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Asymmetric Models for Realized Covariances

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Abstract

We introduce asymmetric effects in the BEKK-type conditional autoregressive Wishart model for realized covariance matrices. The asymmetry terms are specified either by interacting the lagged realized covariances with the signs of the lagged *daily* returns or by using the decomposition of the lagged realized covariance matrix into positive, negative, and mixed semi-covariances, thus relying on the lagged *intra-daily* returns and their signs. We provide a detailed comparison of models with different complexity, for example with respect to restrictions on the parameter matrices. In an extensive empirical study, our results suggest that the asymmetric models outperform the symmetric one in terms of statistical and economic criteria. The asymmetric models using the signs of the daily returns tend to have a better in-sample fit and out-of-sample predictive ability than the models using the signed intra-daily returns.

Keywords: High frequency data; asymmetric volatility; realized covariance; conditional autoregressive Wishart model.

1 Introduction

Forecasts of the covariance matrix of asset returns are a central input to asset pricing, portfolio allocation, and risk management decisions. Such forecasts can be computed using a multivariate generalized autoregressive conditional heteroskedasticity (MGARCH) model (see Bauwens et al. (2006) for a survey) that specifies the unobserved covariance matrix as a function of past (usually daily) returns. Forecasts can also be based on models for realized covariance (RC) matrices, which are 'observable' measures of variances and covariances based on high-frequency (intraday) returns; see, e.g., Andersen et al. (2003), Barndorff-Nielsen and Shephard (2004), Barndorff-Nielsen et al. (2008, 2011). Several types of RC matrix (RCM) models have been introduced in the literature, each facing the need to ensure the positive definite (PD)-ness of the RCM forecasts and to avoid parameter proliferation. In particular, the conditional autoregressive Wishart (CAW) class of models specifies a probability distribution for the RCM such that its conditional expectation is a parametric function of past RC matrices (Gouriéroux et al. (2009)). The parameterization of this function is broadly similar to that of MGARCH models, in particular the BEKK (Baba-Engle-Kent-Kroner)-type (Engle and Kroner (1995), Golosnoy et al. (2012)) and DCC (dynamic conditional correlation)-type (Engle (2002), Bauwens et al. (2012)) specifications.[1]

RCM and MGARCH models are designed to capture the main properties of the time series of covariance matrices of asset returns, corresponding to the clustering and persistence of the volatilities of financial returns. Another stylized fact, specific to stock returns, is the negative correlation between returns and volatilities, initially expounded by Black (1976) and developed by Christie (1982). Based on the Modigliani-Miller framework, these authors explain that an unexpected stock price drop raises the debt-to-equity ratio, i.e., leverage, of a firm, which implies increased riskiness and higher volatility. The alternative interpretation, commonly referred to as the volatility feedback effect, was proposed by French et al. (1987);

[1]Most other RCM models are multivariate generalizations of the heterogenous autoregressive (HAR) model of Corsi (2009), such as the vech-HAR model derived from Chiriac and Voev (2011) and the HAR-DRD model of Oh and Patton (2016).

see also Campbell and Hentschel (1992) and Wu (2001). It is based on the evidence of a positive correlation between future volatility and market risk premium, i.e., in the occurrence of an expected volatility increase, the risk premium increases as well, such that the risk adverse investors sell the stock, putting downward pressure on its price. Whatever interpretation is preferred, it implies that volatility increases more strongly after a negative unexpected return than after a positive one of the same magnitude, what has been named 'asymmetry' in volatility, or (with some misuse of language) 'leverage effect'. In this regard, while this type of asymmetry has been diversely incorporated in the specification of GARCH models, such as the widely used GJR-GARCH model of Glosten et al. (1993), this is less the case for RCM models.[2]

Our contribution consists in developing a class of models for RC matrices based on the BEKK-type CAW model of Golosnoy et al. (2012) to capture the asymmetric responses of the elements of the RCM to shocks, and empirically assessing and comparing these models.

We introduce the 'asymmetry' effect in RC models in two ways. The first one consists of adding terms to the benchmark symmetric specification, which are active if the daily returns at $t-1$ are negative. This is designed in such a way that the conditional variance of an asset at date t is higher if the daily return of the asset is negative at date $t-1$ than if it is positive. Likewise, the covariance at t between two assets is higher if the returns are both negative at date $t-1$ than if at least one of them is positive. The conditional threshold autoregressive Wishart model (CTAW) of Anatolyev and Kobotaev (2018) is of this type. We propose a more flexible model in Section 2.1. The signs of returns to specify an 'asymmetry' effect have also been used (with some noticeable differences) in MGARCH (Kroner and Ng (1998), De Goeij and Marquering (2004), Cappiello et al. (2006), Audrino and Trojani (2011), Francq and Zakoïan (2012), Hafner and Herwartz (2023)), HAR (Qu and Zhang (2022)), and HEAVY models (Bauwens and Xu (2023)).

[2]Univariate realized variance models that include an asymmetric effect have been developed by Corsi and Renò (2012), McAleer and Medeiros (2008), and Patton and Sheppard (2015).

The second way to introduce the 'asymmetry' effect is based on the estimates of realized variances and covariances via the signs of high-frequency returns, i.e., measures known as realized semi-variances and semi-covariances proposed by Bollerslev et al. (2020a) and subsequently used by Bollerslev et al. (2020b) to develop asymmetric MGARCH and realized GARCH models. These authors show that the RCM can be decomposed into the sum of three terms: the positive semi-covariance term, the negative one, and, relevant only for covariances, the mixed one. Instead of assuming that the conditional variance (i.e., the conditional mean of the RCM) of an asset at date t depends on the realized variance of the same asset at date $t-1$, we assume that it depends additively on the realized positive semi-variance (at $t-1$), with a specific coefficient, and on the realized negative semi-variance, with another coefficient. If the latter coefficient is larger than the former, an 'asymmetry' effect is present, corresponding to the leverage effect described previously, whereas if the coefficients are equal, there is no asymmetric effect. A similar specification can be used for a conditional covariance by assuming that it depends linearly on the three semi-covariances (positive, negative, and mixed) with different coefficients instead of the realized covariance (when the three coefficients are equal).[3] The detailed definitions and specifications are provided in Section 2.2.

To perform empirical evaluations of the models, we have built time series of daily returns and RC matrices based on a high-frequency dataset for five stocks (of the banking sector) and an exchange traded fund (ETF) tracking the S&P500 market index. Statistical evaluation criteria consist of the in-sample fit and out-of-sample forecast loss functions (mean squared error and quasi-likelihood). In order to formally determine whether the quality of the forecasts differs significantly across the models, we apply the model confidence set (MCS) procedure of Hansen et al. (2011), which allows us to identify the subset of models that contains the best forecasting models given a pre-specified level of confidence. We also

[3]In Bollerslev et al. (2020b), this specification is used for the conditional mean of the outer product of the daily return vector, as in a traditional MGARCH and a realized GARCH model, instead of being used for the conditional mean of the RC matrix, as we do.

compare the model performances from a portfolio allocation perspective, using as loss functions the standard deviations of the global minimum variance portfolio (GMVP) and of the mean-variance portfolio (MVP).

Both the in-sample and forecasting results essentially underscore that the asymmetric models outperform the symmetric benchmark specification. Such results underline the importance of accounting for the asymmetries in modelling, estimating, and forecasting RC matrices. Furthermore, the results (perhaps surprisingly) indicate that the simple GJR-type asymmetric specification based on daily returns is superior to both the intra-daily asymmetric extensions and more complex models to account for the 'asymmetry' effect in RC matrices.

The rest of the paper is organized as follows. Section 2 introduces the asymmetric extensions of the benchmark symmetric BEKK-CAW model of Golosnoy et al. (2012). Section 3 explains the estimation method. Section 4 provides information on the data used to obtain the empirical results presented in Section 5. Section 6 concludes.

2 Introducing 'asymmetry' effects in the BEKK model

Let us consider the daily RCM C_t, defined as the sum of m outer-products of the intraday return vectors over the day t (Barndorff-Nielsen and Shephard (2004)), i.e.,

$$C_t = \sum_{j=1}^{m} r_{j,t} r'_{j,t}, \tag{1}$$

where $r_{j,t} = (r_{j,t,1}, r_{j,t,2}, \ldots, r_{j,t,n})'$ is a $n \times 1$ vector of returns for the j-th time interval of day t.

To capture the temporal and contemporaneous dependencies of the elements in C_t, the BEKK parameterization adopted by Golosnoy et al. (2012) and inspired by the BEKK-MGARCH model of Engle and Kroner (1995), is used for the conditional covariance matrix S_t, defined as the conditional expectation $\mathbf{E}(C_t|\mathcal{F}_{t-1})$ based on the filtration $\mathcal{F}_{t-1} =$

$\{C_{t-1}, C_{t-2}, ...\}$. The BEKK equation, with one lag of S_t and of C_t, is

$$S_t = CC' + AC_{t-1}A' + BS_{t-1}B', \tag{2}$$

where C is a $n \times n$ full rank lower-triangular matrix and A and B are $n \times n$ parameter matrices, for a total of $n(n+1)/2 + 2n^2$ parameters to be estimated. Given a positive-semidefinite (PSD) S_0, the symmetry and PD-ness of S_t are automatically guaranteed. Following Engle and Kroner (1995), sufficient conditions to identify the benchmark BEKK-CAW model are positive diagonal elements of the matrix C, and that the first diagonal elements of A and of B are positive.

The number of parameters is reduced by imposing that the matrices A and B be diagonal, which leads to a total of $n(n+1)/2+2n$ parameters. In this case, each conditional covariance only depends on the corresponding lagged realized and conditional covariances. Finally, the scalar version, i.e., $A = aI_n$ and $B = bI_n$, requires the estimation of only $n(n+1)/2 + 2$ parameters.

2.1 'Asymmetry' terms based on the signs of daily returns

To introduce 'asymmetry' terms in the BEKK-CAW model (2) that we consider as benchmark, we decompose a RCM additively into several parts based on the signs of daily returns. Accordingly, we augment the filtration to include the information about daily returns, i.e. $\mathcal{F}_{t-1} = \{C_{t-1}, r_{t-1}, C_{t-2}, r_{t-2}, ...\}$

2.1.1 Decomposition of a RCM based on the signs of daily returns

We denote by $r_{t,i}$ the daily return of asset i on day t, by $I_t^- = [1_{\{r_{t,1} \leq 0\}}, \ldots, 1_{\{r_{t,n} \leq 0\}}]'$ the indicator vector of the negative daily returns, and by $I_t^+ = [1_{\{r_{t,1} > 0\}}, \ldots, 1_{\{r_{t,n} > 0\}}]'$ the indicator vector of the positive daily returns. The decomposition of C_t as defined in (1) into positive ($C_{P,t}$), negative ($C_{N,t}$), and mixed ($C_{M,t}$) parts based on the signs of daily returns is then

$$
\begin{aligned}
C_t &= C_{P,t} + C_{N,t} + C_{M,t}, \text{ where} \\
C_{P,t} &= C_t \odot I_t^+ I_t^{+'}, \ C_{N,t} = C_t \odot I_t^- I_t^{-'}, \\
C_{M,t} &= C_t \odot (I_t^+ I_t^{-'} + I_t^- I_t^{+'}),
\end{aligned} \tag{3}
$$

with $\odot$ denoting the Hadamard (element-wise) product of matrices. The decomposition holds because the elements of the matrix $I_t^+ I_t^{+'} + I_t^- I_t^{-'} + I_t^+ I_t^{-'} + I_t^- I_t^{+'}$ are all equal to 1. The positive and negative parts, i.e., $C_{P,t}$ and $C_{N,t}$, are PSD; the qualifier positive (negative) is a shortcut for 'positively (negatively) signed'. So, it does not indicate that the off-diagonal elements of $C_{P,t}$ ($C_{N,t}$) are positive (negative). The diagonal elements of $I_t^+ I_t^{-'}$ and $I_t^- I_t^{+'}$, and therefore of $C_{M,t}$, are always equal to zero, i.e., the mixed part $C_{M,t}$ is necessarily indefinite. We denote by $c_{\bullet,ij,t}$ the (i,j)-th entry of $C_{\bullet,t}$ (where $\bullet$ stands for P, N, or M).

2.1.2 Models using the decomposition based on the signs of daily returns

To include in the symmetric RCM model an 'asymmetry' effect based on the signs of daily returns, we add to (2) a term that uses the negative component $C_{N,t}$. The conditional mean of the RCM dynamic equation is then

$$
S_t = CC' + AC_{t-1}A' + \tilde{A}_N C_{N,t-1} \tilde{A}_N' + BS_{t-1}B', \tag{4}
$$

where $\tilde{A}_N$ is a $n \times n$ parameter matrix. This specification is the same as the conditional threshold autoregressive Wishart (CTAW) model of Anatolyev and Kobotaev (2018) and in the context of MGARCH models, it corresponds to the multivariate version of the GJR univariate GARCH model of Glosten et al. (1993). Using the decomposition of C_t in (3), we parameterize the previous equation equivalently as

$$
S_t = CC' + A_P(C_{P,t-1} + C_{M,t-1})A_P' + A_N C_{N,t-1} A_N' + BS_{t-1}B', \tag{5}
$$

where A_P and A_N are equal to A and $A+\tilde{A}_N$ of (4), respectively. Note that if $A_P = A_N := A$ in (5), this model is equivalent to the benchmark symmetric model (2).

To illustrate this model, we consider two assets ($n = 2$), assuming that $A_P = \text{diag}(a_{P11}, a_{P22})$, $A_N = \text{diag}(a_{N11}, a_{N22})$, and $B = C = 0$, because the corresponding parameters are irrelevant for the asymmetry terms. Then, the dynamic equations of the conditional variances $s_{11,t}$, $s_{22,t}$, and of the conditional covariance $s_{12,t}$ are

$$s_{ii,t} = a_{Pii}^2 c_{P,ii,t-1} + a_{Nii}^2 c_{N,ii,t-1}, \text{ for } i = 1, 2; \tag{6}$$

$$s_{12,t} = a_{P11} a_{P22} (c_{P,12,t-1} + c_{M,12,t-1}) + a_{N11} a_{N22} c_{N,12,t-1}. \tag{7}$$

The leverage effect in the variances corresponds to $a_{Nii}^2 > a_{Pii}^2$ ($i = 1, 2$), since this implies that the conditional variance ($s_{ii,t}$) increases more if the lagged return ($r_{t-1,i}$) is negative than if it is positive, for a given value of the lagged realized variance ($c_{ii,t-1}$), since $c_{P,ii,t-1} = c_{ii,t-1} 1_{\{r_{t-1,i}>0\}}$ and $c_{N,ii,t-1} = c_{ii,t-1} 1_{\{r_{t-1,i}\leq 0\}}$. For the covariance, the leverage effect corresponds to $a_{N11} a_{N22} > a_{P11} a_{P22}$, which is surely true if the effect holds for both variances, but may be true also if it holds only for one variance. Indeed, for a given level of the lagged realized covariance ($c_{12,t-1}$), the conditional covariance ($s_{12,t}$) increases more if the lagged returns ($r_{t-1,1}$ and $r_{t-1,2}$) are negative than if they are positive or of opposite signs.

A more flexible model is obtained by removing the constraint of equal coefficients of $c_{P,12,t-1}$ and $c_{M,12,t-1}$ in (7), resulting in the covariance equation $s_{12,t} = a_{P11} a_{P22} c_{P,12,t-1} + a_{M11} a_{M22} c_{M,12,t-1} + a_{N11} a_{N22} c_{N,12,t-1}$. The corresponding more flexible version of (5) is

$$S_t = CC' + A_P C_{P,t-1} A'_P + A_N C_{N,t-1} A'_N + A_M C_{M,t-1} A'_M + B S_{t-1} B', \tag{8}$$

where $A_M = (a_{Mij})$ is a $n \times n$ parameter matrix. The scalar version of this formulation is proposed by Bollerslev et al. (2020b) in the context of realized GARCH models. If $A_P = A_N = A_M := A$ in (8), this model is equivalent to the benchmark symmetric model (2).

Since $C_{M,t} = C_t \odot (I_t^+ I_t^{-'} + I_t^- I_t^{+'})$, one can split the term $A_M C_{M,t-1} A_M'$ into two terms with different parameter matrices, A_M^+ and A_M^-, creating a more flexible model. Extending the scalar version of Bollerslev et al. (2020b), this is done as follows:

$$S_t = CC' + A_P C_{P,t-1} A_P' + A_N' C_{N,t-1} A_N' + A_M^+ C_{M,t-1}^+ A_M^{+'} + A_M^- C_{M,t-1}^- A_M^{-'} + B S_{t-1} B', \quad (9)$$

where $C_{M,t}^+ = C_t \odot \tau(I_t^+ I_t^{-'})$, $C_{M,t}^- = C_t \odot \tau(I_t^- I_t^{+'})$, and the operator $\tau(\cdot)$ sets the lower triangular part of the matrix argument equal to the upper triangular part.[4] The $\tau(\cdot)$ operator is needed to obtain the symmetry of the $C_{M,t}^+$ and $C_{M,t}^-$ matrices.

We refer to (5) as the tr-BEKK-CAW model ('tr' for 'threshold'), to (8) as the trPNM-BEKK-CAW model, and to (9) as the trPNτM-BEKK-CAW, omitting BEKK-CAW when it is clear that we refer to this class of models. Diagonal and scalar versions are obtained by restricting the parameter matrices in the same way as for the model (2).

2.2 'Asymmetry' terms based on the signs of intra-daily returns

To enable refined intraday asymmetric RC dynamics (Bollerslev et al. (2020b)), we exploit the semi-covariance decomposition of the RCM into several components.

2.2.1 Semi-covariance decomposition of a RCM

Bollerslev et al. (2020a) provide a decomposition of C_t, as defined in (1), into positive (P_t), negative (N_t), and two mixed (M_t^+, M_t^-) realized semi-covariance matrices defined by using the signs of the underlying intraday returns, extending the idea of Barndorff-Nielsen et al.

[4]Following He and Teräsvirta (2002), $\tau(M) = \text{ivech}(\text{vech}(M'))$, where vech$(\cdot)$ stacks the lower triangle of a $n \times n$ matrix into a $n(n+1)/2 \times 1$ vector and ivech$(\cdot)$ is its inverse, thus generating a symmetric matrix. For example, for two assets, if $I_t^+ = (1\,0)'$ and $I_t^- = (0\,1)'$ (the first return is positive, the second is negative), $I_t^+ I_t^{-'} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\tau(I_t{}^+ I_t^{-'}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(2010) to the multivariate setting. The semi-covariance matrices are defined as

$$\begin{aligned} P_t &= \sum_{j=1}^{m} r_{j,t}^{+} r_{j,t}^{+'}; \; N_t = \sum_{j=1}^{m} r_{j,t}^{-} r_{j,t}^{-'}; \\ M_t^{+} &= \sum_{j=1}^{m} r_{j,t}^{+} r_{j,t}^{-'}; \; M_t^{-} = \sum_{j=1}^{m} r_{j,t}^{-} r_{j,t}^{+'}, \end{aligned} \tag{10}$$

where $r_{j,t}^{+} = r_{j,t} \odot \mathrm{I}_{j,t}^{+}$ and $r_{j,t}^{-} = r_{j,t} \odot \mathrm{I}_{j,t}^{-}$ denote the vectors of positive and negative intra-daily returns, with $I_{j,t}^{+} = [1_{\{r_{j,t,1}>0\}}, \ldots, 1_{\{r_{j,t,n}>0\}}]'$ and $I_{j,t}^{-} = [1_{\{r_{j,t,1}\leq 0\}}, \ldots, 1_{\{r_{j,t,n}\leq 0\}}]'$ denoting the corresponding indicator vectors of the signs of intraday returns.

The positive and negative semi-covariance matrices, i.e., P_t and N_t, are PSD; the qualifiers positive and negative do not imply that the off-diagonal elements of these matrices are positive or negative. The mixed components M_t^{+} and M_t^{-} have zero diagonal elements and, thus, are indefinite, with off-diagonal elements that are necessarily negative. Obviously,

$$C_t = P_t + N_t + M_t, \text{ where } M_t = M_t^{+} + M_t^{-}, \tag{11}$$

given that $M_t^{+} = M_t^{-'}$.

The corresponding positive, negative, and mixed terms of the decompositions (3) and (11) generally differ. They may be equal under specific conditions that are unlikely to hold in practice. For example, if $r_{j,t} > 0 \; \forall j$, i.e., the intraday returns (of all stocks) are positive during day t, and the daily return vector r_t used to define $C_{P,t}$ is chosen to be the return over the trading period, i.e., $\sum_{j=1}^{m} r_{j,t}$, then $P_t = C_{P,t} = C_t$. Conversely, if the daily return r_t used to extract the positive daily return indicator vector I_t^{+} is the close-to-close return and thus differs from $\sum_{j=1}^{m} r_{j,t}$ (unless the overnight return is equal to zero), then P_t may differ from $C_{P,t}$. A simple example is developed in Appendix A.

2.2.2 Models using the semi-covariance decomposition

We exploit the realized semi-covariance decomposition of a RCM to define a semi-BEKK-CAW model for S_t, the expectation of C_t conditional on the augmented filtration $\mathcal{F}_{t-1} = \{P_{t-\tau}, N_{t-\tau}, M^+_{t-\tau}, M^-_{t-\tau}, \tau \geq 1\}$,

$$S_t = CC' + A_P P_{t-1} A'_P + A_N N_{t-1} A'_N + A_M M_{t-1} A'_M + B S_{t-1} B', \tag{12}$$

where A_P, A_N, and A_M are $n \times n$ parameter matrices, while P_t, N_t, and M_t denote the positive, negative, and mixed semi-covariance matrices, respectively. If $A_P = A_N = A_M :=$ A, the semi-CAW model is equivalent to the benchmark symmetric model (2).

To illustrate the terms of this model, we write its equations of the bivariate version, eliminating the constant and $BS_{t-1}B'$ terms, and assuming that A_P, A_N, and A_M are lower triangular (LT), i.e., $a_{P12} = a_{N12} = a_{M12} = 0$. The triangularity assumption could be relevant if the first asset is a market index and the second one is a particular stock, so that the market may have an impact on the stock but no impact of the stock on the market is allowed. The corresponding equations are

$$\begin{aligned}
s_{11,t} &= a^2_{P11} p_{11,t-1} + a^2_{N11} n_{11,t-1}, \\
s_{22,t} &= a^2_{P22} p_{22,t-1} + a^2_{N22} n_{22,t-1} \\
&\quad + a^2_{P21} p_{11,t-1} + a^2_{N21} n_{11,t-1} \\
&\quad + 2a_{P21} a_{P22} p_{12,t-1} + 2a_{N21} a_{N22} n_{12,t-1} + 2a_{M21} a_{M22} m_{12,t-1}, \\
s_{12,t} &= a_{P11} a_{P22} p_{12,t-1} + a_{M11} a_{M22} m_{12,t-1} + a_{N11} a_{N22} n_{12,t-1} \\
&\quad + a_{P11} a_{P21} p_{11,t-1} + a_{N11} a_{N21} n_{11,t-1}.
\end{aligned} \tag{13}$$

In each equation, the first line corresponds to the diagonal model (i.e., when both off-diagonal elements of A_P, A_N, and A_M are set to zero). It is clear that in the diagonal model, each conditional variance only depends on the corresponding lagged positive and negative semi-variances. A larger coefficient a_{N11}^2 (a_{N22}^2) of the negative semi-variance than of the positive one a_{P11}^2 (a_{P22}^2) for the first (second) asset corresponds to the 'leverage' effect. In the LT model, several other terms appear in the particular asset conditional variance, i.e., two terms that correspond to the effect of the market positive and negative semi-variances, with the possibility of a cross-leverage effect (if $a_{N21}^2 > a_{P21}^2$), and three terms that correspond to the impacts of the three semi-covariances, with coefficients that can be of any sign.

The conditional covariance equation has three terms that capture the effect of the lagged positive, negative, and mixed semi-covariances ($p_{12,t-1}$, $n_{12,t-1}$, $m_{12,t-1}$). It is possible that the coefficient ($a_{N11}a_{N22} > 0$) of the negative semi-covariance ($n_{12,t-1}$) be larger than that ($a_{P11}a_{P22} > 0$) of the positive one ($p_{12,t-1}$), which can be interpreted as a 'leverage' effect on the conditional covariance. The coefficient $a_{M11}a_{M22}$ of the mixed semi-covariance can be of any sign, since a_{M11} and a_{M22} do not appear squared in the variance equations. The two additional terms of the LT version (in the third line of the covariance equation) correspond to cross-effects of the market semi-variances, with a possibility of a 'cross-leverage' effect if $a_{N11}a_{N21} > a_{P11}a_{P21}$; these coefficients can be of any sign.

Like for the trPNM-BEKK-CAW model (8), the 'semi' model can be made more flexible by splitting the mixed semi-covariance matrix M_t into its two components M_t^+ and M_t^- (the positive and negative mixed semi-covariance matrices, respectively, such that $M_t = M_t^+ + M_t^-$) and applying the τ transformation to each component (Bollerslev et al. (2020a)), i.e.,

$$\begin{aligned} S_t = CC' + BS_{t-1}B' + A_P P_{t-1} A_P' + A_N N_{t-1} A_N' \\ + A_M^+ \tau(M_{t-1}^+) A_M^{+\prime} + A_M^- \tau(M_{t-1}^-) A_M^{-\prime}, \end{aligned} \tag{14}$$

where A_M^+ and A_M^- are $n \times n$ are parameter matrices. We refer to this specification as the semi-τ-BEKK-CAW model. We further discuss the use of the mixed semi-covariance decomposition in Appendix B.

3 Estimation

The estimation of the parameters of the models presented in Section 2 is carried out by maximizing a log-likelihood function (LLF). As in Golosnoy et al. (2012), the latter is based on the assumption that the probability density function of the RC matrices C_t, conditional on the appropriate filtration $\mathcal{F}_{t-1}$, is Wishart, i.e.,

$$C_t|\mathcal{F}_{t-1} \sim W_n(v, S_t(\theta)/v), \tag{15}$$

where $W_n(v, S_t(\theta)/v)$ denotes the n-dimensional central Wishart distribution with v degrees of freedom, with $v \geq n$, and PD $n \times n$ scale matrix $S_t(\theta)/v$, implying $\mathbf{E}(C_t|\mathcal{F}_{t-1}) = S_t(\theta)$; θ is the vector of parameters appearing in the equation defining S_t. For example, for equation (2), θ consists of the elements of C, A, and B. The LLF for T observations is

$$LLF(\theta|C_1, \ldots, C_T) = -\frac{\nu}{2}\sum_{t=1}^{T}\left\{\log|S_t(\theta)| + \text{trace}\left[S_t(\theta)^{-1}C_t\right]\right\}. \tag{16}$$

Bauwens et al. (2012) show that the parameter v can be treated as nuisance parameter, meaning that it can be fixed to an arbitrary value (in practice, 1) to estimate θ. They also show that the Wishart-based LLF provides a quasi-maximum likelihood (QML) estimator for the parameters θ, under suitable conditions, so that the QML estimator is consistent.

The maximization of the LLF is typically difficult due to the dimension of θ, denoted by d_θ, which is of order n^2. For example, in the case of (14), $d_\theta = n(n+1)/2 + 5n^2 (= 201$ if $n = 6)$; in the scalar version of the same model $d_\theta = n(n+1)/2 + 5 (= 26)$ and in the diagonal version, $d_\theta = n(n+1)/2 + 5n (= 51)$. To get rid of the $n(n+1)/2$ parameters of the C matrix in the maximization of the LLF, it is possible to estimate CC' consistently in a first step. In a second step, the remaining parameters are estimated by QML, conditional on the first step estimates. This procedure reduces the number of parameters by $n(n+1)/2$ in the second step. The estimation of C in the first step is called 'covariance targeting'. It is based on writing the constant term (CC') of each dynamic equation for S_t as a function of $\mathbf{E}(C_t)$, and replacing the latter by the sample mean of the C_t matrices. The covariance targeting parameterizations of the models are defined in Appendix C. When targeting is used, it is understood that θ in (16) does not include the elements of C.

4 Data construction and description

For the subsequent empirical analyses, we have constructed the time series of daily RC matrices with the corresponding decompositions (3) and (11) into positive, negative, and mixed matrices, based on a high-frequency data set for the SPDR S&P500 (SPY or Spyder) – an exchange traded fund that tracks the S&P500 index – and five stocks of the banking sector, i.e., Bank of America Corp. (BAC), Citigroup Inc. (C), Goldman Sachs Group Inc. (GS), JPMorgan Chase & Co. (JPM), and Wells Fargo & Co. (WFC).[5]

To avoid the measurement drawbacks due to microstructure effects when sampling returns at very high frequencies, we compute each daily realized (semi-)covariance matrix as the sum of the outer products of the five-minute log-return vectors of the trading period of the day. Given the high liquidity of all the stocks, the effect of non-synchronicity is rather negligible at the chosen frequency; the synchronization was done globally for all the stocks, using the closest prior price. The sample period is January 3, 2012 - December 31, 2021, resulting in 2517 observations.

Table 1 reports, for each asset, the time series means and standard deviations of the realized variances (annualized in percentage, i.e., multiplied by 252 and by 100), and of their 'positive' and 'negative' components used in the two broad classes of asymmetric models. The same statistics for the squared close-to-close and open-to-close log-returns of each asset are also shown in Table 1.

[5]The data provider is the AlgoSeek company (30 Wall Street, New York, NY, 10005, USA). The data provided to us by Algoseek are the prices of the assets, observed every minute during the trading period (9:30-16), compiled from the trades that occurred in sixteen US exchanges and marketplaces.

Table 1: Time series means and standard deviations (between parentheses) of realized variances, their positive and negative decompositions, and squared daily returns

Asset	SPY	BAC	C	GS	JPM	WFC
r^2_{cc}	2.63 (11.98)	9.87 (32.75)	10.44 (40.77)	7.71 (27.07)	7.02 (28.64)	7.92 (30.00)
r^2_{oc}	1.31 (3.75)	5.51 (13.07)	5.47 (13.66)	4.45 (10.60)	3.54 (7.91)	4.23 (11.74)
RV	4.88 (22.65)	5.45 (9.70)	5.78 (14.12)	4.62 (8.35)	3.98 (8.84)	4.63 (11.08)
P	2.41 (11.08)	2.77 (5.36)	2.89 (7.30)	2.36 (4.55)	2.04 (4.81)	2.34 (6.14)
N	2.47 (11.62)	2.68 (4.74)	2.89 (7.30)	2.26 (4.16)	1.95 (4.28)	2.29 (5.41)
C_P-cc	2.15 (9.93)	2.70 (7.18)	2.68 (7.89)	2.29 (6.31)	1.93 (6.32)	2.28 (8.66)
C_N-cc	2.73 (20.65)	2.75 (7.57)	3.10 (12.39)	2.33 (6.37)	2.05 (6.79)	2.35 (7.66)
C_P-oc	2.42 (12.58)	2.74 (7.86)	2.73 (8.74)	2.37 (6.62)	2.00 (6.88)	2.35 (9.02)
C_N-oc	2.45 (19.15)	2.71 (6.86)	3.04 (11.81)	2.24 (6.05)	1.98 (6.23)	2.28 (7.22)

r^2_{cc}: squared close-to-close daily return; r^2_{oc}: squared open-to-close daily return; RV: realized variance; P: positive semi-variance; N: negative semi-variance; C_P: RV if daily return is positive, 0 if negative; C_N: RV if daily return is negative, 0 if positive; the suffixes *-cc* and *-oc* indicate that the different terms of the decomposition (3) are based on the signed daily *close-to-close* and *open-to-close* returns, respectively.

Regarding the statistics reported in Table 1, several comments are worth making:

1. In each row, the time series means are similar between the six assets, except for SPY squared returns. There is more heterogeneity in the standard deviations; in particular, due to more extreme values, those for SPY are larger than for the banks.

2. The average positive semi-variance (P) of each asset is a bit larger than the average positive component (C_P-cc), and the average N is smaller than the corresponding C_N-cc (since $P + N = C_P + C_N = C$, as M and C_M have zero diagonal elements). When the decomposition of the realized variances is based on the open-to-close returns, the averages are closer.

3. C_P-cc is smaller than C_P-oc (by 11 percent for SPY, and about 3 for the other assets), hence C_N-cc is larger than C_N-oc.

4. The average standard deviations of C_P-cc exceed those of P (except for SPY), and the average standard deviations of C_N-cc exceed those of N. Comparing the decompositions based on close-to-close and open-to-close returns, we find that standard deviations are larger, on average, for C_P-oc than for C_P-cc, while the inverse relation holds for the negative part: standard deviations are larger, on average, for C_N-cc than for C_N-oc.

5. For the banking stocks, each average realized variance (covering the open-to-close trading period), is only a fraction of the corresponding average squared close-to-close returns (between 55 and 60%), but it is much closer to the average squared open-to-close returns. Also visible are the larger standard deviations of the time series of squared close-to-close returns compared to those of the realized variances. For SPY, the average realized variance is larger than the average squared returns, and the same relation holds for the standard deviations.

Figures 1-4 show the time series of the realized variances of SPY and JPM, and the components of their decompositions defined by (3) using close-to-close returns, and (11). They illustrate the occurrence of a few extreme values, either isolated (e.g., for JPM, in 2015 and 2018) or clustered (mainly in March 2020, corresponding to the first COVID period in the USA). More generally, the profiles of the series on the two figures relative to the same asset illustrate that the two decompositions differ. This is more visible on Figure 5 that shows a zoom of the realized volatility of SPY and its decompositions during the year 2020, with the very high volatility period starting around the middle of February. One can see that in the decomposition using the signed daily returns (right graphs), the realized volatility (in red on each graph) of each day is fully attributed either to the positively signed component (in blue, top right graph) or to the negatively signed one (in black, bottom right graph). In the decomposition into semi-variances, each realized volatility (in red) is split into a part attributed to the positive semi-variance (in blue, top left graph), and the other to the negative semi-variance (in black, bottom left graph).

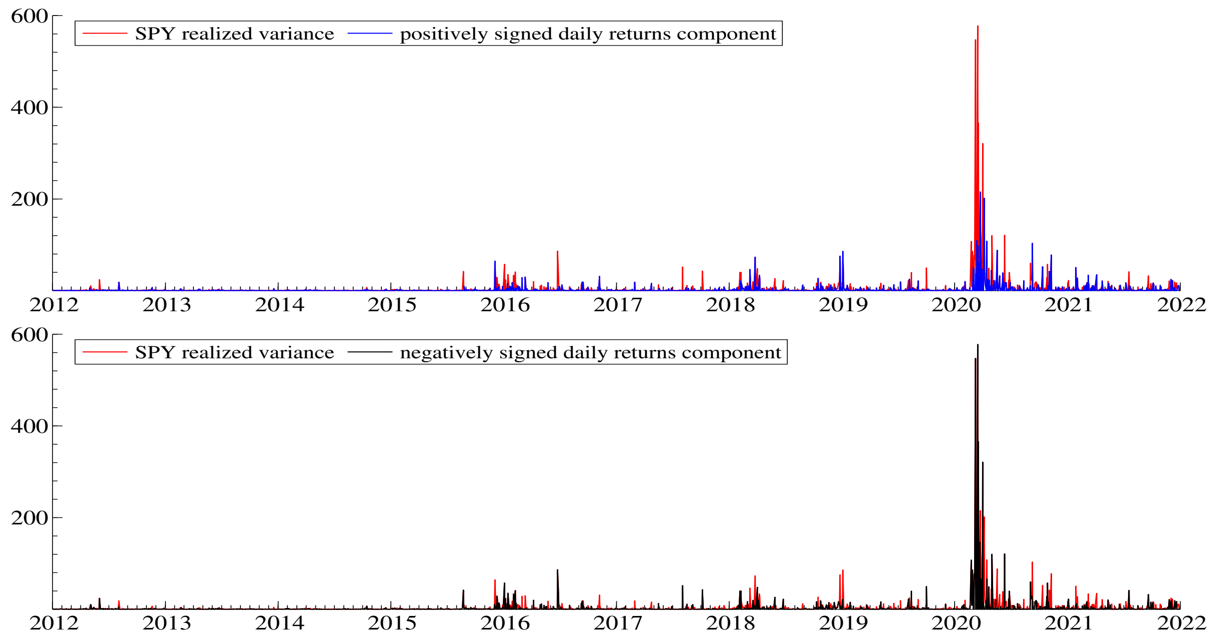

Figure 1: Annualized realized variances of SPY and the terms of their decomposition (3) using the signed daily close-to-close returns

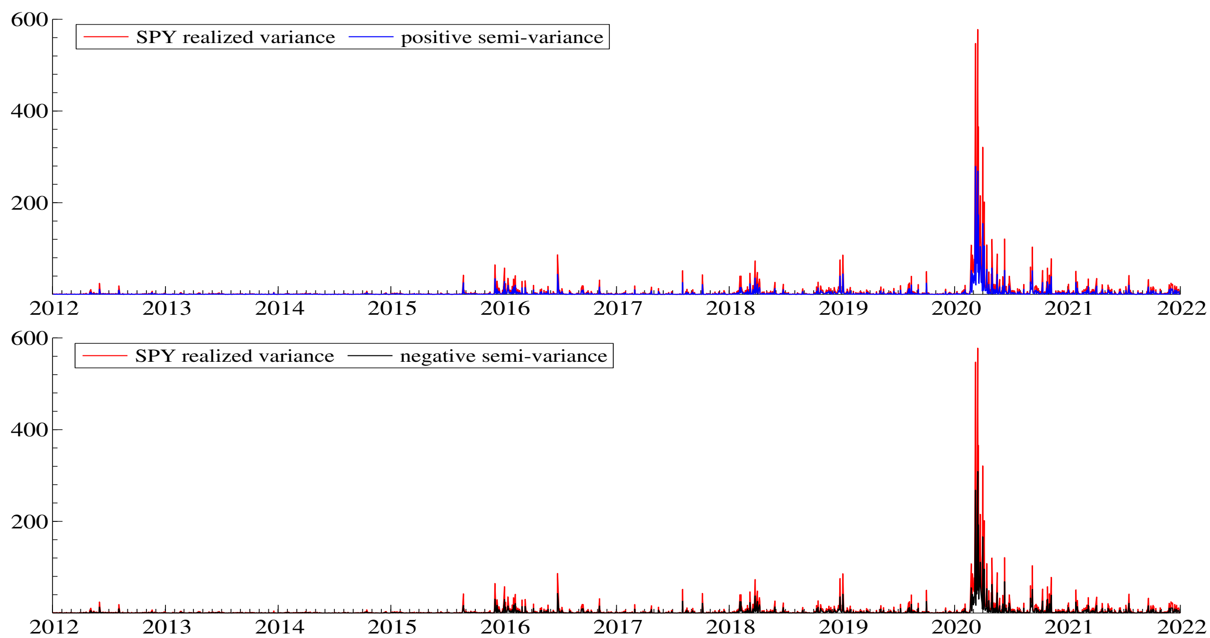

Figure 2: Annualized realized variances of SPY and the terms of their decomposition (11)

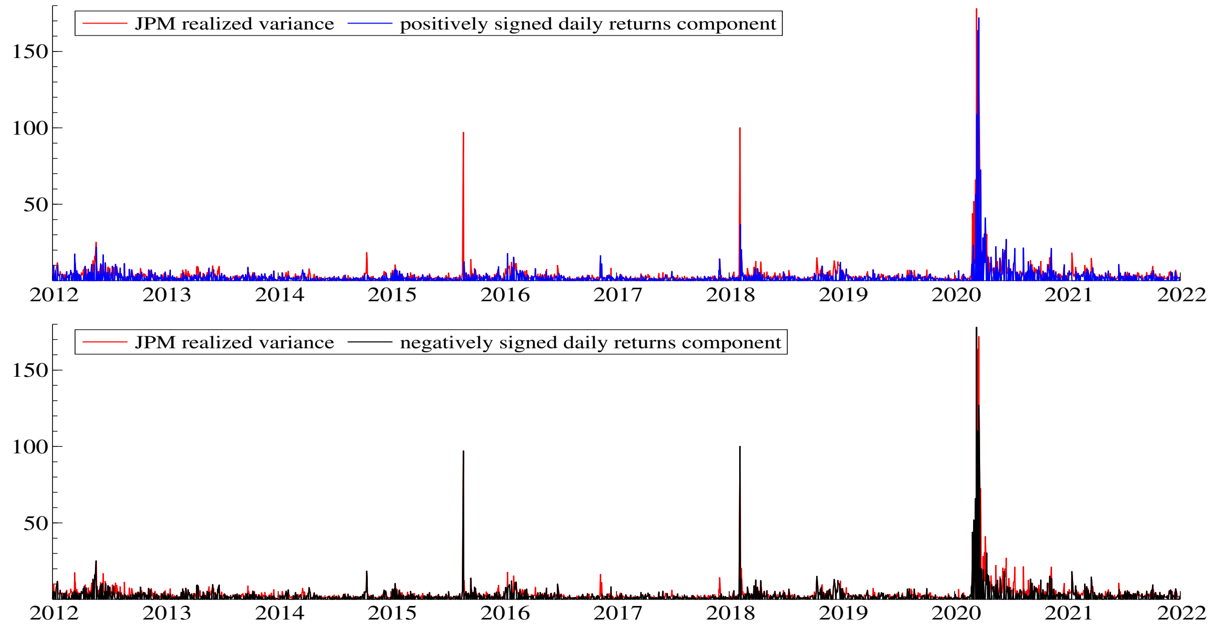

Figure 3: Annualized realized variances of JPM and the terms of their decomposition (3) using the signed daily close-to-close returns

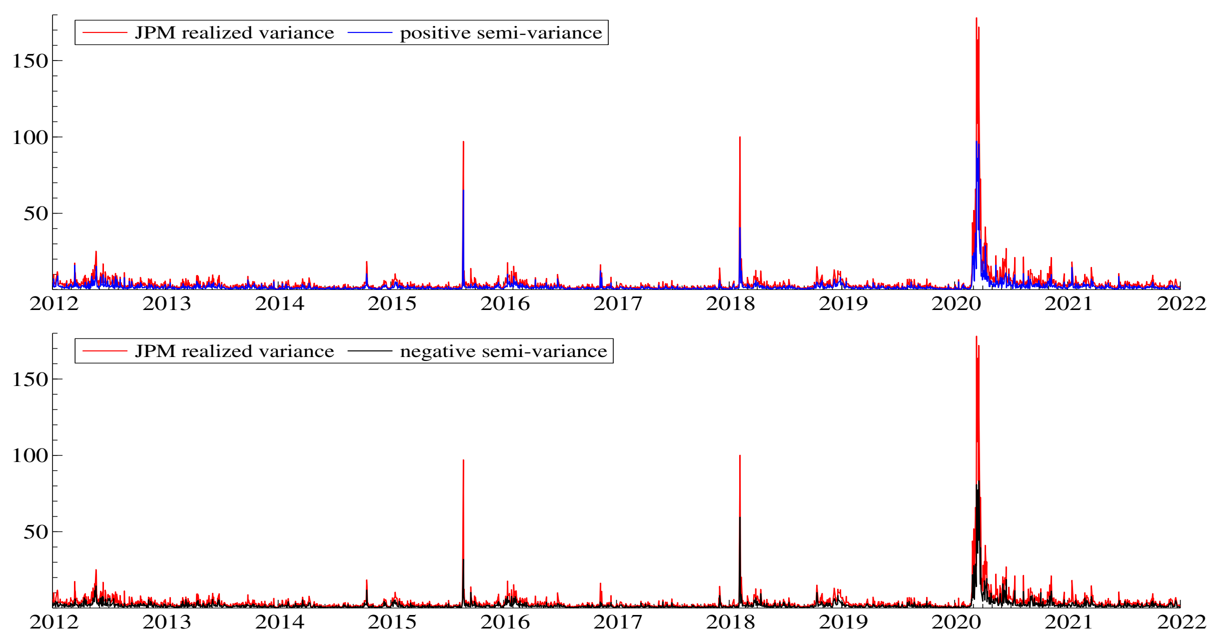

Figure 4: Annualized realized variances of JPM and the terms of their decomposition (11)

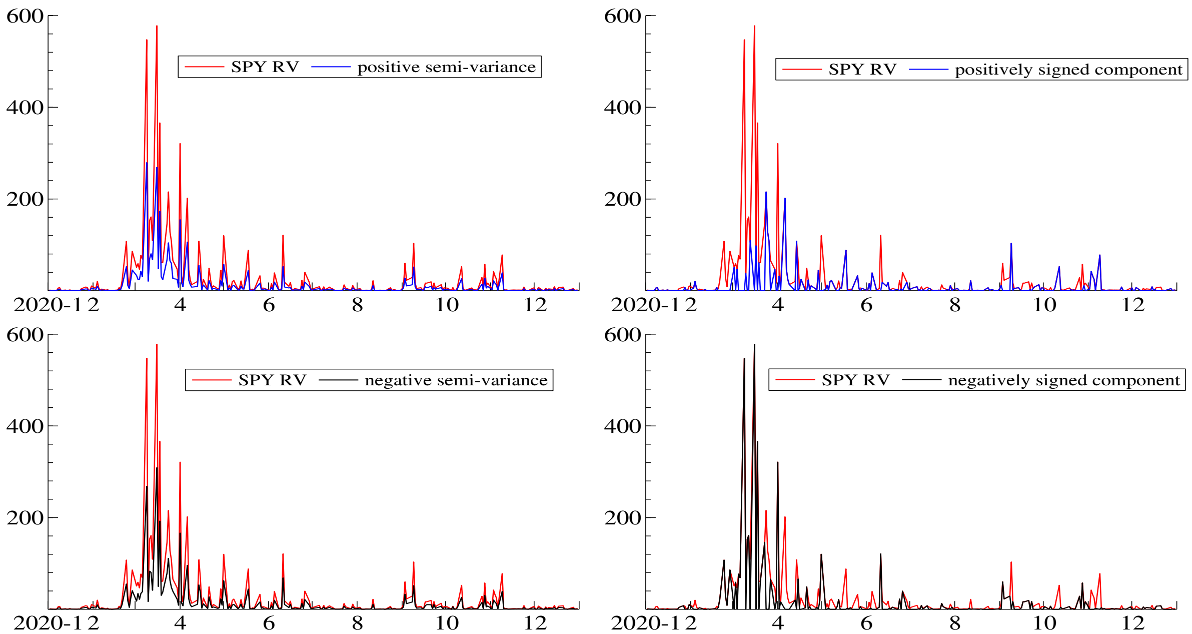

Figure 5: Annualized realized variances of SPY and their decompositions: zoom on the year 2020

Appendix E provides detailed statistics for the realized covariances and the corresponding decompositions, and graphical illustrations. On average, realized covariances are positive and tend to be larger among the five banks than with the Spyder. The difference between the two decompositions is quite obvious for realized covariances: while the off-diagonal elements of M are negative by construction, those of C_M are unrestricted, and their sample averages are positive. This corresponds to P and N being larger than their counterparts, C_P and C_N, respectively.

5 Empirical application

We apply the models defined in Section 2 to the data presented in Section 4. In the following, we first present the estimation results in Section 5.1, then provide forecast comparisons using statistical loss functions in Section 5.2, and economic loss functions in Section 5.3.

5.1 Estimation results

Each BEKK-CAW model is estimated in three versions, namely, scalar, diagonal, and partly lower triangular, the latter being explained below. The models are estimated for six assets on the dataset of 2517 observations described in Section 4, for the period from January 3, 2012, to December 31, 2021. The constant terms are estimated by covariance targeting and the remaining parameters by maximizing the Wishart quasi-likelihood function.

The covariance targeting formulas of the models presented in their general version in Section 2 are given in Appendix C. In the scalar versions, the parameter matrices (except the constant terms) are restricted to be scalar multiple of the identity matrix; see Appendix D for the covariance targeting formulas and the PD-ness conditions. In the diagonal versions, the parameter matrices (except the constant terms) are restricted to be diagonal, so that (like in the scalar versions), there is no impact of the SPY asset on the five banking stocks. To include such effects, we modify each diagonal parameter matrix by adding five parameters in the first column. For example, the matrix A_P in (12) is parameterized as

$$A_P = \begin{pmatrix} a_{P11} & 0 & 0 & 0 & 0 & 0 \\ a_{P21} & a_{P22} & 0 & 0 & 0 & 0 \\ a_{P31} & 0 & a_{P33} & 0 & 0 & 0 \\ a_{P41} & 0 & 0 & a_{P44} & 0 & 0 \\ a_{P51} & 0 & 0 & 0 & a_{P55} & 0 \\ a_{P61} & 0 & 0 & 0 & 0 & a_{P66} \end{pmatrix}, \tag{17}$$

and thus appears as partly lower triangular (PLT). The same extensions are introduced in the matrices A, A_N, A_M, A_M^+, and A_M^-, but not B, which remains diagonal. The variance equation of asset 1 (SPY) is then as in the first equation of (13), and the variance equation of each other asset is as in the second equation of (13). The covariance equations between asset 1 and the other assets are as in the third equation of (13); each of these five covariance equations includes two terms that correspond to the impact of the semi-variances of asset 1, due to the introduction of the parameters $a_{\bullet,j1}$. The ten covariance equations between the assets i and j, $i, j \in \{2, \ldots, 6\}$, do not include these two terms and are of the form

$$s_{ij,t} = a_{Pii} a_{Pjj} p_{ij,t-1} + a_{Mii} a_{Mjj} m_{ij,t-1} + a_{Nii} a_{Njj} n_{ij,t-1}. \tag{18}$$

For the other (than **semi**) models, the variables that multiply the coefficients of the variance and covariance equations are adapted according to the specification of each model presented in Section 2.

The full sets of estimates for each model are reported in Tables 15-17 of Appendix F. In these tables, the estimates of the **tr**, **trPNM**, and **trPN**τ**M** models are obtained with the data based on the decomposition (3) of C_t that uses the *close-to-close* returns. The estimates of the same models with the data based on the decomposition that uses the *open-to-close* returns are reported in Tables 18-20 of Appendix G; these models are designated by **tr**^{oc}, **trPNM**^{oc}, and **trPN**τ**M**^{oc} in the sequel.

5.1.1 In-sample fit comparisons

Table 2 reports for each model the maximized log-likelihood function (LLF), the Akaike information criterion (AIC), and the Bayesian information criterion (BIC). The LLF values are not always comparable, as we explain below. In view of the information criteria, several conclusions can be drawn:

1. According to AIC, the symmetric model does not fit better than any of the asymmetric models of the same version (scalar/diagonal/PLT). According to BIC, this is also the case for the scalar models, while for the diagonal and PLT models, the higher penalty for model complexity leads to a preference for the symmetric model.

2. Comparing only the asymmetric models, for each version and both information criteria, the best fitting model is among the **trPNM**, **trPNM**^{oc}, **tr** and **tr**^{oc} models.

Considering the three versions altogether, the diagonal **trPNM** is the best ftting one (out of 27 models) for AIC, and the scalar **trPNM**^{oc} for BIC.

3. Using the close-to-close returns, instead of the open-to-close ones, improves the AIC in six comparisons (out of nine), and the BIC in five.

4. The extensions of **trPNM** to **trPN**τ**M**, and of **trPNM**^{oc} to **trPN**τ**M**^{oc} do not improve the information criteria. The same result is observed for the extensions of **semi** to **semi-**τ, except for AIC-PLT.

Table 2: Maximum log-likelihood function (LLF), AIC, and BIC values of estimated BEKK-CAW models

	sym	**tr**	**trPNM**	**trPN**τ**M**	**semi**	**semi-**τ	**tr**^{oc}	**trPNM**^{oc}	**trPN**τ**M**^{oc}
LLF-scalar	-518.91	-510.94	-503.38	-503.16	-511.3	-511.24	-512.19	-501.98	-501.98
LLF-diagonal	-493.04	-481.28	-471.95	-470.39	-489.07	-489.061	-479.57	-475.24	-474.45
LLF-PLT	-491.88	-479.38	-466.46	-462.64	-472.03	-459.48	-478.1	-470.54	-467.52
AIC-scalar	9.949	9.944	9.938	9.939	9.945	9.945	9.945	*9.937*	9.938
AIC-diagonal	9.936	9.932	***9.929***	9.933	9.943	9.948	9.931	9.932	9.936
AIC-PLT	9.940	9.938	*9.937*	9.943	9.941	9.940	9.937	9.940	9.946
BIC-scalar	9.954	9.950	9.948	9.951	9.954	9.957	9.951	***9.946***	9.950
BIC-diagonal	9.964	*9.974*	9.985	10.002	9.998	10.017	9.972	9.987	10.005
BIC-PLT	9.979	10.003	10.027	10.058	10.032	10.056	*10.002*	10.030	10.062

LLF values have been shifted by adding 12,000. The underlined values correspond to the best model of each row. The values in bold correspond to the best model for each criterion. The values in italics correspond to the best model in each row, when comparing only the eight asymmetric models.

Table 3 reports the LR test statistics (assumed to be asymptotically chi-squared) and the associated degrees of freedom of the possible nesting tests for each version of the models. The symmetric model (**sym**) is rejected at the one percent level of significance in favour of each nesting asymmetric model for each version of the latter, except in a single case (**sym** vs **semi**, diagonal versions). Between asymmetric models, the **tr** model is rejected against the **trPNM** one, while **tr**^{oc} is rejected against **trPNM**^{oc} only for the scalar version.

Table 3: Likelihood ratio (LR) statistics and their degrees of freedom (between parentheses), for hypotheses making some models restricted cases of other

model			version		
nested	nesting	H_0	scalar	diagonal	PLT
sym	**tr**	$A_P = A_N$ in (5)	15.94 (1)	23.52 (6)	25.00 (11)
sym	**tr**^{oc}	$A_P = A_N$ in (5)	13.44 (1)	26.94 (6)	27.56 (11)
sym	**trPNM**	$A_P = A_N = A_M$ in (8)	31.06 (2)	42.18 (12)	50.84 (22)
sym	**trPNM**^{oc}	$A_P = A_N = A_M$ in (8)	33.86 (2)	35.6 (12)	42.68 (22)
sym	**semi**	$A_P = A_N = A_M$ in (12)	15.22 (2)	<u>7.94</u> (12)	39.70 (22)
tr	**trPNM**	$A_P = A_M$ in (8)	15.12 (1)	18.66 (6)	25.84 (11)
tr^{oc}	**trPNM**^{oc}	$A_P = A_M$ in (8)	20.42 (1)	<u>8.66</u> (6)	<u>15.12</u> (11)

The p-value of the each LR statistic is below 0.012, except those that are underlined, which are larger than 0.10, assuming a chi-squared distribution, with the degrees of freedom reported between parentheses next to the corresponding statistic.

Table 4 reports the likelihood ratio (LR) statistics for testing the null hypothesis of a simpler version against a more complex one that nests it. The scalar version is rejected against the diagonal and the PLT ones for each of the nine models (the largest p-value is 0.012). A diagonal version is not rejected against the corresponding PLT one, except for the two **semi** models.

Table 4: Likelihood ratio tests of scalar versus diagonal, diagonal versus PLT, and scalar versus PLT

	sym	**tr**	**trPNM**	**trPN**τ**M**	**semi**	**semi-**τ	**tr**^{oc}	**trPNM**^{oc}	**trPN**τ**M**^{oc}
LR s/d	51.74	59.32	62.86	65.54	44.46	44.36	65.24	53.48	55.06
df s/d	10	15	20	25	20	25	15	20	25
p s/d	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
LR s/plt	54.06	63.12	73.84	81.04	78.54	103.52	68.18	62.88	68.92
df s/plt	15	25	35	45	35	45	25	35	45
p s/plt	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01	0.012
LR d/plt	2.32	3.8	10.98	15.5	34.08	59.162	2.94	9.4	13.86
df d/plt	5	10	15	20	15	20	10	15	20
p d/plt	0.80	0.96	0.75	0.27	< 0.01	< 0.01	0.98	0.86	0.84

LR s/d, df s/d, p s/d: respectively, likelihood ratio statistic, degrees of freedom, p-value for testing the scalar against the diagonal model; s/plt (d/plt) is for testing the scalar (diagonal) model against the partly lower triangular (plt) one.

5.1.2 Interpretation of the estimates

The estimates reported in the tables of Appendices F and G are difficult to interpret because the conditional variances and covariances are nonlinear functions of the parameters that are estimated, as illustrated, e.g., by the equations (13) and (18) for the **semi** model. In Tables 5-8, we report the estimates in an interpretable way, considering the symmetric model, the three **tr** models using the close-to-close returns (cc), and the two **semi** models.

In Table 5, we show the estimated coefficients (except the constant terms) that appear in the variance equations of all models. Since the diagonal and PLT versions of the different models have six variance equations with different parameters, we report the estimates of the variance equation for SPY (asset 1) and only the averages of the estimates for the five banking assets (numbered 2 to 6). Indeed, the estimates for the banks are close to each other but rather different from those of SPY. For the latter, the diagonal element of B is higher than for the banking stocks, and correspondingly, the parameters of the A_P and A_N matrices are smaller. These differences reflect the fact that the conditional volatility of SPY is smoother than for the banks, while the bank conditional volatilities are more sensitive to the lagged realized volatility values.

By definition, the symmetric (**sym**) model has the same parameters for the $c_{P,ii,t-1}$ and $c_{N,ii,t-1}$ variables, i.e., its variance equations can be written as (6) where $a^2_{Pii} = a^2_{Nii}$. The estimates reported in Table 5 show that for the asymmetric models, these coefficients differ, with a^2_{Nii} in all cases larger than a^2_{Pii}; this is also true for each banking asset (see the tables in Appendix F). Hence, the leverage effect is numerically present in the variance equations. Significance tests for the scalar, diagonal and PLT **tr** models are reported in Appendix H, Table 21, to illustrate that the null hypothesis $a^2_{Nii} \leq a^2_{Pii}$ is rejected at the 5 (and less in some cases) percent level against the hypothesis $a^2_{Nii} > a^2_{Pii}$, except for asset 5.

In Table 6, we report the average estimated coefficients of the additional terms that appear only in the variance equations of the PLT models of the five banking stocks, due to the introduction of the off-diagonal parameters in the first column of the A matrices, as shown in (17). For the **sym**, **tr**, **trPNM**, and **trPN**τ**M** models, because these off-diagonal elements are often much smaller than the diagonal ones (see the estimates in Appendix F), the coefficients (a^2_{Pi1} and a^2_{Ni1}) measuring the impact of the positive and negative components of the lagged realized variance of SPY on the other asset variances are very close to zero and not significant, as reported for the **tr** model in column 8 of Table 21 of Appendix H.

However, for the three asymmetric models of the tr-type, and especially for the **trPNτM** model, the coefficients (in the last five rows of Table 6) corresponding to the impacts of realized covariance components on the variances are not as close to zero as in the first two rows.

Table 5: Variance equations of scalar, diagonal, and PLT BEKK-CAW models: average coefficients of variables present in all models

		Coeff:	b_{ii}^2	a_{Pii}^2	a_{Nii}^2
Model	Version	Asset	$s_{ii,t-1}$	$c_{P,ii,t-1}$	$c_{N,ii,t-1}$
	scalar	1-6	0.70	0.27	0.27
	diagonal	1	0.80	0.18	0.18
sym	diagonal	2-6	0.61	0.33	0.33
	PLT	1	0.80	0.19	0.19
	PLT	2-6	0.61	0.32	0.32
	scalar	1-6	0.71	0.24	0.28
	diagonal	1	0.80	0.14	0.22
tr	diagonal	2-6	0.63	0.30	0.33
	PLT	1	0.80	0.14	0.23
	PLT	2-6	0.63	0.29	0.33
	scalar	1-6	0.72	0.22	0.29
	diagonal	1	0.79	0.15	0.25
trPNM	diagonal	2-6	0.62	0.28	0.37
	PLT	1	0.77	0.16	0.27
	PLT	2-6	0.63	0.29	0.36
	scalar	1-6	0.72	0.22	0.29
	diagonal	1	0.79	0.15	0.28
trPNτM	diagonal	2-6	0.62	0.27	0.41
	PLT	1	0.78	0.15	0.30
	PLT	2-6	0.65	0.25	0.39
			$s_{ii,t-1}$	$p_{ii,t-1}$	$n_{ii,t-1}$
	scalar	1-6	0.70	0.20	0.35
	diagonal	1	0.78	0.15	0.31
semi	diagonal	2-6	0.62	0.18	0.52
	PLT	1	0.79	0.07	0.36
	PLT	2-6	0.61	0.19	0.48
	scalar	1-6	0.70	0.20	0.35
	diagonal	1	0.78	0.16	0.31
semi-τ	diagonal	2-6	0.61	0.21	0.53
	PLT	1	0.79	0.10	0.34
	PLT	2-6	0.60	0.29	0.42

Each row corresponds to the conditional variance equation of the model identified in the first two columns. For a scalar model, the coefficients are the same for assets 1 to 6. For the diagonal and PLT models, the coefficients are reported for assets 2 to 6 as the means of the coefficients for these 5 assets. The coefficients, as defined in the first row, are computed using the estimates reported in Tables 15 for the scalar models, 16 for the diagonal models, and 17 for the PLT models. Each coefficient multiplies the variable written in row 2 below it for the models of the first four blocks and the variable shown above the 'semi' model for the last two blocks. For PLT, assets 2-6, the additional terms of the variance equations are reported in Table 6.

For the **semi** models, the coefficients measuring the impact of the lagged positive and negative semi-variances of SPY are small but not as much as in the other models, and reveal an asymmetric impact (a more important impact of the negative semi-variance than of the positive one). The coefficients in the last five rows are also not close to zero, though their values are less important than the coefficients reported in Table 5 for these models. In brief, these additional terms are relevant in the **semi** models, but not in the other models.

Table 6: Variance equations of PLT BEKK-CAW models: average coefficients of additional terms for assets 2-6

Coefficient	Variable	**sym**	**tr**	**trPNM**	**trPNτM**	Variable	**semi**	**semi-τ**
a_{Pi1}^2	$c_{P,11,t-1}$	0.00000	0.00051	0.00158	0.00205	$p_{11,t-1}$	0.0243	0.0588
a_{Ni1}^2	$c_{N,11,t-1}$	0.00000	0.00013	0.00011	0.00005	$n_{11,t-1}$	0.00241	0.0113
$2a_{Pi1}a_{Pii}$	$c_{P,1i,t-1}$	-0.00153	-0.0226	-0.0413	-0.0455	$p_{1i,t-1}$	-0.119	-0.254
$2a_{Ni1}a_{Nii}$	$c_{N,1i,t-1,}$	-0.00153	0.0120	0.0111	-0.00522	$n_{1i,t-1}$	0.0653	0.132
$2a_{Mi1}a_{Mii}$	$c_{M,1I,t-1,}$	-0.00153	-0.0215	-0.0596	-	$m_{1i,t-1}$	-0.314	-
$2a_{Mi1}^{+}a_{Mii}^{+}$	$c_{M,1i,t-1}^{+}$	-	-	-	0.170	$\tau(m_{1i,t-1}^{+})$	-	-0.157
$2a_{Mi1}^{-}a_{Mii}^{-}$	$c_{M,1i,t-1}^{-}$	-	-	-	-0.294	$\tau(m_{1i,t-1}^{-})$	-	-0.236

The coefficient values for each model are computed using the estimates reported in Table 17 and the coefficients as defined in the first column, corresponding to the variables in the second column for the models in columns 3 to 6, and the variables in column 7 for the last two models.

In Tables 7 and 8, we report the average estimated coefficients of the covariance equations. For the diagonal and PLT versions, we report separately the averages for the five covariance equations between asset 1 and the five banking stocks, and the averages for the ten equations between the five banking stocks. The conditional covariances between the five banks are less smooth and more reactive to lagged realized covariance terms than the covariances between the banks and SPY. Moreover, we find a leverage (or asymmetric) effect in the covariance equations: $a_{Nii}a_{Njj}$ is always larger than $a_{Pii}a_{Pjj}$ and $a_{Mii}a_{Mjj}$ (except or the latter in the diagonal **tr** model). In Appendix H, Table 22 (column 5) reports statistics that indicate the significance of the differences $a_{Nii}a_{Njj} - a_{Pii}a_{Pjj}$ for the scalar, diagonal and PLT **tr** models. The additional terms in these five covariances of the PLT model (last two columns of Table 8) have a quasi-zero impact, except for the **semi** models, where we notice again an asymmetric effect: the positive part of the lagged SPY variance has a smaller (actually negative) impact than the negative part (which has a positive impact).

Table 7: Covariance equations of scalar, diagonal, and PLT models, part 1

Model	Version	Coeff: Asset	$a_{Pii}a_{Pjj}$ $c_{P,ij,t-1}$	$a_{Nii}a_{Njj}$ $c_{N,ij,t-1}$	$a_{Mii}a_{Mjj}$ $c_{M,ij,t-1}$	$a^+_{Mii}a^+_{Mjj}$ $c^+_{M,ij,t-1}$	$a^-_{Mii}a^-_{Mjj}$ $c^-_{M,ij,t-1}$
	scalar	1-6	0.27	0.27	0.27	-	
	diagonal	1-other	0.25	0.25	0.25	-	
sym	diagonal	2-6	0.33	0.33	0.33	-	
	PLT	1-other	0.20	0.20	0.20	-	
	PLT	2-6	0.32	0.32	0.32	-	
	scalar	1-6	0.24	0.28	0.24	-	
	diagonal	1-other	0.20	0.27	0.47	-	
tr	diagonal	2-6	0.30	0.33	0.62	-	
	PLT	1-other	0.20	0.27	0.15	-	
	PLT	2-6	0.30	0.33	0.26	-	
	scalar	1-6	0.22	0.29	0.25	-	
	diagonal	1-other	0.21	0.30	0.21	-	
trPNM	diagonal	2-6	0.28	0.36	0.31	-	
	PLT	1-other	0.22	0.31	0.21	-	
	PLT	2-6	0.28	0.35	0.31	-	
	scalar	1-6	0.22	0.29	-	0.25	0.25
	diagonal	1-other	0.19	0.33	-	0.24	0.25
trPNτM	diagonal	2-6	0.27	0.41	-	0.35	0.34
	PLT	1-other	0.19	0.34	-	0.29	0.19
	PLT	2-6	0.25	0.39	-	0.32	0.32
			$p_{ij,t-1}$	$n_{ij,t-1}$	$m_{ij,t-1}$	$\tau(m^+_{ij,t-1})$	$\tau(m^-_{ij,t-1})$
	scalar	1-6	0.20	0.35	0.23	-	
	diagonal	1-other	0.17	0.40	0.30	-	
semi	diagonal	2-6	0.18	0.52	0.35	-	
	PLT	1-other	0.12	0.41	0.19	-	
	PLT	2-6	0.19	0.48	0.37	-	
	scalar	1-other	0.20	0.35	-	0.22	0.25
	diagonal	1-other	0.18	0.40	-	0.14	0.13
semi-τ	diagonal	2-6	0.21	0.53	-	0.15	0.15
	PLT	1-other	0.17	0.37	-	0.10	0.11
	PLT	2-6	0.28	0.41	-	0.20	0.18

Each row corresponds to the conditional covariance equation of the model identified in the first two columns. For a scalar model, the coefficients are the same for all covariance equations. For the diagonal and PLT '1-other' models, the reported coefficient values are the means of the coefficients of the 5 covariance equations between assets 1 and 2 to 6. For the diagonal and PLT '2-6' models, the reported coefficient values are the means of the coefficients of the 10 covariance equations between assets 2 to 6. The coefficients, as defined in the first row, are computed using the estimates reported in Tables 15 for the scalar models, 16 for the diagonal models, and 17 for the PLT models. Each coefficient multiplies the variable written in row 2 below it for the models of the first four blocks and the variable shown above the 'semi' model for the last two blocks.

Table 8: Covariance equations of scalar, diagonal, and PLT models, part 2

Model	Version	Coeff: Asset	$b_{ii}b_{jj}$ $s_{ij,t-1}$	$a_{P11}a_{Pi1}$ $c_{P,11,t-1}$	$a_{N11}a_{Ni1}$ $c_{N,11,t-1}$
	scalar	1-6	0.70	-	-
	diagonal	1-other	0.70	-	-
sym	diagonal	2-6	0.61	-	-
	PLT	1-other	0.70	-0.0006	-0.0006
	PLT	2-6	0.61	-	-
	scalar	1-6	0.71	-	-
	diagonal	1-other	0.71	-	-
tr	diagonal	2-6	0.63	-	-
	PLT	1-other	0.71	-0.0077	0.0050
	PLT	2-6	0.62	-	-
	scalar	1-6	0.72	-	-
	diagonal	1-other	0.70	-	-
trPNM	diagonal	2-6	0.62	-	-
	PLT	1-other	0.70	-0.0016	0.0049
	PLT	2-6	0.63	-	-
	scalar	1-6	0.72	-	-
	diagonal	1-other	0.70	-	-
trPNτM	diagonal	2-6	0.62	-	-
	PLT	1-other	0.71	-0.0174	-0.0022
	PLT	2-6	0.65	-	-
	Variable		$m_{ij,t-1}$	$p_{11,t-1}$	$n_{11,t-1}$
	scalar	1-6	0.70	-	-
	diagonal	1-other	0.70	-	-
semi	diagonal	2-6	0.62	-	-
	PLT	1-other	0.69	-0.0383	0.0284
	PLT	2-6	0.61	-	-
	scalar	1-6	0.70	-	-
	diagonal	1-other	0.69	-	-
semi-τ	diagonal	2-6	0.61	-	-
	PLT	1-other	0.69	-0.0746	0.0613
	PLT	2-6	0.60	-	-

See note below Table 7.

5.2 Forecast comparisons using statistical loss functions

To compare the out-of-sample forecasting accuracy of the symmetric and asymmetric BEKK-CAW models, we use the James-Stein loss (James and Stein (1976)), which is usually referred to as the multivariate Quasi-Likelihood (QLIK) loss function, i.e.,

$$QL_t(S_t^{(i)}, C_t) = \ln |S_t^{(i)}| + \text{trace}([S_t^{(i)}]^{-1} C_t), \tag{19}$$

where $S_t^{(i)}$ is a forecast by model i of the observed RC matrix C_t that is used as a proxy for the unobservable covariance matrix Σ_t. We also use the squared Frobenius norm loss, defined as

$$FN_t(S_t^{(i)}, C_t) = \sum_{j=1}^{n} \sum_{k=1}^{n} (s_{jk,t}^{(i)} - c_{jk,t})^2 \tag{20}$$

Both loss functions satisfy the conditions for producing a consistent ranking; see Hansen and Lunde (2006) and Laurent et al. (2013).

To jointly compare the forecasts of a set of models, we apply the model confidence set (MCS) procedure of Hansen et al. (2003, 2011), implemented for each loss function. The procedure does not necessarily select a single best model, allowing for the possibility of equal forecasting ability. Hence, a model is not included in the MCS only if it is significantly inferior to other models.

Starting with a set of candidate models $\mathcal{M}_0$, given a loss function, the loss difference between each pair of models in the set is computed at every time point $t = 1, \ldots, T$, so that for models i and j, we get $d_{t,ij} = L_{t,i} - L_{t,j}$, where $L_{t,\cdot} = L_t(S_t^{(\cdot)}, C_t)$ is either the QLIK loss function in (19), or the FN loss function in (20). At each step of the procedure, the null hypothesis of equal predictive accuracy $H_0 : E[d_{t,ij}] = 0$, is tested for $\forall i > j \in \mathcal{M}$, a subset of models $\mathcal{M} \subset \mathcal{M}_0$, with $\mathcal{M} = \mathcal{M}_0$ at the initial step. If H_0 is rejected at a chosen significance level α, the worst-performing model is removed. This process continues until a set of models remains that includes no model that can be rejected at level α.

We adopt the range statistic of Hansen et al. (2011) to test H_0, i.e.,

$$T_R(\mathcal{M}) = \max_{i,j \in \mathcal{M}} \frac{| \overline{d}_{ij} |}{\sqrt{\widehat{\text{var}}(\overline{d}_{ij})}}, \tag{21}$$

where $\overline{d}_{ij} = \frac{1}{T}\sum_{t=1}^{T} d_{t,ij}$, and $\widehat{\text{var}}(\overline{d}_{ij})$ is obtained by a circular block bootstrap approach (Hansen et al. (2003)), which we implement with 10,000 replications and varying block length to verify the robustness of the results.

To compute the forecasts, we estimate each model five times using a rolling window scheme. Starting with the fitting period from January 3, 2012, to June 30, 2020 ($T = 2137$), we compute 76 one-step-ahead forecasts after the last in-sample date. Then, we re-estimate the parameters using the next window of 2137 observations obtained by removing the first 76 observations of the previous window and adding 76 new observations to it, and forecast again the next 76 observations following the end of the estimation sample. This procedure is continued until the end of the sample, resulting in 380 out-of-sample forecasts.

Table 9: Model confidence sets at 90% level of BEKK-CAW models, with QLIK and FN loss functions

		QLIK		FN	
Version	Model	Loss	MCS	Loss	MCS
Scalar	**sym**	12.518	**0.154**	13.916	0.000
	tr	**12.506**	**1.000**	13.828	0.001
	trPNM	12.522	**0.154**	13.969	0.000
	trPNτ**M**	12.521	**0.154**	14.029	0.000
	tr^{oc}	12.519	**0.154**	13.860	0.000
	trPNM^{oc}	12.522	**0.154**	13.969	0.000
	trPNτ**M**^{oc}	12.515	**0.921**	14.043	0.000
	semi	12.509	**0.921**	14.364	0.000
	semi-τ	12.509	**0.921**	14.412	0.000
Diagonal	**sym**	12.556	**0.154**	13.903	0.000
	tr	12.560	0.009	13.728	0.001
	trPNM	12.546	**0.154**	14.874	0.000
	trPNτ**M**	12.544	**0.154**	14.105	0.000
	tr^{oc}	12.561	0.009	13.761	0.001
	trPNM^{oc}	12.542	**0.154**	14.781	0.000
	trPNτ**M**^{oc}	12.549	**0.154**	13.952	0.000
	semi	12.542	**0.154**	14.448	0.000
	semi-τ	12.558	**0.154**	14.814	0.000
PLT	**sym**	12.557	**0.154**	13.896	0.000
	tr	12.554	**0.154**	**13.703**	**1.000**
	trPNM	12.664	0.009	14.884	0.000
	trPNτ**M**	12.614	0.009	14.884	0.000
	tr^{oc}	12.560	0.009	13.748	0.001
	trPNM^{oc}	12.720	0.009	14.782	0.000
	trPNτ**M**^{oc}	12.624	0.009	13.890	0.000
	semi	12.632	0.009	14.585	0.000
	semi-τ	12.679	0.009	14.720	0.000

Column 3 (5) : average value, over the forecast period, of QLIK (FN) losses defined in (19) and (20); bold values identify the minimum loss over the twenty-seven models.
MCS' : p-values of the tests of the MCS procedure when the starting set consists of the twenty-seven models; Column 5 for QLIK loss function and column 7 for FN; bold values identify the models included in the MCS at the 90% confidence level (i.e., p-values larger than 0.10).

Table 9 reports the main results of the MCS procedures obtained using the QLIK and FN loss functions, when the starting set of each MCS procedure consists of the twenty-seven models. For the QLIK loss (see columns 3 and 4 of Table 9), the MCS (at the 90% confidence level) consists of eighteen models: the nine scalar models, seven diagonal models (only **tr** and $\mathbf{tr}^{oc}$ are out), and two PLT models (**sym** and **tr**). The smallest loss is attained by the scalar **tr** model. In brief, the QLIK criterion does not discriminate clearly between the asymmetric and symmetric models, and between the two categories of asymmetric models (daily and intradaily); it penalizes the PLT models that are heavily parameterized.

The last column of Table 9 reports the model confidence set (at the 90% confidence level) using the FN loss function. The MCS consists only of the PLT-**tr** model, which has of course the smallest loss. Clearly, the FN criterion penalizes the symmetric model and the **semi** models, and favours a parsimonious asymmetric daily model.

A common feature of the results for the statistical loss functions is that the smallest loss value is obtained by a **tr** model (scalar for QLIK, PLT for FN). Moreover, the simple **tr** model has the smallest FN loss within each parametrization category (scalar, diagonal and PLT). Thus, the statistical forecasting criteria clearly favour a simple asymmetric model compared to its more generous competitors. The asymmetric version that relies on the signs of daily close-to-close returns tends to outperform the approach based on semi-covariances.

5.3 Forecast comparisons using economic loss functions

Given that the statistical superiority of a model does not automatically translate into superior investment decisions (Fleming et al. (2003)), we evaluate the out-of-sample forecasting performance of all considered models from an economic perspective. We consider two loss functions: the standard deviation of the out-of-sample global minimum variance portfolio (GMVP) and the standard deviation of the out-of-sample minimum variance portfolio (MVP) with a fixed return.

We start by performing a global minimum variance portfolio (GMVP) optimization, where the investor focuses exclusively on reducing the portfolio volatility and ignores the expected returns. Hence, the optimal portfolio weights are independent of the forecasts of mean returns, which tend to be very noisy, and only depend on the covariance matrix forecasts; see, e.g., Chan et al. (2015). We impose short-selling constraints, as advocated by Jagannathan and Ma (2003), which may be beneficial to reduce the risk of estimated optimal portfolios even when the constraints are not necessary. The resulting constrained GMVP optimization problem can be expressed as:

$$\min_{w_t} w_t' \Sigma_t w_t \quad \text{s.t.} \quad w_t' \mathbf{1} = 1, \quad w_t \geq 0, \tag{22}$$

where Σ_t is the unobservable covariance matrix of returns for time t, $\mathbf{1}$ is a (n x 1) vector of ones, and w_t represents a vector of non-negative portfolio weights, i.e., short-selling is not allowed. In practice, Σ_t is replaced by the forecast $S_t^{(i)}$ of model i. Given that the short-selling restrictions prevent an analytical solution for the optimal weights, numerical optimization is used, for which we rely on the MATLAB Financial Toolbox.

The computed weights are applied to the observed returns of the forecasting period, resulting in 380 portfolio returns. The standard deviation of these returns is computed and serves as GMVP loss function. The best model minimizes the portfolio standard deviation. The results of the MCS procedure are presented in columns 3 and 4 of Table 10: the MCS of the twenty-seven models consists only of the scalar **tr** model, i.e. it is the most parsimonious asymmetric model, which has the smallest loss. The symmetric and **semi** models are excluded from the MCS. Again, within each category (scalar, diaonal, PLT), the **tr** model takes the smallest loss value. This preference for the simple asymmetric **tr** model confirms the results using statistical loss functions.

The second loss function is based on the classical mean-variance portfolio (MVP) optimization, which adds to the GMVP optimization problem a constraint that the targeted portfolio return is larger than a pre-set threshold that we set at 3.5% per year. Once the optimal weights are computed, the procedure is the same as for the GMVP loss function. Table 10 provides the results in columns 5 and 6. With this loss criterion, the MCS consists of the PLT **semi** model and seven scalar models: **sym**, **tr**, the three *oc* models, and the two **semi** models. The smallest loss is attained by the scalar **trPN**τ**M**^{oc} model, though the losses of the scalar **tr** and PLT **semi** models are only slightly larger.

Table 10: Model confidence sets at 90% level of BEKK-CAW models, with GMVP and MVP loss functions

		GMVP		MVP	
Version	Model	Loss	MCS	Loss	MCS
Scalar	**sym**	1.536	0.036	1.705	**0.147**
	tr	**1.534**	**1.000**	1.704	**0.956**
	trPNM	1.535	0.036	1.707	0.027
	trPNτ**M**	1.535	0.036	1.707	0.053
	tr^{oc}	1.535	0.036	1.705	**0.750**
	trPNM^{oc}	1.537	0.036	1.705	**0.767**
	trPNτ**M**^{oc}	1.537	0.036	**1.704**	**1.000**
	semi	1.537	0.002	1.706	**0.109**
	semi-τ	1.537	0.002	1.706	**0.109**
Diagonal	**sym**	1.553	0.000	1.718	0.000
	tr	1.547	0.000	1.717	0.000
	trPNM	1.551	0.000	1.720	0.000
	trPNτ**M**	1.556	0.000	1.720	0.000
	tr^{oc}	1.549	0.000	1.717	0.000
	trPNM^{oc}	1.554	0.000	1.720	0.000
	trPNτ**M**^{oc}	1.562	0.000	1.719	0.000
	semi	1.558	0.000	1.719	0.000
	semi-τ	1.559	0.000	1.721	0.000
PLT	**sym**	1.554	0.000	1.719	0.000
	tr	1.548	0.000	1.719	0.000
	trPNM	1.565	0.000	1.717	0.000
	trPNτ**M**	1.563	0.000	1.714	0.000
	tr^{oc}	1.551	0.000	1.718	0.000
	trPNM^{oc}	1.573	0.000	1.708	0.027
	trPNτ**M**^{oc}	1.570	0.000	1.707	0.027
	semi	1.643	0.000	1.704	**0.956**
	semi-τ	1.560	0.000	1.711	0.027

Column 3 (5) : average value, over the forecast period, of GMVP (MVP) losses defined in Section 5.3; bold values identify the minimum loss over the twenty-seven models.
MCS' : p-values of the tests of the MCS procedure when the starting set consists of the twenty-seven models; Column 5 for GMVP loss function and column 7 for MVP; bold values identify the models included in the MCS at the 90% confidence level (i.e., p-values larger than 0.10).

Both economic criteria favour mainly model simplicity (i.e. scalar models). There is no consensus between the economic criteria regarding the systematic inclusion or exclusion of the symmetric model, and between the systematic inclusion or exclusion of one category of models; the same conclusions hold for the statistical criteria. Nevertheless, it is clear that the asymmetric models can be valuable in forecasting, with the **tr** model included in the

MCS of the four loss functions, and even in two of them as the unique 'best' forecasting model.

6 Conclusions

This paper introduces and compares empirically BEKK-CAW models that account for asymmetric dynamics in realized covariance matrices, based on a high-frequency dataset for the S&P500 ETF and five large US banks. While several RC models have revealed that high-frequency data provides important additional information for modelling and forecasting RC matrices, our study is one of the first to document the importance of capturing distinct responses of the conditional variances and covariances to lagged realized (co)variances decomposed additively into components weighted either by signed daily returns or by the signed intra-daily returns.

We summarize our findings in the following:

1. The proposed asymmetric BEKK-CAW models show better in-sample fit and out-of-sample forecasting performance than the benchmark symmetric model.

2. Forecasts of the asymmetric models based on signed daily close-to-close returns dominate the forecasts of the models built upon intraday returns when statistical loss functions or a global minimum variance portfolio loss are used.

3. Extending the simplest asymmetric model based on signed daily close-to-close returns, i.e. the **tr** model, to allow for different coefficient matrices of the positive and mixed components improves the in-sample fit, but not the forecasts.

4. For the construction of asymmetry terms in the **tr** model, it is preferable to use close-to-close returns rather than open-to-close returns.

5. While the scalar restrictions of the parameter matrices are rejected in-sample against

the diagonal alternatives, they show good forecasting performances, in particular the simple scalar **tr** model.

Our second conclusion differs from that of Bollerslev et al. (2020b), who conclude in favour of the models using the decomposition based on the intra-daily returns. There are several explanations for this difference: i) the data sets are different, in terms of asset number, composition, and sample period; ii) Bollerslev et al. (2020b) model the daily covariance matrix, i.e., the covariance matrix of the daily returns, as a function of the decomposed realized covariance matrix of the trading period, whereas we model the realized covariance matrix of the trading period. For forecasting the volatility of crude oil futures, Sévi (2014) concludes that the univariate HAR model of Corsi (2009) does not perform significantly worse than more sophisticated versions that use the components of realized variance as predictors.

Several research tracks are open: i) to develop asymmetric dynamic conditional correlation (DCC)-type models based on the decompositions of the RC matrix; ii) to add HAR-type dynamics to asymmetric BEKK-CAW models to explicitly account for the possible long memory feature of volatility and see how this impacts the forecasting performance; iii) to use the maximum likelihood estimation of the asymmetric models assuming a Matrix-F conditional distribution instead of a Wishart (e.g., Opschoor et al. (2018), Zhou et al. (2022)).

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Appendices

A Impact of daily return definition on decompositions of C_t

Let $n = 2$, $r_{j,t} = (x, y)'/\sqrt{m}$, $\forall j = 1, 2, \ldots, m$, with $x > 0, y > 0$.

Then, $r^+_{j,t} = (x, y)'/\sqrt{m}$, $r^-_{j,t} = (0, 0)'$, and

$$C_t = \sum_{j=1}^{m} r_{j,t} r'_{j,t} = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}, \; P_t = C_t, \; N_t = M_t = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If the daily return r_t is the open-to-close return $\sum_{j=1}^{m} r_{j,t} = \sqrt{m}(x, y)'$, then $I^+_t = (1, 1)'$, $I^-_t = (0, 0)'$, and

$$C_{P,t} = P_t, \; C_{N,t} = N_t, \; C_{M,t} = M_t.$$

If the daily close-to-close return r_t is different from the open-to-close one, e.g., $r_t = (z, 0)'$ with $z > 0$, then $I^+_t = (1, 0)'$, $I^-_t = (0, 1)'$, and

$$C_{P,t} = \begin{pmatrix} x^2 & 0 \\ 0 & 0 \end{pmatrix} \neq P_t, \; C_{N,t} = \begin{pmatrix} 0 & 0 \\ 0 & y^2 \end{pmatrix} \neq N_t, \; C_{M,t} = \begin{pmatrix} 0 & xy \\ xy & 0 \end{pmatrix} \neq M_t.$$

B Relevance of the decomposition of M_t

The adoption of the semi-covariance decomposition in the modelling framework with or without separating the components of the mixed matrix M_t depends on the application context. In a bivariate model of the volatility of a specific asset and of a market index, separating the effect of the two realized semi-covariance matrices M^+_t and M^-_t might be relevant. The off-diagonal elements of the M^+_t and M^-_t matrices are necessarily negative.

The covariance equation of the model (14) for 2 assets (asset 1 is the market portfolio, asset 2 is a stock), assuming that all the A matrices are lower triangular, e.g.,

$$A_M^- = \begin{pmatrix} a_{M11}^- & 0 \\ a_{M21}^- & a_{M22}^- \end{pmatrix}$$

is

$$\begin{aligned} s_{12,t} = & \, a_{P11}a_{P22}p_{12,t-1} + a_{N11}a_{N22}n_{12,t-1} \\ & + a_{M11}^+ a_{M22}^+ [\tau(M_{t-1}^+)]_{12} + a_{M11}^- a_{M22}^- [\tau(M_{t-1}^-)]_{12} \\ & + a_{P11}a_{P21}p_{11,t-1} + a_{N11}a_{N21}n_{11,t-1}, \end{aligned}$$

where $[\tau(M_t^+)]_{12}$ is the (1,2) element of the matrix $\tau(M_t^+)$.

Given the mixed matrix $\tau(M^-)$ implied by a negative market return and a positive stock return, a negative value of the coefficient $a_{M11}^- a_{M22}^-$ implies that the covariance between the asset and the market increases, which is consistent with the stylized fact that in a 'bear' market period, the covariances tend to increase. The opposite holds for a 'bull' period, i.e., given the mixed matrix $\tau(M^+)$ implied by a positive market return and a negative stock return, a positive $a_{M11}^+ a_{M22}^+$ coefficient implies that the covariance declines.

The above arguments readily extend to a partially lower triangular model of higher dimension, such as estimated in the empirical application (see Section 5.1).

C Covariance targeting parameterizations of BEKK-CAW models

To define their targeting parameterizations, the models presented in Section 2 are transformed in vector form, i.e., for $s_t = \text{vech}(S_t)$, where $\text{vech}(\cdot)$ is the operator that stacks the lower triangular part of a symmetric $n \times n$ matrix argument into a $n(n+1)/2 \times 1$ vector. Matrices appearing in the equations specifying S_t, such as C_t, $C_{i,t}$ for $i = P, N, M, ...$, and P_t, N_t, M_t, ..., are transformed in the same way and the corresponding vectors are denoted by lower-case letters (e.g., c_t, $c_{N,t}$, p_t, ...). Adjustment matrices K_i, of order $n(n+1)/2$, for

$i = P, N, M, ...$, as above, are introduced in the targeting terms to account for the difference between the unconditional levels of c_t and the vectors corresponding to the covariances of (signed) daily/intraday returns. In each adjustment matrix, the matrix $\bar{C} = \sum_{t=1}^{T} C_t/T$ appears.

The parameter matrices of the vectorized models are obtained as $\tilde{M} = L_n(M \otimes M)D_n$, where M is a parameter square matrix of order n (e.g., A, A_N, A_P, ...) of the model in matrix format, and L_n and D_n denote the $n(n+1)/2 \times n^2$ elimination and $n^2 \times n(n+1)/2$ duplication matrices, respectively.[6] Hence $\tilde{M}$ is in each case square and of order $n(n+1)/2$. We refer to Noureldin et al. (2012) for details. Each targeting term below is the product of a matrix of order $n(n+1)/2$ and the vector $\text{vech}(\bar{C})$.

Symmetric model (**sym**):

$$s_t = (\mathrm{I}_{n(n+1)/2} - \tilde{A} - \tilde{B})\bar{c} + \tilde{A}c_{t-1} + \tilde{B}s_{t-1}. \tag{C1}$$

Noureldin et al. (2012) prove that the unconditional mean of c_t exists, corresponding to the condition derived in Engle and Kroner (1995), if the eigenvalues of the matrix $\tilde{A}+\tilde{B}$ are less than one in modulus.

As a result, $E(c_t) = (\mathrm{I}_{n(n+1)/2} - (\tilde{A} + \tilde{B}))^{-1}c$, and c can be estimated by $\bar{c}$.

Threshold model (**tr**):

$$s_t = (\mathrm{I}_{n(n+1)/2} - \tilde{A}^* - \tilde{B})\bar{c} + \tilde{A}_P(c_{P,t-1} + c_{M,t-1}) + \tilde{A}_N c_{N,t-1} + \tilde{B}s_{t-1}, \tag{C2}$$

where $c_{P,t} = \text{vech}(C_t \odot I_t^+ I_t^{+'})$, $c_{N,t} = \text{vech}(C_t \odot I_t^- I_t^{-'})$, and $c_{M,t} = \text{vech}(C_t \odot (I_t^+ I_t^{-'} + I_t^- I_t^{+'}))$; $\tilde{A}^* = \sum_{i=1}^{2} \tilde{A}_i K_i$, with $\tilde{A}_i = L_n(A_i \otimes A_i)D_n$, for $i = P, N$,

$K_P = L_n[(\overline{C}_P)^{1/2}\overline{C}^{-1/2} \otimes (\overline{C}_P)^{1/2}\overline{C}^{-1/2} + (\overline{C}_M)^{1/2}\overline{C}^{-1/2} \otimes (\overline{C}_M)^{1/2}\overline{C}^{-1/2}]D_n$,

with $\overline{C}_P = 1/T \sum_{t=1}^{T} C_t \odot I_t^+ I_t^{+'}$, $\overline{C}_M = 1/T \sum_{t=1}^{T} C_t \odot (I_t^+ I_t^{-'} + I_t^- I_t^{+'})$, and

[6] L_n is defined such that for any $n \times n$ matrix Q, $\text{vech}(Q) = L_n \,\text{vec}(Q)$, and D_n such that for any symmetric matrix R, $\text{vec}(R) = D_n \,\text{vech}(R)$ (see e.g., Lütkepohl (1996)), with $\text{vec}(\cdot)$ denoting the operator that stacks the columns of a $n \times n$ matrix into a $n^2 \times 1$ vector.

$K_N = L_n[(\overline{C}_N)^{1/2}\overline{C}^{-1/2} \otimes (\overline{C}_N)^{1/2}\overline{C}^{-1/2}]D_n$, with $\overline{C}_N = 1/T\sum_{t=1}^{T} C_t \odot I_t^- I_t^{-'}$.

Threshold model with PNM terms (**trPNM**):

$$s_t = (\mathrm{I}_{n(n+1)/2} - \tilde{A}^* - \tilde{B})\overline{c} + \tilde{A}_P c_{P,t-1} + \tilde{A}_N c_{N,t-1} + \tilde{A}_M c_{M,t-1} + \tilde{B}s_{t-1}, \tag{C3}$$

where $c_{P,t}$, $c_{N,t}$, and $c_{M,t}$ are defined under (C2); $\tilde{A}^* = \sum_{i=1}^{3}\tilde{A}_i K_i$, with $\tilde{A}_i = L_n(A_i \otimes A_i)D_n$ and $K_i = L_n[(\overline{C}_i)^{1/2}\overline{C}^{-1/2} \otimes (\overline{C}_i)^{1/2}\overline{C}^{-1/2}]D_n$, with $\overline{C}_i$ as under (C2), for $i = P, N, M$.

Threshold model with PNτ(M) terms (**trPNτM**):

$$s_t = (\mathrm{I}_{n(n+1)/2} - \tilde{A}^* - \tilde{B})\overline{c} + \tilde{A}_P c_{P,t-1} + \tilde{A}_N c_{N,t-1} + \tilde{A}_M^+ c_{M,t-1}^+ + \tilde{A}_M^- c_{M,t-1}^- + \tilde{B}s_{t-1}, \tag{C4}$$

where $c_{M,t}^+ = \mathrm{vech}(C_{M,t}^+)$ with $C_{M,t}^+ = C_t \odot \tau(I_t^+ I_t^{-'})$, and $c_{M,t}^- = \mathrm{vech}(C_{M,t}^-)$ with $C_{M,t}^- = C_t \odot \tau(I_t^- I_t^{+'})$; $\tilde{A}^* = \sum_{i=1}^{2}\tilde{A}_i K_i + \tilde{A}_M^+ K_M^+ + \tilde{A}_M^- K_M^-$, with $\tilde{A}_i = L_n(A_i \otimes A_i)D_n$ and $K_i = L_n[(\overline{C}_i)^{1/2}\overline{C}^{-1/2} \otimes (\overline{C}_i)^{1/2}\overline{C}^{-1/2}]D_n$, for $i = P, N$,
$\tilde{A}_M^+ = L_n(A_M^+ \otimes A_M^+)D_n$, $K_M^+ = L_n[(\overline{C}_M^+)^{1/2}\overline{C}^{-1/2} \otimes (\overline{C}_M^+)^{1/2}\overline{C}^{-1/2}]D_n$,
$\tilde{A}_M^- = L_n(A_M^- \otimes A_M^-)D_n$, $K_M^- = L_n[(\overline{C}_M^-)^{1/2}\overline{C}^{-1/2} \otimes (\overline{C}_M^-)^{1/2}\overline{C}^{-1/2}]D_n$, with
$\overline{C}_M^+ = 1/T\sum_{t=1}^{T} C_{M,t}^+$ and $\overline{C}_M^- = 1/T\sum_{t=1}^{T} C_{M,t}^-$.

Semi-covariance model (**semi**):

$$s_t = (\mathrm{I}_{n(n+1)/2} - \tilde{A}^* - \tilde{B})\overline{c} + \tilde{A}_P p_{t-1} + \tilde{A}_N n_{t-1} + \tilde{A}_M m_{t-1} + \tilde{B}s_{t-1}, \tag{C5}$$

where $p_t = \mathrm{vech}(P_t)$, $n_t = \mathrm{vech}(N_t)$, and $m_t = \mathrm{vech}(M_t)$; $\tilde{A}^* = \sum_{i=1}^{3}\tilde{A}_i K_i$, with $\tilde{A}_i = L_n(A_i \otimes A_i)D_n$ and $K_i = L_n[(\bar{i})^{1/2}\overline{C}^{-1/2} \otimes (\bar{i})^{1/2}\overline{C}^{-1/2}]D_n$, for $i = P, N, M$, with $\overline{P} = 1/T\sum_{t=1}^{T} P_t$, $\overline{N} = 1/T\sum_{t=1}^{T} N_t$, and $\overline{M} = 1/T\sum_{t=1}^{T} M_t$.

Semi-covariance model (**semi-τ**):

$$s_t = (\mathrm{I}_{n(n+1)/2} - \tilde{A}^* - \tilde{B})\overline{c} + \tilde{A}_P p_{t-1} + \tilde{A}_N n_{t-1} + \tilde{A}_M^+ m_{t-1}^+ + \tilde{A}_M^- m_{t-1}^- + \tilde{B}s_{t-1}, \tag{C6}$$

where $m_t^+ = \text{vech}(\tau(M_t^+))$ and $m_t^- = \text{vech}(\tau(M_t^-))$;
$\tilde{A}^* = \sum_{i=1}^2 \tilde{A}_i K_i + \tilde{A}_M^+ K_M^+ + \tilde{A}_M^- K_M^-$, K_i is defined as under (C5), for $i = P, N$,
$K_M^+ = L_n[\left(\overline{M}^+\right)^{1/2}\overline{C}^{-1/2} \otimes \left(\overline{M}^+\right)^{1/2}\overline{C}^{-1/2}]D_n$, with $\overline{M}^+ = 1/T\sum_{t=1}^T \tau(M_t^+)$, and $K_M^- = L_n[\left(\overline{M}^-\right)^{1/2}\overline{C}^{-1/2} \otimes \left(\overline{M}^-\right)^{1/2}\overline{C}^{-1/2}]D_n$, with $\overline{M}^- = 1/T\sum_{t=1}^T \tau(M_t^-)$.

D Scalar BEKK-CAW models with covariance targeting

With scalar parameter matrices, it is convenient to write the equations using the matrix format. The equations below are obtained as particular cases of the corresponding equations of Appendix C, when the parameter matrices A, A_N, ..., are scalar, i.e., $A = aI_n$, $A_N = a_N I_n$, ... The largest eigenvalue of a matrix M is denoted by $\rho(M)$; $\rho(M) < 1$ means that the largest eigenvalue is smaller than 1 in modulus.

Symmetric model (**sym**):

$$S_t = (1 - a^2 - b^2)\overline{C} + a^2 C_{t-1} + b^2 S_{t-1}, \tag{D1}$$

$\overline{C} = (1/T)\sum_{t=1}^T C_t$ (PD); $a^2 + b^2 < 1$ (covariance stationarity of S_t and PD target).

Threshold model (**tr**):

$$S_t = (1-b^2)\overline{C} - A^*\overline{C} + a_P^2(I_{t-1}^+ I_{t-1}^{+'} + I_{t-1}^+ I_{t-1}^{-'} + I_{t-1}^- I_{t-1}^{+'}) \odot C_{t-1} + a_N^2(I_{t-1}^- I_{t-1}^{-'}) \odot C_{t-1} + b^2 S_{t-1}, \tag{D2}$$

$A^* = [a_P^2(\overline{C}_P + \overline{C}_M) + a_N^2 \overline{C}_N]\overline{C}^{-1}$; $\overline{C}_P = (1/T)\sum_{t=1}^T C_t \odot I_t^+ I_t^{+'}$ (PSD);
$\overline{C}_N = (1/T)\sum_{t=1}^T C_t \odot I_t^- I_t^{-'}$ (PSD); $\overline{C}_M = (1/T)\sum_{t=1}^T C_t \odot (I_{t-1}^+ I_{t-1}^{-'} + I_{t-1}^- I_{t-1}^{+'})$;
$\rho(A^* + b^2\,\mathrm{I}_n) < 1$ (covariance stationarity of S_t and PD target).

For the models listed below, it seems impossible to derive sufficient conditions to guarantee the PD-ness of S_t. Consequently, we impose no a priori restrictions on the parameters. However, during the estimation, we require that the coefficients jointly behave in such a way that S_t is PD $\forall t$.

Threshold model with PNM terms (**trPNM**):

$$
\begin{aligned}
S_t =& (1-b^2)\overline{C} - A^*\overline{C} + (a_P^2 I_{t-1}^{+} I_{t-1}^{+'} + a_N^2 I_{t-1}^{-} I_{t-1}^{-'} \\
&+ a_M^2 (I_{t-1}^{+} I_{t-1}^{-'} + I_{t-1}^{-} I_{t-1}^{+'})) \odot C_{t-1} + b^2 S_{t-1},
\end{aligned} \tag{D3}
$$

$A^* = (a_P^2 \overline{C}_P + a_N^2 \overline{C}_N + a_M^2 \overline{C}_M) \times \overline{C}^{-1}$; $\rho(A^* + b^2\, \mathrm{I}_n) < 1$ (covariance stationarity of S_t).

Threshold model with PNτ(M) terms (**trPNτM**):

$$
\begin{aligned}
S_t =& (1-b^2)\overline{C} - A^*\overline{C} + (a_P^2 I_{t-1}^{+} I_{t-1}^{+'} + a_N^2 I_{t-1}^{-} I_{t-1}^{-'} \\
&+ (a_M^+)^2 \tau(I_{t-1}^{+} I_{t-1}^{-'}) + (a_M^-)^2 \tau(I_{t-1}^{-} I_{t-1}^{+'})) \odot C_{t-1} + b^2 S_{t-1},
\end{aligned} \tag{D4}
$$

$A^* = (a_P^2 \overline{C}_P + a_N^2 \overline{C}_N + (a_M^+)^2 \overline{C}_M^+ + (a_M^-)^2 \overline{C}_M^-) \times \overline{C}^{-1}$; $\overline{C}_M^+ = (1/T)\sum_{t=1}^T C_t \odot \tau(I_t^+ I_t^{-'})$ (indefinite), with $\overline{C}_M^-$ (indefinite) defined analogously; $\rho(A^* + b^2\, \mathrm{I}_n) < 1$ (covariance stationarity of S_t).

Semi-covariance model (**semi**):

$$
S_t = (1-b^2)\overline{C} - A^*\overline{C} + a_P^2 P_{t-1} + a_N^2 N_{t-1} + a_M^2 M_{t-1} + b^2 S_{t-1}, \tag{D5}
$$

$A^* = (a_P^2 \overline{P} + a_N^2 \overline{N} + a_M^2 \overline{M}) \times \overline{C}^{-1}$; $\overline{P} = (1/T)\sum_{t=1}^T P_t$ (PSD), with $\overline{N}$ (PSD) and $\overline{M}$ (indefinite) defined analogously; $\rho(A^* + b^2\, \mathrm{I}_n) < 1$ (covariance stationarity of S_t).

Semi-covariance model (**semi-τ**):

$$
\begin{aligned}
S_t =& (1-b^2)\overline{C} - A^*\overline{C} + a_P^2 P_{t-1} + a_N^2 N_{t-1} \\
&+ (a_M^+)^2 \tau(M_{t-1}^+) + (a_M^-)^2 \tau(M_{t-1}^-) + b^2 S_{t-1},
\end{aligned} \tag{D6}
$$

$A^* = (a_P^2 \overline{P} + a_N^2 \overline{N} + (a_M^+)^2 \overline{M}^+ + (a_M^-)^2 \overline{M}^-) \times \overline{C}^{-1}$; $\overline{M}^+ = (1/T)\sum_{t=1}^T \tau(M_t^+)$ and $\overline{M}^- = (1/T)\sum_{t=1}^T \tau(M_t^-)$; $\rho(A^* + b^2\, \mathrm{I}_n) < 1$ (covariance stationarity of S_t).

E Statistics and graphs of covariance decompositions

Table 11 shows the time series means and standard deviations of the realized covariances between the six assets and their decomposition into semi-covariances; Table 12 shows the analogous statistics with respect to the decomposition using the signed daily close-to-close returns, and Table 13 the same information when the signed open-to-close returns are used. Several comments can be made:

1. The average realized covariances including SPY show a relative homogeneity, but are smaller than those of the fifteen pairs involving two banks.

2. A noticeable difference is the opposite signs of M and C_M. Each average mixed covariance (M) is negative because the mixed covariances of each day are negative by construction. The fact that each average mixed covariance term (C_M) is positive is not a necessity, but a feature of the data: the sample correlations (and covariances) between the assets are positive, which is not a surprise since five assets are in the same sector and the first one is tracking a market index. Thus, we observe that the inequalities $M_{ij} < 0 < C_{M,ij}$ are confirmed for each asset pair (i, j). Hence, $(P + N)_{ij}$, which is positive, must be larger than $(C_P + C_N)_{ij}$, which is also positive. This is even holding term by term, as stated below.

3. Each average positive semi-covariance (P) is larger than the corresponding average positive component (C_P), and each average N is larger than the corresponding C_N.

4. The two parts M^+ and M^- of M are close; likewise for the two parts C_M^+ and C_M^- of C_M.

5. Comparing the averages of the terms of the decompositions using the close-to-close and open-to-close returns reveals that C_M is larger for open-to-close than for close-to-close, and correspondingly, $C_P + C_N$, but mainly C_N, is smaller. This holds for all pairs, as can be seen in the tables of this Appendix.

Table 11: Time series means and standard deviations (between parentheses) of realized covariances and their decomposition into semi-covariances

Asset Pair	C	P	N	M	M^+	M^-
SPY-BAC	1.58 (4.29)	1.00 (2.91)	0.98 (2.51)	-0.41 (1.77)	-0.20 (0.82)	-0.21 (0.99)
SPY-C	1.64 (4.85)	1.03 (3.21)	1.03 (2.92)	-0.42 (1.99)	-0.21 (1.06)	-0.22 (1.00)
SPY-GS	1.48 (4.08)	0.94 (2.75)	0.91 (2.35)	-0.37 (1.65)	-0.18 (0.70)	-0.20 (0.99)
SPY-JPM	1.42 (4.22)	0.90 (2.94)	0.87 (2.32)	-0.35 (1.59)	-0.17 (0.79)	-0.18 (0.86)
SPY-WFC	1.35 (4.16)	0.89 (2.93)	0.88 (2.59)	-0.42 (1.85)	-0.20 (0.81)	-0.22 (1.08)
BAC-C	4.10 (8.38)	2.18 (4.96)	2.14 (4.59)	-0.22 (0.61)	-0.10 (0.25)	-0.11 (0.38)
BAC-GS	3.20 (6.35)	1.78 (3.64)	1.72 (3.40)	-0.31 (0.53)	-0.15 (0.27)	-0.15 (0.37)
BAC-JPM	3.44 (7.27)	1.84 (4.05)	1.78 (3.73)	-0.18 (0.36)	-0.09 (0.21)	-0.09 (0.18)
BAC-WFC	3.19 (7.05)	1.76 (3.97)	1.71 (3.76)	-0.27 (0.51)	-0.13 (0.25)	-0.14 (0.33)
C-BS	3.37 (7.74)	1.84 (4.27)	1.82 (4.14)	-0.29 (0.51)	-0.15 (0.27)	-0.14 (0.29)
C-JPM	3.54 (8.28)	1.88 (4.63)	1.84 (4.33)	-0.18 (0.49)	-0.09 (0.33)	-0.09 (0.22)
C-WFC	3.30 (8.07)	1.81 (4.51)	1.80 (4.53)	-0.30 (0.91)	-0.14 (0.34)	-0.16 (0.63)
GS-JPM	2.88 (6.47)	1.58 (3.65)	1.53 (3.37)	-0.23 (0.42)	-0.12 (0.26)	-0.12 (0.24)
GS-WFC	2.62 (6.35)	1.51 (3.80)	1.47 (3.36)	-0.35 (0.65)	-0.17 (0.34)	-0.18 (0.42)
JPM-WFC	2.83 (6.55)	1.56 (3.93)	1.50 (3.38)	-0.23 (0.64)	-0.11 (0.27)	-0.12 (0.44)

C: realized covariance; P: positive semi-covariance; N: negative semi-covariance; M: total mixed semi-covariance; M^+ and M^-: positive and negative mixed semi-covariances. See Section 2.2.1 for definitions.

Figures 6 and 7 show the time series of the realized covariances between SPY and JPM, and the components of their decompositions (3) using the close-to-close daily returns and (11). The peaks of the covariances occur at the same periods as those of the variances. The covariances are almost always positive, but a close look at the (identical) top left graphs reveals a few isolated and slightly negative covariances. In the decomposition based on the signed daily returns (Figure 6), these negative values are attributed to one of the three components (as is the case of any positive value). In the decomposition based on the intra-daily returns (Figure 7), positive and negative semi-covariances are positive, and the mixed one is negative (by definition). The two figures illustrate the differences between the two decompositions, in particular in their mixed components.

Table 12: Time series means and standard deviations (between parentheses) of realized covariances and their decomposition into parts signed by the daily *close-to-close* returns

Asset Pair	C	C_P	C_N	C_M	C_M^+	C_M^-
SPY-BAC	1.58 (4.29)	0.60 (3.21)	0.69 (3.00)	0.29 (0.83)	0.14 (0.56)	0.15 (0.65)
SPY-C	1.64 (4.85)	0.59 (2.90)	0.74 (3.63)	0.31 (1.90)	0.13 (0.54)	0.19 (1.84)
SPY-GS	1.48 (4.08)	0.57 (2.86)	0.68 (3.05)	0.23 (0.70)	0.11 (0.46)	0.13 (0.55)
SPY-JPM	1.42 (4.22)	0.55 (3.08)	0.64 (3.01)	0.24 (0.71)	0.11 (0.47)	0.13 (0.55)
SPY-WFC	1.35 (4.16)	0.52 (3.05)	0.60 (2.93)	0.23 (0.77)	0.10 (0.48)	0.13 (0.63)
BAC-C	4.10 (8.38)	1.70 (5.02)	1.91 (6.82)	0.49 (2.94)	0.20 (0.97)	0.29 (2.79)
BAC-GS	3.20 (6.35)	1.30 (4.52)	1.41 (4.93)	0.49 (1.36)	0.23 (0.99)	0.26 (1.00)
BAC-JPM	3.44 (7.27)	1.45 (5.07)	1.57 (5.70)	0.42 (1.30)	0.19 (0.91)	0.22 (0.97)
BAC-WFC	3.19 (7.05)	1.34 (5.35)	1.39 (5.03)	0.47 (1.40)	0.22 (0.92)	0.25 (1.11)
C-BS	3.37 (7.74)	1.34 (4.57)	1.54 (6.24)	0.49 (2.64)	0.27 (2.50)	0.22 (0.93)
C-JPM	3.54 (8.28)	1.40 (4.67)	1.63 (6.35)	0.50 (2.96)	0.28 (2.81)	0.23 (1.01)
C-WFC	3.30 (8.07)	1.30 (4.70)	1.47 (6.24)	0.53 (3.28)	0.29 (3.09)	0.24 (1.17)
GS-JPM	2.88 (6.47)	1.18 (4.42)	1.31 (5.11)	0.39 (1.08)	0.19 (0.79)	0.19 (0.79)
GS-WFC	2.62 (6.35)	1.05 (4.89)	1.11 (4.36)	0.46 (1.30)	0.22 (0.86)	0.24 (1.03)
JPM-WFC	2.83 (6.55)	1.14 (4.68)	1.23 (4.91)	0.45 (1.37)	0.22 (0.90)	0.23 (1.09)

C: realized covariance; C_P: positive part; C_N: negative part; C_M : total mixed part; C_M^+ and C_M^-: positive and negative mixed parts.. See Section 2.1.1 for definitions.

Table 13: Time series means and standard deviations (between parentheses) of realized covariances and their decomposition into parts signed by the daily *open-to-close* returns

Asset Pair	C	C_P	C_N	C_M	C_M^+	C_M^-
SPY-BAC	1.58 (4.29)	0.61 (3.41)	0.63 (2.49)	0.34 (1.46)	0.16 (0.82)	0.18 (1.24)
SPY-C	1.64 (4.85)	0.58 (2.79)	0.67 (3.21)	0.39 (2.67)	0.15 (0.85)	0.24 (1.24)
SPY-GS	1.48 (4.08)	0.57 (2.84)	0.57 (2.47)	0.34 (1.97)	0.16 (0.86)	0.18 (1.79)
SPY-JPM	1.42 (4.22)	0.56 (3.24)	0.56 (2.55)	0.30 (1.44)	0.14 (1.00)	0.16 (1.06)
SPY-WFC	1.35 (4.16)	0.50 (2.87)	0.52 (2.49)	0.33 (2.01)	0.15 (1.10)	0.18 (1.70)
BAC-C	4.10 (8.38)	1.67 (5.23)	1.81 (6.29)	0.62 (3.69)	0.28 (1.34)	0.34 (3.47)
BAC-GS	3.20 (6.35)	1.29 (4.79)	1.30 (4.39)	0.61 (2.13)	0.31 (1.32)	0.30 (1.72)
BAC-JPM	3.44 (7.27)	1.42 (5.35)	1.45 (5.00)	0.57 (2.59)	0.28 (1.71)	0.29 (1.99)
BAC-WFC	3.19 (7.05)	1.29 (5.44)	1.27 (4.67)	0.63 (2.18)	0.31 (1.27)	0.32 (1.83)
C-GS	3.37 (7.74)	1.33 (5.03)	1.42 (5.84)	0.62 (2.76)	0.35 (2.59)	0.27 (1.05)
C-JPM	3.54 (8.28)	1.39 (5.03)	1.51 (6.22)	0.64 (3.53)	0.35 (3.22)	0.29 (1.51)
C-WFC	3.30 (8.07)	1.24 (4.71)	1.37 (6.14)	0.69 (3.50)	0.37 (3.18)	0.32 (1.54)
GS-JPM	2.88 (6.47)	1.15 (4.70)	1.17 (4.53)	0.56 (2.13)	0.27 (1.68)	0.29 (1.36)
GS-WFC	2.62 (6.35)	1.07 (4.99)	1.04 (4.17)	0.51 (1.53)	0.24 (0.92)	0.27 (1.27)
JPM-WFC	2.83 (6.55)	1.13 (4.79)	1.15 (4.56)	0.55 (2.07)	0.28 (1.24)	0.27 (1.70)

C: realized covariance; C_P: positive part; C_N: negative part; C_M : total mixed part; C_M^+ and C_M^-: positive and negative mixed parts.. See Section 2.1.1 for definitions.

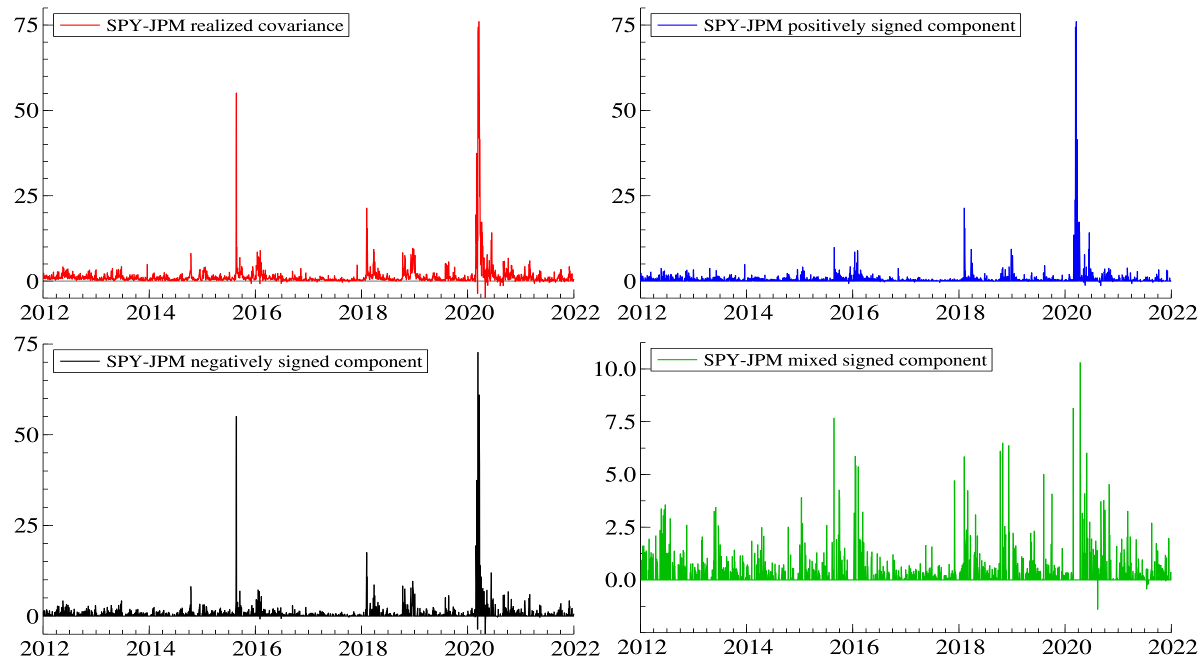

Figure 6: Annualized realized covariances of SPY and JPM and the components of their decomposition (3) using the signed daily close-to-close returns

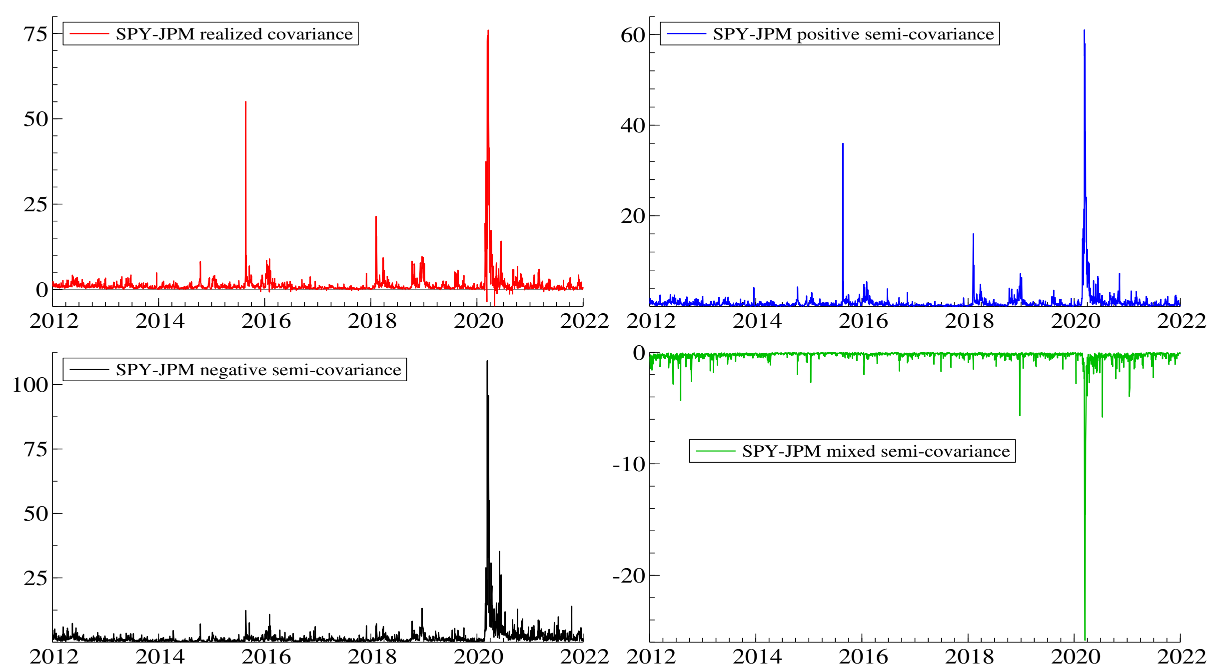

Figure 7: Annualized realized covariances of SPY and JPM and the semi-covariance components of their decomposition (11)

Table 14 shows the means of indicators of the signed daily close-to-close and open-to-close returns; the corresponding statistics of the two types of returns are close. The daily close-to-close returns of SPY are positive for 56% of the days of the sample period, and 51 or 52% for the other stocks (column 2); negative returns occur in the corresponding complementary percentages (column 3). In column 4, the value 0.406 for SPY is indicating that in 40.6% of the days, both SPY and at least one of the five other stocks have positive returns; the other values in the same columns are close to 41%. In column 5, the values fluctuate between 0.333 and 0.38; the latter (for stock C) means that returns of C and at least one of the other assets were simultaneously negative. The values in the last column are the proportions of days when the return of the stock (in the first column) has a different sign than at least one of the other stocks.

Table 14: Time series means of indicators of signed daily returns

Ticker	diag. $I^+I^{+'}$	diag. $I^-I^{-'}$	off-diag. $I^+I^{+'}$	off-diag. $I^-I^{-'}$	$I^+I^{-'} + I^-I^{+'}$
			Close-to-close returns		
SPY	0.557	0.443	0.406	0.333	0.261
BAC	0.521	0.479	0.418	0.373	0.209
C	0.512	0.488	0.418	0.380	0.202
GS	0.521	0.479	0.412	0.368	0.220
JPM	0.515	0.485	0.416	0.376	0.208
WFC	0.512	0.488	0.402	0.365	0.233
			Open-to-close returns		
SPY	0.548	0.452	0.384	0.319	0.297
BAC	0.510	0.490	0.397	0.364	0.239
C	0.514	0.486	0.399	0.362	0.239
GS	0.519	0.481	0.393	0.352	0.255
JPM	0.522	0.478	0.403	0.359	0.238
WFC	0.514	0.486	0.385	0.348	0.267

diag. $I^+I^{+'}$: indicator of positive returns; diag. $I^-I^{-'}$: indicator of negative returns; off-diag. $I^+I^{+'}$ and off-diag. $I^-I^{-'}$:: for the asset indicated in the table row, the average of the five time series means of the off-diagonal elements the indicated matrix. See comments in the text.

F Estimation results of BEKK-CAW models based on the daily close-to-close returns

Table 15: Scalar BEKK-CAW model QML estimates

	sym (D1)	**tr** (D2)	**trPNM** (D3)	**trPNτM** (D4)	**semi** (D5)	**semi-τ** (D6)
a	0.521					
a_P		0.492*	0.466*	0.466*	0.448*	0.448*
a_N		0.529*	0.538*	0.537*	0.594*	0.594*
a_M			0.500*		0.483*	
a_{M^+}				0.497*		0.465*
a_{M^-}				0.503*		0.499*
b	0.836*	0.841*	0.846*	0.846*	0.834*	0.834*
LLF	-12518.91	-12510.94	-12503.38	-12503.16	-12511.30	-12511.24
AIC	9.949	9.944	9.938	9.939	9.945	9.945
BIC	9.954	9.951	9.948	9.951	9.954	9.957

* denotes statistical significance at the 5% level. Each column corresponds to a model; the models are defined in Appendix D, corresponding to the headers in row 1; row 2 refers to the equation numbers in the appendix. The last lines report the obtained maximum value of the log-likelihood function (LLF), and the corresponding Akaike (AIC) and Bayesian information criteria (BIC) values. The models are estimated using the dataset of 2517 observations described in Section 4.

Table 16: Diagonal BEKK-CAW model QML estimates

	sym (C1)	**tr** (C2)	**trPNM** (C3)	**trPNτM** (C4)	**semi** (C5)	**semi-τ** (C6)
A	0.429* 0.568* 0.558* 0.540* 0.581* 0.611*					
A_P		0.374* 0.539* 0.529* 0.518* 0.563* 0.572*	0.393* 0.514* 0.498* 0.501* 0.553* 0.589*	0.382* 0.498* 0.494* 0.462* 0.528* 0.614*	0.390* 0.438* 0.394* 0.404* 0.439* 0.443*	0.399* 0.474* 0.434* 0.423* 0.481* 0.460*
A_N		0.472* 0.570* 0.557* 0.541* 0.575* 0.623*	0.501* 0.593* 0.576* 0.557* 0.610* 0.666*	0.524* 0.623* 0.620* 0.575* 0.647* 0.726*	0.553* 0.696* 0.689* 0.671* 0.755* 0.798*	0.554* 0.708* 0.691* 0.686* 0.751* 0.809*
A_M			0.372* 0.561* 0.545* 0.527* 0.577* 0.589*		-0.505* -0.616* -0.578* -0.590* -0.570* -0.589*	
A_{M^+}				0.409* 0.574* 0.560* 0.538* 0.580* 0.694*		0.351* 0.339* 0.392* 0.391* 0.456* 0.388*
A_{M^-}				0.434* 0.583* 0.583* 0.535* 0.590* 0.622*		0.346* 0.466* 0.407* 0.385* 0.287* 0.383*
B	0.895* 0.780* 0.799* 0.802* 0.773* 0.751*	0.892* 0.790* 0.809* 0.810* 0.782* 0.763*	0.887* 0.791* 0.816* 0.814* 0.778* 0.748*	0.892* 0.806* 0.822* 0.835* 0.797* 0.738*	0.885* 0.790* 0.812* 0.809* 0.769* 0.752*	0.884* 0.784* 0.808* 0.804* 0.768* 0.745*
LLF	-12493.04	-12481.28	-12471.95	-12470.39	-12489.070	-12489.061
AIC	9.937	9.932	9.929	9.933	9.943	9.948
BIC	9.964	9.974	9.985	10.002	9.998	10.017

* denotes statistical significance at the 5% level. In each cell, the first value is the estimate for the market index (SPY), the next ones are for the banking stocks (ordered as BAC, C, GS, JPM, WFC). Each column corresponds to a model; the models are defined in Appendix C, corresponding to the headers in row 1; row 2 refers to the equation numbers in the appendix. The last lines report the obtained maximum value of the log-likelihood function (LLF), and the corresponding Akaike (AIC) and Bayesian information criteria (BIC) values. The models are estimated using the dataset of 2517 observations described in Section 4.

Table 17: Partly lower triangular BEKK-CAW model QML estimates

	sym (C1)	**tr** (C2)	**trPNM** (C3)	**trPNτM** (C4)	**semi** (C5)	**semi-τ** (C6)
A	0.431*					
	-0.002 0.566*					
	-0.004 0.550*					
	-0.004 0.531*					
	0.003 0.594*					
	-0.004 0.604*					
A_P		0.292*	0.403*	0.388*	0.271*	0.311*
		-0.029 0.530*	-0.037 0.518*	-0.043* 0.487*	-0.132* 0.486*	-0.242* 0.583*
		-0.031 0.516*	-0.037 0.496*	-0.045* 0.478*	-0.097 0.383*	-0.248* 0.537*
		-0.021 0.487*	-0.037 0.496*	-0.040* 0.439*	-0.105* 0.402*	-0.206* 0.487*
		-0.010 0.537*	-0.026 0.581*	-0.039 0.517*	-0.101* 0.498*	-0.204* 0.596*
		-0.014 0.482*	-0.056 0.576*	-0.057* 0.588*	-0.271* 0.387	-0.300* 0.459*
A_N		0.475*	0.519*	0.544*	0.596*	0.585*
		0.017 0.570*	0.016 0.588*	0.001 0.615*	0.071* 0.653*	0.129* 0.604*
		0.015 0.551*	0.010 0.564*	-0.001 0.608*	0.037 0.651*	0.122* 0.587*
		0.008 0.533*	0.006 0.548*	-0.007 0.553*	0.046 0.620*	0.098* 0.564*
		0.015 0.590*	0.012 0.631*	0.001 0.647*	0.045* 0.725*	0.096* 0.658*
		0.005 0.618*	0.003 0.648*	-0.014 0.702*	0.039 0.809*	0.079* 0.790*
A_M			0.371*		-0.316*	
			-0.082* 0.562*		0.311* -0.726*	
			-0.073* 0.536*		0.256* -0.604*	
			-0.049* 0.517*		0.251 -0.633*	
			-0.037 0.603*		0.199 -0.678*	
			-0.028 0.574*		0.261* -0.427*	
A_{M^+}				0.506*		0.212*
				0.109* 0.577*		-0.217 0.390*
				0.147* 0.549*		-0.300 0.601*
				0.118* 0.513*		-0.140 0.434*
				0.192* 0.578*		-0.024 0.489*
				0.171* 0.652*		-0.165 0.327*
A_{M^-}				0.335		0.264*
				-0.266* 0.567*		-0.300 0.599*
				-0.300* 0.566*		-0.220 0.353*
				-0.224* 0.508*		-0.283 0.457*
				-0.300* 0.574*		-0.299 0.455*
				-0.212* 0.610*		-0.265 0.255*
B	0.893*	0.892*	0.879*	0.883*	0.886*	0.888*
	0.781*	0.789*	0.794*	0.811*	0.773*	0.770*
	0.806*	0.812*	0.825*	0.829*	0.820*	0.799*
	0.810*	0.818*	0.822*	0.851*	0.813*	0.816*
	0.756*	0.765*	0.756*	0.792*	0.754*	0.757*
	0.758*	0.769*	0.766*	0.752*	0.746*	0.738*
LLF	-12491.88	-12479.38	-12466.46	-12462.64	-12472.03	-12459.48
AIC	9.940	9.938	9.937	9.943	9.941	9.940
BIC	9.979	10.003	10.027	10.058	10.032	10.056

* denotes statistical significance at the 5% level. Each column corresponds to a model; the models are defined in Appendix C, corresponding to the headers in row 1; row 2 refers to the equation numbers in the appendix. For each parameter matrix A, the first column of coefficients gives the impacts of SPY on the banking stocks (ordered as BAC, C, GS, JPM, WFC), and the second column reports the diagonal parameters, with SPY first. The last lines report the obtained maximum value of the log-likelihood function (LLF), and the corresponding Akaike (AIC) and Bayesian information criteria (BIC) values. The models are estimated using the dataset of 2517 observations described in Section 4.

G Estimation results of BEKK-CAW models based on the daily open-to-close returns

Table 18: OC-based scalar BEKK-CAW model QML estimates

	tr (D2)	**trPNM** (D3)	**trPNτM** (D4)
a_P	0.497*	0.463*	0.463*
a_N	0.527*	0.538*	0.538*
a_M		0.500*	
a_{M^+}			0.500*
a_{M^-}			0.501*
b	0.841*	0.849*	0.849*
LLF	-12512.19	-12501.98	-12501.98
AIC	9.945	9.937	9.938
BIC	9.952	9.947	9.950

* denotes statistical significance at the 5% level. Each column corresponds to a model; the models are defined in Appendix D, corresponding to the headers in row 1, with indicator vectors in each specification defined via OC returns; row 2 refers to the equation numbers in the appendix. The last lines report the obtained maximum value of the log-likelihood function (LLF), and the corresponding Akaike (AIC) and Bayesian information criteria (BIC) values. The models are estimated using the dataset of 2517 observations described in Section 4.

Table 19: OC-based diagonal BEKK-CAW model QML estimates

	tr (C2)	**trPNM** (C3)	**trPNτM** (C4)
A_P	0.369* 0.538* 0.530* 0.511* 0.545* 0.573*	0.369* 0.490* 0.482* 0.483* 0.505* 0.578*	0.345* 0.471* 0.484* 0.448* 0.487* 0.594*
A_N	0.472* 0.567* 0.555* 0.534* 0.578* 0.611*	0.482* 0.581* 0.569* 0.549* 0.596* 0.652*	0.482* 0.607* 0.605* 0.558* 0.614* 0.701*
A_M		-0.336* -0.535* -0.524* -0.515* -0.543* -0.588*	
A_{M^+}			-0.322* -0.553* -0.546* -0.516* -0.539* -0.657*
A_{M^-}			-0.377* -0.532* -0.552* -0.514* -0.549* -0.626*
B	0.897* 0.791* 0.809* 0.816* 0.787* 0.767*	0.899* 0.802* 0.821* 0.817* 0.793* 0.752*	0.912* 0.811* 0.821* 0.833* 0.806* 0.741*
LLF	-12479.57	-12475.24	-12474.45
AIC	9.931	9.932	9.936
BIC	9.972	9.988	10.006

* denotes statistical significance at the 5% level. In each cell, the first value is the estimate for the market index (SPY), the next ones are for the banking stocks (ordered as BAC, C, GS, JPM, WFC). Each column corresponds to a model; the models are defined in Appendix C, corresponding to the headers in row 1, with indicator vectors in each specification defined via OC returns; row 2 refers to the equation numbers in the appendix. The last lines report the obtained maximum value of the log-likelihood function (LLF), and the corresponding Akaike (AIC) and Bayesian information criteria (BIC) values. The models are estimated using the dataset of 2517 observations described in Section 4.

Table 20: OC-based PLT BEKK-CAW model QML estimates

	tr		**trPNM**		**trPNτM**	
	(C2)		(C3)		(C4)	
A_P		0.360*		0.383*		0.376*
	-0.020	0.541*	-0.033*	0.493*	-0.049*	0.468*
	-0.018	0.526*	-0.032	0.477*	-0.035*	0.473*
	-0.018	0.510*	-0.034*	0.479*	-0.036*	0.426*
	-0.011	0.563*	-0.028	0.529*	-0.035*	0.495*
	-0.022	0.575*	-0.048*	0.582*	-0.056*	0.602*
A_N		0.474*		0.502*		0.514*
	0.019	0.569*	0.013	0.576*	0.009	0.599*
	0.013	0.551*	0.006	0.556*	-0.005	0.594*
	0.013	0.531*	0.006	0.536*	-0.006	0.534*
	0.020	0.593*	0.011	0.613*	0.001	0.619*
	0.017	0.610*	0.017	0.645*	0.004	0.707*
A_M				-0.346*		
			0.055*	-0.534*		
			0.048*	-0.514*		
			0.041*	-0.506*		
			0.025	-0.564*		
			0.051*	-0.589*		
A_{M^+}						-0.425*
					-0.111*	-0.527*
					-0.081	-0.534*
					-0.063	-0.487*
					-0.059	-0.546*
					-0.056	-0.666*
A_{M^-}						-0.353*
					0.301*	-0.547*
					0.207*	-0.543*
					0.185*	-0.492*
					0.164*	-0.563*
					0.203*	-0.636*
B	0.898*		0.893*		0.895*	
	0.789*		0.814*		0.814*	
	0.812*		0.831*		0.829*	
	0.818*		0.852*		0.855*	
	0.770*		0.776*		0.802*	
	0.768*		0.758*		0.738*	
LLF	-12478.10		-12470.54		-12467.52	
AIC	9.937		9.940		9.946	
BIC	10.002		10.030		10.062	

* denotes statistical significance at the 5% level. Each column corresponds to a model; the models are defined in Appendix C, corresponding to the headers in row 1, with indicator vectors in each specification defined via OC returns; row 2 refers to the equation numbers in the appendix. For each parameter matrix A, the first column of coefficients gives the impacts of SPY on the banking stocks (ordered as BAC, C, GS, JPM, WFC), and the second column reports the diagonal parameters, with SPY first. The last lines report the obtained maximum value of the log-likelihood function (LLF), and the corresponding Akaike (AIC) and Bayesian information criteria (BIC) values. The models are estimated using the dataset of 2517 observations described in Section 4.

H Tests of leverage effects

Table 21: Tests of leverage effects in variance equations of scalar, diagonal and PLT **tr** models

Version	Asset	a_{Pii}^2	a_{Nii}^2	$a_{Nii}^2 \leq a_{Pii}^2$	a_{Pi1}^2	a_{Ni1}^2	$a_{Ni1}^2 \leq a_{Pi1}^2$	$2a_{Pi1}a_{Pii}$	$2a_{Ni1}a_{Nii}$	$2a_{Ni1}a_{Nii} \leq 2a_{Pi1}a_{Pii}$
scalar	1-6	0.24	0.28	3.412^+						
diagonal	1	0.14	0.22	1.661*						
	2	0.29	0.33	2.346^+						
	3	0.28	0.31	2.390^+						
	4	0.27	0.29	2.219*						
	5	0.32	0.33	1.016						
	6	0.33	0.39	2.453^+						
PLT	1	0.13	0.23	1.646*						
	2	0.29	0.32	2.032*	0.0008	0.0003	-0.285	-0.031	0.020	0.849
	3	0.28	0.30	2.175*	0.0010	0.0002	-0.374	-0.033	0.017	0.865
	4	0.26	0.28	2.163*	0.0004	0.0001	-0.326	-0.022	0.008	0.618
	5	0.34	0.35	0.948	0.0001	0.0002	0.180	-0.012	0.017	0.496
	6	0.32	0.38	2.330^+	0.0002	0.0000	-0.146	-0.016	0.006	0.296

The columns 3, 4, 6, 7, 9, and 10 report the estimates of the coefficients indicated in row 1, for the **tr** conditional variance equations identified in the first two columns. The columns 5, 8, and 11 provide the values of the test statistics for testing the significance of the null hypothesis written in row 1 against the complementary alternative hypothesis. The statistics that are significant at the 5% level are denoted by * (critical value: 1.645), at the 1% level by $^+$ (critical value: 2.326).

Table 22: Tests of leverage effects in covariance equations of scalar, diagonal and PLT **tr** modelsl

Version	Asset	$a_{Pii}a_{Pjj}$	$a_{Nii}a_{Njj}$	$a_{Nii}a_{Njj} \leq a_{Pii}a_{Pjj}$	$a_{P11}a_{Pi1}$	$a_{N11}a_{Ni1}$	$a_{N11}a_{Ni1} \leq a_{P11}a_{Pi1}$
diagonal	1-2	0.20	0.27	2.188*			
	1-3	0.20	0.26	2.264*			
	1-4	0.19	0.26	2.109*			
	1-5	0.21	0.27	1.920*			
	1-6	0.21	0.29	2.278*			
	2-3	0.29	0.32	2.769^{+}			
	2-4	0.28	0.31	2.718**			
	2-5	0.30	0.33	1.979*			
	2-6	0.31	0.35	2.838^{+}			
	3-4	0.27	0.30	2.777^{+}			
	3-5	0.30	0.32	1.981*			
	3-6	0.30	0.35	2.864^{+}			
	4-5	0.29	0.31	1.825*			
	4-6	0.30	0.34	2.854^{+}			
	5-6	0.32	0.36	2.212*			
PLT	1-2	0.20	0.27	2.205*	-0.011	0.008	0.811
	1-3	0.19	0.26	2.277*	-0.012	0.007	0.813
	1-4	0.19	0.25	2.142*	-0.008	0.004	0.586
	1-5	0.21	0.28	2.006*	-0.004	0.007	0.513
	1-6	0.21	0.29	2.340^{+}	-0.005	0.002	0.289
	2-3	0.29	0.31	2.474^{+}			
	2-4	0.28	0.30	2.516^{+}			
	2-5	0.31	0.34	1.775*			
	2-6	0.31	0.35	2.619^{+}			
	3-4	0.27	0.29	2.619^{+}			
	3-5	0.31	0.33	1.824*			
	3-6	0.30	0.34	2.611*			
	4-5	0.30	0.31	1.823^{+}			
	4-6	0.29	0.33	2.734^{+}			
	5-6	0.33	0.36	2.150*			

The columns 3, 4, 6, and 7 report the estimates of the coefficients indicated in row 1, for the **tr** conditional covariance equation identified in the first two columns. The columns 5 and 8 provide the values of the test statistics for testing the significance of the null hypothesis written in row 1 against the complementary alternative hypothesis. The statistics that are significant at the 5% level are denoted by * (critical value: 1.645), at the 1% level by $^{+}$ (critical value: 2.326).