of $(1,m)$ and $(n,1)$ exchanges. For the $(1,m)$ exchange

$$\tau^\star(x,\omega) = \arg\sup_{\tau\in\mathcal{S}} \mathbb{E}^x\left[e^{-r\tau}\left(X_1(\tau) - X_2(\tau) - \ldots - X_{m+1}(\tau)\right)\right], \tag{9}$$

this condition takes the form

$$S_{1,m} \supseteq S_{1,m}^- \triangleq \left\{x \in \Re_+^{m+1} : x_1 \geq C_{12}x_2 + \ldots + C_{1,m+1}x_{m+1}\right\} \tag{10}$$

with a similar result for the $(n,1)$ exchange. Note that the sufficient condition (10) does not depend on inter-cost correlations ($\rho_{ij}$ for $i,j > 1$). It is therefore intuitive to think that this condition is too strong (not necessary).[1]

**3.** Hu and Øksendal (1998) provide *necessary* conditions for optimal immediate investment in $(1,m)$ and $(n,1)$ exchanges. Applied to the $(1,m)$ exchange (9), they prove that, having chosen arbitrarily a geometric Brownian $X_u(t,\omega)$ s.t. $X_u(0) = 1$, we have

$$S_{1,m} \subseteq S_{1,m}^+(X_u) \triangleq \left\{x \in \Re_+^{m+1} : x_1 \geq C_{1u}\left(\frac{x_2}{C_{2u}} + \ldots + \frac{x_{m+1}}{C_{m+1,u}}\right)\right\}, \tag{11}$$

where the condition depends on the choice of $X_u$ through the coefficients $C_{iu}$, $i = 1,\ldots,m+1$. Note that as $X_u$ is arbitrarily chosen, we actually have an infinite number

[1] The similar sufficient condition

$$\frac{x_1}{C_{1,n+1}} + \ldots + \frac{x_n}{C_{n,n+1}} \geq x_{n+1}$$

holds for the $(n,1)$ exchange. This condition does not depend on inter-prices correlations ($\rho_{ij}$ for $i,j \leq n$).

of necessary conditions. Because the necessary conditions (11) do not depend on inter-cost correlations, it is intuitive to think that they are too weak (not sufficient).

**4.** Nishide and Rogers (2011) *extend* respectively the works of Olsen and Stensland (1992) and Hu and Øksendal (1998) to $(n, m)$ exchanges (5). Moreover they prove that *the known sufficient conditions for optimal stopping are not necessary.* To be more precise, they characterize the above sufficient and necessary conditions for optimal stopping.

**Sufficient condition.** One has

$$S_{n,m} \supset S_{n,m}^{-} \triangleq \operatorname{conv}\left( \bigcup_{\substack{i=1,\ldots,n \\ j=n+1,\ldots,n+m}} A_{ij} \right) \tag{12}$$

where conv(set) denotes the convex hull of a given set and

$$A_{ij} \triangleq \left\{ x \in \Re_{+}^{n+m} : x_i \geq C_{ij} x_j, \quad x_k = 0 \quad \forall k \neq i, j \right\}.$$

Note that, in (12), $S_{n,m}^{-}$ is *strictly* included in $S_{n,m}$.

**Necessary conditions.** One has

$$S_{n,m} \subseteq S_{n,m}^{+}(X_u, X_v) \triangleq \Bigg\{ x \in \Re_{+}^{n+m} : \\ C_{v1} x_1 + \ldots + C_{vn} x_n \geq C_{vu} \left( \frac{x_{n+1}}{C_{n+1,u}} + \ldots + \frac{x_{n+m}}{C_{n+m,u}} \right) \Bigg\} \tag{13}$$

for any geometric Brownian motion $X_u$ and $X_v$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The sufficient condition (12) is not particularly tractable: it requires to compute the convex hull of the union of $n \times m$ subsets of $\Re_{+}^{n+m}$. This task is analytically cumbersome, with no guarantee that the convex hull has an intuitive representation. In practice it is easier to compute (12) numerically.

The necessary condition (13) is tractable, but it depends on the choice of auxiliary processes $X_u$ and $X_v$. We have no indication on how to efficiently choose these two processes.

Again, note that (12) and (13) depend neither on inter-price nor on inter-cost correlations ($\rho_{ij}$ for $i, j \leq n$ and $\rho_{ij}$ for $i, j > n$, respectively) while obviously the optimal investment rule should involve these correlations. This reflects the weakness of the reward decomposition technique[2] that is used to prove these results and indicates that it is unlikely to give the optimal investment rule.

[2]Consider the optimal stopping problem $\sup_{\tau \in \mathcal{S}} \mathbb{E}^x[g(X_\tau)]$. Decompose the reward by e.g. $g(x) = g_1(x) + g_2(x)$. It follows from the inequality

$$\sup_{\tau \in \mathcal{S}} \mathbb{E}^x[g(X_\tau)] \leq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x[g_1(X_\tau)] + \sup_{\tau \in \mathcal{S}} \mathbb{E}^x[g_2(X_\tau)]$$

that if $x$ simultaneously belongs to the stopping region of $\sup_{\tau \in \mathcal{S}} \mathbb{E}^x[g_1(X_\tau)]$ and $\sup_{\tau \in \mathcal{S}} \mathbb{E}^x[g_2(X_\tau)]$, then $x$ belongs to the stopping region of $\sup_{\tau \in \mathcal{S}} \mathbb{E}^x[g(X_\tau)]$. Thus we can determine sufficient conditions for optimal stopping by decomposing intelligently the reward function (we refer to this method as the reward decomposition technique; see Olsen and Stensland, 1992). What is not obvious is that it allows also to obtain necessary conditions for optimal stopping (see Hu and Øksendal, 1998).

Unlike the results (10), (11), (12) and (13), the expression (8) provided by McDonald and Siegel (1986) relies on the (heuristic) *principle of smooth fit*[3] which states that the value function is $\mathcal{C}^1$ on its entire domain (but fails to be $\mathcal{C}^2$ at the boundary of the continuation region). Two minimal conditions[4] for this principle to hold in great generality appear to be the regularity of the stochastic process[5] and the differentiabibility of the reward function; these conditions hold respectively for (1) and (5). These conditions are however not sufficient: there exist optimal stopping problems with regular stochastic process and differentiable reward function where the smooth fit principle fails to hold (see for instance Peskir, 2007). Brekke and Øksendal (1991) provide (in a wider context) a theorem that allows us to verify if a solution obtained via a smooth fit assumption is indeed correct. This result was applied by Hu and Øksendal (1998) to give the first rigorous proof of McDonald and Siegel (1986) result twelve years after its publication.

It is the aim of this paper to approach the $(n, m)$ exchange heuristically using the principle of smooth fit, just as McDonald and Siegel (1986) and many other authors did or still do.[6] However, we shall see that solving a free boundary problem using the smooth fit principle already leads to difficulties, implying that we cannot even reach Brekke and Øksendal's (1991) verification step.

These difficulties and intuition suggest considering an alternative approach based on a parametrization of the problem by $n + m - 1$ variables: we solve the free boundary problem (using the smooth fit principle) as if we knew with certainty the value of $(n + m - 1)$ components of the multidimensional stochastic process $X$ at the optimal exchange time. This approach were adopted in independent works by Adkins and Paxson (2006) and Gahungu and Smeers (2007,2009). Adkins and Paxson (2006) develop a two uncertainties real options model on optimal renovation. This model can be cast into a $(1, 2)$ exchange problem where one of the two costs is deterministic. In Gahungu and Smeers (2009), a general discussion is developed on the dimensionality issue in exchange problems: it starts from a study of the $(1, 2)$ exchange (with three uncertainties) to finally derive an investment rule for general $(n, m)$ exchanges (see e.g. Rohlfs and Madlener (2010) for an application of this result in a three and a four uncertainties power plants investment problem). From this general result, the model of Adkins and Paxson (2006) can be retrieved as a particular case. Note that none of these papers assess optimality of the introduced rule, neither on mathematical nor numerical grounds.[7]

We find in this paper that the investment rule we derived for $(n, m)$ exchanges in Gahungu and Smeers (2009) is *closed form determinable*, and hence considerably easier to use than conditions (12) and (13) in practice. Note that this rule depends on the entire correlation matrix as the optimal stopping rule should do. If its optimality cannot be rigorously proven, it can however be confronted with well-established characterizations of the stopping region such as the sufficient condition for optimal stopping (12) and the necessary conditions for optimal stopping (13). The tractability of our rule makes it

[3]The principle of *smooth fit* is also called *high contact* in stochastic finance or *smooth pasting condition* in real options literature. It seems to appear for the first time in McKean (1965).

[4]It is easy to construct optimal stopping problems on irregular diffusions such that the smooth fit principle fails to hold. However, it is unclear if these two conditions are necessary for the smooth fit principle to hold.

[5]For a diffusion $X : \Omega \times \Re_+ \to \Re$, define $\tau_c \triangleq \inf\{t > 0 : X_t = c\}$ for $c \in \Re$. $X$ is said regular if $\mathbb{P}^x(\tau_c < +\infty) = 1 \quad \forall x, c \in \Re$.

[6]The smooth fit principle is applied widely in continuous-time real options analysis.

[7]Gahungu (2007) uses this parametrization approach to derive sufficient conditions for optimal stopping in $(n, m)$ exchanges of geometric Brownian motions.

easy to implement on various examples provided in the literature. We show for these examples that our rule is stronger than the necessary investment conditions (13) and weaker than the sufficient condition (12); it is also better in terms of performance. *These examples suggest that the provided rule might be optimal.* We leave the problem of providing a rigorous proof of optimality as an open research issue.

Since this paper is based on an unpublished work (Gahungu and Smeers, 2009), we try to make it as self-contained as possible. This leads to the following structure. Section 2 develops our intuition about the shape of the boundary of the stopping region. In Section 3, we introduce the free boundary problem with smooth fit condition associated to our optimal stopping problem. It comes up that the solution of this free boundary problem is not unique, which leads to a difficulty.

The intuition developed in Section 2 and the difficulties encountered in Section 3 suggest an intuition based stopping rule (given by Definition 2) which is henceforth the candidate stopping rule we analyze. This step is completed in Section 4. Section 5 gives this candidate stopping rule in closed form. Section 6 discusses the interpretation to be given to our candidate investment rule. Section 7 reports numerical results on various examples and Section 8 concludes.

## 2 A trigger for the (n,m) exchange problem

Let $\partial S_{n,m}$ be the boundary of the optimal stopping region $S_{n,m}$ of the optimal stopping problem (5). Intuition suggests that *for all* $(x_2, \ldots, x_{n+m}) \in \Re_+^{n+m-1}$, there must exist a critical value $x_1^\star(x_2, \ldots, x_{n+m})$ such that

$$\left(x_1^\star(x_2, \ldots, x_{n+m}), x_2, \ldots, x_{n+m}\right) \in \partial S_{n,m} \tag{14}$$

with

$$x \in S_{n,m} \Leftrightarrow x_1 \geq x_1^\star(x_2, \ldots, x_{n+m}). \tag{15}$$

As an illustration consider the case $n = 1$ so that we are considering payment of several costs to receive a unique asset $X_1$. Eq. (14) suggests to proceed with the exchange for a critical value of $X_1$ which depends on all the costs. Defining the vector

$$x_{-i} \triangleq (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+m}) \in \Re_+^{n+m-1} \tag{16}$$

condition (15) becomes

$$x \in S_{n,m} \Leftrightarrow x_1 \geq x_1^\star(x_{-1}) \tag{17}$$

where $x_{-1} = (x_2, \ldots, x_{n+m})$ is really understood as the *parameter* of the rule.

Note that the optimal investment rule should satisfy the natural consistency condition of being invariant w.r.t. the parametrization i.e.

$$\forall x \in \Re_+^{n+m}, \quad x_1^\star(x_{-1}) = x_1 \quad \Longleftrightarrow \quad x_j^\star(x_{-j}) = x_j \quad \forall j = 2, \ldots, n+m.$$

Moreover, since the value function of the problem is linearly homogeneous (see Olsen and Stensland, 1992), the optimal investment rule should be linearly homogeneous as well i.e.

$$\forall \alpha > 0, \quad x_1^\star(\alpha x_{-1}) = \alpha x_1^\star(x_{-1}).$$

The next section provides a rule of the form (17) i.e. we provide a mapping $x_1^\diamond(\cdot) : \Re_+^{n+m-1} \to \Re_+$ that associates to any $x_{-1} \in \Re_+^{n+m-1}$ a trigger function $x_1^\diamond(x_{-1})$. This function $x_1^\diamond(\cdot)$ satisfy parametrization invariance and linear homogeneity (see Lemma

2 and 3). It is closed form determinable and depends on the entire correlation matrix (see Proposition 3). This rule induces the stopping region

$$S^{\diamond}_{n,m} \triangleq \left\{x \in \Re^{n+m}_{+} : x_1 \geq x^{\diamond}_1(x_{-1})\right\},$$

the stopping time

$$\tau^{\diamond}(x,\omega) \triangleq \inf\left\{t \geq 0 : X_1(t,\omega) \geq x^{\diamond}_1\left(X_{-1}(t,\omega)\right)\right\}$$

and the performance $J(\tau^{\diamond}, x)$. The fundamental question is whether $x^{\diamond}_1(\cdot)$ is or could be the optimal $x^{\star}_1(\cdot)$.

We start with a free boundary formulation (Problem 1) of the exchange problem (5) where we invoke a principle of smooth fit. Our analysis of this free boundary problem leads to two Propositions that show that its formal resolution is impossible but motivate the formulation of a heuristic algorithm (given in Definition 2) that leads to an explicit formula for $x^{\diamond}_1(x_{-1})$ (see Proposition 3).

# 3 A free boundary formulation

Following McDonald and Siegel (1986) (and the subsequent entire stream of real options literature, see Dixit and Pindyck (1994) for a survey), we try to solve (5) by formulating a free boundary problem.

Define the second order elliptic partial differential ($\partial_i$ denotes the partial derivative w.r.t. the variable $x_i$) operator $\mathcal{L}_X : \mathcal{C}^2(\Re^{n+m}) \to \mathcal{C}^2(\Re^{n+m})$ by

$$\mathcal{L}_X \triangleq \sum_{i=1}^{n+m} \mu_i x_i \partial_i + \frac{1}{2}\sum_{i,j=1}^{n+m} \rho_{ij}\sigma_i\sigma_j x_i x_j \partial_i\partial_j.$$

$\mathcal{L}_X$ is the Dynkin[8] operator associated to the $n+m$ dimensional geometric Brownian motion (1). The boundary of a set $S$ is denoted by $\partial S$; the gradient operator in $\Re^{n+m}$ is noted $\nabla$. Following the reasoning of McDonald and Siegel (1986), we formulate the following problem.

**Problem 1** (The free boundary problem)**.** *Determine the stopping region $S_{n,m}$ and the value function $f \in \mathcal{C}^2(\Re^{n+m}_{+} \setminus \partial S_{n,m})$ such that*

$$\mathcal{L}_X f - rf = 0 \qquad x \in \Re^{n+m}_{+} \setminus S_{n,m} \tag{18}$$

$$f = g \qquad x \in S_{n,m} \tag{19}$$

$$\nabla f = \nabla g \quad x \in \partial S_{n,m}. \tag{20}$$

In Problem 1, (18) expresses standard backward induction before exercising the option and (19) is the natural exercise condition.

There is however no clear motivation for (20)—the multidimensional *smooth-fit condition*—which claims that the value function should be once continuously differentiable everywhere, including at the optimal exercise point ($f \in \mathcal{C}^1(\Re^{n+m}_{+})$). If one manages to find a smooth function $f$ and a set $S_{n,m}$ solving (18),(19) and (20), it is still necessary to apply the verification theorem for optimal stopping (see e.g. Øksendal, 2007, Chapter 10, Theorem 10.4.1) on $f$ and $S_{n,m}$. The next two Propositions show that we will never have to go that far because we are actually not even able to solve Problem 1.

In order to see this we first analyze the partial differential equation (18). Proposition 1 gives its set of acceptable solutions.

[8] $\mathcal{L}_X f(x)$ gives $\lim_{t\to 0}\left[\mathbb{E}^x f(X_t) - f(x)\right]/t$.

**Proposition 1.** *The solution of (18) relevant to the optimal stopping problem* (5) *is*

$$f(x) = a \prod_{i=1}^{n+m} x_i^{\lambda_i} \tag{21}$$

*with* $a \in \Re$ *and* $\lambda \in \Re^{n+m}$ *verifies*

$$Q_{n,m}(\lambda) = 0 \tag{22}$$

*where*

$$Q_{n,m}(\lambda) \triangleq \sum_{i=1}^{n+m} \mu_i \lambda_i + \frac{1}{2} \sum_{i,j=1}^{n+m} \rho_{ij} \sigma_i \sigma_j \left[\lambda_i(\lambda_i - 1)\delta_{ij} + \lambda_i \lambda_j (1 - \delta_{ij})\right] - r \tag{23}$$

*with* $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

*Proof.* It is immediate by direct verification that (21) is a solution of the PDE (18). Then, note that in the case $n = m = 1$, (21) is the value function of the two dimensional case derived by McDonald and Siegel (1986). Thus (21) is the right choice among all possible solutions of the PDE (18). □

**Remark 1.** *The quadratic* (23) *depends on all the correlations between the* $n+m$ *assets of the exchange problem.*

It is clear that except in the trivial case $n = m = 1$, (22) has an infinite number of solutions which constitute a $n + m - 1$ surface. One may wonder if the fact that (21) should verify (19) and (20) for a certain subset $S \in \Re_+^{n+m}$ does not imply some additional conditions on $\lambda$ and on the shape of $S$. In fact we have:

**Proposition 2.** *Suppose that (21) holds for some* $a \neq 0$ *and* $\lambda \in \Re^{n+m}$. *If (19) and (20) hold for some set* $S$, *then:*

*a)*

$$\lambda_i \neq 0, \quad \forall i = 1, \ldots, n+m;$$

*b)*

$$\sum_{i=1}^{n+m} \lambda_i = 1;$$

*c)*

$$\partial S = \partial(S_\lambda) \triangleq \left\{ x \in \Re_+^{n+m} : \frac{x_i}{x_1} = c_i \frac{\lambda_i}{\lambda_1}, \quad i = 2, \ldots, n+m \right\}$$

*where* $c \in \Re^{n+m}$ *was defined by* (3)*;*

*d)*

$$a = \frac{1}{\prod_{i=1}^{n+m} (c_i \lambda_i)^{\lambda_i}}.$$

*Proof.* See Appendix A. □

The two following subsections make two important observations regarding Proposition 2.

### 3.1 Degeneracy

Our first observation is as follows. Proposition 2 do not allow some components of points of $\partial S$ to be zero. Indeed, Prop. 2 a) and c) imply that if (18), (19) and (20) hold for a given set $S$, then points that belong to $\partial S$ are such that

$$x_i \neq 0 \quad \forall i = 1, \ldots, n+m. \tag{24}$$

It is clear that (24) holds for most points of the boundary of the true stopping regions $\partial S_{n,m}$ which are of practical interest. However, for some points of $\partial S_{n,m}$, (24) may fail to hold. These points belong to regions of $\Re_+^{n+m}$ where the exchange problem is degenerate i.e. regions where the exchange problem reduce to an exchange of smaller dimension. The $(n,m)$ exchange problem is degenerate in two cases.

1. Since the geometric Brownian motion is absorbed at zero ($x = 0$, $X$ is a GBM $\Rightarrow X_t^x(\omega) = 0 \; \forall t, \omega$), if $x_i = 0$ for some $i = 1, \ldots, n+m$ then the original problem trivially reduces to a lower dimension exchange problem on assets $X_j$ having a strictly positive initial position for which $x_j \neq 0$. We refer to this situation as *trivial degeneracy.* Note that one can consider $x_i \neq 0$ for all $i = 1, \ldots, n+m$ without loss of generality (w.l.o.g.): if $x_i = 0$ for some $i = 1, \ldots, n+m$, one reduces the dimension of the problem and consider a problem with $x_i \neq 0$ for all $i = 1, \ldots, n'$ with $n' < n+m$. In other words, one can always get rid of trivial degeneracy via a reduction of the problem dimension. Thus, in the following, we assume that $x_i \neq 0$ for all $i = 1, \ldots, n+m$ i.e. that the problem is not trivially degenerate.

2. Assume (w.l.o.g.) that the exchange problem is not trivially degenerate (i.e. $x_i \neq 0$ for all $i = 1, \ldots, n+m$). Assume further that $\max(n,m) \geq 2$. Another type of degeneracy is not trivial. It is introduced as follows.

   Let us extend our previous notation $x_{-1}$ by

   $$x_{-1,-2} \triangleq (x_3, \ldots, x_{n+m}). \tag{25}$$

   Consider the $n+m-1$ dimensional optimal stopping problem obtained by removing asset $X_1$ from the original problem i.e.

   $$\tau^\star(x_{-1}, \omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^{x_{-1}} \left[ e^{-r\tau} \left( X_2(\tau) + \ldots - X_{n+m}(\tau) \right) \right]. \tag{26}$$

   We call $x_2^\star(x_{-1,-2})$ the threshold level of asset 2 such that it is optimal to stop for (26). This is a slight abuse of notation since $x_2^\star(x_{-1,-2})$ should not be confounded with $x_2^\star(x_{-2})$ which is the threshold level of asset 2 such that it is optimal to stop for the original optimal stopping problem

   $$\tau^\star(x, \omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^{x} \left[ e^{-r\tau} \left( X_1(\tau) + \ldots - X_{n+m}(\tau) \right) \right] \tag{27}$$

   using parametrization w.r.t. asset $X_2$ in place of parametrization w.r.t. asset $X_1$.

   Because the geometric Brownian motion is absorbed at zero, we guess that if $x_2 \geq x_2^\star(x_{-1,-2})$, then $x_1^\star(x_{-1}) = 0$. In other words, if $x_{-1,-2}$ belongs to the stopping region of the smaller (i.e. degenerate) problem (26), intuition suggests to stop irrespectively of the value of $X_1$. In such case we say that the problem is *non-trivially* degenerate w.r.t. $X_1$. Since trivial degeneracy can always be eliminated, from now on, unless stated otherwise, when evoking degeneracy, we refer to non trivial degeneracy given by the following general definition.

**Definition 1** (Degeneracy). *For $i = 1, \ldots, n$, we say that the optimal stopping problem* (5) *is degenerate at $x$ w.r.t. asset $X_i$ if $x_i^\star(x_{-i}) = 0$.*

More generally, we say that the optimal stopping problem (5) is degenerate at $x$ if it is degenerate w.r.t. some asset(s). To develop our intuition, let us again consider degeneracy w.r.t asset $X_1$. We hinted that because the geometric Brownian motion is absorbed at zero, to be in the stopping region of (26) should imply degeneracy w.r.t. $X_1$ i.e. $x_2 \geq x_2^\star(x_{-1,-2}) \Rightarrow x_1^\star(x_{-1}) = 0$ (note that, by parametrization invariance, the condition $x_2 \geq x_2^\star(x_{-1,-2})$ is equivalent to $x_i \geq x_i^\star(x_{-1,-i})$ for all $i = 3, \ldots, n$). In fact, the converse is also true i.e. $x_1^\star(x_{-1}) = 0 \Rightarrow x_2 \geq x_2^\star(x_{-1,-2})$ so that the optimal stopping rule should satisfy:

**Lemma 1** (Degeneracy with respect to the asset $i$ for $x^\star$). *Let $x_1^\star(x_{-1})$ be the optimal stopping rule. Assume $n \geq 2$. For any $i = 1, 2, \ldots, n$ one has*

$$x_i^\star(x_{-i}) = 0 \Leftrightarrow x_j \geq x_j^\star(x_{-i,-j}) \qquad \forall j = 1, \ldots, n \quad \textit{such that} \quad j \neq i. \tag{28}$$

*Proof.* See Appendix B. □

Recall that points of $\partial S_{n,m}$ considered in most applications satisfy (24) (because since all the assets are strictly positives, the trader will necessarily exercise the perpetual American option before the problem degenerates). Nevertheless degeneracy has to be introduced to treat the exchange problem in great generality. Later in the discussion, when elaborating about our candidate trigger $x_1^\diamond(x_{-1})$, we will refer to points where the problem degenerate w.r.t. $X_1$ as points $x$ such that $x_1^\diamond(x_{-1}) = 0$. We will be able to prove that our candidate trigger satisfies a weak version of Lemma 1.

### 3.2 On the sign of the components of $\lambda$

Our second observation is as follows. If we require from our economic intuition that the function

$$f(x) = a \prod_{i=1}^{n+m} x_i^{\lambda_i}$$

is positive for all $x \in \Re_+^{n+m}$, we need $a \geq 0$. We see from Prop. 2 d) that this is possible for arbitrary $n$ and $m$ if and only if $c_i \lambda_i \geq 0$ for all $i = 1, \ldots, n+m$. Thus one should have, from the economic point of view,

$$\begin{array}{ll} \lambda_i > 0 & i = 1, \ldots, n \\ \lambda_j < 0 & j = n+1, \ldots, n+m. \end{array} \tag{29}$$

Now the reader can remark that Prop. 2 c) is compatible with (29): if Prop. 2 c) holds, then $\text{sign}(\lambda_i) = -\text{sign}(\lambda_j)$ for all $i = 1, \ldots, n$ and all $j = n+1, \ldots, n+m$. In other words, if Prop. 2 c) holds, then all the $\lambda_i$'s, $i = 1, \ldots, n$ have the same sign and all the $\lambda_j' s$ for $j = n+1, \ldots, n+m$ are of opposite sign. Clearly this last condition is necessary to have (29). Finally note that if Prop. 2 c) holds and $\lambda_1 > 0$ then (29) immediately holds.

### 3.3 Proposition 2 alone leads to difficulties

Proposition 2 states that any function $f(x) = a\prod_{i=1}^{n+m} x_i^{\lambda_i}$ satisfying conditions (22), Prop. 2 a) and Prop. 2 b) solves Problem 1 for a set $S_\lambda$ identified by Prop. 2 c). Since the stopping region $S_{n,m}$ is unique, it would be ideal that a unique vector $\lambda$ satisfied (22), Prop. 2 a) and Prop. 2 b). It is easy to see that this is unfortunately not the case in general: Prop. 2 b) combined with (22) implies that the set of acceptable $\lambda$ is a $n+m-2$ surface. Thus, except if $n = m = 1$ (the McDonald and Siegel (1986) case), $\lambda$ is not fully determinable. This is illustrated in the two following examples.

**Example 1** (The (1,1) exchange)**.** *Assume $f(x) = ax_1^{\lambda_1}x_2^{\lambda_2}$; $Q_{1,1}(\lambda_1, \lambda_2) = 0$. By Prop. 2 b), $\lambda_2 = 1 - \lambda_1$ so $\lambda_1$ should be a root of $Q_{1,1}(\lambda_1, 1-\lambda_1)$. Among the two possible roots of $Q_{1,1}(\lambda_1, 1-\lambda_1)$, one should pick one which is positive and greater than 1 in order to satisfy* (29).[9] *Then, using Prop. 2 c) and d), we successively have*

$$\begin{aligned} \partial S_{1,1} &= \partial(S_{\lambda_1,\lambda_2}) \triangleq \left\{ x \in \Re_+^2 : \frac{x_2}{x_1} = -\frac{\lambda_2}{\lambda_1} = \frac{\lambda_1 - 1}{\lambda_1} \right\} \\ a &= \frac{1}{\lambda_1^{\lambda_1}(-\lambda_2)^{\lambda_2}} = \frac{1}{\lambda_1^{\lambda_1}(\lambda_1 - 1)^{1-\lambda_1}}. \end{aligned}$$

We are unable to determine $\lambda$ for $n + m > 2$, as shown in the following example.

**Example 2** (The (1,2) exchange - Part 1)**.** *Assume now that $f(x) = ax_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3}$; $Q_{1,2}(\lambda_1, \lambda_2, \lambda_3) = 0$. By Prop. 2 b) , $\lambda_3 = 1 - \lambda_1 - \lambda_2$ so the couple $(\lambda_1, \lambda_2)$ verifies $Q_{1,2}(\lambda_1, \lambda_2, 1-\lambda_1-\lambda_2) = 0$. There is however an infinite number of such couples. Prop. 2 c) and d) give*

$$\begin{aligned} \partial S_{1,2} &= \partial(S_{\lambda_1,\lambda_2,\lambda_3}) \triangleq \left\{ x \in \Re_+^3 : \frac{x_2}{x_1} = -\frac{\lambda_2}{\lambda_1}, \frac{x_3}{x_1} = -\frac{1-\lambda_1-\lambda_2}{\lambda_1} \right\} \\ a &= \frac{1}{\lambda_1^{\lambda_1}(-\lambda_2)^{\lambda_2}(-1+\lambda_1+\lambda_2)^{(1-\lambda_2-\lambda_3)}} \end{aligned}$$

*which are undetermined.*

# 4 A definition of the candidate trigger

Example 2 shows that, starting from the candidate value function $f(x) = a\prod_{i=1}^{n+m} x_i^{\lambda_i}$, the standard protocol—which asks to *first* determine completely the vector $\lambda$, and *second*, for this given $\lambda$, to use the $n+m-1$ equations given in Prop. 2 c) as investment conditions—works for $n = m = 1$, but not for problems of higher dimensions.

In fact, the intuition given by (14) even suggests to proceed in the opposite order: given some point $x \in \Re_+^{n+m}$, compute *first* the vector $\lambda(x_2, \dots, x_{n+m}) = \lambda(x_{-1})$ using (22), Prop. 2 b) and $n+m-2$ relations of Prop. 2 c); *second* find $x_1^\diamond(x_2, \dots, x_{n+m}) = x_1^\diamond(x_{-1})$ such that the last equation of Prop. 2 c) holds. In other words, one finds the level $x_1^\diamond(x_{-1})$ for which one can construct a vector $\lambda(x_{-1})$ such that we satisfy the set of conditions.

Let us implement this. Note that Prop. 2 c) can equivalently be stated as

$$\partial S_{n,m} = \left\{ x \in \Re_+^{n+m} : \quad \frac{x_1}{x_{n+m}} = -\frac{\lambda_1}{\lambda_{n+m}}, \right. \tag{30}$$

$$\left. \frac{x_i}{x_{n+m}} = -\frac{c_i\lambda_i}{\lambda_{n+m}}, \quad i = 2, \dots, n+m-1 \right\} \tag{31}$$

[9] Such a root always exists and is unique under the condition $\mu_1 < r$ (recall that $r > 0$). See Appendix E.1.

where we have used the fact that $c_{n+1} = -1$. This formulation of Prop. 2 c) is useful in practice as (31) only depends on the coordinates $x_2, \ldots, x_{n+m}$ of $x_{-1}$. Thus one only needs $x_{-1}$ to compute the $n + m - 2$ relations (31) between the components of $\lambda$. Note also that, with (31), (22) and Prop. 2 b), one has $n + m$ relations for $n + m$ unknowns (the components of the vector $\lambda$). The condition (22) is quadratic in $\lambda$, but the conditions (31) and Prop. 2 b) are linear in $\lambda$. Thus it is easy to eliminate the $n + m - 1$ unknowns $\lambda_2, \ldots, \lambda_{n+m}$ using the $n + m - 1$ equations given by (31) and Prop. 2 b). One can then solve (22) as a quadratic on a single remaining unknown $\lambda_1$. This quadratic has two roots $\lambda_1^{\pm}$; thus there is two possibilities for $\lambda$: $\lambda^+$ obtained taking $\lambda_1 = \lambda_1^+$ and $\lambda^-$ obtained taking $\lambda_1 = \lambda_1^-$.

Choosing the appropriate (economically meaningful) $\lambda$ among these two possibilities is a critical part. Since we excluded trivial degeneracy, it should be clear that (31) implies either

$$\lambda_i > 0 \quad \forall i = 2, \ldots, n \quad \text{and} \quad \lambda_j < 0 \quad \forall j = n+1, \ldots, n+m; \tag{32}$$

or

$$\lambda_i < 0 \quad \forall i = 2, \ldots, n \quad \text{and} \quad \lambda_j > 0 \quad \forall j = n+1, \ldots, n+m. \tag{33}$$

Because of a symmetry consideration[10], for our solution to make sense, among the two vectors $\lambda^+$ and $\lambda^-$, one should have one vector satisfying (32) while the other satisfies (33). Referring to Subsection 3.2, we choose the one that satisfies (32) as the correct $\lambda(x_{-1})$.

Finally, the use of (30) to define the candidate trigger $x_1^\diamond(x_{-1})$ by

$$x_1^\diamond(x_{-1}) = -\frac{\lambda_1}{\lambda_{n+m}} x_{n+m}$$

(note that by (31) we in fact have $x_1^\diamond(x_{-1}) = \frac{\lambda_1}{c_i \lambda_i} x_i$ for all $i = 2, \ldots, n+m$) is meaningful only if $\lambda_1 > 0$ since $x_1^\diamond(x_{-1})$ is negative otherwise. Referring to Subsection 3.1, we deduce that cases $\lambda_1(x_{-1}) \leq 0$ correspond to degenerate exchange problems, and define for these cases $x_1^\diamond(x_{-1}) = 0$. This leads to the following definition of our candidate trigger.

**Definition 2** (A candidate trigger $x_1^\diamond(x_{-1})$)**.** *Let $x \in \Re_+^{n+m}$, $x_i \neq 0$ for all $i = 1, \ldots, n+m$. Let $\lambda(x_{-1}) \in \Re^{n+m}$ be the unique solution of the system:*

*a)* $Q_{n,m}(\lambda) = 0$;

[10] Recall that our original optimal stopping problem was

$$\tau^\star(x, \omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[ e^{-r\tau} \left( \sum_{i=1}^{n+m} c_i X_i(\tau) \right) \right]. \tag{34}$$

Consider the symmetric problem

$$\tau^\star(x, \omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[ e^{-r\tau} \left( \sum_{i=1}^{n+m} (-c_i) X_i(\tau) \right) \right] \tag{35}$$

obtained by the transformation $c_i \mapsto -c_i$ for all $i = 1, \ldots, n+m$. Observe that the conditions (22), (30) and (31) are invariant w.r.t the transformation $c_i \mapsto -c_i$. Thus for a given point $x \in \Re_+^{n+m}$, the two vectors $\lambda_+$ and $\lambda_-$ derived for the original problem (34) are exactly the two vectors derived for the symmetric problem (35). In other words, the couple $(\lambda_+, \lambda_-)$ is stable under the transformation $c_i \mapsto -c_i$. Now, note that $\lambda$ should satisfy (32) for (34) while it should satisfy (33) for (35). Thus, among the two vectors $\lambda^+$ and $\lambda^-$, one should have one vector satisfying (32) while the other satisfies (33).

*b)* $\sum_{i=1}^{n+m} \lambda_i = 1$;

*c)*

$$\frac{x_i}{x_{n+m}} = \frac{c_i \lambda_i}{c_{n+m}\lambda_{n+m}}, \quad i = 2, \ldots, n+m-1.$$

*d)*

$$\lambda_i > 0 \quad \forall i = 2, \ldots, n \quad \text{and} \quad \lambda_j < 0 \quad \forall j = n+1, \ldots, n+m;$$

*Then $x_1^\diamond(x_{-1})$ is defined as follows.*

*i) If $\lambda_1(x_{-1}) > 0$, then $x_1^\diamond(x_{-1}) = c_i \frac{\lambda_1}{\lambda_i} x_i$ for any $i = 2, \ldots, n+m$.*

*ii) If $\lambda_1(x_{-1}) \leq 0$, then $x_1^\diamond(x_{-1}) = 0$ i.e. the problem is degenerate w.r.t. $X_1$.*

The quickest way to grasp Definition 2 is probably to look at how it works on an example. Coming back to Example 2 we proceed as follows.

**Example 3** (The (1,2) exchange - Part 2)**.** *Let $x \in \Re_+^3$ be some vector of state variables. Find $\lambda(x_2, x_3)$ and $x_1^\diamond(x_2, x_3)$ as follows. Def. 2 a) is $Q_{1,2}(\lambda_1, \lambda_2, \lambda_3) = 0$. By Def. 2 b),*

$$\lambda_3 = 1 - \lambda_1 - \lambda_2. \tag{36}$$

*Def. 2 c) implies*

$$\frac{x_2}{x_3} = \frac{\lambda_2}{\lambda_3}. \tag{37}$$

*By* (36) *and* (37) *we find that*

$$\lambda_2(x_2, x_3) = \frac{1 - \lambda_1}{\frac{x_3}{x_2} + 1} = \frac{x_2}{x_3 + x_2}(1 - \lambda_1) \quad \text{and} \tag{38}$$

$$\lambda_3(x_2, x_3) = \frac{x_3}{x_3 + x_2}(1 - \lambda_1). \tag{39}$$

*Now*

$$Q_{1,2}\left(\lambda_1, \frac{x_2}{x_3 + x_2}(1 - \lambda_1), \frac{x_3}{x_3 + x_2}(1 - \lambda_1)\right)$$

*is a simple quadratic in $\lambda_1$. By* (38) *and* (39)*, among the two possible roots of this quadratic, one should pick one which is positive and greater than 1 in order to satisfy Def. 2 d).*[11] *We call this root $\lambda_1(x_2, x_3)$. Thus in this example Def. 2 ii) can never occur*[12] *and $x_1^\diamond(x_2, x_3)$ obtains from Def. 2 i):*

$$x_1^\diamond(x_2, x_3) = -\frac{\lambda_1(x_2, x_3)}{\lambda_2(x_2, x_3)} x_2 = -\frac{\lambda_1(x_2, x_3)}{\lambda_3(x_2, x_3)} x_3$$

*Using* (38) *or* (39)*, one can obtain the very intuitive expression*

$$x_1^\diamond(x_2, x_3) = \left(\frac{\lambda_1(x_2, x_3)}{\lambda_1(x_2, x_3) - 1}\right)(x_2 + x_3). \tag{40}$$

*With $\lambda_1(x_1, x_2) > 1$, the level of asset 1 which triggers investment is higher than the total cost. The policy suggested by* (40) *is to exercise the exchange option at $x \in \Re_+^{n+m}$ if $x_1 \geq x_1^\diamond(x_2, x_3)$, for $x_1^\diamond(x_2, x_3)$ given by* (40)*.*

[11] Such a root always exists and is unique under the conditions $\frac{1}{2}\sigma_i^2 < \mu_i < r$ (recall that $r > 0$). See Appendix E.2.

[12] This is because there is a single price and we excluded trivial degeneracy.

We conclude this section by showing that the candidate trigger $x_1^\diamond(x_{-1})$ defined above (Definition 2) satisfies parametrization invariance and linear homogeneity.

**Lemma 2** (Parametrization invariance)**.**

$$\forall x \in \Re_+^{n+m}, x_1^\diamond(x_{-1}) = x_1 \quad \Longleftrightarrow \quad x_j^\diamond(x_{-j}) = x_j \quad \forall j = 2, \dots, n+m.$$

*Proof.* Let $x \in \Re_+^{n+m}$ be a point such that $x_1 = x_1^\diamond(x_{-1})$. From exclusion of trivial degeneracy we have $x_1 \neq 0$ which implies $x_1^\diamond(x_{-1}) \neq 0$. Therefore, there exists a $\lambda(x_{-1}) \in \Re^{n+m}$ such that Def. 2 a), b), c) and d) hold for $x$ and $\lambda(x_{-1})$; and this $\lambda(x_{-1})$ is such that $\lambda_1(x_{-1}) > 0$ so that we have

$$x_1 = x_1^\diamond(x_{-1}) = c_i \frac{\lambda_1}{\lambda_i} x_i \quad \text{for all} \quad i = 2, \dots, n+m.$$

Thus, more simply, there exists a $\lambda(x_{-1}) \in \Re^{n+m}$ such that

- Def. 2 a) and b) hold
- $$\frac{x_i}{x_1} = c_i \frac{\lambda_i}{\lambda_1}, \quad i = 2, \dots, n+m \tag{41}$$
  (note that (41) is an extension of Def. 2 c) to indice 1)
- $$\lambda_i > 0 \quad \forall i = 1, \dots, n \quad \text{and} \quad \lambda_j < 0 \quad \forall j = n+1, \dots, n+m \tag{42}$$
  (note that (42) is an extension of Def. 2 d) to index 1).

Note that Def. 2 a) and Def. 2 b) are invariant w.r.t. the parametrization (since $Q_{n,m}(\lambda)$ and $\sum_{i=1}^{n+m} \lambda_i$ do not depend on the parametrization we choose). Eq. (42) is also parametrization invariant. Finally, (41) is parametrization invariant since the set

$$\left\{ x \in \Re_+^{n+m} : \frac{x_i}{x_1} = c_i \frac{\lambda_i}{\lambda_1}, \quad i = 2, \dots, n+m \right\}$$

is identical to

$$\left\{ x \in \Re_+^{n+m} : \frac{x_i}{x_j} = \frac{c_i \lambda_i}{c_j \lambda_j}, \quad i = 1, \dots, n+m \quad \text{s.t.} \quad i \neq j \right\} \tag{43}$$

for any $j = 1, \dots, n+m$. Thus the $\lambda(x_{-1})$ that guarantees Def. 2 a), b), (41) and (42) also guarantees Def. 2 a), b), (41′) and (42) with (41′) given by

$$\frac{x_i}{x_j} = \frac{c_i \lambda_i}{c_j \lambda_j}, \quad i = 1, \dots, n+m \quad \text{s.t.} \quad i \neq j \tag{41′}$$

(with parametrization by asset $j$) for any $j = 2, \dots, n+m$. Thus we have $x_j^\diamond(x_{-j}) = x_j$ for all $j = 1, \dots, n+m$, as stated. □

Since we proved parametrization invariance, we can define the investment frontier $\partial S_{n,m}^\diamond$ induced by our rule using parametrization with asset 1, without loss of generality

$$\partial S_{n,m}^\diamond \triangleq \{x \in \Re_+^{n+m} : x_1 = x_1^\diamond(x_{-1})\}. \tag{44}$$

**Lemma 3** (Linear homogeneity)**.** $\forall \alpha > 0$, $x_1^\diamond(\alpha x_{-1}) = \alpha x_1^\diamond(x_{-1})$.

*Proof.* In Def. 2 c) replace $x_{-1}$ by $\alpha x_{-1}$, $\alpha > 0$. One sees that the $\alpha$ cancels out, so that Def. 2 c) remains unchanged. Thus, using Def. 2 a), b) and d), we find $\lambda(\alpha x_{-1}) = \lambda(x_{-1})$. Thus

- if $\lambda_1(x_{-1}) = \lambda_1(\alpha x_{-1}) > 0$, then Def. 2 i) changes to

$$x_1^\diamond(\alpha x_{-1}) = c_i \frac{\lambda_1(\alpha x_{-1})}{\lambda_i(\alpha x_{-1})} \cdot \alpha \cdot x_i = c_i \frac{\lambda_1(x_{-1})}{\lambda_i(x_{-1})} \cdot \alpha \cdot x_i \qquad \forall i = 2, \dots, n+m$$
$$= \alpha x_1^\diamond(x_{-1});$$

- if $\lambda_1(x_{-1}) = \lambda_1(\alpha x_{-1}) \le 0$, then by Def. 2 ii) we have $x_1^\diamond(x_{-1}) = x_1^\diamond(\alpha x_{-1}) = 0$. That completes the proof.

□

## 5 A formula for the trigger $x_1^\diamond(x_{-1})$

The candidate trigger proposed in Definition 2 is in fact determinable in closed form.

**Proposition 3** (A formula for $x_1^\diamond(x_{-1})$)**.** *Define*

$$\begin{aligned}
A(x_{-1}) &\triangleq \sum_{i=2}^{n+m} (-c_i) x_i, \\
B(x_{-1}) &\triangleq -\frac{1}{A} \sum_{j=2}^{n+m} c_j x_j \left( \mu_j - \frac{1}{2}\sigma_j^2 \right), \\
C(x_{-1}) &\triangleq \frac{1}{2A^2} \sum_{i,j \ge 2}^{n+m} \rho_{ij} \sigma_i \sigma_j c_i c_j x_i x_j, \\
D(x_{-1}) &\triangleq \frac{1}{2}\sigma_1^2 + \frac{\sigma_1}{A} \sum_{j=2}^{n+m} \rho_{1j} \sigma_j c_j x_j, \text{ and} \\
\Delta(x_{-1}) &\triangleq (\mu_1 - B - 2C - D)^2 - 4(D+C)(B+C-r).
\end{aligned}$$

*Assume that $\frac{1}{2}\sigma_i^2 < \mu_i < r$ for all $i = 1, \dots, n+m$. Under this assumption, $\Delta(x_{-1}) > 0$ and the real numbers $\lambda_1^+(x_{-1})$ and $\lambda_1^-(x_{-1})$ defined by*

$$\lambda_1^\pm(x_{-1}) \triangleq \frac{-(\mu_1 - B - 2C - D) \pm \sqrt{\Delta}}{2(C+D)}$$

*satisfy $\lambda_1^+(x_{-1}) > 1$ and $\lambda_1^-(x_{-1}) < 1$. Furthermore, $\lambda_1(x_{-1})$ and $x_1^\diamond(x_{-1})$ are given as follows.*

*a) If $A(x_{-1}) > 0$ then $\lambda_1 = \lambda_1^+$ and*

$$x_1^\diamond(x_{-1}) = \left( \frac{\lambda_1^+(x_{-1})}{\lambda_1^+(x_{-1}) - 1} \right) A(x_{-1}).$$

*b) If $A(x_{-1}) < 0$ then $\lambda_1 = \lambda_1^-$ and* *b1) if* $0 < \lambda_1^- < 1$,

$$x_1^\diamond(x_{-1}) = \left(\frac{\lambda_1^-(x_{-1})}{\lambda_1^-(x_{-1}) - 1}\right) A(x_{-1});$$

*b2) if* $\lambda_1^- \leq 0$*, the problem is degenerated w.r.t.* $X_1$ *i.e.* $x_1^\diamond(x_{-1}) = 0$.

*Proof.* See Appendix C □

Note that Proposition 3 do not give $x_1^\diamond(x_{-1})$ if[13] $A(x_{-1}) = 0$ though there is physically no singularity in such configuration of the problem. If you consider e.g. the (2,2) exchange, such a situation appears for $x = (1, 2, 1, 1)$ where we see that $A(x_{-1}) = -2 + 1 + 1 = 0$. In the meantime, we also see that $A(x_{-2}) = -1 + 1 + 1 = 1$. Thus the apparent singularity $A(x_{-1}) = 0$ is a coordinate singularity i.e. a singularity which can be removed by choosing a different parametrization (e.g. parametrization w.r.t. asset $X_2$ in the previous small example). A natural way to prove this claim is to show that the limit of $x_1^\diamond(x_{-1})$ as $A(x_{-1}) \to 0$ is finite. Assuming $A(x_{-1}) \to 0$ one sees that $C >> B, D$ which leads to $Q(\lambda) \approx C\lambda_1^2 - 2C\lambda + C$ and $\lambda_1^\pm \approx 1$. Thus, qualitatively, the limit

$$\lim_{A(x_{-1}) \to 0} x_1^\diamond(x_{-1}) = \lim_{A(x_{-1}) \to 0} \left(\frac{\lambda_1(x_{-1})}{\lambda_1(x_{-1}) - 1}\right) A(x_{-1})$$

multiplies a term which goes arbitrarily close to zero ($A(x_{-1})$) by a term which can be arbitrarily high. We were not able to compute this limit explicitly, but we observed that it converges to a finite value. Thus a simple way to proceed practically when $A(x_{-1}) = 0$ is to compute $x_1^\diamond(x_{-1} + \epsilon)$ for $\epsilon \in \Re^{n+m-1}$ very small. If one feels uneasy with the lack of rigor of this approach, another way to proceed is to change parametrization: one chooses a parametrization w.r.t a price $j = 2, \ldots, n$ such that $A(x_{-j}) \neq 0$ and compute $x_j^\diamond(x_{-j})$.

Recall that the optimal stopping rule $x_1^\star(x_{-1})$ satisfies parametrization invariance, linear homogeneity and the following characterization of degeneracy (see Lemma 1):

$$x_i^\star(x_{-i}) = 0 \Leftrightarrow x_j \geq x_j^\star(x_{-i,-j}) \qquad \forall j = 1, \ldots, n \quad \text{such that} \quad j \neq i. \tag{45}$$

We verified that our candidate trigger $x_1^\diamond(x_{-1})$ satisfies parametrization invariance and linear homogeneity (Lemma 2 and 3 respectively). It would be nice to also prove that $x_1^\diamond(x_{-1})$ satisfies the characterization of degeneracy given by (45) i.e. that we can write Lemma 1 for $x_1^\diamond(x_{-1})$. Using Proposition 3 we were only able to prove the following weaker result.

**Lemma 4** (Degeneracy with respect to the asset $i$ for $x^\diamond$)**.** *Let* $x_1^\diamond(x_{-1})$ *be the candidate trigger. Assume* $n \geq 2$*. For any* $i = 1, 2, \ldots, n$ *one has*

$$x_i^\diamond(x_{-i}) = 0 \Leftarrow x_j = x_j^\diamond(x_{-i,-j}) \qquad \forall j = 1, \ldots, n \quad \textit{such that} \quad j \neq i. \tag{46}$$

*Proof.* See Appendix D. □

Let us close this section by a numerical example which illustrates how to use Proposition 3. Consider the (2,2) exchange

$$\tau^\star(x, \omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-\rho\tau} \left(X_1(\tau) + X_2(\tau) - X_3(\tau) - X_4(\tau)\right)\right] \tag{47}$$

in the following numerical setting

[13]Because if $A(x_{-1}) = 0$ one cannot write Eq. (67): the denominator is equal to zero. Moreover, it is wrong to conclude from a limit argument on Eq. (70) that $x_1^\diamond(x_{-1})$ is zero in this case because, as we shall soon show, $A(x_{-1}) \to 0$ implies $\lambda_1(x_{-1}) \to 1$.

**Dataset 0:** $r = 0.2$, $\mu = (0.1, 0.06, 0.035, 0.12)$, $\sigma = (0.4, 0.1, 0.15, 0.3)$. The correlation matrix is described by $\rho_{12} = 0.25$, $\rho_{13} = 0.35$, $\rho_{14} = -0.5$, $\rho_{23} = -0.25$, $\rho_{24} = 0.2$, $\rho_{34} = -0.55$.

We want to plot the candidate investment threshold $x_1^\diamond(x_{-1}) = x_1^\diamond(x_2, x_3, x_4)$ of this problem for $x_3 = x_4 = 1$ and $x_2 \in [0; 5]$. Before we come to that, let us make the two following observations.

1. Consider the degenerate problem obtained by removing asset 1 from the original one (47), i.e.

$$\tau^\star(x, \omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[ e^{-r\tau} \left( X_2(\tau) - X_3(\tau) - X_4(\tau) \right) \right]. \tag{48}$$

Let $x_2^\diamond(x_{-1,-2}) = x_2^\diamond(x_3, x_4)$ be the candidate investment threshold of problem (48). Using Proposition 3 on this (1,2) exchange, we obtain $x_2^\diamond(1, 1) = 2.69$. Coming back to (47), we deduce from Lemma 4 that $x_1^\diamond(x_2, 1, 1) = 0$ for $x_2 = 2.69$. Consequently, one should observe $x_1^\diamond(x_2, 1, 1) = 0$ for $x_2 \geq 2.69$.

2. If you set $x_2 = 0$ (i.e. trivial degeneracy w.r.t. asset 2), the problem trivially reduces to

$$\tau^\star(x, \omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[ e^{-\rho\tau} \left( X_1(\tau) - X_3(\tau) - X_4(\tau) \right) \right]. \tag{49}$$

Let $x_1^\diamond(x_{-1,-2}) = x_1^\diamond(x_3, x_4)$ be the candidate investment threshold for problem (49). Using Proposition 3 on this (1,2) exchange, we find $x_1^\diamond(1, 1) = 6.06$. Coming back to (47), one should thus observe $x_1^\diamond(0, 1, 1) = 6.06$.

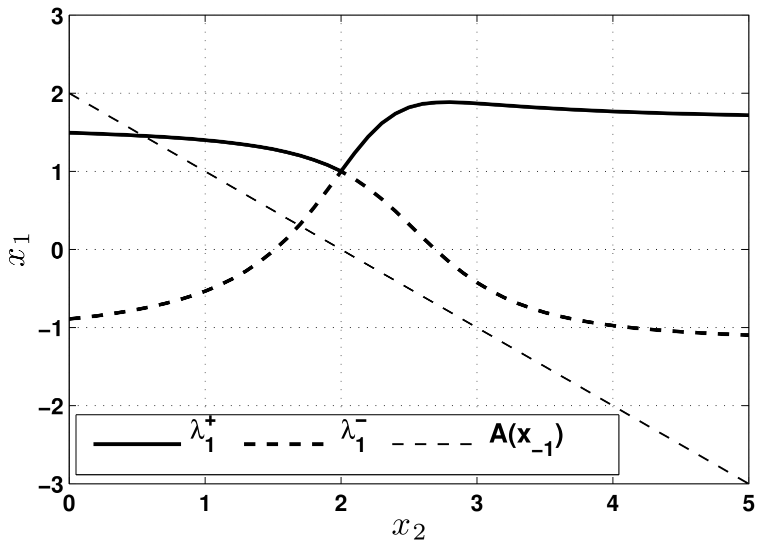


Figure 1: The roots $\lambda_1^+(x_{-1})$ and $\lambda_1^-(x_{-1})$, for $x_3 = x_4 = 1$ and $x_2 \in [0; 5]$.

We now come back to the main objective of the example (i.e. the determination of $x_1^\diamond(x_{-1})$ for $x_3 = x_4 = 1$ and $x_2 \in [0; 5]$) and show that our solution confirms the two observations announced above.

Using Proposition 3, we compute the roots $\lambda_1^+(x_{-1})$ and $\lambda_1^-(x_{-1})$, for $x_3 = x_4 = 1$ and $x_2 \in [0; 5]$. In Figure 1, we plot these roots and observe that $\lambda_1^+(x_{-1}) > 1$ and $\lambda_1^-(x_{-1}) < 1$. We plot also the reduced cost $A(x_{-1}) = -x_2 + x_3 + x_4 = -x_2 + 2$. Thus $A(x_{-1}) > 0$ (resp. $A(x_{-1}) < 0$) corresponds to $x_2 < 2$ (resp. $x_2 > 2$), region for which—following Proposition 3—one sets $\lambda_1(x_{-1}) = \lambda_1^+(x_{-1})$ (resp. $\lambda_1(x_{-1}) = \lambda_1^-(x_{-1})$).

Thus we know $\lambda_1(x_{-1})$ on the entire interval $[0; 5]$; it is plotted in Figure 2. In this Figure we see that $\lambda_1(x_{-1})$ is negative for (approximately) $x_2 \geq 2.7$. Thus for $x_2 \geq 2.7$ the problem is degenerate w.r.t. $X_1$.

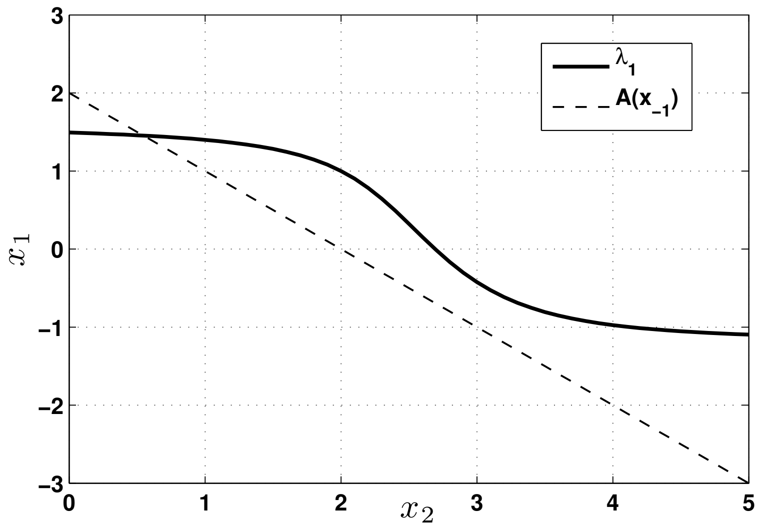


Figure 2: The $\lambda_1(x_{-1})$ of Definition 2 for $x_3 = x_4 = 1$ and $x_2 \in [0; 5]$.

In Figure 3 we finally plot the candidate investment threshold $x_1^\diamond(x_{-1}) = x_1^\diamond(x_2, x_3, x_4)$ of problem (47) for $x_3 = x_4 = 1$ and $x_2 \in [0; 5]$. We see on this figure that $x_1^\diamond(x_2, 1, 1) = 0$ for (approximately) $x_2 \geq 2.7$, and that $x_1^\diamond(0, 1, 1) \approx 6$. Thus this figure confirms the two observations above.

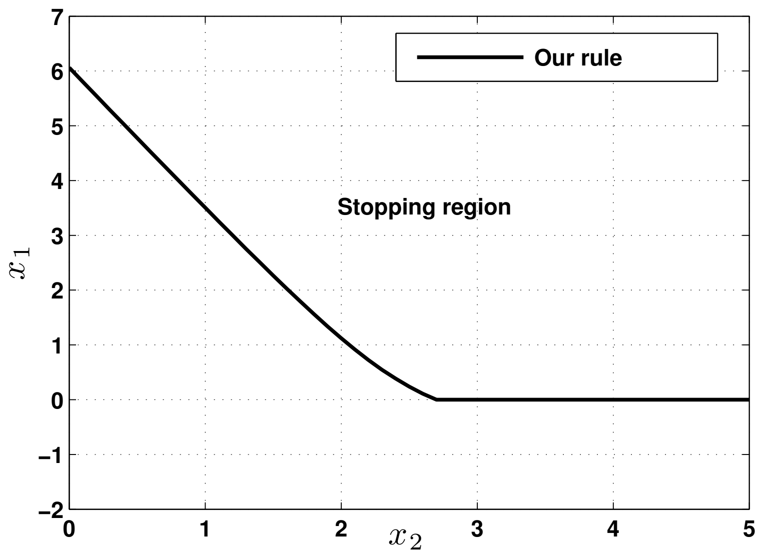


Figure 3: Plot of our candidate trigger $x_1^\diamond(x_{-1}) = x_1^\diamond(x_2, x_3, x_4)$ for problem (47) with $x_3 = x_4 = 1$ and $x_2 \in [0; 5]$. The problem appears to degenerate w.r.t. asset 1 for $x_2 \geq 2.7$.

# 6 Interpretation: a parametrized free boundary problem

Given an initial position $x^0 \in \Re_+^{n+m}$ of the random process $X$, the principle of our algorithm is to test locally if immediate investment should occur. Thus we are using the $x_{-1}^0$ components of $x^0 = (x_1^0, x_{-1}^0)$ as extra boundary conditions to solve a variant of the unsolvable Problem 1. In fact we solve:

**Problem 2** ($x_{-1}^0$ as a parameter of the free boundary problem)**.** *Let $x^0 = (x_1^0, x_{-1}^0) \in \Re_+^{n+m}$. Find the investment trigger $x_1^\diamond(x_{-1}^0)$ advising immediate investment at $x^0$ if $x_1^0 \geq x_1^\diamond(x_{-1}^0)$ by solving the free boundary problem: find $f_{(x_{-1}^0)}(x) \in \mathcal{C}^2(\Re_+^{n+m})$ and $x_1^\diamond(x_{-1}^0)$ s.t.*

$$
\begin{aligned}
(\mathcal{L}_X - r)\, f_{(x_{-1}^0)} &= 0 \qquad x \in \Re_+^{n+m} && (50)\\
f_{(x_{-1}^0)} &= g \qquad x = \left(x_1^\diamond(x_{-1}^0), x_{-1}^0\right) && (51)\\
\nabla f_{(x_{-1}^0)} &= \nabla g \quad\ x = \left(x_1^\diamond(x_{-1}^0), x_{-1}^0\right). && (52)
\end{aligned}
$$

Note that in Problem 2, $x_{-1}^0$ is a parameter,not a variable. The boundary conditions (51) and (52) are conditions on a single point, not on a surface. Thus, inspecting Proposition 2, we see that Problem 2 is solvable in terms of $x_{-1}^0$. In particular, the function $f_{(x_{-1}^0)}(x)$ takes the form

$$f_{(x_{-1}^0)}(x) = a\left(x_{-1}^0\right) \prod_{i=1}^{n+m} x_i^{\lambda_i(x_{-1}^0)}.$$

Note also that even though we parametrize by $x_{-1}^0$, we still solve a $n+m$ dimensional PDE (50) for $f_{(x_{-1}^0)}(x) : \Re_+^{n+m} \to \Re$. We thus do not consider that (locally) $X_{-1}^{x_{-1}^0}$ is deterministic. It thus appears relevant to ask what can be the meaning of $f_{(x^0_{-1})}(x)$; and, more importantly, why could this resolution technique make sense.

Take $x_0 \in \Re_+^{n+m} \setminus S_{n,m}$ with $0 < d\left(x_0, \partial S_{n,m}\right) \leq \epsilon$ where $d$ is the Euclidean distance in $\Re_+^{n+m}$. Since $x_0$ is in the continuation region, one has

$$(\mathcal{L}_X - r)\, f(x) = 0 \tag{53}$$

where $f$ is the value function. Now, as $\epsilon \to 0$, (53) continues to hold and because $X$ is regular (it cannot go to $\infty$ in zero time), one should have

$$\lim_{\epsilon \to 0} X^{x_0}_{\tau^\star(x_0,\omega)} \to x_0 \quad \text{a.s.} \quad \mathbb{P}^{x_0}.$$

This suggests that when solving Problem 2 for $x_0$ close to the stopping region, one can use (locally) $x_0$ as a parameter of the free boundary problem. Note that the key point of this intuitive reasoning is regularity of the stochastic process.

This also indicates that $f_{(x^0_{-1})}(x)$ should not be seen as a good approximation of the value function around $x^0_{-1}$ i.e.

$$f_{(x^0_{-1})}(x_1, x^0_{-1}) \neq \lim_{x_{-1} \to x^0_{-1}} f(x_1, x_{-1}) \qquad \forall x_i \in \Re_+$$

but as a good approximation of the value function $f$ around the optimal investment point corresponding to $x^0_{-1}$, that is,

$$\lim_{x_1 \to x_1^\star(x^0_{-1})} f_{(x^0_{-1})}(x_1, x^0_{-1}) = f\left(x_1^\star(x^0_{-1}), x^0_{-1}\right) = g(x^0).$$

# 7 Numerical examples

This section provides numerical tests for our investment rule $x_1^\diamond(x_{-1})$ given by Proposition 3. Some of the examples used here are taken from the literature, others are new. As a general remark, note that it is difficult (probably impossible) to prove numerically that a given stopping rule is optimal for our class (time-homogeneous and time-continuous) of optimal stopping problems. The reason is as follows. To compute numerically the stopping region, one uses two approximations: a finite horizon and a time-grid. Thus the computation of the true stopping region suffers from errors that make hard to determine whether the rule that we put to the test—in this case, $x_1^\diamond(x_{-1})$—is optimal.

Note that rule $x_1^\diamond(x_{-1})$ is analytic, as well as the sufficient condition (12) and the necessary conditions (13). It is thus possible to compare these rules with a strong degree of confidence because they are unaffected by numerical errors. Needless to say the comparison is limited to examples and the analysis of the Monte Carlo type and hence cannot lead to general claims. The following compares these three rules for several problem configurations. For each example, we plot the sufficient condition (12), the "strictest" necessary condition one can find using a large sample of geometric Brownian motions $X_u$ and $X_v$ in (13) and the stopping rule $x_1^\diamond(x_{-1})$. Then we use Monte Carlo simulations to compare the performance accruing from these different policies. Recall that we defined the performance $J(\tau, x)$ associated to a stopping time $\tau$ and initial point $x$ as

$$J(\tau, x) \triangleq \mathbb{E}^x\left[e^{-r\tau} g(X_\tau)\right].$$

## 7.1 An example from Olsen and Stensland (1992)

Olsen and Stensland (1992, Section 4) treat a particular (1,2) exchange

$$\tau^\star(x,\omega) = \arg\sup_{\tau\in\mathcal{S}} \mathbb{E}^x \left[e^{-r\tau}\left(X_1(\tau) - X_2(\tau) - X_3(\tau)\right)\right]$$

numerically. They use two numerical settings, corresponding to two cases where $x_1$ and $x_2$ are respectively correlated and uncorrelated.

**Dataset 1:** $\mu_1 = \mu_2 = \mu_3 = 0$, $\sigma_1 = \sigma_2 = 0.1$, $\sigma_3 = 0$, $r = 0.1$, $\rho_{12} = \rho_{13} = 0$, $x_3 = 0.5$. The second cost (asset 3) is deterministic. Assets are not correlated. See Olsen and Stensland (1992, p. 48).

**Dataset 2:** $\mu_1 = \mu_2 = 0.01$, $\sigma_1^2 = \sigma_2^2 = 0.02$, $\mu_3 = \sigma_3 = 0$, $\rho_{12} = 0.5$, $\rho_{13} = \rho_{23} = 0$, $r = 0.1$, $x_3 = 0.5$. The second cost (asset 3) is deterministic. The price and the first cost are correlated, but the two costs are not. See Olsen and Stensland (1992, p. 51).

Olsen and Stensland compare the optimal stopping region obtained numerically via backward dynamic programming to the stopping region given by the rule (10) with $n = 1$ and $m = 2$. For their two numerical settings, they assume that the second cost $X_3$ is deterministic, thereby bypassing the fact that their rule does not depend on inter-cost correlations.[14] They verify that the half-space (10) is a subset of the true stopping region; they also find that these two sets are close, but distinct.

To compare the different rules in hand, we generate a cone of necessary conditions by plotting the lines (11) for $10^6$ different geometric Brownian motions $X_u$. This procedure is the natural way to use (11) and (13) in practice. Note that no matter how large the sample of GBMs is chosen, one cannot guarantee that one finds the true weakest and strongest necessary conditions which is the general limitation of these necessary conditions.

The sample of GBMs were randomly generated by extending the decoupled representation of the SDE of $X$ by normal random variables (see Appendix G for details). For each example treated in this paper, we observe that the cone of these lines converges relatively fast with the number of GBMs generated by this method: 1000 processes were enough to have a fair representation of the cone. We thus conjecture that a sample of $10^6$ GBMs is sufficient for a comparison of the different rules. Figure 4 shows how to graphically isolate the weakest and strongest necessary conditions of the sample. The strongest necessary condition is the one closest to the optimal stopping rule and is obviously of considerable practical importance: in Figure 4 it can be optimal to invest only if $x$ is in the area below the strongest necessary condition.

For Dataset 1 and 2 (the two settings of Olsen and Stensland, 1992) we plot the sufficient condition (10), the weakest and strongest necessary conditions (11) as well as our investment rule $x_1^\diamond(x_{-1})$. The horizontal axis is $x_1$ in these figures and the stopping region associated to a given investment rule is thus the area below the rule line. Figures 5 and 6 (resp. 7 and 8) show that the stopping region $S_{1,2}^\diamond$ provided by the rule $x_1^\diamond(x_{-1})$ is such that $S_{1,2}^- \subset S_{1,2}^\diamond \subset S_{1,2}^+$ for Dataset 1 (resp Dataset 2). Thus, $S_{1,2}^\diamond$ could be the true stopping region $S_{1,2}$.

[14] One can always (via adjustment of parameters) reformulate the problem so that one of the cost becomes deterministic (see Olsen and Stensland, 1992, p. 46). In the three assets case (specifically) we thus obtain a problem where the two costs are uncorrelated. However, in problems involving more than 3 assets, this technique only allows to remove one inter-cost correlation.

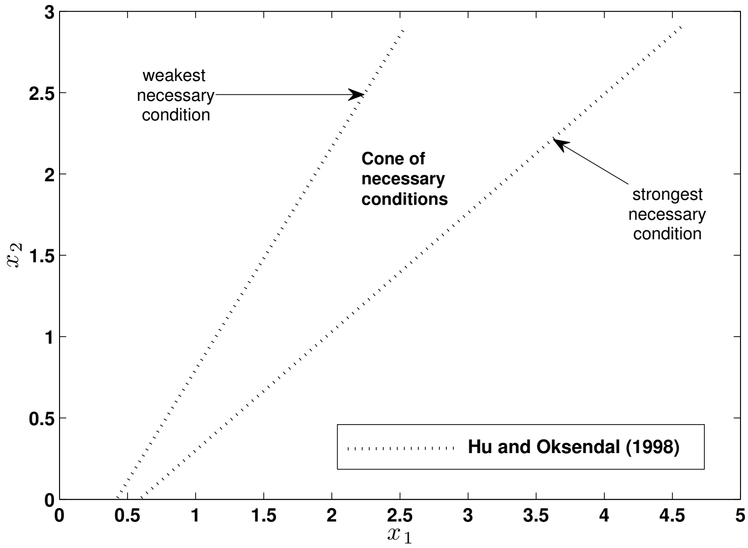


Figure 4: Dataset 1. The cone of necessary conditions obtained from (11) using 1000000 randomly generated geometric Brownian motions $X_u$. We isolate the weakest and strongest necessary condition.

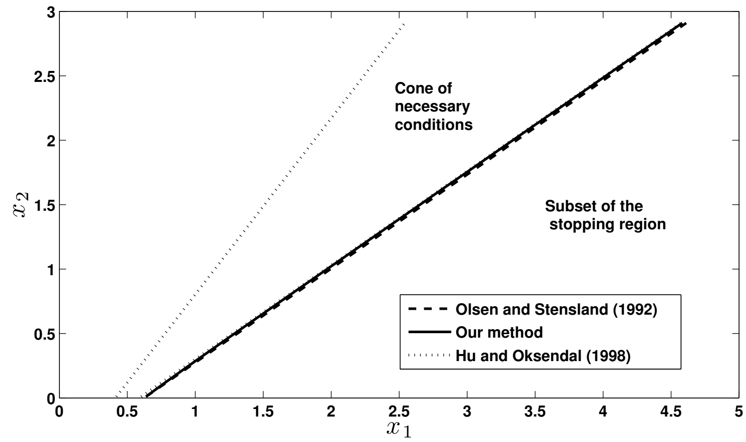


Figure 5: Dataset 1. The three assets are not correlated. It appears that our investment rule $x_1^\diamond(x_{-1})$ lies between the sufficient condition (10) and the necessary condition (11); but one could use a closer look.

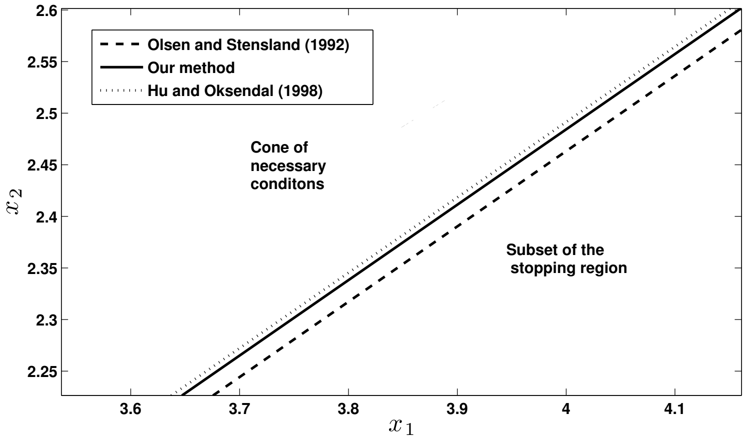


Figure 6: Dataset 1. A closer look: $x_1^\diamond(x_{-1})$ lies between the sufficient condition (10) and the necessary condition (11).

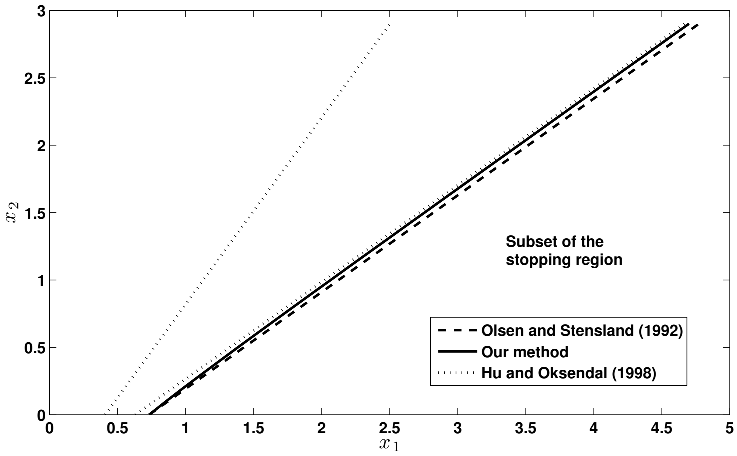


Figure 7: Dataset 2. The price $x_1$ is correlated to the cost $x_2$, but the two costs $x_2$ and $x_3$ are not correlated. It appears that our investment rule $x_1^\diamond(x_{-1})$ lies between the sufficient condition (10) and the necessary condition (11).

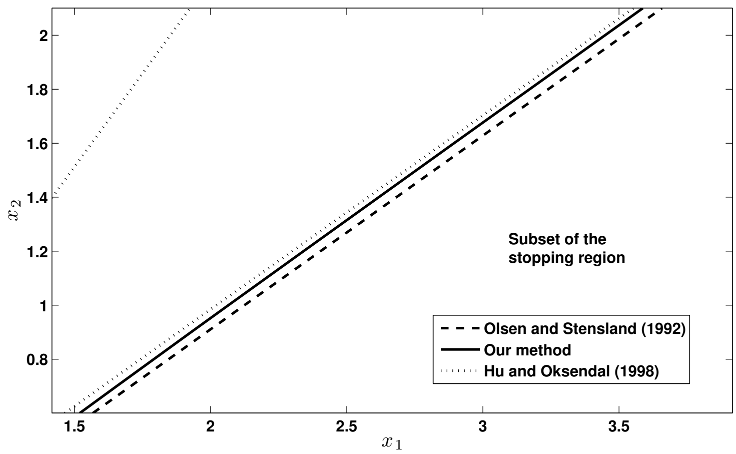


Figure 8: Dataset 2. A closer look: our investment rule $x_1^\diamond(x_{-1})$ lies between the sufficient condition (10) and the necessary condition (11).

One can also compare the performances of the sufficient condition (10), the necessary condition (11) and of our rule $x_1^\diamond(x_{-1})$ using Monte Carlo simulations. We generate sample paths for $X_1(t)$ and $X_2(t)$ (with $X_1(0) = X_2(0) = x_1 = x_2 = 1$; $X_3(t) = x_3 = 0.5$ for all $t$) for $10^6$ scenarios and stop the processes when (10), (11) and $x_1^\diamond(x_{-1})$ hold, respectively (see Appendix F for details on the Monte Carlo procedure). We obtain the performances of these three rules starting the processes at $x = (x_1, x_2, x_3) = (1, 1, 0.5)$.

For Dataset 1 (uncorrelated assets, see Table 1), the average performance of the three triggers (0.0254) is not significantly different. We cannot distinguish the three rules at 99.9% confidence interval (C.I.).

| Rule | Performance | Error (99.9% C.I.) |
|---|---|---|
| Olsen and Stensland (1992) | 0.02543 | $18 \times 10^{-5}$ |
| Hu and Øksendal (1998) | 0.02548 | $14 \times 10^{-5}$ |
| Our rule | 0.02544 | $16 \times 10^{-5}$ |

Table 1: Comparison of the performances at $x = (1, 1, 0.5)$ for Dataset 1.

For Dataset 2 (correlation between the price and one cost, see Table 1), the performance of our rule $x_1^\diamond(x_{-1})$ is better than the performance of the sufficient condition (10) at 99.9% C.I. For the same C.I., one cannot however compare fairly $x_1^\diamond(x_{-1})$ with the strictest necessary condition (11); the two rules appears to have the same performance.

| Rule | Performance | Error | |
|---|---|---|---|
| | | 99 % C.I. | 99.9% C.I. |
| Olsen and Stensland (1992) | 0.03667 | $20 \times 10^{-5}$ | $26 \times 10^{-5}$ |
| Hu and Øksendal (1998) | 0.03719 | $19 \times 10^{-5}$ | $25 \times 10^{-5}$ |
| Our rule | 0.03717 | $20 \times 10^{-5}$ | $25 \times 10^{-5}$ |

Table 2: Comparison of the performances at $x = (1, 1, 0.5)$ for Dataset 2. The price is correlated to one of the costs.

### 7.2 Impact of inter-cost correlation

#### 7.2.1 A (1,2) exchange

We pursue the study of the (1,2) exchange by using different data. We mentioned that the sufficient condition provided by Olsen and Stensland (1992) does not depend on inter-cost correlations. The preceding parameter settings (Datasets 1, 2) overlook the effect of those correlations by assuming that the third cost is deterministic. Introducing a negative correlation between the two costs of this example reduces the uncertainty over the payoff. Our intuition is that it would be optimal to invest earlier i.e. the true stopping region $S_{n,m}$ should be consequently larger than $S^-_{n,m}$. We examine this situation.

**Dataset 3:** $\mu_1 = \mu_2 = \mu_3 = 0.02$, $\sigma_1^2 = \sigma_3^2 = 0.08$, $\sigma_2^2 = 0.12$, $\rho_{12} = 0.8165$, $\rho_{13} = -0.5$, $\rho_{23} = -0.8165$, $r = 0.3$. The two costs (assets 2 and 3) are correlated.

We compare our stopping region $S^\diamond_{1,2}$ with $S^-_{1,2}$ and $S^+_{1,2}$ for Dataset 3 in Figure 9. Because the stopping regions in this example are surfaces in $\Re^3_+$, we enable a 2 dimensional representation by fixing $x_3 = 0.5$. We see that $S^-_{1,2} \subset S^\diamond_{1,2} \subset S^+_{1,2}$. Moreover, we note that, for this parameter setting, these three stopping regions are not close at all. This reveals a case where there is a significant gap between $S^-_{1,2}$ and $S^+_{1,2}$.

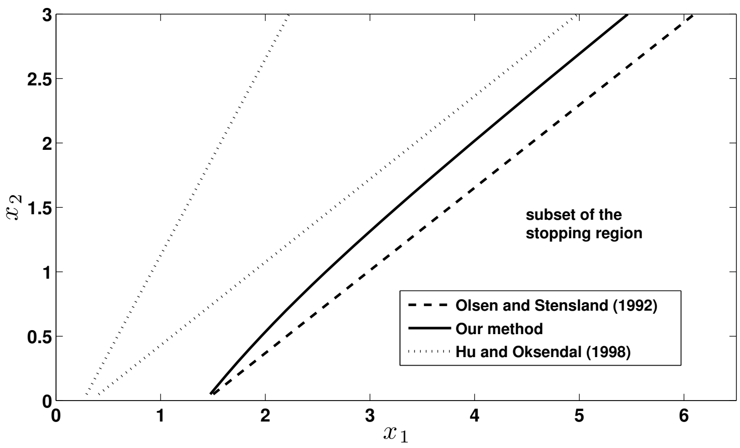


Figure 9: Dataset 3. A comparison of the stopping regions in presence of inter-cost correlation for $x_3 = 0.5$. There is consequent gap between the strictest necessary condition (11) and the sufficient condition (10).

We then use 1000000 Monte Carlo simulations to evaluate the performance of the three associated rules at $x = (1, 1, 0.5)$. Table 3 shows that, at the confidence level 99.9%, the performance (0.068844) of our rule $x_1^\diamond(x_{-1})$ is 6.73% higher than the performance (0.064505) of the sufficient condition (10) and 20.89% higher than the performance (0.056943) of the strictest necessary condition (11).

#### 7.2.2 A (1,3) exchange

Consider the (1,3) exchange

$$\tau^\star(x,\omega) = \arg\sup_{\tau\in\mathcal{S}} \mathbb{E}^x \left[e^{-r\tau}\left(X_1(\tau) - X_2(\tau) - X_3(\tau) - X_4(\tau)\right)\right]$$

under the numerical setting

| Rule | Performance | Error (99.9% C.I.) |
|---|---|---|
| Olsen and Stensland (1992) | 0.06464 | $56 \times 10^{-5}$ |
| Hu and Øksendal (1998) | 0.05694 | $43 \times 10^{-5}$ |
| Our rule | 0.06884 | $53 \times 10^{-5}$ |

Table 3: Comparison of the performances at $x = (1, 1, 0.5)$ for Dataset 3. An example with inter-cost correlation.

**Dataset 4:** $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0.02$, $\sigma_1^2 = 0.08$, $\sigma_2^2 = 0.13$, $\sigma_3^2 = 0.17$, $\sigma_4^2 = 0.11$, $\rho_{12} = 0.7845$, $\rho_{13} = -0.3430$, $\rho_{14} = 0.2132$, $\rho_{23} = -0.3363$, $\rho_{24} = -0.2509$, $\rho_{34} = 0.5119$, $r = 0.3$.

We see from Figure 10 and Table 4 that the same comments generally apply in this new example with inter-cost correlations: $S_{1,3}^- \subset S_{1,3}^\diamond \subset S_{1,3}^+$; these sets are not close and our rule has the better performance.

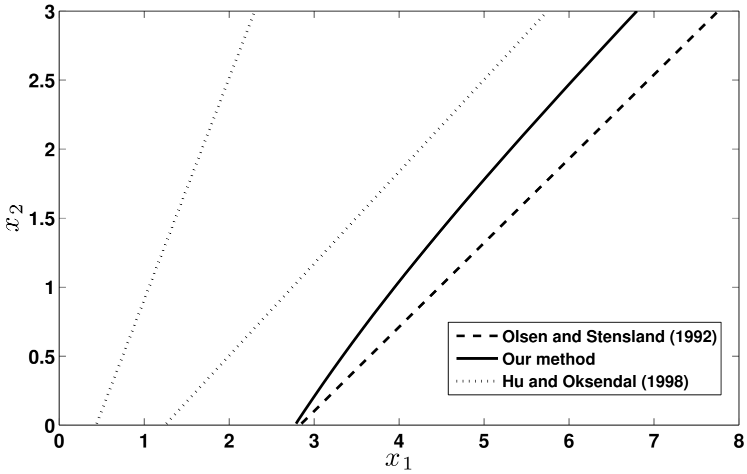


Figure 10: Dataset 4. A comparison of the stopping regions in presence of inter-cost correlations for $x_3 = x_4 = 0.5$. There is a consequent gap between the strictest necessary condition (11) and the sufficient condition (10).

| Rule | Performance | Error (99.9% C.I.) |
|---|---|---|
| Olsen and Stensland (1992) | 0.9703 | $27 \times 10^{-4}$ |
| Hu and Øksendal (1998) | 1 | 0 |
| Our rule | 1.0298 | $22 \times 10^{-4}$ |

Table 4: Dataset 4. Comparison of the performances at $x = (3, 1, 0.5, 0.5)$ for Dataset 4. Note that $x \in S_{1,3}^+$ (see Figure 10). Thus the rule (11) of Hu and Øksendal (1998) advises immediate investment with payoff $3 - 1 - 0.5 - 0.5 = 1$. At the confidence level 99.9%, the performance of our rule $x_1^\diamond(x_{-1})$ is 6.13 % greater than the performance of the sufficient condition (10) and 2.98 % greater than the performance of the strictest necessary condition (11).

### 7.3 An example from Nishide (2010)

The following example is borrowed from the earliest draft (Nishide, 2010) of Nishide and Rogers (2011). Nishide (2010) treats the $(2,2)$ exchange

$$\tau^\star(x,\omega) = \sup_{\tau\in\mathcal{S}} \mathbb{E}^x \left[e^{-r\tau}\left(X_1(\tau)+X_2(\tau)-X_3(\tau)-X_4(\tau)\right)\right]. \tag{54}$$

To compare his results with ours we again use the same data.

**Dataset 5:** $r=0.2$, $\mu_1=\mu_2=0.1$, $\mu_3=\mu_4=0.05$ and the variance-covariance matrix of asset price returns per unit time given by

$$\begin{aligned}\frac{1}{dt}\mathrm{Cov}\left(\frac{dX_i}{X_i},\frac{dX_j}{X_j}\right)_{i,j=1,\dots,4} &= \left(\rho_{ij}\sigma_i\sigma_j\right)_{i,j=1,\dots,4}\\ &= \begin{pmatrix} 0.0125 & 0.01 & 0.0075 & 0.0075\\ 0.01 & 0.0125 & 0.0075 & 0.0075\\ 0.0075 & 0.0075 & 0.0214 & 0.0210\\ 0.0075 & 0.0075 & 0.0210 & 0.0214 \end{pmatrix}.\end{aligned}$$

Nishide (2010) constructs this example because of its symmetry property. In fact, the variance covariance matrix is "almost" singular: its eigenvalues being given by the ordered vector $\lambda = (0.0004, 0.0025, 0.0144, 0.0505)$, we compute $\lambda_{\max}/\lambda_{\min} = \lambda_4/\lambda_1 = 126.25$ and $\lambda_4/\lambda_2 = 20.2$. Thus this 4 dimensional problem is in fact close to a problem of dimension 2, and the sufficient condition (12) and the necessary conditions for optimal stopping (13) are close. Indeed, the author finds that the sufficient condition (12) for immediate investment takes the simple form

$$S^-_{2,2} = \left\{x\in\Re^4_+ : (x_1+x_2)\geq \overline{\mu}(x_3+x_4)\right\} \tag{55}$$

with $\overline{\mu}=1.7249$. On the other hand, the necessary condition (13) is written

$$S^+_{2,2} = \left\{x\in\Re^4_+ : (x_1+x_2)\geq \underline{\mu}(x_3+x_4)\right\} \tag{56}$$

with $\underline{\mu} = 1.7003 < \overline{\mu}$ i.e. $S^-_{2,2}\in S^+_{2,2}$.

The "$\mu$" of our method is defined by

$$\mu^\diamond(x_2,x_3,x_4) \triangleq \frac{x_1^\diamond(x_2,x_3,x_4)+x_2}{x_3+x_4}. \tag{57}$$

It is natural to wish to compute $\mu^\diamond(x_{-1})$ for arbitrary values of $x_{-1}=(x_2,x_3,x_4)$, having in mind that if our candidate $x_1^\diamond(\cdot)$ is the optimal stopping rule $x_1^\star(\cdot)$, then for points $x_{-1}$ such that $x_1^\diamond(x_{-1})>0$ one should have $\mu^\diamond(x_{-1})\in[\underline{\mu};\overline{\mu}]$.[15]

Because of the linear homogeneity of $x_1^\diamond(x_{-1})$ one has $\mu^\diamond(x_2,x_3,x_4) = \mu^\diamond(\frac{x_2}{x_3},1,\frac{x_4}{x_3})$ thus it is sufficient to compute $\mu^\diamond(p_1,1,p_2)$ for different values of $p_1$ and $p_2$.[16] Table

[15] Observing (57), note that for $x_3$ and $x_4$ given, when $x_2$ increases the investment threshold $x_1^\diamond(x_{-1})$ decreases. This compensation effect however breaks when $x_2$ is so high compare to $x_3$ and $x_4$ than the problem degenerates: taking $x_2$ arbitrarily high we see that $\mu^\diamond(x_{-1})$ can become arbitrarily high because $x_1^\diamond(x_{-1})$ is bounded below by zero. Thus we in fact want $\mu^\diamond(x_{-1})$ to range between $\underline{\mu}$ and $\overline{\mu}$ in non degenerate problem configurations.

[16] By linear homogeneity of $x_1^\diamond(x_{-1})$, one has

$$\begin{aligned}\mu(x_1,x_2,x_3) &= \frac{x_1^\diamond(x_2,x_3,x_4)+x_2}{x_3+x_4} = \frac{x_3x_1^\diamond\left(\frac{x_2}{x_3},1,\frac{x_4}{x_3}\right)+x_2}{x_3+x_4}\\ &= \frac{x_1^\diamond\left(\frac{x_2}{x_3},1,\frac{x_4}{x_3}\right)+\frac{x_2}{x_3}}{1+\frac{x_4}{x_3}} = \mu^\diamond\left(\frac{x_2}{x_3},1,\frac{x_4}{x_3}\right).\end{aligned}$$

| | | $p_1$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0.001 | 0.1 | 1 | 10 | 100 | $10^5$ |
| | 0.001 | 1.72481 | 1.72011 | 1.70376 | $\star$ | $\star$ | $\star$ |
| | 0.1 | 1.72444 | 1.72013 | 1.70282 | $\star$ | $\star$ | $\star$ |
| $p_2$ | 1 | 1.72367 | 1.72124 | 1.70565 | $\star$ | $\star$ | $\star$ |
| | 10 | 1.72448 | 1.72403 | 1.72013 | 1.70282 | $\star$ | $\star$ |
| | 100 | 1.72485 | 1.72483 | 1.72465 | 1.72289 | 1.70372 | $\star$ |
| | $10^5$ | 1.72487 | 1.72487 | 1.72487 | 1.72487 | 1.72475 | 1.70378 |

Table 5: Dataset 5. $\mu^\diamond(p_1, 1, p_2)$ for different values of $p_1$ and $p_2$. The '$\star$' indicates degenerate configurations i.e. points where $x_1^\diamond(x_{-1}) = 0$. Note that $\mu^\diamond(p_1, 1, p_2) \in [1.7003; 1.7249]$ for non degenerate configurations.

| Rule | Performance | Error (95% C.I.) |
|---|---|---|
| $\underline{\mu}$ | 0.18968 | $30 \times 10^{-5}$ |
| $\overline{\mu}$ | 0.18959 | $30 \times 10^{-5}$ |
| $\mu$ | 0.18966 | $30 \times 10^{-5}$ |

Table 6: Dataset 5. A comparison of the performances for this example. At the confidence level 95%, the estimation error is around $3.10^{-4}$ and we cannot distinguish the performance of the different methods.

5 gives $\mu^\diamond(p_1, 1, p_2)$ for different values of $p_1$ and $p_2$. We see that $\mu^\diamond(p_1, 1, p_2)$ always range between $\underline{\mu}$ and $\overline{\mu}$. Thus so is $\mu^\diamond(x_{-1})$ i.e. we have $S^-_{2,2} \subset S^\diamond_{2,2} \subset S^+_{2,2}$ for points $x_{-1}$ where the problem is not degenerate w.r.t. asset 1. This suggests that the proposed method could be optimal.

In order to compare the performance of the three investment rules, we use 1000000 Monte Carlo simulations to find the performance associated respectively to $\overline{\mu}$, $\underline{\mu}$ and $\mu$. Table 5 shows that there is no significant differences in performance for this number of simulations, even at the confidence interval 95%. This is not surprising as the trigger rules are analytically close.

# 8 Conclusion

In this paper, we use a heuristic principle of smooth fit to find an investment trigger for the multi-asset exchange of geometric Brownian motions. The method consists of parameterizing the $n + m$ assets exchange by $n + m - 2$ exit conditions. Our solution procedure is easier to apply than those currently known to characterize the stopping region. The use of the parametrization reduces the determination of the immediate investment condition to the problem of finding the roots of a quadratic. It is considerably easier to obtain a trigger by this method than to characterize a superset and a subset of the stopping region.

We tested the solution procedure on various examples, most of which were selected from the literature. For those examples, we show that (a) the stopping region we provide might be the true stopping region: it contains the subset of the stopping region characterized by Olsen and Stensland (1992) and is contained by the superset of the stopping region characterized by Hu and Øksendal (1998); (b) its performance is never lower than the performance of sufficient or necessary conditions associated to these sub- or super-sets, in some case it is significantly higher; (c) its performance is particularly better when there is a strong positive correlation between costs, which comforts the intuition than the sufficient condition provided by Olsen and Stensland (1992) (which does not depend on inter-cost correlation) may be too strong in more general settings.

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# A Proof of Proposition 2

*Proof.* Assume that (21) holds for some $\lambda \in \Re^{n+m}$. If (20) holds for some $S$, then $\forall x \in \partial S$ we have

$$\partial_i f(x) = a\lambda_i x_i^{\lambda_i - 1} \prod_{\substack{j=1 \\ j \neq i}}^{n+m} x_j^{\lambda_j} = c_i \qquad \forall i = 1, \ldots, n+m. \tag{58}$$

Thus if $\lambda_i = 0$ (resp. $x_i = 0$) then $\partial_i f(x) = 0$ and it is impossible to have $\partial_i f(x) = c_i \neq 0$. We have proved Prop. 2 a).

We have also proved that $\forall x \in \partial S$, $x_i \neq 0$ for all $i = 1, \ldots, n+m$. Consequently, from (58) we can write

$$a = \frac{c_i}{\lambda_i x_i^{\lambda_i - 1} \prod_{\substack{j=1 \\ j \neq i}}^{n+m} x_j^{\lambda_j}} \qquad \forall i = 1, \ldots, n+m. \tag{59}$$

By taking $i \neq k$, one has

$$\frac{c_i}{\lambda_i x_i^{\lambda_i - 1} \prod_{\substack{j=1 \\ j\neq i}}^{n+m} x_j^{\lambda_j}} = \frac{c_k}{\lambda_k x_k^{\lambda_k - 1} \prod_{\substack{j=1 \\ j\neq k}}^{n+m} x_j^{\lambda_j}}$$

that is

$$\frac{c_i}{\lambda_i x_i^{\lambda_i - 1} x_k^{\lambda_k} \prod_{\substack{j=1 \\ j\neq i,k}}^{n+m} x_j^{\lambda_j}} = \frac{c_k}{\lambda_k x_k^{\lambda_k - 1} x_i^{\lambda_i} \prod_{\substack{j=1 \\ j\neq i,k}}^{n+m} x_j^{\lambda_j}}$$

which is simply

$$\frac{x_i}{x_k} = \frac{c_k}{c_i}\frac{\lambda_i}{\lambda_k} \qquad \forall i,k = 1,\ldots,n+m \quad \text{such that} \quad i \neq k$$

for all $x \in \partial S$. We have proved Prop. 2 c).

If, in addition, (19) holds, we successively have for $x \in \partial S$ :

$$\frac{c_i}{\lambda_i x_i^{\lambda_i - 1}\left(\prod_{\substack{j=1 \\ j\neq i}}^{n+m} x_j^{\lambda_j}\right)} x_i^{\lambda_i} \left(\prod_{\substack{j=1 \\ j\neq i}}^{n+m} x_j^{\lambda_j}\right) = \sum_{j=1}^{n+m} c_j x_j$$

$$\frac{c_i x_i}{\lambda_i} = \sum_{j=1}^{n+m} c_j \left(x_i \frac{\lambda_j}{\lambda_i}\frac{c_i}{c_j}\right)$$

where we have used Prop. 2 c) to transform the RHS. We thus obtain

$$1 = \sum_{j=1}^{n+m} \lambda_j$$

that is Prop. 2 b).

Prop. 2 d) is easily obtained by simplification of (59) using Prop. 2 b) and Prop. 2 c). □

# B Proof of Lemma 1

*Proof.* To avoid a useless complexity of exposition, we prove the statement for $i = 1$ and $j = 2$. For a given $i$, that the statement holds for any $j = 1,\ldots,n$ such that $j \neq i$ is immediate by parametrization invariance. To prove the result for $i \neq 1$, it suffices to permute subscripts of asset 1 and asset $i$ (the problem is invariant to such permutation for $i = 1,\ldots,n$) and apply the same proof. Note that the geometric Brownian motion is absorbed at zero: if $x_1 = 0$, $X_1^{x_1}(t,\omega) = 0$ for all $t$.

($\Leftarrow$) Let $x \in \Re^+_{n+m}$ such that $x_1 = 0$ and $x_2 \geq x_2^\star(x_{-1,-2})$. Because assets 1 starts and stays at zero, the problem degenerates to

$$\sup_{\tau\in\mathcal{S}} \mathbb{E}^x \left[e^{-r\tau}\left(X_1(\tau) + \ldots - X_{n+m}(\tau)\right)\right] = \sup_{\tau\in\mathcal{S}} \mathbb{E}^x \left[e^{-r\tau}\left(X_2(\tau) + \ldots - X_{n+m}(\tau)\right)\right]. \tag{60}$$

On the other side, because $x_2 \geq x_2^\star(x_{-1,-2})$, one has

$$\sup_{\tau\in\mathcal{S}} \mathbb{E}^x \left[e^{-r\tau}\left(X_2(\tau) + \ldots - X_{n+m}(\tau)\right)\right] = x_2 + \ldots - x_{n+m}. \tag{61}$$

Combining (60) and (61), one has

$$\sup_{\tau\in\mathcal{S}} \mathbb{E}^x \left[e^{-r\tau}\left(X_1(\tau) + \ldots - X_{n+m}(\tau)\right)\right] = 0 + x_2 + \ldots - x_{n+m} = g(x) \tag{62}$$

i.e. it is optimal to stop at $x$. We deduce that it is also optimal to stop for any $x$ such that $x_1 > 0$ and $x_2 \geq x_2^\star(x_{-1,-2})$. Thus if $x_2 \geq x_2^\star(x_{-1,-2})$ then $x_1^\star(x_{-1}) = 0$.

($\Rightarrow$) Let $x_{-1} \in \Re_+^{n+m-1}$ such that $x_1^\star(x_{-1}) = 0$. This is equivalent to: $x = (0, x_{-1})$ belongs to the stopping region $S_{n,m}$. Because, asset 1 starts and stays at zero, the problem degenerates following (60). On the other side, because $x \in S_{n,m}$ one has

$$\sup_{\tau\in\mathcal{S}} \mathbb{E}^x \left[e^{-r\tau}\left(X_1(\tau) + \ldots - X_{n+m}(\tau)\right)\right] = 0 + x_2 + \ldots - x_{n+m} \tag{63}$$

Thus, combining (60) and (63) one has

$$\sup_{\tau\in\mathcal{S}} \mathbb{E}^x \left[e^{-r\tau}\left(X_2(\tau) + \ldots - X_{n+m}(\tau)\right)\right] = x_2 + \ldots - x_{n+m}.$$

which is possible if and only if $x_2 \geq x_2^\star(x_{-1,-2})$. □

# C Proof of Proposition 3

*Proof.* From Def. 2 c), we get

$$\lambda_j(x_{-1}) = \left(\frac{c_i\lambda_i}{x_i}\right)\left(\frac{x_j}{c_j}\right) \qquad i,j = 2,\ldots,n+m. \tag{64}$$

We insert (64) in the homogeneity condition Def. 2 b):

$$\sum_{j=1}^{n+m}\lambda_j \;=\; \lambda_1 + \sum_{j=2}^{n+m}\lambda_j = \lambda_1 + \left(\frac{c_i\lambda_i}{x_i}\right)\sum_{j=2}^{n+m}\frac{x_j}{c_j} = 1$$

This leads to

$$\lambda_1 - 1 = \left(\frac{c_i\lambda_i}{x_i}\right)\sum_{j=2}^{n+m}(-c_j)x_j \quad \forall i = 2,\ldots,n+m \tag{65}$$

$\sum_{j=2}^{n+m}(-c_j)x_j$ can be interpreted as a *reduced cost* (it is not related to the notion of reduced cost in linear programming). Thus we define

$$A(x_{-1}) \triangleq \sum_{j=2}^{n+m}(-c_j)x_j. \tag{66}$$

Assume $A(x_{-1}) \neq 0$. From (65) we obtain

$$\lambda_i(x_{-1}) = \frac{(\lambda_1 - 1)c_i x_i}{A(x_{-1})} \qquad i = 2,\ldots,n+m \tag{67}$$

that is, each component of the vector $\lambda_{-1}(x_{-1})$. We see that Def. 2 d) holds if $(\lambda_1 - 1)/A(x_{-1})$ is positive i.e. if

**Condition 1:** when $A(x_{-1}) > 0$, $\lambda_1 > 1$;

**Condition 2:** when $A(x_{-1}) < 0$, $\lambda_1 < 1$.

Using the $n + m - 1$ relations (67), we can now solve for $\lambda_1$ the quadratic equation Def. 2 a). Substituting (67) in the quadratic form (23), we obtain

$$Q_{n,m}(\lambda_1) = (C + D)\,\lambda_1^2 + (\mu_1 - B - 2C - D)\,\lambda_1 + (B + C - r) \tag{68}$$

with

$$\begin{aligned} B &\triangleq -\frac{1}{A}\sum_{j=2}^{n+m} c_j x_j\left(\mu_j - \frac{1}{2}\sigma_j^2\right) \\ C &\triangleq \frac{1}{2A^2}\sum_{i,j\geq 2}^{n+m}\rho_{ij}\sigma_i\sigma_j c_i c_j x_i x_j \\ D &\triangleq \frac{1}{2}\sigma_1^2 + \frac{\sigma_1}{A}\sum_{j=2}^{n+m}\rho_{1j}\sigma_j c_j x_j. \end{aligned}$$

Note that, like $A$, $B$, $C$ and $D$ are only functions of the parameters $x_{-1}$. The roots of (68) are easily obtained

$$\begin{aligned} \Delta &\triangleq (\mu_1 - B - 2C - D)^2 - 4(D + C)(B + C - r) \\ \lambda_1^{\pm}(x_{-1}) &= \frac{-(\mu_1 - B - 2C - D) \pm \sqrt{\Delta}}{2\,(C + D)}. \end{aligned} \tag{69}$$

One still has to chose, between $\lambda_1^+(x_{-1})$ and $\lambda_1^-(x_{-1})$, which root is the appropriate one.

Note that if the problem is not degenerate (i.e. $x_1^\diamond(x_{-1}) > 0$) the trigger is given by Def. 2 i) i.e.

$$x_1^\diamond(x_{-1}) \;=\; c_i\frac{\lambda_1}{\lambda_i}x_i \qquad \forall i = 2,\ldots,n+m$$

which—using (67)—can be re-expressed as

$$x_1^\diamond(x_{-1}) = \left(\frac{\lambda_1(x_{-1})}{\lambda_1(x_{-1}) - 1}\right)A(x_{-1}). \tag{70}$$

Depending on the sign of $A(x_{-1})$, we thus discuss conditions on $\lambda_1$ such that $x_1^\diamond(x_{-1}) > 0$.

**Condition 3:** If $A(x_{-1}) > 0$, $x_1^\diamond(x_{-1}) > 0$ requires $\lambda_1(x_{-1}) > 1$.

**Condition 4:** If $A(x_{-1}) < 0$, $x_1^\diamond(x_{-1}) > 0$ requires $0 < \lambda_1(x_{-1}) < 1$.

Note that Conditions 3 and 4 ensure Conditions 1 and 2 i.e. that Def. 2 d) holds.

It turns out that under the very reasonable assumptions

$$\frac{1}{2}\sigma_j^2 < \mu_j < r \qquad \forall j = 1, \dots, n+m \tag{71}$$

one can prove that $\lambda_1^+(x_{-1}) > 1$ and $\lambda_1^-(x_{-1}) < 1$ for any $x_{-1} \in \Re_+^{n+m-1}$ (see Appendix E.2). Thus, under assumptions (71), Conditions 3 and 4 require to proceed as follows:

1. If $A(x_{-1}) > 0$, set $\lambda_1(x_{-1}) = \lambda_1^+(x_{-1})$ in (70). We have proved Prop. 3 a).
2. If $A(x_{-1}) < 0$ and $0 < \lambda_1^-(x_{-1}) < 1$, set $\lambda_1(x_{-1}) = \lambda_1^-(x_{-1})$ in (70). We have proved Prop. 3 b1).
3. If $A(x_{-1}) < 0$ and $\lambda_1^-(x_{-1}) \le 0$, we still have to set $\lambda_1(x_{-1}) = \lambda_1^-(x_{-1})$ since by choosing $\lambda_1^+(x_{-1})$ we violate Condition 2. Thus, by Def. 2 ii) we have $x_1^\diamond(x_{-1}) = 0$. We have proved Prop. 3 b2) and the proof is complete.

□

# D Proof of Lemma 4

The proof of Lemma 4 relies on the following result (recall that $x_{-1,-2}$ was defined by Eq. 25).

**Lemma 5.** *Assume $n \ge 2$. Let $x \in \Re_+^{n+m}$. The following two statements are equivalent.*

*a)*

$$x_2 = x_2^\diamond(x_{-1,-2}) \tag{72}$$

*where $x_2^\diamond(x_{-1,-2})$ is the candidate trigger for the degenerate problem*

$$\tau^\star(x_{-1}, \omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^{x_{-1}} \left[ e^{-r\tau} \left( X_2(\tau) + \ldots - X_{n+m}(\tau) \right) \right]$$

*which satisfies Definition 2 for a certain vector $\lambda_{-1}(x_{-1,-2}) \in \Re_+^{n+m-1}$.*

*b) The unique vector $\lambda$ which satisfies Definition 2 a) b) c) and d) for the original problem*

$$\tau^\star(x, \omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^{x} \left[ e^{-r\tau} \left( X_1(\tau) + \ldots - X_{n+m}(\tau) \right) \right]$$

*is $\lambda = (0, \lambda_{-1}(x_{-1,-2})) \in \Re^{n+m}$.*

*Proof.* (⇔) The proof goes both ways.

Let $x \in \Re_+^{n+m}$ be a point such that (72) holds. From exclusion of trivial degeneracy ($x_2 \neq 0$), we have $x_2^\diamond(x_{-1,-2}) \neq 0$. Therefore, there exists a unique $\lambda_{-1}(x_{-1,-2}) = (\lambda_2, \dots, \lambda_{n+m}) \in \Re_+^{n+m-1}$ such that Def. 2 a), b), c) and d) hold for $x$ and $\lambda_{-1}(x_{-1,-2})$; and this $\lambda_{-1}(x_{-1,-2})$ is such that $\lambda_2 > 0$ so that we have

$$x_2 = x_2^\diamond(x_{-1,-2}) = c_i \frac{\lambda_2}{\lambda_i} x_i \quad \text{for} \quad i = 3, \dots, n+m.$$

Thus, more simply, there exists a unique $\lambda_{-1}(x_{-1,-2}) = (\lambda_2, \dots, \lambda_{n+m}) \in \Re_+^{n+m-1}$ s.t.

$$Q_{n-1,m}(\lambda) = 0 \tag{73}$$

$$\sum_{i=2}^{n+m} \lambda_i = 1 \tag{74}$$

$$\frac{x_i}{x_{n+m}} = \frac{c_i \lambda_i}{c_{n+m} \lambda_{n+m}} \quad \text{for} \quad i = 2, \dots, n+m \tag{75}$$

with

$$Q_{n-1,m}(\lambda) \triangleq \sum_{i=2}^{n+m} \mu_i \lambda_i + \frac{1}{2} \sum_{i,j=2}^{n+m} \rho_{ij} \sigma_i \sigma_j \left[ \lambda_i (\lambda_i - 1) \delta_{ij} + \lambda_i \lambda_j (1 - \delta_{ij}) \right] - r. \tag{76}$$

Note that

$$Q_{n,m}(\lambda) = Q_{n-1,m}(\lambda) + S(\lambda)$$

with

$$\begin{aligned} S(\lambda) &\triangleq \mu_1\lambda_1 + \frac{1}{2}\cdot 2\sum_{j=1}^{n+m}\rho_{1j}\sigma_1\sigma_j\left[\lambda_1(\lambda_1-1)\delta_{1j}+\lambda_1\lambda_j(1-\delta_{1j})\right] \\ &= \lambda_1\left(\mu_1+\sum_{j=1}^{n+m}\rho_{1j}\sigma_1\sigma_j\left[(\lambda_1-1)\,\delta_{1j}+\lambda_j(1-\delta_{1j})\right]\right). \end{aligned}$$

It is clear from the expression of $S(\lambda)$ that $\lambda_1 = 0$ is a sufficient condition to have $S(\lambda) = 0$. Furthermore $\lambda_1 = 0$ is necessary to have $\lambda_1 + \sum_{i=2}^{n+m}(\lambda_{-1})_i = 1$. Thus it exists a unique $\lambda_{-1}(x_{-1,-2}) \in \Re^{n+m-1}$ such that (74), (75) and (73) holds if and only if

1) it exists a unique $\lambda(x_{-1}) \in \Re^{n+m}$ such that

$$\begin{aligned} \sum_{i=1}^{n+m}\lambda_i &= 1 \\ Q_{n,m}(\lambda) &= 0 \\ \frac{x_i}{x_{n+m}} &= \frac{c_i\lambda_i}{c_{n+m}\lambda_{n+m}} \quad \text{for} \quad i=2,\ldots,n+m. \end{aligned} \tag{77}$$

2) and this $\lambda$ is given by $\lambda = (0, \lambda_{-1}(x_{-1,-2}))$.

□

We can now prove Lemma 4.

*Proof.* For the reasons invoked in the proof of Lemma 1, it is sufficient to prove the statement for $i = 1$ and $j = 2$. Let $x \in \Re_+^{n+m}$ be such that $x_2 = x_2^\diamond(x_{-1,-2})$. By Lemma 5 (see Appendix D), we have $\lambda_1(x_{-1}) = 0$. Thus by Def. 2 ii) we have $x_1^\diamond(x_{-1}) = 0$. That completes the proof. □

# E The quadratic $Q_{n,m}(\lambda_1)$

## E.1 The roots of $Q_{1,0}(\lambda)$

Consider the following quadratic

$$Q(\lambda) = \frac{1}{2}\sigma^2\lambda^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\lambda - \rho \tag{78}$$

with $\rho > 0$. The roots of the quadratic (78) are required to solve our optimal stopping problem in the one dimensional case. Note that we only assumed the discount rate $\rho$ is positive. We do not (for the time being) make hypothesis on the sign of $\mu$. The discriminant of (78) is

$$\Delta = \left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\rho. \tag{79}$$

It is positive since we assumed $\rho > 0$. Consequently the roots of (78) are the two real numbers

$$\lambda^\pm = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) \pm \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}}. \tag{80}$$

If we impose $\mu < \rho$ (this condition ensures that the discounted geometric Brownian motion do not explode) one finds that

$$\sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}} > \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\mu}{\sigma^2}} = \left|\frac{1}{2} + \frac{\mu}{\sigma^2}\right|.$$

Thus

1)

$$\lambda^+ > \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \left|\frac{1}{2} + \frac{\mu}{\sigma^2}\right|$$

- If $0.5 + \mu/\sigma^2 > 0$, $\lambda^+ > 1$.

- If $0.5 + \mu/\sigma^2 < 0$, $\lambda^+ > -2\mu/\sigma^2 > 1$.

2)

$$\lambda^- < \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \left|\frac{1}{2} + \frac{\mu}{\sigma^2}\right|$$

- If $0.5 + \mu/\sigma^2 > 0$, $\lambda^- < -2\mu/\sigma^2 < 1$.
- If $0.5 + \mu/\sigma^2 < 0$, $\lambda^- < 1$.

Thus, we see that, under the assumptions $\rho > 0$ and $\mu < \rho$ (which are reasonable from an economic point of view), $\lambda^+ > 1$ and $\lambda^- < 1$. Note that if we further assume that $\mu > 0$ (which is also reasonable in the one dimensional case, and make impossible the case $0.5+\mu/\sigma^2 < 0$), we find $\lambda^+ > 1$ and $\lambda^- < 0$.

## E.2 The roots of $Q_{n,m}(\lambda_1)$

Using the analysis in Appendix E.1, we now study the roots of (68) which, we recall, is given by

$$Q_{n,m}(\lambda_1) = (C + D)\,\lambda_1^2 + (\mu_1 - B - 2C - D)\,\lambda_1 + (B + C - r) \tag{68'}$$

with

$$\begin{aligned} B &\triangleq -\frac{1}{A}\sum_{j=2}^{n+m} c_j x_j \left(\mu_j - \frac{1}{2}\sigma_j^2\right) \\ C &\triangleq \frac{1}{2A^2}\sum_{i,j\geq 2}^{n+m} \rho_{ij}\sigma_i\sigma_j c_i c_j x_i x_j \\ D &\triangleq \frac{1}{2}\sigma_1^2 + \frac{\sigma_1}{A}\sum_{j=2}^{n+m} \rho_{1j}\sigma_j c_j x_j. \end{aligned}$$

Observe that if $C + D > 0$ then (68') has the form (78) with

$$\begin{aligned} 0.5\sigma^2 &= C + D \\ \mu &= \mu_1 - B - C \\ \rho &= r - B - C \end{aligned}$$

and one can use the conclusions of Appendix E.1. Therefore the three conditions $C+D > 0$, $r-B-C > 0$, and $\mu_1 - B - C < r - B - C$ will guarantee $\lambda_1^+ > 1$ and $\lambda_1^- < 1$. We now find conditions such that these three conditions hold.

1) The third condition $\mu_1 - B - C < r - B - C$ is easy to work out: we need $r > \mu_1$. And since the problem should be solvable using any parametrization, we will require

$$\mu_j < r \quad \forall j = 1, \ldots, n+m \qquad \textbf{(Assumption 1)}. \tag{81}$$

2) Can we find conditions such that $C + D \geq 0$?

$$C + D = \frac{1}{2A^2}\left[\sum_{i,j\geq 2}^{n+m} \rho_{ij}\sigma_i\sigma_j c_i c_j x_i x_j + A^2\sigma_1^2 + 2A\sum_{j=2}^{n+m} \rho_{1j}\sigma_1\sigma_j c_j x_j.\right]$$

Recall that $A(x_{-1}) \triangleq \sum_{i=2}^{n+m}(-c_i)x_i$. Thus $A^2(x_{-1}) = \sum_{i,j=2}^{n+m} c_i c_j x_i x_j \geq 0$ and $C+D \geq 0$ becomes

$$\sum_{i,j\geq 2}^{n+m} \left[\rho_{ij}\sigma_i\sigma_j c_i c_j x_i x_j + \sigma_1^2 c_i c_j x_i x_j - 2\rho_{1j}\sigma_1\sigma_j c_i c_j x_i x_j\right] \geq 0$$

$$\sum_{i,j\geq 2}^{n+m} \left[\rho_{ij}\sigma_i\sigma_j + \sigma_1^2 - 2\rho_{1j}\sigma_1\sigma_j\right] c_i c_j x_i x_j \geq 0 \tag{82}$$

We were not able to prove that the condition (82) holds for any variance covariance matrix $(\rho_{ij}\sigma_i\sigma_j)_{ij}$, but in a large sample of randomly generated example, it was always the case thus we think that (82) always holds.

3) Can we find conditions such that $r - (B + C) > 0$? Note that

$$r = r\frac{\sum_{i,j=2}^{n+m} c_i c_j x_i x_j}{A^2(x_{-1})} = \frac{\sum_{i,j=2}^{n+m} 2rc_i c_j x_i x_j}{2A^2(x_{-1})}$$

so that

$$\begin{aligned} r - C - B &= \frac{1}{2A^2}\left[\sum_{i,j\geq 2}^{n+m} (2r - \rho_{ij}\sigma_i\sigma_j)\, c_i c_j x_i x_j + 2A\sum_{j=2}^{n+m} c_j x_j \left(\mu_j - \frac{1}{2}\sigma_j^2\right)\right] \\ &= \frac{1}{2A^2}\left[\sum_{i,j\geq 2}^{n+m} (2r - \rho_{ij}\sigma_i\sigma_j)\, c_i c_j x_i x_j + 2\sum_{i,j=2}^{n+m} \left(\mu_j - \frac{1}{2}\sigma_j^2\right) c_i c_j x_i x_j\right] \\ &= \frac{1}{2A^2}\sum_{i,j\geq 2}^{n+m} \left(2(r + \mu_j) - \rho_{ij}\sigma_i\sigma_j - \sigma_j^2\right) c_i c_j x_i x_j. \end{aligned}$$

Thus the conditions

$$r + \mu_j > \frac{1}{2}\left(\sigma_j^2 + \rho_{ij}\sigma_i\sigma_j\right) \qquad \forall i, j = 1, \ldots, n+m \tag{83}$$

guarantee $r - C - B > 0$. Note that we previously required $r > \mu_j \quad \forall j = 1, \ldots, n+m$ (Eq. 81). By further assuming

$$\mu_j > \frac{1}{2}\sigma_j^2 \quad \forall j = 1, \ldots, n+m \qquad \textbf{(Assumption 2)} \tag{84}$$

we have

$$\forall i, j \qquad \frac{1}{2}(\sigma_j^2 + \rho_{ij}\sigma_i\sigma_j) \leq \frac{1}{2}\sigma_j^2 + \frac{1}{2}\sigma_{\max}^2 < \mu_j + r$$

i.e. (83) holds.

Summing up Assumptions 1 and 2, under the conditions

$$\frac{1}{2}\sigma_j^2 < \mu_j < r \qquad \forall j = 1, \ldots, n+m$$

we have $\lambda_1^+ > 1$ and $\lambda_1^- < 1$.

# F Monte Carlo

Random processes were generated starting from the SDE (1) in normal form. Recall that $X_t, \mu \in \Re^{n+m}$. Suppose $\Sigma \in \Re^{(n+m)\times u}$ and $B : \Omega \to \Re^u$ is the $u$-dimensional Brownian motion. (1) is equivalent to

$$dX_t = \mu \cdot \mathrm{diag}(X_t) \cdot dt + \mathrm{diag}(X_t) \cdot \Sigma \cdot dB_t$$

($\mathrm{diag}(X_t) \in \Re^{(n+m)\times(n+m)}$ is the diagonal matrix of diagonal $X_t$) under the condition $\Sigma\Sigma^T = (\rho_{ij}\sigma_i\sigma_j)_{i,j=1,\ldots,n+m}$ that is, if $\Sigma\Sigma^T$ is the variance-covariance matrix of the formulation (1).

To compute the performances, random processes were generated using

$$X_{t+\Delta t} = \mu \cdot \mathrm{diag}(X_t) \cdot \Delta t + \mathrm{diag}(X_t) \cdot \Sigma \cdot \sqrt{\Delta t} \cdot Z$$

with $Z$ a $(n+m)$-dimensional normal random variable, $\Delta t = 0.001$ and a time horizon of $T = 50$.

# G On generating the cone of necessary conditions

The cone of necessary conditions were generated using a wide sample of different geometric Brownian motions.

Consider Dataset 1 in its normal form:

$$B_t \in \Re^2, \quad \mu = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \\ 0 & 0 \end{bmatrix}.$$

Take a three dimensional *normal* random variable $Z : \Omega_z \to \Re^3$. An additional geometric Brownian motion $X_u$ is incorporated in the previous setting by considering the new problem configuration:

$$B_t \in \Re^2, \quad X = \begin{bmatrix} X \\ X_u \end{bmatrix} \quad \mu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Z(1) \end{bmatrix} \quad \Sigma = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \\ 0 & 0 \\ Z(2) & Z(3) \end{bmatrix}.$$

By generating a particular realization $z$ of $Z$, one obtains the particular SDE (see Appendix F)

$$dX_t = d\begin{bmatrix} X \\ X_u \end{bmatrix} = \mu(z) \cdot \mathrm{diag}(X_t) \cdot dt + \mathrm{diag}(X_t) \cdot \Sigma(z) \cdot dB_t.$$

which leads to a variance-covariance matrix $\Sigma\Sigma^T(z)$ and to a superset $S^+_{n,m}(z)$ given by (13). Generating a large sample $\Omega_{Z,N} = \{z_1, \ldots, z_N\}$ of realizations of $Z$ leads to a collection $\{S^+_{n,m}(z_i)\}_{i=1,\ldots,N}$ from which one can isolate the smallest and the larger set.

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