A MUTUALLY EXCITING ROUGH JUMP-DIFFUSION FOR FINANCIAL MODELLING

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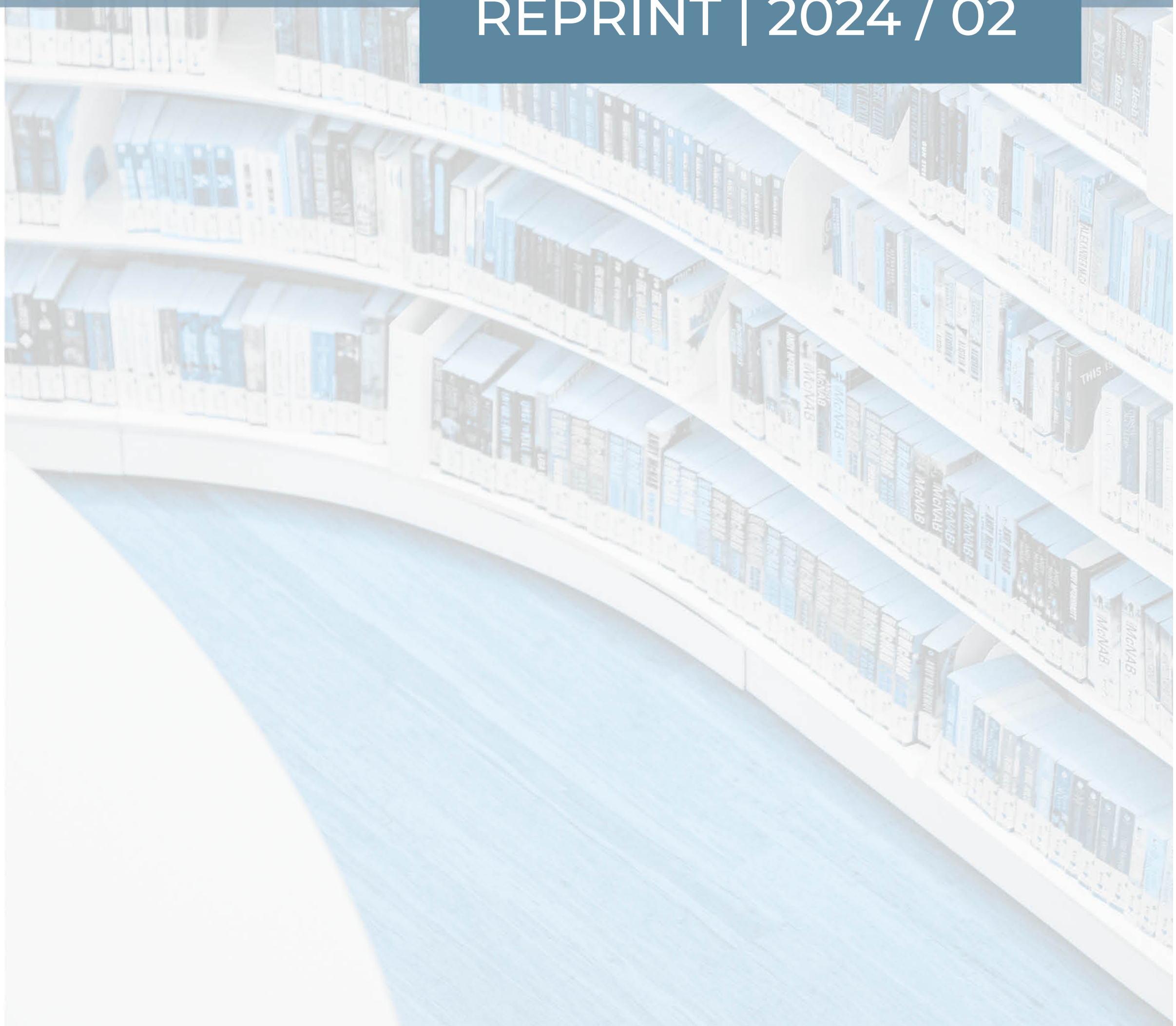

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A mutually exciting rough jump-diffusion for financial modelling

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This article introduces a new class of diffusive processes with rough mutually exciting jumps for modeling financial asset returns. The novel feature is that the memory of positive and negative jump processes is defined by the product of a dampening factor and a kernel involved in the construction of the rough Brownian motion. The jump processes are nearly unstable because their intensity diverges to $+\infty$ for a brief duration after a shock. We first infer the stability conditions and explore the features of the dampened rough (DR) kernel, which defines a fractional operator, similar to the Riemann-Liouville integral. We next reformulate intensities as infinite-dimensional Markov processes. Approximating these processes by discretization and then considering the limit allows us to retrieve the Laplace transform of asset log-return. We show that this transform depends on the solution of a particular fractional integro-differential equation. We also define a family of changes of measure that preserves the features of the process under a risk-neutral measure. We next develop an econometric estimation procedure based on the peak over threshold (POT) method. To illustrate this work, we fit the mutually exciting rough jump-diffusion to time series of Bitcoin log-returns and compare the goodness of fit to its non-rough equivalent. Finally, we analyze the influence of roughness on option prices.

Keywords: self-exciting process, Epidemic Type Aftershock Sequence (ETAS), jump-diffusion, fractional Brownian motion, Riemann-Liouville fractional integral.

1 Introduction

The propensity of financial price jumps to cluster is abundantly documented in the literature. The phenomenon is specifically studied and evidenced by Yu [38] and Maheu and McCurty [30] who respectively examine DJIA and individual stocks returns. More recently, Aït-Sahalia et al. [2] explore international equity market indices on a daily basis and conclude that jump clustering over time is a strong effect for equity market indices.

This phenomenon of jump clustering raises questions about the relevance of classical jump-diffusion dynamics for modeling asset prices. Unsurprisingly, a recent strand of the literature has made significant efforts to develop quantitative methods for modeling jump clustering and investigating its implications for asset or option pricing. A natural way to capture the clustering

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of jumps is by using self-exciting point processes, where the jump arrival intensity depends on the number and sometimes the size of previous shocks on asset log-returns, as shown in Hainaut and Moraux [20]. This approach is closely related to Hawkes self-exciting processes (see Hawkes [25, 26]), which are also commonly used for modeling high-frequency data. Readers interested in high-frequency applications may refer to Giot [15], Bowsher [6], Chavez-Demoulin and McGill [7], Bacry et al. [4], Da Fonseca and Zaatour [10] or Hainaut and Goutte [19], for more recent contributions. Aït-Sahalia and Jacod [3] examine whether such tick data models are compatible with the typical macroscopic continuous-time approaches. The literature on financial applications of self-exciting processes is extensive, and for a detailed review, we recommend Hawkes [28].

In a standard self-exciting model, the jump intensity increases after a shock and revert next to a baseline level. The speed of reversion is determined by a memory kernel. This category of model is also called "Epidemic Type Aftershock Sequence" (ETAS) and is used for modelling earthquakes as in Hawkes and Oakes [27]. In the most common specification, the memory kernel is exponentially decreasing. In this particular case, the jump process is Markov, and we can rely on Itô's calculus to find an analytical expression for its Laplace transform. In a more general setting, we lose the analytical tractability offered by stochastic calculus, except for moments or asymptotic properties. For instance, Muzy et al. [32] derive the moments of stationary processes and study their limit behavior. Stabile and Torrisi [36] study the asymptotic behavior of non-stationary Hawkes process. Cristofaro et al. [9] propose a fractional differential equation that governs the intensity rate of self-exciting process by using the Caputo fractional derivative. Hainaut [21] finds the Laplace transform of self-exciting claims processes for memory kernels that possess a closed-form inverse Fourier transform.

Recently, Jaisson and Rosenbaum [29] observed that nearly unstable Hawkes processes fit high-frequency financial time series well. Under certain conditions, this unstable process asymptotically behaves as a Brownian Volterra process with a kernel, $k(u) = u^{\alpha-1}E_{\alpha,\alpha}(-u^{\alpha})$, where $\alpha \in (0, 1]$ and $E_{\alpha,\alpha}$ is the two-parameter Mittag-Leffler function. Based on this, Chen et al. [8] and Habyarimana et al. [18] use this kernel for defining a fractional Hawkes process. This kernel diverges at zero but the jump process remains stable. In this paper, we incorporate a mutually exciting process with a kernel that diverges at zero to a diffusion, drawing direct inspiration from the literature on fractional Brownian motions and from Hainaut et al. [23].

The fractional Brownian motion (fBm) has dependent increments, unlike the Brownian motion (Bm). This dependence is quantified by the Hurst index, denoted as $H \in (0, 1)$. A value of H greater (or lower) than $1/2$ corresponds to positive (or negative) correlation between increments. For H =1/2, we obtain the Bm with independent increments (for details, see Hainaut [22], chapter 6). FBm's with $H < 1/2$ exhibit highly volatile sample paths and are referred to as 'rough' for this reason. At time $t > 0$, the rough fBm can be expressed as the sum of two integrals, one over $(-\infty, 0)$ and the other over $(0, t)$, with respect to a Bm. The second term is $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dW_s$ where $\alpha \in (0, 1]$ and W_t is a Bm. This integral remains well-defined even when the rough kernel $\frac{u^{\alpha-1}}{\Gamma(\alpha)}$, diverges as u approaches 0. It is also used as an approximation for a rough fBm. In this article, we refer to it as the 'rough Brownian motion' to distinguish it from the true rough fractional Brownian motion for clarity.

FBm's and rough Bm have recently garnered significant attention in the literature. Guo et al. [16] conducted an analytical and numerical study of the fractional Langevin equation driven by fractional Brownian motion. Zeng et al. [39] addressed the stability problem of the fractional order Black-Scholes model driven by fractional Brownian motion. Gatheral et al. [14] proposed a model in which the variance of log-prices is ruled by a rough Bm. Xu and Zhou [37] evaluated perpetual American put options when the underlying asset price follows a sub-mixed fBm. The properties of the rough Heston model, which is based on the rough Bm, are studied by El Euch and Rosenbaum [11] and [12]. They formulated the process characteristic function with fractional Riccati equations.

In this study, we combine a diffusion and a bivariate jump process for both positive and negative shocks with mutual excitation. A noteworthy feature is that the kernels of the jump processes exhibit roughness akin to that of a fractional Brownian motion (fBm) but are dampened by an exponentially decaying function to ensure their stability. This novel process, referred to as the 'mutually exciting rough jump-diffusion' (MERJD), offers several interesting features for modeling high-frequency or highly volatile assets, such as cryptocurrencies. First, in Section 2, we demonstrate that the dampened rough (DR) kernel is a Sonine function, and as a result, it has a conjugate kernel. This property enables us to derive analytical expressions for expected intensities and establish the stability conditions of the bivariate jump process. The MERJD also has an equivalent infinite-dimensional Markov representation, presented in Section 3. By discretizing this representation, we can approximate the Laplace transform of the MERJD. Considering the limit of the finite-dimensional approximation leads to the Laplace transform of the MERJD in Section 4. In a manner similar to El Euch and Rosenbaum [12], this transform depends on the solution of a fractional differential equation (FDE). This FDE involves an operator based on the DR kernel, similar to the left fractional Riemann-Liouville (RL) integral. As detailed in Section 5, there exists a family of changes of measure that preserve the characteristics of the process under a risk-neutral measure, making the MERJD well-suited for pricing financial derivatives. In Section 6, we demonstrate that the log-likelihood of rough mutually exciting jump processes has a closed-form expression. This is combined with a POT method for estimating MERJD parameters using time series data of hourly Bitcoin returns. We conclude with an analysis of the impact of roughness on option prices.

2 A dampened rough kernel

We consider an asset price process, denoted by $(S_t)_{t\geq 0}$ defined on a probability space Ω endowed with the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of all processes involved in the dynamic of S_t and with a probability measure $\mathbb{P}$. The log-return $X_t = \ln \frac{S_t}{S_0}$ is ruled by a mutually exciting rough jump-diffusion (MERJD) that is defined as follows

$$X_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t + \sum_{j=1}^{2} \left(L_t^{(j)} - \mu_j \int_0^t \lambda_s^{(j)} ds \right) , \tag{1}$$

where $\left(L_t^{(1)}\right)_{t\geq 0}$ and $\left(L_t^{(2)}\right)_{t\geq 0}$ are respectively positive and negative point processes, with intensities $\left(\lambda_t^{(1)}\right)_{t\geq 0}$ and $\left(\lambda_t^{(2)}\right)_{t\geq 0}$. $(W_t)_{t\geq 0}$ is a Brownian motion and $\mu, \sigma \in \mathbb{R}$. $\mu_1 \in \mathbb{R}^+$ and $\mu_2 \in \mathbb{R}^-$ are respectively the expected jumps of $L_t^{(1)}$ and $L_2^{(2)}$. These point processes are the sum of random increments, denoted by $J_k^{(1)}$ and $J_k^{(2)}$,

$$L_t^{(j)} = \sum_{k=1}^{N_t^{(j)}} J_k^{(j)} , \; j = 1, 2 .$$

where $\left(N_t^{(j)}\right)_{t\geq 0}$ is the number of shocks observed up to time t. The statistical distributions of $J_k^{(j)} \sim J^{(j)}$ for $j = 1, 2$, are noted $m^{(j)}(\cdot)$. $J^{(1)}$ and $J^{(2)}$ are respectively defined on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$

and $(\mathbb{R}^-, \mathcal{B}(\mathbb{R}^-))$. The jump expectations are $\mu_j = \mathbb{E}\left(J^{(j)}\right)$ and the moment generating functions are

$$\mathcal{J}_j(\omega) = \mathbb{E}\left(e^{\omega J^{(j)}}\right), \tag{2}$$

for $j = 1, 2$.

In the numerical illustration, we examine a MERJD with positive and negative exponentially distributed jumps. The probability density functions (pdf's) of $J^{(1)}$ and $J^{(2)}$ are in this case respectively equal to

$$m^{(1)}(z) = \rho_1 e^{-\rho_1 z} 1_{\{z \geq 0\}}, \, m^{(2)}(z) = -\rho_2 e^{-\rho_2 z} 1_{\{z \leq 0\}}. \tag{3}$$

where $\rho_1 \in \mathbb{R}^+$ and $\rho_2 \in \mathbb{R}^-$. For this choice of density, $\mu_j = \frac{1}{\rho_j}$ and $\mathcal{J}_j(\omega) = \frac{\rho_j}{\rho_j - \omega}$ for $j = 1, 2$, with $\omega < \rho_1$ and $\omega > \rho_2$.

The intensities depend on the past realizations of the counting processes $\left(N_t^{(j)}\right)_{t \geq 0}$ for $j = 1, 2$ as follows:

$$\begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \int_0^{t-} e^{-\beta(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dN_s^{(1)} \\ \int_0^{t-} e^{-\beta(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dN_s^{(2)} \end{pmatrix} \tag{4}$$

where $\alpha \in (0, 1]$, β, $\eta_{i,j} \in \mathbb{R}^+$ for $i, j \in \{1, 2\}$. The stability conditions of these processes are discussed in Proposition 4. The functions $k(u) = e^{-\beta u} \frac{u^{\alpha-1}}{\Gamma(\alpha)}$ is referred to as the 'memory "kernel". This is the product of a dampening term, $e^{-\beta u}$, and the rough kernel, $\frac{u^{\alpha-1}}{\Gamma(\alpha)}$. This kernel is called "rough" because its is involved in the construction of the rough Brownian motion (rBm). This rBm is defined as an integral $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dW_s$ where $(W_s)_{s \geq 0}$ is a Brownian motion. As explained in the following paragraphs, the dampening factor is necessary to prevent the divergence of intensities. When $\alpha = 1$, we obtain a standard bivariate Hawkes process with an exponential kernel. We assume that the dampening and roughness parameters (β and α) are the same for positive and negative jumps. While this assumption may seem restrictive, it is necessary to maintain analytical tractability in subsequent developments.

Before further studying the properties of the MERJD, we focus on the features of the dampened rough kernels. They belong to the class of Sonine functions [35] and they define operators similar to left fractional Riemann-Liouville integrals (defined in Equation 62 of Appendix A).

Definition 1. The kernel $k(u) \in L^1_{loc}(\mathbb{R}^+)$ is a Sonine function if there exists a conjugate kernel $l(u) \in L^1_{loc}(\mathbb{R}^+)$ such that

$$\int_0^t l(t-u)\, k(u)\, du = 1, \, \forall t > 0. \tag{5}$$

Let $\phi \in L^1(\mathbb{R}^+)$, the Sonine operators associated to $k(u)$ and $l(u)$ are defined as

$$(K\phi)(t) = \int_0^t k\,(t-u)\,\phi\,(u)\,du, \, \forall t \geq 0, \tag{6}$$

$$(L\phi)(t) = \int_0^t l\,(t-u)\,\phi\,(u)\,du, \, \forall t \geq 0.$$

We observe a similarity between the Riemann-Liouville integral and $(K\phi)(t) = \int_0^t e^{-\beta(t-u)} \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \phi(u) du$

$$\left(I_{0+}^{\alpha} \phi\right)(t) = \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \phi(u)\, du, \tag{7}$$

The operator K is called the tempered Riemann-Liouville (RL) integral in the literature. Meerschaert et al. [31] study the properties of this operator and of the tempered fractional diffusion.

Remark that kernels $k(u)$ and $l(u)$ are necessary unbounded as $u \to 0$. Therefore, we need to establish conditions on the kernel to ensure the existence of the integral of L_p functions. The kernel $k(u)$ can be expressed as

$$k(u) = \frac{g(u)}{u^{1-\alpha}} \quad , \quad u > 0 \, , \sup_{u \geq 0} |g(u)| < \infty \, ,$$

where $g(u) = \frac{1}{\Gamma(\alpha)} e^{-\beta\, u}$ is bounded over $\mathbb{R}^+$. From the Hardy-Littlewood [24] Sobolev inequality, a sufficient condition is then $\alpha < 1/p$. In this case, the operator acts from $L_p(\mathbb{R})$, $1 < p < 1/\alpha$ into $L_q(\mathbb{R})$ where $1/q = 1/p - \alpha$. Throughout the remainder of this article, we consider L_1-integrands which ensures that $K\phi$ is well defined for $\alpha \in [0, 1)$. We refer to Samko and Cardoso [34] for the necessary conditions of the existence of other integrals with general Sonine kernels.

We denote by $(\mathcal{L}\phi)(z) = \int_0^\infty e^{-zu} \phi(u)\, du$, the Laplace transform of a function, $\phi \in L^1(\mathbb{R}^+)$. By direct calculation, we determine the Laplace transform of the kernel:

$$(\mathcal{L}k)(z) \quad = \quad \frac{1}{(\beta + z)^\alpha}. \tag{8}$$

By Laplace rule for convolution, we find that the Laplace transform of the tempered RL integral of a function $\phi(t)$ is

$$(\mathcal{L}(K\phi))(s) = \frac{\mathcal{L}(\phi)(s)}{(\beta + z)^\alpha} \,. \tag{9}$$

Furthermore, the Sonine condition (5) is rewritten in terms of Laplace transforms of $k(\cdot)$ and $l(\cdot)$:

$$(\mathcal{L}k)(z)\,(\mathcal{L}l)(z) \quad = \quad \frac{1}{z} \,. \tag{10}$$

This last relation allows us to prove the next result.

Proposition 1. *The conjugate kernel $l(\cdot)$ of $k(\cdot)$ satisfying condition (5), is given by*

$$l(u) = \beta^\alpha + \frac{\alpha}{\Gamma(1-\alpha)} \int_u^\infty \frac{e^{-\beta s}}{s^{1+\alpha}} ds \,. \tag{11}$$

Proof. We check that the Laplace transform of $l(\cdot)$ satisfies the condition (10). We first integrate by parts the integral in Eq. (11):

$$\alpha \int_u^\infty \frac{e^{-\beta s}}{s^{1+\alpha}} ds \quad = \quad e^{-\beta u} u^{-\alpha} - \beta \int_u^\infty e^{-\beta s} s^{-\alpha}\, du \,.$$

$(\mathcal{L}l)(z)$ is next rewritten as the sum:

$$\begin{aligned}(\mathcal{L}l)(z) = \beta^\alpha \int_0^\infty e^{-zu} du + \frac{1}{\Gamma(1-\alpha)} \int_0^\infty e^{-(z+\beta)u} u^{-\alpha} du \\ - \frac{\beta}{\Gamma(1-\alpha)} \int_0^\infty \int_u^\infty e^{-\beta s} e^{-zu} s^{-\alpha}\, ds\, du \,.\end{aligned} \tag{12}$$

After a change of variable $v = (z + \beta)u$, we obtain that

$$\int_0^\infty e^{-(z+\beta)u} u^{-\alpha} du = (z + \beta)^{\alpha - 1} \Gamma(1 - \alpha) \,.$$

Using the Fubini's theorem, we change the order of integration in the last integral of Equation (12). We deduce that

$$\begin{aligned}
&\int_0^\infty \int_u^\infty e^{-\beta s} e^{-zu} s^{-\alpha} \, ds \, du \\
&\quad = \frac{1}{z} \int_0^\infty e^{-\beta s} s^{-\alpha} \, ds - \frac{1}{z} \int_0^\infty e^{-(\beta+z)s} s^{-\alpha} \, ds \\
&\quad = \frac{\beta^{\alpha-1}}{z} \Gamma(1-\alpha) - \frac{(\beta+z)^{\alpha-1}}{z} \Gamma(1-\alpha) \, .
\end{aligned}$$

Combining previous results allows us to find that the Laplace transform of the conjugate kernel of $k(\cdot)$ is $(\mathcal{L}l)(z) = \frac{(\beta+z)^\alpha}{z}$, which satisfies the condition (10). $\square$

The tempered RL integral $(K\phi)(t)$ admits an inverse operator, comparable to a fractional derivative.

Proposition 2. *The inverse operator of the tempered RL integral K, is the derivative of its conjugate kernel. For $\phi \in L^1(\mathbb{R}^+)$, it is equal to*

$$\begin{aligned}
\left(K^{-1}\phi\right)(t) &= \frac{d}{dt} (L\phi)(t) \\
&= \frac{d}{dt} \int_0^t l(t-u)\, \phi(u) \, du \, .
\end{aligned} \tag{13}$$

This inverse operator is referred to as "tempered RL derivative".

Proof. This result is a consequence of the Sonine condition. We apply the operator L to $K\phi$ and switch the order of integration. If we do the change of variable $v = s - u$, we have

$$\begin{aligned}
(LK\phi)(t) &= \int_0^t l(t-s) \int_0^s k(s-u)\phi(u) \, du \, ds \\
&= \int_0^t \phi(u) \int_0^{t-u} l(t-s)k(s-u) \, ds \, du \\
&= \int_0^t \phi(u) \int_0^{t-u} l(t-u-v)k(v) \, dv \, du \, .
\end{aligned}$$

From the Sonine condition (5), the inner integral is equal to 1. Deriving both sides with respect to t, leads to the conclusion, $K^{-1}\phi = \frac{d}{dt}(L\phi)$. $\square$

The tempered RL integral and derivative will be later involved in the construction of the Laplace transform of the MERJD, in Section 4. Before doing so, we determine the expected intensities from which we infer conditions of stability.

Proposition 3. *Let us note $u_j(t) = \mathbb{E}_0\left(\lambda_t^{(j)}\right)$ for $j = 1, 2$. The expected intensities at time $t \geq 0$ conditional on the filtration $\mathcal{F}_0$, are given by*

$$\mathbb{E}_0 \begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} (Ku_1)(t) \\ (Ku_2)(t) \end{pmatrix} .$$

where $(Ku_1)(t)$ and $(Ku_2)(t)$ are equal to

$$\begin{cases}
(Ku_1)(t) = \int_0^t \left(\lambda_0^{(1)} + \eta_{12}(Ku_2)(s)\right) e^{-\beta(t-s)}(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\eta_{11}(t-s)^\alpha\right) ds \, , \\
(Ku_2)(t) = \int_0^t \left(\lambda_0^{(2)} + \eta_{21}(Ku_1)(s)\right) e^{-\beta(t-s)}(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\eta_{22}(t-s)^\alpha\right) ds \, ,
\end{cases} \tag{14}$$

and $E_{\alpha,\alpha}(\cdot)$ is the two parametric Mittag Leffler function (see Appendix for definition and properties).

Proof. From Eq (4) and by definition of the tempered RL integral, we have

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} (Ku_1)(t) \\ (Ku_2)(t) \end{pmatrix} . \tag{15}$$

From Eq. (9), the Laplace transform of Ku_j , for $j = 1, 2$, is equal to $(s+\beta)^{-\alpha} U_j(s)$ where $U_j(s) = \int_0^\infty e^{-s\,t} u_j(t)\, dt$. Therefore, the Laplace transform of the system (15) is equal to

$$\begin{cases} U_1(s) &= \frac{\lambda_0^{(1)}}{s} + \eta_{11} (s+\beta)^{-\alpha} U_1(s) + \eta_{12} (s+\beta)^{-\alpha} U_2(s) , \\ U_2(s) &= \frac{\lambda_0^{(2)}}{s} + \eta_{21} (s+\beta)^{-\alpha} U_1(s) + \eta_{22} (s+\beta)^{-\alpha} U_2(s) . \end{cases}$$

Rearranging terms, allows us to express $U_1(s)$ and $U_2(s)$ as follows

$$\begin{cases} \frac{U_1(s)}{(s+\beta)^\alpha} &= \frac{\lambda_0^{(1)}}{s} \frac{1}{((s+\beta)^\alpha - \eta_{11})} + \frac{\eta_{12}}{((s+\beta)^\alpha - \eta_{11})} \frac{U_2(s)}{(s+\beta)^\alpha} , \\ \frac{U_2(s)}{(s+\beta)^\alpha} &= \frac{\lambda_0^{(2)}}{s} \frac{1}{((s+\beta)^\alpha - \eta_{22})} + \frac{\eta_{21}}{((s+\beta)^\alpha - \eta_{22})} \frac{U_1(s)}{(s+\beta)^\alpha} . \end{cases} \tag{16}$$

Given that $\frac{U_j(s)}{(s+\beta)^\alpha} = \mathcal{L}(Ku_j)(s)$ and $\mathcal{L}\left(e^{-\beta t} t^{\alpha-1} E_{\alpha,\alpha}(\pm\eta t^\alpha)\right)(s) = \frac{1}{(s+\beta)^\alpha \mp \eta}$ we obtain the system (14). □

The intensities are by construction unstable since the DR kernel diverges at origin. The next proposition presents the conditions that ensure the existence of counting processes.

Proposition 4. *If the parameters defining $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$ fulfill the following three conditions*

$$\begin{array}{c} \beta^\alpha > \eta_{11} \quad , \quad \beta^\alpha > \eta_{22} , \\ (\beta^\alpha - \eta_{11})(\beta^\alpha - \eta_{22}) > \eta_{12}\eta_{21} , \end{array} \tag{17}$$

the expected intensities admit limits when $t \to \infty$, that are:

$$\begin{pmatrix} \lambda_\infty^{(1)} \\ \lambda_\infty^{(2)} \end{pmatrix} = \lim_{t\to\infty} \mathbb{E}_0 \begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_0^{(1)}(\beta^\alpha - \eta_{22})\beta^\alpha + \eta_{12}\lambda_0^{(2)}\beta^\alpha}{(\beta^\alpha - \eta_{11})(\beta^\alpha - \eta_{22}) - \eta_{12}\eta_{21}} \\ \frac{\lambda_0^{(2)}(\beta^\alpha - \eta_{11})\beta^\alpha + \eta_{21}\lambda_0^{(1)}\beta^\alpha}{(\beta^\alpha - \eta_{11})(\beta^\alpha - \eta_{22}) - \eta_{12}\eta_{21}} \end{pmatrix} . \tag{18}$$

Proof. From Equations (16), we deduce that the Laplace transform $U_1(s)$ of $\mathbb{E}_0\left(\lambda_t^{(1)}\right)$ is equal to

$$\begin{aligned} \frac{U_1(s)}{(s+\beta)^\alpha} &= \frac{\lambda_0^{(1)}}{s\left((s+\beta)^\alpha - \eta_{11}\right)} + \frac{\eta_{12}\lambda_0^{(2)}}{s\left((s+\beta)^\alpha - \eta_{11}\right)\left((s+\beta)^\alpha - \eta_{22}\right)} \\ &\quad + \frac{\eta_{12}\eta_{21}U_1(s)}{\left((s+\beta)^\alpha - \eta_{11}\right)\left((s+\beta)^\alpha - \eta_{22}\right)(s+\beta)^\alpha} . \end{aligned}$$

If we isolate $U_1(s)$, the Laplace transform of $\mathbb{E}_0\left(\lambda_t^{(1)}\right)$ is

$$U_1(s) = \frac{\lambda_0^{(1)}\left((s+\beta)^\alpha - \eta_{22}\right)(s+\beta)^\alpha + \eta_{12}\lambda_0^{(2)}(s+\beta)^\alpha}{s\left(\left((s+\beta)^\alpha - \eta_{11}\right)\left((s+\beta)^\alpha - \eta_{22}\right) - \eta_{12}\eta_{21}\right)} .$$

According to the final value theorem, if $(\mathcal{L}f)(s)$ is the Laplace transform of $f(t)$, then $\lim_{s\to 0^+} s(\mathcal{L}f)(s) = f(\infty)$ if all poles of $s(\mathcal{L}f)(s)$ are in the left half-plane. This property allows us to retrieve $\lambda_\infty^{(1)}$:

$$\begin{aligned} \lim_{t\to\infty} u_1(t) &= \lim_{s\to 0^+} sU_1(s) \\ &= \frac{\lambda_0^{(1)}(\beta^\alpha - \eta_{22})\beta^\alpha + \eta_{12}\lambda_0^{(2)}\beta^\alpha}{(\beta^\alpha - \eta_{11})(\beta^\alpha - \eta_{22}) - \eta_{12}\eta_{21}} . \end{aligned}$$

$\lambda_\infty^{(2)}$ is obtained in the same manner. By definition, these expectations must be positive and bounded and therefore conditions (17) must be fulfilled. □

By construction, the MERJD is not a Markov process.However, we can represent the model as an infinite-dimensional Markov process because the power $x^{\alpha-1}$ has an integral representation. This step is detailed in the next proposition.

Proposition 5. *For $j=1,2$, let us consider a family of auxiliary jump processes $Z_t^{(j,\xi)}$, indexed by $\xi \in \mathbb{R}^+$:*

$$Z_t^{(j,\xi)} = \int_0^t e^{-(\beta+\xi)(t-s)} dN_s^{(j)} . \tag{19}$$

Let us denote $\gamma(d\xi) := \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi$ for $\xi \geq 0$. The intensities $\lambda_t^{(j)}$ are expressed as integrals of $Z_t^{(j,\xi)}$ with respect to $\gamma(d\xi)$:

$$\begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_t^{(1,\xi)} \gamma(d\xi) \\ \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_t^{(2,\xi)} \gamma(d\xi) \end{pmatrix} . \tag{20}$$

Proof. We check by direct integration that $x^{\alpha-1}$ is the following integral

$$x^{\alpha-1} = \int_0^\infty e^{-x\xi} \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi .$$

By construction, the process $Z_t^{(j,\xi)}$ is an Ornstein-Uhlenbeck jump process, reverting toward 0 and ruled by the dynamic

$$dZ_t^{(j,\xi)} = -(\beta+\xi) Z_t^{(j,\xi)} dt + dN_t^{(j)} . \tag{21}$$

This process has a finite expectation for all $\xi \in \mathbb{R}^+$. Furthermore the integrands in the next Equation being bounded functions, we can use the first version of the Fubini's theorem (see e.g. Theorem 64, p207 of Protter [33]) for semimartingales to express the intensities as:

$$\begin{aligned} \begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} &= \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \int_0^{t-} \int_0^\infty e^{-(\xi+\beta)(t-s)} \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi \, dN_s^{(1)} \\ \int_0^{t-} \int_0^\infty e^{-(\xi+\beta)(t-s)} \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi \, dN_s^{(2)} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_t^{(1,\xi)} \gamma(d\xi) \\ \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_t^{(2,\xi)} \gamma(d\xi) \end{pmatrix} . \end{aligned} \tag{22}$$

□

From Equation (21), we check that $Z_t^{(j,\xi)}$ is the sum of $Z_u^{(j,\xi)}$ for $t \geq u$ and of stochastic integral from u to t with respect to $N_s^{(j)}$:

$$Z_t^{(j,\xi)} = Z_u^{(j,\xi)} e^{-(\beta+\xi)(t-u)} + \int_u^t e^{-(\beta+\xi)(t-s)} \, dN_s^{(j)} . \tag{23}$$

Injecting this in Equation (22) allows us to rewrite $\lambda_t^{(j)}$ as the sum of $\lambda_u^{(j)}$ and of linear combinations of integrals from u to t with respect to $N_s^{(j)}$ and $\gamma(d\xi)$:

$$\begin{aligned} \begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} &= \begin{pmatrix} \lambda_u^{(1)} \\ \lambda_u^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \left[\begin{pmatrix} \int_u^t e^{-\beta(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \, dN_s^{(1)} \\ \int_u^t e^{-\beta(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \, dN_s^{(2)} \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_u^{(1,\xi)} \left(e^{-(\beta+\xi)(t-u)} - 1 \right) \gamma(d\xi) \\ \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_u^{(2,\xi)} \left(e^{-(\beta+\xi)(t-u)} - 1 \right) \gamma(d\xi) \end{pmatrix} \right] . \end{aligned} \tag{24}$$

We infer that the infinite-dimensional process $(Y_t)_{t\geq 0} := \left(\left(\lambda_t^{(j)}\right)_{j=1,2}, \left(Z_t^{(j,\xi)}\right)_{j=1,2\,,\,\xi\in\mathbb{R}^+}, \left(N_t^{(j)}\right)_{j=1,2} \right)_{t\geq 0}$ can be rewritten for $u \leq t$ as the sum of Y_u and a vector of increments depending only of evolution of $(Y_t)_{t\geq 0}$ between u and t. Therefore, there exists a probability measure $\mathbb{P}$ on the domain of $(Y_t)_{t\geq 0}$ such that

$$\mathbb{P}(Y_t \in B \,|\, \mathcal{F}_u) \;=\; \mathbb{P}(t-u, Y_u, B)\,,$$

and the process is Markov. A similar rewriting of a non-Markov Hawkes process is used in Hainaut [21] for kernels having a spectral representation. The main differences with this article are that the considered kernels do not diverge at the origin and that equivalent processes to $Z_t^{(j,\xi)}$ are defined in the complex plane. This method is also used in [1] for approximating rough volatility models.

3 Finite-dimensional approximation

In this section, we approximate the integral in (20) on a finite grid of processes $Z_t^{(j,\xi)}$. This method allows us to find the Laplace transform of the log-return $(X_t)_{t\geq 0}$, using Itô's calculus. For this purpose, we approach $\gamma(\cdot)$ by a discrete measure on a finite numbers of atoms. We consider a partition $\mathcal{E}^{(n)} := \{0 < \xi_0^{(n)} < \xi_1^{(n)} < ... < \xi_n^{(n)} < \infty\}$. The mid point of the interval $(\xi_l^{(n)}, \xi_{l+1}^{(n)})$ is

$$b_l \;=\; \frac{\xi_l^{(n)} + \xi_{l+1}^{(n)}}{2}\,, l \in \{0, ..., n-1\} \tag{25}$$

The mass of atoms at b_l, is defined as the integral of $\gamma(\cdot)$ over the interval :

$$w_l = \int_{\xi_l^{(n)}}^{\xi_{l+1}^{(n)}} \gamma(dz)\,, l \in \{0, ..., n-1\}\,. \tag{26}$$

If δ_{b_l} is the Dirac measure located at point b_l, the discrete measure $\gamma_n(\cdot)$ approximating $\gamma(\cdot)$ is defined as

$$\gamma_n(d\xi) \;=\; \sum_{l=0}^{n-1} w_l \delta_{b_l}(\xi)\, d\xi.$$

In a similar manner to [5], the partition $\mathcal{E}^{(n)}$ satisfies three conditions:

1. $\lim_{n\to\infty} \xi_0^{(n)} = 0$ and $\lim_{n\to\infty} \xi_n^{(n)} = \infty$,
2. $\lim_{n\to\infty} \max |\xi_{i+1}^{(n)} - \xi_i^{(n)}| = 0$,
3. $\mathcal{E}^{(n)} \subset \mathcal{E}^{(n+1)}$.

Proposition 6. *Under assumptions (1)-(2)-(3), a non-negative and convex function $g(.)$, $\gamma-$integrable, is such that*

$$\lim_{n\to\infty} \int_0^\infty g(\xi)\gamma_n(d\xi) = \int_0^\infty g(\xi)\gamma(d\xi)\,,$$

and the convergence is monotone increasing.

The sketch of the proof is provided e.g. in [5]. We note $\tilde{Z}_t^{(j,l)} := Z_t^{(j,b_l)}$ for $j = 1,2$ and $l = 1, ...n-1$. Each $\tilde{Z}_t^{(j,l)}$ is mean reverting and ruled by the SDE

$$d\tilde{Z}_t^{(j,l)} = \left(-\left(\beta + b_l\right) \tilde{Z}_t^{(j,l)}\right) dt + d\tilde{N}_t^{(j)} \quad j \in \{1,2\},$$

where $\tilde{N}_t^{(j)}$ is the counting process in the finite-dimensional model. Its intensity, noted $\tilde{\lambda}_t^{(j)}$, is the sum of $\tilde{Z}_t^{(j,l)}$, weighted by the mass of atoms:

$$\begin{pmatrix} \tilde{\lambda}_t^{(1)} \\ \tilde{\lambda}_t^{(2)} \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}_0^{(1)} \\ \tilde{\lambda}_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \tilde{Z}_t^{(1,l)} \\ \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \tilde{Z}_t^{(2,l)} \end{pmatrix}.$$

The next corollary states the convergence of $\tilde{\lambda}_t^{(j)}$ to $\lambda_t^{(j)}$.

Corollary 1. *Under the assumptions of Proposition 6, it holds for $t \geq 0$ that $\lim_{n\to\infty} \tilde{\lambda}_t^{(j)}(\omega) = \lambda_t^{(j)}(\omega)$, almost surely.*

Proof. Let us consider fixed sample paths $\tilde{N}_s^{(j)}(\omega)$ for $s \in [0,t]$ where $\omega \in \Omega$. Thus,

$$\tilde{Z}_t^{(j,\xi)}(\omega) = \int_0^t e^{-(\beta+\xi)(t-s)}\, d\tilde{N}_s^{(j)}(\omega)$$

is a positive function of ξ and is decreasing and convex since

$$\frac{\partial}{\partial \xi} \tilde{Z}_t^{(j,\xi)}(\omega) \leq 0\,, \ \frac{\partial^2}{\partial \xi^2} \tilde{Z}_t^{(j,\xi)}(\omega) \geq 0\,.$$

Therefore, from Proposition 6,

$$\int_0^\infty \tilde{Z}_t^{(j,\xi)}(\omega)\gamma_n(d\xi) \to \int_0^\infty Z_t^{(j,\xi)}(\omega)\gamma(d\xi)\,,$$

with $N_s^{(j)}(\omega) = \tilde{N}_s^{(j)}(\omega)$. □

The infinitesimal dynamics of jump intensities are then equal to

$$\begin{pmatrix} d\tilde{\lambda}_t^{(1)} \\ d\tilde{\lambda}_t^{(2)} \end{pmatrix} = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \left(-\sum_{l=0}^{n-1} \frac{w_l\,(\beta+b_l)}{\Gamma(\alpha)} \begin{pmatrix} \tilde{Z}_t^{(1,l)} \\ \tilde{Z}_t^{(2,l)} \end{pmatrix} dt + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \begin{pmatrix} d\tilde{N}_t^{(1)} \\ d\tilde{N}_t^{(2)} \end{pmatrix} \right) \quad (27)$$

In the discrete framework, the approximate log-return, denoted $\left(\tilde{X}_t\right)_{\geq 0}$ is driven by the following SDE:

$$d\tilde{X}_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t + \sum_{j=1}^{2} \left(d\tilde{L}_t^{(j)} - \mu_j \tilde{\lambda}_t^{(j)} dt\right). \quad (28)$$

Applying Itô's lemma to $\tilde{S}_t = S_0 e^{\tilde{X}_t}$, allows us to show that the approximate asset price follows a geometric dynamic:

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \mu dt + \sigma dW_t + \sum_{j=1}^{2} \left(\left(e^{J^{(j)}} - 1\right) d\tilde{N}_t^{(j)} - \mu_k \tilde{\lambda}_t^{(j)} dt\right). \quad (29)$$

The next proposition provides the Laplace transform of the approximate MERJD conditional on the filtration of $\mathcal{F}_t$, in terms of backward ordinary differential equations (ODE's). To clarify developments, we adopt bold symbols for all vectors, i.e.

$$\tilde{\boldsymbol{\lambda}}_t = \begin{pmatrix} \tilde{\lambda}_t^{(1)} \\ \tilde{\lambda}_t^{(2)} \end{pmatrix}, \ \tilde{\boldsymbol{Z}}_t^{(l)} = \begin{pmatrix} \tilde{Z}_t^{(1,l)} \\ \tilde{Z}_t^{(2,l)} \end{pmatrix}, \ \boldsymbol{\eta}_{.,j} = \begin{pmatrix} \eta_{1j} \\ \eta_{2j} \end{pmatrix}.$$

Proposition 7. *Let $\omega \in \mathbb{R}^+$. The Laplace function of the MERJD log-return $\tilde{X}_t$ at time t, conditional on $\mathcal{F}_t$, is given by the following expression*

$$\mathbb{E}\left(e^{-\omega \tilde{X}_s} \,|\, \mathcal{F}_t\right) = \exp\left(q_0(s-t) + \boldsymbol{q}_\lambda(s-t)^\top \tilde{\boldsymbol{\lambda}}_t + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \boldsymbol{q}_l(s-t)^\top \tilde{\boldsymbol{Z}}_t^{(l)} - \omega \tilde{X}_t\right) \quad (30)$$

where

$$q_0(s-t) = -\omega\left(\mu - \frac{\sigma^2}{2}\right)(s-t) + \omega^2 \frac{\sigma^2}{2}(s-t)\,, \quad (31)$$

and vector of functions $\boldsymbol{q}_\lambda(u), \boldsymbol{q}_l(u) : \mathbb{R}^+ \to \mathbb{R}^2$, solve ODE's:

$$\begin{cases} \frac{d\,q_\lambda^{(j)}(u)}{du} = \omega\mu_j + \left(\mathcal{J}_j(-\omega)\exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)}\left(\boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(u) + q_l^{(j)}(u)\right)\right) - 1\right)\,, \\ \frac{d\,q_l^{(j)}(u)}{du} = -\left(\beta + b_l\right)\left(q_l^{(j)}(u) + \boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(u)\right)\,. \end{cases} \quad (32)$$

with the initial conditions $q_\lambda^{(j)}(0) = 0$ and $q_l^{(j)}(0) = 0$ for $j \in \{1,2\}$ and $l = 0, ..., n-1$. We recall that $\mathcal{J}_j(.)$ is the mgf of jump sizes as defined in equation (2).

Proof. We denote by $f\left(t, \tilde{X}_t, \tilde{\boldsymbol{\lambda}}_t, \left(\tilde{\boldsymbol{Z}}_t^{(l)}\right)_{l=0,\dots,n-1}, \tilde{\boldsymbol{L}}_t\right)$, the Laplace transform $\mathbb{E}\left(e^{-\omega \tilde{X}_s} | \mathcal{F}_t\right)$. $f(\cdot)$ being a conditional expectation, it is also a martingale and $\mathbb{E}\left(df \,|\, \mathcal{F}_t\right) = 0$. From Itô's lemma, we infer that $f(\cdot)$ satisfies a stochastic differential equation (SDE):

$$\begin{aligned} 0 = {} & \partial_t f + \left(\mu - \frac{\sigma^2}{2} - \mu_1 \tilde{\lambda}_t^{(1)} - \mu_2 \tilde{\lambda}_t^{(2)}\right) \partial_{\tilde{X}} f + \frac{\sigma^2}{2} \partial_{\tilde{X}\tilde{X}} f \\ & - \sum_{l=0}^{n-1} (\beta + b_l)\left(\tilde{Z}_t^{(1,l)} \partial_{\tilde{Z}_t^{(1,l)}} f + \tilde{Z}_t^{(2,l)} \partial_{\tilde{Z}_t^{(2,l)}} f\right) \\ & - \sum_{l=0}^{n-1} \frac{w_l(\beta + b_l)}{\Gamma(\alpha)} \sum_{j=1}^{2} \left(\eta_{j1} \tilde{Z}_t^{(1,l)} + \eta_{j2} \tilde{Z}_t^{(2,l)}\right) \partial_{\tilde{\lambda}^{(j)}} f + \\ & \sum_{j=1}^{2} \tilde{\lambda}_t^{(j)} \int_0^\infty f\left(t, \tilde{\boldsymbol{\lambda}}_t + \frac{\sum_{l=0}^{n-1} w_l}{\Gamma(\alpha)} \begin{pmatrix} \eta_{1j} \\ \eta_{2j} \end{pmatrix}, \left(\tilde{Z}_t^{(j,l)} + 1\right)_l, \tilde{X}_t + z\right) - f(.)\, m^{(j)}(dz) \end{aligned} \quad (33)$$

As all processes are affine, we make the Ansatz that $f(\cdot)$ is an exponential affine function of the type

$$f(\cdot) = \exp\left(q_0(s-t) + \boldsymbol{q}_\lambda(s-t)^\top \tilde{\boldsymbol{\lambda}}_t + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \boldsymbol{q}_l(s-t)^\top \tilde{\boldsymbol{Z}}_t^{(l)} - \omega \tilde{X}_t\right)\,. \quad (34)$$

The partial derivatives of $f(\cdot)$ with respect to state variables are given by

$$\begin{aligned} \partial_{\tilde{Z}^{(j,l)}} f &= f \frac{w_l}{\Gamma(\alpha)} q_l^{(j)}(s-t)\,, \\ \partial_{\tilde{\lambda}_j} f &= f\, q_\lambda^{(j)}(s-t)\,, \\ \partial_{\tilde{X}} f &= -\omega f\,,\ \partial_{\tilde{X}\tilde{X}} f = \omega^2 f\,, \end{aligned}$$

whereas, the derivative of f with respect to time is equal to

$$\partial_t f(\cdot) = f(\cdot)\left(\partial_t q_0(s-t) + \partial_t \boldsymbol{q}_\lambda(s-t)^\top \tilde{\boldsymbol{\lambda}}_t + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \partial_t \boldsymbol{q}_l(s-t)^\top \tilde{\boldsymbol{Z}}_t^{(l)}\right)\,.$$

Under the assumption (34), the jump terms in Equation (33) are developed in the following way:

$$f\left(t, \tilde{\boldsymbol{\lambda}}_t + \frac{\sum_{l=0}^{n-1} w_l}{\Gamma(\alpha)} \begin{pmatrix} \eta_{1j} \\ \eta_{2j} \end{pmatrix}, \left(\tilde{Z}_t^{(j,l)} + 1\right)_l, \tilde{X}_t + z\right) - f(\cdot) =$$
$$f(.) \left(\exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(\boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(s-t) + q_l^{(j)}(s-t)\right) - \omega z\right) - 1\right).$$

From previous equations, we infer that $q_0(\cdot)$, $\boldsymbol{q}_\lambda(\cdot)$ and $\boldsymbol{q}_l(\cdot)$ satisfy:

$$\begin{aligned} 0 =\ & \partial_t q_0(s-t) + \partial_t \boldsymbol{q}_\lambda(s-t)^\top \tilde{\boldsymbol{\lambda}}_t + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \partial_t \boldsymbol{q}_l(s-t)^\top \tilde{\boldsymbol{Z}}_t^{(l)} - \omega\left(\mu - \frac{\sigma^2}{2}\right) \\ & + \frac{\omega^2\sigma^2}{2} - \sum_{l=0}^{n-1} \frac{(\beta+b_l)\, w_l}{\Gamma(\alpha)} \left(\tilde{Z}_t^{(1,l)} q_l^{(1)}(s-t) + \tilde{Z}_t^{(2,l)} q_l^{(2)}(s-t)\right) \\ & - \sum_{l=0}^{n-1} \frac{(\beta+b_l)\, w_l}{\Gamma(\alpha)} \sum_{j=1}^{2} \left(\eta_{j1} \tilde{Z}_t^{(1,l)} + \eta_{j2} \tilde{Z}_t^{(2,l)}\right) q_\lambda^{(j)}(s-t) + \omega \sum_{j=1}^{2} \tilde{\lambda}_t^{(j)} \mu_j + \\ & \sum_{j=1}^{2} \tilde{\lambda}_t^{(j)} \left(\int_0^\infty \exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(\boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(s-t) + q_l^{(j)}(s-t)\right) - \omega z\right) - 1\, m^{(j)}(dz)\right). \end{aligned}$$

Grouping and cancelling terms independent of state variables leads to the expression (31) for $q_0(\cdot)$. Cancelling terms multiplying $\tilde{Z}_t^{(j,l)}$ give us for $j = 1, 2$,

$$\begin{aligned} 0 = & \frac{w_l}{\Gamma(\alpha)} \partial_t q_l^{(j)}(s-t) - \frac{w_l\,(\beta+b_l)}{\Gamma(\alpha)} \\ & \times \left(\eta_{1j} q_\lambda^{(1)}(s-t) + \eta_{2j} q_\lambda^{(2)}(s-t) + q_l^{(j)}(s-t)\right), \end{aligned}$$

that is the second equation in (32). The first equation in (32) is obtained by cancelling terms multiplying $\tilde{\lambda}_t^{(j)}$. □

The next corollary states that functions $\boldsymbol{q}_l(u)$ admit an integral representation.

Corollary 2. *The function $q_l^{(j)}(u)$ solving the ODE's in Equation (32) are equal to*

$$q_l^{(j)}(u) = -\int_0^u (\beta+b_l)\, e^{-(\beta+b_l)(u-v)} \boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(v)\, dv \quad , j = 1, 2. \tag{35}$$

This result is checked by deriving the expression of $q_l^{(j)}(u)$ with respect to u. In this way, we find the first ODE in (32). In Hainaut ([22], Chapter 5), the characteristic function of a non-Markov Hawkes process is retrieved by considering the limit of the partition $\mathcal{E}^{(n)}$. We cannot apply the same approach for the dampened rough kernel. The reason is that the limit of the sum of w_l, involved in Equation (32), is not defined when $n \to \infty$:

$$\lim_{n\to\infty} \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi = \infty\,.$$

Nevertheless, we prove in the next section that the Laplace transform of the rough point process seen from time $t = 0$, admits a closed form expression. To establish this result, we need additional intermediate results.

Corollary 3. *The derivative of the function q_λ solves the following equation:*

$$\frac{dq_\lambda^{(j)}(s-t)}{ds} = \omega\mu_j + \mathbb{E}\left(e^{-\omega J^{(j)}}\right)\exp\left(\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\int_0^{s-t} e^{-(\beta+b_l)(s-t-v)}\boldsymbol{\eta}_{.,j}^\top \frac{d\boldsymbol{q}_\lambda(v)}{dv}dv\right) - 1 \quad (36)$$

Proof. As $-\frac{1}{\beta+b_l}\frac{d\,q_l^{(j)}(u)}{du} = q_l^{(j)}(u) + \boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(u)$, the first ODE in (32) may be rewritten as

$$\frac{d\,q_\lambda^{(j)}(u)}{du} = \omega\mu_j + \left(\mathcal{J}_j(-\omega)\exp\left(-\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)\,(\beta+b_l)}\frac{d\,q_l^{(j)}(u)}{du}\right) - 1\right)$$

Given that $\frac{dq_\lambda^{(j)}(s-t)}{ds} = \left.\frac{dq_\lambda^{(j)}(u)}{du}\right|_{u=s-t}$ and $\frac{dq_l^{(j)}(s-t)}{ds} = \left.\frac{dq_l^{(j)}(u)}{du}\right|_{u=s-t}$, this is equivalent to

$$\frac{d\,q_\lambda^{(j)}(s-t)}{ds} = \omega\mu_j + \left(\mathcal{J}_j(-\omega)\exp\left(-\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)\,(\beta+b_l)}\frac{d\,q_l^{(j)}(s-t)}{ds}\right) - 1\right). \quad (37)$$

From the integral representation (35) of $q_l^{(j)}(u)$, we infer that

$$q_l^{(j)}(s-t) = -\int_0^{s-t}(\beta+b_l)\,e^{-(\beta+b_l)\,(s-t-v)}\boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(v)\,dv \quad , j=1,2.$$

After the change of variable $v = s-u$, we rewrite $q_l^{(j)}(s-t)$ as follows:

$$q_l^{(j)}(s-t) = -\int_t^s (\beta+b_l)\,e^{-(\beta+b_l)\,(u-t)}\boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(s-u)\,du,$$

As $q_\lambda(0) = 0$, the derivative of $q_l^{(j)}(s-t)$ with respect to s is given by

$$\frac{dq_l^{(j)}(s-t)}{ds} = -\int_t^s (\beta+b_l)\,e^{-(\beta+b_l)\,(u-t)}\boldsymbol{\eta}_{.,j}^\top \frac{d\boldsymbol{q}_\lambda(s-u)}{ds}du\,.$$

Doing a last change of variable, $v = s-u$, leads to Equation (36). □

From this last proposition, we will infer the Laplace transform of the MERJD by considering the limit.

4 Laplace transform of the MERJD log-return

We now have the necessary tools for computing the Laplace transform of the MERJD process, X_t. The next proposition states that its initial value depends on a function solving a fractional differential equation involving the tempered RL integral and derivative such as defined in Section 2.

Proposition 8. *The Laplace transform of the log-return $(X_s)_{s\geq 0}$, conditional on $\mathcal{F}_0$, for $\omega \in \mathbb{R}^+$, is equal to*

$$\Upsilon_s(-\omega) := \mathbb{E}\left(e^{-\omega X_s}\,|\,\mathcal{F}_0\right) = \exp\left(-\left(\omega\left(\mu - \frac{\sigma^2}{2}\right) - \frac{\omega^2\sigma^2}{2}\right)s + \boldsymbol{q}_\lambda(s)^\top \boldsymbol{\lambda}_0\right), \quad (38)$$

where $q_\lambda^{(j)}(s)$ for $j=1,2$ solves a forward ODE:

$$\frac{dq_\lambda^{(j)}(s)}{ds} = \omega\mu_j + \mathcal{J}_j(-\omega)\exp\left(\boldsymbol{\eta}_{.,j}^\top\left(K\frac{d\boldsymbol{q}_\lambda}{ds}\right)(s)\right) - 1 \quad (39)$$

with the initial condition $q_\lambda^{(j)}(0) = 0$ *and where* $K\frac{dq_\lambda^{(j)}}{ds}$ *is the tempered Riemann-Liouville integral of* $\frac{dq_\lambda^{(j)}}{ds}$*:*

$$\left(K\frac{dq_\lambda^{(j)}}{ds}\right)(s) = \frac{1}{\Gamma(\alpha)}\int_0^s \frac{e^{-\beta(s-u)}}{(s-u)^{1-\alpha}}\frac{dq_\lambda^{(j)}(u)}{du}\,du.$$

An equivalent representation is obtained by defining $\psi^{(j)}(s) := \left(K\frac{dq_\lambda^{(j)}}{ds}\right)(s)$*. this function solves the fractional differential equation*

$$\left(K^{-1}\psi^{(j)}\right)(s) \quad = \quad \omega\mu_j + \mathcal{J}_j(-\omega)\exp\left(\boldsymbol{\eta}_{.,k}^\top\,\boldsymbol{\psi}(s)\right) - 1\,, \tag{40}$$

where $\left(K^{-1}\psi^{(j)}\right)(s) = \frac{dq_\lambda^{(j)}}{ds}(s)$ *is the tempered Riemann-Liouville derivative of* $\psi^{(j)}(s)$*:*

$$\left(K^{-1}\psi^{(j)}\right)(s) = \frac{d}{ds}\int_0^s\left(\beta^\alpha + \frac{\alpha}{\Gamma(1-\alpha)}\int_{s-u}^\infty\frac{e^{-\beta v}}{v^{1+\alpha}}dv\right)\psi^{(j)}(u)\,du\,.$$

Proof. We consider the limit of the exponential term in Equation (36) when the size of the partition $\mathcal{E}^{(n)}$ tends to infinity. By construction, the following limit is well defined and given by:

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\int_0^{s-t} e^{-(\beta+b_l)(s-t-v)}\boldsymbol{\eta}_{.,j}^\top\frac{d\boldsymbol{q}_\lambda(v)}{dv}dv\\
&=\frac{1}{\Gamma(\alpha)}\int_0^{s-t}\int_0^\infty e^{-\xi(s-t-v)}\,\gamma(d\xi)e^{-\beta(s-t-v)}\boldsymbol{\eta}_{.,j}^\top\frac{d\boldsymbol{q}_\lambda(v)}{dv}\,dv\\
&=\boldsymbol{\eta}_{.,j}^\top\int_0^{s-t}\frac{e^{-\beta(s-t-v)}}{\Gamma(\alpha)(s-t-v)^{1-\alpha}}\frac{d\boldsymbol{q}_\lambda(v)}{dv}\,dv\,.
\end{aligned}$$

At time $t=0$, i.e. conditional on $\mathcal{F}_0$, we recognize the the tempered Riemann-Liouville of $\partial_s q_\lambda$:

$$\left(K\frac{d\boldsymbol{q}_\lambda}{ds}\right)(s) \quad = \quad \int_0^s\frac{e^{-\beta(s-u)}}{\Gamma(\alpha)(s-u)^{1-\alpha}}\frac{d\boldsymbol{q}_\lambda(u)}{du}\,du\,, \tag{41}$$

and combining Equations (36) and (41) leads to the fractional equation (39).

$$\frac{dq_\lambda^{(j)}(s)}{ds} \quad = \quad \omega\mu_j + \mathbb{E}\left(e^{-\omega J^{(j)}}\right)\exp\left(\boldsymbol{\eta}_{.,j}^\top\left(K\frac{d\boldsymbol{q}_\lambda}{ds}\right)(s)\right) - 1$$

Given that $K^{-1}K\phi = \phi$, We immediately infer that $\boldsymbol{\psi}(s) = \left(K\frac{d\boldsymbol{q}_\lambda}{ds}\right)(s)$ and Equation (65). ☐

When $\alpha\to 1$, the rough jump process converges toward a standard Hawkes process with an exponential kernel. In this case, the tempered RL integral of $\frac{d\boldsymbol{q}_\lambda}{ds}$ converges toward the following integral

$$\lim_{\alpha\to1}\left(K\frac{d\boldsymbol{q}_\lambda}{ds}\right)(s) \quad = \quad \int_0^s e^{-\beta(s-u)}\frac{d\boldsymbol{q}_\lambda(u)}{du}\,du\,,$$

and from Equation (36), $\frac{dq_\lambda^{(j)}}{ds}$ solves the integro-differential equation:

$$\frac{dq_\lambda^{(j)}(s)}{ds} \quad = \quad \omega\mu_j + \mathcal{J}_j(-\omega)\exp\left(\boldsymbol{\eta}_{.,j}^\top\int_0^s e^{-\beta(s-u)}\frac{d\boldsymbol{q}_\lambda(u)}{du}\,du\right) - 1\,. \tag{42}$$

In practice, we numerically solve Equation (39). We divide $[0, s]$ in n subintervals $[s_k, s_{k+1}]$ of length Δ, for $k = 0, ..., n-1$. We denote by $\boldsymbol{g}(k) := \left.\frac{d\boldsymbol{q}_\lambda(s)}{ds}\right|_{s=s_k}$, the derivative of $\boldsymbol{q}_\lambda$ at time s_k and we next use an explicit approximation of the tempered RL fractional integral :

$$g^{(j)}(k) = \omega\mu_j + \mathbb{E}\left(e^{-\omega J^{(j)}}\right) \exp\left(\frac{\boldsymbol{\eta}_{.,j}^\top}{\Gamma(\alpha)} \sum_{u=0}^{k-1} \frac{e^{-\beta(s_k - s_u)}}{(s_k - s_u)^{1-\alpha}} \boldsymbol{g}(u)\,\Delta \right) - 1\,. \tag{43}$$

The recursion is initialized by setting $g^{(j)}(0) = \omega\mu_j + \mathbb{E}\left(e^{-\omega J^{(j)}}\right) - 1$. We can utilize the previous results to calculate the probability density function of the log-return $(X_t)_{t\geq 0}$. Our approach relies on a discrete fast Fourier transform (DFFT). We invert the characteristic function of the process, which is the Laplace transform (38) evaluated on the imaginary axis. Let's denote the characteristic function of X_s as $\Upsilon_s(i\omega) = \mathbb{E}\left(e^{i\omega X_s} \,|\, \mathcal{F}_0\right)$ for $\omega \in \mathbb{R}$. This function is also the inverse Fourier transform of the probability density function (pdf) $f_s^X(x)$ of $X_s|\mathcal{F}_0$. Therefore, we can retrieve the density by computing the following integral (the Fourier transform, $\mathcal{F}[\cdot]$, of $\Upsilon_s(\cdot)$):

$$\begin{aligned} f_s^X(x) &= \frac{1}{2\pi}\mathcal{F}[\Upsilon_s(i\omega)](x) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Upsilon_s(i\omega) e^{-i\omega x} d\omega\,. \end{aligned} \tag{44}$$

This integral is approximated by discretization with the DFFT algorithm recalled in Appendix B.

5 Change of measure

To avoid arbitrage, derivatives valuation is conducted under an equivalent measure known as the risk-neutral measure. Under this measure, discounted asset prices behave as martingales. However, our model features market incompleteness due to the presence of multiple non-traded risk factors. This incompleteness leads to the existence of various equivalent measures, all of which could potentially serve as candidates for defining a risk-neutral measure.

We begin by considering changes of measure within the finite-dimensional approximation presented in Section 3. Similar to the approach described in Section 4, the dynamics of the MERJD under the transformed measure will be obtained by taking the limit of $\lim_{n\to\infty} \mathcal{E}^{(n)}$. We focus on a family of measure changes induced by exponential martingales in the form:

$$\begin{aligned} \tilde{M}_t &= \exp\left(-\frac{1}{2}\int_0^t \varphi(s)^2 ds - \int_0^t \varphi(s) dW_s\right) \times \\ &\quad \exp\left(\sum_{j=1}^2 \left[\zeta_j \tilde{L}_t^{(j)} + (1 - \mathcal{J}_j(\zeta_j)) \int_0^t \tilde{\lambda}_s^{(j)} ds\right]\right)\,, \end{aligned} \tag{45}$$

where $\varphi(t)$ is a $\mathcal{F}_t$-adapted process such that $\int_0^t |\varphi(s)|^2 ds < \infty$ and $\zeta_j \in \mathbb{R}$ are such that $\mathcal{J}_j(\zeta_j) < \infty$ for $j = 1, 2$. Let us recall that the moment generating function of jumps is denoted by $\mathcal{J}_j(\omega) = \mathbb{E}\left(e^{\omega J^{(j)}}\right)$ for $j = 1, 2$. We can check that $\tilde{M}_t$ is a local martingale. Using the Itô's lemma allows us to infer that

$$\begin{aligned} d\tilde{M}_t &= -\tilde{M}_t \varphi(t) dW_t + \tilde{M}_t \sum_{j=1}^2 (1 - \mathcal{J}_j(\zeta_j))\, \tilde{\lambda}_t^{(j)} dt \\ &\quad + \tilde{M}_t \sum_{j=1}^2 \left(\exp\left(\zeta_j J^{(j)}\right) - 1\right) dN_t^{(j)}\,. \end{aligned}$$

The expectation being null, $\mathbb{E}(d\tilde{M}_t)=0$, the integral of $d\tilde{M}_t$ is a local martingale. We denote by $\mathbb{Q}$ the measure defined by the change of measure (45). This new measure preserves the structure of the MERJD.

Proposition 9. *For $j=1,2$,let us denote by $\tilde{N}_t^{Q(j)}$ the counting processes of intensity*

$$\tilde{\lambda}_t^{Q(j)} = \mathcal{J}_j(\zeta_j)\tilde{\lambda}_t^{(j)}\,, \tag{46}$$

We also define random variables $J^{Q(j)}$, through their moment generating functions under the measure $\mathbb{Q}$:

$$\mathcal{J}_j^{Q}(\omega)=\mathbb{E}^{\mathbb{Q}}\left(e^{\omega J^{Q(j)}}\right) = \frac{\mathcal{J}_j(\omega+\zeta_j)}{\mathcal{J}_j(\zeta_j)}\,,\ j=1,2\,, \tag{47}$$

and processes $\tilde{L}_t^{Q(j)}=\sum_{k=1}^{\tilde{N}_t^{Q(j)}} J_k^{Q(j)}$. Under the measure $\mathbb{Q}$, the dynamic of the log-return is ruled by the following SDE

$$d\tilde{X}_t=\left(\mu-\frac{\sigma^2}{2}-\sigma\varphi(t)\right)dt+\sigma dW_t^Q+ \sum_{j=1}^{2}\left(d\tilde{L}_t^{Q(j)}-\frac{\mu_j}{\mathcal{J}_j(\zeta_j)}\int_0^t\tilde{\lambda}_s^{Q(j)}ds\right)\,, \tag{48}$$

where $dW_t^Q=dW_t+\sigma\varphi(t)dt$. Furthermore, the intensities $\tilde{\lambda}_t^{Q(j)}$ are such that

$$\begin{pmatrix}\tilde{\lambda}_t^{Q(1)}\\ \tilde{\lambda}_t^{Q(2)}\end{pmatrix}=\begin{pmatrix}\mathcal{J}_1(\zeta_1)\tilde{\lambda}_0^{(1)}\\ \mathcal{J}_2(\zeta_2)\tilde{\lambda}_0^{(2)}\end{pmatrix}+\begin{pmatrix}\eta_{11}\mathcal{J}_1(\zeta_1) & \eta_{12}\mathcal{J}_1(\zeta_1)\\ \eta_{21}\mathcal{J}_2(\zeta_2) & \eta_{22}\mathcal{J}_2(\zeta_2)\end{pmatrix}\begin{pmatrix}\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\tilde{Z}_t^{Q(1,l)}\\ \sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\tilde{Z}_t^{Q(2,l)}\end{pmatrix},$$

where $\tilde{Z}_t^{Q(j,l)}=\int_0^t e^{-(\beta+\xi)(t-s)}\,d\tilde{N}_s^{Q(j)}$.

Proof. $\tilde{M}_t$ is product of a Brownian and jump changes of measure. The impact of the Brownian measure change on the drift of $\tilde{X}_t$ being standard, we set $\varphi(t)=0$ and consider a change of measure $\tilde{M}_t=\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t=\exp\left(Y_t\right)$ where

$$dY_t = \sum_{j=1}^{2}\left(1-\mathcal{J}_j(\zeta_j)\right)\tilde{\lambda}_t^{(j)}dt+\sum_{j=1}^{2}\zeta_j\,d\tilde{L}_t^{(j)}\,.$$

Under the $\mathbb{Q}$-measure, the Laplace transform of $\tilde{X}_s$ conditional on $\mathcal{F}_t$ is equal to

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left(e^{-\omega\tilde{X}_s}\,|\,\mathcal{F}_t\right) &= \frac{\mathbb{E}\left(\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_s e^{-\omega\tilde{X}_s}\,|\,\mathcal{F}_t\right)}{\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t}\\ &= e^{-Y_t}\mathbb{E}\left(e^{-\omega\tilde{X}_s+Y_s}\,|\,\mathcal{F}_t\right)\,.\end{aligned}$$

We denote $\mathbb{E}\left(e^{-\omega\tilde{X}_s+Y_s}\,|\,\mathcal{F}_t\right)$ by $f\left(t,\tilde{X}_t,\tilde{\boldsymbol{\lambda}}_t,\left(\tilde{\boldsymbol{Z}}_t^{(l)}\right)_{l=0,\dots,n-1},\tilde{\boldsymbol{L}}_t,Y_t\right)$. From Itô's lemma, $f(\cdot)$

satisfies the following stochastic differential equation (SDE):

$$
\begin{aligned}
0 = {} & \partial_t f + \left(\mu - \frac{\sigma^2}{2} - \mu_1 \tilde{\lambda}_t^{(1)} - \mu_2 \tilde{\lambda}_t^{(2)}\right) \partial_{\tilde{X}} f + \frac{\sigma^2}{2} \partial_{\tilde{X}\tilde{X}} f \\
& + \partial_Y f \sum_{j=1}^{2} \left(1 - \mathcal{J}_j(\zeta_j)\right) \tilde{\lambda}_t^{(j)} - \sum_{l=0}^{n-1} (\beta + b_l) \sum_{j=1}^{2} \tilde{Z}_t^{(j,l)} \partial_{\tilde{Z}_t^{(j,l)}} f \\
& - \sum_{l=0}^{n-1} \frac{w_l (\beta + b_l)}{\Gamma(\alpha)} \sum_{j=1}^{2} \left(\eta_{j1} \tilde{Z}_t^{(1,l)} + \eta_{j2} \tilde{Z}_t^{(2,l)}\right) \partial_{\tilde{\lambda}^{(j)}} f + \sum_{j=1}^{2} \tilde{\lambda}_t^{(j)} \times \\
& \int_0^{\infty} f\left(t, \tilde{\boldsymbol{\lambda}}_t + \sum_{l=0}^{n-1} \frac{w_l \mathcal{J}_k(\zeta_k)}{\Gamma(\alpha)} \boldsymbol{\eta}_{.,j}, \left(\tilde{Z}_t^{(j,l)} + 1\right)_l, \tilde{X}_t + z, Y_t + \zeta_1 z\right) - f(.)\, m^{(j)}(dz)
\end{aligned}
\tag{49}
$$

We do the Ansatz that $f(\cdot)$ is an exponential affine function of risk factors:

$$
f(\cdot) = \exp\left(q_0(s-t) + \boldsymbol{q}_\lambda(s-t)^\top \left(\tilde{\boldsymbol{\lambda}}_t \odot \boldsymbol{\mathcal{J}}(\boldsymbol{\zeta})\right) + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \boldsymbol{q}_l(s-t)^\top \tilde{\boldsymbol{Z}}_t^{(l)} - \omega \tilde{X}_t + q_Y(s-t) Y_t\right),
$$

where $\tilde{\boldsymbol{\lambda}}_t \odot \boldsymbol{\mathcal{J}}(\boldsymbol{\zeta})$ is the element-wise or Hadamard product of $\tilde{\boldsymbol{\lambda}}_t$ and $\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) = (\mathcal{J}_1(\zeta_1), \mathcal{J}_2(\zeta_2))$. The partial derivatives of $f(\cdot)$ with respect to state variables are given by

$$
\begin{aligned}
\partial_{\tilde{Z}^{(j,l)}} f &= f \frac{w_l}{\Gamma(\alpha)} q_l^{(j)}(s-t), \\
\partial_{\tilde{\lambda}_j} f &= \mathcal{J}_j(\zeta_j)\, f\, q_\lambda^{(j)}(s-t), \\
\partial_{\tilde{X}} f &= -\omega f, \ \partial_{\tilde{X}\tilde{X}} f = \omega^2 f, \\
\partial_Y f &= f q_Y(s-t).
\end{aligned}
$$

The derivative of f with respect to time is equal to

$$
\partial_t f = f(\cdot)\left(\partial_t q_0(s-t) + \partial_t \boldsymbol{q}_\lambda(s-t)^\top \tilde{\boldsymbol{\lambda}}_t \odot \boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \partial_t \boldsymbol{q}_l(s-t)^\top \tilde{\boldsymbol{Z}}_t^{(l)} + Y_t \partial_t q_Y(s-t)\right).
$$

The jump terms in Equation (49) become for $j = 1, 2$:

$$
\begin{aligned}
& f\left(t, \tilde{\boldsymbol{\lambda}}_t + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}, \left(\tilde{Z}_t^{(j,l)} + 1\right)_l, \tilde{X}_t + z, Y_t + \zeta_1 z\right) - f(.) = \\
& \quad f(.)\left(\exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(\left(\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}\right)^\top \boldsymbol{q}_\lambda(s-t) + q_l^{(j)}(s-t)\right) - \omega z + \zeta_j z\, q_Y(s-t)\right) - 1\right).
\end{aligned}
$$

Combining the previous equations in Equation (49) allows us to infer that $\partial_t q_Y(s-t) = 0$ and

$q_Y(s-t)=1$. Therefore, $q_0(\cdot)$, $\boldsymbol{q}_\lambda(\cdot)$ and $\boldsymbol{q}_l(\cdot)$ satisfy the relation

$$
\begin{aligned}
0 &= \partial_t q_0(s-t) + \partial_t \boldsymbol{q}_\lambda(s-t)^\top \tilde{\boldsymbol{\lambda}}_t \odot \boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \partial_t \boldsymbol{q}_l(s-t)^\top \tilde{\boldsymbol{Z}}_t^{(l)} \\
&+ Y_t \partial_t q_Y(s-t) - \omega\left(\mu - \frac{\sigma^2}{2} - \mu_1 \tilde{\lambda}_t^{(1)} - \mu_2 \tilde{\lambda}_t^{(2)}\right) + \omega^2 \frac{\sigma^2}{2} \\
&+ \sum_{j=1}^{2} \left(1 - \mathcal{J}_j(\zeta_j)\right) \tilde{\lambda}_t^{(j)} - \sum_{l=0}^{n-1} \frac{(\beta + b_l)\, w_l}{\Gamma(\alpha)} \left(\tilde{Z}_t^{(1,l)} q_l^{(1)}(s-t) + \tilde{Z}_t^{(2,l)} q_l^{(2)}(s-t)\right) \\
&- \sum_{l=0}^{n-1} \frac{(\beta + b_l)\, w_l}{\Gamma(\alpha)} \sum_{j=1}^{2} \mathcal{J}_j(\zeta_j) \left(\eta_{j1} \tilde{Z}_t^{(1,l)} + \eta_{j2} \tilde{Z}_t^{(2,l)}\right) q_\lambda^{(j)}(s-t) + \sum_{j=1}^{2} \tilde{\lambda}_t^{(j)} \times \\
&\int_0^\infty \exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(\left(\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}\right)^\top \boldsymbol{q}_\lambda(s-t) + q_l^{(j)}(s-t)\right) - \omega z + \zeta_j z\, q_Y(s-t)\right) - 1\, m^{(j)}(dz)
\end{aligned}
$$

After grouping terms, we infer that $q_0(s-t)$ is equal to Equation (31). Cancelling terms multiplying $\tilde{Z}_t^{(j,l)}$ give us for $j=1,2$,

$$
0 = \frac{w_l}{\Gamma(\alpha)} \partial_t q_l^{(j)}(s-t) - \frac{w_l\,(\beta + b_l)}{\Gamma(\alpha)} \left(\left(\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}\right)^\top \boldsymbol{q}_\lambda(s-t) + q_l^{(j)}(s-t)\right),
$$

which corresponds to the second Equation (32) with parameters $\boldsymbol{\eta}_{.,k}^\top$ elementwise multiplied by $\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta})$. Cancelling terms multiplying $\tilde{\lambda}_t^{(j)}$ leads to

$$
\partial_t q_\lambda^{(j)}(s-t) \;=\; -\omega\mu_j - \left(\frac{\mathcal{J}_j(\zeta_j - \omega)}{\mathcal{J}_j(\zeta_j)} \exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(\left(\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}\right)^\top \boldsymbol{q}_\lambda(s-t) + q_l^{(j)}(s-t)\right)\right) - 1\right).
$$

After a change of variable, $u = s-t$, we obtain

$$
\frac{d\, q_\lambda^{(j)}(u)}{du} \;=\; \omega\mu_j + \left(\frac{\mathcal{J}_j(\zeta_j - \omega)}{\mathcal{J}_j(\zeta_j)} \exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(\left(\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}\right)^\top \boldsymbol{q}_\lambda(u) + q_l^{(j)}(u)\right)\right) - 1\right),
$$

which is similar to the first Equation in (32). The Laplace transform $\mathbb{E}^{\mathbb{Q}}\left(e^{-\omega \tilde{X}_s} \,|\, \mathcal{F}_t\right)$ has therefore the same form as the one of $\mathbb{E}\left(e^{-\omega \tilde{X}_s} \,|\, \mathcal{F}_t\right)$, provided in Proposition 7 with intensities (46), jumps (47) for $j=1,2$. □

When jumps are exponential random variables with pdf's of Equation (3), the distribution of jumps under $\mathbb{Q}$ remains exponential as stated in the next corollary.

Corollary 4. *If $J^{(j)} \sim expo(\rho_j)$ then for $\zeta_1 \in (-\rho_1, +\infty)$ and $\zeta_2 \in (-\infty, -\rho_2)$, $J^{Q(j)}$ are distributed as exponential random variable with parameters $\rho_j^Q = \rho_j - \zeta_j$, under $\mathbb{Q}$, for $j=1,2$.*

This result is a direct consequence of Equation (47) and combined with $\mathcal{J}_j(\omega) = \frac{\rho_j}{\rho_j - \omega}$ for $j=1,2$, with $\omega \le \rho_1$ and $\omega \ge \rho_2$. Another direct consequence of Proposition 9 is the possibility to identify the familly of measure changes defining a risk neutral measure, i.e. a measure under which the discounted asset price is a martingale. We consider a constant discount rate, denoted by $r \in \mathbb{R}^+$.

Corollary 5. *The equivalent measures $\mathbb{Q}$ defined by the change of measure (45) are risk neutral if*

$$
\varphi(t) \;=\; \frac{\mu - r}{\sigma} + \sum_{j=1}^{2} \frac{\tilde{\lambda}_t^{(j)} \mathcal{J}_j(\zeta_j) \left(\mathbb{E}\left(e^{J^{Q(j)}}\right) - 1\right) - \mu_j \int_0^t \tilde{\lambda}_s^{(j)} ds}{\sigma}, \tag{50}
$$

where r is the discount rate.

Proof. Replacing the expression (50) of $\varphi(t)$ into the dynamic (48) of $\tilde{X}_t$ under $\mathbb{Q}$, allows us to infer that

$$d\tilde{X}_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW_t^Q + \sum_{j=1}^{2} \left(d\tilde{L}_t^{Q(j)} - \tilde{\lambda}_t^{Q(j)} \mathbb{E}\left(e^{J^{Q(j)}} - 1\right) dt\right) . \tag{51}$$

If we remember that the asset price is $\tilde{S}_t = \tilde{S}_0 e^{\tilde{X}_t}$, using Itô's lemma leads to the following SDE:

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = rdt + \sigma dW_t^Q + \sum_{k=1}^{2} \left[\left(e^{J^{Q(k)}} - 1\right) d\tilde{N}_t^{Q(k)} - \mathbb{E}\left(e^{J^{Q(k)}} - 1\right) \tilde{\lambda}_t^{Q(k)} dt\right] . \tag{52}$$

Since $\mathbb{E}^{\mathbb{Q}}\left(\frac{d\tilde{S}_t}{\tilde{S}_t}\right) = rdt$, the discounted price $e^{-rt}\tilde{S}_t$ is a well martingale under $\mathbb{Q}$ and $\mathbb{Q}$ is risk neutral. □

By construction, when the size n of the partition $\mathcal{E}^{(n)}$ tends to $+\infty$, the process $\tilde{X}_t$ converges toward X_t. From Equation (51), we deduce that the MERJD, X_t, under the risk neutral measure is the following sum:

$$X_t = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t^Q + \sum_{j=1}^{2} \left(L_t^{Q(j)} - \mu_j^Q \int_0^t \lambda_s^{Q(j)} ds\right) , \tag{53}$$

where $\mu_j^Q = \left(\mathbb{E}\left(e^{J^{Q(j)}}\right) - 1\right)$ for $j = 1, 2$. The structure of X_t under $\mathbb{Q}$ and $\mathbb{P}$ are similar. Therefore, the Laplace transform of the log-return under $\mathbb{Q}$ is provided by Proposition 8 with updated parameters:

$$\Upsilon_s^Q(\omega) := \mathbb{E}^{\mathbb{Q}}\left(e^{-\omega X_s} \,|\, \mathcal{F}_0\right) = \exp\left(-\left(\omega\left(r - \frac{\sigma^2}{2}\right) - \frac{\omega^2\sigma^2}{2}\right) s + \boldsymbol{q}_\lambda^Q(s)^\top \boldsymbol{\lambda}_t\right) , \tag{54}$$

where $q_\lambda^{Q(j)}(s)$ for $j = 1, 2$ solves a forward ODE:

$$\frac{dq_\lambda^{Q(j)}(s)}{ds} = \omega\mu_j^Q + \mathcal{J}_j^Q(-\omega) \exp\left(\left(\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}\right)^\top \left(K\frac{d\boldsymbol{q}_\lambda^Q}{ds}\right)(s)\right) - 1 \tag{55}$$

with the initial condition $q_\lambda^{Q(j)}(0) = 0$. Inverting the characteristic function by DFFT , as explained in Section 4, allow us to retrieve the pdf of X_t under the risk neutral measure and to price European options on the corresponding asset. The influence of roughness parameters, α and β, on call prices is studied in Section 7.

6 Econometric estimation

Estimating the parameters of a MERJD is a challenging task since jumps are not directly observable. For this reason, we adopt a peak-over-threshold approach for detecting jumps, similar to that in [13].

The discrete record of p asset log-returns, equally spaced with a lag Δ, is noted $\{x_1, x_1, x_2, ..., x_p\}$. The times of observation are $\{s_0, s_1, ..., s_p\}$. We assume that a jump occurs when the log-return is above or below some thresholds. These thresholds are denoted by $g(\alpha_1)$, $g(\alpha_2)$ and depend on two confidence levels, α_1 and α_2. To define thresholds, we first fit a pure Gaussian process: $x_k \sim \mu_g\Delta + \sigma_g W_\Delta$ to time-series. The unbiased estimators of μ_g and σ_g are:

$$\widehat{\mu}_g = \frac{1}{p\Delta}\sum_{j=1}^{p} x_j \qquad \widehat{\sigma}_g^2 = \frac{1}{(p-1)\Delta}\sum_{j=1}^{p}(x_j - \widehat{\mu}_g)^2 . \tag{56}$$

If $\Phi(.)$ denotes the cdf of a standard normal, $g(\alpha_1)$, $g(\alpha_2)$ are set to the α_1 and α_2 percentiles of the Brownian motion: $g(\alpha_i) = \widehat{\mu}_g\Delta + \widehat{\sigma}_g\sqrt{\Delta}\Phi^{-1}(\alpha_i)$ for $i = 1, 2$. The times of the k^{th} jump of $L_t^{(1)}$ and $L_t^{(2)}$ are :

$$\begin{aligned}\tau_k^{(1)} &= \min\{s_j \in \{s_1, ..., s_p\} \,|\, x_j \geq g(\alpha_1)\,,\, s_j \geq \tau_{k-1}^{(1)}\}\,,\\ \tau_k^{(2)} &= \min\{s_j \in \{s_1, ..., s_p\} \,|\, x_j \leq g(\alpha_2)\,,\, s_j \geq \tau_{k-1}^{(2)}\}\,,\end{aligned}$$

where $\tau_0^{(1)} = \tau_0^{(2)} = 0$ and $k \in \mathbb{N}$ is bounded by p. The sample path of $\left(N_t^{(j)}\right)_{t\geq 0}$ for $j = 1, 2$ are approached by the following time series:

$$N^{(j)}(s_i) = \max\{k \in \mathbb{N} \,|\, \tau_k^{(j)} \leq s_i\}\,,\; i = 1, ..., n.$$

The levels of confidence, α_1 and α_2, are optimized such that the skewness and kurtosis of x_i for periods without jumps are close to those of a normal distribution. If the sets of times without jump is noted $\mathcal{T}$, parameter estimates of μ and σ are

$$\begin{cases}\widehat{\sigma} = \frac{1}{(|\mathcal{T}|-1)\Delta}\sum_{s_j\in\mathcal{T}}\left(x_j - \frac{1}{|\mathcal{T}|\Delta}\sum_{s_i\in\mathcal{T}} x_i\right)^2\,,\\ \widehat{\mu} = \frac{1}{|\mathcal{T}|\Delta}\sum_{s_j\in\mathcal{T}} x_j + \frac{\widehat{\sigma}}{2}\,.\end{cases}$$

We next fit the bivariate rough Hawkes process by log-likelihood maximization. We detail the calculation of this log-likelihood. From Equation (4), the sample intensities are equal to

$$\lambda_{t-}^{(j)} = \lambda_0^{(j)} + \sum_{k=1}^{2}\left(\frac{\eta_{jk}}{\Gamma(\alpha)}\sum_{\tau_u^{(k)}<t} e^{-\beta(t-\tau_u^{(k)})}(t-\tau_u^{(k)})^{\alpha-1}\right)\; j = 1, 2\,.$$

These realized intensities are involved in the calculation of the log-likelihood.

Proposition 10. *We denote the Gamma incomplete function by* $\Gamma(\alpha, x) = \int_x^\infty e^{-u}u^{\alpha-1}du$. *The log-likelihood of a sample of observations over* $[0, \mathcal{S}]$ *is defined as:*

$$\ln\mathcal{L} = \sum_{j=1}^{2}\left(-\int_0^{\mathcal{S}}\lambda_s^{(j)}ds + \sum_{k=1}^{N_{\mathcal{S}}^{(j)}}\log\left(\lambda_{\tau_k-}^{(j)}\right)\right)\,, \tag{57}$$

where the integral of the intensity is equal to

$$\int_0^{\mathcal{S}}\lambda_s^{(j)}ds = \lambda_0^{(j)}\mathcal{S} + \sum_{k=1}^{2}\frac{\eta_{jk}}{\beta^\alpha}\sum_{u=1}^{N_{\mathcal{S}}^{(k)}}\left(1 - \frac{\Gamma\left(\alpha, \beta\left(\mathcal{S}-\tau_u^{(k)}\right)\right)}{\Gamma(\alpha)}\right)\,. \tag{58}$$

Proof. From e.g. Embrechts et al. [13], the log-likelihood of the sample is given by Equation (57). Using the expression (4) of λ_t and changing the order of integration, the integral of the intensity becomes:

$$\begin{aligned}\int_0^{\mathcal{S}}\lambda_u^{(j)}du &= \lambda_0^{(j)}\mathcal{S} + \sum_{k=1}^{2}\eta_{jk}\int_0^{\mathcal{S}}\int_0^{u-} e^{-\beta(u-s)}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}dN_s^{(j)}\,du \qquad (59)\\ &= \lambda_0^{(j)}\mathcal{S} + \sum_{k=1}^{2}\eta_{jk}\int_0^{\mathcal{S}}\int_s^{\mathcal{S}} e^{-\beta(u-s)}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}du\,dN_s^{(j)}\,.\end{aligned}$$

The inner integrals are reformulated in terms of Gamma functions by a change of variable $v = \beta(u-s)$

$$\begin{aligned}
&\int_s^{\mathcal{S}} e^{-\beta(u-s)} \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} du \qquad (60)\\
&= \int_s^{\infty} e^{-\beta(u-s)} \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} du - \int_{\mathcal{S}}^{\infty} e^{-\beta(u-s)} \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} du \\
&= \frac{\beta^{-\alpha}}{\Gamma(\alpha)} \int_0^{\infty} e^{-v} v^{\alpha-1} dv - \frac{\beta^{-\alpha}}{\Gamma(\alpha)} \int_{\beta(\mathcal{S}-s)}^{\infty} e^{-v} v^{\alpha-1} dv \\
&= \beta^{-\alpha}\left(1 - \frac{\Gamma(\alpha, \beta(\mathcal{S}-s))}{\Gamma(\alpha)}\right)
\end{aligned}$$

Combining Equations (59) and (60) leads to the expression (58). □

For comparison, we consider a bivariate Hawkes process with an exponential memory kernel:

$$\lambda_t^{(j)} = \lambda_0^{(j)} + \sum_{k=1}^{2} \eta_{jk} \left(\int_0^{t-} e^{-\beta(t-s)} dN_s^{(k)} \right) .$$

The log-likelihood in this case, has the same form as Equation (57) with

$$\int_0^{\mathcal{S}} \lambda_u^{(j)} du = \lambda_0^{(j)} \mathcal{S} + \sum_{k=1}^{2} \eta_{jk} \sum_{u=1}^{N_{\mathcal{S}}^{(k)}} \left(1 - e^{-\beta\left(\mathcal{S} - \tau_u^{(k)}\right)} \right) . \qquad (61)$$

We recall that this corresponds to the rough model with $\alpha = 1$. If we denote by Θ_N, the set of parameters of intensities, their estimates are obtained by maximization of the log-likelihood (57):

$$\widehat{\Theta}_N = \arg\max_{\Theta_N} \ln \mathcal{L}(\Theta_N) .$$

The distribution $m^{(j)}(.)$ of jumps is fitted independently of counting processes. In absence of jumps, the log-return has a normal distribution. If a single jump occurs over Δ, From Chapter 1 of Hainaut [22] Proposition 1.3, the pdf of the sum $\sigma W_\Delta + J^{(j)}$ is equal to:

$$\begin{aligned}
h^{(1)}(z|\sigma, \rho_1) &= 1_{\{z \geq 0\}} \rho_1 \exp\left(\frac{1}{2}(\rho_1\sigma)^2 \Delta - \rho_1 z\right) \Phi\left(\frac{z - \rho_1\sigma^2\Delta}{\sqrt{\Delta}\sigma}\right) , \\
h^{(2)}(z|\sigma, \rho_2) &= -1_{\{z \leq 0\}} \rho_2 \exp\left(\frac{1}{2}(\rho_2\sigma)^2 \Delta - \rho_2 z\right) \left(1 - \Phi\left(\frac{z - \rho_2\sigma^2\Delta}{\sqrt{\Delta}\sigma}\right)\right) .
\end{aligned}$$

where $\Phi(.)$ is the cdf of a standard normal random variable. If $\{J_1^{(j)}, ..., J_{N_t^{(j)}}^{(j)}\}$ for $j = 1, 2$ and Θ_J are respectively the sample of jumps and the set of parameters of $m^{(j)}(.)$, estimates are found by log-likelihood maximization:

$$\widehat{\Theta}_J = \arg\max_{\Theta_J} \sum_{j=1}^{2} \sum_{k=1}^{N_t^{(j)}} h^{(j)}\left(J_k^{(j)} | \widehat{\sigma}, \rho_j\right) .$$

7 Numerical illustration

To illustrate this article, we fit the MERJD to time-series of hourly Bitcoin returns from the 9/2/2018 to 9/2/2023, traded in USD on the platform Gemini. The upper graph of Figure 1 shows hourly returns and the lower plot displays Bitcoin prices. The bitcoin is traded 24h/24h and the time interval between two successive observation is $\Delta = 1/8760$ year.

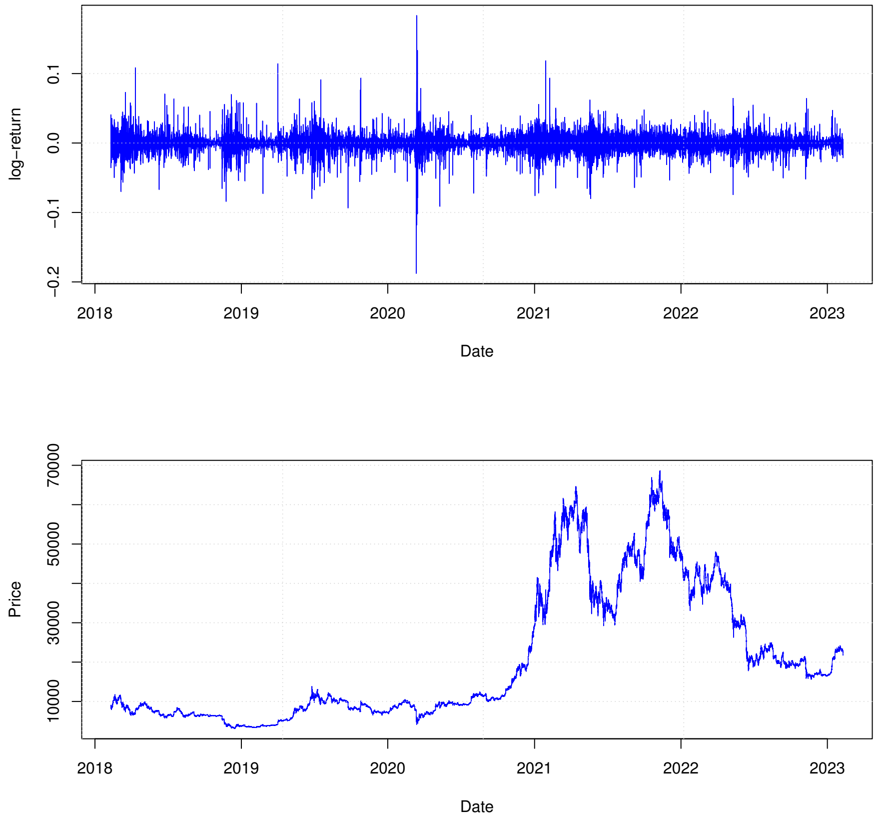


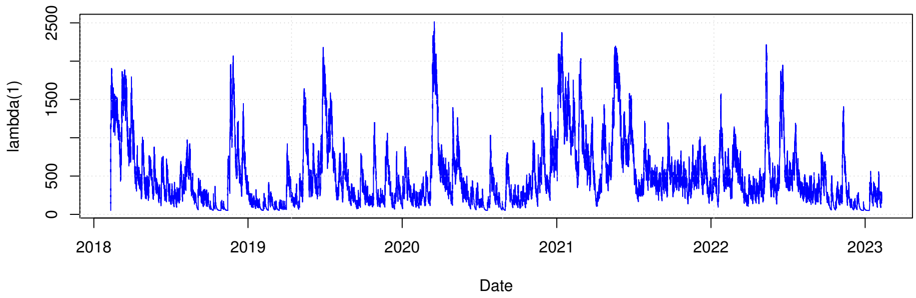
Figure 1: Hourly log-returns of Bitcoin.

Table 1 reports the average and standard deviation of hourly log-returns, such as defined in Equation (56). Jumps are detected with the POT method[1] described in Section 6. The skewness and kurtosis of log-returns for observations without jump are respectively equal to -9.19e-5 and 3.0002. The upper and lower thresholds are close to 1% in absolute values. The estimated Brownian volatility is 0.38% per hour which is large but relevant for cryptocurrencies. Figure 2 shows the sample paths of intensities, reconstructed from jumps detected by the POT method. Peaks of intensities correspond to large fluctuations of Bitcoin.

Parameters	Values	Parameters	Values
$g(\alpha_1)$	-0.9752%	$g(\alpha_2)$	1.0001%
$\widehat{\mu}_g \Delta$	0.0021%	$\widehat{\sigma}_g \sqrt{\Delta}$	0.7974%
$\widehat{\mu}\Delta$	0.0082%	$\widehat{\sigma}\sqrt{\Delta}$	0.3830%

Table 1: Mean and standard deviation of hourly returns. Thresholds for the POT method.

[1] R code available on request.



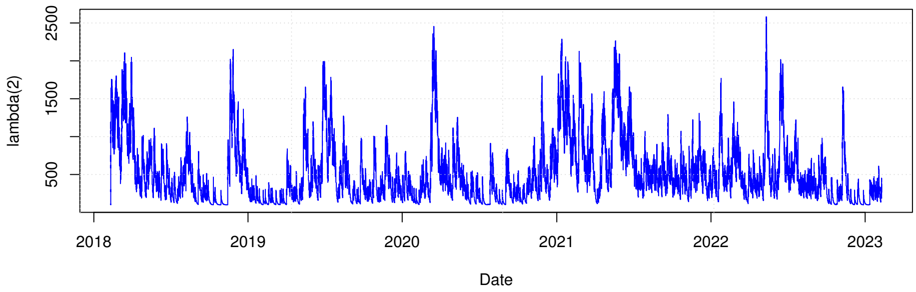


Figure 2: Sample paths of intensities $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$.

The parameter estimates of the bivariate rough jump process are provided in Table 2. The α determines the level of roughness and is around 0.90. The dampening parameter, β is close to 182. The matrix of $\eta_{i,j}$ provides valuable insights into self and mutual excitations between shocks. Negative jumps exhibit self-excitation ($\widehat{\eta}_{22} = 87.13$) but are nearly unaffected by positive jumps, as $\widehat{\eta}_{21}$ is nearly null. In contrast, positive shocks tend to be triggered by negative jumps ($\widehat{\eta}_{11} = 48.68$). In practice, this contagion can be explained by bounce trading strategies. Additionally, the level of self-excitation for positive jumps is almost half that of negative ones ($\widehat{\eta}_{11}$=48.68 whereas $\widehat{\eta}_{22}$= 87.13). The baseline intensity of negative jumps, $\widehat{\lambda}_0^{(2)}$, is twice as high as that of positive shocks, $\widehat{\lambda}_0^{(1)}$.However, the asymptotic expected intensities are similar, with both positive and negative jumps expected to occur at a rate of close to 500 times per year. In absolute values, the average sizes of positive and negative jumps are comparable, approximately $\pm 1.7\%$.

Parameters	Values	Parameters	Values
$\widehat{\alpha}$	0.9061	$\widehat{\beta}$	181.7853
$\widehat{\eta}_{11}$	48.6850	$\widehat{\eta}_{12}$	48.9050
$\widehat{\eta}_{21}$	2.0365	$\widehat{\eta}_{22}$	87.1365
$\widehat{\lambda}_0^{(1)}$	53.8714	$\widehat{\lambda}_0^{(2)}$	101.8721
$\lambda_\infty^{(1)}$	488.9064	$\lambda_\infty^{(2)}$	505.8213
Log-lik. rough Hawkes process, $\mathcal{L}(\widehat{\Theta_N})$: 28046.24			
$\widehat{\rho}_1$	59.4260	$\widehat{\rho}_2$	-57.8427
Log-lik. jumps : 15 458.44			

Table 2: Parameter estimates, rough jump process.

We compare the bivariate rough jump process to its non-rough version ($\alpha = 1$) with parameters in Table 3. The dampening factor, β and levels of self-excitation, η_{11} and η_{22}, are higher in this last model. Similar to the rough process, we observe a one-way contagion of negative jumps on positive ones. Baseline intensities and asymptotic expected intensities are comparable to those of the rough jump process.To assess the relevance of the rough jump model, we perform a log-likelihood ratio test. Under the assumption that $\alpha = 1$, the log of the squared ratio of likelihoods,

$$2\left(\ln \mathcal{L}(\widehat{\Theta}_N) - \ln \mathcal{L}(\widehat{\Theta^h_N})\right) \sim \chi^2_1$$

is (asymptotically) a chi-square random variable with one degree of freedom. Using the figures reported in Tables 2 and 3, we calculate a p-value of 0.0382%. This result confirms that the rough jump model l provides a better goodness of fit compared to its non-rough version.

Parameters	Values	Parameters	Values
$\widehat{\alpha}$	1.0000	$\widehat{\beta}$	221.0378
$\widehat{\eta}_{11}$	107.089	$\widehat{\eta}_{12}$	90.60763
$\widehat{\eta}_{21}$	0.0416	$\widehat{\eta}_{22}$	174.0245
$\widehat{\lambda}_0^{(1)}$	38.0073	$\widehat{\lambda}_0^{(2)}$	106.9566
$\lambda_\infty^{(1)}$	473.9214	$\lambda_\infty^{(2)}$	503.2868
Log-lik. Hawkes process, $\mathcal{L}(\widehat{\Theta^h_N})$: 28 039.94			

Table 3: Parameter estimates, bivariate (non-rough) Hawkes process.

To assess the contribution of jump components to the total volatility of log-return, we set σ to zero. To estimate any potential bias induced by the DFFT, we also cancel the drift by setting $\mu = 0$. Table 4 presents expectations, standard deviations and 5%-95% percentiles of X_t under these conditions for various maturities, with all other parameters equal to their estimate from Table 2). These calculations are performed using the DFFT algorithm outlined in Appendix B, with $x_{max} = 2.2$ and $M = 2^{10}$. We observe a slight bias of 0.20% for the log-return, which theoretically should have zero expectation. Increasing the value of M reduces the bias but prolongs computation time and requires adjusting x_{max} to keep Δ_ω small enough. The standard deviation of X_t ranges from 18% for a one-month horizon up to 41.2% for a half-year horizon. The 90% confidence interval of X_t is broad and expands to $[-70\%, 65\%]$ for a time horizon of 6 months. These statistics confirm that the jump component of X_t significantly contributes to the overall log-return volatility.

Days	Expectation	Standard deviation	percentiles 5%	95%
30	0.002	0.180	-0.32	0.269
60	0.002	0.262	-0.458	0.398
90	0.002	0.315	-0.548	0.484
180	0.002	0.412	-0.703	0.652

Table 4: Statistics of X_t, without Brownian activity and no drift ($\sigma = 0$ and $\mu = 0$). DFFT with $M = 2^{10}$ components (and $x_{max} = 2.2$).

To understand the impact of rough jumps on derivatives, we value European call options using the DFFT and calculate their implied volatility by inverting the Black & Scholes formula. Once more, we set the Brownian volatility and drift to zero to specifically examine the jump components of the log-return. The risk free rate is also set to zero. Options are valued with the parameters of Table 2. We assume that $S_0 = 100$ and consider strikes K from 50 to 150. Expiration dates ranges from 1 to 6 months.

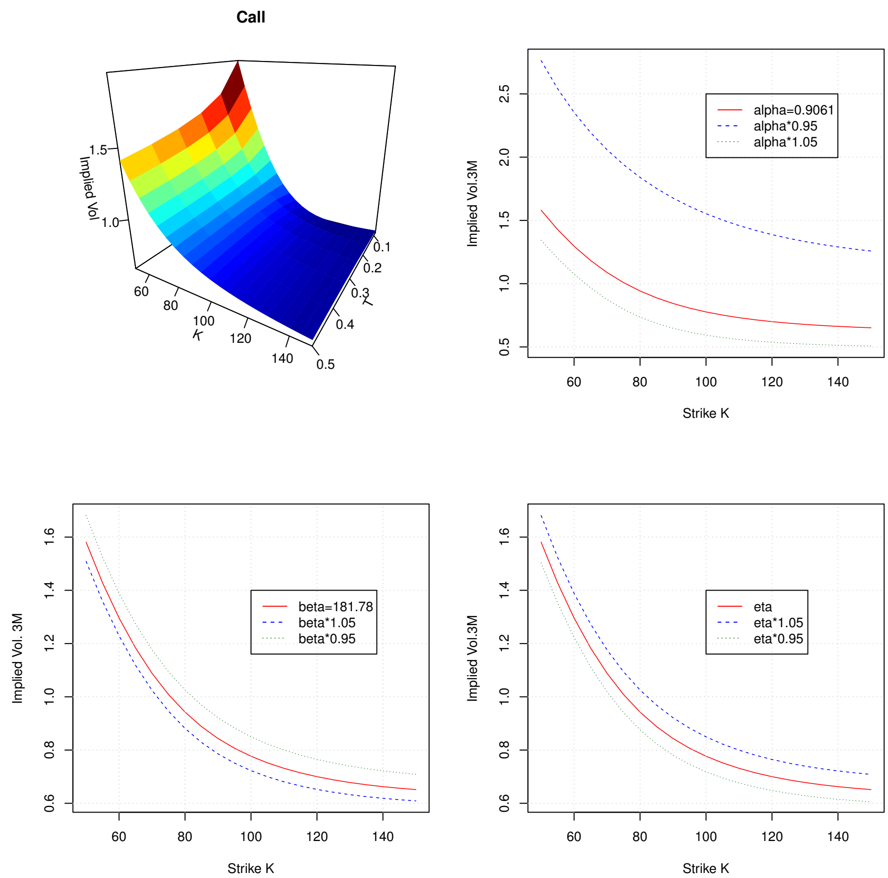


Figure 3: Implied volatility surfaces of European call options ($\sigma = 0$ and $\mu = 0$).

The upper left plot of Figure 3 shows the surface of implied volatilities. They ranges from 61.39% to 195.58% for deep 'out of' and 'in the' money options. The implied volatility generated by jumps seems broad but it is relevant with the history of market data. The implied volatility generated by jumps appears wide in range, but it aligns with historical market data. For context, the BitVol index, which measures the expected 30-day implied volatility in Bitcoin, fluctuates between 60% and 100%, with a peak reaching 168% on March 17, 2020. The upper right plot of Figure 3 illustrates 3 months implied volatilities for an α increased and decreased by 5%. Decreasing this parameter enhances the roughness of the bivariate jump process, resulting in higher implied volatilities. Conversely, increasing the dampening parameter, β, accelerates the reversion of intensities to their baseline values and subsequently reduces implied volatilities, as shown in the left lower plot of Figure 3. The last plot reveals that increasing the parameters of self and mutual excitations, contributes to higher implied volatilities.

8 Conclusions

This paper introduces a novel jump-diffusion process, the MERJD, featuring mutual excitation between positive and negative jumps, governed by a dampened rough (DR) memory kernel. This process exhibits several interesting properties for modeling asset returns with high volatility, particularly in the context of cryptocurrencies.

Even though the memory kernel diverges at zero, the process remains stable under mild conditions. The MERJD can be represented as an infinite-dimensional Markov process. By considering the limit of a finite approximation of this process, we can obtain the Laplace transform of the

MERJD. Given that the DR kernel is a Sonine function, we can define a fractional operator that resembles the Riemann-Liouville derivative. The Laplace transform of the MERJD can then be reformulated in terms of a solution to a fractional differential equation (FDE) using this new operator. This FDE can be solved numerically.

The MERJD is well-suited for pricing derivatives. We have defined a family of changes of measure that preserve the characteristics of the process and determined the conditions under which the new measure is risk-neutral. Numerical analysis indicates that the degree of roughness, as defined by α, significantly increases implied volatilities. In contrast, the dampening parameter β, accelerates the reversion of jump intensities to their baseline values, thereby reducing implied volatilities.

The MERJD can also be utilized for risk management purposes, as parameters under the real measure can be estimated using a peak-over-threshold method. This is feasible because the log-likelihood of the bivariate rough jump process has an analytical expression. The numerical illustration highlights the MERJD's capacity to evaluate price jumps in volatile assets, such as Bitcoin.

Appendix A. Mittag Leffler function

The Mittag-Leffler functions with one and two parameters are respectively defined by

$$\begin{aligned} E_\alpha(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}\,, \\ E_{\alpha,\beta}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}\,, \end{aligned}$$

where $\alpha > 0$ and $\beta \in \mathbb{C}$. The function $u(x) = E_\alpha(\eta x^\alpha)$ is closely related to fractional calculus when $\alpha \in (0,1)$. We denote by $I_{0+}^{\alpha} u$ is the following Riemann-Liouville fractional integral

$$\left(I_{0+}^{\alpha} u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds\,. \tag{62}$$

The left Riemann-Liouville derivative, denoted by $D_{0+}^{\alpha} u(t)$, is the derivative of $I_{0+}^{1-\alpha} u(t)$:

$$\left(D_{0+}^{\alpha} u\right)(t) = \frac{d\left(I_{0+}^{1-\alpha} u\right)(t)}{dt} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^{\alpha}} ds\,,$$

and is such that $D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t)$. The solution of the fractional integral/differential equations

$$\left(D_{0+}^{\alpha} u\right)(t) = \eta\, u(t)$$

is precisely the function $u(x) = E_\alpha(\eta x^\alpha)$. In this article we also use the relation

$$\frac{dE_\alpha(\eta x^\alpha)}{dx} = \eta x^{\alpha-1} E_{\alpha,\alpha}(\eta x^\alpha)\,. \tag{63}$$

From the Laplace's transform of $E_\alpha(\pm x^\alpha)$,

$$\mathcal{L}\left(E_\alpha(\pm x^\alpha)\right) := \int_0^\infty e^{-zx} E_\alpha(\pm x^\alpha)\, dx = \frac{z^{\alpha-1}}{z^\alpha \mp 1}\,, \tag{64}$$

(see Gorenflo et al. [17], pages 40 and 41), we infer that

$$\mathcal{L}\left(E_\alpha(\pm \eta x^\alpha)\right) = \frac{z^{\alpha-1}}{z^\alpha \mp \eta}\,. \tag{65}$$

Appendix B. Fast Fourier's transform

Let M be the number of steps used in the Discrete Fast Fourier's Transform (DFFT) and $\Delta_x = \frac{2x_{max}}{M-1}$, be a step of discretization. Let us denote $\Delta_\omega = \frac{2\pi}{M\,\Delta_x}$ and $\omega_j = (j-1)\Delta_\omega$ for $j = 1, ..., M+1$. The pdf of X_s at points $x_k = -\frac{M}{2}\Delta_x + (k-1)\Delta_x$ for $k = 1, ..., M$ are approached by

$$f_s^X(x_k) = \frac{2}{M\,\Delta_x}\sum_{j=1}^{M}\delta_j \Upsilon_s(i\omega_j)\exp\left(i\left((k-1)\pi\right)\right) \times \exp\left(-i(k-1)(j-1)\frac{2\pi}{M}\right), \quad (66)$$

where$\delta_j = \left(\frac{1}{2}\right)^{1_{\{j_1=1\}}} + 1_{\{j\neq 1\}}$ and $\Upsilon_s(\omega) = \mathbb{E}\left(e^{\omega X_s} \mid \mathcal{F}_0\right)$.

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The author declares that he has no conflicts of interest.

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