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# Risk theory under partial information with applications in Actuarial Science and Finance

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# Contents

# Introduction

A central notion in actuarial mathematics is the notion of *risk*. A risk can be viewed as an event depending on the whims of fate that may or may not take place and that generally brings about some financial loss. The actuary models such an insurance risk by a random variable which represents the random amount of money the insurance company will have to pay out to indemnify the policyholder and/or the third party for the consequences of the occurrence of the insured risk. In most practical situations, the available information about the probability laws of the risks in presence is only partial and it can be interesting to obtain some approximations based on the known incomplete information. If only the first moments of the risks in presence are known, let us for instance quote the approximations done with help of the central-limit theorem, Edgeworth developments, Esscher approximations, normal power formula, etc. Of course, it is always essential to be able to evaluate the quality of these estimates. To that end, the obtention of bounds on the quantities of interest permits to control the approximation error. So, in addition to trying to determine "exactly" the risk quantities under interest, the actuary will obtain accurate lower and upper bounds for it. Then, he will combine these two methods and decide a reasonable estimation of the risk.

In a probabilistic way, instead of working with a completely specified probability law, knowing only partial information about the risks in presence amounts to work with a class of laws compatible with the incomplete available information. Since the '80s, numerous works in the actuarial literature were devoted to this type of approach, determining bounds on various quantities of interest (distribution functions, stop-loss premiums, adjustment coefficients, ruin probabilities, etc.) when the probability law of interest lies on a certain class; see, e.g., TAYLOR (1977), DE VYLDER (1980,1982,1983), DE VYLDER & GOOVAERTS (1982,1983), DE VYLDER, GOOVAERTS, HAEZENDONCK & GARRIDO (1984), GOOVAERTS & DE VYLDER (1980), GOOVAERTS, HAEZENDONCK & DE VYLDER (1982), GOOVAERTS & KAAS (1985), KAAS (1985), KAAS & GOOVAERTS (1985,1986a,b,c,1987), BROCKETT & COX (1985), JANSEN, HAEZENDONCK & GOOVAERTS (1986), HEIJNEN & GOOVAERTS (1986,1989) and HEIJNEN (1990).

The scope of our study is the *reduced moment spaces* of order $s = 1, 2, 3, \ldots$, i.e. classes of probability laws sharing the same first $s-1$ moments (among others, mean, variance and skewness for $s = 4$). Moment spaces have been studied for a long time (see for example KARLIN & STUDDEN (1966)) and they benefit from a great number of interesting properties that make them an appropriate background for the resolution of various problems in risk theory, actuarial science and stochastic finance. As it is often simpler to speak of random variables rather than of distribution functions, in this thesis, we will consider classes of random variables to favor the intuitive contents of the results.

In most practical situations, the random variables that are elements of some moment spaces may be assumed to be non-negative with a bounded support because the upper limit of the financial loss for which the insurance company underwrites is generally fixed by the contract or determined through reinsurance techniques. Sometimes, it is also interesting to restrain to subspaces of these moment spaces. Two examples are the class of unimodal laws (that are easy to study using Khinchine's representation theorem), or the class of "DFR" laws with decreasing failure rates among whose are the models describing claim costs. Another classical constraint relies on the support of the laws in presence. Because they are used to modellize the numbers of claims, counting laws, whose support is composed of all non-negative integers, constitue a very important particular case in actuarial science. Moreover, in practice, one often needs discrete claim distributions. In fact, the great majority of iterative algorithms (as the famous Panjer algorithm for instance) require laws whose support elements have to be multiples of the same span. A more general situation is when the random variables take on values in an arbitrary ordered finite grid of non-negative points. Therefore, in this thesis, we will mainly be interested in discrete risks.

As the actuary feels the need to order the risks, one of the classical problems in actuarial science consists is comparing risks using the so called stochastic order relations. *Stochastic orderings* are probabilistic tools to compare random variables or random vectors. Again, we consider classes of random variables rather than of distribution functions and the reader has to keep in mind that we do not compare the particular versions of the random variables but their respective distributions. Mathematically speaking, stochastic orderings are thus partial order relations defined on sets of probability distributions. The interest of the actuarial literature in the stochastic orderings originated in the seminal papers by Borch (1961), Bühlmann, Gagliardi, Gerber & Straub (1977) and Goovaerts, De Vylder & Haezendonck (1982). Since then, this theory has received in actuarial sciences an increasing attention. A number of actuarial applications can be found in the books by Goovaerts, Kaas, Van Heerwaarden & Bauwelinckx (1990), Kaas, Van Heerwaarden & Goovaerts (1994) and Denuit, Dhaene, Goovaerts & Kaas (2005). The reader is also referred to the books by Shaked & Shanthikumar (1994) and Shaked & Shanthikumar (2006) for a general overview of the stochastic orderings in various fields of applied probability and statistics.

Clearly, there exists many ways to perform such comparisons. In this thesis, we will be interested in some classes of *integral stochastic orderings*. These orderings are defined (or can be defined) by reference to a class $\mathcal{U}_*^{\mathcal{S}}$ of measurable functions $\phi : \mathcal{S} \to \mathbb{R}$ as follows: having two random variables $X$ and $Y$ valued in $\mathcal{S}$, $X$ is smaller than $Y$ in the $\preceq_*^{\mathcal{S}}$-sense, denoted as $X \preceq_*^{\mathcal{S}} Y$, if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for all the functions $\phi$ in $\mathcal{U}_*^{\mathcal{S}}$ for which the expectation exists. Usually, $\mathcal{S}$ is taken to be the union of the supports of $X$ and $Y$. Taking for $\mathcal{U}_*^{\mathcal{S}}$ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}$ with non-negative first derivative ($\phi^{1/} \geq 0$) yields the well-known stochastic dominance $\preceq_{\mathrm{st}}$. Taking for $\mathcal{U}_*^{\mathcal{S}}$ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}$ with non-negative second derivative ($\phi^{2/} \geq 0$) yields the convex order $\preceq_{\mathrm{cx}}$. In the scope of moment spaces, the most profitable approach is to appeal to the *s-convex orders* $\preceq_{s-\mathrm{cx}}^{\mathbb{R}^+}$ defined by Denuit, Lefèvre & Shaked (1998). These relations are generated taking for $\mathcal{U}_*^{\mathcal{S}}$ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}$ with non-negative $s$th derivative ($\phi^{s/} \geq 0$). Moreover, because an $s$-convex comparison of two risks is only possible if they have the same first $s-1$ moments, these orders were specially built to work within moment spaces.

A closely related question in actuarial sciences is the construction of *extrema* with respect to some order relation. Indeed, the actuary sometimes acts in a conservative way by basing his decisions on the least attractive risk that is consistent with the incomplete available information. This can be done by determining in given classes of risks, coherent with the partial known information, the extrema with respect to some stochastic ordering which translates the preferences of the actuary, i.e. the "worst" and the "best" risk consistent with the partial available information. The existence of these extremal laws permits to obtain bounds on the quantities of interest.

Sometimes, it is also interesting to come out the moment space and to appeal to other orders than the $s$-convex ones. In fact, in many applications, extremal distributions with respect to the stochastic order $\preceq_{\text{st}}$ and to the convex order $\preceq_{\text{cx}}$ (that is, respectively with respect to the 1- and 2-convex orders) are needed. Only these stochastic order relations respectively allow to get bounds on distribution functions and on ruin probabilities or stop-loss premiums, for instance. However, it is well-known that two random variables with equal means cannot be ordered with respect to the stochastic dominance relation. Equivalently, two random variables with equal variances cannot be ordered with respect to the convex order. Consequently, to obtain the extremal distributions in the $\preceq_{\text{st}}$- and $\preceq_{\text{cx}}$-sense, only a small part of the available information is needed: no moments or only the mean. In order to take into account all the available information about the risks in presence (and thus to obtain sharper bounds on the aforementioned quantities), other extremal distributions will have to be derived but that do not necessarily belong the considered moment space. It is the scope of Part II.

This thesis is devoted to the derivation of lower and upper bounds on quantities of interest in actuarial science and finance and of the form $\mathbb{E}[\phi(X)]$, for some given $\phi$, when $X$ belongs to a class of random variables satisfying certain moment conditions (i.e. based on the knowledge of some partial information about $X$). Such quantities are for instance distribution functions, stop-loss premiums, adjustment coefficient, ruin probabilities, prices of contingent claims, etc. It is divided in three parts: the first one concerns the use of discrete $s$-convex extremal distributions in the computation of the aforementioned bounds, the second one deals with the computing of these bounds using the particular 1- and 2-convex orderings and taking all the available information into account, and the third one relates to some applications of the discrete $s$-convex orderings in finance.

## Part I: Bounds using $s$-convex stochastic extrema

In many situations, stochastic order relations are used to compare real random variables. Quite recently, a remarkable class of discrete stochastic orderings have been investigated by Denuit & Lefèvre (1997) to compare random variables that are discrete by nature as counts for instance: the class of the *discrete s-convex orderings* among arithmetic random variables valued in some set $\mathcal{N}_n = \{0, 1, 2, \ldots, n\}$, $n \in \mathbb{N}$. Here $s$ is any non-negative integer smaller or equal to $n$. They have been defined as follows. Let $\Delta$ be the first order forward difference operator (with unitary increment) defined for each function $\phi : \mathcal{N}_n \to \mathbb{R}$ by $\Delta\phi(i) = \phi(i+1) - \phi(i)$ for all $i \in \mathcal{N}_{n-1}$. Let $\Delta^k$, $k \in \mathcal{N}_n$, be the $k$-th order forward difference operator defined recursively by $\Delta^k\phi(i) = \Delta^{k-1}\phi(i+1) - \Delta^{k-1}\phi(i)$ for all $i \in \mathcal{N}_{n-k}$ (by convention, $\Delta^1\phi \equiv \Delta\phi$ and $\Delta^0\phi \equiv u$). If $X$ and $Y$ are two random variables valued in $\mathcal{N}_n$, $X$ is said to be smaller than $Y$ with respect to the discrete $s$-convex order if

$$\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)] \text{ for all } \phi \in \mathcal{U}_{s-\text{cx}}^{\mathcal{N}_n} = \{\phi : \mathcal{N}_n \to \mathbb{R} : \Delta^s\phi(i) \geq 0, \;\; \forall \; i \in \mathcal{N}_{n-s}\}\,.$$

In such a case, we write $X \preceq_{s-\text{cx}}^{\mathcal{N}_n} Y$. It can be proved that if $X \preceq_{s-\text{cx}}^{\mathcal{N}_n} Y$ then the $s-1$ first moments of $X$ and $Y$ necessarily match. Consequently, the ordering relation $\preceq_{s-\text{cx}}^{\mathcal{N}_n}$ can only be used to compare the random variables with the same first $s-1$ moments. This motivates to introduce the *moment space* $\mathcal{M}_s\left(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1}\right)$ which contains all random variables valued on $\mathcal{N}_n$ such that the first $s-1$ moments are fixed to $\mu_k$ ($k = 1, \ldots, s-1; s \geq 1$). One remarkable property of $s$-convex orderings is the following: Provided that the moment space satisfies some reasonable conditions (in particular this space is not void), the moment space contains a minimum random variable $X_{\text{min}}^{(s)}$ and a maximum random variable $X_{\text{max}}^{(s)}$ with respect to $\preceq_{s-\text{cx}}^{\mathcal{N}_n}$, i.e. such that

$$X_{\text{min}}^{(s)} \preceq_{s-\text{cx}}^{\mathcal{N}_n} X \preceq_{s-\text{cx}}^{\mathcal{N}_n} X_{\text{max}}^{(s)} \text{ for all } X \text{ in } \mathcal{M}_s\left(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1}\right).$$

Extrema with respect to the discrete version of the $s$-convex orders have been derived by Denuit, Lefèvre & Utev (1999), Denuit & Lefèvre (1997) and Denuit, Lefèvre & Mesfioui (1999) for $s = 1, 2, 3$ and partially for $s = 4$ (maximum). Surprisingly, no explicit formula for $X_{\text{min}}^{(4)}$ was available in the literature. The point is that the argument based on the non-negativity of particular moment matrices is no longer valid for that case. The same phenomenon appears for the derivation of $X_{\min}^{(s)}$ or $X_{\max}^{(s)}$ with $s \geq 5$. Consequently, no general solution was available, contrarily to the continuous case where one disposes of a general methodology based on the search of the zeros of some orthogonal polynomials. In that sense the theory of discrete $s$-convex extremal distributions was limited to the case $s \leq 3$ and partially solved for $s = 4$.

This thesis aims to go beyond this limitation and proposes new arguments, based on a majorant/minorant polynomial method and the so-called "cut-criterion", that allow to derive the explicit extremal distributions for all $s$. However these cases are far more complicated to deal with because a subtle discussion about the points of support of the extremal distribution is needed. To illustrate that point, it is interesting to notice the close connection between the discrete version of the 1-, 2-, 3-convex extrema and the 4-convex maximum and their corresponding continuous extrema, for which a parallel theory is developed when the support of the random variables is the interval $[0, n]$. A comparison between them leads to the conclusion that the discrete extremal distributions can be easily obtained from the corresponding continuous extremal distributions. Indeed, discrete versions of the $s$-convex extrema are obtained by spreading the probability mass of all non-integer support points of the continuous distribution on the two integers that round it.

It is then tempting to conjecture that all discrete extrema can be obtained from their continuous counterparts and this would be a right strategy to solve our problem since an explicit formula for continuous extremal distributions can be written for all $s$. Surprisingly, this conjecture is wrong and the support of the discrete distribution does not appear as the neighbourhood in $\mathcal{N}_n$ of the support of the continuous distribution. It is thus challenging to find the form of the support of the discrete extremal disctribution. This question is addressed in the first chapter of Part I of this thesis where, given a nondegenerated moment space with $s$ fixed moments, explicit formulas for the discrete $s$-convex extremal distributions are derived for general non-negative integer $s$. These results are then applied to compute lower and upper bounds for the probability of extinction in a Galton-Watson branching process and for the Lundberg's coefficient in the classical insurance risk model with discrete claim amounts.

Let us also mention that all the previous results can easily be extended to moment spaces of discrete random variables valued in a finite set $\mathcal{D}_n$ of $(n+1)$ evenly-spaced points, with minimum $e_0$ and separation parameter $h > 0$ say, i.e. $\mathcal{D}_n = \{e_0 + ih : i = 0, \ldots, n\}$. Let $\Delta_h$ be the forward difference operator with increment $h$ defined for each function $\phi : \mathcal{D}_n \to \mathbb{R}$ by $\Delta_h \phi(e_0 + ih) = \phi(e_0 + (i+1)h) - \phi(e_0 + ih)$ for all $i = 0, \ldots, n-1$. Let $\Delta_h^k$, $k \geq 1$, be the $k$-th forward difference operator defined recursively by $\Delta_h^k \phi(e_0 + ih) = \Delta_h^{k-1}\phi(e_0 + (i+1)h) - \Delta_h^{k-1}\phi(e_0 + ih)$ for all $i = 0, \ldots, n-k$ (by convention, $\Delta_h^0 \phi \equiv \phi$). Taking for $\mathcal{U}_*^{\mathcal{S}}$ the class of the functions $\phi : \mathcal{D}_n \to \mathbb{R}$ such that $\Delta_h^s \phi(e_0 + ih) \geq 0$ for all $i = 0, \ldots, n-s$ yields the $\preceq_{s-\mathrm{cx}}^{\mathcal{D}_n}$ order.

Now, a more general situation is when the random variables take on values in an arbitrary (rather than equidistant) ordered finite grid of non-negative points, denoted by $\mathcal{E}_n = \{e_0, \ldots, e_n\}$ say. Stochastic orderings specific for comparing such random variables have been proposed by Denuit, Lefèvre & Utev (1999). They are defined by reference to the concept of divided differences that are built as follows. Let $\phi : \mathcal{S} \to \mathbb{R}$ and $x_0 < x_1 < \ldots < x_s \in \mathcal{S}$. Starting from

$$[x_i]\phi = \phi(x_i), \quad i = 0, \ldots, s,$$

the $s$th divided differences are defined recursively by

$$[x_0, \ldots, x_s]\phi = \frac{[x_1, \ldots, x_s]\phi - [x_0, \ldots, x_{s-1}]\phi}{x_s - x_0} = \sum_{i=0}^{s} \frac{\phi(x_i)}{\prod_{j=0; j \neq i}^{s}(x_i - x_j)}.$$

The order $\preceq_{s-\mathrm{cx}}^{\mathcal{S}}$ can then be defined by taking for $\mathcal{U}_*^{\mathcal{S}}$ the class of all the $s$-convex functions $\phi : \mathcal{S} \to \mathbb{R}$, i.e. the functions $\phi : \mathcal{S} \to \mathbb{R}$ such that $[x_0, \ldots, x_s]\phi \geq 0$ for any $x_0 < x_1 < \ldots < x_s \in \mathcal{S}$. This general approach works whatever the form of the support $\mathcal{S}$ of the random variables to be compared. As for the $\preceq_{s-\mathrm{cx}}^{\mathcal{N}_n}$-orders, the relation $\preceq_{s-\mathrm{cx}}^{\mathcal{S}}$ can only be used to compare random variables with the same first $s-1$ moments and is therefore restricted to moment spaces. The $s$-convex orders on an arbitrary grid are of direct interest in various fields of applications, especially for problems of risky decision making, portfolio selection, insurance premium evaluation and of option pricing. For example, in option pricing, the random process representing the stock price are often assumed to follow binomial/trinomial trees. At each time, the random variable that corresponds to the stock price is thus valued in an arbitrary ordered finite grid of non-negative points. In a binomial model for instance, the stock price is monitored over successive periods of time, at which only two price movements are possible. Usually, it is assumed that the stock price $S_i$ at time $i$ can either move up to a new level $uS_i$ or down to a new level $dS_i$, with $d < 1 < u$. If $S_0$ denotes the initial stock price, the random variable $S_i$ is thus valued in the discrete set $\{S_0u^i, S_0u^{i-1}d, S_0u^{i-2}d^2, \ldots, S_0u^2d^{i-2}, S_0ud^{i-1}, S_0d^i\}$.

At this stage, we could wonder whether there is anything to gain by considering the specific form of the support of the random variables to be compared (instead of viewing all of them valued in $\mathbb{R}^+$). For $s = 1, 2$ we have $\preceq^{\mathcal{S}}_{1-\text{cx}} \Leftrightarrow \preceq^{\mathbb{R}^+}_{1-\text{cx}} \Leftrightarrow \preceq_{\text{st}}$ and $\preceq^{\mathcal{S}}_{2-\text{cx}} \Leftrightarrow \preceq^{\mathbb{R}^+}_{2-\text{cx}} \Leftrightarrow \preceq_{\text{cx}}$ for any $\mathcal{S} \subseteq \mathbb{R}^+$ and the form of the support of the random variables to be compared is thus not relevant in the sense that they can all be seen as valued in $\mathbb{R}^+$. For $s \geq 3$, however, the discrete and real cases are no longer equivalent: Having two random variables valued in $\mathcal{D}_n$, the implication

$$X \preceq^{\mathcal{D}_n}_{s-\text{cx}} Y \Rightarrow X \preceq^{\mathcal{S}}_{s-\text{cx}} Y \Rightarrow X \preceq^{\mathbb{R}^+}_{s-\text{cx}} Y$$

always holds true for all $\mathcal{D}_n \subseteq \mathcal{S} \subset \mathbb{R}^+$, but the reciprocal is false in general. We thus get finer stochastic inequalities taking into account the particular form of the support. For example, in the context of decision analysis, if the decision-maker's preferences agree with some $s$-convex ordering, when comparing two alternatives, it is safer to consider them valued in a smaller set of outcomes rather than in a larger one (because any such comparison can be extended to a larger set but not reciprocally).

In the second chapter of Part I of this thesis, we first prove that the sufficient condition of crossing type established for $\preceq^{\mathbb{R}^+}_{s-\text{cx}}$ is also sufficient for $\preceq^{\mathcal{E}_n}_{s-\text{cx}}$. Then this result is exploited to get the extrema with respect to $\preceq^{\mathcal{E}_n}_{s-\text{cx}}$. Finally, bounds for the eventual ruin probability in the compound risk process model are derived when the first moments of the discrete claim amounts recorded by an insurance company are known. The extrema with respect to $\preceq^{\mathcal{E}_n}_{s-\text{cx}}$ will also be used in the financial applications of Part III of this thesis.

### Part II: Bounds using 1- and 2-convex orderings

As previously mentioned, the $s$-convex extrema are not always the best tool to determine bounds on distribution functions, on ruin probabilities and on stop-loss premiums. To that end, one has to come out the moment space and to appeal to the stochastic dominance $\preceq_{\rm st}$ and to the convex $\preceq_{\rm cx}$ relations. This is the aim of Part II of this thesis. In the first chapter, stochastic extrema among DFR distributions are derived. More explicitly, we derive upper and lower bounds for convex survival functions with known first moments. In particular, all the DFR distributions have concave distribution functions and that makes the class of risks with convex survival functions significant for actuarial applications. In the second chapter, it is showed how to make the best possible use of the information contained in the first few moments of an integer-valued random variable when one is interested in stop-loss premiums.

The problem of deriving bounds on distribution functions has been studied for a long time in the literature. The first one seems to be Markov's fundamental inequality, and since then a number of improvements have been obtained under additional assumptions on the underlying distribution function. For example, about two centuries ago, Gauss derived improved bounds when the distribution function is known to be concave. Considering more recent developments, Sengupta & Nanda (1999) derived a lower bound on a concave distribution function with known mean (they pointed out that no upper bound sharper than unity can be found) and Royden (1953) with known mean and variance. However, no results in the literature seem to be based on the knowledge of more than two moments.

The main idea behind the results of this thesis is to transform a constrained problem (finding bounds on a convex survival function) into an unconstrained one (the same without the convexity condition) using different probability transforms. Once the convexity condition is suppressed, we can use all the existing bounds on survival functions that exist in the literature.

The first approach is quite classical and relies on the Khinchine's representation theorem. Since all the concave distribution functions are unimodal about 0 (they possess decreasing densities), they can be represented as a mixture of uniform distributions. Then, it remains to find bounds on the mixing distribution, the moments of which are easily obtained from the initial ones. Chebyshev-type moment bounds can then be derived for the mixing distribution, leading to bounds for the survival probability under interest.

Another possibility is to take advantage of the remarkable properties of the stationaryexcess operator in the class of continuous concave distribution functions. Applying this operator to the continuous convex survival function under interest yields an unconstrained stop-loss premium, that we can bound using one of the many existing results in the literature. Remark that, in finance, stop-loss premiums represent option prices. Several papers have been devoted to the derivation of bounds on such quantities, and these results directly apply here. We see that this approach yields better bounds than those derived with the help of the Khinchine's representation theorem or with some of the competitors that can be found in the literature. Moreover, these methods have the advantage of being systematic and apply to any number of moments. This approach yields bounds in the stochastic dominance sense on a concave distribution function with known first moments. Of course, as two distributions with identical first moments cannot be ordered with respect to stochastic dominance, the extremal distributions will not share the sequence of moments of the original one.

Extrema with respect to the discrete version of the $s$-convex orders are useful for deriving bounds on $\mathbb{E}[\phi(X)]$ when $\phi$ is $s$-convex and $X$ is valued in $\mathcal{N}_n = \{0, 1, \ldots, n\}$ or, more generally, in an arbitrary grid of points. However, only the convex stochastic order relation allows to get bounds on ruin probabilities or stop-loss premiums. Therefore, we derive in the second chapter of this part the explicit form of the stochastic bounds with respect to the convex order, i.e. the infimum and the supremum with respect to $\preceq_{\text{cx}}$ when the first moments are known. In that respect, the study extends the results obtained by Jansen, Haezendonck & Goovaerts (1986) to discrete random variables. Of course, as two distributions with identical second moments cannot be ordered with respect to the convex stochastic ordering, the extremal distributions will not share the sequence of moments of the original one.

Specifically, we consider random variables with discrete support $\mathcal{N}_n = \{0, 1, \ldots, n\}$, and the class $\mathcal{M}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$ of all random variables with support in $\mathcal{N}_n$ and first $s-1$ moments $\mu_1, \mu_2, \ldots, \mu_{s-1}$ is supposed to be non void. Then, using the knowledge of $(\mu_1, \mu_2, \ldots, \mu_{s-1}, n)$, the aim is to find upper and lower bounds on $\mathbb{E}[(X-d)_+]$ valid for any $X \in \mathcal{M}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$, i.e. to fix $X_{\min,s}$ and $X_{\max,s}$ such that the stochastic inequalities $X_{\min,s} \preceq_{\text{cx}} X \preceq_{\text{cx}} X_{\max,s}$ hold true for any $X \in \mathcal{M}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$.

Results are given explicitely up to $s = 4$ and the method extends easily to any $s \geq 5$.

Securitization of longevity risk, which is undoubtedly one of the major actuarial risk for the next decades, is a good example of application of the previous results. To that end, we consider the random number of survivors in a given cohort when the forces of mortality obey to the Lee-Carter model. This model accounts for the mortality improvements that pose a challenge for the planning of public retirement systems as well as for the private life annuities business. Natural candidates for defining the benefits of longevity bonds or reinsurance treaties covering portfolios of life annuities involve the excess of the actual number of survivors to that expected from a public mortality index. If the expected number of survivors at time $t_0 + d$ is $\overline{n}_d$ then the payoff could be related to $(N - \overline{n}_d)_+$ where $N$ is the number of survivors at time $t_0 + d$ from an initial group of $n$ policyholders aged $x_0$ at time $t_0$. Bounds on $\mathbb{E}[(N - \overline{n}_d)_+]$ are then obtained from the theory developed in this thesis. Bounds for the eventual probability of ruin in the compound Binomial risk process can also easily be computed.

### Part III: Financial applications

The last part of this thesis is devoted to the use in finance of the discrete $s$-convex orders and their extrema.

The first application concerns with the extension to a dynamic setting of the notion of extremal distributions. Specifically, convex bounds on multiplicative processes are derived and extremal elements in the class of risk-neutral probability measures are investigated, leading to bound the prices of contingent claims for incomplete markets.

More precisely, we consider multiplicative discrete-time processes $\{X_n,\ n = 1, 2, \ldots\}$ obtained as follows. Starting from a sequence $\{Y_n,\ n = 1, 2, \ldots\}$ of positive independent random variables, we define recursively the $X_n$'s as $X_{n+1} = X_n Y_{n+1}$ $(n = 1, 2, \ldots)$ with $X_1 = Y_1$. Such a process can be seen as a multiplicative random walk with relative increase $Y_n$ at time $n$ and we considered it because it is widely used in finance to model the price of financial instruments. The problem studied in this part of the thesis is the derivation of processes extremal in the sense that any positive linear combination of the $X_n$'s is bounded in the convex order by the corresponding linear combinations of the components of the extremal processes.

The results are then applied to discrete-time contingent claims pricing models. The underlying assets are assumed to follow a discrete-time process and trading only takes place at some prespecified dates. If there are no arbitrage opportunities then the financial pricing amounts to compute the expectation of the discounted payoff under a risk-neutral probability measure. As we consider an incomplete market framework, the risk-neutral probability measure is not unique and we are in presence of an all class of risk-neutral measures. The class of risk-neutral probability measures can thus be considered as a class of distributions with fixed support and first moment. Extremal elements can then be identified within the set of risk-neutral distributions, leading to bounds on the prices of contingent claims. Examples within a trinomial model (i.e. the simplest discrete and incomplete market model) are discussed and it is seen that, despite their relative simplicity, the extremal processes lead to accurate bounds on option prices. To end with, let us mention that some of the results derived in this thesis are closely related to the work by RÜSCHENDORF (2002). Calling upon lower and upper hedging strategies, it appears that the extremal elements identified here provide bounds on the price of a large class of financial assets.

A financial institution such as an insurance company faces different types of risks, the major one being interest rate fluctuation. If interest rates changes, in either the level of interest rates or the shape of the yield curve, the insurance company is confronted to a risk of losses. Immunization is a technique used by actuaries in asset-liability management to protect the portfolio value against the interest rate risk. An excellent review of immunization theory can be found in PANJER (1998). The second application of this part of the thesis is about the interest risk management of insurance companies or banks. Specifically, classes of $s$-convex relations for arbitrary discrete random variables are used to find extremal strategy of immunization against the interest rate risk in the context of deterministic immunization theory.

The first author to find strategies to protect the portfolio of assets and liabilities against instantaneous variations of the term structure of interest rates seems to be REDINGTON (1952). He found that the strategy to immunize the portfolio value against interest rate fluctuations was to equate the duration of assets to that of the liabilities while requiring the cash flows from the assets to be more spread out than those from liabilities. Nowadays, this theory is still widely used in actuarial practice. However, problems with Redington's theory of immunization are that the yield curves are assumed to be flat and that it allows for arbitrage opportunities. Other models include the one of SHIU (1988) that provides necessary and sufficient conditions so that the change in the portfolio value is non-negative for any convex shift function of the term structure of interest rates: Under the assumption of convex interest shifts a portfolio is immunized if, and only if, a decomposition of asset inflows exists such that each component separately immunizes each liability outflow.

Quite recently, HÜRLIMANN (2002) noticed that there is a quite elementary connection between convex ordering and immunization which leads to an improvement of the technical understanding of this theory and for instance to the derivation of Shiu's necessary and sufficient condition for immunization under arbitrary convex shift factors of the term structure of interest rates. However, in the literature, we did not find any mention of the relationship between financial immunization theory and any other integral stochastic orderings than the convex one. In this thesis, we extend the necessary and sufficient condition for immunization under arbitrary convex shift factors to a necessary and sufficient condition under arbitrary $s$-convex shift factors. To that end, use is made of the $s$-convex integral stochastic orderings on an arbitrary grid. We also demonstrate that in the Nelson-Siegel framework, many shift functions are indeed $s$-convex for realistic values of their parameters. This makes the theoretical results attractive for practical implementation. Moreover, for the class of $s$-convex shifts, we give interesting results that extend some immunization results of the actuarial literature. Precisely, we define an immunization risk measure $R^s(X, Y)$ depending on the $s$th moments of the asset and liability risks and that extends the well-known Shiu measure (using the M-square index). Then, we see that the $s$-convex extremal distributions constitute the appropriate tool to provide immunization strategies that are maximal with respect to $R^s(X, Y)$.

## Plan of the thesis

All the previously mentioned results are gathered in different joint papers, divided in three categories. The first categorie contains papers devoted to the derivation of bounds using the $s$-convex extrema, while the second one contains those devoted to the derivation of bounds using the particular 1- and 2-convex orderings and taking all the available partial information into account. The third categorie is dedicated to financial applications. The papers are assembled as follows:

**Part I: Bounds using $s$-convex stochastic extrema**

- Courtois, C., Denuit, M., & Van Bellegem, S. (2006). Discrete $s$-convex extremal distributions: theory and applications. *Applied Mathematics Letters* **19**, 1367–1377.

- Courtois, C., & Denuit, M. (2006). $S$-convex extremal distributions with arbitrary discrete support. *Working Paper* **WP06-10**, Institute of Actuarial Sciences, UCL.

**Part II: Bounds using 1- and 2-convex orderings**

- Courtois, C., & Denuit, M. (2007). Bounds on convex reliability functions with known first moments. *European Journal of Operational Research* **177**, 365–377.

- Courtois, C., & Denuit, M. (2006). Moment bounds on expected shortfalls with applications. *Working Paper* **WP06-17**, Institute of Actuarial Sciences, UCL.

**Part III: Financial applications**

- Courtois, C., & Denuit, M. (2007). Convex bounds on multiplicative martingales, with applications to pricing in incomplete markets. *Insurance: Mathematics and Economics*, in press.

- Courtois, C., & Denuit, M. (2006). On immunization and $s$-convex extremal distributions. *Working Paper* **WP06-21**, Institute of Actuarial Sciences, UCL.