Vol. 64 (1996), No. 3, pp. 315–324

Journal of Economics
Zeitschrift für Nationalökonomie

Capacity Constraints May Restore the Existence of an Equilibrium in the Hotelling Model

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Received April 3, 1996; revised version received October 5, 1996

In this note, we consider the Hotelling model with linear transportation costs. We show that capacity constraints may restore the existence of an equilibrium for locations inside the first and third quartiles.

Keywords: minimum differentiation, Hotelling, capacity constraints.

JEL classification: L13.

1 Introduction

In a celebrated article, Bertrand (1883) initiated the analysis of strategic price competition by showing how competition between two firms would result in a competitive outcome. On two grounds this result has been challenged in the literature. First, Edgeworth (1925) placed emphasis on the fact that each firm must be able to meet demand at any price above the competitive price in order to obtain the Bertrand result. When firms hold limited levels of production capacities, no equilibrium in pure strategies exists in the model. Second, Hotelling attacked Bertrand, and Edgeworth, on the grounds that "more typical of real situations is the case in which the quantity sold by each merchant is a continuous function of two variables, his own price and his competitor's" (Hotelling, 1929, p. 44). The issue offered by Hotelling consists in introducing differentiation between the two products. However, he solves only one part of the problem. Indeed, when products are too similar, there exists no equilibrium in pure strategies in his model (see d'Aspremont et al., 1979). Moreover, the objection raised by Edgeworth is not addressed by Hotelling, since firms do not face capacity constraints. In fact, the existence of the price equilibrium identified by Hotelling in a model incorporating the presence of capacity constraints remains an open problem. It is the purpose of this note to address this question. In order to do this we study price competition in a model where Hotelling's product-differentiation framework is combined with the presence of capacity constraints.

Since these two canonical models are plagued with the nonexistence of an equilibrium in pure strategies, one would expect that their combination would make things even worse. We suggest hereafter that this is not necessarily the case. More precisely, we show that the introduction of capacity constraints may restore the existence of an equilibrium for locations of the firms at which no equilibrium exists in the original model of Hotelling, namely for locations inside the first and third quartiles.

The basic intuition of our result is summarized as follows. In the Hotelling model, an equilibrium may fail to exist when products are too similar. In this case indeed, undercutting the other's price in order to grab the whole market can be profitable. In this respect, the presence of capacity constraints obviously weakens the incentives to undercut since firms may not be in a position to serve the entire market. However, this is not sufficient to restore the existence of an equilibrium. Indeed, as shown by Edgeworth, the presence of capacity constraints may make it profitable for a firm to deviate by raising its price. In order to see this, consider the following argument. Given some prices such that both firms enjoy a positive market share, one firm, say firm 1, could find it profitable to raise its price, especially if firm 2 is already capacity-constrained. If this is the case, some rationing will appear in firm 2. Therefore, part of the rationed consumers could turn back to firm 1. If these consumers are sufficiently numerous, the upward deviation of firm 1 will be profitable. Summing up, we observe that the presence of capacity constraints generates two countervailing effects. On the one hand, it makes undercutting strategies less profitable, thereby possibly restoring equilibrium. On the other hand, it makes equilibrium existence problematic by generating additional incentives for (upward) deviations. We show hereafter that these two effects may cancel out. A central element in this respect is the level of reservation prices. Indeed, the lower the reservation price, the lower the incentive to deviate upward.

The paper is organized as follows. We present the model in Sect. 2. Section 3 establishes our main result and in Sect. 4 this result is briefly commented.

2 The Model

Let us consider a unit density interval [0, 1]. A continuum of consumers is uniformly distributed in the interval. Two firms are symmetrically located at a distance a from the ends of the interval, i.e., $a \in [0, 1/2]$. They sell a homogeneous good, produced at zero cost. They maximize profits by setting prices noncooperatively.

Consumers have unit demand. We assume further that consuming the product yields a surplus $S > 0$ whereas refraining from consuming any of the two products yields a utility level of 0.[1] When buying one of the products, the consumer goes to the shop and bears a transportation cost which is linear in the distance. Thus, we may define the reservation price of a consumer located at a distance x from the firm by $S - tx$, where t is the unit transportation cost.

Assuming that firm 1 (firm 2) is located at a distance a from the left (right) end of the market, we define the indirect utility of a consumer located at x in the interval [0, 1] as follows:

$u(x, 1) = S - t\,|x - a| - p_1$ when buying product 1.
$u(x, 2) = S - t\,|1 - a - x| - p_2$ when buying product 2.

Demand functions addressed to the firm are derived as follows. We denote the marginal consumer by $\tilde{x}(p_1, p_2)$. By definition, he is indifferent between the two products at prices (p_1, p_2), i.e., he satisfies equation $S - t\tilde{x} - p_1 = S - t(1 - \tilde{x}) - p_2$. Therefore we have $\tilde{x}(p_1, p_2) = (p_2 - p_1 + t)/2t$.

As long as $\tilde{x}(p_1, p_2) \in [a, 1 - a]$, all consumers located to the left of the marginal consumer prefer product 1, and conversely for the consumers located to the right. The demand functions $D_1(p_1, p_2) = \tilde{x}(p_1, p_2)$, $D_2(p_1, p_2) = 1 - \tilde{x}(p_1, p_2)$ hold.

Note that for some pair of prices it may happen that all consumers prefer product 1 to product 2, or product 2 to product 1. The first case occurs whenever $\tilde{x}(p_1, p_2) \geq 1 - a$. The undercutting price for firm 1 is the solution to this equation, i.e., $p_1^{u} = p_2 - t\,|2a - 1|$. The second case occurs whenever $\tilde{x}(p_1, p_2) \leq a$. The undercutting price is thus $p_2^{u} = p_1 - t\,|2a - 1|$.

In this model, the equilibrium in pure strategies is characterized as follows:

- The unique Nash equilibrium is given by $p_1^* = p_2^* = t$ provided $S \geq (3/2)t$ and $a \in [0, 1/4]$.
- When $a > 1/4$, there exists no equilibrium in pure strategies.[2]

1 In the original version of Hotelling's model, the possibility that consumers refrain from buying is not considered. Formally, this implies that S tends to $+\infty$.

2 See d'Aspremont et al. (1979).

Note that the parameter restriction $S \geq (3/2)t$ is made in order to ensure that the market is covered in equilibrium.

3 Capacity Constraints May Restore the Hotelling Equilibrium

In order to show that capacity constraints may restore the existence of the Hotelling equilibrium, we first show how their presence may rule out undercutting strategies. This is Lemma 1. Second, we show that capacity constraints generate the possibility for a firm to deviate from the Hotelling equilibrium with a higher price. However, we show that there exist levels for capacities such that this deviation is not profitable. This is Lemma 2. Finally, in Proposition 1 we show that the conditions of Lemma 1 and 2 may be satisfied simultaneously.

From d'Aspremont et al. (1979) we know that undercutting strategies are not profitable deviations from the Hotelling equilibrium whenever $a \in [0, 1/4]$. Whenever $a \in \left]1/4, 1/2\right]$, undercutting strategies are profitable and an equilibrium does not exist. However, when firms face limited capacities, the incentive to undercut is reduced. Indeed, when undercutting, a firm will be able to serve the market only up to its capacity. In Lemma 1, we formalize this intuition.

Lemma 1: For all symmetric locations $a \in \left]1/4, 1/2\right]$, and for given k_i, undercutting strategies are not profitable whenever $k_i \leq \frac{1}{4a}$.

Proof: The level of profits at the Hotelling equilibrium is equal to $t/2$. When $a > 1/4$ and firms do not face capacity constraints, the best reply against $p_j = t$ consists for firm i in undercutting the other's price with $p_i = 2at$. When firm i holds a limited capacity, the undercutting price is not affected, but the profits are, since demand is now given by the level of capacity k_i. It is then sufficient to solve $2atk_i \leq t/2$ to identify the critical level of capacity below which undercutting strategies are not profitable. We obtain $k_i \leq \frac{1}{4a}$. □

Lemma 1 states that for symmetric locations inside the first and third quartiles, capacity levels should be low enough in order to prevent undercutting strategies. However, this is not sufficient to restore existence of the equilibrium since capacity constraints generate the possibility for one firm to deviate from the equilibrium candidate by raising its price. In Lemma 2, we will derive a condition which is sufficient to ensure that this deviation is not profitable, whatever the location of the firms.

As mentioned previously, the profitability of upward deviations rests on the number of consumers recovered by the firm which creates rationing in the other firm. Therefore, the organization of rationing in the market is of central importance. In this note we follow Kreps and Scheinkman (1983) by assuming an efficient-rationing rule. The main property of this rationing rule is that rationed consumers are those with the lowest reservation price for the product. Note that in a location model, this rationing rule has a very simple interpretation. If we assume that going to the shop takes an amount of time which depends positively on the distance, the efficient-rationing rule organizes rationing on a "first arrived, first served" basis, i.e., the rationed consumers are always those who are located at the largest distance from the firm.

In Fig. 1, we have depicted a configuration with firms located at the limits of the interval. In this case rationed consumers are located in the middle of the interval, namely between $\tilde{x}(p_1, p_2)$ and k_1.

Let us focus first on the case of extreme locations. When firms do not face capacity constraints, the equilibrium is given by (t, t). The first obvious requirement for (t, t) to be an equilibrium in the presence of capacity constraints is $k_1, k_2 \geq 1/2$, i.e., firms must have enough capacity to meet demand at the equilibrium candidate. The intuition behind Lemma 2 is straightforward. Assume that k_1 is slightly greater than 1/2. In this case, $p_2^* = t$ may not be a best reply against $p_1^* = t$. Indeed, if firm 2 increases p_2, its demand decreases, but only up to the point where firm 1 hits k_1, so that $D_2(p_1^*, p_2) = 1 - k_1$. If p_2 is further increased, demand addressed to firm 2 remains unaffected until $\tilde{p}_2 = S - t(1 - k_1)$ since, under efficient rationing, all rationed consumers are still willing to buy product 2 at this price. For k_1 close to

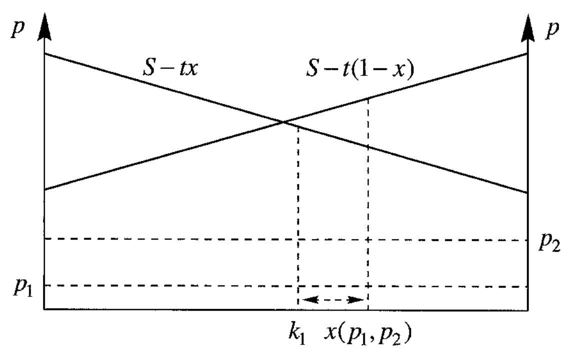

Fig. 1: A constellation of prices and capacity yielding a set of rationed consumers $[k_1, x(p_1, p_2)]$

$1/2$, we may expect that the corresponding profit $\pi_2(t, \tilde{p}_2) = \tilde{p}_2(1-k_1)$ is greater than $\pi_2(t, t) = t/2$.

Thus the profitability of upward deviations depends on the level of capacity of the other firm: the lower the capacity, the greater the incentive to deviate.[3] This provides the first step in establishing Lemma 2. The second step consists in extending the argument to the configurations with inside locations.

Lemma 2: Let us assume that $S \geq (3/2)t$; $k_1, k_2 \geq 1/2$; $k_1 \leq k_2$; $a \in [0, 1/2]$. A sufficient condition ensuring that firms will not deviate upward from the equilibrium candidate is $k_1 \geq 1 - [(S - \sqrt{S^2 - 2t^2})/2t]$.

Proof: Let us focus first on the case of extreme locations. First we show that $p_2 = S - t(1 - k_1)$ is the most profitable deviation for firm 2 against $p_1 = t$. The argument runs as follows. p_2 must be above the level which implies rationing at firm 1, i.e., $p_2 > 2tk_1$. Above this level, raising p_2 raises π_2. Indeed, under the efficient-rationing rule, rationed consumers are located to the right of k_1 (see Fig. 1). All these consumers are willing to buy product 2 as long as $p_2 \leq S - t(1 - k_1)$, which is the reservation price for product 2 for the consumer located in k_1. In other words, demand addressed to firm 2 remains constant while the price is increasing. Raising p_2 above $S - t(1 - k_1)$ is not profitable. Indeed, at this price we reach the monopoly profit function of firm 2. But since $S > (3/2)t$, we are already on the decreasing part of this profit function, so that a higher p_2 yields lower profits.

Second, we derive the condition which ensures that this deviation is not profitable. This amounts to compare $\pi_2^*(p_1^*, p_2^*) = t/2$ to $\pi_2(p_1^*, \tilde{p}_2) = (S-t(1-k_1))(1-k_1)$. We solve $(S-t(1-k_1))(1-k_1) - t/2 \leq 0$, for k_1. The relevant solution is $k_1 \geq 1-[(S-\sqrt{S^2 - 2t^2})/2t]$. Thus, we have shown that there exists no profitable deviation for firm 2 when k_1 is large enough. Note that this analysis extends to all configurations in which $k_2 \geq k_1$. Indeed, if the condition is satisfied for the low-capacity firm, it is also satisfied for the high-capacity one. Therefore, in the presence of asymmetric capacity, we do not need to bother about the deviations of the high-capacity firm.

It then remains to show that this condition can be extended to the case of symmetric locations in the interval. In order to see this, note that the residual demand within an inside-location context cannot be larger

3 Note also that in case of asymmetric capacities, we only need to consider the deviation of the low-capacity firm. If the deviation is profitable to the low-capacity firm, then a profitable deviation also exists for the high-capacity firm.

than the one prevailing for the same level of capacity in the extreme-locations context. Indeed, the only difference could result from the fact that some rationed consumers prefer not to consume rather than buying at the deviating firm. Therefore the conditions ensuring that a firm does not deviate with $\tilde{p}_i = S - t(1 - k_j)$ in the extreme-location game are sufficient to obtain the same property in the inside-location game, since incentives to deviate are at most equal. □

With the help of the two lemmas, we are now able to establish our main result. The following remark is in order at this step. Consider the incentives for deviations of firm 1, given some levels of capacities and locations. Lemma 1 states that the capacity of firm 1 should be low enough in order to prevent undercutting. On the contrary, Lemma 2 states that the capacity of firm 2 should be high enough in order to prevent upward deviations of firm 1. Thus, when we consider the incentives of the two firms simultaneously, we end up with an interval for the capacity of each firm k_i, $i = 1, 2$. The lower bound of the interval for k_i reflects Lemma 2: k_i must be high enough to prevent upward deviations of firm j. The upper bound reflects Lemma 1: k_i must be low enough to prevent undercutting by firm i. When establishing our proposition, we will focus on symmetric capacities as well as symmetric locations.[4]

Proposition 1: Let us assume that $S \geq (3/2)t$, $k_1 = k_2 \geq 1/2$. For all symmetric locations $a \in [0, 1/2]$, it is sufficient for the Hotelling equilibrium to exist that $k_i \in \left[1 - [(S - \sqrt{S^2 - 2t^2})/2t], 1/4a\right]$, $i = 1, 2$. There always exist parameter constellations for which this interval is not empty.

Proof: In order to prove Proposition 1 we simply have to check that our two lemmas may hold simultaneously. In our symmetric setting, this amounts to prove that for all symmetric locations, the interval $\left[1 - [(S - \sqrt{S^2 - 2t^2})/2t], 1/4a\right]$ is not empty. Indeed, in this interval we satisfy the conditions of our two lemmas.

4 Note that the result can be extended to asymmetric locations as well as asymmetric capacities. This point will be discussed in the final section.

Direct computation indicates the following:

$$1 - \frac{S - \sqrt{S^2 - 2t^2}}{2t} \leq k_i(a) = \frac{1}{4a}$$

$$\text{whenever} \quad S \leq \frac{1 - 8a + 24a^2}{4a(4a-1)} t = S^*(a) \ .$$

Thus, whenever $S \leq S^*(a)$, the interval is not empty. Note that $S^*(a)$ tends to $+\infty$ when a tends to $1/4$ and to $(3/2)t$ when a tends to $1/2$. Therefore under the condition that $S \geq (3/2)t$, a set of admissible value for S always exists.[5] □

4 Final Remarks

First, the role of S calls for an explanation. The intuition is that S should not be too high in order to preserve the existence of the equilibrium. Indeed, the higher S, the higher is $\tilde{p}_i$ and thus the more profitable it is to deviate upward.[6] The crucial role assigned to the level of the reservation price in the Hotelling model has been emphasized in Economides (1984). He shows that low reservation prices may restore the existence of an equilibrium in the Hotelling model, even for very similar locations. When S is relatively low, undercutting strategies may become unprofitable because consumers at the ends of the interval would refrain from buying. In the present note, our focus is different. We consider relatively high reservation prices, ensuring that no consumer ever refrains from buying. Then capacity constraints are considered. The intuition here is that a lower value of S enlarges the domain of capacity levels which is robust to upward deviations. Then, for given locations, this domain may include capacity levels which are low enough to prevent undercutting. Summing up, in Economides (1984)

5 Note that the range of capacities for which the interval $\left[1 - [(S - \sqrt{S^2 - 2t^2})/2t], 1/4a\right]$ is not empty shrinks when a tends to $1/2$. At the limit, i.e., when $a = 1/2$, the only value of S which could support (t, t) as an equilibrium is $(3/2)t$.

6 Note that in the original model of Hotelling, S tends to ∞. In this case, the mere presence of a capacity constraint destroys the equilibrium. Indeed, rationed consumers will never refrain from buying, so that, even for very large capacity of the opponent, there exists a price which is high enough to make the deviation profitable.

low reservation prices may restore existence by making undercutting unprofitable whereas in our model, they prevent upward deviations.

Second, it is important to note that, although Proposition 1 is established for the case of symmetric locations and capacities, its extension to asymmetric configurations is not problematic. Consider for instance the point of view of firm 1. Lemma 1 states a condition on k_1 as a function of a_2. Lemma 2 states a condition on the level of k_2, which does not depend on locations. The reverse is true from the point of view of firm 2. In other words, in case of asymmetric locations and capacities we are left with 4 inequalities defining an admissible range for capacities, and thus an upper bound for the reservation price S. In case of asymmetric locations, the restriction on S will depend on the location of the firm which is the closest to the center of the market.

In this note, our ambition was to study price competition between two firms within a framework which incorporates the two major objections raised against Bertrand (1883). We have shown that the presence of capacity constraints may restore the existence of the Hotelling equilibrium, even when products are very similar. The level of the reservation price plays a crucial role in establishing this result. As such, our results provide simple intuitions as to how the presence of capacity constraints affects equilibrium outcomes in the Hotelling model. However, these results are very partial ones. In particular, they are not sufficient to derive solid implications about location choices. Indeed, we have focused the analysis on pure strategy equilibrium. The analysis of location choices calls first for a complete characterization of the price subgames, including those where the equilibrium is in mixed strategies. This was clearly beyond the scope of the present note and is left for future research.

Acknowledgements

Most of this work has been done while the author was postdoctoral fellow at the University of Limburg at Maastricht, whose financial support is gratefully acknowledged. The author is grateful to Jean Gabszewicz and George Norman for comments on a previous version of this note. The detailed comments of two referees greatly improved the content of this paper as well as its presentation. The usual disclaimer applies. This research has been supported by a grant "Actions of Recherche Concertees" 93/98-162 of the Ministry of Scientific Research of the Belgian French-speaking Community.

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