Strategic Stability and Non Cooperative Voting Games: The Plurality Rule

Francesco De Sinopoli*

July, 1998

Abstract

In this paper we show, via some simple examples, that also in the class of games we are dealing with, there are perfect equilibria that are not proper and, moreover, some "proper" outcome is not induced by any stable set. Furthermore, we show that the perfect concept does not appear restrictive enough, since, independently of preferences, it can exclude at most the election of only one candidate. Finally, the stable set's conformity to the iterated dominance principle implies the superiority of this solution concept, even in the peculiar class of plurality games.

*CORE, Voie du Roman Pays 34, 1348 Louvain La Neuve, Belgium. I am deeply indebted to Professor Jean-Francois Mertens for his invaluable encouragements, comments and suggestions in every step of the present research. I thank Fabrizio Germano for having read a first draft of the present work, giving useful advices. The usual disclaimer applies.

1 Introduction

An interesting application of strategic stability is offered by non cooperative voting games. With the plurality rule, even if every voter has the same preference order on the various alternatives, voting for the least preferred candidate is a Nash equilibrium, if there are more than three voters. Given the irrationality of such behavior, it is necessary to use some refinement of the Nash concept that excludes this "bad" outcome. In this paper we apply the general model of a one stage voting procedure defined by Myerson and Weber [7] to the plurality rule case. In this model, given the set of candidates $K = (1, ..., k)$, each voter submits a ballot, which is a vector of k components. An electoral system is then defined by the set of possible ballots that each voter can submit and by the election rule that, given the ballots cast, selects the winner from the set K. Hence, with the plurality rule, every voter has the same strategy space, and each pure strategy is a vector with all zeros except for a one in position c which represents the vote for candidate c, while abstention can be represented by the zero vector. With plurality, the election rule selects the candidate that receives the largest total number of votes. In case of ties, to preserve the symmetry of the voters, we allow an equal probability lottery among the winners.

The set of candidates, the electoral system, the set of voters and the utility vectors with k components (representing for each voter his payoff for all the possible results of the election) define the associated normal form game. This resulting game is highly non generic, since the same outcome arises from many different pure strategy combinations.

However, in [2] it is shown that, for generic utility vectors, under plurality rule, an equilibrium that induces a mixed distribution over the outcomes (i.e. with two or more candidates elected with positive probability) is isolated. This result immediately implies that, for generic plurality games, all equilibria in the same stable set are outcome equivalent.

In this paper we show that the solution concept of perfect equilibria is not restrictive enough, in this context, since independently of the voters' preferences, it can exclude at most the election of only one candidate. Simple examples show that the proper equilibrium is a refinement of the perfect concept, even in this class of games, but also that there are cases where some outcome selected by this concept is not induced by any stable set. This is so because the proper equilibrium does not satisfy the iterated dominance principle.

From that we deduce the superiority of the stable set as solution concept, even in this class of games.

2 The plurality rule

Given the set of candidates $K = (1, ..., k)$ and the set of voters $N = (1, ..., n)$, the plurality rule determines the strategy space of each player. More precisely, since each voter can cast his vote for each candidate or he can abstain, we have that the pure strategy set of each player i is:

$$V^i = V = \{1, ..., k, 0\},$$

where each $c \in K$ is a vector of k components with all zeros except for a one in position c which represents the vote for candidate c, while 0 is the zero vector representing the abstention[1]. Denoting $K_0 = K \cup \{0\}$, the strategy space of each player is:

$$\Sigma = \Delta(K_0).$$

In order to determine the winner, we do not need to know the ballots cast by all the voters, it is enough simply to know their sum. Given a pure strategy vector $v \in V^n$, let $\omega = \sum_{i=1}^{n} v^i$. Clearly ω is a k-dimensional vector, and each coordinate represents the total number of votes obtained by the corresponding candidate; then, denoting the probability that candidate c is elected if v is played by $p(c \mid v)$, we have:

$$p(c \mid v) = \begin{cases} 0 & if\ \exists m \in K\ s.t\ \omega_c < \omega_m \\ \frac{1}{q} & if\ \omega_c \geq \omega_m\ \forall m \in K\ and \\ & \#\ \{d \in K\ s.t.\ \omega_c = \omega_d\} = q \end{cases}. \tag{1}$$

Hence, given the utility vectors $\{u^i\}_{i \in N}$, where $u^i = (u^i_1, ..., u^i_k)$ and each u^i_c represents the payoff that player i gets if candidate c is elected, we have a normal form game; for each pure strategy combination v, the payoff of player i is given by:

$$U^i(v) = \sum_{c \in K} p(c \mid v)\, u^i_c. \tag{2}$$

[1] In order to simplify the notation we denote both a candidate and the strategy of voting for him by the same symbol.

Clearly we can extend (1) and (2) to mixed strategies. Under a mixed strategy σ we have:

$$p(c \mid \sigma) = \sum \sigma(v)\, p(c \mid v)$$

and

$$U^i(\sigma) = \sum_{c \in K} p(c \mid \sigma)\, u_c^i,$$

where, as usual, $\sigma(v)$ denotes the probability of the (pure) strategy combination v under σ.

Since the election rule depends only upon the sum of the votes cast, the payoff functions and the best reply correspondence also have this property.

Then, defining $\Omega^{-i} = \left\{ \varpi \mid \exists v \in V^n \ s.t. \ \sum_{j \neq i} v^j = \varpi \right\}$, we can get the following definition of weak dominance, where by (c^i, ϖ) we denote every strategy combination where player i votes for c, and the behavior of the others is summarized by the vector ϖ:

Definition 1 *A ballot c weakly dominates, for player i, a ballot $c^{|}$, and we write $cD^ic^{|}$, if:*

$$\begin{aligned} U^i\left(c^i, \varpi\right) &\geq U^i\left(c^{i^{|}}, \varpi\right) \ \forall \varpi \in \Omega^{-i} \\ U^i\left(c^i, \varpi\right) &> U^i\left(c^{i^{|}}, \varpi\right) \ for \ some \ \varpi. \end{aligned} \quad (3)$$

A ballot c is said to be dominant for player i if $cD^ic^{|}$ for each $c^{|} \in K_o - \{c\}$. A ballot c is dominated if there exists a strategy that dominates it.

In the following discussion, we will often refer to the next trivial proposition:

Proposition 2 *In a plurality game where each player has a strict preference order over the set K, denoting $M_i = \arg\max_{c \in K} u_c^i$, $m_i = \arg\min_{c \in K} u_c^i$, we have that:*

$$M_i D^i 0 D^i m_i \quad (4)$$

Furthermore with three voters and three candidates, denoting $L_i = K - \{M_i\} - \{m_i\}$, if $\left(u_{M_i}^i - u_{L_i}^i\right) \geq \left(u_{L_i}^i - u_{m_i}^i\right)$ then:

$$M_i D^i L_i$$

and if $\left(u^i_{M_i} - u^i_{L_i}\right) \leq \left(u^i_{L_i} - u^i_{m_i}\right)$ *then:*

$$L_i D^i m_i.$$

Proof. We don't give a proof of the first obvious and well known result here (cf. [1]). For the second part: with three candidates every voter has the same strategy set

$$V^i = \{(1,0,0), (0,1,0), (0,0,1), (0,0,0)\},$$

then:

$$\Omega^{-i} = \left\{ \begin{array}{c} (0,0,0), (1,0,0), (0,1,0), (0,0,1), (2,0,0), \\ (1,1,0), (1,0,1), (0,2,0), (0,1,1), (0,0,2) \end{array} \right\}.$$

Let us suppose (w.l.o.g.) that $u^i_1 > u^i_2 > u^i_3$. Independently of the exact values of u^i it is easy to see that voting for the first candidate is, for player i, a best reply to every element of $\Omega^{-i} - (0,1,1)$. It is trivial to see that:

$U^i(1^i, (0,1,1)) = \frac{1}{3}(u^i_1 + u^i_2 + u^i_3)$ and $U^i(2^i, (0,1,1)) = u^i_2$.

Simple algebra, with the fact that voting for the first candidate is a strict best reply to $(0,0,0)$ gives the first result.

Analogously it is easy to see that voting for the second candidate is always a better reply to $\Omega^{-i} - (1,1,0)$ than voting for the third, independently of the exact values of u^i and, furthermore, $U^i(2^i, (1,1,0)) = u^i_2$ and $U^i(3^i, (1,1,0)) = \frac{1}{3}(u^i_1 + u^i_2 + u^i_3)$. Since, under the abstention of the other two voters, voting for the second candidate is strictly better than voting for the third, the last result follows. ■

Remark 1 *Analogous results can be obtained for more than three candidates, but with more than three voters only the relations (4) will persist.*

2.1 Perfect equilibria

As mentioned above, even if every voter has the same preference order, voting for the least preferred candidate is a Nash equilibrium. This follows from the main drawback of this solution concept, that is it admits the use of dominated strategies. To avoid such irrational behavior, the concept of perfect equilibrium was introduced by Selten[2] [8].

[2] The following definition, due to Myerson (cf. [6]), is equivalent to the original one.

Definition 3 *A completely mixed strategy σ^{ε} is an $\varepsilon-$perfect equilibrium if*

$$\begin{aligned} \forall i &\in N,\, \forall v^{i}, v^{i^{|}} \in V^{i} \\ if\ U^{i}\left(v^{i}, \sigma^{\varepsilon}\right) &> U^{i}\left(v^{i^{|}}, \sigma^{\varepsilon}\right)\ then \\ \sigma^{\varepsilon}\left(v^{i^{|}}\right) &\leq \varepsilon. \end{aligned}$$

A strategy combination σ is a perfect equilibrium if there exists a sequence $\{\sigma^{\varepsilon}\}$ of $\varepsilon-$perfect equilibria converging (for $\varepsilon \to 0$) to σ.

Since a perfect equilibrium does not contain dominated strategies, and voting for the least preferred candidates is dominated, such a concept eliminates the "bad" outcome described above. However, for plurality games, this solution concept does not appear sufficiently restrictive, since it excludes, independently of preferences, at most the election of only one candidate. This is basically the result of the next proposition. Before stating it, we need the definition of a Condorcet loser:

Definition 4 *A candidate c is a strict Condorcet loser if*

$$\#\left\{i \in N \mid c \succ_{i} c^{|}\right\} < \#\left\{i \in N \mid c^{|} \succ_{i} c\right\} \quad \forall c^{|} \in K - \{c\}. \tag{5}$$

A candidate c is a weak Condorcet loser if:

$$\#\left\{i \in N \mid c \succ_{i} c^{|}\right\} \leq \#\left\{i \in N \mid c^{|} \succ_{i} c\right\} \quad \forall c^{|} \in K - \{c\}. \tag{6}$$

With the above definition, it is easy to show, under the mild assumption that no voter is indifferent between two candidates, that if a candidate is not a strict Condorcet loser then there exists a perfect equilibrium that leads to his election with positive probability, and furthermore if the candidate is not a weak Condorcet loser either such probability is equal to one.

Proposition 5 *In a plurality game with more than 4 voters, if everyone has a strict preference order over the set K, then, for every candidate $c \in K$ who is not a strict Condorcet loser, there exists a perfect equilibrium σ with $p(c \mid \sigma) \geq \frac{1}{2}$. Furthermore, if c is not a weak Condorcet loser either then this probability is equal to 1 .*

Proof. Take a candidate c that is not a Condorcet loser and let $c^{\prime}$ be the candidate for which (5) (resp. (6)) does not hold. Divide the voters in two groups:

$$\begin{aligned} N^{P}(c,c^{\prime}) &= \left\{ i\in N \mid c\succ_{i} c^{\prime}\right\} \\ N^{\bar{P}}(c,c^{\prime}) &= \left\{ i\in N \mid c^{\prime}\succ_{i} c\right\}. \end{aligned}$$

Consider the strategy combination σ^{ε} where, for $i\in N^{P}(c,c^{\prime})$,

$$\sigma^{i^{\varepsilon}}=(1-\varepsilon-(k-1)\,\varepsilon^{2})c+\varepsilon c^{\prime}+\varepsilon^{2}\,(0+1+...+k)\,.$$

and for $i\in N^{\bar{P}}(c,c^{\prime})$

$$\sigma^{i^{\varepsilon}}=(1-\varepsilon-(k-1)\,\varepsilon^{2})c^{\prime}+\varepsilon c+\varepsilon^{2}\,(0+1+...+k)\,.$$

It is obvious that, for ε sufficiently close to zero, σ^{ε} is an $\varepsilon-perfect$ equilibrium. In fact, for each voter, the probability of being decisive between c and $c^{\prime}$ is infinitely greater than the probability of any other circumstance where he is decisive. Hence, each voter uses in σ^{ε} only his best reply with probability greater than ε. Since the sequence $\{\sigma^{\varepsilon}\}_{\varepsilon}$ converges, for ε going to zero, to the equilibrium where every voter that prefers c to $c^{\prime}$ votes for c, and every other votes for $c^{\prime}$, we have the results. ■

The above proposition implies that the concept of perfect equilibrium is not very powerful, in this class of games, since independently of the voters' preferences, it excludes at most the election of only one candidate. From this we deduce that some refinement of the perfect concept has to be used to get more sensible solutions.

2.2 Proper equilibria

In this section we give a simple example that shows that, even in the class of plurality games, there are perfect equilibria that are not proper. Initially, we review the definition of the proper concept, introduced by Myerson [6].

Definition 6 *A completely mixed strategy σ^{ε} is an $\varepsilon-proper$ equilibrium, if*

$$\begin{aligned} \forall i &\in N,\ \forall v^{i},v^{i^{\prime}}\in V^{i} \\ if\ U^{i}\left(v^{i},\sigma^{\varepsilon}\right) &> U^{i}\left(v^{i^{\prime}},\sigma^{\varepsilon}\right)\ then \\ \sigma^{\varepsilon}\left(v^{i^{\prime}}\right) &\leq \varepsilon\cdot\sigma^{\varepsilon}\left(v^{i}\right). \end{aligned}$$

A strategy combination σ is a proper equilibrium, if there exists a sequence $\{\sigma^{\varepsilon}\}$ of $\varepsilon -$ proper equilibria converging (for $\varepsilon \to 0$) to σ.

If we compare the above definition with the definition of the perfect concept, it is evident that every proper equilibrium is also perfect, while the converse is not true. The following example with three voters and three candidates shows that this fact, namely that some perfect equilibrium is not proper, also holds in the class of games we are dealing with.

Example I

$$
\begin{aligned}
u^1 &= (3,1,0) \\
u^2 &= (2,3,0) \\
u^3 &= (2,3,0)\,.
\end{aligned}
$$

For simplicity, let a denote the first candidate, b the second and c the third. By proposition 2 we know that player 1 has a dominant strategy (a) while player 2 and 3 have two undominated strategies (a and b). The game has two undominated and perfect equilibria:

$$
\begin{aligned}
a^* &= (a,a,a) \\
b^* &= (a,b,b)\,,
\end{aligned}
$$

but only the second one is proper. To see that the equilibrium a^* is perfect, it is enough to consider the following strategy combination σ^{ε}:

$$\sigma_i^{\varepsilon} = (1-\varepsilon-2\varepsilon^2)a + \varepsilon c + \varepsilon^2 b + \varepsilon^2 0 \ \forall i \in N.$$

It is easy to see that, for ε sufficiently close to zero, this is an $\varepsilon-$*perfect* equilibrium. In fact, for player 2 the probability (under $(\sigma_1^{\varepsilon}, \sigma_3^{\varepsilon})$) that candidates a and c take one vote each is infinitely greater than the probability of any other circumstance under which his vote matters. Hence voting for a is his best reply to $(\sigma_1^{\varepsilon}, \sigma_3^{\varepsilon})$. Analogously for the third player, while voting for a is dominant for player 1, then σ^{ε} is an $\varepsilon-$*perfect* equilibrium. Since for ε going to zero, σ^{ε} converges to a^* we have the perfection of this equilibrium. It is easy to see that a^* is not proper.

For convergence and dominance relationships, the strategies of players 2 and 3 in the sequence of $\varepsilon -$*proper* equilibria converging to a^* must have the following form:

$$\sigma_i^{\varepsilon} = \delta_0 a + \delta_1 b + \delta_2 0 + \delta_3 c \ \ \textit{with} \ \delta_{n+1} \leq \varepsilon \delta_n \ \ i = 2,3.$$

Hence player 1 prefers to vote for c than for b, thus his $\varepsilon - proper$ strategy has to be of the following form:

$$\sigma_1^\varepsilon = \delta_0 a + \delta_1 0 + \delta_2 c + \delta_3 b \ \ with \ \delta_{n+1} \leq \varepsilon \delta_n.$$

But it is easy to check that, for the above specified strategies, 2 and 3 prefer voting for b than voting for a, hence a^* cannot be a proper equilibrium.

By existence of proper equilibrium (cf. [6]) we deduce than b^* is the only proper, and hence perfect, equilibrium of the game.

2.2.1 A drawback of the proper concept

In the context of voting games, the main drawback of the proper concept appears to be that it fails to satisfy the iterated dominance principle. Since the pioneering work of Farquharson [3], authors involved in voting theory have long used this principle (under the name of sophisticated voting) to solve these kind of games. Farquharson defined a voting game as "determinate" if the iterated elimination of dominated strategies isolates an outcome. Even if not every plurality game is determinate, it appears sensible to ask the solution to select the "sophisticated outcome" when the voting game is determinate (cf. also [4] for an in-depth discussion about this principle). The following example, with three players and three candidates, shows us that the proper concept does not satisfy this requirement in plurality games either.

Example II

$$\begin{aligned} u^1 &= (3,2,0) \\ u^2 &= (3,2,0) \\ u^3 &= (0,2,3)\,. \end{aligned}$$

It is easy to calculate the set of undominated equilibria. It is given by the following two components:

$$\begin{aligned} A &= (a, a, \gamma b + (1-\gamma)c \mid 0 \leq \gamma \leq 1) \\ B &= (b, b, \gamma b + (1-\gamma)c \mid 0 \leq \gamma \leq 1)\,. \end{aligned}$$

To find out the proper equilibria, we use proposition 2 that implies the following dominance relationships:

$$\begin{aligned} \text{players 1 and 2} &: aD0Dc \text{ and } bDc \\ \text{player 3} &: cD0Da \text{ and } bDa \end{aligned}$$

Since every proper equilibrium is undominated, we can proceed to analyze A and B.

Consider the A component.

For $\gamma < \frac{1}{4}$ at equilibrium, for player 1 and 2, the strategy b is strictly better than 0, hence also nearby. This implies that the $\varepsilon - proper$ strategies of player 1 and 2 will be of the following form:

$$\sigma^{\varepsilon} = \delta_0 a + \delta_1 b + \delta_2 0 + \delta_3 c \;\; with \; \delta_{n+1} \leq \varepsilon \delta_n.$$

But for these strategies of the first two voters, player 3 strictly prefer b to c, hence contradicting $\gamma < \frac{1}{4}$.

It is possible to apply the same kind of arguments to the case $\gamma > \frac{1}{4}$. In this case the probability of player 1 and 2 trembling towards the abstention is infinitely greater than the one of trembling towards b. Hence player 3 prefers c to b, contradicting $\gamma > \frac{1}{4}$.

So the only proper equilibrium in A can be given by $\gamma = \frac{1}{4}$.

The following strategies show us that $\gamma = \frac{1}{4}$ is in fact a proper equilibrium[3]:

$$\begin{aligned} \sigma_1^{\varepsilon} &= (1-\varepsilon-\varepsilon^2)a + (1-x)\varepsilon 0 + x\varepsilon b + \varepsilon^2 c \qquad (7) \\ \sigma_2^{\varepsilon} &= (1-\varepsilon-\varepsilon^2)a + (1-x)\varepsilon 0 + x\varepsilon b + \varepsilon^2 c \\ \sigma_3^{\varepsilon} &= (1-\varepsilon-\varepsilon^2-\varepsilon^3)\left(\frac{1}{4}b + \frac{3}{4}c\right) + (1-y)\varepsilon b + y\varepsilon c + \varepsilon^2 0 + \varepsilon^3 a \\ with\; x &= \frac{3+8\left(\varepsilon-\varepsilon^2-\varepsilon^3\right)}{5-2\varepsilon-8\varepsilon^2} \;\; and\; y = \frac{11-2\varepsilon-4(1+x)\varepsilon^2-\varepsilon^4}{8+4\varepsilon-20\varepsilon^2}. \end{aligned}$$

The values of x and y assure that, along all the sequence, the strategies 0 and b are equivalent for player 1 and 2 likewise the strategies b and c for player 3. Since every σ^{ε} is an $\varepsilon^{\shortmid} - proper$ equilibrium, we have that

$$a^* = \left(a, a, \frac{1}{4}b + \frac{3}{4}c\right)$$

[3] The stability of the set A, cf. below, implies that A contains a proper equilibrium. Hence the conditions under (7) are not necessary to claim that $\gamma = \frac{1}{4}$ is a proper equilibrium.

is a proper equilibrium.

Let us now analyze the B component; for convergence and weak dominance, we get that in every $\varepsilon - proper$ equilibrium converging to B, the strategies of players 1 and 2 have the following form:

$$\sigma^{\varepsilon} = \delta_0 b + \delta_1 a + \delta_2 0 + \delta_3 c \ \ with \ \delta_{n+1} \leq \varepsilon \delta_n$$

and this implies that the strategy b gives player 3 a higher payoff than c. Hence, the only equilibrium in B that can be proper is the one corresponding to $\gamma = 1$. As a matter of fact, it is proper. It is easy to check that the following strategy is an $\varepsilon' - proper$ equilibrium:

$$\begin{aligned}
\sigma_1^{\varepsilon} &= \left(1 - \varepsilon^2 - \varepsilon^3 - \varepsilon^4\right) b + \varepsilon^2 a + \varepsilon^3 0 + \varepsilon^4 c \\
\sigma_2^{\varepsilon} &= \left(1 - \varepsilon^2 - \varepsilon^3 - \varepsilon^4\right) b + \varepsilon^2 a + \varepsilon^3 0 + \varepsilon^4 c \\
\sigma_3^{\varepsilon} &= \left(1 - \varepsilon - \varepsilon^2 - \varepsilon^3\right) b + \varepsilon c + \varepsilon^2 0 + \varepsilon^3 a.
\end{aligned}$$

Hence, the game has two proper equilibria:

$$\begin{aligned}
a^* &= \left(a, a, \frac{1}{4} b + \frac{3}{4} c\right) \\
b^* &= (b, b, b).
\end{aligned}$$

This game is determinate. In fact, by proposition 2 we know that abstention and voting for the less preferred candidate is dominated for every player. Eliminating 0 and c for players 1 and 2, and 0 and a for player 3, we obtain a reduced game where the strategy c is dominated for player 3. Hence, eliminating this strategy too, a becomes dominant for the first two players. As a result the game is determinate and the sophisticated outcome is the election of the candidate a, while b is also elected in a proper equilibrium. Furthermore, the result of this example is robust. If the player has the above preference order on the candidates, for all payoff vectors such that:

$$\begin{aligned}
u_1^1 - u_2^1 &< u_2^1 - u_3^1 \\
u_1^2 - u_2^2 &< u_2^2 - u_3^2 \\
u_3^3 - u_2^3 &< u_2^3 - u_1^3,
\end{aligned}$$

we obtain that the equilibrium $b^* = (b, b, b)$ is proper, even if the sophisticated outcome coincides with the election of candidate a.

2.3 Stable sets

The explicit definition of stable set [5] goes beyond the purpose of the present analysis. For this reason we refer to [5] for such definition and we list here the properties of this solution concept that we are going to use:

i)Stable sets always exist.

ii)Stable sets are connected sets of normal form perfect (hence undominated) equilibria.

iii)A stable set contains a stable set of every game obtained by eliminating a strategy that is used with minimum probability in any $\varepsilon-perfect$ equilibrium close to it.

These properties, with the fact that for generic plurality games an equilibrium that induces a mixed distributions over the outcomes is isolated (cf. [2]), immediately imply:

Proposition 7 *For generic plurality games, if the sophisticated outcome exists, it is the only stable outcome (i.e. the only outcome induced by elements of stable sets).*

Proof. The fact that, for generic plurality games, an equilibrium that induces a mixed distribution over the outcomes is isolated and property (ii) imply that all elements belonging to the same stable set are outcome equivalent. Moreover, property (iii) implies that every stable set contains a stable set of a game obtained by iterated elimination of dominated strategies. Thus, if the game is determinate, every stable set "contains" also the sophisticated outcome. Hence the result. ■

Now let us come back to example II. In order to find out the stable sets of this game, we do not use, as mentioned above, their explicit definition, since it is extremely difficult to actually compute them, except for simple cases. However in many games, as this one, their properties are sufficient to obtain the exact solution. In this example, by (i) and (ii) we know that every stable set is contained in A or B. By (iii), it follows that every stable set contains a stable set of a game obtained by iterated elimination of dominated strategies. Hence the same argument made to claim that this game is determinate leads to the conclusion that every stable set is contained in A and it contains the equilibrium (a, a, b).

It is also clear that in any $\varepsilon-perfect$ equilibrium close to A, the strategy a is the only best reply for 1 and 2. Hence b is used with minimum probability. Eliminating this strategy for 1 and 2, the strategy b becomes dominated for player 3. Hence from (iii) we deduce that the equilibrium (a, a, c) also has to be contained in every stable set; then (ii) implies that the game has a unique stable set $S \equiv A$.

We can find many applications of proposition 7. The following example shows that in a classical spatial model with three candidates, if there is a strong majority against one candidate (i.e. if a candidate is the least preferred for more than $\frac{2}{3}$ of the voters) the only stable outcome coincides with the candidate most preferred by the median voter.

Example III

Let $K = \{L, R, M\}$, with each candidate being represented by a number between 0 and 1. Suppose that there are n voters equidistant on $[0, 1]$, where the utility of a voter x if the elected candidate is k is a negative transformation of the distance between x an k[4], i.e. $u_k^x = f_x(\mid x - k \mid)$ with $f_x^{'} < 0$, and let us assume that $L < M < R$. Define:

$$\begin{aligned} a &= \frac{L+M}{2} \\ b &= \frac{L+R}{2} \\ c &= \frac{R+M}{2}. \end{aligned}$$

For generic positioning of the three candidates, we have that no voter is indifferent between two candidates and, hence, we will have the following preference orders:

For $0 \leq x < a$ $\quad L \succ_x M \succ_x R$
For $a < x < b$ $\quad M \succ_x L \succ_x R$
For $b < x < c$ $\quad M \succ_x R \succ_x L$
For $c < x \leq 1$ $\quad R \succ_x M \succ_x L.$

By proposition 5, we know that for generic $L < M < R$ there are at least two candidates elected with positive probability in a perfect equilibrium. However for $b \notin \left[\frac{1}{3}, \frac{2}{3}\right]$ the game is determinate and the sophisticated outcome coincides with the election of the candidate most preferred by the median voter. Hence, for generic positioning such that $b \notin \left[\frac{1}{3}, \frac{2}{3}\right]$, there is only one stable outcome and it coincides with the sophisticated one. We limit the analysis to $b \in \left[0, \frac{1}{3}\right)$, the other case being symmetric.

[4]This condition is necessary to assure that for generic positioning of candidates the resulting game is generic.

By proposition 2, the strategies of abstaining and voting for the least preferred candidate are dominated. Then we get that, in the reduced game, the players have the following strategies:

$0 \le x < b$ $\quad L, M$

$b < x \le 1$ $\quad R, M.$

In this reduced game, for $b < \frac{1}{3}$, if the number of players is sufficiently large[5], there is no chance of candidate L being elected. Hence voting for L is dominated, as well as R for $b < x < c$ and M for $c < x \le 1$. Then the sophisticated outcome is given by the following strategies:

$0 \le x < c$ $\quad M$

$c < x \le 1$ $\quad R$

Hence for $c < \frac{1}{2}$ the candidate elected is R, while for $c > \frac{1}{2}$ it is M.

3 Conclusion

In this paper we have shown that the perfect equilibrium concept is, in this context, not restrictive enough since, independently of preferences, it excludes at most the election of only one candidate. The proper concept, despite being a strict refinement of the perfect one, still presents some weakness. Even in this context there are proper outcome that are not induced by any stable set. This is so because the proper concept does not satisfy the iterated dominance principle. We can so deduce the superiority of the stable set solution concept, even in the class of plurality games. We hope that further researches will lead to a deeper characterization, in this class of games, of the stable sets. For example it would be useful to find a property that helps us to determine the solution of example III for values of b between $\frac{1}{3}$ and $\frac{2}{3}$.

Two questions naturally arise. If there is a Condorcet winner (i.e. a candidate that defeats every other in a pairwise competition), does strategic stability force his election? Does strategic stability imply the Duverger's law (i.e. only two candidates take a positive amount of votes)?

Both the answers are negative. In fact, for the first point, consider the following situation: there are $2n$ voters and three candidates (a, b and c). Candidate a is the most preferred by everyone and n voters prefer b to c, while the other n voters prefer c to b. If $n \ge 2$ the strategy where everyone

[5] That is if there are strictly less then $\frac{1}{3}$ of the voters located between 0 and b.

in the first group of voters votes for b and everyone in the second group votes for c is a strict equilibrium, hence strategically stable in every conceivable sense, while a is clearly the Condorcet winner.

For the second point consider the case where there are $3n$ voters and three candidates (a, b and c). Suppose that for n voters the most preferred candidate is a, for other n it is b and for the rest it is c. If for every voter the difference, in utility, between his most preferred candidate and his second preferred one is strictly greater than the difference between the second and the third one, the strategy where each voter votes for his most preferred candidate is a strict equilibrium, hence strategically stable.

Another extension of the present work could be the study of alternative voting procedures such as Borda or approval.

References

[1] Brams S.J. (1994): "Voting procedures" in *Handbook of Game Theory* vol.2, ed. by Aumann R.J. and S. Hart, Elsiever.

[2] De Sinopoli F. (1998): "Two Results About Generic Non Cooperative Voting Games with Plurality Rule", CORE Discussion Paper 9834.

[3] Farquharson R. (1969) *Theory of voting*, New Haven: Yale University Press.

[4] Kohlberg E. and J.F. Mertens (1986): "On the Strategic Stability of Equilibria" *Econometrica* vol.54, pp.1003-1037.

[5] Mertens J.F. (1989): "Stable Equilibria-A reformulation. Part I: Definition and basic properties" *Math. Oper. Res.* vol.14, pp.575-625.

[6] Myerson R.B. (1978): "Refinements of the Nash equilibrium concept" *Int. J. of Game Theory* vol.7, pp.73-80.

[7] Myerson R.B. and R.J. Weber (1993): "A Theory of Voting Equilibria" *Amer. Pol. Sc. Rev.* vol.87, pp.102-114.

[8] Selten R. (1975): "Reexamination of the perfectness concept for equilibrium points in extensive games" *Int. J. of Game Theory* vol.4, pp.24-55.