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Abstract

Some behaviors, ideas or technologies spread and become persistent in society, whereas others vanish. This paper analyzes the role of social influence in determining such distinct collective outcomes. Agents are assumed to acquire information from others through a certain sampling process that generates an *influence network*, and they use simple rules to decide whether to adopt or not depending on the observed sample. We characterize, as a function of the primitives of the model, the *diffusion threshold* (i.e., the spreading rate above which the adoption of the new behavior becomes persistent in the population) and the *endemic state* (i.e., the fraction of adopters in the stationary state of the dynamics). We find that the new behavior will easily spread in the population if there is a high correlation between how influential (visible) and how easily influenced an agent is, which is determined by the sampling process and the adoption rule. We also analyze how the density and variance of the out-degree distribution affect the diffusion threshold and the endemic state.

Keywords: social influence, networks, diffusion threshold, endemic state.

JEL Classification: C73, L14, O31, O33

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1 Introduction

The proliferation of internet-based communication and interactivity over the past decade has led to new consumer patterns, innovative marketing approaches, and even unconventional ways of running political campaigns (e.g., Godes and Mayzlin, 2004, Salganick et al., 2006, Willimas and Gulati, 2008). A central assumption underlying these new strategies is that individuals influence each other when making decisions. Other relevant social phenomena such as crime activities, religious fundamentalism, cultural fads, life-style habits, or even epidemics also share this logic (e.g., Aguirre et al., 1988, Glaeser et al., 1996, Young and Burke, 2001). As a result we have witnessed the arousal of a tremendous interest in the study of social networks, leading some to herald the arrival of a "science of networks" (e.g., Watts, 2007).

This paper analyzes how social influence determines the spread of new behaviors in an interconnected society, a question that lies at the foundations of the theory of networks (see, e.g., Goyal, 2007, Vega-Redondo, 2007, Jackson, 2008). A distinctive feature of this work will be to consider a reiterative sampling process, thus leading to an evolving influence network, rather than assuming a fixed network of interactions, as it happens in most of the related literature. In doing so, we shall aim to develop a tractable theoretical model that could help the testing of specific predictions.[1]

As people may differ in the information they posses regarding the behavior of others, we introduce heterogeneity in our model by assigning to each agent an out-degree (or information level) indicating the number of agents observed by them before making a decision. We define a dynamic process in which agents repeatedly sample from the population a subset of agents, observe their choices regarding the new behavior, and decide whether or not to adopt. The influence network so determined specifies who is influenced by whom at different time periods. We make two crucial assumptions with respect to the sampling process. On one hand, it is assumed that sampling is directional, i.e., agent i sampling agent j does not necessarily imply that j samples i.[2] On the other hand, some agents are sampled more often than others (i.e., are more "visible") and this is related to their out-degree in a way specified by the sampling process. More precisely, apart

[1] Many papers dealing with diffusion on fixed networks (although randomly generated) are theoretically intractable and thus rely on extensive simulations studies or mean-field approximations of the models (e.g., Pastor-Satorrás and Vespignani, 2001,Watts, 2002, Watts and Dodds, 2007, Jackson and Rogers, 2007, and López-Pintado, 2008a).

[2] Internet plays a crucial role in generating such directed influence structures (e.g., individuals with popular websites or blogs who are observed by many others but do not necessarily observe many others).

from out-degree, agents are characterized by their in-degree, that is, the number of agents sampling them. The correlation between in-degree (or visibility level) and out-degree (information level) is determined by the sampling process as described below.

The family of sampling processes considered encompasses a wide variety of options characterized by a parameter $\alpha \in [0, 1]$, which determines the correlation existing between the out-degree and the in-degree. For ease of exposition, two polar (and extreme) cases are singled out. First, the case in which all agents are equally visible ($\alpha = 0$). Here, agents sample *uniformly* from the population according to their variable out-degrees. Second, the situation in which an agent's in-degree is perfectly aligned with her out-degree ($\alpha = 1$). In other words, an agent with out-degree k is k times more visible than an agent with out-degree 1. When $\alpha = 1$ the model essentially coincides with the mean-field approximation of an (undirected) random network model (e.g., Pator-Satorrás and Vespignani, 2001, Jackson and Rogers, 2007 and López-Pintado, 2008a). The current work, thus, helps understand the extent and nature of such approximations. The case $\alpha = 0$ resembles the model introduced by Galeotti and Goyal (2009). These authors also use a directed sampling process to analyze the optimal targeting strategy of a firm who wants to introduce a new product in a population anticipating the effect of word of mouth.

In our model, agents use simple rules to decide whether or not to adopt the new behavior. The probability of adopting depends exclusively on the number of adopters and non-adopters in an agent's sample, and not on who specifically has adopted. Apart from this simplification, the class of rules analyzed here is quite general and expands models described in previous work. For instance, in the susceptible-infected-susceptible (SIS) model, enunciated by epidemiologists to analyze the spread of a disease in a population (e.g., Bailey, 1975), a susceptible agent becomes infected at a constant rate from each interaction with an infected agent, whereas the transition from infected to susceptible depends on an exogenous rate of recovery. As a result, the adoption rule exclusively depends on the absolute number of infected interactions. We extend the SIS model to allow for more general adoption rules (e.g., rules that depend on the relative number of adopters) capturing features in the process of adoption that might not be relevant for the diffusion of a disease, but that seem fundamental for diffusion of behavior or information (see also López-Pintado, 2008a).[3]

[3] Alternatively, several authors have addressed the issue of strategic interactions and networks incorporating incomplete information and characterizing the Nash-Bayes equilibrium of the resulting network game. These models assume that agents know their own degree and the degree distribution of the population, but have incomplete information about the precise structure of the social network in which they

In this paper we present an evolving influence network model and analyze the long-run state of the adoption dynamics. We characterize the diffusion threshold (i.e., the value for the spreading rate of the new behavior above which adoption by a significant fraction of the population occurs) providing its closed-form solution. We also (implicitly) characterize the endemic fraction of adopters (i.e., the fraction of adopters in the stationary state of the dynamics) and perform a comparative static analysis. Roughly speaking, we find that the new behavior will easily spread in the population if there is a high correlation between how influential (visible) and how easily influenced an agent is, which is determined by the sampling process and the adoption rule. We also analyze how the density and variance of the out-degree distribution affect the diffusion threshold and the endemic state. To this end, we mostly focus on the extreme case where $\alpha = 0$ and compare the performance of populations characterized by out-degree distributions ordered according to First Order Stochastic Dominance and Mean Preserving Spread.

The paper is organized as follows. Section 2 introduces the model. Section 3 presents the results of the paper, whereas Section 4 concludes. For a smooth passage we defer all the proofs to the Appendix.

2 The Model

2.1 The Influence Network

There is a unit measure of agents $N = [0, 1]$. Each agent $i \in N$ is characterized by her out-degree k_i which determines the number of agents whose behavior i observes (and hence is influenced by). Some agents have more access to information than others or simply wish to make a more informed decision. Therefore, we assume that the population is characterized by an out-degree distribution denoted by $P(k)$.[4] Let us define the in-degree

are embedded As in this paper, the results crucially depend on the degree distribution (see e.g., Jackson and Yariv, 2007 for a dynamic approach and Galeotti et al. 2010 for a static approach). Young (2009) also analyzes diffusion of behavior in a population but, unlike what we do here, he builds on the literature initiated by Granovetter (1978) and studies the case where agents are heterogenous with respect to the adoption rule but homogeneous with respect to their (out-) degree. Moreover, instead of concentrating on the stationary state of the dynamics as we do in this paper, he focuses on the evolution over time of the fraction of adopters.

[4]For simplicity in some of the proofs, let us assume that $P(k)$ has a finite support and that the degree of agents is at least 3. More precisely, $P(k) = 0$ if either $k \leq 2$ or $k \geq K$, where K is a finite upperbound of degrees.

(or visibility) of an agent as the number of agents sampling this agent. The model takes as primitives the out-degree distribution and a certain sampling process which determines the in-degree. The family of sampling processes considered allows for a wide variety of options, each associated to a parameter $\alpha \in [0,1]$. Formally, the *α-sampling process* indicates that an agent with out-degree k is sampled with probability

$$\frac{k^\alpha P(k)}{\sum k^\alpha P(k)}.$$

Note that, if $\alpha = 0$, this probability becomes $P(k)$. In this case, agents are selected completely at random and thus the probability of observing an agent with out-degree k is simply the fraction of agents with such an out-degree. We refer to this situation as the *homogeneous-visibility* case, since in this context all agents would have the same in-degree. [5] If, on the other hand, $\alpha = 1$ then agents with out-degree equal to k also have in-degree equal to k. We refer to this situation, in which the visibility level is perfectly aligned with information level as the *information-visibility* case. [6]

We can then define an *influence network* as a result of combining an out-degree distribution and a sampling process. Formally, the *P_α-influence network* is the network obtained when the α-sampling process described above is imposed to a population with out-degree distribution $P(k)$.

2.2 The Adoption Rule

Assume the existence of a new behavior (or product) spreading in a population over time. In a given period t, agents can either be active or passive with respect to this behavior. Let i be a passive agent with out-degree k_i. Assume that at a spreading rate $\nu \geq 0$ an

[5] Galeotti and Goyal (2009) analyze a similar framework although with significant differences. First of all, they focus on the targeting strategy of a firm, which has incomplete information about the network structure, and thus can only rely on the out-degree distribution to estimate the returns associated with each possible strategy. In our model there is no explicit firm and therefore no intentional targeting strategy (although one could easily include similar features). Second, whereas we analyze the whole process of information transmission, Galeotti and Goyal (2009) mostly focus on the case where information only spreads over two periods. Finally, we assume that becoming an adopter is a reversible decision (see the description of the adoption rule in Section 2.2), whereas in Galeotti and Goyal (2009) the decision is irreversible.

[6] The case $\alpha = 1$ is such that out-degrees coincide with in-degrees, however, this does not imply that agents observe each other as it occurs in an undirected network framework. This is an additional assumption implicit in a mean-field approximation approach commonly used in the epidemiology literature which had not yet been pointed out.

agent considers the possibility of adopting the new behavior. To make a decision she samples k_i agents following the sampling process defined above. Assume there are a_i active agents sampled by i at t.[7] The rate of adoption of i is given by $f_{k_i}(a_i)$, where $f_{k_i}(\cdot)$ is what we define as the adoption rule.[8] Formally, an *adoption rule* is a function $f_{k_i} : [0, 1, 2...k_i] \to \mathbb{R}_+$ satisfying two conditions:

(1) f_{k_i} is non-decreasing

(2) $f_{k_i}(0) = 0$

Condition (1) implies that the rate of adoption increases with the number of adopters.[9] Condition (2) implies that in order to adopt one needs to sample at least one agent who has already done so.

We assume that an active agent becomes passive again at some constant rate $\delta > 0$, which is independent of the behavior of others. Let us define the *effective spreading rate* by $\lambda = \frac{\nu}{\delta}$ which will be one of the crucial parameters of the model. Note that the higher the value of λ the more contagious the behavior is.

A plausible interpretation for the transition from passive to active is the following. At an exogenous rate ν any given agent i becomes interested in adopting the behavior or product (e.g., due to the objective quality of the product, or the presence of mass media advertisements). The agent's final decision, however, depends critically on the influence exerted by the agents in her sample characterized by the adoption rule $f_{k_i}(a_i)$. We can assume that the product is not indefinitely durable and it becomes obsolete at a certain rate δ.[10]

Two types of adoption rules are singled out:

(1) *Viral rules.* These adoption rules depend exclusively on the absolute number of adopters, i.e., $f_k(a) = f_{k'}(a)$ for all k and k'. The so-called SIS model of diffusion studied in epidemiology (e.g., Pastor-Satorrás and Vespignani, 2001) simply corresponds to a viral

[7] For ease of notation we avoid now the subscript t that will be included later once the dynamics is specified.

[8] We define rates instead of probabilities because we consider a continuous time dynamics. The intuition should be that in a small increment of time dt, the probability of adopting the product is $\nu f_{k_i}(a_i)dt$.

[9] In doing so, we are implicitly assuming the existence of incentives for coordination on the same action. The opposite phenomenon, i.e., the existence of incentives to "anticoordinate" has also been analyzed elsewhere (e.g., Bramoullé and Kranton, 2007, López-Pintado, 2008b).

[10] Alternatively, we could have assumed that the transition from active to passive also depends on the behavior of others. This assumption has been considered in related models of diffusion where, unlike what has been assumed here, an agent's choice in a certain period does not depend on whether the agent is currently active or passive, but exclusively on the behavior of neighbors (e.g., López-Pintado, 2006, 2008b, Watts, 2002 and Jackson and Yariv, 2006).

rule where adoption depends linearly on the number of infected agents in the sample, i.e., $f_k(a) = a$.

(2) *Persuasive rules.* These adoption rules depend on the relative number of adopters and thus $f_k(a)$ can actually be reinterpreted as a function of $\frac{a}{k}$. These rules represent situations where there is some persuasion in favor and against adoption by adopters and non-adopters, respectively. A stylized case which lies in this category is the *Imitation rule,* where an agent simply chooses randomly one of her sampled agents and imitates her behavior. In such a case $f_k(a) = \frac{a}{k}$.

2.3 The Adoption Dynamics and the Stationary States

Let $\rho_k(t)$ denote the frequency of active agents among those with out-degree k at time t. Thus, $\rho(t) = \sum_k P(k)\rho_k(t)$ is the total frequency of active agents in the population at time t. The adoption dynamics is then described as follows:

$$\frac{d\rho_k(t)}{dt} = -\rho_k(t)rate_k^{1\to 0}(t) + (1-\rho_k(t))rate_k^{0\to 1}(t),$$

where $rate_k^{0\to 1}(t)$ is the rate at which a passive agent with out-degree k becomes active and $rate_k^{1\to 0}(t)$ stands for the reverse transition. As mentioned above $rate_k^{1\to 0}(t) = \delta$. As for $rate_k^{0\to 1}(t)$ we need a piece of additional notation. Let $\theta(t)$ be the probability that a sampled agent is active. Given the sampling process described above we find that

$$\theta(t) = \frac{1}{\langle k^\alpha\rangle}\sum_k k^\alpha P(k)\rho_k(t) \tag{1}$$

where, for simplicity, we denote $\langle k^\alpha\rangle = \sum_k k^\alpha P(k)$. It follows from here that

$$rate_k^{0\to 1}(t) = \sum_{a=0}^{k} \nu f_k(a)\binom{k}{a}\theta(t)^a(1-\theta(t))^{(k-a)}.$$

Let $r_k(\theta(t)) = \sum_{a=0}^{k} f_k(a)\binom{k}{a}\theta(t)^a(1-\theta(t))^{(k-a)}$, then the dynamics can be rewritten as

$$\frac{d\rho_k(t)}{dt} = -\rho_k(t)\delta + (1-\rho_k(t))\nu r_k(\theta).$$

In a stationary state $\frac{d\rho_k(t)}{dt} = 0$ and therefore

$$\rho_k = \frac{\lambda r_k(\theta)}{1+\lambda r_k(\theta)}. \tag{2}$$

Combining (1) and (2) we obtain the following fixed-point equation whose solutions correspond to the stationary values of θ (denoted by θ^*)

$$\theta = H_\lambda(\theta), \tag{3}$$

where

$$H_\lambda(\theta) = \frac{1}{\langle k^\alpha \rangle} \sum_k k^\alpha P(k) \frac{\lambda r_k(\theta)}{1+\lambda r_k(\theta)}. \tag{4}$$

The frequency of adopters in the stationary state (ρ^*) is subsequently determined by

$$\rho^* = \sum_k P(k) \frac{\lambda r_k(\theta^*)}{1+\lambda r_k(\theta^*)}. \tag{5}$$

Recall that the transition from active to passive is always possible. Therefore, the concept of a stationary state only refers to stationary values of ρ and θ and not to the identities of the agents choosing each action.

3 Results

In this section, we determine the threshold for the effective spreading rate above which diffusion to a positive fraction of the population occurs. Formally, let A_λ be the set of effective spreading rates for which an infinitely small fraction of initial active agents spreads the behavior to a positive fraction of the population. In other words, λ belongs to A_λ if a *finite* number of of initial adopters can spread the behavior to an *infinite* number of agents. Then, we define the *diffusion threshold* λ^*as the highest lower bound of such a set, i.e., $\lambda^* = \inf A_\lambda$.[11]

The following lemma, which is interesting on its own, will be used to characterize the diffusion threshold.

Lemma 1 *The expected in-degree of an agent with out-degree k in a P_α-influence network is given by $\frac{k^\alpha}{\langle k^\alpha \rangle}\langle k \rangle$.*

This result formally establishes the relationship between the number of agents an agent is influenced by (out-degree) and the number of agents influenced by this agent (in-degree). This lemma shows, in particular, that if $\alpha = 0$ (homogeneous-visibility case) all agents are (in expected terms) equally influential (or visible) and, thus, the expected number of individuals influenced by any given agent is $\langle k \rangle$. If $\alpha = 1$ (information-visibility

[11]Note that if $A_\lambda = \varnothing$ then $\lambda^* = \infty$

case), however, the expected number of individuals influenced by an individual coincides with her out-degree. Finally, if α lies somewhere in between 0 and 1, there exists a positive correlation between in and out-degree, but this correlation is not perfect. [12]

The main result of this section comes next.

Theorem 1 *Given a P_α-influence network and an adoption rule f, the diffusion threshold is given by*

$$\lambda^* = \frac{\langle k^\alpha \rangle}{\sum_k k^{\alpha+1} P(k) f_k(1)}$$

Note that the diffusion threshold depends on the adoption rule through $f_k(1)$ (instead of $f_k(a)$) because in the initial stages of the dynamics, the probability of sampling more than one active agent is insignificant in comparison with sampling just one active agent. In particular, for the SIS adoption rule the diffusion threshold is

$$\lambda^* = \frac{\langle k^\alpha \rangle}{\langle k^{\alpha+1} \rangle}$$

which depends on the out-degree distribution $P(k)$ and the sampling process characterized through α. Nevertheless, for other adoption rules such as the Imitation rule the diffusion threshold is

$$\lambda^* = 1$$

which is independent of the $P_\alpha-$influence network. Note that the diffusion threshold crucially depends on the adoption rule specified by the model and therefore testing which rules match best which applications is an important empirical question.

Beyond the diffusion threshold, we also analyze the endemic state of the dynamics. To fix ideas, we say that the adoption dynamics has reached an *endemic state* with a fraction of adopters ρ^* if this fraction of adopters remains constant in the upcoming periods. In particular, ρ^* is obtained as the solution of the system of equations (3) and (5) derived before. The next result provides a necessary condition over the adoption rule for which the endemic fraction of adopters is unique.[13]

[12] One could easily extend the results to other values of α. For instance, if we allow for $\alpha > 1$ the in-degree distribution would be more skewed than the out-degree distribution, making high out-degree agents have an even higher in-degree. If, instead, $\alpha < 0$ the correlation between out-degree and in-degree would be negative (e.g. geniuses who listen to no one but that everybody listens to). For simplicity, we have decided to concentrate on the values ranging between the two focal points of $\alpha = 0$ and $\alpha = 1$.

[13] This result is a generalization of Proposition 1 in López-Pintado (2008a).

Theorem 2 *Consider a $P_\alpha-$ influence network and an adoption rule $f_k(a)$ (weakly) concave with respect to a, for all k. If $\lambda > \lambda^*$ there exists a unique positive endemic fraction of adopters. Otherwise the unique endemic state is such that $\rho^* = 0$. Moreover, at $\lambda = \lambda^*$ there exists a first order phase transition.*

The skeleton of the proof is the following. Algebraic computations allow showing that if for all k the adoption rule f_k is (weakly) concave with respect to a then $H_\lambda(\theta)$ is an increasing and a concave function of θ, where $H_\lambda(0) = 0$. Therefore, the fixed point equation $\theta = H_\lambda(\theta)$ has either no positive solution (when $H'_\lambda(0) \leq 1$) or just one positive solution (when $H'_\lambda(0) > 1$). The value of the spreading rate λ separating these two cases is obtained from the equation $H'_\lambda(0) = 1$. As expected, the threshold value for λ obtained here coincides with the diffusion threshold λ^* provided in Theorem 1. Due to the continuity of $H_\lambda(\theta)$ as a function if λ, it is also straightforward to show that the transition from a zero to a positive fraction of adopters occurs smoothly and thus $\rho^*(\lambda)$ converges to 0 when $\lambda \to \lambda^*$ (see Figure 1). This continuous transition is what we refer to as a first order phase transition.

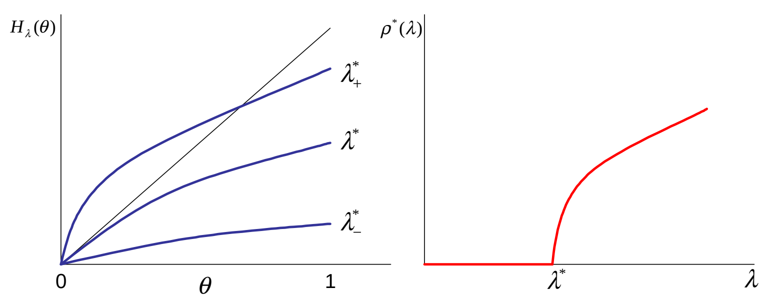

Figure 1: The graph in the left hand side represents $H_\lambda(\theta)$ for a (weakly) concave adoption rule when (i) λ equals the diffusion threshold λ^* (ii) λ is above the diffusion threshold ($\lambda = \lambda^*_+$) and (iii) λ is below the diffusion threshold ($\lambda = \lambda^*_-$). The graph in the right hand side represents the corresponding fraction of adopters in the endemic state ρ^* as a function of λ, highlighting the first order phase transition occurring at $\lambda = \lambda^*$.

There are many adoption rules satisfying the concavity assumption at the statement of Theorem 2 (e.g., the SIS and Imitation rules). Other relevant rules (e.g., $f_k(a) = a^2$ or $f_k(a) = (\frac{a}{k})^2$) do not. A persuasive rule that also violates the assumption is the deterministic threshold rule satisfying that agents adopt with probability 1 if and only if the fraction of adopters in the sample ($\frac{a}{k}$) is above a certain threshold (see e.g., Morris, 2000, Watts, 2002, López-Pintado, 2006, López-Pintado and Watts, 2008 and Young, 2009 for papers where the deterministic threshold rule, or a slightly modified version of it, has been analyzed). In general, non-concave rules can exhibit multiple endemic states with different corresponding fraction of adopters. Moreover, continuity of $\rho^*(\lambda)$ at $\lambda = \lambda^*$ is not guaranteed.

3.1 The Role of the Sampling Process (α)

One of the main objectives of this paper is to understand how diffusion depends on the correlation between information and visibility. For this purpose, the next result takes as given a certain out-degree distribution $P(k)$ and adoption rule f_k, for every k, and analyzes how the diffusion threshold depends on the sampling process, characterized through the parameter α.

Proposition 1 *Given a P_α-influence network and an adoption rule f_k, for every k, the following statements hold:*

(i) If $kf_k(1)$ is increasing with respect to k the diffusion threshold decreases with α.

(ii) If $kf_k(1)$ is decreasing with respect to k the diffusion threshold increases with α.

(iii) If $kf_k(1)$ is constant with respect to k, the diffusion threshold does not depend on α.

The distinction between a *visible* (or influential) agent and an *easily influenced* agent is crucial for understanding the proposition. Note that, it is obviously the case that diffusion will be enhanced whenever influential agents are also easily influenced. The first simply refers to agents that are sampled by many others (i.e., have high in-degrees), whereas the second refers to agents that are early adopters of the dynamics. Whether an agent is or not an early adopter depends on two features. On one hand, the out-degree k (i.e., how many agents somebody observes) determines the chances of finding an adopter. On the other hand, the adoption rule f_k specifies the probability of becoming an adopter given the composition of the sample. As Proposition 1 suggests $kf_k(1)$ (a joint measure of both features) is roughly the rate at which an agent with degree k adopts in the initial (and crucial for determining future success) stages of the adoption dynamics. Proposition 1 does not characterize all possible adoption rules but points out the existence of two distinctive types of rules: the ones where early adopters are agents with high out-degrees, and the ones where early adopters are agents with low out-degree (cases (i) and (ii) respectively in the proposition). If high out-degree agents adopt early on, then diffusion is helped if these agents are also influential which occurs precisely for higher values of α. On the contrary, if high out-degree agents adopt later on, then diffusion is helped if these agents are less influential (i.e., for low values of α). Finally, case (iii) in the proposition corresponds to rules in which adopting early on or not is independent of the out-degree and consequently independent on α.

There are examples of adoption rules in each of the cases established by Proposition 1. For instance, all viral adoption rule satisfy (i). Some persuasive adoption rules, however, as for example the rule $f_k(a) = (\frac{a}{k})^2$, satisfy (ii), whereas, other persuasive rules, such as the Imitation rule (i.e., $f_k(a) = \frac{a}{k}$), satisfy (iii).[14]

To further investigate the effect of the sampling process on diffusion, the next result assumes a certain out-degree distribution $P(k)$ and a concave adoption rule f_k, for all k, and analyzes how the endemic fraction of adopters depends on α. Note that, the (unique) endemic fraction of adopters is $\rho^{*\cdot} = 0$ for values of the spreading rate below the diffusion threshold, and it is the unique positive solution ρ^* of the system of equations determined by (3) and (5) whenever the spreading rate is above the diffusion threshold.

Proposition 2 *Given a $P_\alpha-$ influence network and an adoption rule $f_k(a)$ (weakly) concave with respect to a, for every k, the following statements hold:*

(i) If, for all $\theta \in [0,1]$, $r_k(\theta)$ is increasing with respect to k, the endemic fraction of adopters ρ^* increases with respect to α.

(ii) If, for all $\theta \in [0,1]$, $r_k(\theta)$ is decreasing with respect to k for all $\theta \in [0,1]$, the endemic fraction of adopters ρ^* decreases with respect to α.

(iii) If, for all $\theta \in [0,1]$, $r_k(\theta)$ is constant with respect to k for all $\theta \in [0,1]$, the endemic fraction of adopters ρ^* does not depend on α.

Recall that $r_k(\theta)$ is the rate at which an individual with out-degree k adopts as a function of θ. For values of θ infinitely small $r_k(\theta)$ can be approximated by $kf_k(1)$, which is the relevant measure used in the computation of the diffusion threshold and the results obtained in Proposition 1. Regarding the endemic fraction of adopters, conditions on $r_k(\theta)$ must hold for all values of θ which leads to the above result.

As a consequence of Proposition 2 one finds that all (concave) viral adoption rules satisfy (i) in the proposition and thus, the fraction of adopters in the endemic state

[14] An illustration of how the adoption rule $f_k(a) = (\frac{a}{k})^2$ could be derived is the following. Assume agents obtain utility 0 if they decide not to adopt the new behavior and a utility of $u(a) = (\frac{a}{k})^2 - c$ if they decide to adopt, where $\frac{a}{k}$ is the fraction of adopters in the sample and c is the cost of adopting. Assume also that c is uniformly distributed $c \sim U\ [0,1]$. The probability of adopting would then be $Pr(c \leq (\frac{a}{k})^2) = (\frac{a}{k})^2$.

increases with the (positive) correlation between out-degree and visibility.[15] The Imitation rule, however, satisfies (iii) and thus, the fraction of adopters in the endemic state is independent of α. Indeed, for such a case, it is straightforward to show that $\rho^* = 0$ if $\lambda \leq 1$ and $\rho^* = 1 - \frac{1}{\lambda}$ otherwise.

3.2 The Role of the out-degree Distribution ($P(k)$)

Another aspect of the model that has been the focus of most of the related literature is the effect on the diffusion outcomes of variations in the out-degree distribution. Does having more information about the behavior of others help or harm diffusion? Does heterogeneity favor diffusion? These questions are partially answered in the following section. To do so, we take as given the α-sampling process and compare populations with different out-degree distributions. We denote by $\lambda^*(P_\alpha)$ the diffusion threshold obtained for a certain $P_\alpha-$influence network.

Proposition 3 *Given two influence networks $\widetilde{P_\alpha}$ and P_α and an adoption rule f_k, for every k, the following statements hold:*

(i) If $\widetilde{P}(k)$ First Order Stochastic Dominates $P(k)$ and $k^{\alpha+1}f_k(1)$ is decreasing with respect to k then $\lambda^*(P_\alpha) \leq \lambda^*(\widetilde{P_\alpha})$

(ii) If $\widetilde{P}(k)$ is a Mean Preserving Spread of $P(k)$ and $k^{\alpha+1}f_k(1)$ is convex with respect to k then $\lambda^*(\widetilde{P_\alpha}) \leq \lambda^*(P_\alpha)$

The first part of Proposition 3 suggests, contrary to the basic intuition, that for certain adoption rules the lower the density of the influence network the easier it is to spread the behavior in the population. Note that the result applies to some convex adoption rules such as $f_k(a) = (\frac{a}{k})^2$ for which early adopters coincide with low out-degree agents, whereas all viral rules, as well as other persuasive rules (e.g., the Imitation rule), are not contemplated in this result. The counterpart of (i) where $\lambda^*(P_\alpha) > \lambda^*(\widetilde{P_\alpha})$, although more intuitive, is not straightforward to show. Indeed Jackson and Rogers (2007) concentrate on the SIS model and find support for such inequality if in addition to $\widetilde{P}(k)$ FOSD $P(k)$ it also holds that $\frac{1}{\langle k \rangle_{\widetilde{P}}} k\widetilde{P}(k)$ FOSD $\frac{1}{\langle k \rangle_P} kP(k)$.

As for the second part of the proposition, note that, there are many adoption rules that satisfy the condition provided therein . In particular, all viral adoption rules, as well as a large number of persuasive rules (including the Imitation rule, among others) satisfy

[15] Note that, for viral adoption rules $r_k(\rho) = \sum_{a=0}^{k} f(a) \binom{k}{a} \rho^a (1-\rho)^{(k-a)}$ is increasing as a function of k since $f(a)$ is an increasing function of a.

the convexity of $k^{\alpha+1} f_k(1)$. Here, we compare influence networks with the same average out-degree but with different variance. We find that, for a large range of adoption rules, the diffusion threshold is lower for networks with larger variance.[16]

3.2.1 The Homogeneous-Visibility Case ($\alpha = 0$)

In order to obtain further comparative statics results we have concentrated on the case of $\alpha = 0$ which is significantly simpler than the remaining cases where $0 < \alpha \leq 1$. The reason is that, in such a case, the value of θ (probability of sampling an adopter) coincides with the overall fraction of adopters in the population ρ, that is $\theta = \rho$. The diffusion threshold is simply

$$\lambda^* = \frac{1}{\sum_k k f_k(1) P(k)}$$

and, if $f_k(a)$ is a concave function of a, for every k, the endemic fraction of adopters ρ^* is the unique positive solution of the following fixed point equation

$$\rho = \sum_k P(k) \frac{\lambda r_k(\rho)}{1 + \lambda r_k(\rho)}.$$

All agents are now equally influential and thus heterogeneity among them is only due to the amount of information they have about the behavior of others. The following propositions explain the effect on the diffusion threshold of a FOSD shift and a MPS of the out-degree distribution.

Proposition 4 *Given two influence networks $\widetilde{P_0}$ and P_0, where $\widetilde{P}(k)$ FOSD $P(k)$, and an adoption rule f, the following statements hold:*

(i) if $k f_k(1)$ is increasing with respect to k then $\lambda^*(\widetilde{P_0}) \leq \lambda^*(P_0)$

(ii) if $k f_k(1)$ is decreasing with respect to k then $\lambda^*(P_0) \leq \lambda^*(\widetilde{P_0})$

(iii) if $k f_k(1)$ is constant with respect to k then $\lambda^*(\widetilde{P_0}) = \lambda^*(P_0)$

Proposition 5 *Given two influence networks $\widetilde{P_0}$ and P_0, where $\widetilde{P}(k)$ is a MPS of $P(k)$, and an adoption rule f_k, the following statements hold:*

(i) if $k f_k(1)$ is convex with respect to k then $\lambda^*(\widetilde{P_0}) \leq \lambda^*(P_0)$

(ii) if $k f_k(1)$ is concave with respect to k then $\lambda^*(P_0) \leq \lambda^*(\widetilde{P_0})$

(iii) if $k f_k(1)$ is linear with respect to k then $\lambda^*(\widetilde{P_0}) = \lambda^*(P_0)$

[16] The result that heterogeneity in the network enhances diffusion can be considered as a generalization of the main finding in the mean-field model presented by Pastor-Satorrás and Vespignani (2001), who focused on the SIS adoption rule and the case $\alpha = 1$.

Note that all viral adoption rules satisfy conditions (i) and (iii) in Propositions 4 and 5, respectively. Therefore, the higher the density of the influence network the lower its diffusion threshold. Moreover, two populations with the same average out-degree but different variance have the same diffusion threshold since $\lambda^* = \frac{1}{f(1)\langle k \rangle}$. Regarding persuasive adoption rules, further properties of the rule are necessary in order to determine the results. For example, when $f_k(a) = \sqrt{(\frac{a}{k})}$, the higher the density of the network the lower the diffusion threshold whereas the opposite holds when $f_k(a) = (\frac{a}{k})^2$. Furthermore, for the adoption rule $f_k(a) = \sqrt{(\frac{a}{k})}$ the higher the variance, the higher the diffusion threshold whereas the opposite holds when $f_k(a) = (\frac{a}{k})^2$.

In addition, we also analyze the effect of varying the out-degree distribution on the endemic state. To this end, denote by $\rho^*(P_\alpha)$ to the endemic fraction of adopters obtained for a $P_\alpha-$influence network .

Proposition 6 *Given two influence networks $\widetilde{P_0}$ and P_0, where $\widetilde{P}(k)$ FOSD $P(k)$, and a (weakly) concave adoption rule $f_k(a)$ with respect to a, for every k, the following statements hold:*

(i) If $r_k(\rho)$ is increasing with respect to k for any $\rho \in [0,1]$ then $\rho^*(P_0) \leq \rho^*(\widetilde{P}_0)$

(ii) If $r_k(\rho)$ is decreasing with respect to k for any $\rho \in [0,1]$ then $\rho^*(\widetilde{P_0}) \leq \rho^*(P_0)$

(iii) If $r_k(\rho)$ is constant with respect to k for any $\rho \in [0,1]$ then $\rho^*(\widetilde{P}_0) = \rho^*(P_0)$

Proposition 7 *Given two influence networks $\widetilde{P_0}$ and P_0, where $\widetilde{P}(k)$ is a MPS of $P(k)$, and a (weakly) concave adoption rule $f_k(a)$ with respect to a, the following statements hold:*

(i) If $\frac{\lambda r_k(\rho)}{1+\lambda r_k(\rho)}$ is convex with respect to k for any $\rho \in [0,1]$ then $\rho^*(P_0) \leq \rho^*(\widetilde{P}_0)$

(ii) If $\frac{\lambda r_k(\rho)}{1+\lambda r_k(\rho)}$ is concave with respect to k for any $\rho \in [0,1]$ then $\rho^*(\widetilde{P_0}) \leq \rho^*(P_0)$

(iii) If $\frac{\lambda r_k(\rho)}{1+\lambda r_k(\rho)}$ is linear with respect to k for any $\rho \in [0,1]$ then $\rho^*(\widetilde{P}_0) = \rho^*(P_0)$.

Viral adoption rules satisfy (i) in Proposition 6 and thus the higher the density of the influence network, the higher the endemic fraction of adopters. Regarding the effect of a MPS of the out-degree distribution for viral adoption rules, the result is not conclusive and depends on the further properties of the rule. Nevertheless, for the specific case of the SIS rule it is straightforward to show that it satisfies (ii) in Proposition 7 and thus, the higher the variance of the out-degree distribution, the lower the endemic fraction of the adopters. Figure 2 summarizes the qualitative results obtained for the SIS rule, both regarding the diffusion threshold and the endemic fraction of adopters. The SIS rule when $\alpha = 1$ has been previously analyzed by Pastor-Satorrás and Vespignani (2001) and

Jackson and Rogers (2007). If one compares the two extreme cases ($\alpha = 1$ and $\alpha = 0$), the more striking difference is that homogeneity in the out-degree distribution increases the endemic fraction of adopters for all values of λ when $\alpha = 0$, whereas it, instead, decreases the endemic fraction of adopters (at least for low range of values of λ) when $\alpha = 1$. The intuition for such a result is the following. The SIS rule is such that (alike all viral rules) agents with high out-degree are more easily influenced. In fact the rate of adoption at any given moment in time $r_k(\rho)$ is increasing with k. Due to the concavity of $r_k(\rho)$ as a function of k the effect of having high out-degree exhibits decreasing returns to scale. This leads to the result that a population where all agents have roughly the same average degree helps diffusion more than a more heterogeneous out-degree distribution. In the case $\alpha = 1$ the argument does not follow since agents with a high out-degree are particularly valuable for spreading the product; not only they adopt early but also once they adopt, they are very influential and spread the infection further. This second advantageous effect prevails and therefore degree distribution with high variance favor diffusion.[17]

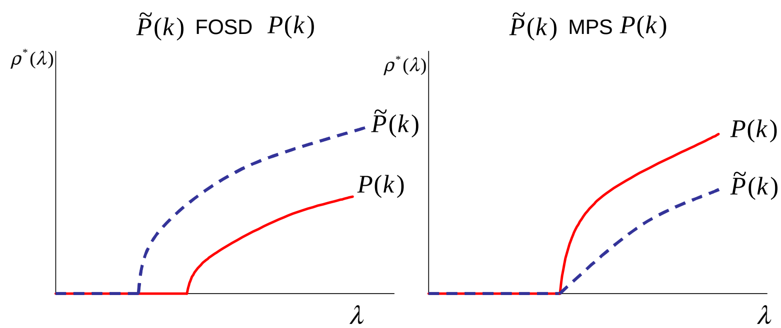

Figure 2: The graphs plot the endemic fraction of adopters ρ^* as a function of the spreading rate λ, focusing on the effects of a FOSD shift (graph on the left) and a MPS (graph on the right) of the degree distribution for the SIS adoption rule and $\alpha = 0$.

4 Concluding Remarks

Nowadays, more complex influence structures have replaced traditional patterns generated exclusively through standard personal interactions. We have proposed in this paper a

[17]These results resemble those found by Galeotti and Goyal (2009). For instance, analogously to Propositions 4 and 5, they also find that if the adoption rate is an increasing (decreasing) function of the out-degree a FOSD shift of the out-degree distribution would increase (decrease) the profits of the firm wanting to spread the new product. Moreover, similarly to what we find in Propositions 6 and 7, they find that if the adoption rate is convex (concave) with respect to out-degree, a MPS would increase (decrease) the profits of the firm.

stylized model to analyze some of the implications of such complexities. We have modeled the influence structure by means of an explicit sampling process characterized by the correlation between the out-degree (information level) and in-degree (visibility level) of agents. Surprisingly, we have observed that an increase in such a correlation may favor or harm diffusion; the effect actually depends on the specific details of the adoption process. Two types of adoption rules can be singled out: those for which high out-degree agents are the early adopters and those for which low out-degree agents are the early adopters. In the former case, diffusion is eased if the high out-degree are also influential, whereas in the latter case, the opposite holds. We have also shown that an increase in both the level and dispersion of information has a strong impact on the results, hence questioning the hypothesis that more dense and heterogeneous networks always favor diffusion.

The current work could contribute to gain further insight into the dynamics of social processes, pointing out possible directions for empirical studies of value for understanding diffusion in the real world. Influence networks, however, are not formed completely at random. Therefore, one might consider enriching our model to account for clustering and community structures. There has already been significant work analyzing network formation in a semi-random framework.[18] The study of diffusion on such more realistic networks seems to be a fertile and promising area of research.

5 Acknowledgments

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[18] In particular, Jackson and Rogers (2007b) formalize the idea that individuals find others through their current friends. Moreover, Currarini et al. (2009) introduce homophyly based networks capturing the well-known phenomenon that individuals interact more often with similar others.

6 Appendix

Proof of Lemma 1: The probability that any agent samples another agent with out-degree k is equal, by assumption, to

$$\frac{k^\alpha P(k)}{\sum_h h^\alpha P(h)}.$$

Note that $\frac{P(h)}{P(k)}$ determines the relative size of the population of agents with out-degree h with respect to the population of agents with out-degree k. For example, $\frac{P(h)}{P(k)} = 2$ means that the size of the population with out-degree h is twice as large as the size of the population with out-degree k. Therefore, the expected number of links an agent with out-degree k receives from agents with out-degree 1 is

$$\frac{k^\alpha P(k)}{\sum_h h^\alpha P(h)} 1 \frac{P(1)}{P(k)}.$$

Analogously, the expected number of links an agent with out-degree k receives from agents with out-degree 2 is

$$\frac{k^\alpha P(k)}{\sum_h h^\alpha P(h)} 2 \frac{P(2)}{P(k)}$$

and so on and so forth.

Thus, the expected number of links pointing to an agent with out-degree k equals

$$\sum_l \frac{k^\alpha P(k)}{\sum_h h^\alpha P(h)} l \frac{P(l)}{P(k)} = \frac{k^\alpha \langle h \rangle}{\langle h^\alpha \rangle}.$$

This lemma is used in the proof of the following theorem.

Proof of Theorem 1: If an active agent is observed by another agent in an influence network, we say that there is an *active link* between them. It is not difficult to show (with the help of Lemma 1) that the expected number of new active links generated by an initial active link is given by

$$\sum_k \frac{kP(k)}{\langle k \rangle} \nu f_k(1) \frac{1}{\delta} \frac{k^\alpha}{\langle k^\alpha \rangle} \langle k \rangle$$

where $\frac{kP(k)}{\langle k \rangle}$ is the probability that an agent, say j, sampling an initial adopter has out-degree k and $\nu f_k(1)$ is the rate at which this agent adopts. While this agent is active (i.e., during an interval of time equal to $\frac{1}{\delta}$) the number of new active links generated on average is $\frac{k^\alpha}{\langle k^\alpha \rangle}\langle k \rangle$, which is the average number of individuals sampling agent j since j has out-degree k. Therefore, the number of new active links originated by one active link is greater than 1 if and only if

$$\lambda > \frac{\langle k^\alpha \rangle}{\sum_k k^{\alpha+1} P(k) f_k(1)} \tag{6}$$

To complete the proof let us show that diffusion occurs if and only if condition (8) holds. Consider the discrete approximation of the dynamics. Let us show that if there is diffusion then condition (8) must hold, or analogously, that if condition (8) does not hold there is no diffusion. Assume that initially there is a finite number of adopters N_0 and let i be one of them. Let r_0^i be the number of individuals influenced by this initial adopter (i.e., in period 0). Note that r_0^i is also the number of active links generated by this initial adopter. If condition (8) does not hold then the expected number of active links generated by i decreases with time. In a discrete version of the dynamics this implies that the number of active links in period 1 generated by i is such that $r_0^i > r_1^i$. The same argument is valid to show that $r_1^i > r_2^i$, and so on. Therefore, there must exist a period $\overline{t^i}$ above which the number of active links is zero (i.e., $r_t^i = 0$ for all $t \geq \overline{t^i}$). Thus, for $t > max_{i \in N_0}\{\overline{t^i}\}$ it holds that $\rho_t = 0$ and thus $\rho^* = 0$. A similar reasoning can be used to show the reverse implication; if condition (8) holds then $\rho^* \neq 0$. In this case the sequence $\{r_t^i\}_{t \geq 0}$ is increasing and thus converges to infinity.

Proof of Theorem 2: In order to find the stationary fraction of active agents ρ^* one must first find the stationary values of the parameter θ, denoted by θ^*. Indeed, $\rho^* \neq 0$ if and only if $\theta^* \neq 0$. It is straightforward to show that $0 \leq H(\theta) < 1$ for all $\theta \in [0,1]$. We also have that $H(0) = 0$ which implies that $\theta = 0$ is a stationary state of the dynamics for all values of λ. Let us now determine the values of λ for which there also exists a non-null stationary state. To this end, let us first show that H is increasing and concave. Note that

$$\frac{dH(\theta)}{d\theta} = \frac{1}{\langle k^\alpha \rangle} \sum_k k^\alpha P(k) \frac{\lambda \frac{dr_k(\theta)}{d\theta}}{(1+\lambda r_k(\theta))^2},$$

where

$$\begin{aligned} \frac{dr_k(\theta)}{d\theta} &= \sum_{a=0}^{k} f_k(a) \binom{k}{a} \left(a\theta(1-\theta)^{(k-a)} - \theta(k-a)(1-\theta)^{(k-a-1)}\right) \\ &= \sum_{a=0}^{k-1} ((a+1) f(k,a+1) \binom{k}{a+1} - (k-a) f_k(a) \binom{k}{a})\theta(1-\theta)^{(k-a-1)} \end{aligned} \tag{7}$$

and since

$$(a+1)\binom{k}{a+1} = (k-a)\binom{k}{a} = \frac{k!}{a!(k-a-1)!},$$

then

$$\frac{dr_k(\theta)}{d\theta} = \sum_{a=0}^{k-1} \frac{k!}{a!(k-a-1)!}(f(k,a+1) - f_k(a))\theta(1-\theta)^{(k-a-1)}$$

which is non-negative given condition (1) imposed on the adoption rule f. Therefore $H(\theta)$ is non-decreasing. To show that $H(\theta)$ is concave we must take the second derivative of $H(\theta)$. That is

$$\frac{d^2H(\theta)}{d^2\theta} = \frac{1}{\langle k^\alpha \rangle}\sum_k k^\alpha P(k) \frac{\lambda^2 \frac{d^2 r_k(\theta)}{d^2\theta}(1+\lambda r_k(\theta)) - 2(\lambda \frac{dr_k(\theta)}{d\theta})^2}{(1+\lambda r_k(\theta))^3},$$

where

$$\begin{aligned}\frac{d^2r_k(\theta)}{d^2\theta} &= \sum_{a=0}^{k-1} \frac{k!}{a!(k-a-1)!}(f(k,a+1) - f_k(a)) \\ &\quad (a\theta(1-\theta)^{(k-a-1)} - \theta(k-a-1)(1-\theta)^{(k-a-2)})\end{aligned}$$

or equivalently

$$\begin{aligned}\frac{d^2r_k(\theta)}{d^2\theta} &= \sum_{a=0}^{k-2} \frac{k!(a+1)}{(a+1)!(k-a-2)!}(f(k,a+2) - f(k,a+1))\theta(1-\theta)^{(k-a-2)} \\ &\quad - \frac{k!(k-a-1)}{a!(k-a-1)!}(f(k,a+1) - f_k(a))\theta(1-\theta)^{(k-a-2)} \\ &= \sum_{a=0}^{k-2}((f_k(a+2) - f(k,a+1)) - (f(k,a+1) - f_k(a))) \\ &\quad \frac{k!}{a!(k-a-2)!}\theta(1-\theta)^{(k-a-2)}.\end{aligned}$$

Since $f_k(a)$ is concave with respect to a then $\frac{d^2r_k(\theta)}{d^2\theta} \leq 0$ which in turn shows that $H(\theta)$ is concave. Finally, notice that, if $H(\theta)$ is non-decreasing and concave, there exists a (unique) non-null stationary state of the dynamics if and only if

$$\frac{dH(\theta)}{d\theta}\rfloor_{\theta=0} > 1,$$

and

$$\frac{dH(\theta)}{d\theta}\rfloor_{\theta=0} = \lambda \frac{1}{\langle k^\alpha \rangle}\sum_{k\geq 1} k^\alpha P(k) f_k(1) > 1 \Leftrightarrow \lambda > \lambda^* = \frac{\langle k^\alpha \rangle}{\sum_{k\geq 1} k^\alpha P(k) f_k(1)}.$$

Moreover, if $\lambda \leq \lambda^*$the unique stationary value for θ is 0.

Proof of Proposition 1: It is straightforward to show that $\lambda^*(\alpha)$ is a continuous and derivable function of α. We then demonstrate that if $kf(1,k)$ is an increasing (decreasing) function of k then $\frac{d\lambda^*(\alpha)}{d\alpha} \leq 0$ $\left(\frac{d\lambda^*(\alpha)}{d\alpha} \geq 0\right)$ and that if $kf(1,k)$ is constant then $\frac{d\lambda^*(\alpha)}{d\alpha} = 0$. Note that

$$\frac{d\lambda^*(\alpha)}{d\alpha} = \frac{\langle k^\alpha(\log k)\rangle\langle k^{\alpha+1}f(1,k)\rangle - \langle k^\alpha\rangle\langle k^{\alpha+1}f(1,k)(\log k)\rangle}{\langle k^{\alpha+1}f(1,k)\rangle^2}$$

where for ease of notation we use $\langle g(k)\rangle$ to be $\sum_k g(k)P(k)$ for any function $g(k)$. Let us characterize the sign of $\langle k^\alpha(\log k)\rangle\langle k^{\alpha+1}f(1,k)\rangle - \langle k^\alpha\rangle\langle k^{\alpha+1}f(1,k)(\log k)\rangle$. It is straightforward to show that for any given k, the coefficient (multiplying) $P(k)^2$ in the expression $\langle k^\alpha(\log k)\rangle\langle k^{\alpha+1}f(1,k)\rangle - \langle k^\alpha\rangle\langle k^{\alpha+1}f(1,k)(\log k)\rangle$ is 0. Let us now compute the coefficient of $P(k)P(\overline{k})$ for any $k \neq \overline{k}$. Assume without loss of generality that $\overline{k} < k$, the coefficient is

$$k^\alpha(\log k)\overline{k}^{\alpha+1}f(1,\overline{k}) + \overline{k}^\alpha(\log \overline{k})k^{\alpha+1}f(1,k) - k^\alpha\overline{k}^{\alpha+1}f(1,\overline{k})(\log \overline{k}) - \overline{k}^\alpha k^{\alpha+1}f(1,k)(\log k)$$

which simplifies to

$$(k^\alpha\overline{k}^{\alpha+1}f(1,\overline{k}) - \overline{k}^\alpha k^{\alpha+1}f(1,k))(\log k - \log \overline{k}).$$

The sign of the above expression coincides with the sign of

$$\overline{k}f(1,\overline{k}) - kf(1,k)$$

which completes the proof.

Proof of Proposition 2: The following fixed point equation determines the endemic value of θ

$$\theta = \frac{1}{\langle k^\alpha\rangle}\sum_k k^\alpha P(k)\frac{\lambda r_k(\theta)}{1+\lambda r_k(\theta)}.$$

The endemic state for θ depends on the value of α. To show the monotonicity of the fixed point value $\theta^*(\alpha)$ (taken as fixed all other primitives of the model) one must evaluate the monotonicity of $\frac{1}{\langle k^\alpha\rangle}\sum_k k^\alpha P(k)\frac{\lambda r_k(\theta)}{1+\lambda r_k(\theta)}$ as a function of α. Note that if $\frac{1}{\langle k^\alpha\rangle}\sum_k k^\alpha P(k)\frac{\lambda r_k(\theta)}{1+\lambda r_k(\theta)}$ is increasing (decreasing) as a function of α (for all $\theta \in [0,1]$) then $\theta^*(\alpha)$ must be increasing (decreasing) as well. The monotonicity of $\frac{1}{\langle k^\alpha\rangle}\sum_k k^\alpha P(k)\frac{\lambda r_k(\theta)}{1+\lambda r_k(\theta)}$ is determined by the sign of the following expression

$$\langle k^\alpha(\log k)\frac{\lambda r_k(\theta)}{1+\lambda r_k(\theta)}\rangle\langle k^\alpha\rangle - \langle k^\alpha\frac{\lambda r_k(\theta)}{1+\lambda r_k(\theta)}\rangle\langle k^\alpha(\log k)\rangle \tag{8}$$

It is straightforward to show that for any given k, the coefficient (multiplying) $P(k)^2$ in the expression (8) is 0. Let us now compute the coefficient of $P(k)P(\overline{k})$ for any $k \neq \overline{k}$. Assume without loss of generality that $\overline{k} < k$, the coefficient is

$$k^{\alpha}(\log k)\frac{\lambda r_k(\theta)}{1+\lambda r_k(\theta)}\overline{k}^{\alpha}+\overline{k}^{\alpha}(\log \overline{k}^{\alpha})\frac{\lambda r_{\overline{k}}(\theta)}{1+\lambda r_{\overline{k}}(\theta)}k^{\alpha}-k^{\alpha}\frac{\lambda r_k(\theta)}{1+\lambda r_k(\theta)}\overline{k}^{\alpha}(\log \overline{k})-\overline{k}^{\alpha}\frac{\lambda r_{\overline{k}}(\theta)}{1+\lambda r_{\overline{k}}(\theta)}k^{\alpha}(\log k)$$

which simplifies to

$$k^{\alpha}\overline{k}^{\alpha}\left(\frac{\lambda r_k(\theta)}{1+\lambda r_k(\theta)}-\frac{\lambda r_{\overline{k}}(\theta)}{1+\lambda r_{\overline{k}}(\theta)}\right)(\log k - \log \overline{k}).$$

The sign of the above expression coincides with the sign of

$$\frac{\lambda r_k(\theta)}{1+\lambda r_k(\theta)}-\frac{\lambda r_{\overline{k}}(\theta)}{1+\lambda r_{\overline{k}}(\theta)}$$

or analogously with the sign of

$$r_k(\theta) - r_{\overline{k}}(\theta)$$

which completes the proof.

Proof of Proposition 3: Notice that if $\widetilde{P}(k)$ FOSD $P(k)$ then for any increasing function $u(k)$ we have that

$$\sum_k u(k)P(k) \leq \sum_k u(k)\widetilde{P}(k).$$

Since k^{α} is increasing then $\sum_k k^{\alpha}P(k) \leq \sum_k k^{\alpha}\widetilde{P}(k)$. By assumption $k^{\alpha+1}f_k(1)$ is decreasing and therefore $\sum_k k^{\alpha+1}P(k)f_k(1) \geq \sum_k k^{\alpha+1}\widetilde{P}(k)f_k(1)$. Both inequalities together imply that $\lambda^*(P_{\alpha}) \leq \lambda(\widetilde{P_{\alpha}})$ which completes the first part of the proof.

Regarding the second part of the proof, it is the case that if $\widetilde{P}$ is a MPS of P then for any concave function $u(k)$

$$\sum_k u(k)\widetilde{P}(k) \leq \sum_k u(k)P(k).$$

Notice that if $k^{\alpha+1}f_k(1)$ is convex then

$$\sum_k k^{\alpha+1}P(k)f_k(1) \leq \sum_k k^{\alpha+1}\widetilde{P}(k)f_k(1)$$

and since k^{α} is concave then

$$\sum_k k^{\alpha}\widetilde{P}(k) \leq \sum_k k^{\alpha}P(k).$$

These two inequalities together imply that $\lambda^*(\widetilde{P}_\alpha) \leq \lambda^*(P_\alpha)$.

Proof of Proposition 4: It is immediate to show that the diffusion threshold equals

$$\lambda^* = \frac{1}{\sum_k kP(k)f_k(1)}$$

for a P_0-random network. Note that, if $kf_k(1)$ is increasing then $\sum_k kP(k)f_k(1) \leq \sum_k k\widetilde{P}(k)f_k(1)$ which implies that $\lambda^*(\widetilde{P}_\alpha) \leq \lambda^*(P_\alpha)$. If $kf_k(1)$ is decreasing then $\sum_k kP(k)f_k(1) \geq \sum_k k\widetilde{P}(k)f_k(1)$ which implies that $\lambda^*(P_\alpha) \leq \lambda^*(\widetilde{P}_\alpha)$. Finally $\sum_k kP(k)f_k(1) = \sum_k k\widetilde{P}(k)f_k(1)$ if $kf_k(1)$ is constant and thus $\lambda^*(P_\alpha) = \lambda^*(\widetilde{P}_\alpha)$ in such a case.

Proof of Proposition 5: If $kf_k(1)$ is convex then $\sum_k kP(k)f_k(1) \leq \sum_k k\widetilde{P}(k)f_k(1)$ which implies that $\lambda^*(P_\alpha) \leq \lambda^*(\widetilde{P}_\alpha)$. If $kf_k(1)$ is concave then $\sum_k kP(k)f_k(1) \geq \sum_k k\widetilde{P}(k)f_k(1)$ which implies that $\lambda^*(P_\alpha) \leq \lambda^*(\widetilde{P}_\alpha)$. Finally, $\sum_k kP(k)f_k(1) = \sum_k k\widetilde{P}(k)f_k(1)$ if $kf_k(1)$ is a linear function of k and thus $\lambda^*(P_\alpha) = \lambda^*(\widetilde{P}_\alpha)$ in such a case.

Proof of Proposition 6: The fraction of adopters ρ^* is computed as the solution of equation

$$\rho = \sum_k P(k)\frac{\lambda r_k(\rho)}{1+\lambda r_k(\rho)}. \tag{9}$$

Note that if $r_k(\rho)$ is increasing as a function of k for all ρ then $\frac{\lambda r_k(\rho)}{1+\lambda r_k(\rho)}$ is also an increasing function of k for all ρ. Therefore,

$$\sum_k P(k)\frac{\lambda r_k(\rho)}{1+\lambda r_k(\rho)} \leq \sum_k \widetilde{P}(k)\frac{\lambda r_k(\rho)}{1+\lambda r_k(\rho)}$$

for all ρ, which in particular implies that the value of ρ that solves equation (10) is smaller or equal for the out-degree distribution $P(k)$ than for $\widetilde{P}(k)$. The proofs of (ii) and (iii) go along the same lines.

Proof of Proposition 7: If $\frac{\lambda r_k(\rho)}{1+\lambda r_k(\rho)}$ is a convex function of k for all ρ then

$$\sum_k P(k)\frac{\lambda r_k(\rho)}{1+\lambda r_k(\rho)} \leq \sum_k \widetilde{P}(k)\frac{\lambda r_k(\rho)}{1+\lambda r_k(\rho)}$$

for all ρ, which in particular implies that the value of ρ that solves equation (10) is smaller or equal for the out-degree distribution $P(k)$ than for $\widetilde{P}(k)$. The proofs of (ii) and (iii) go along the same lines.

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