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The rough Hawkes process

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This article introduces a new class of self-excited jump processes, with a dampened rough (DR) kernel. The memory of this process is driven by the product of an exponential decreasing function and a kernel involved in the construction of the rough Brownian motion. This process, called rough Hawkes process, is nearly unstable since its intensity diverges to $+\infty$ for a very brief duration when a jump occurs. Firstly, we find the conditions that ensure the stability of the process and provide the closed form expression of the expected intensity. We next reformulate the intensity as an infinite dimensional Markov process. Approaching these processes by discretization and next considering the limit leads to the Laplace's transform of the point process. This transform depends on the solution of an elegant fractional integro-differential equation. The fractional operator is defined by the DR kernel and is similar to the left-fractional Riemann-Liouville integral. We provide a simple method for computing the Laplace's transform. This is easily invertible by discrete Fourier's transform for retrieving the probability density of the process. We also modify the Ogata's algorithm to manage the instability of the process. We conclude by presenting the log-likelihood of the rough Hawkes process and fit it to hourly Bitcoin log-returns from the 9/2/18 to the 9/2/23.

Keywords: self-excited process, Hawkes process, point process.

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1 Introduction

Self-excited processes offer a natural solution to introduce contagion between shocks in time-series. In this approach, the occurrence of a jump depends on the history of previous shocks. This dynamic was initially introduced by Hawkes [12, 13] and Hawkes and Oakes [14] to model earthquake aftershocks. The literature on these processes is vast and we refer to Hawkes [15] for a review. In its standard version, the intensity of the self-excited processes, that is akin to the instantaneous probability of a shock, increases as soon as a jump of price occurs. The influence of this jump on the intensity next decays according to a memory function of time, called kernel. In the most common and simplest specification, the kernel is exponential and the pair, jump and intensity processes, is Markov. In this case, the moment generating function (mgf) admits an analytical solution, found by Itô's calculus. In a general setting, we lose most of the analytical tractability offered by stochastic calculus as the claim-intensity process is not Markov anymore. Apart from moments or asymptotic properties, very few results are available in the literature. For instance, Muzy et al. [17] find the first moments of stationary processes and their limit behaviour. Stabile and Torrisi [21] study the asymptotic behavior of non-stationary hawkes. Hainaut [9] establishes the mgf of self-excited claim processes with memory functions that admit a spectral representation. Jaisson and Rosenbaum [16] remark that nearly unstable Hawkes processes often fit high-frequency finance data properly. They show that under certain conditions, the limiting law of an unstable process corresponds to a Brownian Volterra process with a kernel, $k(u) = u^{\alpha-1}E_{\alpha,\alpha}(-u^{\alpha})$, where $\alpha \in (0, 1]$ and $E_{\alpha,\alpha}$ is the Mittag-Leffler function. Chen et al. [2] and Habyarimana et al. [8] use the same kernel for defining the fractional Hawkes process. Despite that this kernel diverges at the origin, the process remains stable and the expected intensity and number of jumps admit a closed-form expression. In this article, we study another type of Hawkes process with a diverging kernel at the origin, directly inspired from the literature on fractional and rough Brownian motions.

Unlike the regular Brownian motion (Bm), the fractional Brownian motion (fBm) has dependent increments. This dependence is measured on a scale from zero to one by the Hurst parameter or index, $H \in (0, 1)$. A value of 1/2 corresponds to the Bm with independent increments. A value of H greater (resp. lower) than 1/2 corresponds to positive (resp. negative) correlation between increments. We refer the reader to chapter 6 of [10] for a detailled introduction to fBm. The sample paths of fBm with $H < 1/2$, exhibit a high ruggedness compared to the Bm and are called rough for this reason. The fBm at time $t > 0$ admits an integral representation with respect to a Bm over $(-\infty, t]$. A rough process is defined as this integral truncated to the positive time axis: $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dW_s$ where $\alpha \in (0, 1]$ and W_t is a Bm. This stochastic integral is well-posed even if the rough kernel $k(u) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}$, diverges when $u \to 0$. Rough asset dynamics recently received a great deal of attention in the fractional literature and especially in finance because they are consistent with empirical behaviour of stock price volatility. Gatheral et al. [6] propose a model in which the variance of stock prices is driven by a fractional Brownian motion (fBm) with a Hurst index $H < 1/2$. The properties of this model are studied in details by El Euch and Rosenbaum [3] and [4], who express the characteristic function of the rough process in terms of a fractional Ricatti equation. In this article, we define a Hawkes process with a rough kernel, dampened by an exponential decaying function to ensure its stability.

The contributions of this article are multiple. At our knowledge, this work is the first to study the properties of a Hawkes process with a dampened rough kernel. We infer the closed-form expressions of expected intensity and number of jumps. We next find the conditions that guarantee the stability of the process. We build an equivalent infinite dimensional Markov representation. Approaching this by discretization allows us to approximate the Laplace's transform of the rough Hawkes by solving forward ordinary differential equations (ODE's). As the dampened rough kernel belongs to the family of Sonine function, it admits a conjugate kernel and we can define a similar operator to the left fractional Riemann-Liouville (RL) integral, called dampened RL integral. Considering the limit of the finite dimensional approximation leads to the Laplace's transform of the rough Hawkes process. In a similar manner to El Euch and Rosenbaum [4], this transform is expressed in terms of a fractional differential equation involving the dampened RL integral. Finally, we discuss practical aspects such as simulation and statistical estimation.

The outline of this article is as follows. Section 2 introduces the rough Hawkes process and presents the properties of the dampened rough kernel. We find next the first moment of the process and the conditions of stability. We close the section by reformulating the rough Hawkes process as an infinite dimensional process. Section 3 develops a finite dimensional approximation. In this setting, we express the Laplace's transform of the rough Hawkes process in terms of backward and forward ODE's. In the fourth section, we consider the limit of forward ODE's when the size of the finite dimensional approximation tends to $+\infty$. We obtain an elegant formulation of the Laplace's transform depending on a fractional equation, involving the dampened RL integral. Section 5 presents a modified version of the Ogata's algorithm to simulate the rough Hawkes process. In Section 6, we provide the closed form expression of the log-likelihood. We conclude by fitting the rough process to the time-series of excessive negative jumps in the hourly Bitcoin return. The model is benchmarked to an exponential Hawkes process.

2 A dampened rough kernel

We consider a point process $(L_t)_{t\geq 0}$ defined on a probability space Ω, that is a self-excited process with an intensity $(\lambda_t)_{t\geq 0}$. This process is the sum of random increments , noted J_k,

$$L_t \;=\; \sum_{k=1}^{N_t} J_k \,,$$

where $(N_t)_{t\geq 0}$ is the number of jumps or events observed up to time t. The statistical distribution of $J_k \sim J$ is denoted by $m(.)$ and is defined on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ to ensure the positivity of the intensity. The natural filtration of (L_t, N_t, λ_t) and the probability measure are respectively denoted by $\mathcal{F}_t = \sigma\left((L_s, N_s, \lambda_s)\ ,\ s \leq t\right)$ and $\mathbb{P}$. The intensity depends upon the past realizations of the counting process $(N_t)_{t\geq 0}$ in the following way:

$$\lambda_t \;=\; \lambda_0 + \eta \int_0^{t-} e^{-\beta(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dN_s \,, \tag{1}$$

where $\alpha \in (0,1]$, $\beta, \eta \in \mathbb{R}^+$. We will discuss later the conditions ensuring the stability of the intensity. The function $k(u) = e^{-\beta u} \frac{u^{\alpha-1}}{\Gamma(\alpha)}$ is called the memory kernel in the rest of the article. This is the product of a dampening term, $e^{-\beta u}$, and of the rough kernel, $\frac{u^{\alpha-1}}{\Gamma(\alpha)}$. This rough kernel presents several interesting features. Jaisson and Rosenbaum [16] have shown that the limit of nearly unstable Hawkes processes is a Brownian Volterra process with this rough kernel. On the other hand, Gatheral et al. [6] have provided empirical arguments that stock price variances are well fitted by a rough Brownian motion. This rough Brownian motion is defined as the integral $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dW_s$ where $(W_s)_{s\geq 0}$ is a Brownian motion. This naturally leads us to consider a Hawkes process with a rough kernel. As we will see in the next paragraphs, a dampening factor is nevertheless required to avoid the divergence of the intensity process. The presence of this dampening factor also implies that the rough model with $\alpha = 1$ is a standard Hawkes process with an exponential kernel.

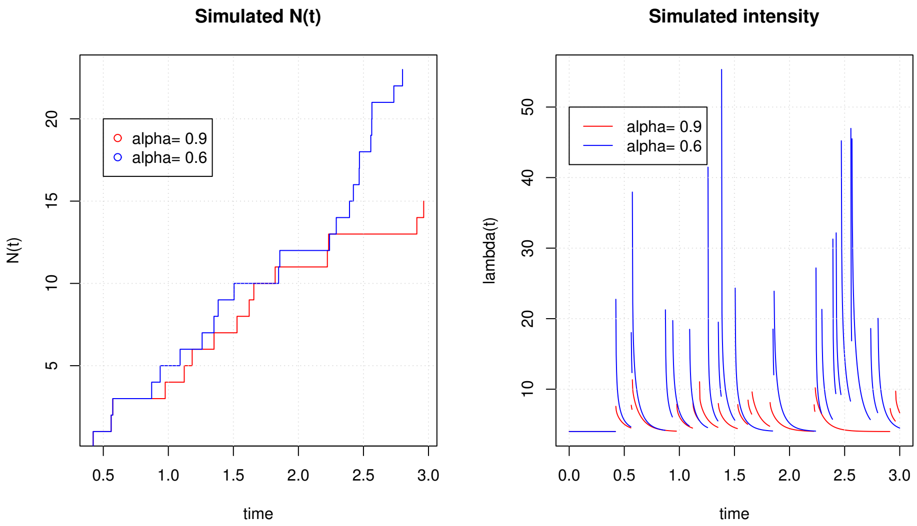

Figure 1: Comparison of two simulated samples: $\eta = 2$, $\beta = 10.80$, $\lambda_0 = 4$ and $\alpha \in \{0.60\,,\,0.90\}$.

Figure 1 shows two simulated sample paths of the rough Hawkes process. The seed of the generator of random numbers and parameters excepted α, are the same in both cases. The left reveals that the number of events observed over the same time horizon is higher for $\alpha = 0.6$ than for $\alpha = 0.9$. The algorithm used for this simulation is detailed in Section 5. In both cases, the intensity quickly reverts to the baseline λ_0.

The dampened rough kernel has several interesting features. Firstly, it belongs to the family of Sonine functions [20], defined hereafter. Secondly, we can associate an operator similar to the left fractional Riemann-Liouville integral (see Equation 52).

Definition 1. A kernel $k(u) \in L^1_{loc}(\mathbb{R}^+)$ is a Sonine function if there exists a conjugate kernel $l(u) \in L^1_{loc}(\mathbb{R}^+)$ such that

$$\int_0^t l(t-u)\,k(u)\,du = 1\,,\,\forall t \geq 0. \tag{2}$$

Let $\phi \in L^1(\mathbb{R}^+)$, the Sonine operators associated to $k(u)$ and $l(u)$ are defined as

$$(K\phi)(t) = \int_0^t k\,(t-u)\,\phi\,(u)\,du\ ,\ \forall t \geq 0, \tag{3}$$
$$(L\phi)(t) = \int_0^t l\,(t-u)\,\phi\,(u)\,du\ ,\ \forall t \geq 0.$$

Given the similarity between $K\phi$ and $I^{\alpha}_{0+}\phi$, we call the operator K as the dampened Riemann-Liouville (RL) integral. Notice that by definition, the kernels $k(u)$ and $l(u)$ are necessary unbounded as $u \to 0$. We have to introduce some conditions on the kernel to guarantee the existence of the integral on L_p functions. In this article, $k(u)$ has the form

$$k(u) = \frac{g(u)}{u^{1-\alpha}} \quad , \quad x > 0\,, \sup_{u\geq 0}|g(u)| < \infty\,,$$

where $g(u) = \frac{1}{\Gamma(\alpha)}e^{-\beta\, u}$ is a bounded function and $\alpha \in (0,1)$. From the Hardy-Littlewood [11] Sobolev inequality, a sufficient condition is $\alpha < 1/p$. In this case, the operator acts from $L_p(\mathbb{R})$, $1 < p < 1/\alpha$ into $L_q(\mathbb{R})$ where $1/q = 1/p - \alpha$. In the remainder of this article, we consider L_1-integrands which ensures that the operator $K\phi$ is well defined for $\alpha \in (0,1)$. We refer the refer to Samko and Cardoso [19] for the necessary conditions for the existence of other integrals with general Sonine kernels.

If we denote by $(\mathcal{L}\phi)\,(z) = \int_0^\infty e^{-zu}\phi(u)\,du$, the Laplace's transform of a function, $\phi \in L^1(\mathbb{R}^+)$, we infer by direct calculation that

$$(\mathcal{L}k)\,(z) \quad = \quad \frac{1}{(\beta+z)^{\alpha}}\,. \tag{4}$$

Furthermore, the Sonine condition (2) may be rewritten in terms of Laplace's transforms of $k(.)$ and $l(.)$:

$$(\mathcal{L}k)\,(z)\,(\mathcal{L}l)\,(z) \quad = \quad \frac{1}{z}\,. \tag{5}$$

This last relation is the key to prove the next result.

Proposition 1. *The conjugate kernel $l(.)$ of $k(.)$ satisfying condition (2), is :*

$$l(u) = \beta^{\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\int_u^\infty \frac{e^{-\beta s}}{s^{1+\alpha}}ds\,, \tag{6}$$

Proof. We will check that the Laplace's transform of $l(.)$ fulfills the condition (5). Firstly, we integrate by parts the integral in Eq. (6):

$$\alpha\int_u^\infty \frac{e^{-\beta s}}{s^{1+\alpha}}ds \quad = \quad e^{-\beta u}u^{-\alpha} - \beta\int_u^\infty e^{-\beta s}s^{-\alpha}\,du\,.$$

This allows us to rewrite $(\mathcal{L}l)\,(z)$ as the sum:

$$(\mathcal{L}l)\,(z) = \beta^{\alpha}\int_0^\infty e^{-zu}du + \frac{1}{\Gamma(1-\alpha)}\int_0^\infty e^{-(z+\beta)u}u^{-\alpha}du \tag{7}$$
$$- \frac{\beta}{\Gamma(1-\alpha)}\int_0^\infty\int_u^\infty e^{-\beta s}e^{-zu}s^{-\alpha}\,ds\,du\,.$$

If we perform a change of variable $v = (z+\beta)u$, we immediately infer that

$$\int_0^\infty e^{-(z+\beta)u} u^{-\alpha} du = (z+\beta)^{\alpha-1}\Gamma(1-\alpha)\,.$$

Next, we change the order of integration in the last integral of Equation (7) using the Fubini's theorem. We obtain that

$$\begin{aligned}&\int_0^\infty \int_u^\infty e^{-\beta s} e^{-zu} s^{-\alpha}\, ds\, du \\ &\quad = \int_0^\infty e^{-\beta s} s^{-\alpha}\, ds - \frac{1}{z}\int_0^\infty e^{-(\beta+z)s} s^{-\alpha}\, ds \\ &\quad = \frac{\beta^{\alpha-1}}{z}\Gamma(1-\alpha) - \frac{(\beta+z)^{\alpha-1}}{z}\Gamma(1-\alpha)\,.\end{aligned}$$

Combining previous intermediate results allows us to infer that the Laplace's transform of the conjugate kernel of $k(.)$ is equal to $(\mathcal{L}l)(z) = \frac{(\beta+z)^\alpha}{z}$, which fulfills the condition (5). □

The dampened RL integral $(K\phi)(t)$ admits an inverse operator, provided in the next proposition and comparable to a fractional derivative.

Proposition 2. *The inverse operator of the dampened RL integral K, is the derivative of its conjugate kernel. For $\phi \in L^1(\mathbb{R}^+)$, it is equal to*

$$\begin{aligned}\left(K^{-1}\phi\right)(t) &= \frac{d}{dt}(L\phi)(t) \\ &= \frac{d}{dt}\int_0^t l(t-u)\,\phi(u)\, du\,.\end{aligned} \tag{8}$$

This inverse operator is called the dampened RL derivative.

Proof. This result is a direct consequence of the Sonine condition. We apply the operator L to $K\phi$ and permut the order of integration. We next perform the change of variable $v = s-u$:

$$\begin{aligned}(LK\phi)(t) &= \int_0^t l(t-s)\int_0^s k(s-u)\phi(u)\, du\, ds \\ &= \int_0^t \phi(u)\int_0^{t-u} l(t-s)k(s-u)\, ds\, du \\ &= \int_0^t \phi(u)\int_0^{t-u} l(t-u-v)k(v)\, dv\, du\end{aligned}$$

From the Sonine condition (2), we deduce that the inner integral is equal to 1. Differentiating both sides with respect to t, allows to conclude that $K^{-1}\phi = \frac{d}{dt}(L\phi)$. □

The dampened RL integral and derivative will later be involved in the definition of the Laplace's transform of point and intensity processes. Before exploring this, we find the expectation of the intensity which involves the Mittag-Leffler function with one and two parameters (see Appendix A).

Proposition 3. *The expected intensity at time $t \geq 0$ conditionally to the filtration $\mathcal{F}_0$, is equal to*

$$\mathbb{E}_0\left(\lambda_t\right) = \lambda_0 E_\alpha\left(\eta t^\alpha\right) e^{-\beta t} + \beta\lambda_0 \int_0^{t-} E_\alpha\left(\eta(t-s)^\alpha\right) e^{-\beta(t-s)} ds\,. \tag{9}$$

If $\alpha \in (0,1]$ and $\beta \in \mathbb{R}^+$ are such that $\lim_{t\to\infty} E_\alpha(\eta t^\alpha) e^{-\beta t} \to 0$ and $\beta^\alpha \geq \eta$ then

$$\lambda_\infty := \lim_{t\to\infty} \mathbb{E}_0\left(\lambda_t\right) = \lambda_0 \frac{\beta^\alpha}{\beta^\alpha - \eta}. \tag{10}$$

Proof. Let us denote by $u(t) = e^{\beta t}\mathbb{E}_0\left(\lambda_t\right)$, then Eq (1) is rephrased as follows

$$u(t) = \lambda_0 e^{\beta t} + \frac{\eta}{\Gamma(\alpha)} \int_0^{t-} (t-s)^{\alpha-1} u(s) ds\,. \tag{11}$$

From Gorenflo et al. ([7], page 63 Theorem 4.2), this equation admits an unique solution that is

$$u(t) = \lambda_0 e^{\beta t} + \int_0^{t-} \eta(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\eta\left(t-s\right)^\alpha\right) e^{\beta s}\lambda_0 ds \tag{12}$$

If we remind Equation (53) in Appendix A, we have that

$$\begin{aligned}\frac{dE_\alpha\left(\eta(t-s)^\alpha\right)}{ds} &= -\left.\frac{dE_\alpha(\eta x^\alpha)}{dx}\right|_{x=(t-s)} \\ &= -\eta(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\eta(t-s)^\alpha\right)\,.\end{aligned}$$

This allows us to rewrite Equation (12) as follows

$$\begin{aligned}\mathbb{E}_0\left(\lambda_t\right) &= \lambda_0 + \lambda_0 \int_0^{t-} \eta(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\eta\left(t-s\right)^\alpha\right) e^{-\beta(t-s)} ds \\ &= \lambda_0 - \lambda_0 \int_0^{t-} \frac{dE_\alpha\left(\eta(t-s)^\alpha\right)}{ds} e^{-\beta(t-s)} ds\,.\end{aligned} \tag{13}$$

Combining Equations (12) and (13) leads to the result (9). We next Integrate by parts and obtain that

$$\begin{aligned}&\int_0^{t-} \frac{dE_\alpha\left(\eta(t-s)^\alpha\right)}{ds} e^{-\beta(t-s)} ds \\ &= \left(1 - E_\alpha\left(\eta t^\alpha\right) e^{-\beta t}\right) - \beta \int_0^{t-} E_\alpha\left(\eta(t-s)^\alpha\right) e^{-\beta(t-s)} ds\,.\end{aligned} \tag{14}$$

By assumption, the limit of the first term when $t \to \infty$ is equal to

$$\lim_{t\to\infty} \left(1 - E_\alpha\left(\eta t^\alpha\right) e^{-\beta t}\right) = 1\,.$$

The limit of the second term in Eq. (14) is the Laplace's transform of $E_\alpha\left(\eta x^\alpha\right)$ (see Eq. 54 in Appendix A):

$$\begin{aligned}\lim_{t\to\infty} \int_0^{t-} E_\alpha\left(\eta(t-s)^\alpha\right) e^{-\beta(t-s)} ds &= \int_0^\infty e^{-\beta x} E_\alpha\left(\eta x^\alpha\right) dx \\ &= \frac{\beta^{\alpha-1}}{\beta^\alpha - \eta}\,.\end{aligned}$$

Therefore

$$\lambda_\infty = \lambda_0 - \lambda_0 \left(1 - \frac{\beta^\alpha}{\beta^\alpha - \eta}\right) = \lambda_0 \frac{\beta^\alpha}{\beta^\alpha - \eta} ,$$

which is well defined if $\beta^\alpha > \eta$. □

The previous proposition reveals that the intensity diverges to $+\infty$ if one of the two following conditions are breached:

$$\lim_{t\to\infty} E_\alpha(\eta t^\alpha) e^{-\beta t} \to 0 \quad \text{and} \quad \beta^\alpha \geq \eta \,. \tag{15}$$

We deduce from the first relation that a Hawkes process with a non-dampened rough kernel ($\beta = 0$) has an explosive intensity. This observation motivates the multiplication of the rough kernel by a dampening factor.

It is interesting to compare asymptotic expected intensities of exponential and rough Hawkes processes. In the exponential process, $\alpha = 1$ and the intensity of the counting process denoted by N_t^h, has an exponential decaying kernel:

$$\lambda_t^h \;=\; \lambda_{h,0} + \eta_h \int_0^{t-} e^{-\beta_h (t-s)} dN_s^h \,, \tag{16}$$

where $\eta_h, \beta_h \in \mathbb{R}^+$. The condition $\beta_h - \eta_h > 0$ ensures the stability of the process. The expected intensity is in this case given by

$$\mathbb{E}_0\left(\lambda_t^h\right) \;=\; \frac{\lambda_{h,0}}{\beta_h - \eta_h}\left(\beta_h - \eta_h e^{-(\beta_h - \eta_h)t}\right) ,$$

whereas the asymptotic intensity is

$$\lambda_\infty^h := \lim_{t\to\infty} \mathbb{E}_0\left(\lambda_t^h\right) = \lambda_{h,0} \frac{\beta_h}{\beta_h - \eta} . \tag{17}$$

A comparison with Equation (10) reveals that the rough Hawkes process observed behaves at long term like an exponential Hawkes process with a parameter β^α instead of β_h. In Section 6, we will compare the goodness of fit of rough and exponential Hawkes processes.

Corollary 1. *The expected number of events at time t, conditionally to $\mathcal{F}_0$ is equal to*

$$\mathbb{E}_0\left(N_t\right) = \lambda_0 \left(e_\alpha(t) + \int_0^t e_\alpha(u)\, du\right) , \tag{18}$$

where $e_\alpha(t)$ is the incomplete Laplace's transform of $E_\alpha\left(\eta u^\alpha\right)$:

$$e_\alpha(u) = \int_0^u E_\alpha\left(\eta s^\alpha\right) e^{-\beta s} ds \,. \tag{19}$$

Proof. By construction, the expectation of a counting process is the expectation of the integrated intensity. If we remember Eq. (9), this integral is developped as follows:

$$\begin{aligned}
\mathbb{E}_0\left(N_t\right) &= \int_0^t \mathbb{E}_0\left(\lambda_u\right) du \\
&= \lambda_0 \int_0^t E_\alpha\left(\eta u^\alpha\right) e^{-\beta u} du + \lambda_0 \int_0^t \int_0^{u-} E_\alpha\left(\eta (u-s)^\alpha\right) e^{-\beta(u-s)} ds\, du \\
&= \lambda_0 e_\alpha(t) + \lambda_0 \int_0^t \int_0^{u-} E_\alpha\left(\eta (v)^\alpha\right) e^{-\beta v} dv\, du \,.
\end{aligned}$$

□

The integral $e_\alpha(t)$ cannot be calculated in closed-form but is easily computable by discretizing the integral. Figure 2 displays expected intensities and number of events, computed with the same parameters used for simulating the sample paths in Figure 1. For both values of α, we observe a quick convergence of $\mathbb{E}_0(\lambda_t)$ toward λ_∞ (respectively equal to 5.23 and 7.69 for $\alpha = 0.9$ and $\alpha = 0.6$) . Both plots confirm our first intuition,

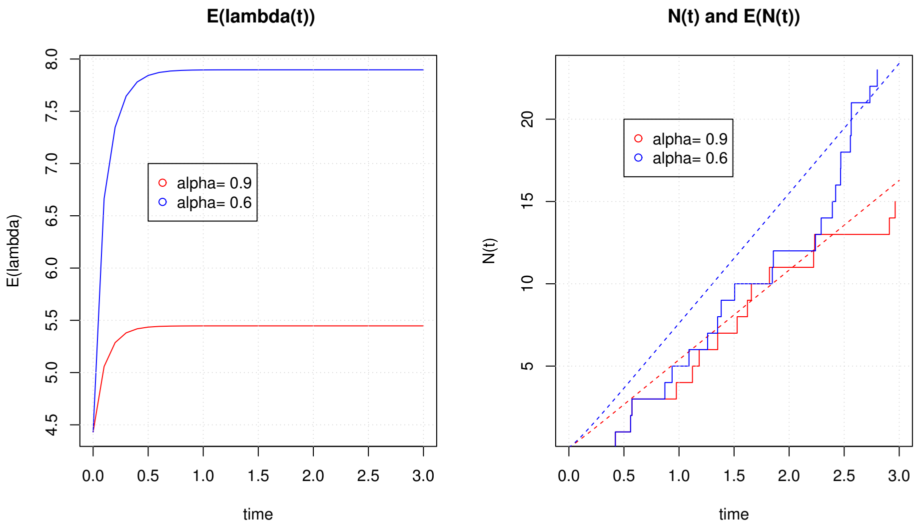

Figure 2: Left plot: computation of $\mathbb{E}_0(\lambda_t)$. Right plot: comparison of counting process of Figure 1 to their expectation. $\eta = 2$, $\beta = 10.80$, $\lambda_0 = 4$ and $\alpha \in \{0.60\,,\,0.90\}$.

By construction, the rough Hawkes process is not Markov since its intensity λ_t cannot be rewritten as a function of λ_s for $0 \leq s \leq t$. Nevertheless, we can reformulate the model as an infinite dimensional Markov process because the power $x^{\alpha-1}$ admits an integral representation. We detail this in the next proposition.

Proposition 4. *Let us define a family of auxiliary jump processes $Z_t^{(\xi)}$, indexed by $\xi \in \mathbb{R}^+$ and defined as*

$$Z_t^{(\xi)} = \eta \int_0^t e^{-(\beta+\xi)(t-s)} dN_s \,. \tag{20}$$

If we denote by $\gamma(d\xi) := \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi$ for $\xi \geq 0$, the intensity λ_t is rewritten as an integral of $Z_t^{(\xi)}$ with respect to $\gamma(d\xi)$:

$$\lambda_t = \lambda_0 + \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_t^{(\xi)} \gamma(d\xi) \,. \tag{21}$$

Proof. We can check by direct integration that $x^{\alpha-1}$ admits an integral representation

$$x^{\alpha-1} = \int_0^\infty e^{-x\xi} \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi \,.$$

On the other hand, the process $Z_t^{(\xi)}$ is an Ornstein-Uhlenbeck jump process reverting toward 0, with the infinitesimal dynamic

$$dZ_t^{(\xi)} = -(\beta+\xi) Z_t^{(\xi)} dt + \eta dN_t \,. \tag{22}$$

This process having a finite expectation for all $\xi \in \mathbb{R}^+$,we can rewrite the intensity using the Fubini's theorem:

$$\begin{aligned}\lambda_t &= \lambda_0 + \eta \int_0^t e^{-\beta(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dN_s \\ &= \lambda_0 + \frac{\eta}{\Gamma(\alpha)} \int_0^t \left(\int_0^\infty e^{-(\beta+\xi)(t-s)} \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi \right) dN_s \\ &= \lambda_0 + \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_t^{(\xi)} \gamma(d\xi) \,.\end{aligned}$$

□

By definition, the processes $\left(\lambda_t, L_t, \left(Z_t^{(\xi)}\right)_{\xi \in \mathbb{R}^+}\right)$ is Markov. A similar reformulation of a non-Markov Hawkes process into an infinite Markov one in used in Hainaut [9] for processes with kernels admitting a spectral representation. The main differences with the current article are that the considered kernels do not diverge at the origin and that equivalent processes to $Z_t^{(\xi)}$ are defined in the complex plane. We will also see that the divergence at the origin of the dampened rough kernel prevent us to express the Laplace's transform in terms of backward ordinary equations.

3 Finite dimensional approximation

In this section , we approach the integral in (21) on a finite grid of processes $Z_t^{(\xi)}$. This method makes possible to use the Itô's calculus to find the Laplace's transform of $(L_t)_{t \geq 0}$. For this purpose, we approximate $\gamma(.)$ by a discrete measure on a finite numbers of atoms. Let us define the partition $\mathcal{E}^{(n)} := \{0 < \xi_0^{(n)} < \xi_1^{(n)} < ... < \xi_n^{(n)} < \infty\}$. The mid point of each interval $(\xi_l^{(n)}, \xi_{l+1}^{(n)})$ is denoted by

$$b_l = \frac{\xi_l^{(n)} + \xi_{l+1}^{(n)}}{2} \,, l \in \{0, ..., n-1\} \tag{23}$$

The mass of atoms is defined as the integral of $\gamma(.)$ over the interval :

$$w_l = \int_{\xi_l^{(n)}}^{\xi_{l+1}^{(n)}} \gamma(dz) \,, l \in \{0, ..., n-1\} \tag{24}$$

we note $\tilde{Z}_t^{(l)} := Z_t^{(b_l)}$ for $l = 1, ...n-1$. Each $\tilde{Z}_t^{(l)}$ is mean reverting and ruled by the SDE

$$d\tilde{Z}_t^{(l)} = \left(-(\beta + b_l) \tilde{Z}_t^{(l)} \right) dt + \eta \, d\tilde{N}_t \quad ,$$

where $\tilde{N}_t$ is the counting process in the finite dimensional model. Its intensity, noted $\tilde{\lambda}_t$, is the sum of $\tilde{Z}_t^{(l)}$, weighted by the mass of atoms:

$$\tilde{\lambda}_t = \lambda_0 + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \tilde{Z}_t^{(l)} \,.$$

Its differential is given by

$$d\tilde{\lambda}_t = -\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} (\beta + b_l)\, \tilde{Z}_t^{(l)} dt + \eta \left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \right) d\tilde{N}_t \,. \tag{25}$$

The next proposition provides the joint Laplace's transform of $\tilde{L}_s$ and $\tilde{\lambda}_s$ conditionally to the filtration of $\mathcal{F}_t$, in terms of backward ordinary differential equations (ODE's).

Proposition 5. *Let $\omega_1, \omega_2 \in \mathbb{R}^+$. The joint Laplace's function of jump and intensity processes at time t, conditionally to $\mathcal{F}_t$, is given by the following expression*

$$\mathbb{E}\left(e^{-\omega_1 \tilde{L}_s - \omega_2 \tilde{\lambda}_s} \,|\, \mathcal{F}_t \right) \;=\; \exp\left(q_\lambda(t,s)\tilde{\lambda}_t + \sum_{l=0}^{n-1} q_l(t,s) \frac{w_l}{\Gamma(\alpha)} \tilde{Z}_t^{(l)} - \omega_1 \tilde{L}_t \right) \tag{26}$$

where functions $q_\lambda(t,s), q_l(t,s) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, solves the ODE's:

$$\begin{cases} \partial_t q_\lambda(t,s) = -\left(\mathbb{E}\left(e^{-\omega_1 J}\right) \exp\left(\eta \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(q_\lambda(t,s) + q_l(t,s) \right) \right) - 1 \right) , \\ \\ \partial_t q_l(t,s) = (\beta + b_l) \left(q_l(t,s) + q_\lambda(t,s) \right) . \end{cases} \tag{27}$$

with the terminal conditions $q_\lambda(s,s) = -\omega_2$ and $q_l(s,s) = 0$ for $l = 0, ..., n-1$.

Proof. Let us denote the Laplace's transform by $f\left(t, \lambda_t, \left(Z_t^{(\xi)}\right)_{\xi \in \mathbb{R}^+}, L_t\right) = \mathbb{E}\left(e^{-\omega_1 \tilde{L}_s - \omega_2 \tilde{\lambda}_s} | \, \mathcal{F}_t\right)$. By definition $f(.)$ is a conditional expectation and then a martingale. This implies that $\mathbb{E}\left(df \,|\, \mathcal{F}_t\right) = 0$. From the Itô's lemma, $f(.)$ satisfies the following stochastic differential equation (SDE):

$$\begin{aligned} 0 = \partial_t f(.) - \sum_{l=0}^{n-1} (\beta + b_l)\, \tilde{Z}_t^{(l)}\, \partial_{\tilde{Z}_t^{(l)}} f(.) - \partial_{\tilde{\lambda}} f(.) \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} (\beta + b_l)\, \tilde{Z}_t^{(l)} \\ + \tilde{\lambda}_t \int_0^\infty f\left(t, \tilde{\lambda}_t + \eta \left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \right), \left(\tilde{Z}_t^{(l)} + \eta \right)_{l=0,\dots,n-1}, \tilde{L}_t + z \right) - f(.)\, m(dz) \,. \end{aligned} \tag{28}$$

We do the ansatz that $f(.)$ is an exponential affine function that looks like

$$f(.) = \exp\left(q_0(t,s) + q_\lambda(t,s)\tilde{\lambda}_t + \sum_{l=0}^{n-1} q_l(t,s) \frac{w_l}{\Gamma(\alpha)} \tilde{Z}_t^{(l)} - \omega_1 \tilde{L}_t \right) . \tag{29}$$

The partial derivatives of $f(.)$ with respect to state variables and times are given by

$$\begin{aligned} \partial_{\tilde{Z}^{(l)}} f(.) &= f(.) \frac{w_l}{\Gamma(\alpha)} q_l(t,s) \\ \partial_{\tilde{\lambda}} f(.) &= f(.) q_\lambda(t,s) \,, \end{aligned}$$

and the derivative with respect to time is

$$\partial_t f(.) \;=\; f(.) \left(\partial_t q_0(t,s) + \partial_t q_\lambda(t,s) \tilde{\lambda}_t + \sum_{l=0}^{n-1} \partial_t q_l(t,s) \frac{w_l}{\Gamma(\alpha)} \tilde{Z}_t^{(l)} \right) .$$

Under assumption (29), the jump term in Equation (28) becomes

$$f\left(t,\tilde{\lambda}_t+\eta\left(\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\right),\left(\tilde{Z}_t^{(l)}+\eta\right)_{l=0,\dots,n-1},\tilde{L}_t+\,z\right)-f(.)=$$
$$f(.)\left(\exp\left(\eta\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\left(q_\lambda(t,s)+q_l(t,s)\right)-\omega_1\,z\right)-1\right).$$

Combining the previous equations allows us to infer that $q_0(t,s)$, $q_\lambda(t,s)$ and $q_l(t,s)$ satisfy the relation

$$\begin{aligned}0&=\partial_t q_0(t,s)+\partial_t q_\lambda(t,s)\tilde{\lambda}_t+\sum_{l=0}^{n-1}\partial_t q_l(t,s)\frac{w_l}{\Gamma(\alpha)}\tilde{Z}_t^{(l)}\\&-\sum_{l=0}^{n-1}(\beta+b_l)\,\tilde{Z}_t^{(l)}\frac{w_l}{\Gamma(\alpha)}q_l(t,s)-q_\lambda(t,s)\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}(\beta+b_l)\,\tilde{Z}_t^{(l)}\\&+\tilde{\lambda}_t\int_0^\infty\left(\exp\left(\eta\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\left(q_\lambda(t,s)+q_l(t,s)\right)-\omega_1\,z\right)-1\right)m(dz)\,.\end{aligned}\tag{30}$$

Grouping terms allows us to infer that $q_0(t,s)=0$ and

$$0=\partial_t q_\lambda(t,s)+\left(e^{\eta\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}(q_\lambda(t,s)+q_l(t,s))}\mathbb{E}\left(e^{-\omega_1\,J}\right)-1\right),$$
$$0=\sum_{l=0}^{n-1}\partial_t q_l(t,s)\frac{w_l}{\Gamma(\alpha)}-\sum_{l=0}^{n-1}(\beta+b_l)\,\frac{w_l}{\Gamma(\alpha)}q_l(t,s)-q_\lambda(t,s)\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}(\beta+b_l)\,.$$

This last equation is fulfilled if

$$\partial_t q_l(t,s)\;=\;(\beta+b_l)\left(q_l(t,s)+q_\lambda(t,s)\right)\,.$$

□

The next corollary states that the function $q_l(t,s)$ admits an integral representation.

Corollary 2. *The function $q_l(t,s)$ solving the second ODE in Equation (27) is equal to*

$$q_l(t,s)=-\int_t^s(\beta+b_l)\,e^{-\,(\beta+b_l)\,(u-t)}q_\lambda(u,s)\,du\,.\tag{31}$$

This result is checked by deriving the expression of $q_l(t,s)$ with respect to t. We immediately retrieve the first ODE in (27). In Hainaut ([10], Chapter 5), the characteristic function of the non-Markov Hawkes process L_t, is retrieved by increasing the size n of the partition $\mathcal{E}^{(n)}$, up to infinity. Unfortunately, we cannot apply the same approach for the dampened rough kernel. Indeed, the limit of the sum of w_l, involved in Equation (27), is not defined when $n\to\infty$ because

$$\lim_{n\to\infty}\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\;=\;\frac{1}{\Gamma(\alpha)}\int_0^\infty\frac{\xi^{-\alpha}}{\Gamma(1-\alpha)}d\xi=\infty\,.$$

Nevertheless, we can convert the backward ODE's (27) into forward ones, which admit a limit when $n\to\infty$. To establish these forward ODE's, we need the next result.

Corollary 3. *For any $s, s' \in \mathbb{R}^+$ and $v \in \mathbb{R}^+$ such that $v \leq \min(s, s')$, the following equality holds*

$$q_\lambda(s - v, s) = q_\lambda(s' - v, s') \,. \tag{32}$$

Proof. By definition the equality (32) is true for $v = 0$, $q_\lambda(s, s) = q_\lambda(s', s') = -\omega_2$. Let us then assume that the property (32) holds for all $r \in [0, v]$ where $0 < v \leq s$. From previous equations, we notice that

$$\frac{\partial q_\lambda(s-v,s)}{\partial v} = -\left.\frac{\partial q_\lambda(t,s)}{\partial t}\right|_{t=s-v} =$$
$$\left(\mathbb{E}\left(e^{-\omega_1 J}\right)\exp\left(\eta\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\left(q_\lambda(s-v,s)+q_l(s-v,s)\right)\right)-1\right)$$

where

$$q_l(s-v,s) = -\left(\beta+b_l\right)\int_{s-v}^{s} e^{-\,(\beta+b_l)\,(u-(s-v))}q_\lambda(u,s)\,du\,.$$

We rewrite the integrals in the previous equation using the change of variable $u = s - r$,

$$\int_{s-v}^{s} e^{-\,(\beta+b_l)\,(u-(s-v))}q_\lambda(u,s)\,du \;=\; \int_0^v e^{-(\beta+b_l)\,(v-r)}q_\lambda(s-r,s)\,dr$$

Since $q_\lambda(s - r, s) = q_\lambda(s' - r, s')$ for all $r \in [0, v]$, we infer that

$$\begin{aligned}\int_0^v e^{-(\beta+b_l)\,(v-r)}q_\lambda(s-r,s)\,dr &= \int_0^v e^{-(\beta+b_l)\,(v-r)}q_\lambda(s'-r,s')\,dr \\ &= \int_{s'-v}^{s'} e^{-\,(\beta+b_l)\,(u-(s'-v))}q_\lambda(u,s')\,du\end{aligned}$$

and $q_l(s - v, s) = q_l(s' - v, s')$. Therefore we have well

$$\frac{\partial q_\lambda(s-v,s)}{\partial v} = \frac{\partial q_\lambda(s'-v,s')}{\partial v}$$

and conclude that the equality (32) also holds for $v + dv$. □

A direct consequence of this last corollary is that $q_\lambda(t, s) = q_\lambda(s - t)$ and $q_l(t, s) = q_l(s - t)$. Then, we infer that

$$\begin{aligned}\partial_t q_\lambda(t,s) &= -\partial_s q_\lambda(t,s)\,, \\ \partial_t q_l(t,s) &= -\partial_s q_l(t,s)\,.\end{aligned}$$

This allows to rewrite the Laplace's transform in terms of a forward differential equation ruling $q_\lambda(t, s)$.

Proposition 6. *Let $\omega_1, \omega_2 \in \mathbb{R}^+$. The joint Laplace's function of jump and intensity processes at time t, conditionally to $\mathcal{F}_t$, is given by the following expression*

$$\mathbb{E}\left(e^{-\omega_1\tilde{L}_s-\omega_2\tilde{\lambda}_s}\,|\,\mathcal{F}_t\right) \;=\; \exp\left(q_\lambda(t,s)\tilde{\lambda}_t+\sum_{l=0}^{n-1}q_l(t,s)\frac{w_l}{\Gamma(\alpha)}\tilde{Z}_t^{(l)}-\omega_1\tilde{L}_t\right) \tag{33}$$

where $q_\lambda(t,s)$ *solves a forward ODE:*

$$\partial_s q_\lambda(t,s) = \mathbb{E}\left(e^{-\omega_1 J}\right)\exp\left(-\eta\left(\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)\,(\beta+b_l)}\partial_s q_l(t,s)\right)\right)-1 \tag{34}$$

and

$$\partial_s q_l(t,s) = -\partial_s\int_t^s (\beta+b_l)\,e^{-\,(\beta+b_l)\,(u-t)}q_\lambda(u,s)\,du\,, \tag{35}$$

with the initial condition $q_\lambda(t,t)=-\omega_2$ *and* $q_l(s,s)=0$ *for* $l=0,...,n-1$.

Proof. From equation (27) and as $\partial_t q_l(t,s)=-\partial_s q_l(t,s)$, we have that

$$q_l(t,s)+q_\lambda(t,s)=\frac{1}{\beta+b_l}\partial_t q_l(t,s)=-\frac{1}{\beta+b_l}\partial_s q_l(t,s)\,.$$

On the other hand, the backward ODE ruling $q_\lambda(.,.)$ is rewritten as follows

$$\partial_t q_\lambda(t,s) \;=\; -\left(\mathbb{E}\left(e^{-\omega_1 J}\right)\exp\left(\eta\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)\,(\beta+b_l)}\partial_t q_l(t,s)\right)-1\right).$$

As $\partial_t q_\lambda(t,s)=-\partial_s q_\lambda(t,s)$ and $\partial_t q_l(t,s)=-\partial_s q_l(t,s)$, this is equivalent to

$$\partial_s q_\lambda(t,s) \;=\; \left(\mathbb{E}\left(e^{-\omega_1 J}\right)\exp\left(-\eta\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)\,(\beta+b_l)}\partial_s q_l(t,s)\right)-1\right).$$

□

This last proposition is the conerstone that allows us to find the Laplace's transform of the dampened rough Hawkes process in the next section.

4 Laplace's transform of the rough Hawkes process

We now dispose of necessary tools for retrieving the Laplace's transform of the rough Hawkes process. The next proposition states that its initial value depends on a function solving a fractional differential equation involving the dampened RL integral and derivative.

Proposition 7. *The Laplace's transform of the rough point process* $(L_t)_{t\geq 0}$, *conditionally to* $\mathcal{F}_0$, *for* $\omega\in\mathbb{R}^+$, *is equal to*

$$\Upsilon_s(\omega):=\mathbb{E}\left(e^{-\omega L_s}\,|\,\mathcal{F}_0\right) \;=\; \exp\left(q_\lambda(s)\lambda_0\right) \tag{36}$$

where $q_\lambda(s)$ *solves a forward ODE:*

$$\frac{dq_\lambda(s)}{ds} \;=\; \mathbb{E}\left(e^{-\omega J}\right)\exp\left(\eta\left(K\frac{dq_\lambda}{ds}\right)(s)\right)-1 \tag{37}$$

with the initial condition $q_\lambda(0)=0$ *and where* $K\frac{dq_\lambda}{ds}$ *is the dampened Riemann-Liouville integral of* $\frac{dq_\lambda}{ds}$*:*

$$\left(K\frac{dq_\lambda}{ds}\right)(s)=\frac{1}{\Gamma(\alpha)}\int_0^s\frac{e^{-\,\beta\,(s-u)}}{(s-u)^{1-\alpha}}\frac{dq_\lambda(u)}{du}\,du.$$

An equivalent representation is obtained by defining $\psi(s) := \left(K\frac{dq_\lambda}{ds}\right)(s)$. this function solves the fractional differential equation

$$\left(K^{-1}\psi\right)(s) \;=\; \mathbb{E}\left(e^{\omega J}\right)\exp\left(\eta\,\psi(s)\right)-1\,, \tag{38}$$

where $\left(K^{-1}\psi\right)(s)=\frac{dq_\lambda}{ds}(s)$ is the dampened Riemann-Liouville derivative of $\psi(s)$:

$$\left(K^{-1}\psi\right)(s)=\frac{d}{ds}\int_0^s\left(\beta^\alpha+\frac{\alpha}{\Gamma(1-\alpha)}\int_{s-u}^\infty\frac{e^{-\beta v}}{v^{1+\alpha}}dv\right)\psi\left(u\right)du\,.$$

Proof. As in this case, $q_\lambda(s,s)=0$, we develop the forward derivative of $q_l(t,s)$ provided in Equation (35), as follows

$$\begin{aligned}\partial_s q_l(t,s) &= -\partial_s\int_t^s\left(\beta+b_l\right)e^{-\left(\beta+b_l\right)(u-t)}q_\lambda(u,s)\,du\\ &= -\int_t^s\left(\beta+b_l\right)e^{-\left(\beta+b_l\right)(u-t)}\partial_s q_\lambda(u,s)\,du\,.\end{aligned}$$

We have proven in the previous section that $q_\lambda(u,s)$ is in fact a function of $s-u:\,q_\lambda(s-u)$, if we perform the change of variable $v=s-u$, the derivative $\partial_s q_l(t,s)$ can be rewritten as

$$\partial_s q_l(t,s)=-\int_0^{s-t}\left(\beta+b_l\right)e^{-\left(\beta+b_l\right)(s-t-v)}\frac{dq_\lambda(v)}{dv}\,dv\,.$$

Furthermore, we have seen that $\partial_s q_\lambda(t,s)=\frac{dq_\lambda(s-t)}{ds}$. Then, we reformulate the forward ODE (34) as:

$$\frac{dq_\lambda(s-t)}{ds} \;=\; \mathbb{E}\left(e^{-\omega J}\right)\exp\left(\eta\left(\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\int_0^{s-t}e^{-\left(\beta+b_l\right)(s-t-v)}\frac{dq_\lambda(v)}{dv}\,dv\right)\right)-1\,. \tag{39}$$

We next consider the limit of the term in the exponential when the size of the partition $\mathcal{E}^{(n)}$ tends to infinity. By construction, the following limit is well defined:

$$\begin{aligned}&\lim_{n\to\infty}\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\int_0^{s-t}e^{-\left(\beta+b_l\right)(s-t-v)}\frac{dq_\lambda(v)}{dv}\,dv\\ &=\frac{1}{\Gamma(\alpha)}\int_0^{s-t}\int_0^\infty e^{-\xi\,(s-t-v)}\,\gamma(d\xi)e^{-\beta\,(s-t-v)}\frac{dq_\lambda(v)}{dv}\,dv\\ &=\int_0^{s-t}\frac{e^{-\beta\,(s-t-v)}}{\Gamma(\alpha)(s-t-v)^{1-\alpha}}\frac{dq_\lambda(v)}{dv}\,dv\,.\end{aligned}$$

At time $t=0$, i.e. conditionally to $\mathcal{F}_0$, we recognize the the dampened Riemann-Liouville of $\partial_s q_\lambda$:

$$\left(K\frac{dq_\lambda}{ds}\right)(s) \;=\; \int_0^s\frac{e^{-\beta\,(s-u)}}{\Gamma(\alpha)(s-u)^{1-\alpha}}\frac{dq_\lambda(u)}{du}\,du\,, \tag{40}$$

and combining Equations (39) and (40) leads to the fractional equation (37).

$$\frac{dq_\lambda(s)}{ds} \;=\; \mathbb{E}\left(e^{-\omega J}\right)\exp\left(\eta\left(K\frac{dq_\lambda}{ds}\right)(s)\right)-1$$

Given that $K^{-1}K\phi=\phi$, We immediately infer that $\psi(s)=\left(K\frac{dq_\lambda}{ds}\right)(s)$ and Equation (55). □

When $\alpha \to 1$, the rough process converges toward a Hawkes process with an exponential kernel. In this case, the dampened RL integral of $\frac{dq_\lambda}{ds}$ converges toward the following integral

$$\lim_{\alpha\to 1}\left(K\frac{dq_\lambda}{ds}\right)(s) \;=\; \int_0^s e^{-\beta\,(s-u)}\frac{dq_\lambda(u)}{du}\,du\,,$$

and from Equation (39), $\frac{dq_\lambda}{ds}$ solves the integro-differential equation:

$$\frac{dq_\lambda(s)}{ds} \;=\; \mathbb{E}\left(e^{-\omega J}\right)\exp\left(\eta\int_0^s e^{-\beta\,(s-u)}\frac{dq_\lambda(u)}{du}\,du\right)-1\,. \tag{41}$$

This formulation is not standard in the literature. For this reason, we show in Appendix C, that $e^{q_\lambda(s)\lambda_0}$ with $\frac{dq_\lambda}{ds}$ satisfying the above equation, does also correspond to the Laplace's transform an exponential Hawkes process.

In practice, we solve numerically Equation (37). We divide $[0,s]$ in n subintervals $[s_j,s_{j+1}]$ of length Δ, for $j=0,...,n-1$. We denote by $g(k):=\left.\frac{dq_\lambda(s)}{ds}\right|_{s=s_k}$, the differential of q_λ at time s_k and we next use an explicit approximation of the dampened RL fractional integral :

$$g(k)=\mathbb{E}\left(e^{-\omega J}\right)\exp\left(\frac{\eta}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\frac{e^{-\beta\,(s_k-s_j)}}{(s_k-s_j)^{\,1-\alpha}}\,g(j)\,\Delta\right)-1\,. \tag{42}$$

The recursion is initialized by setting $g(0)=\mathbb{E}\left(e^{\omega J}\right)-1$. We can exploit previous results to compute the probability density functions of the point process $(L_t)_{t\geq 0}$. Our approach is based on a discrete fast Fourier's transform (DFFT). It consists to invert the characteristic function of the process, that is the Laplace's transform (36) valued on the imaginary axis. Let us denote the characteristic function of L_s by $\Upsilon_s(i\omega)=\mathbb{E}\left(e^{i\,\omega L_s}\,|\,\mathcal{F}_0\right)$ for $\omega\in\mathbb{R}$. This is also the inverse Fourier's transform of the probability density function (pdf) $f_s^L(x)$ of $L_s|\mathcal{F}_0$. Therefore, this density can be retrieved by computing the following integral (the Fourier's transform, $\mathcal{F}[.]$, of $\Upsilon_s(.)$):

$$\begin{aligned} f_s^L(x) &= \frac{1}{2\pi}\mathcal{F}[\Upsilon_s(i\omega)](x) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Upsilon_s(i\omega)e^{-i\omega x}d\omega\,. \end{aligned} \tag{43}$$

This integral is approximated by discretization with the DFFT algorithm recalled in Appendix B.

	Moments of L_T		Quantiles of L_T	
α	$\mathbb{E}(L_T)$	$\sqrt{\mathbb{V}(L_T)}$	5%	95%
0.9	0.144	0.067	0.047	0.267
0.8	0.159	0.075	0.051	0.298
0.7	0.179	0.087	0.055	0.341
0.6	0.209	0.106	0.063	0.408
0.5	0.250	0.134	0.071	0.502
0.4	0.298	0.167	0.078	0.624

Table 1: Expectation, standard deviation, 5% and 95% quantiles of L_T for $\alpha = 0.4$ up to $\alpha = 0.9$.

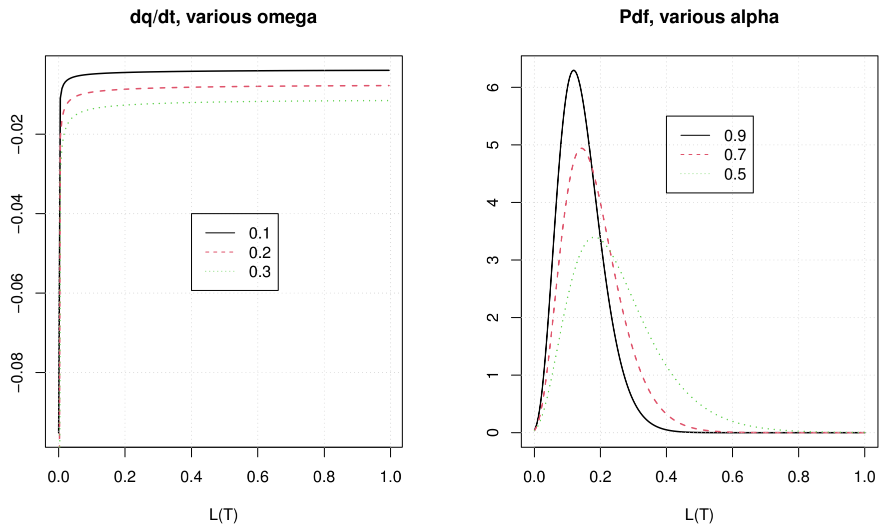

Figure 3: Left plot, $\left.\frac{dq_\lambda(s)}{ds}\right|_{s=s_k}$ for r $\omega \in \{0.1\,,\,0.2\,,\,0.3\}$. Right plot, pdf's of L_1 for $\alpha \in \{0.5\,,\,0.7\,,\,0.9\}$.

The recursion (42) and the inversion by DFFT is illustrated in Figure 3. The left plot shows the derivative $\left.\frac{dq_\lambda(s)}{ds}\right|_{s=s_k}$ for $\omega \in \{0.1\,,\,0.2\,,\,0.3\}$. We consider a dampened rough process with parameters $\alpha = 0.6$, $\eta = 2$, $\beta = 6.35$ and $\lambda_0 = 10$. Jumps are exponential random variables parametrized by $\rho = 100$. In this case, $\mathbb{E}\left(e^{-\omega J}\right) = \frac{\rho}{\rho+\omega}$, for $\omega > -\rho$. We consider $n = 200$ steps of time. The graph reveals a very fast increase of the differential over a short time horizon. After this, the $\left.\frac{dq_\lambda(s)}{ds}\right|_{s=s_k}$'s slowly converge to values close to zero. The right plot of Figure 3 presents the pdf's $f_1^L(x)$ of L_1 for various levels of α (all other parameters remaining unchanged). These distributions are computed with $M = 2^8$ discretization steps of the Fourier's integral (43). This graph emphasizes the impact of α on the distribution shape of L_T. Decreasing α clearly raises the mean and variance of L_T. This is confirmed by table 1 which provides the moments, standard deviations, 5% and 95% quantiles of L_T for $\alpha = 0.4$ up to $\alpha = 0.9$.

5 Simulation

The thinning procedure of Ogata [18] to sample a standard Hawkes processes is based on two conditions. The first one is that between two jumps occurring at times τ_k and τ_{k+1}, the counting process behaves locally like an non-homogeneous Poisson process. This is well the case for the rough Hawkes process that has an intensity

$$\lambda(t) \quad = \quad \lambda_0 + \frac{\eta}{\Gamma(\alpha)} \sum_{j=1}^{k} e^{-\beta(t-\tau_j)} (t-\tau_j)^{\alpha-1} \, t \geq \tau_k \,. \tag{44}$$

The second condition is that $\lambda(t)$ is a decreasing function, bounded by $\lambda^* = \lambda(\tau_k) < \infty$. In this case, the exponential random time, τ with pdf $f_\tau(t) = \lambda^* e^{-\lambda^* t}$ and cdf $F_\tau(t) = 1 - e^{-\lambda^* t}$ is well defined. Because $F_\tau(t)$ and $F_\tau^{-1}(t)$ admit a closed-form expression, we can use the inverse transform technique to sample waiting times. We know that $F_\tau^{-1}(\tau)$ is an uniform random variable $\mathcal{U}_{[0,1]}$. Therefore, sampling a waiting interval τ for a Poisson process is done by:

$$\text{sampling } U \sim \mathcal{U}_{[0,1]} \text{ and setting } s = -\frac{1}{\lambda^*} \ln U \,. \tag{45}$$

On the other hand, the thinning property of Poisson processes states that a Poisson process with an intensity λ can be split into two independent processes with intensities λ_1 and λ_2, so that $\lambda = \lambda_1 + \lambda_2$. A jump is respectively caused by the first or the second processes with probabilities $\frac{\lambda_1}{\lambda}$ and $\frac{\lambda_2}{\lambda}$. From this property, we can see that we can simulate of a non-homogeneous Poisson process with the intensity function $\lambda(t)$ by thinning a homogeneous Poisson process with the intensity $\lambda^* \geq \lambda(t)$ for all $t \geq 0$.

Unfortunately, the intensity (44) is by construction not bounded at the instant of jump. Nevertheless, $\lambda(t)$ is bounded over $[\tau_k + \epsilon\,,\, \infty)$ by $\lambda_\epsilon^* = \lambda(\tau_k + \epsilon) < \infty$ for any infinitesimal $\epsilon > 0$. This allows us to sample the rough Hawkes process with the Algorithm 1.

To summarize, let us consider that we start our time counter at $T = \tau_k$. We sample an inter-arrival time τ, with Equations (45) and $\lambda_\epsilon^* = \lambda(T + \epsilon)$. Next, we update the time counter $T = T + \tau$. We accept or reject this inter-arrival time according to the ratio of the true event rate to the thinning rate λ_ϵ^* (step 5 of the Algorithm). If accepted, we record the $(k+1)$ event time as $\tau_{k+1} = T$. Otherwise, we repeat the sampling of an inter-arrival time until one is accepted. Notice that even if an inter-arrival time is rejected, the time counter T is updated. As similar approach is used by Chen et al. [2] for simulating fractional Hawkes processes whose kernel diverges to ∞ at zero. The illustration in Figure 1 is computed with this algorithm and $\epsilon = 10^{-10}$.

6 Estimation

We estimate the rough Hawkes process by log-likelihood maximization. Given that the intensity reaches $+\infty$ for an extremely brief moment after a jump, we detail the calculation of this log-likelihood. We assume to observe the process over the time interval $[0, \mathcal{T}]$. The k^{th} jump time of L_t is denoted by τ_k for $k = 1, ..., N_{\mathcal{T}}$. From Equation (1), the sample intensity at time τ_{k-} is equal to

$$\lambda_{\tau_{k-}} \quad = \quad \lambda_0 + \frac{\eta}{\Gamma(\alpha)} \sum_{j=1}^{k-1} e^{-\beta(\tau_{k-} - \tau_j)} (\tau_{k-} - \tau_j)^{\alpha-1} \,.$$

Algorithm 1 Sampling algorithm of N jumps of the rough Hawkes process.

Set current time $T=0$, $\epsilon>0$ and jump counter $k=1$

While $k \leq N$

1. Set the upper bound of Poisson intensity $\lambda_\epsilon^* = \lambda(T+\epsilon)$ with Equation (44).
2. Sample $U \sim \mathcal{U}_{[0,1]}$, and set $\tau = -\frac{1}{\lambda_\epsilon^*}\ln U$
3. Update current time: $T = T+\tau$.
4. Sample $R \sim \mathcal{U}_{[0,1]}$.
5. If $R \leq \frac{\lambda(T-)}{\lambda_\epsilon^*}$ then , $\tau_k = T$, $k=k+1$
6. Otherwise reject the sample and return to step 1.

We will see in the next proposition that this realized intensity is involved in calculation of the log-likelihood.

Proposition 8. *We denote the Gamma incomplete function by* $\Gamma(\alpha,x) = \int_x^\infty e^{-u}u^{\alpha-1}du$. *The log-likelihood of a sample of observations over* $[0,\mathcal{T}]$ *is defined as:*

$$\ln\mathcal{L} \quad = \quad -\int_0^{\mathcal{T}} \lambda_s ds + \sum_{k=1}^{N_{\mathcal{T}}} \log\left(\lambda_{\tau_k-}\right) \ , \tag{46}$$

where the integral of the intensity is equal to

$$\int_0^{\mathcal{T}} \lambda_s ds \quad = \quad \lambda_0\mathcal{T} + \frac{\eta}{\beta^\alpha}\sum_{k=1}^{N_{\mathcal{T}}}\left(1-\frac{\Gamma\left(\alpha,\beta\left(\mathcal{T}-\tau_k\right)\right)}{\Gamma(\alpha)}\right) \ . \tag{47}$$

Proof. From e.g. Embrechts et al. [5], the log-likelihood of the sample is given by Equation (46). Using the expression (1) of λ_t and changing the order of integration, we develop the integral of the intensity as follows:

$$\begin{aligned}\int_0^{\mathcal{T}} \lambda_u du \quad &= \quad \lambda_0\mathcal{T} + \eta\int_0^{\mathcal{T}}\int_0^{u-} e^{-\beta(u-s)}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}dN_s\, du \\ &= \quad \lambda_0\mathcal{T} + \eta\int_0^{\mathcal{T}}\int_s^{\mathcal{T}} e^{-\beta(u-s)}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\, du\, dN_s\end{aligned} \tag{48}$$

The inner integral is reformulated in terms of Gamma functions by performing the change of variable $v = \beta\left(u-s\right)$

$$\begin{aligned}&\int_s^{\mathcal{T}} e^{-\beta(u-s)}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\, du \\ &= \int_s^{\infty} e^{-\beta(u-s)}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\, du - \int_{\mathcal{T}}^{\infty} e^{-\beta(u-s)}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\, du \\ &= \frac{\beta^{-\alpha}}{\Gamma(\alpha)}\int_0^\infty e^{-v}v^{\alpha-1}\, dv - \frac{\beta^{-\alpha}}{\Gamma(\alpha)}\int_{\beta(\mathcal{T}-s)}^\infty e^{-v}v^{\alpha-1}\, dv \\ &= \beta^{-\alpha}\left(1-\frac{\Gamma\left(\alpha,\beta\left(\mathcal{T}-s\right)\right)}{\Gamma(\alpha)}\right)\end{aligned} \tag{49}$$

Combining Equations (48) and (49) leads to the expression (47). □

To benchmark our model, we consider a Hawkes process with an exponential memory kernel, such as defined by Equation (16). The log-likelihood in this case, has the same form as Equation (46) with an intensity computed iteratively by

$$\lambda^h_{\tau_{k-}} = \lambda_{h,0}\left(1 - e^{-\beta_h(\tau_k - \tau_{k-1})}\right) + e^{-\beta_h((\tau_k - \tau_{k-1})}\left(\lambda^h_{\tau_{(k-1)-}} + \eta_h\right) , \tag{50}$$

whereas the integral of intensity is equal to

$$\int_0^{\mathcal{T}} \lambda^h_u du = \lambda_{h,0}\mathcal{T} + \frac{\eta_h}{\beta_h}\sum_{k=1}^{N^h_{\mathcal{T}}}\left(1 - e^{-\beta_h(\mathcal{T} - \tau_k)}\right) . \tag{51}$$

We recall that this corresponds to the rough model with $\alpha = 1$. We respectively denote by Θ_N and Θ^h_N, the set of parameters defining the exponential and rough processes. Their estimates, noted $\widehat{\Theta}_N$ or $\widehat{\Theta^h}_N$ are obtained by maximization of log-likelihoods

$$\widehat{\Theta_N} = \arg\max_{\Theta_N} \ln \mathcal{L}(\Theta_N) .$$

The distribution $m(.)$ of jumps is fitted independently of counting processes. If $\{j_1, ..., j_{N_t}\}$ and Θ_J are respectively the sample of jumps and the set of parameters of $m(.)$, estimates are found by log-likelihood maximization:

$$\widehat{\Theta_J} = \arg\max_{\Theta_J} \sum_{k=1}^{N_t} \ln\left(m\left(j_k \,|\, \Theta_J\right)\right) .$$

To conclude this section, we fit the model to time-series of negative jumps in Bitcoin returns. The dataset contains hourly Bitcoin log-returns from the 9/2/2018 to 9/2/2023, traded in USD on the exchange platform Gemini, as illustrated in Figure 4. We adopt a "peak over threshold" approach for detecting jumps. If the Bitcoin log-return falls below a certain threshold, we assume that a jump has occurred. We consider three thresholds: -1.0%, -1.5% and -2.0%. For each of them, we fit an exponential and a rough Hawkes process to jump times.

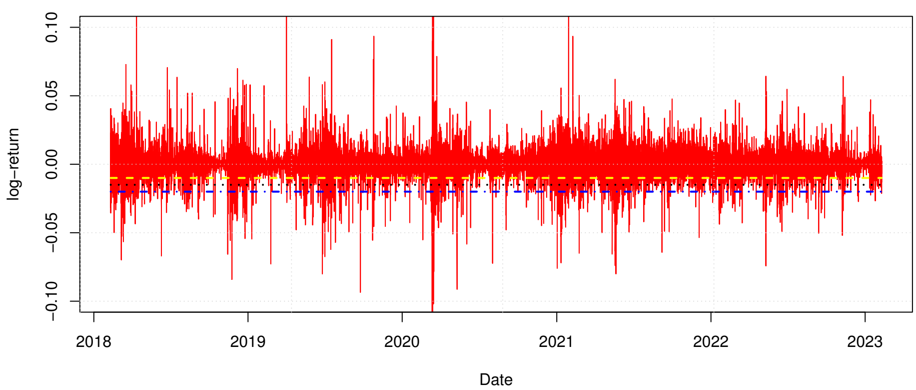

Figure 4: Hourly log-returns of Bitcoin. The three dotted lines are the thresholds -1.0%, -1.5% and -2.0%

Threshold	α	β	η	λ_0	λ_∞	Log-lik.	p-value
-1.0%		145.392	145.391	51.407	518.318 e4	13521.892	
	0.940	166.518	99.039	94.894	496.177	13548.497	0.0000
-1.5%		173.005	128.627	64.284	250.605	6084.924	
	0.846	125.548	45.568	58.997	250.811	6087.705	0.0184
-2.0%		158.551	111.093	40.93	136.741	2962.557	
	0.844	115.773	39.788	38.269	136.765	2964.171	0.0724

Table 2: Parameter estimates, asymptotic intensities, log-likelihoods and p-values of the test with the null hypothesis H_0 ; $\alpha = 1$.

Results of the estimation procedure are provided in Table (2).The first four columns provide parameter estimates. The fifth and sixth columns report the asymptotic intensities and log-likelihoods. In order to check the relevance of the rough Hawkes model, we perform a log-likelihood ratio test. Under the assumption that $\alpha = 1$, the statistic

$$2\left(\ln \mathcal{L}(\widehat{\Theta}_N) - \ln \mathcal{L}(\widehat{\Theta_N^h})\right) \sim \chi_1^2$$

is (asymptotically) a chi-square random variable with one degree of freedom. We conclude from the observation of p-values of this test that the rough Hawkes model better explains jumps than the exponential Hawkes process, for -1% and -1.5% thresholds. For larger thresholds, jumps are too scarce and the rough model has no added value. We explain this by the relatively long average duration between two jumps (i.e. several hours). For such durations, the behaviour of the kernel is mainly ruled by the dampening factor. The rough process behaves then like an exponential Hawkes process in this case. This is confirmed by another test in which we fit the rough and exponential processes to time series of negative shocks in daily log-returns of the Eurostoxx 50. The comparison of log-likelihoods also reveals that the rough model does not outperform the exponential process. This is again explained by the minimum 1 day delay between two successive jumps. Beyond this period of time, the kernel behaves like a decreasing exponential. Therefore, we do not recommend to fit a rough Hawkes process to time series of low frequency events. We also notice that the estimated exponential Hawkes process, for a -1% threshold, is nearly unstable as λ_∞ is abnormally high. This observation is in line with conclusions of Jaisson and Rosenbaum [16] who remark that nearly unstable Hawkes processes often fit high-frequency finance data properly.

7 Conclusions

As detailed in this article, the rough Hawkes process presents several interesting properties and a sufficient level of analytical tractability for many future applications. Even if its kernel diverges at origin, the process remains stable under mild conditions. The expected intensity and number of jumps admit closed form expression. When α tends to one, the process converges to a classical Hawkes process with an exponential kernel. It is also possible to rewrite it as an infinite dimensional process. Considering the limit of a finite approximation allows us to retrieve the Laplace's transform of the rough Hawkes process. This is expressed in terms of a solution of particular kind of fractional differential equation. This equation is solved numerically and the density of the rough Hawkes process is retrieved by DFFT. The rough process can be simulated by a modified Ogata's algorithm. Finally, the log-likelihood of the rough process has an analytical expression which allows us to fit it to time-series.

This article paves the way to further research. We can for instance develop a multivariate extension in order to replicate contagion of shocks between different time-series. An application of this model would be the modelling of the order book of stock prices, at high frequency. Another possible development is to consider jumps of intensity proportional to jumps of L_t.

Appendix A. Mittag Leffler function

The Mittag-Leffler functions with one and two parameters are respectively defined by

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)},$$
$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha > 0$ and $\beta \in \mathbb{C}$. The function $u(x) = E_\alpha(\eta x^\alpha)$ is closely related to fractional calculus when $\alpha \in (0,1)$. We denote by $I^\alpha_{0+}u$ is the following Riemann-Liouville fractional integral

$$\left(I^\alpha_{0+}u\right)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(s)}{(t-s)^{1-\alpha}}ds\,. \tag{52}$$

The left Riemann-Liouville derivative, denoted by $D^\alpha_{0+}u(t)$, is the derivative of $I^{1-\alpha}_{0+}u(t)$:

$$\left(D^\alpha_{0+}u\right)(t) = \frac{d\left(I^{1-\alpha}_{0+}u\right)(t)}{dt} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{u(s)}{(t-s)^\alpha}\,ds\,,$$

and is such that $D^\alpha_{0+}I^\alpha_{0+}u(t) = u(t)$. The solution of the fractional integral/differential equations

$$\left(D^\alpha_{0+}u\right)(t) = \eta\, u(t)$$

is precisely the function $u(x) = E_\alpha(\eta x^\alpha)$. In this article we also use the relation

$$\frac{dE_\alpha(\,\eta x^\alpha)}{dx} = \eta x^{\alpha-1}E_{\alpha,\alpha}(\,\eta x^\alpha)\,. \tag{53}$$

From the Laplace's transform of $E_\alpha(\pm x^\alpha)$,

$$\mathcal{L}\left(E_\alpha(\pm x^\alpha)\right) := \int_0^\infty e^{-zx}E_\alpha(\pm x^\alpha)\,dx = \frac{z^{\alpha-1}}{z^\alpha \mp 1}\,, \tag{54}$$

(see Gorenflo et al. [7], pages 40 and 41), we infer that

$$\mathcal{L}\left(E_\alpha(\pm\eta x^\alpha)\right) = \frac{z^{\alpha-1}}{z^\alpha \mp \eta}\,. \tag{55}$$

Appendix B. Fast Fourier's transform

Let M be the number of steps used in the Discrete Fast Fourier's Transform (DFFT) and $\Delta_x = \frac{x_{max}}{M}$, be a step of discretization. Let us denote $\Delta_\omega = \frac{2\pi}{M\,\Delta_x}$ and $\omega_j = -\frac{M}{2}\Delta_\omega + (j-1)\Delta_\omega$ for $j = 1, ..., M+1$. The values of $f_s^L(.)$ at points $x_k = (k-1)\Delta_x$ for $k = 1, ..., M$ are approached by

$$
\begin{aligned}
f_s^L(x_k) \quad &= \quad \Delta_\omega \sum_{j=1}^{M+1} \delta_j \Upsilon_s\left(i\omega_j\right) \exp\left(i\left((k-1)\pi\right)\right) \\
&\quad \times \exp\left(-i(k-1)(j-1)\frac{2\pi}{M}\right) ,
\end{aligned}
\tag{56}
$$

where$\delta_j = \left(\frac{1}{2}\right)^{1_{\{j_1=1\}}} + 1_{\{j\neq 1\}}$.

Appendix C. Laplace's transform of an exponential Hawkes process

We rewrite the intensity of a Hawkes process with an exponential kernel, such as described by Equation (16) as the following sum $\lambda_t^h = \lambda_{h,0} + Z_t$, where $Z_t = \eta_h \int_0^{t-} e^{-\beta_h(t-s)} dN_s$. Z_t is solution of the SDE:

$$
dZ_t \quad = \quad -\beta_h\, Z_t\, dt + \eta_h\, dN_t\,.
$$

Proposition 9. *The Laplace's transform of the point process $L_t^h = \sum_{k=0}^{N_t^h} J_k$, conditionally to $\mathcal{F}_0$, for $\omega \in \mathbb{R}^+$, is equal to*

$$
\mathbb{E}\left(e^{-\omega L_s^h} \,|\, \mathcal{F}_0\right) \quad = \quad \exp\left(q_h(s)\lambda_0\right)\,, \tag{57}
$$

where $q_h(s)$ solves a forward ODE:

$$
\frac{dq_h(s)}{ds} \quad = \quad \mathbb{E}\left(e^{-\omega J}\right) \exp\left(\eta_h \left(\int_0^s e^{-\,\beta_h\,(s-u)} \frac{dq_h(u)}{du}\, du\right)\right) - 1\,. \tag{58}
$$

Proof. The function $f\left(t, Z_t, L_t^h\right) = \mathbb{E}\left(e^{-\omega L_s^h} | \mathcal{F}_t\right)$ solves the partial differential equation:

$$
0 = \partial_t f(.) - \beta_h Z_t \partial_\lambda f(.) + (\lambda_0 + Z_t) \int_0^\infty f\left(t, Z_t + \eta_h\,, L_t^h + z\right) - f(.)\, m(dz)\,.
$$

We can prove that $f(.)$ is an exponential affine function that is

$$
f(.) = \exp\left(q_h(t,s)\lambda_{h,0} + q_z(t,s) Z_t - \omega L_t^h\right)\,. \tag{59}
$$

where q_h and q_z solve the backward ODE's:

$$
\begin{cases}
\partial_t q_h(t,s) & = -\left(e^{q_z(t,s)\eta_h}\mathbb{E}\left(e^{-\omega J}\right) - 1\right)\,, \\
\partial_t q_z(t,s) & = \beta_h q_z(t,s) - \left(e^{q_z(t,s)\eta_h}\mathbb{E}\left(e^{-\omega J}\right) - 1\right)\,.
\end{cases}
$$

When $t=0$, $f(.)=\exp\left(q_h(0,s)\lambda_{h,0}\right)$ and q_0, q_h solve the forward ODE's:

$$\begin{cases}\partial_s q_h(0,s) & =\left(e^{q_z(0,s)\eta_h}\mathbb{E}\left(e^{-\omega J}\right)-1\right)\,,\\ \partial_s q_z(0,s) & =-\beta_h q_z(0,s)+\left(e^{q_z(0,s)\eta_h}\mathbb{E}\left(e^{-\omega J}\right)-1\right)\,.\end{cases} \tag{60}$$

If we insert the expression of $\partial_s q_h(0,s)$ in the second equation, we have

$$\partial_s q_z(0,s) \;=\; -\beta_h q_z(0,s)+\partial_s q_h(0,s)\,,$$

which admits the solution:

$$q_z(0,s)=\int_0^s e^{-\beta(s-u)}\partial_u q_h(0,u)du\,.$$

Combining this with the first Equation of (60) leads to the result (58). □

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