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# STRATEGIC COMPLEMENTS AND SUBSTITUTES IN BILATERAL OLIGOPOLIES

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**Abstract**

This note analyzes the effect of product complementarity in a bilateral oligopoly. We show that offers of traders on the two sides of the market are strategic complements (substitutes) if and only if the two goods are substitutes (complements). The outcome of the bilateral oligopoly game converges monotonically to the competitive equilibrium, as the elasticity of substitution between the goods decreases to -$\infty$.

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# 1 Introduction

The strategic market games introduced by Shapley and Shubik (Shapley (1976) and Shubik (1973)) offer a simple description of an exchange economy where all traders behave strategically. A specific example of Shapley-Shubik games, recently introduced by Gabszewicz and Michel (1997) can be viewed as an extension of the Cournot oligopoly model, where both sides of the market (buyers and sellers) are strategic players. In this "bilateral oligopoly", there are two types of traders and two commodities, and each type of trader is endowed with one unit of the two commodities and wants to consume both. General conditions for existence and uniqueness of an interior equilibrium have recently been studied by Bloch and Ghosal (1997). (See also Cordella and Gabszewicz (1998) for a counterexample).

In this note, we investigate further the structure of bilateral oligopolies by analyzing the relation between product complementarity and the Nash equilibrium of the strategic market game, in the simple setting of a CES (constant elasticity of substitution) utility function. We show that the offers of traders on the two sides of the market are strategic complements (substitutes) if and only if the two goods are substitutes (complements)[1]. Furthermore, we prove that the outcome of the game converges monotonically to the competitive equilibrium, as the elasticity of substitution between the goods decreases to minus infinity.

# 2 The model

We consider an economy with two goods, labeled $x$ and $y$, and two types of traders. There are $n \geq 2$ traders of type I (who are endowed with one unit of good $x$) and $n$ traders of type II (who are endowed with one unit of good $y$). Preferences for the two types of traders are identical, and given by the CES utility function

$$U(x, y) = (x^\rho + y^\rho)^{\frac{1}{\rho}}, \text{ with } \rho \leq 1.$$

This utility function encompasses both situations where the goods are substitutes ($\rho \geq 0$) and complements ($\rho \leq 0$). When $\rho = 1$, the goods are perfect substitutes; when $\rho = 0$, the utility function is Cobb-Douglas, and when $\rho \rightarrow -\infty$, the goods are perfect complements. This exchange

[1] See Bulow, Geneakoplos and Klemperer (1985) for the original definition of strategic substitutes and complements in traditional oligopoly models.

economy has a unique competitive equilibrium given by $p^\star = (1,1)$ and $(x^\star, y^\star) = \left(\frac{1}{2}, \frac{1}{2}\right)$, for traders of types I and II.

We analyze a strategic market game where each trader offers a fraction of the commodity he owns on the market. Hence, the strategy spaces for the two types of traders are

$$\begin{aligned} S_I &= \{q_i \in \mathbb{R} \,|\, 0 \leq q_i \leq 1\} \\ S_{II} &= \{b_j \in \mathbb{R} \,|\, 0 \leq b_j \leq 1\}. \end{aligned}$$

The final allocations obtained by the traders are given by

$$\begin{aligned} (x_i,\, y_i) &= \left(1 - q_i,\, q_i \frac{\sum b_j}{\sum q_i}\right), \text{ for traders of type I} \\ (x_j,\, y_j) &= \left(b_j \frac{\sum q_i}{\sum b_j},\, 1 - b_j,\right), \text{ for traders of type II,} \end{aligned}$$

with corresponding utility levels

$$\begin{aligned} U_i\,(x_i,\, y_i) &= \left((1 - q_i)^\rho + \left(q_i \frac{\sum b_j}{\sum q_i}\right)^\rho\right)^{\frac{1}{\rho}} \\ U_j\,(x_j,\, y_j) &= \left(\left(b_j \frac{\sum q_i}{\sum b_j}\right)^\rho + (1 - b_j)^\rho\right)^{\frac{1}{\rho}}. \end{aligned}$$

## 3 The equilibrium

To compute the reaction functions of the two types of traders, we solve the maximization problem faced by agents of type I

$$\underset{0 \leq q_i \leq 1}{\text{Max}} \; U_i\,(q_i,\, b_j)\,.$$

We obtain

$$\frac{\partial\, U_i}{\partial\, q_i} = A\,(q_i) \cdot B\,(q_i)\,, \text{ where}$$

$$A\,(q_i) = -\,(1 - q_i)^{\rho - 1} + \left(\textstyle\sum b_j\right)^\rho \frac{\sum\limits_{k \neq i} q_k}{\left(\sum q_i\right)^{\rho+1}} q_i^{\rho - 1}$$

$$B\,(q_i) = \left[(1 - q_i)^\rho + \left(q_i \frac{\sum b_j}{\sum q_i}\right)^\rho\right]^{\frac{1}{\rho} - 1} > 0.$$

Next note that $\lim_{q_i \to 0} A(q_i) = +\infty$ and $\lim_{q_i \to 1} A(q_i) = -\infty$, and

$$\begin{aligned}\frac{\partial A}{\partial q_i} &= (\rho - 1)(1 - q_i)^{\rho - 2} + \left(\sum b_j\right)^{\rho} \sum_{k \neq i} q_k \left[\frac{q_i^{\rho - 2}\left((\rho - 1)\sum q_i - (\rho + 1) q_i\right)}{\left(\sum q_i\right)^{\rho + 2}}\right] \\ &< 0.\end{aligned}$$

We conclude that there exists $\bar{q}_i$ such that $\forall q_i \leq \bar{q}_i$, $\frac{\partial U_i}{\partial q_i} \geq 0$ and $\forall q_i > \bar{q}_i$, $\frac{\partial U_i}{\partial q_i} < 0$.

Hence, the maximization problem has a unique interior solution given by

$$A(q_i) = \left[-(1 - q_i)^{\rho - 1} + \left(\sum b_j\right)^{\rho} \frac{\sum_{k \neq i} q_k}{\left(\sum q_i\right)^{\rho + 1}} q_i^{\rho - 1}\right] = 0.$$

We now show that in equilibrium all traders on the same side of the market adopt the same strategy. Suppose by contradiction that for two traders $i$ and $k$ of type I, $q_i \neq q_k$. Without loss of generality, let $q_k > q_i$. The following two equations must hold

$$\begin{aligned}-(1 - q_i)^{\rho - 1} + \left(\sum b_j\right)^{\rho} \frac{\sum_{t \neq i} q_t}{\left(\sum q_i\right)^{\rho + 1}} q_i^{\rho - 1} &= 0 \\ -(1 - q_k)^{\rho - 1} + \left(\sum b_j\right)^{\rho} \frac{\sum_{t \neq k} q_t}{\left(\sum q_i\right)^{\rho + 1}} q_k^{\rho - 1} &= 0.\end{aligned}$$

Thus, we have

$$\frac{(1 - q_i)^{\rho - 1}}{\sum_{t \neq i} q_t q_i^{\rho - 1}} = \frac{(1 - q_k)^{\rho - 1}}{\sum_{t \neq k} q_t q_k^{\rho - 1}}.$$

Since $q_k > q_i$, we have $\sum_{t \neq i} q_t > \sum_{t \neq k} q_t$ , which implies that

$$\left(\frac{q_i}{1 - q_i}\right)^{1 - \rho} > \left(\frac{q_k}{1 - q_k}\right)^{1 - \rho}$$

yielding

$$q_i > q_k,$$

contradicting the assumption.

Since the equilibrium is symmetric among traders, on the same side of the market, we may denote by $q$ and $b$ the offers of traders of type I and type II on the market. The reaction functions are given by

$$\begin{aligned} \frac{\partial U_i}{\partial q_i} &= -q(1-q)^{\rho-1} + \frac{(n-1)}{n} b^\rho = 0 \\ \frac{\partial U_i}{\partial b_j} &= -b(1-b)^{\rho-1} + \frac{(n-1)}{n} q^\rho = 0. \end{aligned} \quad (1)$$

**Lemma 1** : *The offers of traders on the two sides of the market are strategic complements (substitutes) if and only if the goods are substitutes (complements).*

**Proof.** : By implicit differentiation, we get

$$\frac{\partial q_i}{\partial b_j} = \frac{\rho b^{\rho-1}\frac{(n-1)}{n}}{(1-q)^{\rho-2}[1-\rho q]}.$$

Hence,

$$\frac{\partial q_i}{\partial b_j} \geq 0 \text{ iff } \rho \geq 0.$$

In the Appendix, we show that the system of equations (1) characterizes a unique symmetric Nash equilibrium, where all traders adopt the same strategy.

**Proposition 1** : *The strategic market game has a unique interior Nash equilibrium. All traders adopt the same strategy:*

$$q^\star = b^\star = \frac{1}{\left(\frac{n}{n-1}\right)^{\frac{1}{1-\rho}} + 1}.$$

**Lemma 2** : *As the degree of substitution of the two goods increases the equilibrium offers $q^\star$ and $b^\star$ decrease.*

**Proof.** : We compute $\dfrac{\partial b^\star}{\partial \rho} = -\dfrac{1}{(1-\rho)^2} \dfrac{\log\left(\frac{n}{n-1}\right)\left(\frac{n}{n-1}\right)^{\frac{1}{1-\rho}}}{\left[1+\left(\frac{n}{n-1}\right)^{\frac{1}{1-\rho}}\right]^2} < 0.$

We obtain for a linear utility function ($\rho \to 1$) $q^\star = b^\star = 0$, for a Cobb-Douglas utility function ($\rho = 0$), $q^\star = b^\star = \frac{n-1}{2n-1}$. As the goods become perfect complements ($\rho \to -\infty$), $q^\star = b^\star = \frac{1}{2}$, the equilibrium outcome converges to the competitive equilibrium.

# 4 Discussion

In this note, we analyze the Nash equilibrium of a bilateral oligopoly when the degree of substitution between the two goods is parametrized by the constant elasticity of substitution $\rho$. We first show that the offers of traders on the two sides of the market are strategic substitutes (or complements) if and only if the goods are complements (substitutes). To understand this result, consider the behavior of an agent $i$ of type I. If the offer $b_j$ of agents of type II increases, the amount of good $y$ in agent $i$'s allocation, $y_i$, increases. If the two goods are substitutes, this decreases the marginal utility of $x_i$ and induces trader $i$ to increase her offer $q_i$. If, on the other hand, the two goods are complements, this increases the marginal utility of $x_i$ and induces trader $i$ to reduce her offer. Note that this effect is not related to the traditional analysis of strategic substitutes and complements in oligopoly.

We also show that, as the complementary between the two goods increases, the equilibrium offers on the two sides of the market increase. The intuition underlying this result is easily grasped. When the complementarity increases, the marginal utility of good $y$ to traders of type I increases. Hence, for any fixed offer $b_j$ of traders of type II, the offer $q_i$ increases. In equilibrium, both offers $b_j$ and $q_i$ are increasing with the degree of complementarity between goods.

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# A Appendix

## A.1 Proof of Proposition 1.

The symmetric offers given in the Proposition clearly satisfy the system of equations (1). We show that this equilibrium is unique and globally stable. Following Dixit (1986), a sufficient condition for uniqueness of equilibrium is

$$\left|\frac{\partial q}{\partial b}\right|\left|\frac{\partial b}{\partial q}\right| \leq 1.$$

By implicit differentiation,

$$\frac{\partial q}{\partial b} = \frac{\rho\frac{n-1}{n}b^{\rho-1}}{(1-q)^{\rho-2}(1-\rho q)}.$$

At equilibrium, $(1-q)^{\rho-2} = \frac{(1-q)^{\rho-1}}{1-q} = \frac{\frac{n-1}{n} b^{\rho}}{q\,(1-q)}$. Hence, we get

$$\frac{\partial q}{\partial b} = \frac{\rho q\,(1-q)}{b\,(1-\rho q)}.$$

A symmetric computation gives

$$\frac{\partial b}{\partial q} = \frac{\rho b\,(1-b)}{q\,(1-\rho b)}.$$

Hence,

$$\left|\frac{\partial q}{\partial b}\right| \left|\frac{\partial b}{\partial q}\right| = \frac{\rho^2\,(1-b)\,(1-q)}{(1-\rho q)\,(1-\rho b)}$$

and

$$\left|\frac{\partial q}{\partial b}\right| \left|\frac{\partial b}{\partial q}\right| \leq 1 \Longleftrightarrow \rho + 1 \geq \rho\,(b+q)\,. \tag{2}$$

Next, note that the system of equations (1) gives

$$\left(\frac{1-q}{1-b}\right)^{\rho-1} = \left(\frac{b}{q}\right)^{\rho+1}. \tag{3}$$

If $\rho \geq -1$, equation (3) has a unique solution: $b = q$. So, the only case to consider is $\rho \leq -1$. Inequality (2) then becomes

$$b + q \geq 1 + \frac{1}{\rho}. \tag{4}$$

At the first step, we show that $b + q \leq 1$. Equation (3) gives

$$\frac{b}{q} = \left(\frac{1-q}{1-b}\right)^{\frac{\rho-1}{1+\rho}}.$$

Without loss of generality, suppose $b \geq q$. Then $1 - q \geq 1 - b$, and since $\frac{\rho-1}{1+\rho} \geq 1$,

$$\frac{b}{q} = \left(\frac{1-q}{1-b}\right)^{\frac{\rho-1}{1+\rho}} \geq \frac{1-q}{1-b},$$

implying $b + q \leq 1$.

Next suppose without loss of generality that $b \geq q$. Then,

$$(1-q)^{\rho-1} = \frac{n-1}{n}\frac{b^{\rho}}{q} \geq \frac{n-1}{n} b^{\rho-1}$$

yielding

$$\left(\frac{b}{1-q}\right)^{1-\rho} \geq \frac{n-1}{n} \geq \frac{1}{2}$$

or,

$$\frac{b}{1-q} \geq \left(\frac{1}{2}\right)^{\frac{1}{1-\rho}}$$

or finally,

$$2^{\frac{1}{1-\rho}} b + q \geq 1.$$

Now,

$$b + q - \left(1 + \frac{1}{\rho}\right) = \left(2^{\frac{1}{1-\rho}} b + q - 1\right) + \left[b\left(1 - 2^{\frac{1}{1-\rho}}\right) - \frac{1}{\rho}\right].$$

To show inequality (4), it thus suffices to show that $b\left(1 - 2^{\frac{1}{1-\rho}}\right) - \frac{1}{\rho} \geq 0$. Since this expression is decreasing in $b$, a sufficient condition for inequality (4) is

$$1 - \frac{1}{\rho} \geq 2^{\frac{1}{1-\rho}}.$$

It is easy to see that this inequality is satisfied for all $\rho \leq -1$, since $\Phi(\rho) = 1 - \frac{1}{\rho} - 2^{\frac{1}{1-\rho}}$ is a strictly increasing function and $\lim_{\rho \to -\infty} \Phi(\rho) = 0$. This concludes the proof.