DICTATORIAL DOMAINS *

Navin Aswal

University of Minnesota, Minneapolis, USA

Shurojit Chatterji

Indian Statistical Institute, New Delhi, India

and

Arunava Sen

Indian Statistical Institute, New Delhi, India

July 5, 1999

*This paper supercedes an earlier paper "Restricted Domains and the Gibbard-Satterthwaite Theorem" by Navin Aswal and Arunava Sen.

ABSTRACT

In this paper, we introduce the notion of a linked domain and prove that a non-manipulable social choice function defined on such a domain must be dictatorial. This result not only generalizes the Gibbard-Satterthwaite Theorem but also demonstrates that the equivalence between dictatorship and non-manipulability is far more robust than suggested by that theorem. We provide an application of this result in a particular model of voting. We also provide a necessary condition for a domain to be dictatorial and characterize dictatorial domains in the cases where the number of alternatives is three and four.

1 INTRODUCTION

In this paper, we pursue a line of inquiry which has as its starting point, the celebrated Gibbard-Satterthwaite Theorem (GS Theorem) (Gibbard [5] and Satterthwaite [12]). The GS theorem is an important result in the theory of incentives. It states that a surjective social choice function defined over an unrestricted domain of preferences is non-manipulable if and only if it is dictatorial (provided that there are at least three alternatives). In other words, it is impossible to design a non-trivial incentive scheme defined over an unrestricted domain where individuals do not have the opportunity to profitably misrepresent their preferences.

The negative conclusion of the GS Theorem depends critically on the domain of preferences which are considered to be admissible. If these preferences are restricted, it is well-known that possibility results can emerge. For example, if preferences are restricted to be single-peaked, then the majority voting rule is non-manipulable. However, the precise "frontier" between possibility and impossibility results in terms of the structure of the domain is not clearly understood and appears to be a formidable problem. In this paper we make an attempt to address some of these issues. Specifically, we examine the structure of dictatorial domains, i.e. those domains which have the property that all non-manipulable social functions defined over them (satisfying the weak requirement of unanimity) are dictatorial. Our primary conclusion is that the equivalence between non-manipulability and dictatorship is far more robust than indicated by the GS Theorem.

We consider strict orderings of the elements of a finite set of alternatives. The central concept in our paper is that of a *linked domain* of preferences; the main result states that all non-manipulable and unanimity satisfying social choice functions defined over such domains are dictatorial (provided that there are at least three alternatives). Linked domains are easy to describe. Two alternatives a and b are said to be *connected* if there exists an admissible preference ordering where a is ranked first and b second and another where b is first and a second. A domain is linked if we can arrange all the alternatives in a sequence which satisfies the following property: the second alternative is connected to the first and every alternative after the second is connected to at least two others before it in the sequence. It is immediately apparent that the assumption that a domain is linked is significantly weaker than the assumption that the domain is universal. Of course, the latter domain is a special case of the former so that our result is a generalization of the GS Theorem. One way to emphasize the relative generality of linked domains is to observe that they only places restrictions on the way alternatives are ranked first and second in the orderings which constitute the domain. Thus linked domains can be much smaller in size than the universal domain. We demonstrate the existence of one which has exactly $4M - 6$ alternatives where M is the cardinality of the set of alternatives. In contrast the universal domain is of size $M!$ which is a polynomial of order M.

We apply our main result to a model of voting introduced first by Barberá, Sonnenchein and Zhou [4]. There are N voters who have to elect some subset of a set of L candidates. Voters preferences over these alternatives are assumed to be separable; in other words, candidates do not impose externalities on other candidates by either their presence or absence. We consider a variant of this model where preferences remain separable but certain alternatives may not be feasible. For example, it may be the case that at least one candidate has to be elected or that no more than K can be elected and so on. We show by applying our main result that typically under these circumstances (i.e depending on what alternatives remain feasible) non-manipulability implies dictatorship. We note that these results cannot be obtained from the GS result as corollaries because there is a significant preference restriction.

Our result regarding linked domains is however not a characterization result for dictatorial domains. The task of finding a necessary and sufficient condition for a domain to be dictatorial appears to be a difficult one. We are nonetheless able to obtain a simple necessary condition which leads immediately to a characterization when there are three alternatives. We also provide a complete characterization in the case where there are four alternatives. The condition is quite intricate and the arguments give a flavour of the complexities involved in obtaining such a condition in the general case. Another conclusion that can be drawn from this analysis is that it is not sufficient in general to restrict attention to the first and second ranked alternatives in a preference ordering as is done for linked domains.

The paper is organized as follows. Section 2 introduces the model while Section 3 contains the main results. In Section 4 we consider the application to the constrained voting model while Section 5 deals with the special cases of three and four alternatives. Section 6 concludes while Appendix contains the proof of Theorem 5.3.

2 THE MODEL

Let $I = \{1, ..., N\}$ denote the set of individuals. Let A denote the set of alternatives. We assume that A is finite with $|A| = M$. Let $\mathbb{P}$ denote the set of strict orderings[1] of the elements of A. An admissible domain is a set $\mathbb{D}$ with $\mathbb{D} \subset \mathbb{P}$. A typical preference ordering will be denoted by P_i where $a_j P_i a_k$ will signify that a_j is preferred (strictly) to a_k under P_i. A preference profile is an element of the set $\mathbb{D}^N$. Preference profiles will be denoted by $P, \bar{P}, P'$ etc and their ith components as $P_i, \bar{P}_i, P'_i$ respectively with $i = 1, ..., N$. Let $(\bar{P}_i, P_{-i})$ denote the preference profile where the ith component of the profile P is replaced by $\bar{P}_i$.

A *Social Choice Function* or (SCF) f is a mapping $f : \mathbb{D}^N \rightarrow A$.

A SCF f is *manipulable* if there exists an individual i, an admissible profile P, and an admissible ordering $\bar{P}_i$ such that $f(\bar{P}_i, P_{-i}) P_i f(P)$.

A SCF is *non-manipulable* or *strategyproof* if it is not manipulable.

In this framework, it is assumed that an individual's preference ordering is his private information. A strategyproof SCF has the property that all individuals have a strong incentive to reveal this information truthfully. In particular, they cannot profit by lying, irrespective of the beliefs that they hold about the announcements of other individuals.

For all $P_i \in \mathbb{P}$ and $k = 1, ..., M$, let $r_k(P_i)$ denote the kth ranked alternative in P_i, i.e. $r_k(P_i) = a$ implies that $|\{b \neq a | b P_i a\}| = k - 1$.

A SCF is *unanimous* if $f(P) = a_j$ whenever $a_j = r_1(P_i)$ for all individuals $i \in I$.

Throughout the paper, we will assume that SCFs under consideration satisfy unanimity. This is an extremely weak assumption which merely requires an outcome which is first-ranked by all individuals (if it exists) to be chosen

[1] A strict ordering is a complete, transitive and antisymmetric binary relation.

by the SCF. It is well known that a SCF which is surjective and strategyproof must satisfy unanimity. We could therefore have proved our result assuming the weaker property of surjectivity, but since unanimity is a property which natural and appealing, we assume it directly.

A special class of SCFs is described below.

A SCF f is *dictatorial* if there exists an individual i such that, for all profiles P, $f(P) = r_1(P_i)$.

A dictatorial SCF clearly represents an ethically unsatisfactory procedure for making collective decisions. However, they are strategyproof. The dictator clearly has no incentive to lie because that can only be disadvantageous. On the other hand, individuals other than the dictator have no role to play in outcome selection and therefore cannot benefit from lying. Unfortunately, there is a large class of preference domains where strategyproofness implies dictatorship, so that there is no escape from this unpleasant dilemma. These domains which we define formally below, are the objects of our study.

DEFINITION 2.1 The domain $\mathbb{D} \subset \mathbb{P}$ is *dictatorial* if for all SCFs $f : \mathbb{D}^N \to A$ satisfying unanimity, [f is strategyproof] $\Rightarrow$ [f is dictatorial].

In the next section we proceed with our analysis of dictatorial domains.

3 THE MAIN RESULT

We begin by introducing the notion of a linked domain which is central to our paper. For what follows, we fix a domain $I\!\!D$.

DEFINITION 3.1 : A pair of alternatives a_j, a_k are *connected*, denoted $a_j \sim a_k$ if there exists $P_i, \bar{P}_i \in I\!\!D$ such that $r_1(P_i) = a_j$, $r_2(P_i) = a_k$, $r_1(\bar{P}_i) = a_k$, and $r_2(\bar{P}_i) = a_j$

It is clear that the relation $\sim$ is symmetric i.e. $a_j \sim a_k$ implies that $a_k \sim a_j$.

DEFINITION 3.2 : Let $B \subset A$ and let $a_j \notin B$. Then a_j is *linked to* B if there exists $a_k, a_r \in B$ such that $a_j \sim a_k$ and $a_j \sim a_r$.

DEFINITION 3.3 : The Domain $I\!\!D$ is *linked* if there exists a one to one function $\sigma : A \to A$ such that

(i) $a_{\sigma(1)} \sim a_{\sigma(2)}$

(ii) $a_{\sigma(j)}$ is linked to $\{a_{\sigma(1)}, a_{\sigma(2)}, .., a_{\sigma(j-1)}\}$, $j = 3, .., M$.

REMARK 3.1 : Let $I\!\!D$ denote a domain which is linked and let $\bar{I\!\!D}$ contain $I\!\!D$. Then $\bar{I\!\!D}$ is also linked.

We illustrate the notion of a linked domain by means of several examples.

EXAMPLE 3.1 : Let $\bar{I\!\!D}$ denote a domain which has the following property : for all $a_j, a_k \in A$, there exists $P_i \in I\!\!D$ such that $r_1(P_i) = a_j$ and $r_2(P_i) = a_k$.

It is clear that the domain described above is linked. The function σ can be an arbitrary one to one function.

EXAMPLE 3.2 : The domain $\mathbb{P}$ is linked. This follows from the trivial observation that the domain in Example 3.1 is linked and Remark 3.1.

EXAMPLE 3.3 : Let $A = \{a_1, a_2, ..., a_M\}$ and let $\mathbb{D}$ be a domain which induces the following connectivity structure: $a_1 \sim a_2$; $a_j \sim a_1$ and $a_j \sim a_2$, for all $j = 3, ..., M$. This domain satisfies conditions (i) and (ii) of Definition 3.3 and is therefore linked. Later, we shall use this domain to put an upper bound on the size of dictatorial domains. Note that this domain involves $2M - 3$ "connections". Since each connection requires 2 orderings, we can construct such a domain using exactly $4M - 6$ orderings.

Example 3.3 also illustrates a general procedure which can be used to construct numerous linked domains. We can start with a "connectivity" structure satisfying conditions (i) and (ii) and then use it to construct a domain by "filling in" alternatives ranked from 3 to M, arbitrarily.

We now give an instance of a domain which is it not linked. This is the classical single-peaked domain. It is well known that this domain is non-dictatorial and it is instructive to recognize the reason why it is not linked.

EXAMPLE 3.4 : Let $>$ be a strict ordering of the elements of A. The domain $\mathbb{D}$ is *single peaked* if, for all $P_i \in \mathbb{D}$ and $a_j \in A$ such that $a_j = r_1(P_i)$, we have

(i) $a_r > a_k > a_j$ implies that $a_k P_i a_r$

(ii) $a_j > a_k > a_r$ implies that $a_k P_i a_r$.

An immediate consequence of single peakedness is that two alternatives a_j and a_k can be connected only if they are contiguous with respect to the ordering $>$, i.e. if there does not exist an alternative a_r such that either $a_j > a_r > a_k$ or $a_k > a_r > a_j$. Suppose that the single peaked domain is linked and assume without loss of generality that the function σ in Definition 3.3 is the identity function. Then we must have three alternatives a_1, a_2 and a_3 such that $a_1 \sim a_2$, $a_2 \sim a_3$ and $a_1 \sim a_3$. Since every alternative has exactly two contiguous alternatives (one on either "side"), if a_1 is contiguous with a_2 and a_2 with a_3, then a_1 cannot be contiguous with a_3. Therefore the single peaked domain is not linked. Informally speaking, a linked domain requires at least one triple of alternatives which are connected (provided there are at least three alternatives) but this does not exist in a single peaked domain.

Before we state our main result, we need another definition.

DEFINITION 3.5 : A domain $\mathbb{D}$ is *minimally rich* if, for all $a \in A$, there exists $P_i \in \mathbb{D}$ such that $r_1(P_i) = a$. Observe that a linked domain is minimally rich.

Our main result is the following.

THEOREM 3.1 : Assume $M \geq 3$. If $\mathbb{D}$ is a linked domain, then it is dictatorial.

Our first step in the proof, Proposition 1, reduces the dimension of the problem from an arbitrary number of individuals to two individuals and is of independent interest. Results of this nature already exist in the literature. Kalai and Muller [6] (see also Muller and Satterthwaite [10]) prove a related "reduction result" but the voting procedures that they consider satisfy an additional (and very restrictive) assumption called "independence of non-optimal alternatives". The problem of designing a non-manipulable SCF can then be transformed into an equivalent Arrovian aggregation problem. Barberá and Peleg [3] also use a similar induction result in their proof of the GS Theorem but they assume unrestricted domain. Our result, on the other hand, remains valid for domains which satisfy the weak requirement of minimal richness. It is closely related to Theorem 4 in Kim and Roush [7]. Our proof is different and we provide the entire argument for the sake of completeness.

PROPOSITION 3.1 : Let $\Omega \subset I\!\!P$ be a minimally rich domain. Then, the following two statements are equivalent

(a) $f : \Omega^2 \to A$ is strategyproof and satisfies unanimity $\Rightarrow f$ is dictatorial

(b) $f : \Omega^N \to A$ is strategyproof and satisfies unanimity $\Rightarrow f$ is dictatorial, $N \geq 1$.

PROOF : (b) $\Rightarrow$ (a) is trivial. We now show that $(a) \Rightarrow$(b). Let $f : \Omega^N \to A$ be a non-manipulable SCF satisfying unanimity. Pick $i, j \in I$ and construct a SCF $g : \Omega^2 \to A$ as follows: for all $P_i, P_j \in \Omega^2$, $g(P_i, P_j) = f(P_i, P_j, ..., P_j)$.

Since f satisfies unanimity, it follows immediately that g satisfies this property. We claim that g is non-manipulable. If i can manipulate g at (P_i, P_j), then i can manipulate f at $(P_i, P_j, ..., P_j)$ which contradicts the assumption that f is non-manipulable. Suppose j can manipulate g, i.e. there exists $P_i, P_j, \bar{P}_j \in \Omega$ such that $b = g(P_i, \bar{P}_j) P_j g(P_i, P_j) = a$ where $b \neq a$. Now consider the sequence of outcomes obtained when individuals other than i progressively switch preferences from P_j to $\bar{P}_j$. Let $f(P_i, \bar{P}_j, P_j, ..., P_j) = a_1$. If a and a_1 are distinct, then aP_ja_1 since f is non-manipulable. Let $f(P_i, \bar{P}_j, \bar{P}_j, P_j, ..., P_j) = a_2$. Again, since f is non-manipulable, $a_1P_ja_2$ whenever a_1 and a_2 are distinct . Since P_j is transitive, aP_ja_2. Continuing in this manner to the end of the sequence, we obtain aP_jb which contradicts our initial assumption.

Since g is strategyproof and satisfies unanimity, statement (a) applies, so that either i or j is a dictator. Let $O_{-i}(P_i) = \{a \in A | a = f(P_i, P_{-i})$ for some $P_{-i} \in \Omega^{N-1}\}$. We claim that $O_{-i}(P_i)$ is either a singleton or the set A. Suppose i is the dictator in the SCF g. Let $P_i \in \Omega$ with $r_1(P_i) = a$. Since g satisfies unanimity, it follows that $g(P_i, P_j) = a$ where $r_1(P_j) = a$. Therefore $a \in O_{-i}(P_i)$. Suppose there exists $b \neq a$ such that $b \in O_{-i}(P_i)$, i.e. there exists $P_{-i} \in \Omega^{N-1}$ such that $f(P_i, P_{-i}) = b$. Let $\bar{P}_j \in \Omega$ such that $r_1(\bar{P}_j) = b$ (we are again using minimal richness). Observe that $f(P_i, \bar{P}_j, ..., \bar{P}_j) = b$ (progressively switch preferences of individuals j other than i from P_j to $\bar{P}_j$ and note that the outcome at each stage must remain b; otherwise an individual who can shift the outcome from b will manipulate). Therefore, $g(P_i, \bar{P}_j) = b$. This contradicts the assumption that i is the dictator. Therefore, $O_{-i}(P_i)$ is a singleton. Suppose j is the dictator. Then $A = \{a \in A | g(P_i, P_j) = a \text{ for some } P_j \in \Omega\} \subseteq O_{-i}(P_i)$, so that $O_{-i}(P_i) = A$.

We now complete the proof by induction on N. It is trivially true when $N = 1$. Suppose it is true for all societies of size less than or equal to $N - 1$. Consider the case where there are N individuals. Pick $i \in I$. From the earlier argument, either $O_{-i}(P_i)$ is a singleton or the set A. Suppose the latter case holds. Fix $P_i \in \Omega$ and define a SCF $g : \Omega^{N-1} \to A$ as follows : $g(P_{-i}) = f(P_i, P_{-i})$ for all $P_{-i} \in \Omega^{N-1}$. Since $O_{-i}(P_i) = A$, g satisfies unanimity because it is strategyproof and its range is A. Applying the induction hypothesis, it follows that there exists an individual $j \neq i$ who is a dictator. We need to show that the identity of this dictator does not depend on P_i. Suppose that there exists $P_i, \bar{P}_i \in \Omega$ such that the associated dictators are j and k respectively. Pick $a, b \in A$ such that aP_ib $(a \neq b)$. Since Ω is minimally rich, there exists $P_j, P_k \in \Omega$ such that $r_1(P_j) = b$ and $r_1(P_k) = a$. Let P_{-i} be the $N-1$ profile where j has the ordering P_j and k has the ordering P_k. Then $f(P_i, P_{-i}) = b$ and $f(\bar{P}_i, P_{-i}) = a$ and i manipulates at (P_i, P_{-i}). Therefore f is dictatorial. Suppose then that $O_{-i}(P_i)$ is a singleton. We claim that $O_{-i}(P_i)$ must be a singleton for all $P_i \in \Omega$. Suppose not, i.e. there exists $\bar{P}_i \in \Omega$ such that $O_{-i}(\bar{P}_i) = A$. From our earlier argument, there exists an individual $j \neq i$ who is a dictator. But this would imply that $O_{-i}(\bar{P}_i)$ is a singleton. Therefore, it must be the case that $O_{-i}(P_i)$ is a singleton for all $P_i \in \Omega$. But this implies that individual i is a dictator. ■

PROOF OF THEOREM 3.1 : In view of Proposition 3.1, we only need to show that if $\mathbb{D}$ is linked, then $f : \mathbb{D}^2 \to A$ is strategyproof and satisfies unanimity $\Rightarrow$ f is dictatorial. The proof extensively employs the option set technique of Barberá and Peleg [3].

In what follows, we assume that $I = \{1, 2\}$. Let $\mathbb{D}$ be a linked domain and assume without loss of generality that the function σ in Definition 3.3 is the identity function. Let $\mathbb{D}^j = \{P_i \in \mathbb{D} | r_1(P_i) = a_j \text{ for some } a_j \in A\}, j = 1, \cdots, M$. For all $P_1, P_2 \in \mathbb{D}$, let $O_2(P_1) = \{a \in A | a = f(P_1, P_2) \text{ for some } P_2 \in \mathbb{D}\}$ and $O_1(P_2) = \{a \in A | a = f(P_1, P_2) \text{ for some } P_1 \in \mathbb{D}\}$. We note that for all $P_1, P_2 \in \mathbb{D}$, $f(P_1, P_2) = \max(P_2, O_2(P_1)) = \max(P_1, O_1(P_2))$.[2] (For a proof of this statement, see Barberá and Peleg [3]).

Our proof consists in establishing two steps.

STEP 1 : There exists $j \in \{1, 2\}$ such that $a_1, a_2, a_3 \in O_j(P_i)$ for all $P_i \in \mathbb{D}^1 \cup \mathbb{D}^2 \cup \mathbb{D}^3$.

STEP 2 : If $a_1, a_2, ...a_{l-1} \in O_j(P_i)$ where $P_i \in \mathbb{D}^1 \cup \mathbb{D}^2 .... \cup \mathbb{D}^{l-1}$, then $a_1, a_2, ..., a_{l-1}, a_l \in O_j(P_i)$ where $P_i \in \mathbb{D}^1 \cup \mathbb{D}^2 .... \cup \mathbb{D}^{l-1} \cup \mathbb{D}^l$, $l = 4, ..., M$.

We proceed to establish Step 1 through a sequence of lemmas. First note that since $\mathbb{D}$ is linked and σ is the identity function, we have $a_1 \sim a_2$, $a_2 \sim a_3$ and $a_3 \sim a_1$.

LEMMA 3.1 : Let $P_1, \bar{P}_1 \in \mathbb{D}^j$ with $j = 1, 2, 3$. Then $O_2(P_1) \cap \{a_1, a_2, a_3\} = O_2(\bar{P}_1) \cap \{a_1, a_2, a_3\}$.

PROOF : Suppose not. Assume without loss of generality that $r_1(P_1) = r_1(\bar{P}_1) = a_1$. Since f satisfies unanimity, a_1 both $O_2(\bar{P}_1)$ and $O_2(P_1)$. Since the Lemma is assumed to be false, we can further assume w.l.o.g that $a_2 \in O_2(P_1) - O_2(\bar{P}_1)$. Since $a_1 \sim a_2$, there exists $P_2 \in \mathbb{D}$ such that $r_1(P_2) = a_2$ and $r_2(P_2) = a_1$. Therefore, $f(P_1, P_2) = a_2$ and $f(\bar{P}_1, P_2) = a_1$. But then individual 1 manipulates at (P_1, P_2). ■

LEMMA 3.2 : Let $P_1 \in \mathbb{D}^j$ with $j = 1, 2, 3$. Then $O_2(P_1) \cap \{a_1, a_2, a_3\}$ is

[2] For all $P_i \in \mathbb{D}$ and $B \subset A$, let max (P_i, B) denote the maximal element in B according to P_i.

either a singleton or the set $\{a_1, a_2, a_3\}$.

PROOF : Suppose not. Assume $r_1(P_1) = a_1$. Assume w.l.o.g that $a_2 \notin O_2(P_1)$ and $a_3 \in O_2(P_1)$. Lemma 3.1 and the hypothesis that $a_1 \sim a_2$ imply that we can assume $r_2(P_1) = a_2$. Pick $P_2 \in \mathbb{D}$ such that $r_1(P_2) = a_2$ and $r_2(P_2) = a_3$ (feasible, since $a_2 \sim a_3$). Then $f(P_1, P_2) = a_3$. Let $\bar{P}_1 \in \mathbb{D}$ be such that $r_1(\bar{P}_1) = a_2$. It cannot be the case that $f(\bar{P}_1, P_2) = a_2$ because then individual 1 would manipulate at (P_1, P_2). But this contradicts the fact that f satisfies unanimity. ■

LEMMA 3.3 : Either $O_2(P_1) \cap \{a_1, a_2, a_3\}$ is a singleton for all $P_1 \in \mathbb{D}^1 \cup \mathbb{D}^2 \cup \mathbb{D}^3$ or $O_2(P_1) \cap \{a_1, a_2, a_3\} = \{a_1, a_2, a_3\}$ for all $P_1 \in \mathbb{D}^1 \cup \mathbb{D}^2 \cup \mathbb{D}^3$.

PROOF : Suppose not. In view of Lemma 3.2, we can assume w.l.o.g that $O_2(P_1) \cap \{a_1, a_2, a_3\} = \{a_1\}$ and $O_2(\bar{P}_1) \cap \{a_1, a_2, a_3\} = \{a_1, a_2, a_3\}$ for some $P_1, \bar{P}_1 \in \ ID^1 \cup \mathbb{D}^2 \cup \mathbb{D}^3$. Clearly $a_1 = r_1(P_1)$. We can assume w.l.o.g that $a_2 = r_1(\bar{P}_1)$ (it follows from Lemma 3.1 that $a_1 \neq r_1(\bar{P}_1)$). We can also assume (using Lemma 3.1) that $a_1 = r_2(\bar{P}_1)$. Let $P_2 \in \mathbb{D}$ be such that $r_1(P_2) = a_3$. Then $f(\bar{P}_1, P_2) = a_3$ and $f(P_1, P_2) = a_1$. Since $a_1 \bar{P}_1 a_3$, individual 1 manipulates at $(\bar{P}_1, P_2)$. ■

LEMMA 3.4 : If $O_2(P_1) \cap \{a_1, a_2, a_3\}$ is a singleton for all $P_1 \in \mathbb{D}^1 \cup \mathbb{D}^2 \cup \mathbb{D}^3$, then $O_1(P_2) \cap \{a_1, a_2, a_3\} = \{a_1, a_2, a_3\}$ for all $P_2 \in \mathbb{D}^1 \cup \mathbb{D}^2 \cup \mathbb{D}^3$. If on the other hand, $O_2(P_1) \cap \{a_1, a_2, a_3\} = \{a_1, a_2, a_3\}$ for all $P_1 \in \mathbb{D}^1 \cup \mathbb{D}^2 \cup \mathbb{D}^3$, then $O_1(P_2)$ is a singleton for all $P_2 \in \mathbb{D}^1 \cup \mathbb{D}^2 \cup \mathbb{D}^3$.

PROOF : Replacing $O_2(P_1)$ by $O_1(P_2)$ in Lemmas 3.1-3.3, it follows that

$O_1(P_2) \cap \{a_1, a_2, a_3\}$ is either a singleton or the set $\{a_1, a_2, a_3\}$ for all $P_2 \in \mathbb{D}^1 \cup \mathbb{D}^2 \cup \mathbb{D}^3$.

Pick $\bar{P}_1$ such that $r_1(\bar{P}_1) = a_1, r_2(\bar{P}_1) = a_2$ and $\bar{P}_2$ such that $r_1(\bar{P}_2) = a_2, r_2(\bar{P}_2) = a_1$.

Suppose first that these sets are both singletons for all $P_1, P_2 \in \mathbb{D}^1 \cup \mathbb{D}^2 \cup \mathbb{D}^3$. Clearly, $O_1(\bar{P}_2) \cap \{a_1, a_2, a_3\} = \{a_2\}$ and $O_2(\bar{P}_1) \cap \{a_1, a_2, a_3\} = \{a_1\}$. Since $f(\bar{P}_1, \bar{P}_2) = \max(\bar{P}_1, O_1(\bar{P}_2))$, we have $f(\bar{P}_1, \bar{P}_2) = a_2$. But, since $f(\bar{P}_1, \bar{P}_2) = \max(\bar{P}_2, O_2(\bar{P}_1))$, we have $f(\bar{P}_1, \bar{P}_2) = a_1$. We have a contradiction to the assumption that f is singlevalued.

Suppose next that $O_1(P_2) \cap \{a_1, a_2, a_3\} = O_2(P_1) \cap \{a_1, a_2, a_3\}$ for all $P_1, P_2 \in \mathbb{D}^1 \cup \mathbb{D}^2 \cup \mathbb{D}^3$. Pick $\bar{P}_1, \bar{P}_2$ as before. Since $f(\bar{P}_1, \bar{P}_2) = \max(\bar{P}_1, O_1(\bar{P}_2))$, we have $f(\bar{P}_1, \bar{P}_2) = a_1$. But, since $f(\bar{P}_1, \bar{P}_2) = \max(\bar{P}_2, O_2(\bar{P}_1))$, we have $f(\bar{P}_1, \bar{P}_2) = a_2$. We have a contradiction to the assumption that f is singlevalued. ■

Lemmas 3.1 - 3.4 establish Step 1. We assume without loss of generality that individual j in the statement of Step 1 is individual 1. We now turn to Step 2. In view of the assumption that $j = 1$ in Step 1, we will henceforth write the option set $O_1(P_2)$ as $O_(P_2)$.

STATEMENT $*$: Pick an integer l in the set $\{4, .., M\}$. Since a_l is linked to $\{a_1, ..., a_{l-1}\}$, there exists $a_i, a_j \in \{a_1, ..., a_{l-1}\}$ such that $a_l \sim a_i$ and $a_l \sim a_j$.

LEMMA 3.5 : Pick l in the set $\{4, .., M\}$ and let $P_2 \in \mathbb{D}^i \cup \mathbb{D}^j$ where a_i and a_j are specified in $(*)$. Then, $a_l \in O(P_2)$.

PROOF : We first claim that if $a_l \in O(\bar{P}_2)$ for some $\bar{P}_2 \in \mathbb{D}^j$, then $a_l \in O(P_2)$ for all $P_2 \in \mathbb{D}^j$. Suppose not, i.e. suppose $a_l \in O(\bar{P}_2) - O(P_2)$ for some $\bar{P}_2, P_2 \in \mathbb{D}^j$. Let $\bar{P}_1$ be such that $r_1(\bar{P}_1) = a_l, r_2(\bar{P}_1) = a_j$ (that such a $\bar{P}_1$ exists follows from $(*)$). Since $a_j \in O(P_2)$, it follows that $f(\bar{P}_1, P_2) = a_j$. But $f(\bar{P}_1, \bar{P}_2) = a_l$. Since $a_j \ \bar{P}_2 \ a_l$,individual 2 manipulates at $(\bar{P}_1, \bar{P}_2)$.

We next claim that $a_l \in O(P_2)$ for all $P_2 \in \mathbb{D}^j$. Suppose $a_l \notin O(\bar{P}_2)$ for some $\bar{P}_2 \in \mathbb{D}^j$. In view of the claim in the previous paragraph, we can assume w.l.o.g that $r_1(\bar{P}_2) = a_j, r_2(\bar{P}_2) = a_l$. Let $\bar{P}_1$ be such that $r_1(\bar{P}_1) = a_l$ and $r_2(\bar{P}_1) = a_i$ where a_i is as specified in Statement$*$. Since $a_l \notin O(\bar{P}_2)$ but $a_i \in O(\bar{P}_2)$ by hypothesis, we have $f(\bar{P}_1, \bar{P}_2) = a_i$. But then 2 can manipulate by announcing $P_2 \in \mathbb{D}^l$. By unanimity, $f(\bar{P}_1, P_2) = a_l$ and $a_l \ \bar{P}_2 \ a_i$ so that 2 manipulates at $(\bar{P}_1, \bar{P}_2)$.

The previous arguments imply that $a_i \in O(P_2)$, $P_2 \in \mathbb{D}^j$. By an identical argument, $a_i \in O(P_2)$, $P_2 \in \mathbb{D}^i$. This proves Lemma 3.5. ■

LEMMA 3.6 : $a_l \in O(P_2)$ for all $P_2 \in \mathbb{D}^1 \cup \mathbb{D}^2 \cup ... \cup \mathbb{D}^l$.

PROOF : In view of Lemma 3.5, we need only consider the case where $P_2 \in \mathbb{D}^r$ where $a_r \in \{a_1, .., a_{l-1}\}$ and $a_r \neq a_i, a_j$. Suppose $a_l \notin O(\bar{P}_2)$ for some $\bar{P}_2 \in \mathbb{D}^r$. Let $P_2^{'} \in \mathbb{D}^j$ and let $\bar{P}_1$ be such that $r_1(\bar{P}_1) = a_l, r_2(\bar{P}_1) = a_j$. From Lemma 3.5, it follows that $f(\bar{P}_1, P_2^{'}) = a_l$. On the other hand, since $a_l \notin O(\bar{P}_2)$ by assumption and $a_j \in O(\bar{P}_2)$ by hypothesis, $f(\bar{P}_1, \bar{P}_2) = a_j$. Since $a_j \ P_2^{'} \ a_l$, individual 2 manipulates at $(\bar{P}_2, P_2^{'})$. This completes the proof of Lemma 3.6.■

LEMMA 3.7 : For all $P_2 \in \mathbb{D}^l$, it must be true that $a_i, a_j \in O(P_2)$.

PROOF : Suppose $a_j \notin O(\bar{P}_2)$ for some $\bar{P}_2 \in \mathbb{D}^l$. Using $(*)$, we can pick P_1, P_2 such that $r_1(P_1) = a_j$, $r_2(P_1) = a_l$ and $r_1(P_2) = a_i$, $r_2(P_2) = a_l$. Since $a_j \in O(P_2)$ by hypothesis, $f(P_1, P_2) = a_j$. Since $a_l \in O(\bar{P}_2)$ and $a_j \notin O(\bar{P}_2)$ by assumption, it must be the case that $f(P_1, \bar{P}_2) = a_l$. But $a_l \ P_2 \ a_j$. Therefore 2 manipulates at (P_1, P_2). By an identical argument, if follows that $a_i \in O(P_2)$ for all $P_2 \in \mathbb{D}^i$. ■

LEMMA 3.8 : For all $a_r, a_s \in \{a_1, ..., a_{l-1}\}$, if $a_r \sim a_s$ and $a_r \in O(P_2)$, $P_2 \in D^l$, then $a_s \in O(P_2)$.

PROOF : Suppose not, i.e. let $a_r, a_s \in \{a_1, ..., a_{l-1}\}$, $a_r \sim a_s$ and let $\bar{P}_2 \in \mathbb{D}^l$ such that $a_r \in O(\bar{P}_2)$ but $a_s \notin O(\bar{P}_2)$. Let $P_2^{'} \in \mathbb{D}^r$ and let P_1 be such that $r_1(P_1) = a_s$, $r_2(P_1) = a_r$ (such a P_1 exists since by assumption $a_r \sim a_s$). Since $a_r, a_s \in \{a_1, ..., a_{l-1}\}$, $a_s \in O(P_2^{'})$, by hypothesis so that $f(P_1, P_2^{'}) = a_s$. Since $a_r \in O(\bar{P}_2)$ and $a_s \notin O(\bar{P}_2)$ by assumption, $f(P_1, \bar{P}_2) = a_r$. But $a_r P_2^{'} a_s$. Therefore 2 manipulates at (P_1, P_2). ■

LEMMA 3.9 : for all $a_r \in \{a_1, ..., a_{l-1}\}$ and $P_2 \in \mathbb{D}^l$, it must be true that $a_r \in O(P_2)$.

PROOF : Pick $P_2 \in \mathbb{D}^l$. From Lemma 3.7, we know that $a_i, a_j \in O(P_2)$. Pick $a_r \in \{a_1, ..., a_{l-1}\}$. Since $\mathbb{D}$ is Linked, there must exist a sequence $b_0, b_1, ..., b_t \in A$ such that $b_0 = a_j, b_t = a_r$ and $b_0 \sim b_1$, $b_1 \sim b_2$,..., $b_{t-1} \sim b_t$. (This follows since for all $a_r \in \{a_1, ..., a_{l-1}\}$, there exists $a_s \in \{a_1, ..., a_{l-1}\}$ such that $a_r \sim a_s$). Applying Lemma 3.8 repeatedly, it follows that $a_r \in O(P_2)$. ■

Lemmas 3.5 - 3.9 establish Step 2. Combining the two steps it follows that for all $a_j \in A$ and $P_2 \in \mathbb{D}$, we have $a_j \in O(P_2)$. This immediately implies that individual 1 is a dictator and completes the proof of the Theorem. ■

We now state some consequences of Theorem 3.1.

COROLLARY 3.1 : (Barberá-Peleg [3]) : Assume $M \geq 3$ and let $\mathbb{D}$ be as described in Example 3.1. The SCF $f : \mathbb{D}^N \rightarrow A$ is strategyproof $\Leftrightarrow$ f is dictatorial.

Barberá -Peleg [3] did not formally claim to have proved the result above. However, their proof of the GS Theorem (at least in the case where N=2) remains valid when it is assumed that the domain is the one described in Example 3.1.

Another corollary is the Gibbard-Satterthwaite Theorem.

COROLLARY 3.2 (Gibbard [2], Satterthwaite [6]) : Assume $|A| \geq 3$. The SCF $f : \mathbb{P}^N \rightarrow A$ is strategyproof $\Leftrightarrow$ f is dictatorial.

Both these corollaries follow immediately from an application of Theorem 3.1 and our earlier observation that the domains described in Examples 3.1 and 3.2 are linked.

Our final result in this section is an upper bound on the smallest (in terms of cardinality) dictatorial domain.

PROPOSITION 3.2 : There exists a dictatorial domain $\mathbb{D}$ such that $|\mathbb{D}| = 4M - 6$.

This result is an immediate consequence of our remark that the domain constructed in Example 3.3 is linked. Dictatorial domains can therefore be very "small". Observe that the size of the universal domain assumed in the Gibbard-Satterthwaite Theorem, is a polynomial of order M. The domain where the top pair can be chosen arbitrarily (the domain in Example 3.1) is of cardinality at least $M^2 - M$. As Proposition demonstrates, there are dictatorial domains which are even smaller in the sense that they are linear in M. An interesting question is whether the bound obtained in Proposition 3.2 is tight. We are unable to answer this question generally because Theorem 3.1 provides only a sufficient condition for a domain to be dictatorial. We note however, that since SCFs must satisfy unanimity, the minimal size of domains under consideration is M.

4 AN APPLICATION : VOTING UNDER CONSTRAINTS

In this section we apply the main result of the previous section to a model of voting under constraints. The basic model is due to Barberá, Sonnenschein and Zhou (BSZ)[4] and is intended to represent a situation where members are being elected to a club (for instance). There are L candidates in an election and there are N voters. Any subset (including the null set) of the set of candidates can be elected. Voters have preferences over all such subsets i.e. over 2^L alternatives. These preferences are assumed to be separable (defined formally below) which implies that each voter has an unambiguous opinion of whether a given candidate should be elected or not. BSZ demonstrate that the non-manipulability assumption characterizes a class of SCFs which they refer to as *voting by committee.* Each candidate is considered in isolation and her election decided on the basis of a voting rule such as majority voting. These results have been extended and generalized further in Barberá, Gul and Stachetti [1] and Le-Breton and Sen [8],[9] and Serizawa [13].

A critical feature of the model described above is that the set of alternatives is a product set. In this section we consider a variant of this model where we allow for the possibility that certain alternatives are infeasible so that the product structure of the set of alternatives is destroyed. For instance, it may be the case that at least K candidates have to be elected or between K_1 and K_2 candidates have to be elected, at least 1 candidate has to be elected and so on. An immediate consequence of such an assumption is that SCFs where decisions on candidates are made individually, are no longer admissible. But are there "other" strategyproof SCFs in these circumstances? We are able to show that, in certain interesting cases, the answer to this question is "no". Under different assumptions on the feasible set of alternatives (now no longer a product set), we show that non-manipulability implies dictatorship. Moreover, we derive these results by suitable applications of Theorem 3.1. An important observation in this context is that the Gibbard-Satterthwaite result cannot be applied because preferences are restricted by the assumption that they are separable.

Barberá, Masso and Neme [3], investigate a general problem of constrained voting. They consider product domains and assume that preferences are Multi-dimensional single-peaked. Their main result states that if the maximal elements of preferences lie in the feasible set, then non-manipulable SCFs are " generalized median voting schemes" which satisfy a complicated condition called the intersection property. Our formulation does not require maximal elements of preferences to be feasible. We are able to show that in the specific model of BSZ, constraints on the feasible set typically leads to dictatorship.

We now proceed to details.

The set of individuals, or the set of voters is once again, the set $I = \{1, 2, .., N\}$. The set of candidates is the set is $\{1, 2, .., L\}$. The set of alternatives A, is the set of all subsets of the set of candidates. Thus $A = \{0, 1\}^L$ and a typical element a of the set A is an L-tuple whose jth component denoted by a_j is either 0 or 1. Then a represents the set where candidate j belongs if a_j is 1 and does not belong if a_j is 0 with $j = 1, 2, ., L$

DEFINITION 4.1 : An ordering P_i of the elements of A is *separable* if, for all $a_j, b_j \in \{0, 1\}$, $x_{-j}, y_{-j} \in \{0, 1\}^{L-1}$ and $j = 1, 2, .., L$, if $(a_j, x_{-j})P_i(b_j, x_{-j})$, then $(a_j, y_{-j})P_i(b_j, y_{-j})$.

An ordering is separable if a candidate does not exert "externality effects" on other candidates. For example, preferences of the following kind are ruled out: " I prefer candidate 1 to be included rather than excluded if candidate 2 is included but excluded when 2 is also excluded ". Clearly separable preferences induce unambiguous preference over the inclusion/exclusion of every candidate.

Let $\bar{I\!\!D}$ denote the set of separable orderings over the set A.

Let B be a subset of A. A SCF is a mapping $f : \bar{I\!\!D}^N \to B$. We assume without loss of generality that this mapping is onto.

We describe below a particular class of subsets of A which are of special interest to us.

Let K_1 and K_2 be integers lying between 0 and L with $K_1 \leq K_2$. We shall let the set $B(K_1, K_2) = \{a \in A | \sum a_j = r, K_1 \leq r \leq K_2\}$. We shall refer to sets of the kind $B(K_1, K_2)$ as *interval subsets* of the set A. Thus $B(K_1, K_2)$ denotes the set of alternatives where between K_1 and K_2 candidates are elected. A strict interval subset of A is an interval subset which is a strict subset of A. Observe that this implies that it is not the case that $K_1 = 0$ and $K_2 = L$.

Preferences over an arbitrary subset B can be induced in a natural manner from the preferences over A as specified by the domain $\bar{\mathbb{D}}$. The domain of the induced preferences over B is denoted $\bar{\mathbb{D}}^B$. Lemma 4.1 below shows that the value of a strategyproof SCF f defined on $\bar{\mathbb{D}}$ can depend only the preferences that it induces on B. We show that whenever B is an interval subset, or is of the form $B = A - \{a\}$, where $a \in A$, it is the case that $\bar{\mathbb{D}}^B$ satisfies the requirements of a linked domain. Hence any strategyproof f defined on $\bar{\mathbb{D}}$ but whose range is B must be dictatorial.

We now state our results formally.

THEOREM 4.1 : Let B be a strict interval subset of A with $L \geq 3$. Then the SCF $f : \bar{\mathbb{D}}^N \to B$ is strategyproof only if it is dictatorial.

THEOREM 4.2 : Let $L \geq 2$. Pick $a \in A$ and let $B = A - \{a\}$. Then the SCF $f : \bar{\mathbb{D}}^N \to B$ is strategyproof only if it is dictatorial.

The second result is perhaps more striking than the first. It states that even if a single alternative is removed from the feasible set, then all the possibility results disappear and we are left with dictatorship.

Before we can apply 3.1 in the previous context, we need some preliminary results. The first one is quite standard and states that the value of a strategyproof SCF at any profile can depend only on preferences over the set of alternatives which are in the range of the SCF.

We say that two preference profiles P and $\bar{P}$ agree on a subset X of A if, for all i and all $a, b \in X$, $aP^i b$ if and only if $a\bar{P}^i b$.

LEMMA 4.1 : Let f be a strategyproof SCF whose range is the set B. Let P and $\bar{P}$ be two admissible preference profiles which agree on B. Then $f(P) = f(\bar{P})$.

PROOF : It clearly suffices to show that $f(P) = f(\bar{P}_1, P_{-1})$. Suppose without loss of generality that this is not true and $f(P) = a \neq b = f(\bar{P}_1, P_{-1})$. Clearly, $a, b \in X$. If $bP_1 a$, then 1 manipulates at P. If on the other hand, $aP_1 b$, then $a\bar{P}_1 b$ and 1 manipulates at $(\bar{P}_1, P_{-1})$. ∎

The next lemma describes a useful procedure to generate separable orderings.

Let $a \in A$ and let r be an integer lying between 0 and L. We say that outcome b is an r-variant of a if there exists $S \subset \{1, 2, .., L\}$ with $|S| = r$ such that (i) $a_j \neq b_j$ for all $j \in S$ and (ii) $a_j = b_j$ for all $j \notin S$.

Thus an r-variant of an alternative a is an alternative where exactly r components in a have been flipped. The next lemma states that separable preference orderings can be generated as follows: pick a maximal alternative a and then simply ensure that an alternative b which is a "lower" r-variant from a than another alternative c is ranked above c. We can imagine creating separable orderings by arranging various "blocks". We begin with an arbitrary alternative

a. Below it we place the block of the L alternatives which are 1-variants of a. Within this block, any ordering of alternatives is permissible. Then consider the block of $\frac{L(L-1)}{2}$ alternatives which are 2-variants of a. Once again alternatives within this block can be ordered in an arbitrary way. Proceeding in this way, we arrange blocks of increasing r-variants of a until we reach the last ranked alternative which must be the unique element which is the L variant of a.

LEMMA 4.2 : Let $a \in A$ and P_i be an ordering over A such that (i) $r_1(P_i) = a$ and (ii) for all $b, c \in A$, if b is an r-variant of a and c is an s-variant of a and $r < s$, then bP_ic. Then P_i is a separable ordering.

PROOF : Pick $j \in \{1, .., L\}$ and $x_{-j} \in \{0, 1\}^{L-1}$. Suppose that (a_j, x_{-j}) is an r variant of a. Then (b_j, x_{-j}) (where $a_j \neq b_j$) must be an $r+1$ variant of a. Therefore $(a_j, x_{-j})P^i(b_j, x_{-j})$ by hypothesis. Since x_{-j} was chosen arbitrarily, the proof is complete. ∎

PROOF OF THEOREM 4.1 : Let B be the interval subset $B(K_1, K_2)$. In view of Lemma 4.1, we can restrict attention only to preferences over feasible alternatives. Thus, for any ordering P_i, $r_k(P_i) = a$ implies that $a \in B$ and $|\{bP_ia, b \in B\}| = k - 1$.

Let s be an integer between 0 and L. Let (s) denote the set
$\{a \in A | \sum a_j = s\}$.

CLAIM 1 : Let s and t be consecutive integers no less than K_1 and no greater than K_2. Let $a \in (s)$. Then there exists $b, c \in (t)$ such that $a \sim b$ and $a \sim c$.

In order to prove the claim, assume w.l.o.g that $s < t$. Assume also w.l.o.g that a is the alternative where the first s components are 1 and the remaining $L-s$ are 0. Let b be the alternative where the first t components are 1 and the remaining components are 0. Let c be the alternative where the first $t-1$ components and the $t+1$th component is 1 and the other components are 0. Observe that both b and c are 1-variants of a. It follows from Lemma 4.2 that there exist separable orderings where a is first and b and c are either second or third. We can also find separable orderings where either b or c is first and a is second. This establishes the claim.

CLAIM 2 : For all $a, b \in (K_1)$, we have $a \sim b$, provided $K_1 \neq 0$. Similarly, for all $a, b \in (K_2)$, we have $a \sim b$ provided $K_2 \neq L$.

In order to verify the claim pick $a, b \in (K_1)$. There exists $c \in (K_1 - 1)$ such that a and b are 1-variants of c. It follows from Lemma 4.2 that we can find separable orderings such that c is first and a and b are ranked either second or third. Since B is an interval subset, c is not feasible. Therefore $a \sim b$. An identical argument holds in the case where $a, b \in (K_2)$ except that we pick c in the set $(K_2 + 1)$. Thus the claim is established.

We now complete the proof of Theorem 4.1 by showing that preferences over B constitute a linked domain. Since B is a strict interval subset, either $K_1 \neq 0$ or $K_2 \neq L$. Assume w.l.o.g that the former holds. Arrange the elements of B in the following way. First arrange alternatives in (K_1) in an arbitrary way, followed by alternatives in $(K_1 + 1)$ and so on until (K_2). The order in which these alternatives are picked defines the permutation function σ in Definition 3.3. It is easy to check that Claims 1 and 2 imply that conditions (i) and (ii) of Definition 3.3 are satisfied by this choice of σ. Since $L > 3$ by assumption, the domain is linked and the result follows by applying Theorem 3.1. ■

PROOF OF THEOREM 4.2 : Assume w.l.o.g that the alternative discarded from A is $(0,0,0,...0)$, i.e. $B = A - (0,0,...0)$.

We first claim that all alternatives in the set (1) are connected. To see this observe that all $a, b \in (1)$ are 1-variants of $(0,0...0)$. Therefore, using Lemma 4.2, we can construct an ordering where $(0,0...0)$ is maximal followed by a and b. Since a and b were chosen arbitrarily, we have $a \sim b$.

Our next claim is that for all alternatives $a \in (s)$, $s = 1,2,..,L-1$, there exists an integer t where s and t are consecutive and a is connected to at least two alternatives in (t). To establish this, it suffices to observe that since $L \geq 2$, each alternative in (s) has at least two 1-variants in the set (t) where s, t are consecutive integers. (Of course, a minor qualification has to be made in the case where $s = L-1$, when t must be chosen to be $L-2$ rather than L because (L) has only one alternative.) Now, using Lemma 4.2, we can pick a separable ordering where a is first ranked and b is second where $b \in (t)$ and is a 1-variant of a. Since b is a 1-variant of a, we can also find an ordering where b is first and a second.

We can now complete the proof of Theorem 4.2. Order the alternatives in B in the following manner. First choose all alternatives in (1) (in some arbitrary order), then (2), (3),...,(L). Define a permutation function σ on the set B from this order. In view of the claims established in the last two paragraphs, it is easy to check that σ satisfies conditions (i) and (ii) of Definition 3.3. Therefore the domain is linked and the result follows from Theorem 4.1. ■

We provide a diagrammatic illustration of the proof of Theorem 4.2 in the case where $L = 3$. Figure 1 shows the connections between various alternatives in the set A. An alternative is a vertex in the graph and two vertices are connected only if they are connected. Observe that the domain of preferences over A is not linked because there does not exist a triple of alternatives which are mutually linked - a necessary condition for the existence of a linked domain. Of course, we know from BSZ that there are numerous strategyproof SCFs (including ones which are anonymous and neutral) over this domain.

Figure 2 illustrates the connections between alternatives once $(0,0,0)$ has been excluded from the set of feasible outcomes. Now the set of 1-variants from $(0,0,0)$ are mutually connected. Thus $(1,0,0) \sim (0,1,0) \sim (0,0,1)$. The other connections excluding the ones which involve $(0,0,0)$ remain. Consider the sequence of alternatives : $(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,0,1),$ $(1,1,1)$. It is easy to verify from the diagram that this sequence satisfies conditions (i) and (ii) in Definition 3.3. Therefore the domain is linked.

5 A NECESSARY CONDITION AND SOME SPECIAL CASES

The linked domain condition introduced in Section 3 is only a sufficient condition for a domain to be dictatorial. Unfortunately, it appears that a necessary and sufficient condition for a domain to be dictatorial is likely to be extremely complicated. In this section we provide an elementary necessary condition and show that it is necessary and sufficient for the case where $M = 3$. We also obtain a necessary and sufficient condition for the case $M = 4$.

Our first result, Theorem 5.1 below describes a "minimal variation" needed in a domain for it to be dictatorial.

DEFINITION 5.1 : A domain $\mathbb{D}$ has the *unique seconds* property if there exists a pair of alternatives $a_j, a_k \in A$ such that for any preference ordering $P_i \in D$ where $a_j = r_1(P_i)$, it is the case that $a_k = r_2(P_i)$.

Thus the unique seconds property is satisfied if there exists a pair of outcomes a_j, a_k such that whenever a_j is first ranked, a_k is second. We show that a domain satisfying this property is non-dictatorial.

THEOREM 5.1. : If a domain satisfies the unique seconds property, then it is non-dictatorial.

PROOF : Let $\mathbb{D}$ be a domain satisfying the unique seconds property. Suppose in particular that whenever, a_j is ranked first, a_k is ranked second. Define a two -person SCF f on this domain as follows. Fix individuals 1 and 2 and let $f(P) = r_1(P_1)$ whenever $r_1(P_1) \neq a_j$. If $r_1(P_1) = a_j$, let $f(P)$ be the outcome that 2 prefers in the set $\{a_j, a_k\}$. It is readily verified that f is non-dictatorial and satisfies unanimity. We claim that it is also non-manipulable. To see this observe that individual 2 is always maximizing over the options offered to him by 1. Individual 1 always gets his maximal outcome except perhaps when his maximal outcome is a_j. Since the domain has the unique seconds property, then the outcome is his second ranked outcome a_k provided that a_k is prefered to a_j by 2. Observe that under these circumstances there is no announcement by 1 which will change the outcome to a_j. ■

Theorem 5.1 is sufficient to completely characterize dictatorial domains when $M = 3$.

THEOREM 5.2 : Let $M = 3$. Then $\mathbb{D}$ is a dictatorial domain if and only if $\mathbb{D} = \mathbb{P}$.

PROOF : The sufficiency follows from the Gibbard-Satterthwaite Theorem (Corollary 3.2). Consider a domain $\mathbb{D} \subset \ IP$. Observe then that the unique seconds property must be satisfied for some alternative. But then $\mathbb{D}$ cannot be dictatorial in view of Theorem 5.1. ■

We next investigate the structure of dictatorial domains when $M = 4$. This case is considerably more complicated. In what follows we assume that the set of alternatives is now $A = \{a, b, c, d\}$.

DEFINITION 5.2 : The domain $\mathbb{D}$ satisfies *Condition* α if one of the two conditions below is satisfied.

(i) Each alternative is connected to least two alternatives and one alternative is connected to three.

(ii) Each alternative is connected to exactly two others. Furthermore, for all pairs $x, w \in A$ such that x and w are not connected either (a) there exists P_i, such that $x = r_1(P_i)$, and $w \neq r_4(P_i)$ or (b) there exists P_i, such that $x = r_2(P_i)$ and $w \neq r_3(P_i)$ holds.

Condition (i) is self-explanatory. Condition (ii) is more subtle. It applies to the case shown in Figure 3. As in the earlier figures, each vertex in the graph is an alternative and an edge represents a connection between two alternatives. Now consider two vertices such as b and c which are not connected. Then it must be the case that either there is an ordering where b is ranked first, and c

is not ranked last or there is an ordering where b is second and c is not third. A similar condition applies to the other pair a and d which are not connected.

THEOREM 5.3 : Let $M = 4$. Then $\mathbb{D}$ is dictatorial if and only if it satisfies Condition α.

PROOF : If (i) of Condition α holds, then the domain is linked and Theorem 3.1 applies. If on the other hand, there exists an alternative which is not connected to at least two others, then Theorem 5.1 holds and the domain is not dictatorial. Therefore the only remaining case of interest relates to (ii). We need to show that if (ii) holds then $\mathbb{D}$ is dictatorial. If on the other hand, it does not hold, then we have to construct a strategyproof, non-dictatorial unanimity satisfying SCF on the domain. The proof of the first part is in the Appendix. Here we only show that (ii) is necessary for $\mathbb{D}$ to be dictatorial.

Assume that $\mathbb{D}$ is such that connectivity structure of alternatives is given by that in Figure 3. Assume further that (ii) is violated with respect to b and c; i.e. for all orderings where b is ranked first, c is last and that for all orderings where b is second, c is third. We construct a two person SCF f on this domain in terms of the option set $O_2(P_1)$ that individual 1 offers 2. Recall that for all P_1, P_2, $f(P_1, P_2) = max(P_2, O_2(P_1))$.

$O_2(P_1) = \{a, c\}$ for P_1 such that $r_1(P_1) = a$

$O_2(P_1) = \{c\}$ for P_1 such that $r_1(P_1) = c$

$O_2(P_1) = \{a, b, c, d\}$ for P_1 such that $r_1(P_1) = b$

$O_2(P_1) = \{d, c\}$ for P_1 such that $r_1(P_1) = d$

We claim that f is strategyproof. We consider 4 cases separately.

Case A : Individual 1 has preferences P_1 with $r_1(P_1) = a$. If P_2 has b as the top ranked alternative, then by hypothesis, c must be last, so that $f(P_1, P_2) = a$ and 1 would have no reason to manipulate. If P_2 has c as the top ranked alternative then the outcome is c for any manipulation by 1 since every option Set contains c. So here 1 cannot manipulate. Finally, if P_2 has d on top, then $f(P_1, P_2) = c$ since by hypothesis c must be preferred to a. Here, agent 1 may want to manipulate to get b or a. The only way for 1 to do so is to announce P_1' which has b is first and thereby offer the option Set $\{a, b, c, d\}$. But now $f(P_1', P_2) = d$ which is worse than c for agent 1. So here too, 1 would not manipulate.

Case B : Individual 1 has preferences P_1 with $r_1(P_1) = b$. Suppose $r_1(P_2) = a$. Then $f(P_1, P_2) = a$. If a is second in P_1, then 1 cannot manipulate because there is no way for him to offer b without also offering a. Since c must be last in P_1 by hypothesis, the only case to consider is where d is second and a is third in P_1. Here 1 may want to manipulate in order to get d. But d cannot be second in P_2. If either b or c is second, then cP_2d and manipulation by 1 cannot lead to d. If $r_1(P_2) = b$, then $f(P_1, P_2) = b$ and 1 will not want to manipulate. If $r_1(P_2) = c$ then 1 cannot manipulate because he cannot change the outcome from c. If $r_1(P_2) = d$, the argument is exactly as in the case where a is first in P_1, with the roles of a and d reversed.

Case C : Individual 1 has preferences P_1 with $r_1(P_1) = c$. This case is easy because $f(P_1, P_2) = c$ for all P_2 so that 1 has no incentive to manipulate.

Case D : Individual 1 has preferences P_1 with $r_1(P_1) = d$. It is easy to verify that the arguments of Case A apply with the roles of a and d interchanged.

These arguments establish that f is strategyproof. Since f is clearly non-dictatorial, we have established that if (ii) of Condition α is violated, then the domain is non-dictatorial. ■

We end this section with the remark that Condition α demonstrates that a necessary and sufficient condition for a domain to be dictatorial will involve restrictions on the way that alternatives are ranked in positions three, four and above. Thus the linked domain approach which places restrictions only on the way alternatives are ranked first and second is inadequate in providing a complete characterization.

6 CONCLUSION

In this paper, we introduce the notion of a linked domain and prove that a strategyproof social choice function defined on such a domain must be dictatorial. This result not only generalises the Gibbard-Satterthwaite Theorem but demonstrates also that the dictatorship result remains valid for domains much smaller than the universal domain. We use this result to derive some new results in a model of voting. We also provide a necessary condition for a domain to be dictatorial and completely characterize dictatorial domains in the case where there are three and four alternatives.

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7 APPENDIX

In this appendix we complete the proof of Theorem 5.3. We show that if a domain $\mathbb{D}$ satisfies part (ii) of Condition α then $\mathbb{D}$ is dictatorial.

The following facts about two person strategyproof SCF's will be used.

We will consider the option set $O_2(P_1)$ that individual one offers the second individual. To simplify notation, we omit the subscript and write $O(P_1)$.

FACT 1 : Let P_1 be a preference ordering such that $x \in O(P_1)$. Let $z \sim x$ and $z \ P_1 \ x$. Then whenever any option set contains z, it must contain x.

PROOF : Suppose not. Then $\exists P_1^{'}$ such that $z \in O(P_1^{'})$ and $x \notin O(P_1^{'})$. Since $z \sim x$, there exists P_2 such that $r_1(P_2) = x$ and $r_2(P_2) = z$. Now $f(P_1, P_2) = x$ while $f(P_1^{'}, P_2) = z$. Consequently, 1 manipulates from P_1 to $P_1^{'}$ and gets the outcome z that is preferred to x, a contradiction. ■

FACT 2 : Let P_1 be any preference ordering. Then $r_1(P_1) \in O(P_1)$. Suppose that $x \in O(P_1)$. Let $z \sim x$ and $z \ P_1 \ x$. Then $z \in O(P_1)$.

PROOF : By unanimity it must be the case that $r_1(P_1) \in O(P_1)$. Suppose $z \notin O(P_1)$. Since $z \sim x$, there exists P_2 such that $r_1(P_2) = z$ and $r_2(P_2) = x$. Now $f(P_1, P_2) = x$ while by unanimity, $F(P_1^{'}, P_2) = z$ where $P_1^{'}$ is any preference ordering where $r_1(P_1^{'}) = z$, a contradiction. ■

The set of alternatives is $A = \{a, b, c, d\}$ and we work with the connectivity structure specified by Figure 3 in the text. We will adopt the following notation. We will let P^x denote a preference ordering with $r_1(P^x) = x$ while y and z will denote the elements that are ranked second for some preference ordering with x as the top ranked alternative. The alternative w will denote the element that is never ranked second to x. For example, if $x = a$, then y and z can be either b or c while d is w.

Lemma 1 below shows that every strategyproof SCF defined on $\mathbb{D}$ must have the convenient 'Tops Only' property, i.e. the option sets and hence the values the SCF takes are determined by the top ranked alternative of the individuals preference orderings.

LEMMA 1 : Let P^x and $P^{'x}$ be two preference orderings such that x is the top ranked alternative for both. Then $O(P^x) = O(P^{'x})$.

PROOF : We first show that $O(P^x) \subset O(P^{'x})$. Suppose $y \in O(P^x)$. x belongs to both $O(P^x)$ and $O(P^{'x})$. Since $x \sim y$, by Fact 1 y must belong to $O(P^{'x})$. Identical arguments apply if $z \in O(P^x)$. Now suppose $w \in O(P^x)$. Then $r_2(P^x)$ is either y or z. Assume without loss of generality that $r_2(P^x) = y$. Since $y \sim w$, by Fact 2, $y \in O(P^x)$. As proved above, $y \in O(P^{'x})$ as well. By applying Fact 1 again, one concludes $w \in O(P^{'x})$.

Identical arguments imply that $O(P^{'x}) \subset O(P^{'x})$. ■

Lemma 2 below specifies conditions under which $O(P^x)$ must be the set A.

LEMMA 2 : (a) If $w \in O(P^x)$, then $O(P^x) = A$. (b) If $y \in O(P^x)$ and $\exists\ P^{'x}$

such that $y = r_4(P'^x)$, then $O(P^x) = A$.

PROOF : We first show (a). Suppose $w \in O(P^x)$. Let $y = r_2(P^x)$. Since $y \sim w$, by Fact 2 it follows that $y \in O(P^x)$. Let P'^x be the preference ordering where $z = r_2(P'^x)$. By an identical argument, $z \in O(P'^x)$. The conclusion now follows from Lemma 1.

We now show (b). Suppose $y \in O(P^x)$. Then by Lemma 1 $y \in O(P'^x)$. By Fact 2, $w \in O(P'^x)$. So by (a), $O(P'^x) = A$ and the conclusion follows again from Lemma 1. ■.

Lemma 2 will be used repeatedly in the arguments below.

PROOF OF THEOREM 5.3 : The proof is divided into two parts. In Part 1, we show that if (ii)(a) of (α) holds for every pair x, w, $\mathbb{D}$ must be dictatorial. In Part 2 we consider the case where for some pair x, w, (ii)(a) of (α) does not hold, but for any such pair, (ii)(b) of (α) holds, and show then that $\mathbb{D}$ must again be dictatorial.

PART 1.

Given any pair x, w there exists by hypothesis P^x such that $w \neq r_4(P^x)$. By Lemma 1, the option set $O(P^x)$ coincides with every other option set $O(P'^x)$ where P'^x is an arbitrary preference ordering with x as the top ranked alternative. We first establish the following statement.

STEP 1 : If $\exists x \in A$ such that $O(P^x) \neq \{x\}$, then $O(P^x) = A$.

To verify Step 1, assume it is not true. So $O(P^x)$ is neither $\{x\}$ nor A. By hypothesis $w \neq r_4(P^x)$. Assume without loss of generality that $y = r_4(P^x)$. If $y \in O(P^x)$, then by Lemma 2(b), $O(P^x) = A$, a contradiction. If $w \in O(P^x)$, then by Lemma 2(a), $O(P^x) = A$, a contradiction. So for the claim in Step 1 to be untrue, it must be the case that $O(P^x) = \{x, z\}$ and $z \neq r_4(P'^x)$ for any P'^x (including P^x) that has x as the top ranked alternative. (Or else by Lemma 2(b) one gets that $O(P^x) = A$, a contradiction).

Assume without loss of generality that $x = a$ and $z = c$. Let P^{a1} denote the preference ordering such that $a = r_1(P^{a1}), b = r_2(P^{a1}), c = r_3(P^{a1})$. (We know such a preference ordering must exist, since $a \sim b$ and from the earlier discussion we are considering the case where $a = r_1(P^a) \Rightarrow c \neq r_4(P^a)\ \forall\ P^a$.) We are considering the case $O(P^{a1}) = \{a, c\}$.

We next claim that $O(P^b) = A$. Suppose instead that $O(P^b) = \{a, b\}$. Then $f(P^b, P^c) \in \{a, b\}$ while $f(P^{a1}, P^c) = c$ and 1 would manipulate from P^{a1} to P^b, a contradiction. Identical arguments apply if $O(P^b) = \{b\}$. If $c \in O(P^b)$, then by Lemma 2(a), $O(P^b) = A$. Suppose now that $O(P^b) = \{a, b, d\}$. Consider the pair b, c. By hypothesis, it cannot be the case that $c = r_4(P^b)$ for all preference orderings P^b. So $\exists\ P'^b$ such that $r_4(P'^b)$ is either a or d. In either case, by Lemma 2(b) $O_2(P^b) = A$. The only remaining case is $O(P^b) = \{b, d\}$. Let P^{b1} be the preference order where $r_2(P^{b1}) = a$. Given that $O(P^b) = \{b, d\}$, it must be the case that $d = r_3(P^{b1})$ or else by Lemma 2(b) $O(P^b) = A$. Now let P^{a2} be the preference ordering where $r_2(P^{a2}) = c$ and $r_3(P^{a2}) = d$. (Such a preference ordering must exist or else we obtain that both P^{a1} and P^{a2} have d ranked last, which is ruled out by hypothesis). Now $f(P^{b1}, P^{a2}) = d$ and consequently 1 manipulates from P^{b1} to P^{a2} and ensures the outcome a by unanimity where a is preferred to d under P^{b1}, a contradiction. We have thus proved that $O(P^b) = A$.

We next claim that $a = r_2(P^b) \Rightarrow d = r_3(P^b)$. Suppose not. Then $c = r_3(P^b)$. Since $c \in O(P^b) = A$ and $c \in O(P^a) = \{a, c\}$, by Fact 1, it must be the case that $d \in O(P^a)$, a contradiction. Therefore, $\exists$ P^{b2} such that $d = r_2(P^{b2})$ and $c = r_3(P^{b2})$. (If not, c is last for all P^b with $d = r_2(P^b)$. But c is also last for all P^b with $a = r_2(P^b)$, which implies c is last for all P^b, which is ruled out by hypothesis).

We claim that $O(P^d) = A$. To verify this claim, observe that it must be the case that $a \in O(P^d)$. (If not, $f(P^{b2}, P^a) = a$ while $f(P^d, P^a) \in \{b, c, d\}$ and so 1 manipulates from P^{b2} to P^d and ensures an outcome that is preferred to a, its lowest ranked alternative). But then, by Lemma 2(a) it follows that $O(P^d) = A$.

Let P^{d2} be such that $c = r_2(P^{d2})$. Then it must be the case that $b = r_3(P^{d2})$. (If not, then a P^{d2} b, $b \in O(P^{d2})$ would imply by Fact 1 that $b \in O(P^a)$, a contradiction.)

We claim that $O(P^c) = A$. To verify this claim, observe first that since $a \in O(P^b) = A$ and $a = r_4(P^{b2})$, a must belong to $O(P^c)$.(Suppose not. Then $f(P^c, P^a) \in \{b, c, d\}$ while $f(P^{b2}, P^a) = a$. So 1 manipulates from P^{b2} to P^c and ensures an outcome that is preferred to a, its lowest ranked alternative). If $b \in O(P^c)$, then by Lemma 2(a), $O(P^c) = A$. Suppose, instead that $O(P^c) = \{a, c, d\}$. By hypothesis $\exists$ P^{c1} s.t $b \neq r_4(P^{c1})$. So $r_4(P^{c1}) \in \{a, d\}$ in which case again by Lemma 2(b), $O(P^c) = A$. The last possibility is that $O(P^c) = \{a, c\}$. Recall that P^{b2} satisfies $d = r_2(P^{b2})$ and $c = r_3(P^{b2})$. Now $f(P^{d2}, P^{b2}) = b$ while $f(P^c, P^{b2}) = c$. But c P^{d2} b. So 1 manipulates from P^{d2} to P^c, a contradiction. So it must be the case that $O(P^c) = A$.

Since $d \in O(P^c) = A$, $c \sim d$ and $c \in O(P^a)$, by Fact 1 it must be the case that $d \in O(P^a)$, a contradiction to the supposition that $O(P^a) = \{a, c\}$. This completes the proof of Step 1.

STEP 2 : If there exists $x \in A$ such that $O(P^x) = A$, then $O(P^v) = A$, $\forall$ $v \in A$.

To verify Step 2, assume here too without loss of generality that $x = a$. We will show first that $O(P^b) = A$. Consider P^{a1} (which has the property that $b = r_2(P^{a1})$). Since $b \sim d$, $d \in O(P^b)$ for all P^b. If $\exists$ P^b such that $d = r_4(P^b)$, then by Lemma 2(b) $O(P^b) = A$, and we are done. So we consider the case $a = r_2(P^b) \Rightarrow d = r_3(P^b)$. This is the configuration of P^{b1} defined earlier. Thus P^b with $a = r_2(P^b)$ must coincide with P^{b1}. If $a \in O(P^b)$ then too $O(P^b) = A$. To see why, observe that by hypothesis, there must exist P^{b2} such that $a = r_4(P^{b2})$. (If not, c is always last ranked for all P^b, which is ruled out by hypothesis). Now by Lemma 2(b), $O(P^b) = A$. If $c \in O(P^b)$, then again by Lemma 2(a), $O(P^b) = A$. So it remains to consider the case where $O(P^b) = \{b, d\}$.

Likewise, if $\exists$ P^a with $c = r_4(P^a)$, it must be the case that $c \in O(P^b)$. (If not, 1 manipulates from P^a to P^b whenever 2 has a preference ordering with c as the top ranked alternative.) Then by Lemma 2(a), $O(P^b) = A$. So we consider the case where any P^a with $b = r_2(P^a)$ coincides with P^{a1} which has $d = r_4(P^{a1})$. By hypothesis, there must exist P^{a2} such that $c = r_2(P^{a2})$ and $b = r_4(P^{a2})$. Now it must be the case that $b \in O(P^c)$. (If not, 1 manipulates from P^a to P^c whenever 2 has a preference ordering with b as the top ranked alternative.) Then again by Lemma 2(a), $O(P^c) = A$. If $\exists$ P^c such that $d = r_2(P^c)$ and $b = r_3(P^c)$, then by Fact 1, $a \in O(P^b)$ and we are done.

So it remains to consider the case where for any P^c such that $d = r_2(P^c)$, it is the case that $a = r_3(P^c)$ and so $b = r_4(P^c)$. By hypothesis, since b cannot be last for all P^c, there must exist P^{c1} such that $a = r_2(P^{c1})$ and $d = r_4(P^{c1})$. Now $f(P^{a1}, P^{c1}) = c$. Since we have assumed that $O(P^b) = \{b, d\}$, it is the case that $f(P^{b1}, P^{c1}) = b$. Since $b\ P^{a1}\ c$, 1 manipulates, a contradiction. So we have proved that $O(P^b) = A$.

We show $O(P^c) = A$. Consider P^a with the property that $c = r_2(P^a)$. Since $c \sim d$, by Fact 1 it follows that $d \in O(P^c)$ for all P^c. Suppose $O(P^c) \neq A$. As in the case of $O(P^b)$ above, the only other possibility is $O(P^c) = \{c, d\}$. Consider P^{c1} with $a = r_2(P^{c1})$. Now consider any P^a with the property that $c = r_2(P^a)$. If for such a P^a, $b = r_4(P^a)$, then b must belong to $O(P^c)$ (or else 1 manipulates from P^a to P^c whenever 2 has a preference ordering P^b) and so by Lemma 2(a) it follows that $O(P^c) = A$ and we are done. So consider the case where for any such P^a, $d = r_4(P^a)$. Here too there must exist P^{a3} such that $b = r_2(P^{a3})$ and $c = r_4(P^{a3})$ (If not, it is the case that for all P^a, d is ranked last, which is ruled out.) Now $f(P^{c1}, P^{a3}) = d$ since the Option Set $O(P^c)$ is assumed to contain only c and d and $d\ P^{a3}\ c$. By manipulating to P^{a3} from P^{c1}, 1 can ensure the outcome a by unanimity and $a\ P^{c1}\ d$. So 1 manipulates, a contradiction. Thus it must be the case that $O(P^c) = A$.

Finally, we show $O(P^d) = A$. Consider P^c with $d = r_2(P^c)$. Since $b \in O(P^c) = A$ and $b \sim d$, by Fact 1 $b \in O(P^d)$. Likewise consider P^b with $d = r_2(P^b)$. Since $c \in O(P^b) = A$ and $c \sim d$, by Fact 1 $c \in O(P^d)$. By hypothesis, there must exist P^d such that $r_4(P^d) \neq a$. So by Lemma 2(b), $O(P^d) = A$.

STEP 3 : Either 1 or 2 is a dictator.

Suppose 1 is not a Dictator. Then there exists $O(P^x) \neq \{x\}$. By Step 1, $O(P^x) = A$. By Step 2, $O(P^v) = A \; \forall \; v \in A$ and consequently 2 is a Dictator.

This completes Part 1 of the proof. ■

PART 2

We now consider the case where (ii)(a) of (α) does not hold for some pair, but for any such pair (ii)(b) of (α) holds. Assume without loss of generality that the pair b, c violates (ii)(a) of (α) and the preference ordering P^{a1} satisfies the requirement of (ii)(b) of (α), i.e. $b = r_2(P^{a2})$ but $c = r_4(P^{a2})$.

Steps 1,2 and 3 below will refer to the configuration (*) below

P^{a1}	P^{a2}	P^{b1}	P^{b2}	P^{c1}	P^{c2}	P^{d1}	P^{d2}
a	a	b	b	c	c	d	d
b	c	a	d	a	d	b	c
d	.	d	a	.	.	.	.
c	.	c	c	.	.	.	.

STEP 1 : $O(P^b)$ is either $\{b\}$ or A.

To verify this step, assume $O(P^b)$ is neither $\{b\}$ nor A. Suppose $c \in O(P^b)$. Then by Lemma 2(a) $O(P^b) = A$, a contradiction. Assume $d \in O(P^b)$. Then $O(P^{a1})$ must contain d or c (if not, 1 manipulates from P^{b1} to P^a), which implies that $O(P^a) = A$. Since $d \sim c$, by Fact 1 $d \in O(P^b) \Longrightarrow c \in O(P^b)$ and so by Lemma 2(a) it follows that $O(P^b)$ is A, a contradiction.

We consider now the only remaining case $O(P^b) = \{a, b\}$.

Suppose there exists P^d s.t $b = r_2(P^d)$ and $a = r_3(P^d)$. We claim that $O(P^d) = A$. (To verify this claim, observe that if $O(P^d) = \{d\}$, then P^{b2} manipulates to P^d when 2 has a preference P^a. If $a \in O(P^d)$, then the conclusion follows from Lemma 2(a). If $c \in O(P^d)$, then the conclusion follows from Lemma 2(b)). Since $a \sim c$, $c \in O(P^d) = A$, $a \in O(P^b) \implies c \in O(P^b)$ and so by Lemma 2(a) it follows that $O(P^b) = A$, which contradicts the supposition that $O(P^b) = \{a, b\}$.

For the remainder of this verification we consider the remaining case where $b = r_2(P^d) \implies c = r_3(P^d)$. This preference ordering is denoted P^{d1}. This case has two sub cases.

(i) Assume $\exists P^{d2}$ such that $r_2(P^{d2}) = c$ and $r_3(P^{d2}) = a$. We claim that $O(P^d)$ must be A. (If $O(P^d) = \{d\}$, then P^{b2} manipulates to P^d when 2 has a preference P^a. If $a \in O(P^d)$, then the conclusion follows from Lemma 2(a). If $b \in O(P^d)$, then the conclusion follows from Lemma 2(b) since $b = r_4(P^{d2})$. The only remaining case is $O(P^d) = \{c, d\}$. Then $f(P^{b2}, P^{a1}) = a$ while $f(P^d, P^{a1}) = d$. So 1 manipulates at P^{b2}, a contradiction. So $O(P^d)$ must be A). This implies that $O(P^c)$ must be A. (Since c is higher than a in P^{d2}, by Fact 1 $a \in O(P^c)$. Since a is higher than b in P^{d2}, again by Fact 1 $b \in O_2(P^c)$, which by Lemma 2(a) gives the conclusion $O(P^c) = A$). This in turn implies that $O(P^a) = A$.($O(P^a)$ cannot be $\{a\}$ for then 1 would manipulate at P^{c1} whenever 2 has a preference P^b. If $c \in O(P^a)$, then by Lemma 2(b), the conclusion follows. If $\exists P^a$ s.t $b = r_4(P^a)$, then again the conclusion follows from Lemma 2(b). So suppose $b = r^3(P^{a2})$. Now $O(P^a)$ might be $\{a, b\}$. But $f(P^{c1}, P^{d2}) = d$ while $f(P^a, P^{d2}) = a$ and so, 1 manipulates from P^{c1}

to P^a, a contradiction). Finally, since $b \sim d$, by Fact 1, $d \in O(P^b)$ and so $c \in O(P^b)$ which implies that $O_2(P^b) = A$ which contradicts the supposition that $O(P^b) = \{a, b\}$.

(ii) Assume (i) does not hold and so $c = r_2(P^d) \Longrightarrow b = r_3(P^d)$. Notice now that the pair d, a violates (ii)(a) of (α). So it must be the case that (ii)(b) of (α) holds. This implies that there must exist P^{c^2} such that $d = r_2(P^{c2})$ and $b = r_3(P^{c2})$

One now has the configuration

P^{a1}	P^{a2}	P^{b1}	P^{b2}	P^{c1}	P^{c2}	P^{d1}	P^{d2}
a	a	b	b	c	c	d	d
b	c	a	d	a	d	b	c
d	.	d	a	.	b	c	b
c	.	c	c	.	a	a	a

We claim $O(P^d) = A$. (It cannot be either of $\{d\}$ $\{b, d\}$, $\{d, c\}$, $\{b, d, c\}$ for in all cases $f(P^{b2}, P^{a1}) = a$ and by manipulating to P^d, 1 can ensure b or d, both of which are ranked higher than a in P^{b2}). We now claim that $O(P^c) = A$. (Since $c \sim a$ and c is higher than a in P^{d1}, Fact 1 implies $a \in O(P^c)$ and since $a = r_4(P^{c2})$, by Lemma 2(b), $O(P^c) = A$). We next claim that $O(P^a) = A$. (To verify this claim, observe that if $\exists P^c$ such that $r_2(P^c) = a$ and $r_3(P^c) = b$ then by Fact 1 $b \in O(P^a)$ and again by Fact 1 $d \in O(P^a)$ and so by Lemma 2(a) it follows that $O(P^a) = A$. So suppose there does not exist P^c such that $r_2(P^c) = a$ and $r_3(P^c) = b$. Then it must be the case that P^{c1} satisfies $r_2(P^{c1}) = a$ and $r_3(P^{c1}) = d$. Now $O(P^a)$ cannot be $\{a\}$ for then 1 at P^{c1} would manipulate to P^a whenever 2 has the preference P^b. If $c \in O(P^a)$, then by Lemma 2(b), $O(P^a) = A$. So assume the only remaining possibility $O(P^a) = \{a, b\}$. If $\exists P^a$ s.t $b = r_4(P^a)$, then $O(P^a) = A$ by Lemma 2(b), a contradiction. If there does not exist P^a s.t $b = r_4(P^a)$, it must be the case that P^{a2} satisfies $r_2(P^{a2}) = c$ and $r_3(P^{a2}) = b$. Now $f(P^{d1}, P^{c2}) = c$ while $f(P^a, P^{c2}) = b$. So 1 manipulates from P^{d1} to P^a, a contradiction. So in all cases, $O(P^a) = A$.) Now by Fact 1 $d \in O(P^b)$ and by Fact 1 again $c \in O(P^b)$, and so $O(P^b) = A$ which contradicts the supposition that $O(P^b) = \{a, b\}$. This completes the verification of Step 1.

STEP 2 : Assume $O_2(P^b) = \{b\}$. Then 1 is a dictator.

To verify this step, we assume it is not true. If 1 is not a Dictator, $\exists$ a strategyproof f such that $O(P^b) = \{b\}$ but $\exists\ x$ s.t $O(P^x)$ contains at least two elements.

We claim that it cannot be the case that $O(P^a) = \{a\}$ and $O(P^d) = \{d\}$.

To verify this claim assume on the contrary that $O(P^a) = \{a\}$ and $O(P^d) = \{d\}$. Then $O(P^c)$ must contain at least two elements since by hypothesis 1 is not a dictator. By Lemma 1, it must contain an element which is ranked third for some P^c. Let $\bar{P}^c$ be the preference ordering s.t its third ranked element is contained in $O(\bar{P}^c)$. Then either a or d is $r_2(\bar{P}^c)$. (i) Suppose first that $a = r_2(\bar{P}^c)$. Then $O(P^a)$ cannot be $\{a\}$ for then 1 manipulates at $\bar{P}^c$ to P^a when 2 has a preference ordering whose top ranked alternative is $r_3(\bar{P}^c)$. (ii) Suppose $d = r_2(\bar{P}^c)$. Identical arguments to (i) above imply then that $O(P^d)$ cannot be $\{d\}$.

Now suppose $O(P^a) \neq \{a\}$. It cannot contain c. (Indeed then $f(P^{a1}, P^c) =$

c, $f(P^b, P^c) = b$ since $O(P^b) = \{b\}$ and 1 manipulates from P^{a1} to P^c, a contradiction) . It cannot contain d, for then $O(P^a) = A$ by Lemma 2(a) which in turn implies by Fact 1 that $O(P^b) \neq \{b\}$, a contradiction. So it must be the case that $O(P^a) = \{a, b\}$. Analogously, if $O(P^d) \neq \{d\}$, it must be $O(P^d) = \{b, d\}$.

To summarize, one of the following must hold :

$$[O(P^a) = \{a, b\}] \quad [O(P^d) = \{b, d\}].$$

We consider without loss of generality that $O(P^a) = \{a, b\}$ for the remainder of the verification. Analogous arguments apply if we were to assume instead that $O(P^d) = \{b, d\}$.

We continue to refer to the configuration (*).

If $O(P^a) = \{a, b\}$ then, $c = r_2(P^a) \implies b = r_3(P^a)$, i.e. P^{a2} must have $b = r_3(P^{a2})$.

Since $c = r_2(P^{a2})$, $O(P^c)$ cannot be $\{c\}$ or $\{a, c\}$, for then 1 would manipulate from P^{a2} to P^c whenever 2 has a preference ordering P^b. If $O(P^c)$ contains b then $O(P^c) = A$. So $O(P^c)$ must contain d. If $d = r_4(P^{c1})$ where $a = r_2(P^{c1})$, $O_2(P^b)$ must contain d, (or else 1 manipulates at P^{c1} to P^b whenever 2 has a preference ordering P^d), which contradicts the supposition that $O(P^b) = \{b\}$.

So it remains to consider the case that $d = r_3(P^{c1})$. It must then be the case that $a \in O(P^c)$. (If not, $O(P^c) = \{d, c\}$. Now $f(P^{c1}, P^{a1}) = d$ while $f(P^a, P^{a1}) = a$ and 1 manipulates from P^{c1} to P^a). If $a = r_4(P^{c2})$, then $a \in O(P^b)$ (or else 1 manipulates at P^{c2} to P^b whenever 2 has a preference P^a) which contradicts the supposition that $O(P^b) = \{b\}$. We now assume the case that remains, namely $a = r_3(P^{c2})$. Notice that in this case that $b = r_4(P^c)$ for all P^c. Hence (ii)(a) of (α) is violated by the pair c, b. For (ii)(b) of (α) to hold it must be the case that there exist P^{d2} such that $r_3(P^{d2}) = a$ (since we have already assumed P^{a2} where $c = r_2(P^{a2})$ has $r_3(P^{a2}) = b$). We finally claim that $O(P^c) = A$. If not, then $O(P^c) = \{c, d\}$. (As observed earlier it cannot be $\{a, c\}$ or $\{c\}$. If it contains b then by Lemma 2(a) it is A.) Now $f(P^{c1}, P^{d2}) = d$ while $f(P^{a1}, P^{d2}) = a$ and so 1 manipulates from P^{c1} to P^{a1}, a contradiction. Thus $O(P^c) = A$ as claimed. Now $b \in O(P^d)$. (If not, then 1 manipulates at P^c to P^d whenever 2 has a preference ordering P^b). Since $b = r_4(P^{d2})$ in the case under consideration, by Lemma 2(b), $O(P^d) = A$ which implies by Fact 1 that $a \in O(P^b)$, a contradiction to the supposition that $O(P^b) = \{b\}$. We have shown that if 1 is not a Dictator, in all cases we contradict the supposition that $O(P^b) = \{b\}$. So 1 is a Dictator. This completes the verification of Step 2.

STEP 3 : Assume $O_2(P^b)$ is A. Then 2 is a Dictator.

We first claim that $O(P^a)$ is A. Suppose not. Then $O(P^a)$ cannot be $\{a\}$ or $\{a, b\}$. (In either case $f(P^{b1}, P^d) = d$ and $f(P^a, P^d) \in \{a, b\}$ and so 1 would manipulate from P^{b1} to P^a). So $O(P^a)$ must contain either d or c. In either case by Lemma 2, $O(P^a)$ is A.

We next claim that $O(P^d)$ is A. $O(P^d)$ cannot be either of $\{d, b\}$, $\{b, d, c\}$, $\{d\}$. (In either case $f(P^{b2}, P^{a1}) = a$ and by deviating to P^d, individual 1 can ensure $f(P^d, P^{a1}) \in \{b, d\}$. So 1 would manipulate.) Therefore, it must be that $O_2(P^d)$ is A.

We finally claim that $O(P^c)$ is A. From the previous claim we know that $a \in O(P^d)$ and so by Fact 1 $a \in O(P^c)$. We also know that $b \in O(P^a)$ which implies by Fact 1 that $b \in O(P^c)$. Thus by Lemma 2(a), $O(P^c)$ is A.

This completes the verification of Step 3 and Part 11 of the proof of the Theorem. ■.