Conditioning of Integrable Determinantal Point Processes

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Determinantal Point Processes and Operators of Integrable Form

Introduction

The present thesis deals with topics at the intersection of three domains, namely (stochastic) point processes, the theory of traces/determinants of operators and lastly integrable structures. The latter only has a precise definition when it is restricted to a certain domain, such as differential geometry, PDE theory, quantum physics etc.; the particular structure of interest here consists of integral operators of integrable form, i.e. whose kernel admits a certain representation, and the point processes they induce via determinants, which are then called determinantal. Point processes can be understood as random discrete measures or equivalently as probability measures on a space of discrete measures, typically on $\mathbb{R}$. The main topic of this thesis is motivated by the operation of conditioning arising in probability theory, which roughly means that, by taking into account additional information, a new probability measure is created from the former one $\mathbb{P}$. Here we will condition a point process on what we call a randomly incomplete observation of its points, which then changes the probability distribution of the remaining unobserved points then denoted by $\mathbb{P}_{|}$ (compared to the precise notation $\mathbb{P}^{\theta}_{|\mathbf{v}}$ used below, we have removed in this section the observation's data $(\theta, \mathbf{v})$ for the sake of clarity). When we deal with a determinantal point process $\mathbb{P}$ induced by an integral operator K (see Section , Example 0.0.2 below for the precise statement), the following question arises naturally:

(1) Is the conditioned point process $\mathbb{P}_{|}$ determinantal as well?

We show that under mild conditions this is indeed the case; this then induces a "conditioning" $\mathrm{K} \mapsto \mathrm{K}_{|}$ at the level of operators, where $\mathrm{K}_{|}$ induces $\mathbb{P}_{|}$. In turn this raises a slightly more practical question:

(2) How can we obtain $\mathrm{K}_{|}$ from K?

Under stronger assumptions which constrain our study to observations resulting in finitely many points, this transformation can be described using a kind of resolvent procedure well-known in operator theory, and the kernel of the resolvent operator so-obtained can be explicitly given using the classical notions of Fredholm determinants/minors of trace-class operators. Returning to integrability, assuming that the kernel of K admits an integrable representation in terms of (f, g) (see (0.0.75) below) we are naturally drawn to wonder

(3) Does the integral operator $\mathrm{K}_{|}$ admits an integrable representation $(f_{|}, g_{|})$?

along with a once again more practical pondering

(4) How can we obtain $(f_{|}, g_{|})$ from (f, g)?

The former question is rather technical: the algebraic computations leading to the identification of potential candidates for $(f_{|}, g_{|})$ are rather simple, yet verifying that these are indeed well defined demands proper assumptions on both the observation and the domain of K. The latter question to construct $(f_{|}, g_{|})$ in terms of (f, g) reveals a connection to the theory of Riemann-Hilbert problems; those arise in complex analysis, and deal with holomorphic functions having certain singularities, namely poles with associated Laurent coefficients, together with boundary values along a curve, which differ depending on the side from which we approach this curve; those quantities encoding the singularities are then required to be related by a certain relations. Here we show that the transition $(f, g) \mapsto (f_{|}, g_{|})$ can be characterized by the unique solution to such a Riemann-Hilbert problem, which is a generalisation of a classical result used many times during the last two decades. A summary of the context as well as the questions which arise and that we answer is given by the following diagram:

$$
\begin{tikzcd}
\text{Probability measure } \mathbb{P} \arrow[r, "\text{Conditioning on}", "\text{an observation}"'] & \text{New probability measure } \mathbb{P}_{|} \arrow[d, "\text{(1) Is it}"', "\text{determinantal?}"] \\
\text{Integral operator } \mathrm{K} \arrow[u, "\text{Induces}"] \arrow[r, "\text{(2) How to obtain}", "\mathrm{K}_{|} \text{ from } \mathrm{K}?"'] & \text{New integral operator } \mathrm{K}_{|} \arrow[d, "\text{(3) Is it of}"', "\text{integrable form?}"] \\
\begin{array}{c}\text{Integrable structure}\\ (f, g)\end{array} \arrow[u, "\text{Induces}"] \arrow[r, "\text{(4) How to obtain}", "(f_{|}, g_{|}) \text{ from } (f, g)?"'] & \begin{array}{c}\text{New integrable structure}\\ (f_{|}, g_{|})\end{array}
\end{tikzcd}
$$

This thesis is informally divided into two parts: Chapter and then Chapters 1, 2 and 3. The first is an informal introduction to the concept of point process, Fredholm determinant and operators of integrable form. We thereafter examine simple historical examples which gave rise to the notion of (determinantal) point process. This takes us from the early 20th century and the foundations of probability theory with its applications to insurance, as well as Fredholm's study of integral equations and its generalisation of determinants to operators, to the late 1990s and the connection of integrable probabilistic models to integrable ordinary/partial differential equations. That part is concluded by a collection of fundamental mathematical definitions and results about (determinantal) point processes used throughout this thesis. The second part consists of our original contributions, the first and main chapter being an adaptation of our second article [58] wherein we answer precisely the aforementioned questions. It is then followed by our third article [60] wherein we utilised this machinery to generalise well-known results concerning the Airy point process as well as the Painlevé II and Korteweg-de Vries equations. Finally we conclude with our first and stand-alone paper [59] concerning asymptotic results for determinantal point process related to orthogonal groups.

Historical examples

Point processes

Point processes can be roughly understood as a collection (more precisely a set) of random points $\{X_j\}_{j=1:J}$ (J can be finite or infinite, but the latter case is more complicated), say on the real line $X_j \in \mathbb{R}$, with the particularity that they are indistinguishable: if $x, y \in \mathbb{R}$, then we cannot distinguish between realizations, say where $(X_1, X_2) = (x, y)$ and $(X_1, X_2) = (y, x)$, an this translates into symmetries of the finite-dimensional distribution (see [65] and [66] for an introduction to the general theory). These points can for instance represent times as in insurance, energy levels as in nuclear physics or eigenvalues as in random matrix theory. The concept has evolved substantially over time and is now formulated as $\mathbb{N}$-valued random measures, or equivalently as a probability measure on the space of purely atomic measures. In the following examples however, we first construct the random points $\{X_j\}_{j=1:J}$ and then define the associated point process ξ on $\mathbb{R}$, which is done via the formula

$$\xi = \sum_{j=1}^{J} \delta_{X_j}, \tag{0.0.1}$$

where the Dirac measure δ_y is such that for $B \subset \mathbb{R}$

$$\delta_y(B) = \mathbf{1}_{\{y \in B\}} = \begin{cases} 1 & y \in B \\ 0 & y \notin B \end{cases}. \tag{0.0.2}$$

We then emphasise how probabilistic quantities formulated in terms of the points are translated in the context of point processes and introduce informally important quantities arising in the study of point processes.

Early 20th century: The beginning of mathematical insurance ([84])

- Life insurance and one-point processes

We model the duration of each human life by (independent copies of) the same positive random variable $T > 0$ with survival function for $t > 0$

$$F(t) = \mathbb{P}(T > t). \tag{0.0.3}$$

Now from an insurance point of view we are interested in the remaining life time of a client when they enter the contract: let $x > 0$ be the insured's age when the contract starts, then the random variable of interest is $T_x := \max\{T - x, 0\}$ with survival function

$$\mathbb{P}(T_x > t) = F(t + x), \qquad \mathbb{P}(T_x = 0) = 1 - F(x). \tag{0.0.4}$$

However, in order to obtain accurate predictions and pricing so as to avoid bankruptcy, we have to take into account the fact that they survive until the beginning of the contract, hence we need to consider the conditional survival function

$$F(t|x) = \mathbb{P}(T_x > t \mid T_x > 0) = \frac{F(t + x)}{F(x)}. \tag{0.0.5}$$

Assuming that F is absolutely continuous with density f

$$-\partial_t F(t) = f(t), \tag{0.0.6}$$

then so is $F(\cdot|x)$:

$$F(t|x) = 1 - \int_0^t f(s|x)\mathrm{d}s, \tag{0.0.7}$$

with density

$$f(t|x) = \frac{f(t + x)}{F(x)}. \tag{0.0.8}$$

Now it is very practical from a modelling point of view in (0.0.4) to have the same function F but simply shifted, unfortunately this is not the case for $F(\cdot|x)$ nor $f(\cdot|x)$, yet if we introduce the so-called force of mortality $\mu(t|x)$ which is such that

$$F(t|x) = \exp\left(-\int_0^t \mu(s|x)\mathrm{d}s\right), \tag{0.0.9}$$

it is easily seen that

$$\mu(t|x) = \frac{\partial_t(1 - F(t|x))}{F(t|x)} = \frac{f(t|x)}{F(t|x)} = \frac{f(t + x)}{F(t + x)} =: \mu(t + x). \tag{0.0.10}$$

The name of mortality, and in more general contexts hazard rate function, stems from the following heuristic interpretation:

$$\mu(t|x) \approx \mathbb{P}(T_x = t \mid T_x \geq t), \tag{0.0.11}$$

meaning that $\mu(t)$ is the likelihood to die at time t having survived up to time t, and this is more intuitive to model than the likelihood to die at time t, which is $f(t)$, since we expect μ to be increasing in contrast to f.

Now let us study the point process ξ_x on $(0,\infty)$ induced by T_x, defined as above by

$$\xi_x = \delta_{T_x}. \tag{0.0.12}$$

We thus have for any interval $[a,b] \subset (0,\infty)$ that its measure is a Bernoulli random variable

$$\xi_x([a,b]) = \mathbf{1}_{\{a \le T_x \le b\}} \in \{0,1\}, \tag{0.0.13}$$

with parameter

$$\mathbb{P}(\xi_x([a,b]) = 1) = \mathbb{P}(a \le T_x \le b) = \int_a^b f(x+s)\mathrm{d}s. \tag{0.0.14}$$

The survival function and the probability density are now respectively expressed as the so-called one-gap probability, i.e. the probability of measure of a certain interval being zero

$$\mathbb{P}(\xi_x([0,t]) = 0) = \mathbb{P}(T_x > t) = F(t+x), \tag{0.0.15}$$

and the one-point correlation function $\rho_x(s)$, which is the likelihood that there is a point at s, is in this case $f(x+s)$, i.e. for any continuous function ψ with compact support in $(0,\infty)$ it holds

$$\mathbb{E}\int_{\mathbb{R}} \psi \mathrm{d}\xi_x = \mathbb{E}\psi(T_x) = \int_{\mathbb{R}} \psi(s) f(x+s)\mathrm{d}s. \tag{0.0.16}$$

Finally, in order to make an analogy later on, let us notice that the force of mortality is given by the logarithmic derivative with respect to the "external" parameter x of the gap probability

$$\mu(t|x) = -\frac{\partial_x \mathbb{P}(\xi_x([0,t]) = 0)}{\mathbb{P}(\xi_x([0,t]) = 0)}. \tag{0.0.17}$$

Under a very popular model, the so-called Gompertz-Makeham model, the force of mortality is given by

$$\mu(t|x) = A + c^{-2}Be^{c(t+x)} \tag{0.0.18}$$

where $A \ge 0$ is related to the "accidental" mortality while $B, c > 0$ take into account the "biological" one, and it satisfies the second-order differential equation

$$-\partial_x^2 \mu(x|t) + c^2 \mu(x|t) = c^2 A. \tag{0.0.19}$$

• Non-life insurance and Poisson point process ([4])

In non-life insurance, we have to deal with several (in theory possibly infinite) events, which can be car accidents, house fires, etc.. We thus have positive random variables

$$\{T_n\}_{n\in\mathbb{N}} \subset [0,\infty), \qquad T_n < T_{n+1} \quad \forall n \in \mathbb{N}, \tag{0.0.20}$$

where T_0 designates the start of the contract and T_n denotes the time of incident n for $n \geq 1$. In order to describe the probabilistic model, we will here take advantage of the ordered structure and use the so-called conditional intensity approach: to this end introduce for $n \geq 1$ the inter-arrival times between incidents

$$\Delta T_n := T_n - T_{n-1}, \tag{0.0.21}$$

Once we know T_0, which does not have much to do with incidents happening and thus its modelling is ignored here, we can describe the distribution of ΔT_1, and once we know $T_0, \Delta T_1$, or equivalently T_0, T_1, we can describe the distribution of ΔT_2, and so on and so forth. Thus we have to fix for each $n \geq 1$ a function F_n of $n+1$ variables which will give the value of the following conditional probabilities:

$$\mathbb{P}(\Delta T_n > t \mid T_0 = t_0, ..., T_{n-1} = t_{n-1}) = F_n(t; t_0, ..., t_{n-1}). \tag{0.0.22}$$

Since $\Delta T_n > 0$, we can adopt the description from life insurance in terms of hazard rate functions: let $\lambda_n(t; t_0, ..., t_{n-1})$ be such that

$$F_n(t; t_0, ..., t_{n-1}) = \exp\left(-\int_0^t \lambda(s; t_0, ..., t_{n-1}) \mathrm{d}s\right), \tag{0.0.23}$$

then it is the likelihood that the n-th incident occurs at time $t + t_{n-1}$ knowing all the preceding incidents times $T_0 = t_0, ..., T_{n-1} = t_{n-1}$ and that it is yet to occur:

$$\lambda_n(t; t_0, ..., t_{n-1}) \approx \mathbb{P}(\Delta T_n = t \mid T_0 = t_0, ..., T_{n-1} = t_{n-1}, \Delta T_n \geq t). \tag{0.0.24}$$

The advantage of this approach remains the same: this likelihood is more intuitively modelled than other quantities. For instance in car insurance, we can expect that the hazard is memoryless, meaning that the so-called Markov property is satisfied:

$$\lambda_n(t; t_0, ..., t_{n-1}) = \lambda_1(t; t_{n-1}). \tag{0.0.25}$$

This assumption means that people do not change the way they drive after an accident. Another assumption could be that the exact time of entering the contract and the subsequent time of each incident do not matter, so if $t_0, ..., , t_n$ are all shifted in the same way, then the risk remains unchanged. This leads to the time invariance property: for any $s \geq 0$ we should have

$$\lambda_n(t; t_0 + s, ..., t_{n-1} + s) = \lambda_n(t; t_0, ..., t_{n-1}). \tag{0.0.26}$$

Combining both yields that for a certain function $\lambda \geq 0$ we have for all $n \geq 1$ and $t_0, ..., t_{n-1}$

$$\lambda_n(t; t_0, ..., t_{n-1}) = \lambda(t). \tag{0.0.27}$$

We now fix such a function and explain how the sequence $\{T_n\}_{n \in \mathbb{N}}$ can be understood as a very special kind of point process, namely a Poisson point process, here with intensity λ (with respect to Lebesgue's measure). Proceeding as in life insurance, we can construct the random discrete measure

$$\xi = \sum_{n\geq 1} \delta_{T_n}, \tag{0.0.28}$$

which for any interval $[a, b] \subset (0, \infty)$ counts the number of incidents happening between time a and b:

$$\xi([a, b]) = \# \{n \in \mathbb{N} \mid a \leq T_n \leq b\}. \tag{0.0.29}$$

Note that now the order of the T_n's does not matter any more. The name Poisson point process with intensity λ comes from the fact that for any interval $[a, b] \subset (0, \infty)$, the distribution of $\xi([a, b])$ is Poisson with parameter $\int_a^b \lambda$: for any $n \in \mathbb{N}$

$$\mathbb{P}(\xi([a, b]) = n) = \frac{\left(\int_a^b \lambda\right)^n}{n!} e^{-\int_a^b \lambda}. \tag{0.0.30}$$

More is true: if for $j = 1, ..., m$, the intervals $[a_j, b_j]$ are disjoint, then we have independence:

$$\mathbb{P}(\xi([a_j, b_j]) = n_j,\ j = 1 : m) = \prod_{j=1}^{n} \mathbb{P}(\xi([a_j, b_j]) = n_j). \tag{0.0.31}$$

This can also be expressed as follows: for $m \in \mathbb{N}_*$, let ρ_m denote the m-point correlation function of the point process, which means that $\rho_m(x_1, ..., x_m)$ is the likelihood that there are points at $x_1, ..., x_m$, then it is given by the simple formula

$$\rho_m(x_1, ..., x_m) = \prod_{j=1}^{m} \lambda(x_j). \tag{0.0.32}$$

Given that independence is a desired feature of many models and the law of rare events, this explains why the Poisson point process and its derived constructions are widely spread in applications.

1950-1960s : Nuclear physics and random matrix theory

As alluded before, any probability density (say with respect to the Lebesgue measure) π_n of $(x_1, ..., x_n) \in I^n$ (say on an interval $I \subset \mathbb{R}$) which is symmetric (to account for indistinguishability), i.e. satisfies for any permutation $\sigma \in S_n$ of n elements

$$\pi_n(x_{\sigma(1)}, ..., x_{\sigma(n)}) = \pi_n(x_1, ..., x_n), \tag{0.0.33}$$

induces a point process on I defined by

$$\xi = \sum_{j=1}^{n} \delta_{x_j}. \tag{0.0.34}$$

We give two important examples of such point processes that later will be revealed as determinantal. Contrary to the Poisson point process which features independence, here the points will repel each other because $\pi_n(x_1, ..., x_n) = 0$ whenever $x_i = x_j$ for some $i \neq j$.

- The Circular Unitary Ensemble $\mathrm{CUE_n}$ (see [81, Chapter 2])

Consider a unitary matrix $U \in \mathrm{U}(n)$ of size $n \geq 1$, i.e. which satisfies

$$U^* = U^{-1}, \tag{0.0.35}$$

where * indicates complex conjugation and transposition. It is well known that $\mathrm{U}(n)$ is a compact Lie group and therefore admits a unique Haar probability measure $\mathrm{d}\mathbb{P}(U)$. This means that for any $V \in \mathrm{U}(n)$ there holds

$$\mathrm{d}\mathbb{P}(UV) = \mathrm{d}\mathbb{P}(U) = \mathrm{d}\mathbb{P}(VU), \qquad \mathbb{P}(\mathrm{U}(n)) = 1. \tag{0.0.36}$$

Weyl's integration formula states that the spectral decomposition $U = VDV^*$, $V \in \mathrm{U}(n), D = \mathrm{diag}(e^{2\pi i x_1}, ..., e^{2\pi i x_n})$ induces random variables V, D which are independent, and that the probability density of $(x_1, ..., x_n) \in [0,1)^n$ is

$$\pi_n^{\mathrm{CUE}}(x_1, ..., x_n) = \frac{1}{n!} \prod_{1 \leq l < k \leq n} \left| e^{2\pi i x_k} - e^{2\pi i x_l} \right|^2. \tag{0.0.37}$$

We mention as well that expectations of multiplicative statistics in the CUE yield Toeplitz determinants: using the Vandermonde determinant we have

$$\pi_n^{\mathrm{CUE}}(x_1, ..., x_n) = \frac{1}{n!} \left| \det_{l,k=1:n} (e^{2\pi i (l-1) x_k}) \right|^2, \tag{0.0.38}$$

so that for $\phi \in L^1(0,1)$ we can write Andréief's formula in this particular case (see(0.0.134) below for the general one) as

$$\frac{1}{n!} \int_{[0,1)^n} \left| \det(e^{2\pi i (l-1) x_k}) \right|^2 \prod_{j=1}^n \phi(x_j) \mathrm{d}x_j = \det_{l,k=1:n} \left(\int_{[0,1)} e^{-2\pi i (l-k) x} \phi(x) \mathrm{d}x \right); \tag{0.0.39}$$

in the right-hand side lies the so-called Toeplitz determinant of the symbol ϕ:

$$D_n[\phi] := \det_{l,k=1:n} \left(\int_{[0,1)} e^{-2\pi i (l-k) x} \phi(x) \mathrm{d}x \right), \tag{0.0.40}$$

therefore we have proved that

$$\mathbb{E}_n^{\mathrm{CUE}} \prod_{j=1}^n \phi(x_j) = D_n[\phi]. \tag{0.0.41}$$

- The Gaussian Unitary Ensemble $\mathrm{GUE_n}$ (see [81, Chapter 1])

As a model for Hamiltonians of heavy atomic nuclei, Wigner proposed to use large random hermitian matrices with or without appropriate additional symmetries depending on the Hamiltonian's ones. Here we consider no additional symmetries of the Hamiltonian, which means that we only require invariance under conjugation by unitary matrices. Let $H \in \mathrm{Her}(n)$ be an Hermitian matrix of size $n \geq 1$, i.e. satisfying

$$H^* = H, \tag{0.0.42}$$

then for any $U \in \mathrm{U}(n)$, $H \in \mathrm{Her}(n)$, we indeed have $UHU^* \in \mathrm{Her}(n)$. We can then consider a "Gaussian" probability measure on $\mathrm{Her}(n)$ respecting this invariance under unitary conjugation:

$$\mathrm{d}\mathbb{P}(H) \propto e^{-\mathrm{tr}H^2}\mathrm{d}H, \tag{0.0.43}$$

where

$$\mathrm{d}H = \prod_{i\leq j} \mathrm{d}H_{ij} \tag{0.0.44}$$

is the $\frac{n(n+1)}{2}$-dimensional Lebesgue measure on $\mathrm{Her}(n) \simeq \mathbb{R}^{\frac{n(n+1)}{2}}$. Once again the spectral decomposition $H = VDV^*$, $V \in \mathrm{U}(n), D = \mathrm{diag}(x_1, ..., x_n))$ induces random variables V, D which are independent. The probability density of $(x_1, ..., x_n) \in \mathbb{R}^n$ can be computed and equals

$$\pi_n^{\mathrm{GUE}}(x_1, ..., x_n) = \frac{1}{n! Z_n^{\mathrm{GUE}}} \prod_{1\leq l<k\leq n} (x_k - x_l)^2 \prod_{j=1}^n e^{-x_j^2}, \tag{0.0.45}$$

for some appropriate normalisation constant $Z_n^{\mathrm{GUE}} > 0$. As a consequence of Andréief's formula again, multiplicative statistics now produce Hankel determinants: for $\phi \in L^\infty(\mathbb{R})$ we have

$$\mathbb{E}_n^{\mathrm{GUE}} \prod_{j=1}^n \phi(x_j) = \frac{H_n[\phi e^{-\cdot^2}]}{H_n[e^{-\cdot^2}]}, \tag{0.0.46}$$

where the Hankel determinant of the symbol ψ is defined by

$$H_n[\psi] := \det_{l,k=1:n} \left(\int_{\mathbb{R}} \psi(x) x^{l+k-2} \mathrm{d}x \right). \tag{0.0.47}$$

It is also readily seen that $Z_n^{\mathrm{GUE}} = H_n[e^{-\cdot^2}]$.

- Determinantal structure of correlation functions

It is to be noted that the appearance of Toeplitz and Hankel determinants is not really the justification for the name determinantal point process. Rather it comes from the following: consider for a general π_n as above its associated correlation functions $\{\rho_m\}_{m=1:n}$ defined for $1 \leq m \leq n$ by

$$\rho_m(x_1, ..., x_m) = \frac{n!}{(n-m)!} \int_{I^{n-m}} \pi_n(x_1, ..., x_m, y_1, ..., y_{n-m}) \mathrm{d}y_1 ... \mathrm{d}y_{n-m}. \tag{0.0.48}$$

Once again $\rho_m(x_1, ..., x_m)$ is the likelihood that there are points at $x_1, ..., x_m$. It was a remarkable observation of Wigner and Dyson (see [81, 124]) that we can write these as determinants of a single correlation kernel $K_n : I^2 \to \mathbb{C}$:

$$\rho_m(x_1, ..., x_m) = \det_{l,k=1:m} \big(K_n(x_l, x_k)\big), \tag{0.0.49}$$

where for the GUE ($I = \mathbb{R}$)

$$K_n^{\mathrm{GUE}}(x, y) = \sum_{j=0}^{n-1} h_j(x) h_j(y), \tag{0.0.50}$$

and $h_j(x) = H_j(x) e^{-\frac{x^2}{2}}$ is the j-th Hermite function, with H_j the normalized Hermite polynomial of degree j, while for the CUE ($I = [0, 1)$)

$$K_n^{\mathrm{CUE}}(x, y) = \sum_{m=0}^{n-1} e^{2\pi i m(x-y)}. \tag{0.0.51}$$

1970-1980s: Point processes, fermions and determinants

It is during this time that work from Macchi [122] and Lenard [118] began to introduce rigorous general definitions of determinantal point processes and study some of their fundamental properties in terms of their correlation functions. We leave the technical details for later and focus on a model proposed by Macchi coming from physics and encompassing essentially all the point processes having a fixed finite number of points. Consider a stationary quantum system consisting of n fermions at zero temperature and thermal equilibrium. Each fermion has the possibility to be in n different states, and we denote $\{\psi_j\}_{j=1:n} \subset L^2(I)$ their wave-functions, say describing the positions in an interval $I \subset \mathbb{R}$, which we assume are orthonormal:

$$\int_I \overline{\psi_l} \psi_k = \delta_{lk}. \tag{0.0.52}$$

In order to describe the system's state, we form the so-called Slater determinant Ψ_n of the states $\{\psi_j\}_{j=1:n}$:

$$\Psi_n(x_1, ..., x_n) := \frac{1}{\sqrt{n!}} \det_{l,k=1:n} \big(\psi_l(x_k)\big). \tag{0.0.53}$$

This is actually a candidate for the wave function of the system, yet not all states can be described in this manner. Andréief's formula (0.0.134) below applies again and tells us that

$$\pi_n(x_1, ..., x_n) := |\Psi_n(x_1, ..., x_n)|^2 \tag{0.0.54}$$

is a symmetric probability density on I^n and therefore induces a point process on I having exactly n points. Macchi draws attention to the so-called coincidence intensities $\{\rho_m\}_{m=1:n}$ of this point process (which turned out to be the same as correlation functions), which satisfy the following equality for $B \subset I$:

$$\mathbb{E}_n \xi(B)^{[m]} = \int_{B^m} \rho_m(x_1, ..., x_m) \mathrm{d}x_1 ... \mathrm{d}x_m, \tag{0.0.55}$$

where the factorial power is $l^{[k]} = \frac{l!}{(l-k)!}$. Exactly as before, it turns out that

$$\rho_m(x_1, ..., x_m) = \det_{l,k=1:m} (K_n(x_l, x_k)), \tag{0.0.56}$$

where

$$K_n(x, y) = \sum_{j=1}^{n} \psi_j(x) \overline{\psi_j(y)}. \tag{0.0.57}$$

So far we have not really seen quantum mechanics; if we deal with the state being position, we have to consider the stationary Schrödinger equation, i.e. take $\psi_k \in L^2(I)$ such that there exists $\lambda_k \in \mathbb{R}$ with

$$\left(-\partial_x^2 + V(x)\right) \psi_k(x) = \lambda_k \psi_k(x). \tag{0.0.58}$$

Surprisingly, this is the case for both of the examples mentioned above: for the CUE_n we take (with periodic boundary conditions $\psi_k(0) = \psi_k(1)$)

$$V(x) = 0, \qquad \psi_k(x) = e^{2\pi i k x}, \qquad \lambda_k = (2\pi k)^2, \tag{0.0.59}$$

while for the GUE_n, we take

$$V(x) = x^2 \qquad \psi_k(x) = h_k(x) \qquad \lambda_k = 2k + 1. \tag{0.0.60}$$

As a generalisation of Toeplitz and Hankel determinants, we have now that expectations of multiplicative statistics are given by Gram determinants:

$$\mathbb{E}_n \prod_{j=1}^{n} \phi(x_j) = \det_{l,k=1:n} \left(\int_I \phi \overline{\psi_l} \psi_k \right). \tag{0.0.61}$$

Yet another way to look at that relation and which actually holds in greater generality, is that for $\phi : I \to [0, 1]$ (see [135], [6, Chapter 11])

$$\mathbb{E}_n \prod_{j=1}^{n} (1 - \phi(x_j)) = \det \left(1 - \mathrm{M}_{\sqrt{\phi}} \mathrm{K}_n \mathrm{M}_{\sqrt{\phi}}\right), \tag{0.0.62}$$

where right-hand side is a Fredholm determinant (which we explore briefly in the section hereafter), M_ϕ is the operator on $L^2(I)$ of multiplication by ϕ, i.e.

$\mathrm{M}_\phi[\psi](x) = \phi(x)\psi(x)$, and finally K_n is the integral operator with kernel K_n, meaning that

$$\mathrm{K}_n[\psi](x) = \int_I K_n(x,y)\psi(y)\mathrm{d}y, \tag{0.0.63}$$

which is nothing but the orthogonal projection on the subspace of $L^2(I)$ spanned by $\{\psi_j\}_{j=1:n}$.

Traces and determinants of operators

Towards the beginning of the 20th century arose the question of determinants of infinite matrices and integral operators (see [89] for an introduction to the general theory from a modern viewpoint). Let us first review some basic results for matrices and see some of their generalisations. To a matrix $\mathrm{M} = (M_{lk})_{l,k=1:N} \in \mathbb{C}^{N\times N}$ we can associate the determinant $\det M$ and the trace $\operatorname{tr} M$, one of their many equivalent definitions being

$$\det \mathrm{M} = \sum_{\sigma\in S_N} \epsilon(\sigma) \prod_{j=1}^{N} M_{j\sigma(j)}, \qquad \operatorname{tr}\mathrm{M} = \sum_{j=1}^{N} M_{jj}, \tag{0.0.64}$$

where S_N denotes the set of all permutations of N elements and $\epsilon(\sigma) \in \{\pm 1\}$ is the sign of the permutation σ. These have lots of interesting properties, e.g. det is multiplicative while tr is linear:

$$\det(\mathrm{M}_1\mathrm{M}_2) = \det \mathrm{M}_1 \det \mathrm{M}_2, \qquad \operatorname{tr}(\alpha\mathrm{M}_1 + \beta\mathrm{M}_2) = \alpha \operatorname{tr}\mathrm{M}_1 + \beta \operatorname{tr}\mathrm{M}_2. \tag{0.0.65}$$

Another is that they are spectral quantities: they can be expressed in terms of the (complex) eigenvalues $\{\lambda_j\}_{j=1:N}$ of M as

$$\det(1+\mathrm{M}) = \prod_{j=1}^{N}(1+\lambda_j), \qquad \operatorname{tr}\mathrm{M} = \sum_{j=1}^{N} \lambda_j, \tag{0.0.66}$$

where for each j there exists $v_j \in \mathbb{C}^{N\times 1}$ such that

$$\mathrm{M}v_j = \lambda_j v_j. \tag{0.0.67}$$

Note that $1+\lambda_j$ is then an eigenvalue of $1+\mathrm{M}$, we made the switch for a reason we now explain: informally, if $N=\infty$, then the question of convergence arises. From the theory of series and infinite products, we would at least need

$$\sum_{j=1}^{\infty} |\lambda_j| < \infty; \tag{0.0.68}$$

the reason for the absolute value is because any permutation of the eigenvalues should give the same trace and determinant (the spectrum is roughly the set of eigenvalues and is therefore unordered). With this in mind and going back to the finite N case, it turns out that there are also several interesting formulae for $\det(1+\mathrm{M})$, $\mathrm{M}\in\mathbb{C}^{N\times N}$, one of which is

$$\det(1+\mathrm{M}) = \sum_{n=0}^{N} \frac{1}{n!} \sum_{i_1,\ldots,i_n=1}^{N} \det_{l,k=1:n}(M_{i_l i_k}). \tag{0.0.69}$$

Now this is not very practical since we are expressing a determinant in terms of other determinants, yet if the left-hand side is to denote the determinant of an infinite matrix, the right-hand side expresses it as a series of determinants of finite size matrices. It can be shown that if all the non zero singular values of an infinite matrix M (which can be roughly understood as the absolute values of the eigenvalues, but actually dominate them) are countable and form a convergent series then the previous expression is well-defined and is equal to the expected infinite product (cf.. Lidskii's theorem [89, Chapter 4])

$$\det(1+\mathrm{M}) := \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1,\dots,i_n=1}^{\infty} \det_{l,k=1:n} (M_{i_l i_k}) = \prod_{j=1}^{\infty} (1+\lambda_j), \tag{0.0.70}$$

and the same results holds for the trace

$$\operatorname{tr} \mathrm{M} := \sum_{j=1}^{\infty} M_{jj} = \sum_{j=1}^{\infty} \lambda_j \tag{0.0.71}$$

Fredholm is well-known for his study of equations involving integral operators K (which can be thought of as infinite matrices with continuous indices), i.e. defined via a kernel $K : I^2 \to \mathbb{C}$ as

$$\mathrm{K}[\psi](x) = \int_I K(x,y)\psi(y)\mathrm{d}y, \tag{0.0.72}$$

which led him to consider compact operators for which there are only countably non-zero singular values. Among these are the trace-class operators which can be defined through the condition that their singular values moreover form a convergent series. For such an integral operator K (with a "nice" integral kernel) the continuous analogues of the above formulae hold (the first is known as Fredholm's formula [89, Chapter 6])

$$\det(1+\mathrm{K}) = \sum_{n\in\mathbb{N}} \frac{1}{n!} \int_{I^n} \det_{l,k=1:n} (K(x_l,x_k)) \prod_{j=1}^{n} \mathrm{d}x_j, \tag{0.0.73}$$

$$\operatorname{tr} \mathrm{K} = \int_I K(x,x)\mathrm{d}x. \tag{0.0.74}$$

Operators of integrable form and Riemann-Hilbert problems

Integrable operators, as well as Riemann-Hilbert problems, require a curve in the complex plane as domain (for all this subsection, see [6, Chapter 5], [68, Chapter 7] and [81, Chapter 9] for an introduction). For the sake of simplicity, we focus on the case where this curve is the real line $\mathbb{R}$. An integral operator K on $L^2(\mathbb{R})$ is then said to be of integrable form if its integral kernel $K : \mathbb{R}^2 \to \mathbb{C}$ admits the following representation in terms of certain functions $f : \mathbb{R} \to \mathbb{C}^{1\times p}$, $g : \mathbb{R} \to \mathbb{C}^{p\times 1}$ satisfying $fg = 0$:

$$K(x,y) = \frac{f(x)g(y)}{x-y} = \frac{\sum_{j=1}^{p} f_j(x)g_j(y)}{x-y}. \tag{0.0.75}$$

Note that such a representation (f, g) is not unique: for any (constant) invertible $M \in \mathbb{C}^{p\times p}$ we have that (fM^{-1}, Mg) also represents K. An example of a famous operator of integrable form is given by the so-called sine kernel:

$$K^{\sin}(x, y) = \operatorname{sinc}(x - y), \qquad \operatorname{sinc}(z) = \frac{\sin \pi z}{\pi z}; \tag{0.0.76}$$

it is readily seen that an integrable representation is

$$f^{\sin}(x) = \frac{1}{\sqrt{\pi}} \begin{pmatrix} -\cos \pi x & \sin \pi x \end{pmatrix}, \qquad g^{\sin}(y) = \frac{1}{\sqrt{\pi}} \begin{pmatrix} \sin \pi y \\ \cos \pi y \end{pmatrix}. \tag{0.0.77}$$

The sine kernel arises as the so-called 'bulk'-scaling limit of the $\mathrm{GUE_n}$, which is actually also induced by an operator of integrable form: the Christoffel-Darboux formula applied to the Hermite polynomials implies that for some explicit constant $c_n > 0$

$$K_n^{\mathrm{GUE}}(x, y) = c_n \frac{h_n(x)h_{n-1}(y) - h_{n-1}(x)h_n(y)}{x - y}. \tag{0.0.78}$$

At the level of operators, we can informally write K as a composition

$$\mathrm{K} = \mathrm{M}_f \pi \mathrm{H}_{\mathbb{R}} \mathrm{M}_g, \tag{0.0.79}$$

involving the multiplication operators $\mathrm{M}_g : L^2(\mathbb{R}) \to L^2(\mathbb{R})^{p\times 1}$, $\mathrm{M}_f : L^2(\mathbb{R})^{p\times 1} \to L^2(\mathbb{R})$ and the Hilbert transform on the real line $\mathrm{H}_{\mathbb{R}}$ (here extended to an operator on $L^2(\mathbb{R})^{p\times 1}$), the latter being defined for ψ smooth with compact support as

$$\mathrm{H}_{\mathbb{R}}[\psi](x) = \mathrm{p.v}\frac{1}{\pi} \int_{\mathbb{R}} \frac{\psi(y)}{x - y} \mathrm{d}y := \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{\{|x-y|>\epsilon\}} \frac{\psi(y)}{x - y} \mathrm{d}y, \tag{0.0.80}$$

and it can be shown that it indeed extends to an operator on $L^2(\mathbb{R})$. Closely connected to the Hilbert transform and taking us naturally to complex analysis is the Stieltjes transform: denoting $\mathrm{Hol}(U)$ the space of holomorphic functions on U, it is the operator $\mathrm{C}_{\mathbb{R}} : L^2(\mathbb{R}) \to \mathrm{Hol}(\mathbb{C} \setminus \mathbb{R})$ defined for $\psi \in L^2(\mathbb{R})$ as

$$\mathrm{C}_{\mathbb{R}}[\psi](z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\psi(w)}{w - z} \mathrm{d}w, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{0.0.81}$$

In order to state the relationship between $\mathrm{H}_{\mathbb{R}}$ and $\mathrm{C}_{\mathbb{R}}$, as well as the basic idea of a Riemann-Hilbert problem, we need the concept of boundary values along the curve $\mathbb{R}$: for $h \in \mathrm{Hol}(\mathbb{C} \setminus \mathbb{R})$, we define for $w \in \mathbb{R}$

$$h^{\pm}(w) := \lim_{z \to w^{\pm}} h(z), \tag{0.0.82}$$

where the sign $\pm$ refers to the imaginary part of $z \in \mathbb{C} \setminus \mathbb{R}$ as it approaches $w \in \mathbb{R}$. The Sokhotski–Plemelj theorem states that if $\psi \in L^2(\mathbb{R})$, then the limits

$$\mathrm{C}_{\mathbb{R}}^{\pm}[\psi](w) = \lim_{z \to w^{\pm}} \mathrm{C}_{\mathbb{R}}[\psi](z) \tag{0.0.83}$$

exist for almost every $w \in \mathbb{R}$ and define functions $\mathrm{C}^{\pm}_{\mathbb{R}}[\psi] \in L^2(\mathbb{R})$; moreover we have the Sokhotski–Plemelj formula:

$$\mathrm{C}^{\pm}_{\mathbb{R}} = \pm\frac{1}{2} + \frac{i}{2}\mathrm{H}_{\mathbb{R}}. \tag{0.0.84}$$

These are of utmost importance in the study of Riemann-Hilbert problems; indeed in their simplest form, given $\psi \in L^2(\mathbb{R})$ we need to find $h \in \mathrm{Hol}(\mathbb{C} \setminus \mathbb{R})$ such that 1) $h^{\pm} \in L^2(\mathbb{R})$ are related by $h^+ = h^- + \psi$ and 2) h has the following behaviour as $z \to \infty$: $h(z) = o(1)$. From the formulae above we can see that the solution is given by $h = \mathrm{C}_{\mathbb{R}}[\psi]$.

1990s: The method of Its, Izergin, Korepin and Slavnov (IIKS)

It is straightforward to see that operators of integrable form constitute a vector space; slightly more difficult yet still easily feasible is to convince oneself that they constitute an algebra. Surprisingly, more is true: they are stable under taking the resolvent, which is defined, provided that $1 - \mathrm{K}$ is invertible, by

$$\mathrm{R} := (1 - \mathrm{K})^{-1}\mathrm{K}. \tag{0.0.85}$$

Its, Izergin, Korepin and Slavnov (IIKS) [98] encountered this construction when dealing with certain determinants of operators of integrable form. They showed that R admits an integrable representation

$$R(x, y) = \frac{F(x)G(y)}{x - y} = \frac{\sum_{j=1}^{p} F_j(x)G_j(y)}{x - y} \tag{0.0.86}$$

where

$$F = (1 - \mathrm{K})^{-1}[f], \qquad G = (1 - \mathrm{K})^{-\mathrm{t}}[g], \tag{0.0.87}$$

and K is extended to vector-valued functions component-wise. Moreover (F, G) can be obtained from (f, g) and the solution to a Riemann-Hilbert problem. Let us denote $I_p \in \mathbb{C}^{p\times p}$ the identity matrix, then we obtain after some computations using the expression of K as a composition of operators that

$$F(x) = (1 - \mathrm{M}_f \pi \mathrm{H}_{\mathbb{R}} \mathrm{M}_g)^{-1}[f](x) = f(x)(1 - \mathrm{H}_{\mathbb{R}}\mathrm{M}_{\pi g f})^{-1}[I_p](x), \tag{0.0.88}$$

which reveal that we have to solve for a matrix–valued function χ solution to the singular value integral equation $(1 - \mathrm{H}_{\mathbb{R}}\mathrm{M}_{\pi g f})\,[\chi] = I_p$. Similarly we have $G = vg$ where v solves $(1 - \mathrm{H}_{\mathbb{R}}\mathrm{M}_{\pi g f})^{\mathrm{t}}\,[v] = I_p$. Now Deift and Zhou [73], generalising IIKS, showed that solving this type of singular integral equation is equivalent to solving a certain type of Riemann-Hilbert problems. In our case we need to find Y such that

1. $Y \in \mathrm{Hol}(\mathbb{C} \setminus \mathbb{R})^{p\times p}$;

2. Y admits boundary values $Y_\pm \in I_p + L^2(\mathbb{R})^{p\times p}$, which are related by

$$Y_+ = Y_-(1+J), \qquad J(w) := 2\pi i g(w) f(w) \in \mathbb{C}^{p\times p}; \tag{0.0.89}$$

3. As $z \to \infty$ we have the asymptotic behaviour

$$Y(z) = I_p + o(1). \tag{0.0.90}$$

With this Riemann-Hilbert problem, we can obtain (F, G) from (f, g):

$$F(x) = f(x) Y_\pm(x)^{-1}, \qquad G(y) = Y_\pm(y) g(y). \tag{0.0.91}$$

1990s-2000s: Tracy-Widom's equations and Borodin-Deift, Hubert -Kapaev's reformulation

In the 1990s, Tracy and Widom [137] investigated Fredholm determinants associated to the Airy point process, which is induced by the operator K^{Ai} on $L^2(\mathbb{R})$ whose kernel is the so-called Airy kernel:

$$K^{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{Ai}(x+t)\mathrm{Ai}(y+t)\mathrm{d}t, \tag{0.0.92}$$

where Ai is the Airy function, a particular solution to Airy's differential equation

$$\partial_t^2 \mathrm{Ai}(t) = t\mathrm{Ai}(t). \tag{0.0.93}$$

Using this we can show that K^{Ai} admits the following integrable representation:

$$f^{\mathrm{Ai}}(x) = \begin{pmatrix} -\mathrm{Ai}'(x) & \mathrm{Ai}(x) \end{pmatrix}, \qquad g^{\mathrm{Ai}}(y) = \begin{pmatrix} \mathrm{Ai}(y) \\ \mathrm{Ai}'(y) \end{pmatrix}, \tag{0.0.94}$$

with $\mathrm{Ai}'(t) = \partial_t \mathrm{Ai}(t)$. The family of Fredholm determinants that they investigated gave rise to the so-called Tracy-Widom distribution, whose cumulative distribution function is defined in terms of K^{Ai} as

$$F_{TW}(s) = \det(1 - \mathrm{K}_s^{\mathrm{Ai}}), \tag{0.0.95}$$

where $\mathrm{K}_s^{\mathrm{Ai}} = \mathrm{M}_{\mathbf{1}_{(s,\infty)}} \mathrm{K}^{\mathrm{Ai}} \mathrm{M}_{\mathbf{1}_{(s,\infty)}}$. Setting $v(s) := \partial_s^2 \log F_{TW}(s)$, they proved that there exists a unique solution to the following generalisation of Airy's differential equation:

$$\left[\partial_s^2 + 2v(s) - s\right] u(s) = 0, \qquad u(s) \sim \mathrm{Ai}(s) \qquad s \to +\infty. \tag{0.0.96}$$

Furthermore, it turns out that

$$v(s) = -u(s)^2, \tag{0.0.97}$$

so that u actually solves the Painlevé II equation:

$$\partial_s^2 u(s) = su(s) + 2u(s)^3. \tag{0.0.98}$$

Note the resemblance with (0.0.19). The core properties yielding their result are twofold: on the one hand Jacobi's variational formula for determinants makes the connection with the method of Its, Izergin, Korepin and Slavnov, as

$$\partial_s \log\det(1 - \mathrm{K}_s^{\mathrm{Ai}}) = -R_s^{\mathrm{Ai}}(s,s) = F_s^{\mathrm{Ai}}(s)\partial_y G_s^{\mathrm{Ai}}(s) \tag{0.0.99}$$

where R_s^{Ai} is the kernel of the resolvent $\mathrm{R}_s^{\mathrm{Ai}} := (1-\mathrm{K}_s^{\mathrm{Ai}})^{-1}\mathrm{K}_s^{\mathrm{Ai}}$ and $(F_s^{\mathrm{Ai}}, G_s^{\mathrm{Ai}})$ is its integrable representation associated with the one of K_s^{Ai} given by $(f_s^{\mathrm{Ai}}, g_s^{\mathrm{Ai}}) := (\mathbf{1}_{(s,\infty)} f^{\mathrm{Ai}}, g^{\mathrm{Ai}}\mathbf{1}_{(s,\infty)})$; on the other hand the differential equation is rooted in the following differential system for $(f^{\mathrm{Ai}}, g^{\mathrm{Ai}})$

$$\partial_x f^{\mathrm{Ai}}(x) = -f^{\mathrm{Ai}}(x)\begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}, \qquad \partial_y g^{\mathrm{Ai}}(y) = \begin{pmatrix} 0 & 1 \\ y & 0 \end{pmatrix} g^{\mathrm{Ai}}(y), \tag{0.0.100}$$

which in turn will lead to a differential system for $(F_s^{\mathrm{Ai}}, G_s^{\mathrm{Ai}})$. While Tracy and Widom worked with the formulae (0.0.87), Borodin-Deift [31] and Hubert-Kapaev [95] approached the problem using Riemann-Hilbert problems and the formulae (0.0.91), while making connection to the Schlesinger system of equations arising in isomonodromic deformation of Painlevé differential systems (see e.g. [81]).

Precise definitions and fundamental results

We consider a measurable space $(\Lambda, \mathcal{B}_\Lambda)$, where Λ is a complete separable metric space and $\mathcal{B}_\Lambda$ its Borel σ-algebra. We will be mainly interested in $\Lambda = \mathbb{R}$ with the Lebesgue measure or $\Lambda = \mathbb{T}$ the unit circle in the complex plane with the arc length measure, and the reader may prefer to keep only these examples in mind for the sake of simplicity. We denote by $\mathcal{N}(\Lambda)$ the set of boundedly finite Borel counting measures on Λ (a.k.a. the space of configurations); we can represent such a configuration $\xi \in \mathcal{N}(\Lambda)$ as

$$\xi = \sum_{j \in J} \delta_{x_j}, \tag{0.0.101}$$

where J is a countable index set and $x_j \in \Lambda$. From the space $\mathcal{N}(\Lambda)$ we can construct the cylinder set σ-algebra $\mathcal{C}(\Lambda)$ generated by the so-called cylinder sets C, i.e. sets of the form

$$C = \bigcap_{i=1}^{n} \{\xi \in \mathcal{N}(\Lambda) \ : \ \xi(B_i) = k_i\}, \tag{0.0.102}$$

where $B_1, \dots, B_n \in \mathcal{B}_\Lambda$ are disjoint and $n, k_1, \dots, k_n$ are non-negative integers. It turns out that $\mathcal{N}(\Lambda)$ can be made into a Polish space whose Borel σ-algebra is precisely $\mathcal{C}(\Lambda)$, hence $(\mathcal{N}(\Lambda), \mathcal{C}(\Lambda))$ is a standard Borel space (see [65]).

A point process $\mathbb{P}$ on Λ is by definition a probability measure on $(\mathcal{N}(\Lambda), \mathcal{C}(\Lambda))$. We will mostly be interested in simple point processes $\mathbb{P}$, meaning that they satisfy the additional property that for all $x \in \Lambda$ we have $\mathbb{P}$-a.s.

$$\xi(\{x\}) \leq 1. \tag{0.0.103}$$

This means that in (0.0.101) we have $\mathbb{P}$-a.s. distinct points: $x_i \neq x_j$ for $i \neq j$, hence we can identify such a counting measure ξ by its support $\operatorname{supp} \xi$. Recall (see e.g.[66, Section 9.4]) that a simple point process on Λ is characterized uniquely by its Laplace functional

$$\mathcal{L} : B_+(\Lambda) \to \mathbb{R}^+ : \phi \mapsto \mathcal{L}[\phi], \qquad \mathcal{L}[\phi] = \mathbb{E} e^{-\int_\Lambda \phi \mathrm{d}\xi} = \mathbb{E} e^{-\sum_{x \in \operatorname{supp} \xi} \phi(x)}, \tag{0.0.104}$$

where $B_+(\Lambda)$ is the space of bounded non-negative measurable functions $f : \Lambda \to [0, +\infty)$ with bounded support.

We will also always work with a reference measure on $(\Lambda, \mathcal{B}_\Lambda)$: we assume that the latter is endowed with a boundedly finite positive Borel measure μ, i.e. satisfying $\mu(B) < \infty$ for any bounded $B \in \mathcal{B}_\Lambda$. For the two examples we keep in mind we take for $\Lambda = \mathbb{R}$ the Lebesgue measure while for $\Lambda = \mathbb{T}$ the arc-length measure.

Now let us see how we can encode more concretely the point process, or equivalently assert its existence using more simple quantities. For disjoint sets $B_1, \dots, B_n \in \mathcal{B}_\Lambda$ and non negative integers $k_1, \dots, k_n$ such that $\sum_{j=1}^n k_j = m$, the m-th *Jánosssy measure* of $\mathbb{P}$ (encoding its finite dimensional distributions) associated to $B \in \mathcal{B}_\Lambda$ is the (symmetric) Borel measure on B^m given by

$$J_m^B(B_1^{k_1} \times \cdots \times B_n^{k_n}) = \prod_{j=1}^n k_j! \, \mathbb{P}\big(\xi(B) = m, \ \xi(B_j) = k_j \text{ for } j = 1:n\big), \tag{0.0.105}$$

where $\sum_{j=1}^n k_j = m$ and $\sqcup_{j=1}^n B_j = B$. Note that up to the combinatorial prefactor, this is the probability of a typical cylinder set (0.0.102). Note also that the Jánossy measures form a collection of local quantities: there exists one for each set B and each positive integer m. If the point process is finite, which means that $\mathbb{P}$-a.s

$$\xi(\Lambda) < \infty, \tag{0.0.106}$$

then $\mathbb{P}$ is completely determined by the global family $\{J_m^\Lambda\}_{m \geq 1}$, whereas if the point process if $\mathbb{P}$-a.s infinite, then trivially $J_m^\Lambda = 0$ for all m and we typically only have local quantities. Another approach which leads to global quantities is due to Macchi [122]: we define the m-th *factorial moment measure* M_m of $\mathbb{P}$ as the (symmetric) Borel measure M_m on Λ^m such that

$$M_m(B_1^{k_1} \times \cdots \times B_n^{k_n}) = \mathbb{E} \prod_{j=1}^n \xi(B_j)^{[k_j]}, \quad \text{with} \quad l^{[k]} = \frac{l!}{(l-k)!}. \tag{0.0.107}$$

Now although the Jánossy measures exist for at least all bounded Borel sets B, the factorial moment measures might actually not; yet when these exist and are sufficiently regular, they provide a simple way to describe the point process and assert its existence. Let us give more details and examples: recall that we already assumed the existence of a boundedly finite positive Borel measure μ on Λ; we assume the following regularity of $\mathbb{P}$ with respect to μ:

1. the point process $\mathbb{P}$ is μ-simple, i.e. for μ-a.e. $x \in \Lambda$, $\mathbb{P}(\xi(\{x\}) \leq 1) = 1$;
2. $\mathbb{P}$ admits correlation functions of all orders with respect to μ, i.e. for any positive integer m there exists a (symmetric) boundedly integrable functions $\rho_m : \Lambda^m \to [0, \infty)$ with respect to the measure $\mu^{\otimes m}$ on Λ^m such that
$$\mathrm{d}M_m = \rho_m \mathrm{d}^m \mu;$$
3. for any bounded $B \in \mathcal{B}_\Lambda$, there exists $\epsilon_B > 0$ such that
$$\sum_{m=1}^{\infty} \frac{(1+\epsilon_B)^m}{m!} M_m(B^m) < \infty.$$

Under these assumptions, it is a classical fact [118, 135] that the correlation functions $\{\rho_m\}_{m \geq 1}$ uniquely determine the point process $\mathbb{P}$ and induce boundedly finite factorial moment measures of all orders. We also have [122] that for each bounded set $B \in \mathcal{B}_\Lambda$, there exist boundedly integrable and symmetric functions $j_m^B : \Lambda^m \to [0, +\infty)$ called Jánossy densities and such that $\mathrm{d}J_m^B = j_m^B \mathrm{d}^m \mu$. Note that j_m^B, as a Radon-Nikodym derivative, is only defined on B^m, however under the above assumptions we have the identity [122, 66]

$$j^B(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} \rho(\mathbf{x} \sqcup \mathbf{y}) \mathrm{d}^n \mu(\mathbf{y}), \tag{0.0.108}$$

which allows to extend j_m^B to Λ^m, since the series converges in the space of boundedly integrable functions on Λ^m. Here we adopted the conventions

$$\begin{aligned} \Lambda^0 = B^0 = \{\emptyset\}, \quad J_0^B(\emptyset) = j^B(\emptyset) = \mathbb{P}(\xi(B) = 0), \\ M_0(\emptyset) = \rho(\emptyset) = 1, \quad \mu^{\otimes 0} = \delta_\emptyset. \end{aligned} \tag{0.0.109}$$

and have abbreviated

$$j^B(\mathbf{x}) := j_m^B(x_1, \ldots, x_m), \qquad \rho(\mathbf{x}) := \rho_m(x_1, \ldots, x_m),$$

because we interpret $\mathbf{x}$ either as a vector with $m \geq 0$ components $x_1, \ldots, x_m$ or as a configuration $\{x_1, \ldots, x_m\}$ of $m \geq 0$ (not necessarily distinct) points; $\rho(\mathbf{x} \sqcup \mathbf{y})$ then means $\rho_{m+n}(x_1, ..., x_m, y_1, ..., y_n)$ with $\mathbf{x} = \{x_1, \ldots, x_m\}$, $\mathbf{y} = \{y_1, \ldots, y_n\}$. This notation in which we neglect the order of the variables is justified because ρ_m and j_m^B are symmetric in their variables. Moreover, if the third assumption holds also globally, i.e. for $B = \Lambda$, (thus implying that the point process is finite) we have

$$\rho(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j^\Lambda(\mathbf{x} \sqcup \mathbf{y}) \mathrm{d}^n \mu(\mathbf{y}). \tag{0.0.110}$$

Example 0.0.1. Poisson point processes

$\mathbb{P}$ is a Poisson point process if and only if it satisfies the second and third assumption above and

$$\rho_m(x_1, ..., x_m) = \prod_{j=1}^{m} \rho_1(x_j), \tag{0.0.111}$$

in which case ρ_1 is called the intensity (function with respect to μ) of the process. More generally its intensity measure is given by ν defined as

$$\mathrm{d}\nu := \rho_1 \mathrm{d}\mu. \tag{0.0.112}$$

The factorial moment measures are then simply given by

$$M_m = \nu^{\otimes m} \tag{0.0.113}$$

while using (0.0.108) we obtain that for any bounded $B \in \mathcal{B}_\Lambda$

$$j^B(\mathbf{x}) = \rho(\mathbf{x}) e^{-\nu(B)}, \tag{0.0.114}$$

or in other words that the Jánossy measures are equal to

$$J_m^B = e^{-\nu(B)} \nu^{\otimes m}. \tag{0.0.115}$$

This is consistent with what we obtained when discussing non-life insurance: from this and the definition of Jánossy measures we deduce that for $B_1, ..., B_n$ disjoint and $k_1, ..., k_n$ non negative integer it holds

$$\mathbb{P}(\xi(B_j) = k_j,\ j = 1 : n) = \prod_{j=1}^{n} \frac{\nu(B_j)^{k_j}}{k_j!} e^{-\nu(B_j)}, \tag{0.0.116}$$

meaning that $(\xi(B_1), ..., \xi(B_n))$ is a vector of independent random variables, Poisson distributed with parameters $(\nu(B_1), ..., \nu(B_n))$. Note that this point process is simple if and only if ν is non-atomic. Finally, let us compute the Laplace functional: let $\phi \in B_+(\Lambda)$ and assume up to an approximation argument that $\phi = \sum_{j=1}^{m} \alpha_j \mathbf{1}_{B_j}$ with disjoint and bounded B_j, then by independence and using the expression for the moment generating function of the Poisson distribution we obtain

$$\begin{aligned}
\mathcal{L}[\phi] &= \mathbb{E} \exp\left(-\int_\Lambda \phi \mathrm{d}\xi\right) = \mathbb{E} \exp\left(\sum_{j=1}^{m} \alpha_j \xi(B_j)\right) = \prod_{j=1}^{m} \mathbb{E} \exp\left(-\alpha_j \xi(B_j)\right) \\
&= \prod_{j=1}^{m} e^{\nu(B_j)(e^{-\alpha_j} - 1)} = \exp\left(\sum_{j=1}^{m} \nu(B_j)(e^{-\alpha_j} - 1)\right) \\
&= \exp\left(-\int_\Lambda (1 - e^{-\phi}) \mathrm{d}\nu\right).
\end{aligned} \tag{0.0.117}$$

Example 0.0.2. Determinantal point processes

A point process satisfying the above assumptions is determinantal if and only if there exists a correlation kernel $K : \Lambda^2 \to \mathbb{C}$ inducing a locally trace-class operator K on $L^2(\Lambda, \mu)$ such that

$$\rho_m(x_1, \ldots, x_m) = \det_{i,j=1:m} \left(K(x_i, x_j) \right). \tag{0.0.118}$$

Some authors do not require that the kernel induces a locally trace-class operators, however all the meaningful results do require that assumption. We explain how the Laplace functional turns out to be a Fredholm determinant:

$$\mathcal{L}[\phi] = \det \left(1 - \mathrm{M}_{\sqrt{1-e^{-\phi}}} \mathrm{K} \mathrm{M}_{\sqrt{1-e^{-\phi}}} \right), \tag{0.0.119}$$

with M_ϕ the multiplication operator by $\phi \in L^\infty(\Lambda, \mu)$ on $L^2(\Lambda, \mu)$, and the determinant is given by Fredholm's formula

$$\det\left(1 - \mathrm{L}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det_{i,j=1:n} \left(L(x_i, x_j) \right) \prod_{j=1}^{n} \mathrm{d}\mu(x_j). \tag{0.0.120}$$

Note that the kernel K might not be well defined on the diagonal of Λ^2, however we can always assume that $K(x, x)$ is chosen such that for any bounded Borel set B the following holds (see [135]):

$$\mathrm{Tr}\ \mathrm{K}|_{L^2(B,\mu)} = \int_B K(x, x) \mathrm{d}\mu(x).$$

For notational convenience, let us introduce a change of variable in the Laplace functional and define the *average multiplicative functional*

$$L[\phi] := \mathbb{E} \prod_{x \in \operatorname{supp} \xi} (1 - \phi(x)) = \mathcal{L}[-\log(1 - \phi)], \tag{0.0.121}$$

for $\phi : \Lambda \to [0, 1]$ measurable and with bounded support, such that $L[\phi] = \det\left(1 - \mathrm{M}_{\sqrt{\phi}} \mathrm{K} \mathrm{M}_{\sqrt{\phi}}\right)$ if $\mathbb{P}$ is the DPP with kernel of the operator K. Now let us distribute the product in such a way that

$$\prod_{x \in \operatorname{supp} \xi} (1 - \phi(x)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{x_1 \neq \ldots \neq x_n} \phi(x_1) \ldots \phi(x_n), \tag{0.0.122}$$

and introduce the measures $\xi^{[n]}$ on Λ^n as (remember that the points in the support of ξ are almost surely distinct)

$$\xi^{[n]} = \sum_{x_1 \neq \ldots \neq x_n} \delta_{x_1} \otimes \ldots \otimes \delta_{x_n}, \qquad \xi = \sum_{j \in J} \delta_{x_j}. \tag{0.0.123}$$

A moment of thought reveals that these satisfy for $B_1, ..., B_m$ disjoint, $k_1, ..., k_m$ non-negative integers with $\sum_{j=1}^m k_j = n$

$$\xi^{[n]}(B_1^{k_1} \times ... \times B_m^{k_m}) = \prod_{j=1}^m \xi(B_j)^{[k_j]}, \tag{0.0.124}$$

so that their expectations are actually the factorial moment measures (cf. (0.0.107)):

$$\mathbb{E}\xi^{[n]} = M_n. \tag{0.0.125}$$

We thus deduce that

$$\begin{aligned} L[\phi] &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \mathbb{E} \int_{\Lambda^n} \phi(x_1)...\phi(x_n) \mathrm{d}\xi^{[n]}(x_1, ..., x_n) \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_{\Lambda^n} \rho_n(x_1, ..., x_n) \prod_{j=1}^n \phi(x_j) \mathrm{d}\mu(x_j), \end{aligned} \tag{0.0.126}$$

and recognize Fredholm's formula (0.0.120) for the determinant given the structure of the correlation functions. Now we do not prove it, but the Jánossy densities also enjoy a similar determinantal structure: we actually have

$$j_m^B(x_1, ..., x_m) = \det(1 - \mathrm{M}_{\mathbf{1}_B} \mathrm{K} \mathrm{M}_{\mathbf{1}_B}) \det_{i,j=1:m} \left(R_B(x_i, x_j) \right), \tag{0.0.127}$$

where $R_B(x, y)$ is the integral kernel of the operator R_B defined by

$$\mathrm{R}_B := (1 - \mathrm{K}\mathrm{M}_{\mathbf{1}_B})^{-1} \mathrm{K}, \tag{0.0.128}$$

provided $\det(1 - \mathrm{M}_{\mathbf{1}_B}\mathrm{K}\mathrm{M}_{\mathbf{1}_B}) \neq 0$, or equivalently $1 - \mathrm{K}\mathrm{M}_{\mathbf{1}_B}$ is invertible, and we have a more explicit identity

$$R_B(x, y) = \frac{\displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_{B^n} \det_{i,j=1:n} \begin{pmatrix} K(x, y) & (K(x, w_j)) \\ (K(w_i, y)) & (K(w_i, w_j)) \end{pmatrix} \prod_{j=1}^n \mathrm{d}\mu(w_j)}{\det(1 - \mathrm{M}_{\mathbf{1}_B}\mathrm{K}\mathrm{M}_{\mathbf{1}_B})}. \tag{0.0.129}$$

Note that if we had instead plugged the expression for the correlation of a Poisson point process (0.0.111) we would have obtained

$$L[\phi] = \exp\left(- \int_\Lambda \phi \mathrm{d}\nu \right), \tag{0.0.130}$$

which is consistent given the change between $\mathcal{L}$ and L. We conclude by mentioning a very practical result due to Macchi [122] and Soshnikov [135] establishing the existence of the point process potentially induced by an operator: if K is a hermitian locally trace-class operator on $L^2(\Lambda, \mu)$, then it induces a (unique) point process if and only if $0 \leq \mathrm{K} \leq 1$. It is still an open question which conditions on an non-Hermitian locally trace-class operators to impose in order for it to induce a determinantal point process; examples have arisen in the literature but had to be treated case by case.

Example 0.0.3. Biorthogonal ensembles (BiOEs).

Let us explore a particular kind of determinantal point process arising quite frequently in applications: these are called Biorthogonal ensembles (BiOEs) and are such that for a certain positive integer n there exist $\{\varphi_j\}_{j=1:n}, \{\psi_j\}_{j=1:n} \subset L^2(\Lambda, \mu)$ such that the n-th global Jánossy density is given by

$$j_n^\Lambda(\mathbf{x}) = \frac{1}{Z_n} \det_{l,k=1:n} \left(\varphi_l(x_k)\right) \det_{l,k=1:n} \left(\psi_l(x_k)\right), \tag{0.0.131}$$

where $Z_n > 0$ is a normalisation constant so that

$$\int_{\Lambda^n} j_n^\Lambda \mathrm{d}^n \mu = n!. \tag{0.0.132}$$

Note at once that by definition of Jánossy measure/density this implies

$$\mathbb{P}(\xi(\Lambda) = n) = 1, \tag{0.0.133}$$

hence we have a finite point process with exactly n points. To show that it is determinantal, there are two ways: the first relies on Andréief's formula (see [6, Chapter 6]), which states that

$$\frac{1}{n!} \int_{\Lambda^n} \det_{l,k=1:n} (\varphi_l(x_k)) \det_{l,k=1:n} (\psi_l(x_k)) \, \mathrm{d}\mu(x_1)...\mathrm{d}\mu(x_n) = \det_{l,k=1:n} \left(\int_\Lambda \varphi_l \psi_k \mathrm{d}\mu \right), \tag{0.0.134}$$

while the second make use of the integrating-out lemma (0.0.136) (see [81, Chapter 5]), which for an integral kernel K_n satisfying

$$\int_\Lambda K_n(x,w)K_n(w,y)\mathrm{d}\mu(w) = K_n(x,y), \qquad \int_\Lambda K_n(x,x)\mathrm{d}\mu(x) = n, \tag{0.0.135}$$

ensures that

$$\int_\Lambda \det_{l,k=1:m} (K_n(x_l,x_k))\mathrm{d}\mu(x_j) = (n-m+1) \det_{l,k=1:m,\neq j} (K_n(x_l,x_k)). \tag{0.0.136}$$

Note that this is equivalent to the operator K_n induced by K_n being a projection of rank n. Now let us first compute the average multiplicative function $L[\phi]$ for $\phi : \Lambda \to [0,1]$: thanks to Andréief's formula we compute that

$$Z_n = \det_{l,k=1:n} \left(\int_\Lambda \varphi_l \psi_k \mathrm{d}\mu \right), \tag{0.0.137}$$

while applying it with the measure $(1-\phi)\mathrm{d}\mu$ instead shows that

$$L[\phi] = \int_\Lambda j_n^\Lambda(\mathbf{x}) \prod_{j=1}^n (1-\phi(x_j))\mathrm{d}\mu(x_j) = \frac{1}{Z_n} \det_{l,k=1:n} \left(\int_\Lambda \varphi_l \psi_k (1-\phi) \mathrm{d}\mu \right). \tag{0.0.138}$$

Using the multiplicativity of the determinant as well as a well-known formula for the Fredholm determinant of finite-rank perturbation of the identity (see [89, Chapter 1]) reveals that

$$\begin{aligned} L[\phi] &= \det\left((\delta_{l,k})_{l,k=1:n} - \left(\int_\Lambda \varphi_l \psi_k \mathrm{d}\mu\right)^{-1}_{l,k=1:n} \left(\int_\Lambda \varphi_l \sqrt{\phi}\sqrt{\phi}\psi_k \mathrm{d}\mu\right)\right) \\ &= \det\left(1 - \mathrm{M}_{\sqrt{\phi}}\mathrm{K}_n\mathrm{M}_{\sqrt{\phi}}\right), \end{aligned} \tag{0.0.139}$$

where K_n is the operator with integral kernel

$$K_n(x,y) = \begin{pmatrix}\psi_1(x) & \cdots & \psi_n(x)\end{pmatrix} \left(\int_\Lambda \varphi_l\psi_k \mathrm{d}\mu\right)^{-1}_{l,k=1:n} \begin{pmatrix}\varphi_1(y) \\ \vdots \\ \varphi_n(y)\end{pmatrix}. \tag{0.0.140}$$

Note that K_n satisfies the assumption of the integrating out lemma (0.0.136). Let us then use the latter to compute the correlation functions: up to using a Gram-Schmidt procedure, we can assume that

$$\int_\Lambda \varphi_l\psi_k \mathrm{d}\mu = \delta_{lk}, \tag{0.0.141}$$

with the orthonormalization ensured by the fact that if we have biorthogonality then the Gram determinant reduces to

$$Z_n = \det_{l,k=1:n}\left(\int_\Lambda \varphi_l\psi_k\mathrm{d}\mu\right) = \prod_{j=1}^n \int_\Lambda \varphi_j\psi_j \mathrm{d}\mu. \tag{0.0.142}$$

Therefore the Jánossy density becomes by multiplicativity of the determinant

$$j_n^\Lambda(\mathbf{x}) = \det_{l,k=1:n}\left(\sum_{j=1}^n \varphi_j(x_l)\psi_j(x_k)\right) = \det_{l,k=1:n}(K_n(x_l,x_k)). \tag{0.0.143}$$

Now Macchi's relations (0.0.110) simplify here to

$$\rho_m(\mathbf{x}) = \frac{1}{(n-m)!}\int_{\Lambda^{n-m}} j_n^\Lambda(\mathbf{x}\sqcup\mathbf{y})\mathrm{d}^{n-m}\mu(\mathbf{y}), \tag{0.0.144}$$

since $j_m^\Lambda = 0$ for any $m \neq 0$. Note that we already encountered it in (0.0.48). Using the integrating out lemma (0.0.136) repeatedly $n-m$ times yields that indeed

$$\rho_m(\mathbf{x}) = \det_{l,k=1:m}(K_n(x_l,x_k)). \tag{0.0.145}$$

We recover the point processes considered by Macchi by merely taking $\varphi_j = \overline{\psi}_j$, which are then orthogonal projection of rank n.

Example 0.0.4. DPPs induced by projections.

Inspired by the previous example, we can consider a determinantal point process induced by a projection $\mathrm{K} = \mathrm{P}_{H,J}$, where

$$H = \operatorname{ran}\mathrm{K}, \qquad J^{\perp} = \ker\mathrm{K}, \qquad H \oplus J^{\perp} = L^2(\Lambda, \mu). \tag{0.0.146}$$

Taking

$$H = \operatorname{span}\{\psi_j\}_{j=1:n}, \qquad J = \operatorname{span}\{\overline{\varphi_j}\}_{j=1:n}, \tag{0.0.147}$$

we recover the previous example. For such point processes, the expected number of points is

$$\mathbb{E}\xi(\Lambda) = \int_\Lambda K(x,x)\mathrm{d}\mu(x) = \operatorname{rank}\mathrm{K}, \tag{0.0.148}$$

while its variance is given by

$$\begin{aligned}\mathbb{V}\xi(\Lambda) &= \mathbb{E}\xi(\Lambda)^{[2]} + \mathbb{E}\xi(\Lambda) - \mathbb{E}^2\xi(\Lambda)\\ &= \int_\Lambda \left(K(x,x) - \int_\Lambda K(x,y)K(y,x)\mathrm{d}\mu(y)\right)\mathrm{d}\mu(x)\\ &= \operatorname{tr}(\mathrm{K}-\mathrm{K}^2) = 0,\end{aligned} \tag{0.0.149}$$

whence it holds

$$\mathbb{P}(\xi(\Lambda) = \operatorname{rank}\mathrm{K}) = 1. \tag{0.0.150}$$

Here if K is an infinite rank projection some approximation procedure is needed. Examples where $\operatorname{rank}\mathrm{K} = \infty$ include the sine kernel (0.0.76) and the Airy kernel (0.0.92). Conversely, using the Macchi-Soshnikov theorem, Soshnikov proved that a self-adjoint operator induces a point process with a fixed total number of points if and only if it is orthogonal projection with rank equal to the number of points.

Chapter 1

Conditioning on randomly incomplete configurations

This chapter retakes my second paper [58] in collaboration with Tom Claeys, where we answer the questions discussed in the introduction of this thesis. Most of the results are my own original research, except for the section on rigidity.

Abstract

For a broad class of point processes, including determinantal point processes, we construct associated marked and conditional ensembles, which allow to study a random configuration in the point process, based on information about a randomly incomplete part of the configuration. We show that our construction yields a well behaving transformation of sufficiently regular point processes. In the case of determinantal point processes, we explain that special cases of the conditional ensembles already appear implicitly in the literature, namely in the study of unitary invariant random matrix ensembles, in the Its-Izergin-Korepin-Slavnov method to analyse Fredholm determinants, and in the study of number rigidity. As applications of our construction, we show that a class of determinantal point processes induced by orthogonal projection operators, including the sine, Airy, and Bessel point processes, satisfies a strengthened notion of number rigidity, and we give a probabilistic interpretation of the Its-Izergin-Korepin-Slavnov method.

1.1 Introduction

1.1.1 Background and motivation

Determinantal point processes (DPPs) are point processes whose correlation functions can be written as determinants of a correlation kernel, and for which average multiplicative statistics are Fredholm determinants. Prominent examples of DPPs are the eigenvalue distributions of a large class of random matrix ensembles, distributions of particles in asymmetric exclusion processes and tiling models, distributions of non-intersecting random paths, and the zeros of Gaussian analytic functions. They are special cases of repulsive point processes, in which one can study relevant probabilistic quantities through the analysis of the correlation kernel and associated Fredholm determinants [6, 94, 103, 121, 122, 135].

A groundbreaking discovery for the development of random matrix theory and more generally the study of DPPs has been the observation of Wigner and Dyson and their collaborators in the 1960s that energy levels of heavy nuclei can be accurately modelled by eigenvalues of random matrices. Despite his spectacular contributions, when Dyson looked back at his work on heavy nuclei in 2002 during the MSRI program *Recent Progress in Random Matrix Theory and Its Applications*, he explained [75] that the practical implications of his work on random matrices in nuclear physics were disappointing, because detectors were imperfect, and missing or spurious energy levels corrupted the data. Inspired by this, Dyson raised the question to develop error-correcting code for random matrix eigenvalues: given an imperfect observed spectrum of a random matrix, can one detect missing or spurious eigenvalues? This would not be possible for point processes with independent points, because the positions of a fraction of the points in the process do not carry any information about the other points. In strongly correlated point configurations such as random matrix eigenvalues or DPPs, one can however hope to extract information based on incomplete data. According to [75], Dyson did not suggest this direction of research because of its importance in nuclear physics, but purely because he believed it would lead to interesting mathematics. This question has been explored by Bohigas and Pato [23, 24] using randomly thinned random matrix eigenvalues, and has been picked up in the mathematics literature with the study of random thinnings of DPPs [19, 33, 34, 36, 37, 50, 51, 52, 53, 82], but a general mathematical theory for extracting information from the observation of randomly thinned DPPs has not been developed so far.

However, in the same spirit of attempting to extract information about DPPs from a partial observation, the remarkable property of number rigidity has recently been investigated. Informally, a point process is said to be number rigid if the configuration of points outside any bounded set determines almost surely the number of points inside the set. Important DPPs like the sine, Airy, and Bessel point processes arising in random matrix theory, are known to be number rigid [43, 86, 88, 121], and in the case of the sine process, the distribution of the points inside a bounded set, conditioned on the configuration of points outside the set, has been studied and proved to converge to the sine process when the size of the interval grows [112].

In this work, for any sufficiently regular point process, and in particular for any DPP, we introduce a family of marked and conditional point processes which allow to formalize the following question: *Given a randomly incomplete sample of the point process, what can we say about the missing points?* Although these point processes have, to the best of our knowledge, not been introduced and studied on a general basis, special cases of them do already appear in the literature in various contexts, as we will explain in more detail later; firstly, unitarily invariant Hermitian random matrix ensembles are a special case of conditional ensembles associated to the Gaussian Unitary Ensemble (GUE); secondly, special cases of the conditional ensembles arise naturally in the Its-Izergin-Korepin-Slavnov (IIKS) [98] method to characterize Fredholm determinants via Riemann-Hilbert problems; and finally, special cases of the conditional ensembles have been studied in relation to number rigidity.

Our objectives are:

1. to construct the marked and conditional ensembles rigorously;
2. to prove that the conditional ensembles define a well-behaving transformation which preserves the structure of DPPs and of several interesting subclasses of DPPs;
3. to introduce a refined notion of number rigidity and to show that important DPPs like the sine, Airy, and Bessel DPPs satisfy this notion of rigidity;
4. to illustrate that the IIKS method provides an effective framework to study the conditional ensembles via Riemann-Hilbert methods.

1.1.2 DPPs: generalities and main examples

Consider a measure space $(\Lambda, \mathcal{B}_\Lambda, \mu)$, with Λ a complete separable metric space, $\mathcal{B}_\Lambda$ the Borel σ-algebra, and μ a locally[1] finite positive Borel measure on Λ, i.e. satisfying $\mu(B) < \infty$ for any bounded $B \in \mathcal{B}_\Lambda$. We will be mainly interested in $\Lambda = \mathbb{R}$ with the Lebesgue measure or Λ the unit circle in the complex plane with the arc length measure, and the reader may prefer to keep only these examples in mind for the sake of simplicity. Let $\mathbb{P}$ be a simple point process on Λ, i.e. a probability measure on the set $\mathcal{N}(\Lambda)$ of locally finite point configurations in Λ (see Section 1.2 for a more precise definition of the probability space), such that there are a.s. no points with multiplicity > 1. We can represent such a configuration $\xi \in \mathcal{N}(\Lambda)$ as a locally finite counting measure

$$\xi = \sum_{j \in J} \delta_{x_j},$$

where J is a countable index set, and $x_j \in \Lambda$, $x_i \neq x_j$ when $i \neq j$. Recall (see e.g. [66, Section 9.4]) that a simple point process on Λ is characterized uniquely by its Laplace functional

$$\mathcal{L} : B_+(\Lambda) \to \mathbb{R}^+ : f \mapsto \mathcal{L}[f], \qquad \mathcal{L}[f] = \mathbb{E} e^{-\sum_{x \in \operatorname{supp} \xi} f(x)} = \mathbb{E} e^{-\int_\Lambda f \mathrm{d}\xi},$$

where $B_+(\Lambda)$ is the space of bounded non-negative measurable functions $f : \Lambda \to [0, +\infty)$ with bounded support.

[1] Here and for the rest of this paper, whenever we say that a property holds locally, we mean that it holds for any bounded Borel set.

Some of our results hold for any sufficiently regular point process, but our main focus will be on DPPs, for which the correlation functions $\rho_k : \Lambda^k \to [0, +\infty)$ (see again Section 1.2 for details) of all orders exist and can be written in terms of a correlation kernel $K(x_i, x_j)$ in determinantal form:

$$\rho_k(x_1, \dots, x_k) = \det \left(K(x_i, x_j)\right)_{i,j=1}^{k}. \tag{1.1.1}$$

If $K : \Lambda^2 \to \mathbb{C}$ is the kernel of a locally trace class operator K on $L^2(\Lambda, \mu)$, then the Laplace functional is a Fredholm determinant:

$$\mathcal{L}[f] = \det\left(1 - \mathrm{M}_{\sqrt{1-e^{-f}}}\mathrm{K}\mathrm{M}_{\sqrt{1-e^{-f}}}\right), \tag{1.1.2}$$

with M_g the multiplication operator with $g \in L^\infty(\Lambda, \mu)$ on $L^2(\Lambda, \mu)$, and the determinant is given by Fredholm's formula

$$\det\left(1 - \mathrm{M}_{\sqrt{g}}\mathrm{K}\mathrm{M}_{\sqrt{g}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \det_{l,k=1:n} \left(\sqrt{g(x_l)} K(x_l, x_k) \sqrt{g(x_k)}\right) \prod_{j=1}^{n} \mathrm{d}\mu(x_j). \tag{1.1.3}$$

Note that the kernel K might not be well defined on the diagonal of Λ^2, however we can always assume that $K(x, x)$ is chosen such that for any bounded Borel set B the following holds (see [135]):

$$\mathrm{Tr}\ \mathrm{K}|_{L^2(B,\mu)} = \int_B K(x, x)\mathrm{d}\mu(x).$$

For notational convenience, let us introduce a change of variable in the Laplace functional and define the *average multiplicative functional*

$$L[\phi] := \mathbb{E} \prod_{x \in \operatorname{supp} \xi} (1 - \phi)(x) = \mathcal{L}[-\log(1 - \phi)], \tag{1.1.4}$$

for $\phi : \Lambda \to [0, 1]$ measurable and with bounded support, such that $L[\phi] = \det\left(1 - \mathrm{M}_{\sqrt{\phi}}\mathrm{K}\mathrm{M}_{\sqrt{\phi}}\right)$ if $\mathbb{P}$ is the DPP with kernel of the operator K.

Besides DPPs, it will be insightful to keep in mind the example of a Poisson point process with bounded locally integrable intensity $\rho : \Lambda \to [0, +\infty)$, for which

$$\rho_k(x_1, \dots, x_k) = \prod_{j=1}^{k} \rho(x_j). \tag{1.1.5}$$

In Sections 1.3–1.5, we will consider some important subclasses of DPPs, which we already define now.

Example 1.1.1. Orthogonal polynomial ensembles (OPEs). *Let N be a positive integer and consider the point process consisting of configurations of N real points $x_1, \dots, x_N$ with joint probability distribution*

$$\frac{1}{Z_N} \Delta(x_1, \dots, x_N)^2 \prod_{j=1}^{N} w(x_j)\mathrm{d}x_j, \qquad \Delta(x_1, \dots, x_N) = \prod_{1 \le i < j \le N} (x_j - x_i), \tag{1.1.6}$$

where Z_N is a normalization constant, and $w(x)$ is a non-negative integrable weight function decaying sufficiently fast as $x \to \pm\infty$, such that all the moments $\int_{\mathbb{R}} x^k w(x)dx$, $k \in \mathbb{N}$, exist. If $w(x) = e^{-2Nx^2}$, this is the distribution of the (re-scaled, such that the eigenvalues follow a semi-circle law on $[-1,1]$) eigenvalues of a random matrix from the GUE. If $w(x) = x^\alpha e^{-Nx} 1_{(0,+\infty)}(x)$ with $\alpha > -1$, it is the distribution of the eigenvalues of a random matrix in the Laguerre-Wishart ensemble. More generally, if w takes the form $w(x) = e^{-NV(x)}$ with V real analytic and growing sufficiently fast at $\pm\infty$, (1.1.6) is the eigenvalue distribution of a random matrix in the unitary invariant ensemble

$$\frac{1}{\widehat{Z}_N} e^{-N\mathrm{Tr}V(M)} \mathrm{d}M,$$

with $\mathrm{d}M$ the Lebesgue measure on the space of $N \times N$ Hermitian matrices, and $\widehat{Z}_N$ a normalization constant.

Similarly, let N be a positive integer and consider the point process consisting of configurations of N points $e^{it_1}, \dots, e^{it_N}$ on the unit circle in the complex plane with joint probability distribution

$$\frac{1}{Z_N} \prod_{1 \le j < k \le N} |\Delta(e^{it_1}, \dots, e^{it_N})|^2 \prod_{j=1}^{N} w(e^{it_j}) \mathrm{d}t_j, \qquad t_j \in [0, 2\pi), \tag{1.1.7}$$

where Z_N is a normalization constant, and $w(e^{it})$ is a non-negative integrable weight function. If $w(e^{it}) = 1$, this is the distribution of the eigenvalues of a random matrix from the Circular Unitary Ensemble (CUE), or in other words a Haar distributed $N \times N$ unitary matrix.

It is well-known that the above OPEs are DPPs, with correlation kernel K_N built out of orthogonal polynomials on the real line or on the unit circle. We will study these ensembles in more detail in Section 1.4.

Example 1.1.2. DPPs induced by orthogonal projection operators. *Consider a DPP with correlation kernel K whose associated integral operator K on $L^2(\Lambda, \mu)$, defined by*

$$\mathrm{K}f(x) = \int_\Lambda K(x,y) f(y) \mathrm{d}\mu(y), \tag{1.1.8}$$

is a locally trace class orthogonal projection onto a closed vector subspace H $L^2(\Lambda, \mu)$. As we will see, the OPEs from Example 1.1.1 are of this form, and the associated projection operators are then of rank N. We recall from [135] that a DPP defined by the kernel of a Hermitian locally trace class operator K has the property that the number of particles is a.s. equal to N, i.e. $\mathbb{P}(\xi(\Lambda) = N) = 1$, if and only if K is a projection operator of rank N. We will also consider DPPs induced by infinite rank projection operators. Such DPPs arise for instance when taking scaling limits of the kernels K_N from Example 1.1.1: we mention the DPPs defined by the sine kernel, the Airy kernel, the edge Bessel kernel, and the bulk Bessel kernel [68, 110]. More complicated kernels associated to Painlevé equations and hierarchies (see [74] for an overview), arising as double scaling limits of OPEs, are also of this form. We will consider such DPPs and derive rigidity results for some of them in Section 1.3.

Example 1.1.3. DPPs with integrable kernels. *In line with the terminology of Its, Izergin, Korepin, and Slavnov [98], we say that a kernel $K(x,y)$ is k-integrable if it can be written in the form*

$$K(x,y) = \frac{\sum_{j=1}^{k} f_j(x)g_j(y)}{x-y} \qquad \text{with } \sum_{j=1}^{k} f_j(x)g_j(x) = 0, \tag{1.1.9}$$

for some functions $f_j, g_j : \Lambda \to \mathbb{C}$, $j = 1, \ldots, k$. The previous examples of OPEs on the real line and on the unit circle are 2-integrable, and so are the sine point process, the Airy point process, and the Bessel point processes.

There are however many DPPs with integrable kernels that are not induced by projection operators. Indeed, if a kernel $K(x,y)$ defines a DPP on Λ, then any kernel of the form $\phi(x)K(x,y)$ with $\phi : \Lambda \to [0,1]$ measurable also defines a DPP, namely the random thinning of the original DPP realized by removing each particle x in the support of a random point configuration ξ independently with probability $1-\phi(x)$ [114]. If $K(x,y)$ is of integrable form, it is easy to see that the same is true for $\phi(x)K(x,y)$, but even if $K(x,y)$ defines an orthogonal projection operator, $\phi(x)K(x,y)$ in general does not define a projection operator. DPPs with integrable kernels will be our topic of interest in Section 1.5.

1.1.3 Marking and conditioning: informal construction and statement of results

For any sufficiently regular point process $\mathbb{P}$, we can construct an associated marked point process in which we assign a random mark to each point independently. If the random mark is a Bernoulli random variable taking the value 0 or 1, then the marked point process is a point process on $\Lambda \times \{0,1\}$, in which we interpret the points with mark 1 as visible or observed particles, and the points with mark 0 as invisible or unobserved particles. Concretely, we mark the points in the DPP by introducing a measurable marking function $\theta : \Lambda \to [0,1]$, and by assigning mark 1 to particle x in a configuration of the DPP with probability $\theta(x)$, and mark 0 with probability $1-\theta(x)$. We denote the resulting marked point process as $\mathbb{P}^\theta$. The random marking splits a configuration ξ on Λ into configurations ξ_0 and ξ_1, where ξ_b is the configuration ξ restricted to the points with mark b. We denote $\mathbb{P}_b^\theta$, $b = 0,1$, for the marginal probability distribution of ξ_b, which is a random position-dependent thinning of the ground process $\mathbb{P}$. We will introduce these marked point processes in detail in Section 1.2, and gather some of their general properties in Proposition 1.2.2. The point processes in which we are most interested here, are point processes obtained as conditional ensembles of this marked point process, by conditioning on the (observed) configuration of mark 1 points.

In the remaining part of this section, for the sake of simplicity, we will present our main results about these conditional ensembles only in the case where $\mathbb{P}$ is a DPP. We note however that most of our results hold for more

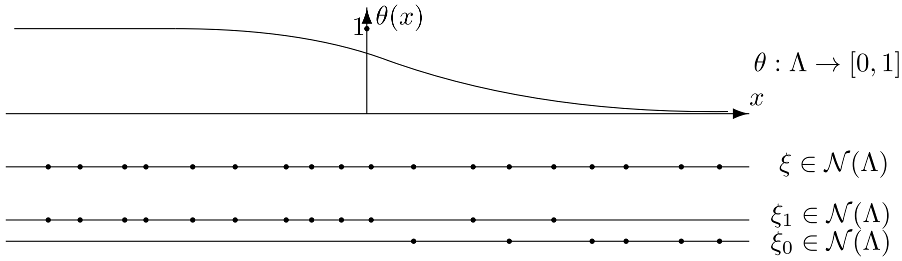

Figure 1.1: Illustration of the marked point process $\mathbb{P}^\theta$: at the top, we see the graph of a possible marking function θ; in the middle, a possible configuration ξ corresponding to the point process $\mathbb{P}$; at the bottom, possible associated mark 0 and mark 1 configurations ξ_0 and ξ_1 corresponding to $\mathbb{P}^\theta$.

general point processes. The theorems stated below are thus special cases of more general results, stated in full generality and proved in later sections.

In the simplest case, we condition on the event that no points have mark 1 (in other words, there are no observed particles). If this event has non-zero probability, then the resulting conditional point process, which we will denote as $\mathbb{P}^\theta_{|\emptyset}$, is defined in the classical sense, and configurations in this point process have support in $\Lambda \times \{0\}$. Hence, by omitting the marks, we can identify configurations in this point process with configurations on Λ, and identify $\mathbb{P}^\theta_{|\emptyset}$ with a point process on Λ. The following result about the point process transformation $\mathbb{P} \mapsto \mathbb{P}^\theta_{|\emptyset}$, which is part of the more general Theorem 1.2.4 in Section 1.2, will be fundamental for our concerns.

Theorem 1.1.1. *Let $\mathbb{P}$ be the DPP with kernel K of a locally trace class operator K and let $\theta : \Lambda \to [0,1]$ be measurable and such that $\mathrm{M}_{\sqrt{\theta}+1_B}\mathrm{K}\mathrm{M}_{\sqrt{\theta}+1_B}$ is trace class for any bounded Borel set B, and*

$$\mathbb{P}^\theta(\xi_1(\Lambda) = 0) = \det(1 - \mathrm{M}_{\sqrt{\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\theta}}) > 0.$$

Then $\mathbb{P}^\theta_{|\emptyset}$ is also a DPP, defined by the kernel of the $L^2(\Lambda, \mu)$-operator

$$\mathrm{M}_{1-\theta}\mathrm{K}(1 - \mathrm{M}_\theta \mathrm{K})^{-1}. \tag{1.1.10}$$

Remark 1.1.2. *If the locally trace-class operator K is self-adjoint, it induces a DPP if and only if $0 \le \mathrm{K} \le 1$ [135]. In this case, the condition that $\mathrm{M}_{\sqrt{\theta}+1_B}\mathrm{K}\mathrm{M}_{\sqrt{\theta}+1_B}$ is trace class for any bounded Borel set B, is equivalent to the integrability condition $\int_\Lambda (\sqrt{\theta(x)} + 1_B(x))^2 K(x,x)\mathrm{d}\mu(x) < \infty$, which is automatically satisfied whenever $\int_\Lambda \theta(x)K(x,x)d\mu(x) < \infty$, if $K(x,x)$ is locally integrable. The trace class condition is then practical to verify in concrete situations. However, for non self-adjoint operators K, $\operatorname{tr}\mathrm{K} < \infty$ does not imply K being trace class, and then the condition that $\mathrm{M}_{\sqrt{\theta}+1_B}\mathrm{K}\mathrm{M}_{\sqrt{\theta}+1_B}$ is trace class cannot be verified directly by computing a trace. In such cases, one rather tries to prove that an operator is a composition of Hilbert-Schmidt operators, to prove that it is trace class.*

Remark 1.1.3. *Since*

$$\mathrm{K}(1-\mathrm{M}_\theta\mathrm{K})^{-1} = (1-\mathrm{K}\mathrm{M}_\theta)^{-1}\mathrm{K} = \mathrm{K} + \mathrm{K}\mathrm{M}_{\sqrt{\theta}}(1-\mathrm{M}_{\sqrt{\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\theta}})^{-1}\mathrm{M}_{\sqrt{\theta}}\mathrm{K},$$

the operator $\mathrm{K}(1-\mathrm{M}_\theta\mathrm{K})^{-1}$ *indeed exists provided that* $\det(1-\mathrm{M}_{\sqrt{\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\theta}}) > 0$. *If* K *is self-adjoint, the operator* (1.1.10) *is in general not self-adjoint, however the operator*

$$\mathrm{M}_{\sqrt{1-\theta}}\mathrm{K}(1-\mathrm{M}_\theta\mathrm{K})^{-1}\mathrm{M}_{\sqrt{1-\theta}}$$

is self-adjoint, and it is readily verified that this operator induces the same DPP $\mathbb{P}^\theta_{|\emptyset}$. *If* K *is a projection, then it is easily seen that* (1.1.10) *is equal to the conjugation* $(1-\mathrm{M}_\theta\mathrm{K})\mathrm{K}(1-\mathrm{M}_\theta\mathrm{K})^{-1}$ *of* K.

The probability to observe a given non-empty finite configuration of points in the marked point process will typically be zero, but we can still, $\mathbb{P}^\theta$-a.s., condition on such events by making use of disintegration and reduced Palm measures (see Section 1.2 for details). Given a mark 1 configuration $\mathbf{v} = \{v_1, \dots, v_m\}$, we will denote this conditional ensemble, which we will define properly in Section 1.2.4 below, as $\mathbb{P}^\theta_{|\mathbf{v}}$. Before stating our main result about $\mathbb{P}^\theta_{|\mathbf{v}}$ in the case where $\mathbb{P}$ is a DPP, we need to introduce the reduced Palm measure $\mathbb{P}_v$ of $\mathbb{P}$ associated to a point $v \in \Lambda$. This represents the conditional ensemble obtained by first conditioning $\mathbb{P}$ on the event $v \in \operatorname{supp}\xi$, and then removing the point v from the configuration. If $\mathbb{P}$ is the DPP with kernel K and if $K(v,v) > 0$, then [131] $\mathbb{P}_v$ is also a DPP, with kernel

$$K_v(x,y) = \frac{\det\begin{pmatrix} K(x,y) & K(x,v) \\ K(v,y) & K(v,v) \end{pmatrix}}{K(v,v)}. \tag{1.1.11}$$

Similarly, we can condition $\mathbb{P}$ on the presence of a finite number of distinct points $\mathbf{v} = \{v_1, \dots, v_m\}$. This is consistent in the sense that the reduced Palm measure $\mathbb{P}_{\mathbf{v}} = \mathbb{P}_{v_1,\dots,v_m}$ is, for $\mu^{\otimes m}$-a.e. $\mathbf{v} \in \Lambda^m$ such that $\det(K(v_\ell, v_k))_{\ell,k=1}^m > 0$, equal to the measure $\big((\mathbb{P}_{v_1})_{v_2}\cdots\big)_{v_m}$ obtained by iteratively conditioning on $v_1, \dots, v_m$, for any chosen order of the points. Let us for notational convenience write $K(\mathbf{v},\mathbf{v})$ for the $m \times m$ matrix $\big(K(v_\ell, v_k)\big)_{\ell,k=1}^m$, $K(x,\mathbf{v})$ for the row vector$(K(x,v_k))_{k=1}^m$, and $K(\mathbf{v},y)$ for the column vector $(K(v_\ell,y))_{\ell=1}^m$. If $\mathbb{P}$ is a DPP with kernel K and if $\det K(\mathbf{v},\mathbf{v}) > 0$, then $\mathbb{P}_{\mathbf{v}}$ is the DPP with kernel given by

$$K_{\mathbf{v}}(x,y) = \frac{\det\begin{pmatrix} K(x,y) & K(x,\mathbf{v}) \\ K(\mathbf{v},y) & K(\mathbf{v},\mathbf{v}) \end{pmatrix}}{\det K(\mathbf{v},\mathbf{v})}, \tag{1.1.12}$$

which defines a finite rank perturbation of K. Let us also set for consistency the convention that when $\mathbf{v} = \emptyset$, $\mathbb{P}_\emptyset = \mathbb{P}$ and $K_\emptyset = K$.

In analogy to and as a generalisation of Theorem 1.1.1, we have the following result, which is part of the more general Theorem 1.2.7 below.

Theorem 1.1.4. *If $\mathbb{P}$ is the DPP with locally trace class operator* K *and* $\theta \in L^\infty(\Lambda, \mu)$ *is such that* $\mathrm{M}_{\sqrt{\theta}+1_B}\mathrm{K}\mathrm{M}_{\sqrt{\theta}+1_B}$ *is trace class for any bounded Borel set* B*, then for* $\mathbb{P}^\theta$*-a.e.* ξ_1*, writing* $\mathbf{v} = \operatorname{supp}\xi_1$*, we have* $\det\left(1 - \mathrm{M}_{\sqrt{\theta}}\mathrm{K}_{\mathbf{v}}\mathrm{M}_{\sqrt{\theta}}\right) \neq 0$*, and* $\mathbb{P}^\theta_{|\mathbf{v}}$ *is also a DPP, defined by the* $L^2(\Lambda, \mu)$*-operator*

$$\mathrm{M}_{1-\theta}\mathrm{K}_{\mathbf{v}}\left(1 - \mathrm{M}_\theta \mathrm{K}_{\mathbf{v}}\right)^{-1}. \tag{1.1.13}$$

Remark 1.1.5. *This result implies that the class of DPPs is stable under the transformation* $\mathbb{P} \mapsto \mathbb{P}^\theta_{|\mathbf{v}}$*. More is actually true: as we will see, each of the subclasses of DPPs defined in Examples 1.1.1–1.1.3 are also stable, and in Assumptions 1.2.1 below, we will define a larger class of (not necessarily determinantal) point processes which is stable under this transformation.*

Section 1.2 will be devoted to the rigorous construction of the marked and conditional point processes $\mathbb{P}^\theta, \mathbb{P}^\theta_{|\emptyset}, \mathbb{P}^\theta_{|\mathbf{v}}$, and to the proofs of (generalisations of) the results stated above.

We should note that in the case where θ is the indicator function of a subset of Λ, all the above results are well-known, see e.g. [29, 44, 45].

1.1.4 Rigidity

In Section 1.3, we will study conditional ensembles corresponding to infinite configurations of mark 1 points $\delta_{\mathbf{v}} := \sum_j \delta_{v_j} \in \mathcal{N}(\Lambda)$. In such cases, the disintegration theorem implies that one can still define $\mathbb{P}^\theta_{|\mathbf{v}}$. If $B \in \mathcal{B}_\Lambda$ is bounded and if $\theta = 1_{B^c}$ is the indicator function of the complement of B, then $\mathbb{P}^\theta_{|\mathbf{v}} = \mathbb{P}^{1_{B^c}}_{|\mathbf{v}}$ is connected to the notion of number rigidity in the following manner. A point process $\mathbb{P}$ is said to be (number) rigid if for any bounded $B \in \mathcal{B}_\Lambda$, the conditional ensemble $\mathbb{P}^{1_{B^c}}_{|\mathbf{v}}$ has for $\mathbb{P}^{1_{B^c}}_1$-a.e. $\delta_{\mathbf{v}}$ a deterministic number of points, or in other words if there exists a $\mathcal{C}(\Lambda)$-measurable function

$$\ell : \mathcal{N}(\Lambda) \to \mathbb{N} \cup \{0, \infty\} : \delta_{\mathbf{v}} \mapsto \ell_{\mathbf{v}} \quad \text{such that} \quad \mathbb{P}^{1_{B^c}}_{|\mathbf{v}}\left(\xi(B) = \ell_{\mathbf{v}}\right) = 1.$$

This property is trivially satisfied for DPPs defined by kernels of finite rank orthogonal projections, since the number of particles in these DPPs is deterministic. Remarkably, a wide class of DPPs defined by kernels of infinite rank locally trace class orthogonal projections are also known to be number rigid [86, 88, 43]. Conversely, it is known [87] that a DPP can only be number rigid if it is defined by a projection. The construction of marked and conditional ensembles naturally suggests the following stronger notion of rigidity, which requires that given a.e. configuration of mark 1 points, the number of mark 0 point is deterministic.

Definition 1.1.6. A point process $\mathbb{P}$ is *marking rigid* if for any Borel measurable $\theta : \Lambda \to [0, 1]$, there exists a Borel measurable function

$$\ell : \mathcal{N}(\Lambda) \to \mathbb{N} \cup \{0, \infty\} : \delta_{\mathbf{v}} = \sum_i \delta_{v_i} \mapsto \ell_{v_1, v_2, \ldots} =: \ell_{\mathbf{v}}$$

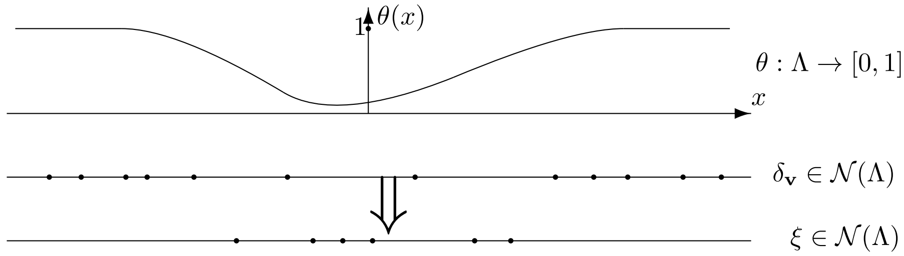

Figure 1.2: Illustration of marking rigidity: at the top, we see the graph of a possible marking function θ; at the bottom, a possible configuration of observed points $\delta_{\mathbf{v}}$ and a possible configuration ξ in the conditional ensemble $\mathbb{P}^{\theta}_{|\mathbf{v}}$. If $\mathbb{P}$ is marking rigid, then the marking function θ and the observed configuration $\delta_{\mathbf{v}}$ a.s. determine the number of points (6 in the picture) in the unobserved configuration ξ.

such that the following holds: for $\mathbb{P}^{\theta}_{1}$-a.e. $\delta_{\mathbf{v}}$,

$$\mathbb{P}^{\theta}_{|\mathbf{v}}\left(\xi(\Lambda) = \ell_{\mathbf{v}}\right) = 1.$$

Here $\xi(\Lambda)$ denotes the number of points of a random configuration ξ in the set Λ.

The following result is a special case of Theorem 1.3.5.

Theorem 1.1.7. *Let $\mathbb{P}$ be a DPP induced by a locally trace class orthogonal projection* K *such that the following holds: for any $\epsilon > 0$ and for any bounded $B \in \mathcal{B}_\Lambda$, there exists a bounded measurable function $f : \Lambda \to [0, +\infty)$ with bounded support such that*

$$f|_B = 1, \qquad \operatorname{Var} \int_\Lambda f \mathrm{d}\xi < \epsilon,$$

where Var *denotes the variance with respect to $\mathbb{P}$. Then, $\mathbb{P}$ is marking rigid.*

Remark 1.1.8. *It is well-known that the existence of a function f as in the above statement for any bounded $B \in \mathcal{B}_\Lambda$ and $\epsilon > 0$, implies number rigidity of the point process $\mathbb{P}$ [86, 88], and it is also known that such f exists if $\mathbb{P}$ is a DPP with sufficiently regular* 2*-integrable kernel defining an orthogonal projection, such as the sine, Airy, and Bessel point processes [43]. We thus prove that these point processes are marking rigid.*

Remark 1.1.9. *The above result is trivial for DPPs induced by finite rank orthogonal projections, which a.s. have a deterministic number of points. For DPPs associated to infinite rank orthogonal projections, which have a.s. configurations with an infinite number of points, it is striking that the observation of a random (possibly infinite) part of a configuration determines a.s. the number of unobserved points.*

1.1.5 Orthogonal polynomial ensembles

In Section 1.4, we will focus on the OPEs from Example 1.1.1, and we will show that conditional ensembles of OPEs are also OPEs, but with a deformed weight function, see Proposition 1.4.1. As a consequence, for $\Lambda = \mathbb{R}$, we show that a large class of OPEs on the real line, which are eigenvalue distributions of unitarily invariant Hermitian random matrices, are in fact conditional ensembles of the GUE. We also give explicit expressions for the marginal distribution of the mark 0 points, given the number of mark 1 points. These are in general not DPPs, but do have a special structure involving Hankel determinants.

1.1.6 DPPs with integrable kernels and Riemann-Hilbert problems

In Section 1.5, we will consider DPPs associated to integrable kernels. We will show how we can characterize the kernels of the associated conditional ensembles in terms of Riemann-Hilbert problems via the IIKS method, and explain how this opens the door for asymptotic analysis and for deriving integrable differential equations associated to the conditional ensembles $\mathbb{P}^{\theta}_{|\mathbf{v}}$. We will also be able to interpret Jacobi's identity for Fredholm determinants in terms of the conditional measure $\mathbb{P}^{\theta}_{|\emptyset}$.

1.2 Construction of marked and conditional processes

1.2.1 Preliminaries

We consider a measurable space $(\Lambda, \mathcal{B}_\Lambda)$, where Λ is a complete separable metric space and $\mathcal{B}_\Lambda$ its Borel σ-algebra. We denote by $\mathcal{N}(\Lambda)$ the set of locally finite Borel counting measures on Λ, and by $\mathcal{C}(\Lambda)$ the σ-algebra generated by cylinder sets of the form

$$C = \bigcap_{i=1}^{n} \{\xi \in \mathcal{N}(\Lambda) \ : \ \xi(B_i) = k_i\},$$

where $B_1, \dots, B_n \in \mathcal{B}_\Lambda$ are disjoint and $n, k_1, \dots, k_n$ are non-negative integers. Note that we can identify $\mathcal{N}(\Lambda)$ with the space of locally finite sets of points, counted with multiplicity. For configurations of distinct points, this means that we identify the counting measure ξ with its support. We consider a point process $\mathbb{P}$ on Λ, i.e. a probability measure on the complete separable metric space $(\mathcal{N}(\Lambda), \mathcal{C}(\Lambda))$.

For disjoint sets $B_1, \dots, B_n \in \mathcal{B}_\Lambda$ and non negative integers $k_1, \dots, k_n$ such that $\sum_{j=1}^{n} k_j = m$, the m-th *factorial moment measure* M_m of $\mathbb{P}$ is the symmetric measure on Λ^m given by

$$M_m(B_1^{k_1} \times \cdots \times B_n^{k_n}) = \mathbb{E}\xi(B_1)^{[k_1]} \dots \xi(B_n)^{[k_n]}, \quad \text{with} \quad l^{[k]} = \frac{l!}{(l-k)!}, \tag{1.2.1}$$

if the average exists. Similarly, the m-th *Jánossy measure* of $\mathbb{P}$ (encoding its finite dimensional distributions) associated to $B \in \mathcal{B}_\Lambda$ is the symmetric measure on B^m given by

$$J_m^B(B_1^{k_1} \times \cdots \times B_n^{k_n}) = \prod_{j=1}^{n} k_j! \mathbb{P}\big(\xi(B) = m,\ \xi(B_j) = k_j \text{ for } j = 1, ..., n\big),$$

where $\sum_{j=1}^n k_j = m$ and $\sqcup_{j=1}^n B_j = B$.

Throughout this section, we will impose the following regularity assumptions on the point process $\mathbb{P}$ on Λ.

Assumptions 1.2.1.
There exists a locally finite positive Borel measure μ on Λ such that:

1. *the point process $\mathbb{P}$ is simple, i.e. for μ-a.e. $x \in \Lambda$, $\mathbb{P}(\xi(\{x\}) \leq 1) = 1$;*
2. *$\mathbb{P}$ admits correlation functions of all orders, i.e. for any positive integer m there exists a (symmetric) locally integrable function $\rho_m : \Lambda^m \to [0, +\infty)$ with respect to the measure $\mu^{\otimes m}$ on Λ^m such that*
$$\mathrm{d}M_m = \rho_m \mathrm{d}^m \mu;$$
3. *for any bounded $B \in \mathcal{B}_\Lambda$, there exists $\epsilon_B > 0$ such that*
$$\sum_{m=1}^{\infty} \frac{(1 + \epsilon_B)^m}{m!} M_m(B^m) < \infty.$$

Under these assumptions, it is a classical fact [118, 135] that the correlation functions ρ_m uniquely determine the point process $\mathbb{P}$. We also have [122] that for every bounded $B \in \mathcal{B}_\Lambda$, there exist locally integrable Jánossy densities $j_m^B : \Lambda^m \to [0, +\infty)$ such that $\mathrm{d}J_m^B = j_m^B \mathrm{d}^m \mu$. Note that j_m^B is only defined on B^m, however under Assumptions 1.2.1, we have the identity

$$j^B(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{B^n} \rho(\mathbf{x} \sqcup \mathbf{y}) \mathrm{d}^n \mu(\mathbf{y}), \tag{1.2.2}$$

which allows to extend j_m^B to Λ^m, since the series converges in the space of locally integrable functions on Λ^m. Here we abbreviated

$$j^B(\mathbf{x}) := j_m^B(x_1, \dots, x_m), \qquad \rho(\mathbf{x}) := \rho_m(x_1, \dots, x_m),$$

because we interpret $\mathbf{x}$ either as a vector with m components $x_1, \dots, x_m$ or as a configuration $\{x_1, \dots, x_m\}$ of m (not necessarily distinct) points; $\rho(\mathbf{x} \sqcup \mathbf{y})$ then means $\rho_{m+n}(x_1, ..., x_m, y_1, ..., y_n)$ with $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{y} = (y_1, \dots, y_n)$. This notation in which we neglect the order of the variables is justified because ρ_m and j_m^B are symmetric in their variables. Moreover, if Assumptions 1.2.1 (3) holds also globally, i.e. for $B = \Lambda$, we have

$$\rho(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j^\Lambda(\mathbf{x} \sqcup \mathbf{y}) \mathrm{d}^n \mu(\mathbf{y}). \tag{1.2.3}$$

The above formulas continue to hold for $m = 0$ by adopting the conventions

$$\Lambda^0 = B^0 = \{\emptyset\}, \quad J_0^B(\emptyset) = j^B(\emptyset) = \mathbb{P}(\xi(B) = 0),$$
$$M_0(\emptyset) = \rho(\emptyset) = 1, \quad \mu^{\otimes 0} = \delta_\emptyset.$$

Let us note first that the Poisson point process with locally bounded intensity $\rho : \Lambda \to [0, +\infty)$ on (Λ, μ) satisfies Assumptions 1.2.1 if μ is non-atomic, with correlation functions given by (1.1.5). Our interest goes in particular to DPPs, characterized by the kernel $K : \Lambda^2 \to \mathbb{C}$ of a locally trace class operator K on $L^2(\Lambda, \mu)$. These point processes are simple [135], and the correlation functions are locally integrable and given by

$$\rho(x_1, \ldots, x_m) = \det\left(K(x_j, x_k)\right)_{j,k=1}^m .$$

The average multiplicative functional is a Fredholm determinant, recall (1.1.2)–(1.1.4). In particular, by (1.1.3), we have for any $\epsilon > 0$ and bounded $B \in \mathcal{B}_\Lambda$ that

$$\sum_{m=0}^{\infty} \frac{(1+\epsilon)^m}{m!} M_m(B^m) = \det(1 + (1+\epsilon)\mathrm{M}_{1_B}\mathrm{K}\mathrm{M}_{1_B}) < \infty.$$

Hence, we can conclude that Assumptions 1.2.1 are satisfied when $\mathbb{P}$ is a DPP induced by a locally trace class operator K.

1.2.2 Bernoulli marking

Given a point process satisfying Assumptions 1.2.1 and a measurable function $\theta : \Lambda \to [0, 1]$, we now construct a marked point process $\mathbb{P}^\theta$ on $\Lambda \times \{0, 1\}$, by assigning to each point $x \in \Lambda$ independently a random Bernoulli variable which takes the value 1 with probability $\theta(x)$, and the value 0 with probability $1 - \theta(x)$. Let us define the measures ν_x^θ and μ^θ respectively on $\{0, 1\}$ and $\Lambda_{\{0,1\}} := \Lambda \times \{0, 1\}$ as

$$\nu_x^\theta = (1-\theta(x))\delta_0 + \theta(x)\delta_1, \qquad \mathrm{d}\mu^\theta(x; b) = \mathrm{d}\nu_x^\theta(b)\mathrm{d}\mu(x), \qquad x \in \Lambda,\ b \in \{0, 1\}. \tag{1.2.4}$$

This marked point process $\mathbb{P}^\theta$ satisfies Assumptions 1.2.1 with Λ replaced by $\Lambda_{\{0,1\}}$ and μ by μ^θ. The correlation functions are then simply given by

$$\rho_m^\theta((x_1, b_1), \ldots, (x_m, b_m)) = \rho_m(x_1, \ldots, x_m), \tag{1.2.5}$$

with respect to the measure μ^θ, and hence do not depend on the marks. As a direct consequence of the expression for the correlation functions, if the ground process $\mathbb{P}$ is determinantal and induced by the operator K on $L^2(\Lambda, \mu)$ with kernel $K : \Lambda^2 \to \mathbb{C}$, then the marked point process $\mathbb{P}^\theta$ is also determinantal, induced by the operator K^θ on $L^2(\Lambda_{\{0,1\}}, \mu^\theta)$ with kernel

$$K^\theta((x, b_x), (y, b_y)) := K(x, y), \tag{1.2.6}$$

which is independent of the marks.

Now for $b \in \{0,1\}$ and for a marked configuration $\xi_{0,1} \in \mathcal{N}(\Lambda_{\{0,1\}})$, we define $\xi_b \in \mathcal{N}(\Lambda)$ by

$$\xi_b(B) = \xi_{0,1}(B \times \{b\}), \quad B \in \mathcal{B}_\Lambda, \tag{1.2.7}$$

i.e. ξ_b is the configuration of points with mark b, or equivalently

$$\xi_b = \sum_{j:b_j=b} \delta_{x_j}, \quad \text{when} \quad \xi_{0,1} = \sum_j \delta_{(x_j,b_j)} = \xi_0 \otimes \delta_0 + \xi_1 \otimes \delta_1. \tag{1.2.8}$$

As explained in the introduction, we interpret ξ_1 as the configuration of observed particles and ξ_0 as the configuration of unobserved particles. If we define the Borel measures μ_b^θ on Λ for $b \in \{0,1\}$ by

$$\mathrm{d}\mu_b^\theta(x) = \theta_b(x)\mathrm{d}\mu(x), \qquad \theta_1 = \theta, \quad \theta_0 = 1-\theta, \tag{1.2.9}$$

then the point processes $\mathbb{P}_b^\theta$, $b = 0,1$, obtained from $\mathbb{P}^\theta$ via transportation through the maps $\xi_{0,1} \mapsto \xi_b$, or in other words the marginal distributions of the mark b configurations, also satisfy Assumptions 1.2.1 with correlations functions $\rho_b^\theta(\mathbf{x}) = \rho(\mathbf{x})$ with respect to μ_b^θ and Jánossy densities for any bounded $B \in \mathcal{B}_\Lambda$ given by

$$j_b^{\theta,B}(\mathbf{x}) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_{B^n} \rho(\mathbf{x} \sqcup \mathbf{y}) \mathrm{d}^n \mu_b^\theta(\mathbf{y}). \tag{1.2.10}$$

Both point processes $\mathbb{P}_0^\theta$ and $\mathbb{P}_1^\theta$ on Λ are random independent thinnings of the ground point process $\mathbb{P}$. If the ground process is determinantal and induced by the kernel of a locally trace class operator K on $L^2(\Lambda,\mu)$, then so is $\mathbb{P}_b^\theta$ with the same kernel, but now with the corresponding operator acting on $L^2(\Lambda,\mu_b^\theta)$ [114].

Summarizing the above, we have proved the following result.

Proposition 1.2.2. *Let $\mathbb{P}$ satisfy Assumptions 1.2.1, and let $\theta : \Lambda \to [0,1]$ be measurable.*

1. *The marked point process $\mathbb{P}^\theta$ satisfies Assumptions 1.2.1 with Λ replaced by $\Lambda_{\{0,1\}}$ and μ by μ^θ; for $b = 0,1$, the component $\mathbb{P}_b^\theta$ satisfies Assumptions 1.2.1 with μ replaced by μ_b^θ ; in both cases the correlation functions are the same as those of the ground process $\mathbb{P}$.*

2. *If $\mathbb{P}$ is the DPP with kernel K on (Λ,μ), then $\mathbb{P}^\theta$ is the DPP with kernel K^θ on $(\Lambda_{\{0,1\}},\mu^\theta)$. For $b = 0,1$, the component $\mathbb{P}_b^\theta$ is the DPP with kernel K on (Λ,μ_b^θ).*

Remark 1.2.3. *Observe the analogy with the corresponding result if $\mathbb{P}$ is the Poisson point process with intensity $\rho : \Lambda \to [0,+\infty)$ with respect to μ. Then $\mathbb{P}^\theta$ is the Poisson point process with intensity $\rho^\theta(x,b) = \rho(x)$ on $\Lambda_{\{0,1\}}$ with respect to μ^θ, and $\mathbb{P}_b^\theta$ is the Poisson point process on Λ with intensity ρ with respect to μ_b^θ.*

1.2.3 Conditioning on an empty observation

Let us now assume, in addition to Assumptions 1.2.1, that the probability to have no mark 1 particles is non-zero, i.e.

$$\mathbb{P}^\theta(\xi_1(\Lambda) = 0) = \mathbb{E} \prod_{x \in \operatorname{supp} \xi} (1 - \theta(x)) = L[\theta] > 0, \tag{1.2.11}$$

where we recall the definition of $L[.]$ from (1.1.4). Then, we can condition $\mathbb{P}^\theta$ on the event $\xi_1(\Lambda) = 0$ in the classical sense and identify it with a point process on Λ by identifying $\xi_{0,1}$ with ξ_0, to obtain the conditional point process $\mathbb{P}^\theta_{|\emptyset}$ on Λ defined by

$$\mathbb{P}^\theta_{|\emptyset}(\xi \in C) = \frac{\mathbb{P}^\theta(\xi_{0,1} \text{ is such that } \xi_0 \in C,\ \xi_1(\Lambda) = 0)}{\mathbb{P}^\theta(\xi_1(\Lambda) = 0)}, \qquad C \in \mathcal{C}(\Lambda). \tag{1.2.12}$$

We write $L^\theta_{|\emptyset}$ for the average multiplicative functional (1.1.4) corresponding to the probability $\mathbb{P}^\theta_{|\emptyset}$. For K locally trace class on (Λ, μ) such that $\mathrm{M}_{\sqrt{\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\theta}}$ is trace class and $\det(1 - \mathrm{M}_{\sqrt{\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\theta}}) > 0$, let us introduce $K^\theta_{|\emptyset}$ as the kernel of the integral operator

$$\mathrm{K}(1 - \mathrm{M}_\theta \mathrm{K})^{-1} : L^2(\Lambda, \mu) \to L^2(\Lambda, \mu),$$
$$\mathrm{K}(1 - \mathrm{M}_\theta \mathrm{K})^{-1} f(x) = \int_\Lambda K^\theta_{|\emptyset}(x, y) f(y) \mathrm{d}\mu(y),$$

and $\mathrm{K}^\theta_{|\emptyset}$ as the operator with the kernel $K^\theta_{|\emptyset}$ on $L^2(\Lambda, \mu^\theta_0)$,

$$\mathrm{K}^\theta_{|\emptyset} : L^2(\Lambda, \mu^\theta_0) \to L^2(\Lambda, \mu^\theta_0), \qquad \mathrm{K}^\theta_{|\emptyset} f(x) = \int_\Lambda K^\theta_{|\emptyset}(x, y) f(y) \mathrm{d}\mu^\theta_0(y).$$

Theorem 1.2.4. *Let $\theta : \Lambda \to [0, 1]$ be measurable and let $\mathbb{P}$ be such that $L[\theta] > 0$.*

1. *The point process $\mathbb{P}^\theta_{|\emptyset}$ is well-defined and has average multiplicative functional*
$$L^\theta_{|\emptyset}[\phi] = \frac{L[1 - (1 - \phi)(1 - \theta)]}{L[\theta]}.$$
If in addition $\mathbb{P}$ satisfies Assumptions 1.2.1 and there exists $\epsilon > 1$ such that $L[-\epsilon\theta] < \infty$, then so does $\mathbb{P}^\theta_{|\emptyset}$, with correlations functions with respect to μ^θ_0 given by
$$\rho^\theta_{|\emptyset}(\mathbf{x}) = \frac{j_1^{\theta,\Lambda}(\mathbf{x})}{\mathbb{P}^\theta(\xi_1(\Lambda) = 0)}.$$

2. *If $\mathbb{P}$ is the DPP with kernel K of a locally trace class operator K and the operator $\mathrm{M}_{\sqrt{\theta + 1_B}}\mathrm{K}\mathrm{M}_{\sqrt{\theta + 1_B}}$ is trace class for any bounded Borel set B, then $\mathbb{P}^\theta_{|\emptyset}$ is the DPP on (Λ, μ) with kernel of the integral operator (1.1.10) acting on $L^2(\Lambda, \mu)$, or equivalently the DPP on (Λ, μ^θ_0) with kernel $\mathrm{K}^\theta_{|\emptyset}$. Moreover, if K is self-adjoint then $\mathrm{K}^\theta_{|\emptyset}$ is self-adjoint.*

Proof. 1. By definition of conditional probability, for $\phi : \Lambda \to \mathbb{R}^+$ measurable, we have

$$L^\theta_{|\emptyset}[\phi] = \mathbb{E}^\theta_{|\emptyset} \prod_{u \in \operatorname{supp} \xi_0} (1 - \phi(u)) = \frac{\mathbb{E} \prod_{x \in \operatorname{supp} \xi} (1 - \phi(x))(1 - \theta(x))}{\mathbb{P}^\theta(\xi_1(\Lambda) = 0)}$$
$$= \frac{\mathbb{E} \prod_{x \in \operatorname{supp} \xi} (1 - \phi(x))(1 - \theta(x))}{\mathbb{E} \prod_{x \in \operatorname{supp} \xi} (1 - \theta(x))} = \frac{L[1 - (1 - \phi)(1 - \theta)]}{L[\theta]}.$$

Now if $\mathbb{P}$ is simple, then so is $\mathbb{P}^\theta$ and a fortiori so is $\mathbb{P}^\theta_{|\emptyset}$, and the inequality for $\phi \geq 0$

$$L^\theta_{|\emptyset}[-\phi] \leq \frac{L[-\phi]}{L[\theta]}$$

shows that $\mathbb{P}^\theta_{|\emptyset}$ satisfies the third of Assumptions 1.2.1 whenever $\mathbb{P}$ does. It thus remains to compute the correlation functions. Note first that $L[-\epsilon\theta] < \infty$ implies that $\mathbb{P}^\theta_1$ satisfies the third of Assumptions 1.2.1 with $B = \Lambda$, so that the global Jánossy densities $j^{\theta,\Lambda}_1$ are well-defined and given by (1.2.2). The computations hereafter then involve absolutely convergent series, and all the needed results of integration theory may be applied. Let $\eta = 1 - (1 - \theta)(1 - \phi) = \theta + (1 - \theta)\phi$, then

$$L[\eta] = \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_{\Lambda^n} \rho_n(\mathbf{x}) \prod_{j=1}^n \eta(x_j) \mathrm{d}^n \mu(\mathbf{x}).$$

Writing $\mathbf{x} = \mathbf{y} \sqcup \mathbf{z}$ and using the symmetry of the measure $\rho_n(\mathbf{x})\mathrm{d}^n\mu(\mathbf{x})$ yields that each integral is equal to

$$\sum_{l=0}^n \binom{n}{l} \int_{\Lambda^n} \rho_n(\mathbf{y} \sqcup \mathbf{z}) \prod_{j=1}^l (1 - \theta(y_j))\phi(y_j) \prod_{i=1}^{n-l} \theta(z_i) \mathrm{d}^n \mu(\mathbf{y} \sqcup \mathbf{z}),$$

so that

$$L[\eta] = \sum_{l \geq 0} \frac{(-1)^l}{l!} \int_{\Lambda^l} \left[\sum_{n \geq l} \frac{(-1)^{n-l}}{(n-l)!} \int_{\Lambda^{n-l}} \rho(\mathbf{y} \sqcup \mathbf{z}) \mathrm{d}^{n-l} \mu^\theta_1(\mathbf{z}) \right] \times$$
$$\prod_{j=1}^l (1 - \theta(y_j))\phi(y_j) \mathrm{d}^l \mu^\theta_0(\mathbf{y}).$$

We recognize expression (1.2.2) for $j^{\theta,\Lambda}_1$ in the integral. Dividing the previous equation by $\mathbb{P}^\theta(\xi_1(\Lambda) = 0) = L[\theta]$, we get an expression for $L^\theta_{|\emptyset}[\phi]$, and when $\phi = -1_B$, this implies the existence of all factorial moment measure $M^\theta_{m|\emptyset}$ of $\mathbb{P}^\theta_{|\emptyset}$, given the estimate $M^\theta_{m|\emptyset}(B^m) \leq L^\theta_{|\emptyset}[-1_B]$. Replacing ϕ by $w\phi$ for $w \in \mathbb{C}$ with a small enough modulus, we obtain a power series in w and we can read off the expressions for the correlation functions $\mathrm{d}M^\theta_{m|\emptyset} = \rho^\theta_{m|\emptyset} \mathrm{d}^m \mu^\theta_0$ by looking at each power of w.

2. Let $(B_n)_{n\in\mathbb{N}}$ be an exhausting increasing sequence of bounded Borel subsets of Λ, and let $\mathrm{K}_n = \mathrm{M}_{1_{B_n}}\mathrm{K}\mathrm{M}_{1_{B_n}}$. By (1.1.2)–(1.1.4), the associated conditional ensemble $(\mathbb{P}_n)^{\theta}_{|\emptyset}$ has average multiplicative functional equal to

$$\frac{\det\left(1-\mathrm{M}_{\phi+\theta-\phi\theta}\mathrm{K}_n\right)}{\det(1-\mathrm{M}_\theta\mathrm{K}_n)} = \det\left[\left((1-\mathrm{M}_\theta\mathrm{K}_n)-\mathrm{M}_\phi\mathrm{M}_{1-\theta}\mathrm{K}_n\right)(1-\mathrm{M}_\theta\mathrm{K}_n)^{-1}\right] = \det\left[1-\mathrm{M}_\phi\mathrm{M}_{1-\theta}\mathrm{K}_n(1-\mathrm{M}_\theta\mathrm{K}_n)^{-1}\right],$$

and it follows that $(\mathbb{P}_n)^{\theta}_{|\emptyset}$ is also determinantal on (Λ,μ) with kernel of the integral operator $\mathrm{M}_{1-\theta}\mathrm{K}_n(1-\mathrm{M}_\theta\mathrm{K}_n)^{-1}$. The left hand side in the above identity is equal to $\frac{\det\left(1-\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}}\mathrm{K}_n\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}}\right)}{\det(1-\mathrm{M}_{\sqrt{\theta}}\mathrm{K}_n\mathrm{M}_{\sqrt{\theta}})}$ and as $n\to\infty$, it converges to

$$\frac{\det\left(1-\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}}\right)}{\det(1-\mathrm{M}_{\sqrt{\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\theta}})} = L^{\theta}_{|\emptyset}[\phi],$$

since $\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}}\mathrm{K}_n\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}}$ and $\mathrm{M}_{\sqrt{\theta}}\mathrm{K}_n\mathrm{M}_{\sqrt{\theta}}$ converge in trace norm to $\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}}$ and $\mathrm{M}_{\sqrt{\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\theta}}$, since the latter two operators are trace class. Indeed, $\mathrm{M}_{\sqrt{\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\theta}} = \mathrm{M}_{\sqrt{\theta}+1_B}\mathrm{K}\mathrm{M}_{\sqrt{\theta}+1_B}$ with $B=\emptyset$; now $\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}}$ can be decomposed, with $B=\operatorname{supp}\phi$, as

$$\begin{gathered}\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}}1_B\mathrm{K}1_B\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}} + 1_{B^c}\mathrm{M}_{\sqrt{\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\theta}}1_{B^c}\\ +1_{B^c}\mathrm{M}_{\sqrt{\theta}}\mathrm{K}1_B\mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}} + \mathrm{M}_{\sqrt{\phi+\theta-\phi\theta}}1_B\mathrm{K}\mathrm{M}_{\sqrt{\theta}}1_{B^c}\end{gathered}$$

and it is easy to see that each term is trace class.
Similarly, the right hand side converges as $n\to\infty$ to

$$\det\left[1-\mathrm{M}_{\sqrt{\phi}}\mathrm{M}_{1-\theta}\mathrm{K}(1-\mathrm{M}_\theta\mathrm{K})^{-1}\mathrm{M}_{\sqrt{\phi}}\right],$$

since $\mathrm{M}_{\sqrt{\phi}}\mathrm{M}_{1-\theta}\mathrm{K}_n(1-\mathrm{M}_\theta\mathrm{K}_n)^{-1}\mathrm{M}_{\sqrt{\phi}}$ converges in trace norm to the operator $\mathrm{M}_{\sqrt{\phi}}\mathrm{M}_{1-\theta}\mathrm{K}(1-\mathrm{M}_\theta\mathrm{K})^{-1}\mathrm{M}_{\sqrt{\phi}}$ (note that we need the condition that $\mathrm{M}_{\sqrt{\theta}+1_B}\mathrm{K}\mathrm{M}_{\sqrt{\theta}+1_B}$ is trace class for any bounded Borel set B here again, in order to have $\mathrm{M}_{\sqrt{\phi}}\mathrm{K}\mathrm{M}_{\sqrt{\theta}}$, $\mathrm{M}_{\sqrt{\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\phi}}$ trace class). Thus, $\mathbb{P}^{\theta}_{|\emptyset}$ is the DPP with kernel of the operator $\mathrm{M}_{1-\theta}\mathrm{K}(1-\mathrm{M}_\theta\mathrm{K})^{-1}$ on $L^2(\Lambda,\mu)$, or equivalently the DPP on (Λ,μ^{θ}_0) with kernel $K^{\theta}_{|\emptyset}$.
If K is self-adjoint on $L^2(\Lambda,\mu)$, then so is

$$\mathrm{K}(1-\mathrm{M}_\theta\mathrm{K})^{-1} = \mathrm{K}+\mathrm{K}\mathrm{M}_{\sqrt{\theta}}(1-\mathrm{M}_{\sqrt{\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\theta}})^{-1}\mathrm{M}_{\sqrt{\theta}}\mathrm{K},$$

as the sum of two self-adjoint operators, hence the kernel $K^{\theta}_{|\emptyset}$ defines a self-adjoint operator on $L^2(\Lambda,\mu^{\theta}_0)$ as well.

□

Remark 1.2.5. *If $\mathbb{P}$ is the Poisson point process with intensity ρ on Λ with respect to μ, then $\mathbb{P}^{\theta}_{|\emptyset}$ is the Poisson point process with the same intensity ρ on Λ, but with respect to μ^{θ}_{0}. Hence, $\mathbb{P}^{\theta}_{|\emptyset}$ is equal to $\mathbb{P}^{\theta}_{0}$, and as it should be, the fact that there are no mark 1 points does not give any further information about the mark 0 points.*

Remark 1.2.6. *Theorem 1.1.1 is a restatement of the second part of the above result.*

1.2.4 Conditioning on a finite mark 1 configuration ξ_1

For non-empty configurations ξ_1 of points with mark 1, the situation is more involved. Here we need to assume that θ is such that there exists $\epsilon > 0$ such that

$$L[-(1+\epsilon)\theta] = \mathbb{E} \prod_{x \in \operatorname{supp} \xi} (1 + (1+\epsilon)\theta(x)) < \infty, \tag{1.2.13}$$

where the average is with respect to the ground process $\mathbb{P}$. This condition ensures, by (1.2.1), that $\mathbb{P}^{\theta}_{1}$ satisfies Assumptions 1.2.1 (3) also for $B = \Lambda$, and in particular that

$$\mathbb{E}^{\theta}\xi_1(\Lambda) = \mathbb{E} \sum_{x \in \operatorname{supp} \xi} \theta(x) \le \mathbb{E} \prod_{x \in \operatorname{supp} \xi} (1 + \theta(x)) < \infty.$$

This implies that the number of observed particles $\xi_1(\Lambda)$ is finite for $\mathbb{P}^{\theta}$-a.e. ξ_1. Based on such an observed configuration ξ_1, we would like to obtain information about the configuration ξ_0 of points with mark 0. To this end, we want to define a point process $\mathbb{P}^{\theta}_{|\mathbf{v}}$ on $\Lambda \times \{0\}$ representing the restriction to $\Lambda \times \{0\}$ of the conditioning of $\mathbb{P}^{\theta}$ on an observation $\mathbf{v} = \{v_1, \ldots, v_m\}$, or more precisely on ξ_1 being equal to $\delta_{\mathbf{v}} := \sum_{j=1}^{m} \delta_{v_j}$. We can then identify $\mathbb{P}^{\theta}_{|\mathbf{v}}$ with a point process on Λ by omitting the marks 0. The probability to observe given points $\mathbf{v}$ with mark 1 will typically be zero, such that we cannot use classical conditional probability to construct the conditional point processes.

Conditioning on m mark 1 points

Let us assume that $\mathbb{P}^{\theta}(\xi_1(\Lambda) = m) > 0$. Then we can condition $\mathbb{P}^{\theta}$ on the event $\xi_1(\Lambda) = m$ in the classical sense. Now, we want to construct a family of conditional point processes $\left\{\mathbb{P}^{\theta}_{|\mathbf{v}}\right\}_{\mathbf{v} \in \mathbf{\Lambda^m}}$, which is consistent in the sense that averaging the $\mathbb{P}^{\theta}_{|\mathbf{v}}$-probability of an event $\xi_0 \in C \in \mathcal{C}(\Lambda)$ over the positions of the m-point configuration $v_1, \ldots, v_m$ (with respect to the probability $\mathbb{P}^{\theta}_{1}(.|\xi_1(\Lambda) = m)$) is equal to the $\mathbb{P}^{\theta}(.|\xi_1(\Lambda) = m)$-probability of the event $\xi_0 \in C$. In other words, we average $v_1, \ldots, v_m$ with respect to the joint probability distribution

$$\mathrm{d}\pi^{\theta}_{1,m}(\mathbf{v}) = \mathrm{d}\pi^{\theta}_{1,m}(v_1, \ldots, v_m) := \frac{j^{\theta,\Lambda}_{1,m}(v_1, \ldots, v_m)}{m!\mathbb{P}^{\theta}(\xi_1(\Lambda) = m)} \prod_{j=1}^{m} \mathrm{d}\mu^{\theta}_{1}(v_j), \tag{1.2.14}$$

for $v_1, \dots, v_m$, where $j_{1,m}^{\theta,\Lambda}$ is the m-th order global Jánossy density of the measure $\mathbb{P}_1^\theta$ for the mark 1 configuration (which exists if (1.2.13) holds), and we will need consistency in the sense that

$$\int_{\Lambda^m} \mathbb{P}_{|\mathbf{v}}^\theta(\xi \in C) \mathrm{d}\pi_{1,m}^\theta(\mathbf{v}) = \mathbb{P}^\theta\left(\xi_0 \in C | \ \xi_1(\Lambda) = m\right), \qquad C \in \mathcal{C}(\Lambda). \quad (1.2.15)$$

Preliminaries on reduced Palm measures

As explained in Section 1.1, to construct $\mathbb{P}_{|\mathbf{v}}^\theta$, we need reduced local Palm distributions. Given a point process $\mathbb{P}$ satisfying Assumptions 1.2.1 and $m \in \mathbb{N}$, there exists a family of point processes $\{\mathbb{P}_\mathbf{w}\}_{\mathbf{w} \in \Lambda^m}$, which represent the conditioning of $\mathbb{P}$ on m points $\mathbf{w} = \{w_1, \dots, w_m\} \subset \operatorname{supp} \xi$, reduced by mapping $\xi \in \mathcal{N}(\Lambda)$ to its restriction $\xi|_{\Lambda \setminus \mathbf{w}}$. We need the following fundamental properties (see e.g. [66]) of these m-th order reduced Palm measures.

1. For any $C \in \mathcal{C}(\Lambda)$, the map $\mathbf{w} \in \Lambda^m \mapsto \mathbb{P}_\mathbf{w}(C)$ is $\mathcal{B}_{\Lambda^m}$-measurable.
2. For $\mu^{\otimes m}$-a.e. $\mathbf{w} \in \Lambda^m$ such that $\rho(\mathbf{w}) > 0$, the reduced Palm measure $\mathbb{P}_\mathbf{w}$ satisfies Assumptions 1.2.1, and its correlation functions $\rho_\mathbf{w}$ with respect to μ are given by (see [131])
$$\rho_\mathbf{w}(\mathbf{x}) = \frac{\rho(\mathbf{x} \sqcup \mathbf{w})}{\rho(\mathbf{w})}. \quad (1.2.16)$$
3. Writing $\delta_\mathbf{w} = \sum_{j=1}^m \delta_{w_j}$, we have for any measurable $\psi : \Lambda^m \times \mathcal{N}(\Lambda) \to \mathbb{R}^+$ that the disintegration
$$\mathbb{E} \sum \psi(\mathbf{w}; \xi - \delta_\mathbf{w}) = \int_{\Lambda^m} \mathbb{E}_\mathbf{w} \psi(\mathbf{w}, \xi) \rho(\mathbf{w}) \mathrm{d}^m \mu(\mathbf{w}) \quad (1.2.17)$$
holds, where the sum at the left is over all ordered m-tuples $\mathbf{w} = (w_1, \dots, w_m)$ of distinct points in $\operatorname{supp} \xi$ and where $\mathbb{E}_\mathbf{w}$ is the average with respect to $\mathbb{P}_\mathbf{w}$.

In particular, the second property implies that if $\mathbb{P}$ is determinantal with kernel K, then for $\mu^{\otimes m}$-a.e. $\mathbf{w} \in \Lambda^m$ such that $\det K(\mathbf{w}, \mathbf{w}) > 0$, the reduced Palm measure $\mathbb{P}_\mathbf{w}$ is determinantal and induced by the kernel $K_\mathbf{w}$ given by (1.1.12), or equivalently by

$$K_\mathbf{w}(x, y) = K(x, y) - K(x, \mathbf{w}) K(\mathbf{w}, \mathbf{w})^{-1} K(\mathbf{w}, y), \quad (1.2.18)$$

where we used the block determinant formula

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\left(A - BD^{-1}C\right) \det D, \quad (1.2.19)$$

and where similarly as before, $K(\mathbf{w}, \mathbf{w})$ represents an $m \times m$ matrix, $K(x, \mathbf{w})$ a row vector, and $K(\mathbf{w}, y)$ a column vector.

Construction of the conditional ensembles

We will now apply the above properties of reduced Palm measures to the point process $\mathbb{P}^\theta(.|\xi_1(\Lambda)=m)$, the marked point process conditioned on observing exactly m particles. If $\mathbb{P}^\theta(\xi_1(\Lambda)=m)>0$, this point process indeed satisfies Assumptions 1.2.1. Setting $\mathbf{w}=\{(v_1,1),\dots,(v_m,1)\}$ and $\mathbf{v}=\{v_1,\dots,v_m\}$, we define $\mathbb{P}^\theta_{|\mathbf{v}}$ as the m-th order reduced local Palm distribution of $\mathbb{P}^\theta(.|\xi_1(\Lambda)=m)$ associated to the points $\mathbf{w}$. This is a point process on $\Lambda_{\{0,1\}}$ whose configurations have a.s. no points in $\Lambda\times\{1\}$; hence we can identify $\mathbb{P}^\theta_{|\mathbf{v}}$ with a point process on Λ by omitting the marks. Before we prove some important properties of the conditional ensembles $\mathbb{P}^\theta_{.|\mathbf{v}}$, let us mention that another intuitive way of defining them would be to first take the Palm measure of $\mathbb{P}^\theta$ at $\mathbf{w}=((v_1,1),...,(v_m,1))$ and then condition on there being no other particles with mark 1. The third item of the next result shows that this is indeed equivalent to our definition, and when $\mathbb{P}$ is a DPP, it has the advantage that it allows us to define the DPP $\mathbb{P}^\theta_{|\mathbf{v}}$ without need to pass via the (in general not determinantal) point process $\mathbb{P}^\theta(.|\xi_1(\Lambda)=m)$. Thus for K a locally trace class operator on (Λ,μ) such that $\mathrm{M}_\theta\mathrm{K}$ is trace class and $\det(1-\mathrm{M}_\theta\mathrm{K}_\mathbf{v})>0$, let us introduce $K^\theta_{|\mathbf{v}}$ as the kernel of the integral operator

$$\mathrm{K}_\mathbf{v}(1-\mathrm{M}_\theta\mathrm{K}_\mathbf{v})^{-1}:L^2(\Lambda,\mu)\to L^2(\Lambda,\mu),$$
$$\mathrm{K}_\mathbf{v}(1-\mathrm{M}_\theta\mathrm{K}_\mathbf{v})^{-1}f(x)=\int_\Lambda K^\theta_{|\mathbf{v}}(x,y)f(y)\mathrm{d}\mu(y),$$

and $\mathrm{K}^\theta_{|\mathbf{v}}$ as the operator with the kernel $K^\theta_{|\mathbf{v}}$ on $L^2(\Lambda,\mu^\theta_0)$,

$$\mathrm{K}^\theta_{|\mathbf{v}}:L^2(\Lambda,\mu^\theta_0)\to L^2(\Lambda,\mu^\theta_0),\qquad \mathrm{K}^\theta_{|\mathbf{v}}f(x)=\int_\Lambda K^\theta_{|\mathbf{v}}(x,y)f(y)\mathrm{d}\mu^\theta_0(y).$$

Theorem 1.2.7. *Let $\mathbb{P}$ satisfy Assumptions 1.2.1, and let $\theta:\Lambda\to[0,1]$ be measurable and such that (1.2.13) holds. Let $m\geq 0$ be such that $\mathbb{P}^\theta(\xi_1(\Lambda)=m)>0$. The family of point processes $\left\{\mathbb{P}^\theta_{|\mathbf{v}}\right\}_{\mathbf{v}\in\Lambda^m}$ satisfies the following properties.*

1. *For any $C\in\mathcal{C}(\Lambda)$, the map $\mathbf{v}\in\Lambda^m\mapsto\mathbb{P}^\theta_{|\mathbf{v}}(C)$ is $\mathcal{B}_{\Lambda^m}$-measurable.*
2. *For any Borel measurable $\phi:\mathcal{N}(\Lambda_{\{0,1\}})\to[0,+\infty)$, with $\delta_\mathbf{v}=\sum_{j=1}^m\delta_{v_j}$, we have the disintegration*
$$\mathbb{E}^\theta\left[\phi(\xi_{0,1})\mid\xi_1(\Lambda)=m\right]=\int_{\Lambda^m}\mathbb{E}^\theta_{|\mathbf{v}}\phi(\xi\otimes\delta_0+\delta_\mathbf{v}\otimes\delta_1)\mathrm{d}\pi^\theta_{1,m}(\mathbf{v}),\quad(1.2.20)$$
where $\pi^\theta_{1,m}$ is given by (1.2.14).
3. *For $\pi^\theta_{1,m}$-a.e. $\mathbf{v}\in\Lambda^m$, the point process $\mathbb{P}^\theta_{|\mathbf{v}}$ satisfies Assumptions 1.2.1, its correlation functions $\rho^\theta_{|\mathbf{v}}$ with respect to μ^θ_0 are given by*
$$\rho^\theta_{|\mathbf{v}}(\mathbf{x})=\frac{j_1^{\theta,\Lambda}(\mathbf{x}\sqcup\mathbf{v})}{j_1^{\theta,\Lambda}(\mathbf{v})},$$

and its average multiplicative functional is given by

$$L^\theta_{|\mathbf{v}}[\phi_0] = \frac{L_\mathbf{v}[1-(1-\theta)(1-\phi_0)]}{L_\mathbf{v}[\theta]}, \tag{1.2.21}$$

where $L_\mathbf{v}$ denotes the average multiplicative functional of the reduced Palm measure $\mathbb{P}_\mathbf{v}$ of the ground process $\mathbb{P}$ on Λ associated to the points $\mathbf{v}$.

4. *If $\mathbb{P}$ is the DPP on (Λ,μ) with kernel of the operator K on $L^2(\Lambda,\mu)$, and $\mathrm{M}_{\sqrt{\theta}+1_B}\mathrm{K}\mathrm{M}_{\sqrt{\theta}+1_B}$ is trace class for any bounded Borel set B, then for $\pi^\theta_{1,m}$-a.e. $\mathbf{v}\in\Lambda^m$, $\mathbb{P}^\theta_{|\mathbf{v}}$ is the DPP on (Λ,μ) with kernel $(1-\theta)(x)K^\theta_{|\mathbf{v}}(x,y)$ of the operator $\mathrm{M}_{1-\theta}\mathrm{K}_\mathbf{v}\left(1-\mathrm{M}_\theta\mathrm{K}_\mathbf{v}\right)^{-1}$ on $L^2(\Lambda,\mu)$, or equivalently the DPP on (Λ,μ^θ_0) with kernel $K^\theta_{|\mathbf{v}}$. Moreover, if K on $L^2(\Lambda,\mu)$ is self-adjoint, then the operator $\mathrm{K}^\theta_{|\mathbf{v}}$ with kernel $K^\theta_{|\mathbf{v}}$ on $L^2(\Lambda,\mu^\theta_0)$ is self-adjoint.*

Proof.
(1) This follows directly from the corresponding general property of reduced Palm measures.
(2) Applying (1.2.17) to $\mathbb{P}=\mathbb{P}^\theta(.|\xi_1(\Lambda)=m)$ and

$$\begin{aligned}\psi:\Lambda^m_{\{0,1\}}\times\mathcal{N}(\Lambda_{\{0,1\}})&\to[0,+\infty)\\ (\mathbf{w},\xi_{0,1})&\mapsto\begin{cases}\phi(\xi_{0,1}+\delta_\mathbf{w}) & \text{if } \xi_1(\Lambda)=0 \text{ and } \mathbf{w}\in(\Lambda\times\{1\})^m,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

with $\phi:\mathcal{N}(\Lambda_{\{0,1\}})\to[0,+\infty)$, and denoting $\mathbf{w}=((v_1,1),\dots,(v_m,1))$, $\mathbf{v}=(v_1,\dots,v_m)$, we obtain a family of point processes $\left\{\mathbb{P}^\theta_{|\mathbf{v}}\right\}_{\mathbf{v}\in\Lambda^m}$ on Λ which is such that

$$\mathbb{E}^\theta\left[\phi(\xi_{0,1})\mid\xi_1(\Lambda)=m\right]=\int_{\Lambda^m}\mathbb{E}^{\theta|m}_\mathbf{w}\phi(\xi\otimes\delta_0+\delta_\mathbf{v}\otimes\delta_1)\mathrm{d}\pi^\theta_{1,m}(\mathbf{v}),$$

where we used the symmetry of ψ and the fact that there are $m!$ ordered m-tuples $\mathbf{w}$ in $\operatorname{supp}\xi_{0,1}$ at the left, and the fact that the m-point correlation function of $\mathbb{P}^\theta\left(.|\xi_1(\Lambda)=m\right)$ evaluated at $\mathbf{w}$ is equal to $\frac{j^{\theta,\Lambda}_{1,m}(\mathbf{v})}{\mathbb{P}^\theta(\xi_1(\Lambda)=m)}$ (by (1.2.10)) at the right. Using (1.2.14), we obtain the required disintegration.

(3) Let us apply (1.2.20) to the multiplicative statistic

$$\phi(\xi_{0,1})=\prod_{(x,b)\in\operatorname{supp}\xi_{0,1}}(1-\phi_b(x)),$$

where $\phi_0,\phi_1:\Lambda\to(-\infty,1]$ are Borel measurable and ϕ_0 has bounded support. If $\phi_1=0$, the disintegration implies that for $\pi^\theta_{1,m}$-a.e. $\mathbf{v}\in\Lambda^m$, $\mathbb{P}^\theta_{|\mathbf{v}}$ satisfies Assumptions 1.2.1 *(3)*, thereby justifying the computations hereafter involving series and integrals.

The right hand side of (1.2.20) is then equal to

$$\int_{\Lambda^m} L^\theta_{|\mathbf{v}}[\phi_0]\prod_{j=1}^m(1-\phi_1(v))\mathrm{d}\pi^\theta_{1,m}(\mathbf{v})$$

$$= \frac{\displaystyle\int_{\Lambda^m} L^\theta_{|\mathbf{v}}[\phi_0]\prod_{j=1}^m(1-\phi_1(v))j_1^{\theta,\Lambda}(\mathbf{v})\mathrm{d}^m\mu_1^\theta(\mathbf{v})}{m!\mathbb{P}^\theta(\xi_1(\Lambda)=m)}$$

$$= \frac{\displaystyle\int_{\Lambda^m} L^\theta_{|\mathbf{v}}[\phi_0]\prod_{j=1}^m(1-\phi_1(v))\left(\sum_{n=0}^\infty\frac{(-1)^n}{n!}\int_{\Lambda^n}\rho(\mathbf{u}\sqcup\mathbf{v})\mathrm{d}^n\mu_0^\theta(\mathbf{u})\right)\mathrm{d}^m\mu_1^\theta(\mathbf{v})}{m!\mathbb{P}^\theta(\xi_1(\Lambda)=m)}$$

$$= \frac{\displaystyle\int_{\Lambda^m} L^\theta_{|\mathbf{v}}[\phi_0]L_{\mathbf{v}}[\theta]\prod_{j=1}^m(1-\phi_1(v))\rho(\mathbf{v})\mathrm{d}^m\mu_1^\theta(\mathbf{v})}{m!\mathbb{P}^\theta(\xi_1(\Lambda)=m)},$$

by (1.2.14), (1.2.10), and (1.2.16).

The left hand side of (1.2.20) is equal to

$$\frac{\displaystyle\sum_{n=0}^\infty\frac{(-1)^n}{n!}\int_{\Lambda^m}\int_{\Lambda^n}\prod_{k=1}^n\phi_0(u_k)\rho(\mathbf{u}\sqcup\mathbf{v})\mathrm{d}^n\mu_0^\theta(\mathbf{u})\prod_{j=1}^m(1-\phi_1(v_j))\mathrm{d}^m\mu_1^\theta(\mathbf{v})}{m!\mathbb{P}^\theta(\xi_1(\Lambda)=m)} =$$

$$\frac{\displaystyle\int_{\Lambda^m}\left(\sum_{n=0}^\infty\frac{(-1)^n}{n!}\int_{\Lambda^n}\prod_{k=1}^n\phi_0(u_k)\rho_{\mathbf{v}}(\mathbf{u})\mathrm{d}^n\mu_0^\theta(\mathbf{u})\right)\prod_{j=1}^m(1-\phi_1(v_j))\rho(\mathbf{v})\mathrm{d}^m\mu_1^\theta(\mathbf{v})}{m!\mathbb{P}^\theta(\xi_1(\Lambda)=m)}$$

$$= \frac{\displaystyle\int_{\Lambda^m} L_{\mathbf{v}}[1-(1-\theta)(1-\phi_0)]\prod_{j=1}^m(1-\phi_1(v_j))\rho(\mathbf{v})\mathrm{d}^m\mu_1^\theta(\mathbf{v})}{m!\mathbb{P}^\theta(\xi_1(\Lambda)=m)}.$$

Since both sides are equal for any choice of ϕ_1, we can conclude that (1.2.21) holds. To compute the correlation functions, we note that the transformation under consideration is the composition of taking the Palm measure and then conditioning on observing no particles, as the form of the average multiplicative functional reveals. Since for $\pi^\theta_{1,m}$-a.e $\mathbf{x}\in\Lambda^m$ one has $\rho(\mathbf{x})\geq j_1^{\theta,\Lambda}(\mathbf{x})>0$ by (1.2.10), the result follows from the corresponding one in Theorem 1.2.4 after noticing that the Jánossy densities of the Palm measure are given by $j^B_{\mathbf{v}}(\mathbf{x})=\frac{j^B(\mathbf{x}\sqcup\mathbf{v})}{\rho(\mathbf{v})}$, while recalling the convention $j^B(\emptyset)=\mathbb{P}(\xi(B)=0)$.

(4) This follows after a straightforward computation from *(3)* and (1.1.4). If K is self-adjoint then so is $\mathrm{K}_{\mathbf{v}}$, thus the result follows again from *(3)* and Theorem 1.2.4. □

Remark 1.2.8. *The disintegration in part (2) of the above result is more general than* (1.2.15)*: it suffices indeed to take*

$$\phi(\xi_{0,1})=1_C(\xi_0)1_{\{\xi_1(\Lambda)=m\}}(\xi_{0,1}),$$

to recover (1.2.15).

Remark 1.2.9. *For the Poisson point process on Λ with intensity ρ, the above result is again trivial. We then have that $\mathbb{P}^\theta_{|\mathbf{v}} = \mathbb{P}^\theta_{|\emptyset} = \mathbb{P}^\theta_0$, in other words the positions of the mark 1 points do not carry any information about the mark 0 points.*

Remark 1.2.10. *The last part of the above result implies Theorem 1.1.4.*

1.3 Number rigidity and DPPs corresponding to projection operators

1.3.1 DPPs induced by orthogonal projections

Let $\mathbb{P}$ be a DPP on Λ, defined by a correlation kernel K with respect to a locally finite positive Borel measure μ which is such that the associated operator $\mathrm{K} : L^2(\Lambda, \mu) \to L^2(\Lambda, \mu)$ is a locally trace class orthogonal projection, i.e. $0 \le \mathrm{K} \le 1$ and $\mathrm{K}^2 = \mathrm{K}$, onto a closed subspace H of $L^2(\Lambda, \mu)$. The rank of K can be finite or infinite, but the results in this section will only be non-trivial in the infinite rank case. We assume here that the kernel $K : \Lambda^2 \to \mathbb{C}$ of K is such that $\mathrm{K}f(x)$ is defined for every $x \in \Lambda$ and for every $f \in L^2(\Lambda, \mu)$. Note that this is true whenever $K(x, .) \in L^2(\Lambda, \mu)$ for every $x \in \Lambda$, by the Cauchy-Schwarz inequality. Classical examples of admissible point processes are the sine, Airy, and Bessel point processes on the real line.

By Proposition 1.2.2, the marked point process associated to $\mathbb{P}$ with marking function θ is the DPP on $(\Lambda_{\{0,1\}}, \mu^\theta)$ with correlation kernel $K^\theta((x, b), (x', b')) = K(x, x')$, where we recall that μ^θ is given by (1.2.4). The induced operator K^θ acting on $L^2(\Lambda_{\{0,1\}}, \mu^\theta)$ is the orthogonal projection operator onto the space

$$H^\theta := \left\{ h_{0,1} \in L^2(\Lambda_{\{0,1\}}, \mu^\theta) : h_{0,1}(., 0) = h_{0,1}(., 1) \in H \right\},$$

and it is straightforward to verify that $\dim H^\theta = \dim H$.

As in Section 1.2, let us consider the conditional measure obtained by conditioning the marked point process on a configuration of mark 1 points. Under the assumptions that $\mathbb{P}^\theta(\xi_1(\Lambda) = m) > 0$ and $\mathrm{M}_{\sqrt{\theta} + 1_B} \mathrm{K} \mathrm{M}_{\sqrt{\theta} + 1_B}$ is trace class for any bounded Borel set B, we know from Theorem 1.2.7 (4) that for $\pi^\theta_{1,m}$-a.e. $\mathbf{v} \in \Lambda^m$, the conditional measure $\mathbb{P}^\theta_{|\mathbf{v}}$ is the DPP induced by the operator

$$\mathrm{M}_{1-\theta} \mathrm{K}_{\mathbf{v}} (1 - \mathrm{M}_\theta \mathrm{K}_{\mathbf{v}})^{-1} = (1 - \mathrm{M}_\theta) \mathrm{K}_{\mathbf{v}} (1 - \mathrm{M}_\theta \mathrm{K}_{\mathbf{v}})^{-1}$$

on $L^2(\Lambda, \mu)$. Moreover, from (1.1.12), it is straightforward to verify that $\mathrm{K}_{\mathbf{v}}$ is the orthogonal projection on the subspace

$$H_{\mathbf{v}} = \overline{\{h \in H : h(v) = 0 \ \forall v \in \mathbf{v}\}}. \tag{1.3.1}$$

Consequently, since $\mathrm{K}_{\mathbf{v}}^2 = \mathrm{K}_{\mathbf{v}}$, the $L^2(\Lambda,\mu)$-operator $\mathrm{M}_{1-\theta}\mathrm{K}_{\mathbf{v}}(1-\mathrm{M}_\theta\mathrm{K}_{\mathbf{v}})^{-1}$ inducing $\mathbb{P}^\theta_{|\mathbf{v}}$ is equal to a conjugation of $\mathrm{K}_{\mathbf{v}}$,

$$(1-\mathrm{M}_\theta\mathrm{K}_{\mathbf{v}})\mathrm{K}_{\mathbf{v}}(1-\mathrm{M}_\theta\mathrm{K}_{\mathbf{v}})^{-1},$$

and this implies that it is a (not necessarily self-adjoint) projection onto the subspace

$$H^\theta_{\mathbf{v}} := \overline{(1-\mathrm{M}_\theta\mathrm{K}_{\mathbf{v}})H_{\mathbf{v}}} = \overline{(1-\mathrm{M}_\theta)H_{\mathbf{v}}}, \tag{1.3.2}$$

with dimension equal to that of $H_{\mathbf{v}}$, and that the $L^2(\Lambda,\mu_0^\theta)$-operator $\mathrm{K}^\theta_{|\mathbf{v}}$ is the orthogonal projection onto $H_{\mathbf{v}}$. Indeed, $\mathrm{K}^\theta_{|\mathbf{v}}$ is Hermitian, and for $h \in H_{\mathbf{v}}$, we have

$$\mathrm{K}^\theta_{|\mathbf{v}}h = \mathrm{K}_{\mathbf{v}}(1-\mathrm{M}_\theta\mathrm{K}_{\mathbf{v}})^{-1}\mathrm{M}_{1-\theta}h = \mathrm{K}_{\mathbf{v}}(1-\mathrm{M}_\theta\mathrm{K}_{\mathbf{v}})^{-1}\mathrm{M}_{1-\theta}\mathrm{K}_{\mathbf{v}}h = \mathrm{K}_{\mathbf{v}}h = h. \tag{1.3.3}$$

Let us now consider the more general case where $\mathbb{P}$ is induced by a not necessarily Hermitian projection operator, say $\mathrm{K} = \mathrm{P}_{H,J}$ is the unique linear projection with range H and kernel $J^\perp$, where H, J are closed subspaces of $L^2(\Lambda,\mu)$ such that $H \oplus J^\perp = L^2(\Lambda,\mu)$. Note that the adjoint projection is given by $\mathrm{P}^*_{H,J} = \mathrm{P}_{J,H}$. Since $\phi \in L^2(\Lambda,\mu)$ can be identified with $\phi \in L^2(\Lambda,\mu_0^\theta)$, we can also see H, J as subspaces of $L^2(\Lambda,\mu_0^\theta)$. Examples of DPPs induced by non-Hermitian projections are biorthogonal ensembles and their scaling limits like the Pearcey DPP.

Proposition 1.3.1. *If* $\mathrm{K} = \mathrm{P}_{H,J}$*, then*

$$\mathrm{K}^\theta_{|\mathbf{v}} = \mathrm{P}_{H_{\mathbf{v}},J_{\mathbf{v}}},$$

where $H_{\mathbf{v}}, J_{\mathbf{v}}$ *are seen as closed subspaces of* $L^2(\Lambda,\mu_0^\theta)$*.*

Proof. First we recall that the transformation $\mathrm{K}^\theta_{|\mathbf{v}}$ is obtained by first taking the reduced Palm measure and then conditioning on $\xi_1 = \emptyset$ in view of Theorem 1.2.7, so that it suffices to prove the result separately in the cases $\theta = 0$ and $\mathbf{v} = \emptyset$. The case $\theta = 0$ is straightforward from (1.1.12), while for $\mathbf{v} = \emptyset$, $\mathrm{K}^\theta_{|\emptyset}$ is a projection with range H by (1.3.3) (with $\mathbf{v} = \emptyset$; observe that these equalities continue to hold when K is not self-adjoint). Finally, to identity the kernel, it suffices to apply the previous reasoning to $(\mathrm{K}^\theta_{|\emptyset})^* = (\mathrm{K}^*)^\theta_{|\emptyset}$. □

DPPs induced by projections have the property that the number of points in a configuration is almost surely equal to the rank of the projection [135]. If $\mathbb{P}$ has configurations with a deterministic number of points, it is obvious that the same must hold for $\mathbb{P}^\theta_{|\mathbf{v}}$, for any finite configuration $\mathbf{v}$. Since the projection $P_{H_{\mathbf{v}},J_{\mathbf{v}}}$ is also defined for infinite configurations $\mathbf{v}$, it is natural to ask whether the DPP induced by this projection can in such a situation still be interpreted as the conditional DPP $\mathbb{P}^\theta_{|\mathbf{v}}$. This is not true in general, see e.g. [46], but we will see below that $\mathbb{P}^\theta_{|\mathbf{v}}$ is under suitable assumptions induced by an orthogonal projection, albeit not necessarily equal to $P_{H_{\mathbf{v}},J_{\mathbf{v}}}$.

1.3.2 Disintegration

We first show that the family of conditional ensembles $\left\{\mathbb{P}^\theta_{|\mathbf{v}}\right\}_{\delta_\mathbf{v}\in\mathcal{N}(\Lambda)}$ exists under general conditions, and then we rely on results from [40] to prove that $\mathbb{P}^\theta_{|\mathbf{v}}$ is a DPP induced by a Hermitian operator $\mathrm{K}^\theta_{|\mathbf{v}}$ if $\mathbb{P}$ is a DPP induced by an orthogonal projection.

Proposition 1.3.2. *Let $\theta : \Lambda \to [0,1]$ be measurable, and let $\mathbb{P}$ satisfy Assumptions 1.2.1. There exists a family of point processes $\left\{\mathbb{P}^\theta_{|\mathbf{v}}\right\}_{\delta_\mathbf{v}\in\mathcal{N}(\Lambda)}$ such that the following conditions hold.*

1. *The map $\delta_\mathbf{v} \in \mathcal{N}(\Lambda) \mapsto \mathbb{P}^\theta_{|\mathbf{v}}(C)$ is $\mathcal{C}(\Lambda)$-measurable for any $C \in \mathcal{C}(\Lambda)$, and the disintegration*

$$\mathbb{P}^\theta_0(C) = \int_{\mathcal{N}(\Lambda)} \mathbb{P}^\theta_{|\mathbf{v}}(C)\mathrm{d}\mathbb{P}^\theta_1(\delta_\mathbf{v})$$

   holds for any $C \in \mathcal{C}(\Lambda)$.

2. *If $\mathbb{P}$ is a DPP induced by an orthogonal projection with kernel $K : \Lambda^2 \to \mathbb{R}$ such that $\mathrm{K}f(x)$ is defined for every $x \in \Lambda$ and for every $f \in L^2(\Lambda, \mu)$, then for $\mathbb{P}^\theta_1$-a.e. $\delta_\mathbf{v}$, $\mathbb{P}^\theta_{|\mathbf{v}}$ is a DPP induced by a Hermitian locally trace class operator $\mathrm{K}^\theta_{|\mathbf{v}}$.*

Proof. Let us define $\widehat{\mathbb{P}}^\theta_{|\mathbf{v}}$ by disintegrating $\mathbb{P}^\theta$ with respect to the surjective mapping

$$r : \mathcal{N}(\Lambda_{\{0,1\}}) \to \mathcal{N}(\Lambda \times \{1\}) : \xi_0 \otimes \delta_0 + \xi_1 \otimes \delta_1 \mapsto \xi_1 \otimes \delta_1.$$

The disintegration theorem then implies that the map $\delta_\mathbf{v} \mapsto \widehat{\mathbb{P}}^\theta_{|\mathbf{v}}(\widetilde{C})$ is Borel measurable for any $\widetilde{C} \in \mathcal{C}(\Lambda_{\{0,1\}})$, and that

$$\mathbb{P}^\theta(\widetilde{C}) = \int_{\mathcal{N}(\Lambda_{\{0,1\}})} \widehat{\mathbb{P}}^\theta_{|\mathbf{v}}(\widetilde{C})\mathrm{d}\mathbb{P}^\theta(r^{-1}(\delta_\mathbf{v} \otimes \delta_1)).$$

Taking $\widetilde{C} = C \otimes \{\delta_0\} \subset \mathcal{N}(\Lambda \times \{0\})$ with $C \in \mathcal{C}(\Lambda)$ and defining $\mathbb{P}^\theta_{|\mathbf{v}}$ on Λ as $\mathbb{P}^\theta_{|\mathbf{v}}(C) := \widehat{\mathbb{P}}^\theta_{|\mathbf{v}}(\widetilde{C})$, this becomes

$$\mathbb{P}^\theta_0(C) = \int_{\mathcal{N}(\Lambda)} \mathbb{P}^\theta_{|\mathbf{v}}(C)\mathrm{d}\mathbb{P}^\theta_1(\delta_\mathbf{v}),$$

and part *(1)* of the theorem is proved.

Part (2) follows directly upon applying [40, Lemma 1.11] to the marked point process $\mathbb{P}^\theta$ and $W = \Lambda \times \{1\}$. □

Remark 1.3.3. *It is important to note that the operator $\mathrm{K}^\theta_{|\mathbf{v}}$ is not necessarily a projection in part (2) of the above result.*

1.3.3 Marking rigidity

We will now further refine our assumptions on $\mathbb{P}$, in order to obtain a sufficient condition for $\mathbb{P}$ to be marking rigid. Let us emphasize that we will not need $\mathbb{P}$ to be a DPP. However, DPPs induced by integrable orthogonal projection operators will provide our main example of point processes which satisfy the assumption below.

Assumptions 1.3.4. *$\mathbb{P}$ satisfies Assumptions 1.2.1 and is such that the following holds: for any $\epsilon > 0$ and for any bounded $B \in \mathcal{B}_\Lambda$, there exists a bounded measurable function $f : \Lambda \to [0, +\infty)$ with bounded support such that*

$$f|_B = 1, \qquad \mathrm{Var} \int_\Lambda f \mathrm{d}\xi < \epsilon,$$

where Var *denotes the variance with respect to $\mathbb{P}$.*

Theorem 1.3.5. *Let $\mathbb{P}$ satisfy Assumptions 1.3.4.*

1. *If for any measurable $\theta : \Lambda \to [0, 1]$, $\mathbb{P}^\theta(\xi_0(\Lambda) < \infty)$ is either 0 or 1, then $\mathbb{P}$ is marking rigid.*

2. *Let $\mathbb{P}$ be a DPP induced by a locally trace class orthogonal projection with kernel $K : \Lambda^2 \to \mathbb{C}$ such that $\mathrm{K}f(x)$ is defined for every $x \in \Lambda$ and for every $f \in L^2(\Lambda, \mu)$. Then $\mathbb{P}$ is marking rigid, and for any measurable $\theta : \Lambda \to [0, 1]$ such that $\mathrm{M}_{1-\theta}\mathrm{K}$ is trace class, the conditional ensemble $\mathbb{P}^\theta_{|\mathbf{v}}$ is for $\mathbb{P}^\theta_1$-a.e. $\delta_\mathbf{v}$ induced by a finite rank orthogonal projection $\mathrm{K}^\theta_{|\mathbf{v}}$.*

Proof. Let us first consider the case where θ is such that $\mathbb{P}^\theta(\xi_0(\Lambda) < \infty) = 0$, when $\mathbb{P}^\theta$-a.s., we have $\xi_0(\Lambda) = \infty$. Then, by Proposition 1.3.2,

$$1 = \mathbb{P}^\theta_0(\xi(\Lambda) = \infty) = \int_{\mathcal{N}(\Lambda)} \mathbb{P}^\theta_{|\mathbf{v}}(\xi(\Lambda) = \infty) \mathrm{d}\mathbb{P}^\theta_1(\delta_\mathbf{v}),$$

and consequently $\mathbb{P}^\theta_{|\mathbf{v}}(\xi(\Lambda) = \infty) = 1$ for $\mathbb{P}^\theta_1$-a.e. $\delta_\mathbf{v}$.

We now assume that $\mathbb{P}^\theta(\xi_0(\Lambda) < \infty) = 1$. Let $\Lambda_1 \subset \Lambda_2 \subset \cdots$ be an exhausting sequence of bounded Borel subsets of Λ. First, we observe that for $\mathbb{P}^\theta$-a.s. ξ_0, $\xi_0(\Lambda \setminus \Lambda_n) = 0$ for n sufficiently large. Secondly, we take a sequence of positive numbers $\epsilon_1, \epsilon_2, \ldots$ which converges to 0 as $n \to \infty$, and we observe that by Assumptions 1.3.4, there exists a sequence of bounded measurable functions $f_1, f_2, \ldots$ with bounded support such that $f_n|_{\Lambda_n} = 1$, and such that $\mathrm{Var} \int_\Lambda f_n \mathrm{d}\xi < \epsilon_n$. This implies by Chebyshev's inequality that

$$\mathbb{P}\left(\left|\int_\Lambda f_n \mathrm{d}\xi - \mathbb{E}\int_\Lambda f_n \mathrm{d}\xi\right| \geq \delta\right) \leq \delta^{-2}\mathrm{Var}\int_\Lambda f_n \mathrm{d}\xi \leq \delta^{-2}\epsilon_n \to 0,$$

as $n \to \infty$ for any $\delta > 0$, and hence that there exists a subsequence $\left(f_{n_j}\right)_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty}\left(\int_\Lambda f_{n_j} \mathrm{d}\xi - \mathbb{E}\int_\Lambda f_{n_j} \mathrm{d}\xi\right) = 0, \qquad \text{for } \mathbb{P}\text{-a.e. } \xi,$$

or equivalently with $\hat{f}_n(x,b) = f_n(x)$

$$\lim_{j\to\infty}\left(\int_{\Lambda_{\{0,1\}}} \hat{f}_{n_j}\mathrm{d}\xi_{0,1} - \mathbb{E}^\theta \int_{\Lambda_{\{0,1\}}} \hat{f}_{n_j}\mathrm{d}\xi_{0,1}\right) = 0, \qquad \text{for } \mathbb{P}^\theta\text{-a.e. } \xi_{0,1}.$$

For any $\xi_{0,1} \in \mathcal{N}(\Lambda_{\{0,1\}})$, we can write $\xi_0(\Lambda)$ as

$$\begin{aligned}
&\int_{\Lambda_{\{0,1\}}} 1_{\{b=0\}}\mathrm{d}\xi_{0,1}(x,b) = \int_{\Lambda_{\{0,1\}}} \hat{f}_{n_j}\mathrm{d}\xi_{0,1} + \int_\Lambda (1-f_{n_j})\mathrm{d}\xi_0 - \int_\Lambda f_{n_j}\mathrm{d}\xi_1 \\
&= \left(\int_{\Lambda_{\{0,1\}}} \hat{f}_{n_j}\mathrm{d}\xi_{0,1} - \mathbb{E}^\theta \int_{\Lambda_{\{0,1\}}} \hat{f}_{n_j}\mathrm{d}\xi_{0,1}\right) + \int_\Lambda (1-f_{n_j})\mathrm{d}\xi_0 \\
&\qquad\qquad + \mathbb{E}\int_\Lambda f_{n_j}\mathrm{d}\xi - \int_\Lambda f_{n_j}\mathrm{d}\xi_1.
\end{aligned}$$

Taking the limit $j \to \infty$, the part between parentheses at the right converges to 0 $\mathbb{P}^\theta$-a.s., and the term $\int_\Lambda (1-f_{n_j})\mathrm{d}\xi_0$ vanishes $\mathbb{P}^\theta$-a.s. for sufficiently large j, since $1-f_{n_j}$ vanishes on $\operatorname{supp}\xi_0 \subset \Lambda_{n_j}$. The other terms on the bottom line are deterministic or depend only on ξ_1. We can conclude that

$$\begin{aligned}
1 &= \mathbb{P}^\theta\left(\xi_0(\Lambda) = \lim_{j\to\infty}\left(\mathbb{E}\int_\Lambda f_{n_j}\mathrm{d}\xi - \int_\Lambda f_{n_j}\mathrm{d}\xi_1\right)\right) \\
&= \int_{\mathcal{N}(\Lambda)} \mathbb{P}^\theta_{|\mathbf{v}}\left(\xi(\Lambda) = \lim_{j\to\infty}\left(\mathbb{E}\int_\Lambda f_{n_j}\mathrm{d}\xi - \int_\Lambda f_{n_j}\mathrm{d}\delta_{\mathbf{v}}\right)\right)\mathrm{d}\mathbb{P}^\theta_1(\delta_{\mathbf{v}}),
\end{aligned}$$

by Proposition 1.3.2, and it follows that

$$\mathbb{P}^\theta_{|\mathbf{v}}\left(\xi(\Lambda) = \lim_{j\to\infty}\left(\mathbb{E}\int_\Lambda f_{n_j}\mathrm{d}\xi - \int_\Lambda f_{n_j}\mathrm{d}\delta_{\mathbf{v}}\right) =: \ell_{\mathbf{v}}\right) = 1 \quad \text{for } \mathbb{P}^\theta_1\text{-a.e. } \delta_{\mathbf{v}}.$$

By Proposition 1.3.2 (1), the map $\delta_{\mathbf{v}} \mapsto \mathbb{P}^\theta_{|\mathbf{v}}(\xi(\Lambda) = \ell)$ is $\mathcal{C}(\Lambda)$-measurable for any $\ell \in \mathbb{N}\cup\{0,\infty\}$, hence the preimage of 1 under this map is in $\mathcal{C}(\Lambda)$. But by definition, this is the same as the preimage of ℓ under the map $\delta_{\mathbf{v}} \mapsto \ell_{\mathbf{v}}$, hence $\delta_{\mathbf{v}} \mapsto \ell_{\mathbf{v}}$ is $\mathcal{C}(\Lambda)$-measurable. We can conclude that $\mathbb{P}$ is marking rigid, and part (1) of the theorem is proved.

For part (2), it follows from [135, Theorem 4] that $\mathbb{P}^\theta(\xi_0(\Lambda) < \infty) = 0$ if $\operatorname{tr} \mathrm{M}_{1-\theta}\mathrm{K} = \infty$, while $\mathbb{P}^\theta(\xi_0(\Lambda) < \infty) = 1$ if $\operatorname{tr} \mathrm{M}_{1-\theta}\mathrm{K} < \infty$. We then know that $\mathbb{P}$ is marking rigid from part (1), and it follows that, for any measurable $\theta : \Lambda \to [0,1]$ such that $\operatorname{tr} \mathrm{M}_{1-\theta}\mathrm{K} < \infty$ and for $\mathbb{P}^\theta_1$-a.e. $\delta_{\mathbf{v}}$, there exists a finite number $\ell_{\mathbf{v}}$ such that $\mathbb{P}^\theta_{|\mathbf{v}}(\xi(\Lambda) = \ell_{\mathbf{v}}) = 1$. By Proposition 1.3.2 (2), $\mathbb{P}^\theta_{|\mathbf{v}}$ is a DPP induced by a Hermitian locally trace class operator, hence again by [135, Theorem 4], it is induced by an orthogonal projection $\mathrm{K}^\theta_{|\mathbf{v}}$ of rank $\ell_{\mathbf{v}}$. □

Remark 1.3.6. *The above proof gives us more information about $\xi_0(\Lambda)$. Depending on θ, this number is either a.s. infinite, or a.s. equal to*

$$\ell_{\mathbf{v}} = \lim_{j\to\infty} \mathbb{E}\int_\Lambda f_{n_j}\mathrm{d}(\xi - \delta_{\mathbf{v}}).$$

It is surprising that this number does not depend explicitly on the marking function θ. Of course, the configurations $\delta_{\mathbf{v}}$ for which it holds implicitly encode information about θ.

1.4 OPEs on the real line or on the unit circle

1.4.1 OPEs on the real line

Let us consider the N-point OPE on the real line defined by (1.1.6). It is well-known that (1.1.6) is a DPP on $(\mathbb{R}, w(x)\mathrm{d}x)$, with kernel

$$K_N(x,y) = \sum_{j=0}^{N-1} p_j(x)p_j(y), \tag{1.4.1}$$

where p_j is the normalized orthogonal polynomial of degree j with positive leading coefficient on the real line with respect to the weight $w(x)$. From the orthogonality of the polynomials, it follows that the integral operator K_N with kernel K_N acting on $L^2(\mathbb{R}, w(x)\mathrm{d}x)$, defined by

$$\mathrm{K}_N f(y) = \int_{\mathbb{R}} K_N(x,y)f(y)w(y)\mathrm{d}y, \tag{1.4.2}$$

is the orthogonal projection onto the N-dimensional space

$$H_N := \{p : p \text{ is a polynomial of degree } \leq N-1\}.$$

Alternatively, by the Christoffel-Darboux formula, we can write the correlation kernel in 2-integrable form

$$K_N(x,y) = \gamma_N \frac{p_N(x)p_{N-1}(y) - p_N(y)p_{N-1}(x)}{x-y}, \tag{1.4.3}$$

with $\gamma_N = \frac{\kappa_{N-1}}{\kappa_N}$, where κ_n is the leading coefficient of p_n, or equivalently $\kappa_n^{-1} = \int_{\mathbb{R}} p_n(x)x^n w(x)\mathrm{d}x$. See e.g. [68] for more background and details about these ensembles.

1.4.2 OPEs on the unit circle

For general integrable weight functions w, (1.1.7) is a DPP on the unit circle $\{z = e^{it}\}$ with respect to $w(e^{it})\mathrm{d}t$, with correlation kernel

$$K_N(e^{it}, e^{is}) = \sum_{j=0}^{N-1} \varphi_j(e^{it})\overline{\varphi_j(e^{is})}, \tag{1.4.4}$$

where φ_j is the normalized orthogonal polynomial of degree j with positive leading coefficient on the unit circle with respect to the weight $w(e^{it})$. The associated integral operator K_N is the orthogonal projection onto the space

$$H_N := \{\varphi : \varphi \text{ is a polynomial of degree } \leq N-1\}$$

Alternatively, by the Christoffel-Darboux formula for orthogonal polynomials on the unit circle, we have the 2-integrable form

$$K_N(e^{it}, e^{is}) = \frac{e^{iN(t-s)}\varphi_N(e^{is})\overline{\varphi_N(e^{it})} - \varphi_N(e^{it})\overline{\varphi_N(e^{is})}}{1 - e^{i(t-s)}}.$$

For the uniform weight $w = 1$, we have $\varphi_j(z) = (2\pi)^{-\frac{1}{2}} z^j$ and thus, after conjugation of the operator K_N, the kernel can be taken to be

$$K_N(e^{it}, e^{is}) = \frac{1}{2\pi} \frac{\sin \frac{N(t-s)}{2}}{\sin \frac{t-s}{2}}.$$

In the scaling limit where $t-s = \frac{2\pi(u-v)}{N}$ and $N \to \infty$, $\frac{2\pi}{N} K_N(e^{it}, e^{is})$ converges to the sine kernel

$$K^{\sin}(u, v) = \frac{\sin \pi(u-v)}{\pi(u-v)}, \tag{1.4.5}$$

uniformly for u, v in compact subsets of the real line. See e.g. [81, 124] for details.

1.4.3 Conditional ensembles associated to OPEs

From Proposition 1.3.1, it follows that the conditional ensemble $\mathbb{P}^{\theta}_{|\mathbf{v}}$ is the DPP, on $(\mathbb{R}, (1-\theta(x))w(x)\mathrm{d}x)$ or on the unit circle with measure $(1-\theta(e^{it}))w(e^{it})\mathrm{d}t$, with kernel $\widetilde{K}_N$ of the orthogonal projection onto the space

$$H_{N,\mathbf{v}} := \{p : p \text{ is a polynomial of degree} \leq N-1, \text{ and } p(v) = 0 \ \forall v \in \mathbf{v}\}.$$

Let us now define

$$w^{\theta}_{|\mathbf{v}}(x) := (1-\theta(x))w(x) \prod_{v \in \mathbf{v}} |x - v|^2$$

for the real line, and

$$w^{\theta}_{|\mathbf{v}}(e^{it}) := (1-\theta(e^{it}))w(e^{it}) \prod_{v \in \mathbf{v}} |e^{it} - v|^2$$

for the unit circle. In the case of the real line, we then have $\widetilde{K}_N(x, y) = \prod_{v \in \mathbf{v}}(x-v)(y-v)K_n(x, y)$, with $n := N - \#\mathbf{v}$ and with K_n the Christoffel-Darboux kernel (1.4.1) with N replaced by n and w by $w^{\theta}_{|\mathbf{v}}$. It follows that we can also see $\mathbb{P}^{\theta}_{|\mathbf{v}}$ as a DPP on $(\mathbb{R}, w^{\theta}_{|\mathbf{v}}(x)\mathrm{d}x)$ with kernel K_n, which implies that it is the n-point OPE

$$\frac{1}{Z_n} \Delta(\mathbf{x})^2 \prod_{j=1}^{n} w^{\theta}_{|\mathbf{v}}(x_j)\mathrm{d}x_j, \qquad \Delta(\mathbf{x}) = \prod_{1 \leq i < j \leq N} (x_j - x_i).$$

In the case of the unit circle, we obtain similarly that $\mathbb{P}^{\theta}_{|\mathbf{v}}$ is the n-point OPE

$$\frac{1}{Z_n} |\Delta(\mathbf{e^{it}})|^2 \prod_{j=1}^{n} w^{\theta}_{|\mathbf{v}}(e^{it_j})\mathrm{d}t_j, \qquad \Delta(\mathbf{e^{it}}) = \prod_{1 \leq l < k \leq N} (e^{it_k} - e^{it_j}), \quad t_j \in [0, 2\pi).$$

Summarizing the above, we have proved the following result.

Proposition 1.4.1. *If $\mathbb{P}$ is the N-point OPE with weight w on the real line or the unit circle and $n := N - \#\mathbf{v} > 0$, then $\mathbb{P}^{\theta}_{|\mathbf{v}}$ is the n-point OPE with weight $w^{\theta}_{|\mathbf{v}}$ on the real line or the unit circle.*

1.4.4 Unitary invariant ensembles and scaling limits

The above form of the conditional ensembles has the remarkable consequence that any unitary invariant ensemble (1.1.6) with $V(x) \geq x^2$, can be constructed theoretically from the GUE: to see this, consider the conditional ensemble $\mathbb{P}^{\theta}_{|\emptyset}$ with $w(x) = e^{-Nx^2}$ the Gaussian weight in (1.1.6), and with $\theta(x) = 1 - e^{-N(V(x)-x^2)} \in [0,1]$. The latter has the joint probability distribution (1.1.6), but now with weight

$$w^{\theta}_{|\emptyset}(x) := (1-\theta(x))e^{-Nx^2} = e^{-NV(x)}.$$

This is of course not of any practical use for N large, because the event on which we condition then has very small probability unless $V(x)$ is close to x^2 (note that there exist algorithms to generate DPPs in general, see e.g. [94]). Nevertheless, it is striking that the GUE encodes any of the above unitary invariant ensembles via marking and conditioning.

This becomes even more surprising if we look at scaling limits of the correlation kernels. It is a classical fact that the GUE converges to the sine point process in the bulk scaling limit and to the Airy point process in the edge scaling. It is also understood that conditioning on an eigenvalue and scaling around this eigenvalue leads to the bulk Bessel point process, and that conditioning on a gap leads to the hard edge Bessel kernel. But unitary invariant ensembles admit for special choices of V also more complicated limit processes, associated to Painlevé equations and hierarchies [74]. In fact, it follows from the above that these Painlevé point processes are already encoded in the GUE eigenvalue distribution, if one combines a suitable conditioning with taking scaling limits.

1.4.5 Marginal distribution of mark 0 points with known number of mark 1 points.

The construction of the conditional ensembles in Section 1.2 passed through the marked point process conditioned on having m mark 1 particles. In this case, we can make these ensembles more explicit. Indeed, from (1.1.6), we obtain that the marginal distribution of the mark 0 particles, conditioned on having exactly m mark 1 particles, is given by

$$\frac{1}{Z_{N,m}}|\Delta(\mathbf{u})|^2 \left(\int_{\mathbb{R}^m} |\Delta(\mathbf{v})|^2 \prod_{k=1}^{m}\left(\prod_{\ell=1}^{n}|u_\ell - v_k|^2\right)\theta(v_k)w(v_k)\mathrm{d}v_k\right) \times \prod_{j=1}^{n}(1-\theta(u_j))w(u_j)\mathrm{d}u_j,$$

where

$$Z_{N,m} = \int_{\mathbb{R}^n} |\Delta(\mathbf{u})|^2 \left(\int_{\mathbb{R}^m} |\Delta(\mathbf{v})|^2 \prod_{k=1}^{m} \left(\prod_{\ell=1}^{n} |u_\ell - v_k|^2 \right) \theta(v_k) w(v_k) \mathrm{d}v_k \right) \times \prod_{j=1}^{n} (1-\theta(u_j)) w(u_j) \mathrm{d}u_j.$$

By Heine's formula, the v-integral can be written as a Hankel determinant in the case of the real line, and as a Toeplitz determinant in the case of the unit circle. For the real line, defining the Hankel determinant as

$$H_m(f) = \det \left(f_{j+k} \right)_{j,k=0}^{m-1}, \qquad f_\ell = \int_{\mathbb{R}} x^\ell f(x) \mathrm{d}x,$$

we have

$$\frac{1}{Z'_{N,m}} |\Delta(\mathbf{u})|^2 \; H_m \left(\theta.w. \prod_{j=1}^{n} (. - u_j)^2 \right) \prod_{j=1}^{n} (1-\theta(u_j)) w(u_j) \mathrm{d}u_j, \tag{1.4.6}$$

with $Z'_{N,m} = \int_{\mathbb{R}^n} |\Delta(\mathbf{u})|^2 \; H_m \left(\theta.w. \prod_{j=1}^{n} (. - u_j)^2 \right) \prod_{j=1}^{n} (1-\theta(u_j)) w(u_j) \mathrm{d}u_j$. For the unit circle, defining the Toeplitz determinant as

$$T_m(g) = \det \left(g_{j-k} \right)_{j,k=0}^{m-1}, \qquad g_\ell = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\ell t} g(e^{it}) \mathrm{d}t,$$

we obtain

$$\frac{1}{Z'_{N,m}} |\Delta(\mathbf{e^{it}})|^2 \; T_m \left(\theta.w. \prod_{j=1}^{n} |. - e^{it_j}|^2 \right) \prod_{j=1}^{n} (1-\theta(e^{it_j})) w(e^{it_j}) \mathrm{d}t_j, \tag{1.4.7}$$

with

$$Z'_{N,m} = \int_{(0,2\pi)^n} |\Delta(\mathbf{e^{it}})|^2 \; T_m \left(\theta.w. \prod_{j=1}^{n} |. - e^{it_j}|^2 \right) \prod_{j=1}^{n} (1-\theta(e^{it_j})) w(e^{it_j}) \mathrm{d}t_j.$$

Similar formulas hold for the marginal distributions of the mark 1 points. Alternatively, by [7, Theorem 3.2], we can write both densities, with either $x_j = u_j$, $\mathrm{d}\mu(x_j) = \mathrm{d}x_j$ or $x_j = e^{it_j}$, $\mathrm{d}\mu(x_j) = \mathrm{d}t_j$, as

$$\frac{1}{Z''_{N,m}} \det \left(K_N^\theta(x_l, x_k) \right)_{l,k=1}^{n} \prod_{j=1}^{n} (1-\theta(x_j)) w(x_j) \mathrm{d}\mu(x_j),$$

with a new normalization constant $Z''_{N,m}$ obtained in a similar manner, and where $K_N^\theta(x,y)$ is the kernel inducing the point processes (1.1.6)/(1.1.7) with $N = n+m$ particles and weight function θw. There is no reason to believe that these marginal distributions are in general DPPs, but they do have a special Hankel or Toeplitz determinant structure. In particular, probabilities can be expressed in terms of integrals of Toeplitz or Hankel determinants, which can in some cases be computed asymptotically as $m \to \infty$. Similar integrals of Toeplitz and Hankel determinants appear in the study of moments of moments in random matrix ensembles, connected to the study of extreme values of characteristic polynomials, see e.g. [11, 83].

1.5 Integrable DPPs

In this section, we will consider DPPs $\mathbb{P}$ on curves Λ in the complex plane with k-integrable kernels of the form (1.1.9). For simplicity, let us assume that Λ is a smooth closed curve on $\mathbb{C} \cup \{\infty\}$ without self-intersections, that the functions $f_1, \ldots, f_k$ and $g_1, \ldots, g_k$ are smooth functions on Λ, and that the reference measure is smooth with respect to $\mathrm{d}z$, i.e. $\mathrm{d}\mu(z) = h(z)\mathrm{d}z$ with h smooth (say C^∞, even if one can proceed with less regularity if needed) on Λ. Even if $\mathrm{d}z$ is not a positive measure on Λ, by mapping $K(x,y)$ to $K(x,y)h(y)$, we can then work with a kernel

$$K(x,y) = \frac{\mathbf{f}(x)^T \mathbf{g}(y)}{x-y} = \frac{\mathbf{g}(y)^T \mathbf{f}(x)}{x-y},$$

with column vectors $\mathbf{f} = (f_j)_{j=1}^k, \mathbf{g} = (g_j)_{j=1}^k$, with respect to the complex measure $\mathrm{d}z$, and with the associated integral operator K acting on $L^2(\Lambda, \mathrm{d}z)$.

1.5.1 General integrable kernels

Let us first show that the Palm kernels $K_{\mathbf{v}}$ are also of k-integrable form.

Proposition 1.5.1. *For any $\mathbf{v} = \{v_1, \ldots, v_m\}$ such that $\det K(\mathbf{v},\mathbf{v}) > 0$, the kernel of the reduced Palm measure $\mathbb{P}_{\mathbf{v}}$ is of k-integrable form $K_{\mathbf{v}}(x,y) = \frac{\mathbf{f}_{\mathbf{v}}(x)^T \mathbf{g}_{\mathbf{v}}(y)}{x-y}$, and the j-th entries of $\mathbf{f}_{\mathbf{v}}$ and $\mathbf{g}_{\mathbf{v}}$ are given by*

$$f_{\mathbf{v},j}(x) = \frac{1}{\det K(\mathbf{v},\mathbf{v})} \det \begin{pmatrix} f_j(x) & K(x,\mathbf{v}) \\ f_j(\mathbf{v}) & K(\mathbf{v},\mathbf{v}) \end{pmatrix},$$
$$g_{\mathbf{v},j}(y) = \frac{1}{\det K(\mathbf{v},\mathbf{v})} \det \begin{pmatrix} g_j(y) & g_j(\mathbf{v}) \\ K(\mathbf{v},y) & K(\mathbf{v},\mathbf{v}) \end{pmatrix},$$

where $K(\mathbf{v},\mathbf{v})$ represents the $m \times m$ matrix with (i,j)-entry equal to $K(v_i,v_j)$, $K(x,\mathbf{v})$, $g_j(\mathbf{v})$ represent m-dimensional row vectors with l-entry respectively equal to $K(x,v_\ell), g_j(v_\ell)$, and $K(\mathbf{v},y)$, $f_j(\mathbf{v})$ represent m-dimensional column vectors with entries equal to $K(v_i,y), f_j(v_i)$.

Proof. Using the block determinant formula (1.2.19), we have that $f_{\mathbf{v},j}$ as defined in the statement of the proposition is given by

$$f_{\mathbf{v},j}(x) = f_j(x) - K(x,\mathbf{v})K(\mathbf{v},\mathbf{v})^{-1} f_j(\mathbf{v}).$$

Now let $\mathbf{v} = \mathbf{v}' \sqcup \{v\}$ and assume without loss of generality that $\det K(\mathbf{v}', \mathbf{v}') > 0$, then using again the block determinant formula (1.2.19) one obtains

$$\begin{aligned}
f_{\mathbf{v},j}(x) &= \frac{1}{\det K(\mathbf{v},\mathbf{v})} \det \begin{pmatrix} f_j(x) & K(x,v) & K(x,\mathbf{v}') \\ f_j(v) & K(v,v) & K(v,\mathbf{v}') \\ f_j(\mathbf{v}') & K(\mathbf{v}',v) & K(\mathbf{v}',\mathbf{v}') \end{pmatrix} \\
&= \frac{\det K(\mathbf{v}',\mathbf{v}')}{\det K(\mathbf{v},\mathbf{v})} \times \\
&\det \left(\begin{pmatrix} f_j(x) & K(x,v) \\ f_j(v) & K(v,v) \end{pmatrix} - \begin{pmatrix} K(x,\mathbf{v}') \\ K(v,\mathbf{v}') \end{pmatrix} K(\mathbf{v}',\mathbf{v}')^{-1} \begin{pmatrix} f_j(\mathbf{v}') & K(\mathbf{v}',v) \end{pmatrix} \right) \\
&= \frac{1}{K_{\mathbf{v}'}(v,v)} \det \begin{pmatrix} f_{\mathbf{v}',j}(x) & K_{\mathbf{v}'}(x,v) \\ f_{\mathbf{v}',j}(v) & K_{\mathbf{v}'}(v,v) \end{pmatrix},
\end{aligned}$$

which implies that $\mathbf{f}_{\mathbf{v}} = (\mathbf{f}_{\mathbf{v}'})_v$, and similarly for $\mathbf{g}_{\mathbf{v}}$. Since also $\mathrm{K}_{\mathbf{v}} = (\mathrm{K}_{\mathbf{v}'})_v$, it now suffices to prove the result for $\mathbf{v} = \{v\}$. We then easily verify by (1.2.18) that

$$\begin{aligned}
(x-y)K_v(x,y) &= \mathbf{f}(x)^T\mathbf{g}(y) - ((x-v)+(v-y))\frac{K(x,v)K(v,y)}{K(v,v)} \\
&= \mathbf{f}(x)^T\mathbf{g}(y) - \mathbf{f}(x)^T\mathbf{g}(v)\frac{K(v,y)}{K(v,v)} - \frac{K(x,v)}{K(v,v)}\mathbf{f}(v)^T\mathbf{g}(y) \\
&= \left(\mathbf{f}(x) - \frac{K(x,v)}{K(v,v)}\mathbf{f}(v)\right)^T \left(\mathbf{g}(y) - \frac{K(v,y)}{K(v,v)}\mathbf{g}(v)\right),
\end{aligned}$$

since $\mathbf{f}^T\mathbf{g} = 0$. To complete the proof, it remains to check that $\mathbf{f}_v^T\mathbf{g}_v = 0$, but this follows from a similar computation:

$$\begin{aligned}
\mathbf{f}_v(x)^T\mathbf{g}_v(x) &= -\mathbf{f}(x)^T\mathbf{g}(v)\frac{K(v,x)}{K(v,v)} - \frac{K(x,v)}{K(v,v)}\mathbf{f}(v)^T\mathbf{g}(x) \\
&= K(x,v)\frac{\mathbf{f}(v)^T\mathbf{g}(x)}{K(v,v)} - \frac{K(x,v)}{K(v,v)}\mathbf{f}(v)^T\mathbf{g}(x) = 0.
\end{aligned}$$

□

Next, we will explain how the kernel of the point process $\mathbb{P}^\theta_{|\mathbf{v}}$ on (Λ, μ_0^θ), with kernel of the operator $\mathrm{K}(1-\mathrm{M}_\theta\mathrm{K})^{-1}$, can be characterized in terms of a RH problem. For this, we rely on the IIKS method developed in [98, 70].

In what follows, we assume that the entries of $\sqrt{\theta}\mathbf{g}$ and $\sqrt{\theta}\mathbf{f}$ are smooth, bounded and integrable functions on Λ which decay as $z \to \infty$, and that their derivatives are also bounded and integrable.

Let us consider the following RH problem.

RH problem for Y

(a) $Y : \mathbb{C} \setminus \Lambda \to \mathbb{C}^{k\times k}$ is analytic; we mean by this that every entry of the matrix is analytic in $\mathbb{C} \setminus \Lambda$.

(b) Y has continuous boundary values $Y_\pm$ when Λ is approached from the left $(+)$ or right $(-)$, with respect to the orientation chosen for Λ, and they are related by

$$Y_+(z) = Y_-(z) J_Y(z), \qquad J_Y(z) = I_k - 2\pi i \theta(z) \mathbf{f}_\mathbf{v}(z) \mathbf{g}_\mathbf{v}(z)^T, \qquad z \in \Lambda, \tag{1.5.1}$$

where I_k is the $k \times k$ identity matrix.

(c) As $z \to \infty$, $Y(z) \to I_k$ uniformly.

The following is a consequence of results from, e.g., [70, Section 2], see also [98] and [21].

Proposition 1.5.2. *Suppose that $\mathrm{M}_{\sqrt{\theta}+1_B}\mathrm{K}_\mathbf{v}\mathrm{M}_{\sqrt{\theta}+1_B}$ is trace class on the space $L^2(\Lambda, \mathrm{d}z)$ for any bounded Borel set B, and that $\det(1-\mathrm{M}_{\sqrt{\theta}}\mathrm{K}_\mathbf{v}\mathrm{M}_{\sqrt{\theta}}) \neq 0$.*

1. *The RH problem for Y is uniquely solvable, and the solution $Y(z)$ is invertible for any $z \in \mathbb{C} \setminus \Lambda$.*

2. *The DPP $\mathbb{P}^\theta_{|\mathbf{v}}$ on $(\Lambda, (1-\theta)\mathrm{d}z)$ is characterized by the k-integrable kernel*

$$K^\theta_{|\mathbf{v}}(x,y) = \frac{\mathbf{f}^\theta_{|\mathbf{v}}(x)^T \mathbf{g}^\theta_{|\mathbf{v}}(y)}{x-y} \tag{1.5.2}$$

where

$$\mathbf{f}^\theta_{|\mathbf{v}} = Y_\pm \mathbf{f}_\mathbf{v}, \qquad \mathbf{g}^\theta_{|\mathbf{v}} = Y_\pm^{-T}\mathbf{g}_\mathbf{v}, \tag{1.5.3}$$

and the above expressions are independent of the choice $\pm$ of boundary value, with $Y_\pm^{-T}$ denoting the inverse transpose of the matrix $Y_\pm$. Consequently

$$K^\theta_{|\mathbf{v}}(x,y) = \frac{1}{x-y}\mathbf{g}_\mathbf{v}(y)^T Y_\pm(y)^{-1} Y_\pm(x)\mathbf{f}_\mathbf{v}(x). \tag{1.5.4}$$

Proof. Observe first that, because of the assumptions and Proposition 1.5.1, $\theta(x)\mathbf{f}_\mathbf{v}(x)\mathbf{g}_\mathbf{v}(x)^T$ is also smooth, in $L^2(\Lambda, \mathrm{d}z)$, and decaying as $x \to \infty$, $x \in \Lambda$. We then set $\mathrm{A} = \mathrm{M}_{\sqrt{\theta}}\mathrm{K}_\mathbf{v}\mathrm{M}_{\sqrt{\theta}}$ and $V = I_k - 2\pi i \theta \mathbf{f}_\mathbf{v}\mathbf{g}_\mathbf{v}^T$, and apply [70, Lemma 2.12]: this result states that

$$(1-\mathrm{A})^{-1} - 1 = \mathrm{A}(1-\mathrm{A})^{-1} = \mathrm{M}_{\sqrt{\theta}}\mathrm{K}_\mathbf{v}\left(1-\mathrm{M}_\theta\mathrm{K}_\mathbf{v}\right)^{-1}\mathrm{M}_{\sqrt{\theta}}$$

has kernel $\frac{\mathbf{F}(x)^T\mathbf{G}(y)}{x-y}$ with

$$\mathbf{F} = Y_\pm\sqrt{\theta}\mathbf{f}_\mathbf{v}, \qquad \mathbf{G} = Y_\pm^{-T}\sqrt{\theta}\mathbf{g}_\mathbf{v}.$$

Hence, if θ has no zeros on Λ, the operator $\mathrm{K}_\mathbf{v}(1-\mathrm{M}_\theta\mathrm{K}_\mathbf{v})^{-1}$ has kernel $\frac{\mathbf{f}^\theta_{|\mathbf{v}}(x)^T\mathbf{g}^\theta_{|\mathbf{v}}(y)}{x-y}$ on $L^2(\Lambda, \mathrm{d}z)$ with

$$\mathbf{f}^\theta_{|\mathbf{v}} = \frac{1}{\sqrt{\theta}}\mathbf{F}, \qquad \mathbf{g}^\theta_{|\mathbf{v}} = \frac{1}{\sqrt{\theta}}\mathbf{G},$$

and the result follows from Theorem 1.2.7. If θ has zeros on Λ, the result does not directly follow, but it is readily seen that one can follow the proof of [70, Lemma 2.12] to prove the result also in this case. $\square$

Remark 1.5.3. *The smoothness and decay of the entries of $\theta \mathbf{f}\mathbf{g}^T$ are assumptions we make to avoid technical complications, and which guarantee smooth boundary values $Y_\pm$ and uniform convergence at infinity. One can also proceed with less regularity, but then care must be taken about the sense of the boundary values of Y, which are not necessarily continuous, and about the convergence at infinity, which is not necessarily uniform, see e.g. [68, 70] for general theory of RH problems.*

The above results imply that given $K, \theta, \mathbf{v}$, we obtain $K^\theta_{|\mathbf{v}}$ by first computing $K_{\mathbf{v}}$, and then solving the RH problem for Y. Next, we explain how to bypass this procedure by characterizing $K^\theta_{|\mathbf{v}}$ directly in terms of a RH problem which depends in a simple explicit way on $K, \theta, \mathbf{v}$, without need to go through the transformation $K \mapsto K_{\mathbf{v}}$. For that purpose, let us construct a rational matrix-valued function R, which will allow us to connect $\mathbf{f}, \mathbf{g}$ with $\mathbf{f}_{\mathbf{v}}, \mathbf{g}_{\mathbf{v}}$.

For a singleton $\mathbf{v} = \{v\}$, we observe that

$$\mathbf{f}_v(x) = \mathbf{f}(x) - \mathbf{f}(v)\frac{K(x,v)}{K(v,v)} = \left(I_k - \frac{R_1}{x-v}\right)\mathbf{f}(x), \qquad R_1 = \frac{\mathbf{f}(v)\mathbf{g}(v)^T}{K(v,v)},$$

and similarly since $R_1^2 = 0$,

$$\mathbf{g}_v(x)^T = \mathbf{g}(x)^T\left(I_k + \frac{R_1}{x-v}\right) = \mathbf{g}(x)^T\left(I_k - \frac{R_1}{x-v}\right)^{-1}.$$

For the general case $\mathbf{v} = \{v_1, ..., v_m\}$, we inductively define the matrices R_j for $j = 1, ..., m$ by

$$R_j = \frac{\mathbf{f}_{v_1,...,v_{j-1}}(v_j)\mathbf{g}_{v_1,...,v_{j-1}}(v_j)^T}{K_{v_1,...,v_{j-1}}(v_j, v_j)},$$

satisfying $R_j^2 = 0$, and for $z \in \mathbb{C} \setminus \{v_1, \ldots, v_m\}$

$$R(z) = \left(I_k + \frac{R_1}{z-v_1}\right)\left(I_k + \frac{R_2}{z-v_2}\right)\cdots\left(I_k + \frac{R_m}{z-v_m}\right). \tag{1.5.5}$$

Then R has determinant identically equal to 1, and

$$\mathbf{f}_{\mathbf{v}}(x) = R(x)^{-1}\mathbf{f}(x), \qquad \mathbf{g}_{\mathbf{v}}(x)^T = \mathbf{g}(x)^T R(x),$$

so that we can rewrite the jump matrix J_Y as

$$J_Y(x) = I_k - 2\pi i\theta(x)\mathbf{f}_{\mathbf{v}}(x)\mathbf{g}_{\mathbf{v}}(x)^T = R(x)^{-1}\left(I_k - 2\pi i\theta(x)\mathbf{f}(x)\mathbf{g}(x)^T\right)R(x). \tag{1.5.6}$$

Note that although the construction of R uses a certain order of $v_1, ..., v_m$, the result only depends on the unordered set $\mathbf{v}$, as it can be checked that

$$R(z) = I_k + \frac{\mathbf{f}(\mathbf{v})}{z-\mathbf{v}}K(\mathbf{v},\mathbf{v})^{-T}\mathbf{g}(\mathbf{v})^T = \left(I_k - \mathbf{f}(\mathbf{v})K(\mathbf{v},\mathbf{v})^{-T}\left(\frac{\mathbf{g}(\mathbf{v})}{z-\mathbf{v}}\right)^T\right)^{-1},$$

where $\frac{\mathbf{f}(\mathbf{v})}{z-\mathbf{v}}$ and $\mathbf{g}(\mathbf{v})$ are the $k \times m$ matrices whose i-th columns are $\frac{\mathbf{f}(v_i)}{z-v_i}$ and $\mathbf{g}(v_i)$, and similarly for the others. It turns out that R is a rational function which can also be characterized by a discrete RH problem (in fact, R^{-1} is the solution to the below RH problem for U with $\theta = 0$). Let us now define

$$U(z) = Y(z)R(z)^{-1}, \tag{1.5.7}$$

then U satisfies the following RH problem.

RH problem for U

1. Each entry of $U : \mathbb{C} \setminus \Lambda \to \mathbb{C}^{k\times k}$ is analytic.
2. On $\Lambda \setminus \mathbf{v}$, U has continuous boundary values $U_\pm$ which satisfy the jump condition
$$U_+ = U_-(I_k - 2\pi i\theta \mathbf{f}\mathbf{g}^T),$$
while for each $v \in \mathbf{v}$, the residue $\rho_U(v) = \lim_{z\to v}(z - v)U(z)$ is well-defined and given by
$$\rho_U(v) = -\lim_{z\to v} U(z)\frac{\mathbf{f}(v)\mathbf{g}(v)^T}{K(v, v)}.$$
3. As $z \to \infty$, $U(z) \to I_k$ uniformly.

Conditions (1) and (3) are immediately verified. To check the jump relation (2) for U, it suffices to use (1.5.7) and (1.5.6). For the residues of U, observe that it is sufficient to verify the condition for $v = v_1$ by the iterative construction of R. We then have by (1.5.5), (1.5.7), $R_1^2 = 0$, and the fact that $Y_+(v_1) = Y_-(v_1)$ (since $\mathbf{f}_\mathbf{v}(v_1) = \mathbf{g}_\mathbf{v}(v_1) = 0$) that

$$\begin{aligned}
\lim_{z\to v_1}(z - v_1)U(z) &= -Y_\pm(v_1)\mathrm{Res}_{z=v_1} R^{-1}\\
&= -Y_\pm(v_1)\left(I_k - \frac{R_m}{v_1 - v_m}\right)\cdots\left(I_k - \frac{R_2}{v_1 - v_2}\right)R_1\\
&= -\lim_{z\to v_1} Y(z)\left(I_k - \frac{R_m}{z - v_m}\right)\cdots\left(I_k - \frac{R_1}{z - v_1}\right)R_1\\
&= -\lim_{z\to v} U(z)\frac{\mathbf{f}(v)\mathbf{g}(v)^T}{K(v, v)}.
\end{aligned}$$

In conclusion, we have the following result.

Proposition 1.5.4. *Suppose that* $\mathrm{M}_{\sqrt{\theta}+1_B}\mathrm{K}_\mathbf{v}\mathrm{M}_{\sqrt{\theta}+1_B}$ *is trace class on the space* $L^2(\Lambda, \mathrm{d}z)$ *for any bounded Borel set* B*, and that* $\det(1-\mathrm{M}_{\sqrt{\theta}}\mathrm{K}_\mathbf{v}\mathrm{M}_{\sqrt{\theta}}) \neq 0$*. There exists a unique solution* U *to the RH problem for* U *which is furthermore invertible and satisfies*

$$\mathbf{f}^\theta_{|\mathbf{v}} = U_\pm\mathbf{f}, \qquad \mathbf{g}^\theta_{|\mathbf{v}} = U_\pm^{-T}\mathbf{g},$$

and

$$K^\theta_{|\mathbf{v}}(x, y) = \frac{1}{x - y}\mathbf{g}(y)^T U_\pm(y)^{-1}U_\pm(x)\mathbf{f}(x).$$

1.5.2 Integrable kernels characterized by a RH problem

The above RH characterization of $K^\theta_{|\emptyset}$ and $K^\theta_{|\mathbf{v}}$ is particularly useful in cases where the kernel K of the DPP $\mathbb{P}$ itself can also be characterized in terms of a RH problem. In such a case, the IIKS method allows to transform the RH problem to an *undressed* RH problem which is in a form amenable to asymptotic analysis and to derive differential equations [21, 22, 98].

Such a RH characterization is available for many important 2-integrable DPPs, like OPEs and the DPPs characterized by the Airy kernel, the sine kernel, the Bessel kernels, the confluent hypergeometric kernels, and kernels connected to Painlevé equations. Multiple orthogonal polynomial ensembles [109] and their scaling limits like Pearcey and tac node kernels are examples of k-integrable kernels with $k > 2$, which can also be characterized through a $(k \times k)$ RH problem.

Let us illustrate this in the case $k = 2$.

Suppose that we can write

$$K(x,y) = \frac{\mathbf{f}(x)^T \mathbf{g}(y)}{x-y}, \quad \mathbf{f}(x) = \frac{w(x)}{2\pi i}\Psi_\pm(x)\begin{pmatrix}1\\0\end{pmatrix}, \quad \mathbf{g}(y)^T = \begin{pmatrix}0 & 1\end{pmatrix}\Psi_\pm(y)^{-1}, \tag{1.5.8}$$

for a smooth bounded function $w : \Lambda \to [0, +\infty)$, where Ψ satisfies a RH problem of the following form.

RH problem for Ψ

1. $\Psi : \mathbb{C} \setminus \Lambda \to \mathbb{C}^{2\times 2}$ is analytic.
2. Ψ has continuous boundary values $\Psi_\pm$, and they are related by
$$\Psi_+(z) = \Psi_-(z)\begin{pmatrix}1 & w(z)\\ 0 & 1\end{pmatrix}, \qquad z \in \Lambda,$$
for some smooth bounded function $w : \Lambda \to \mathbb{C}$.
3. For some $\Psi_\infty : \mathbb{C} \setminus \Lambda \to \mathbb{C}^{2\times 2}$ such that $\det \Psi_\infty(z) = 1$, we have
$$\Psi(z) = \left(I_2 + O(z^{-1})\right)\Psi_\infty(z),$$
uniformly as $z \to \infty$.

Then, it is straightforward to show that $\det \Psi(z) \equiv 1$, hence $\Psi(z)$ is an invertible matrix for every $z \in \mathbb{C} \setminus \Lambda$, and that there is only one solution to the RH problem for Ψ.

The third RH condition would be trivially valid with $\Psi_\infty = \Psi$, but as we illustrate in examples below, one usually prefers to specify a simpler explicit function Ψ_∞ to describe the asymptotic behaviour of Ψ, in order to facilitate further analysis of the RH problem. Observe that the RH conditions imply that the first column of Ψ and the second row of Ψ^{-1} extend to entire functions in the complex plane, and hence that $\mathbf{f}/w$ and $\mathbf{g}$ extend to entire functions.

Let us define

$$\Psi^\theta_{|\mathbf{v}} = U\Psi. \tag{1.5.9}$$

Then, $\Psi^\theta_{|\mathbf{v}}$ is invertible and it is the unique solution to the following RH problem.

RH problem for $\Psi^\theta_{|\mathbf{v}}$

1. Each entry of $\Psi^\theta_{|\mathbf{v}} : \mathbb{C} \setminus \Lambda \to \mathbb{C}^{2\times 2}$ is analytic.
2. $\Psi^\theta_{|\mathbf{v}}$ has continuous boundary values $\Psi^\theta_{|\mathbf{v}\pm}$ on $\Lambda \setminus \mathbf{v}$ and they are related by
$$\Psi^\theta_{|\mathbf{v}+}(z) = \Psi^\theta_{|\mathbf{v}-}(z) \begin{pmatrix} 1 & w(z)(1-\theta(z)) \\ 0 & 1 \end{pmatrix},$$
while as $z \to v \in \mathbf{v}$ we have
$$\Psi^\theta_{|\mathbf{v}}(z) = \mathcal{O}(1)(z-v)^{\sigma_3}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
3. As $z \to \infty$, we have the uniform asymptotics
$$\Psi^\theta_{|\mathbf{v}}(z) = \left(I_2 + \mathcal{O}(z^{-1})\right)\Psi_\infty(z).$$

The first and the third conditions are immediate from the corresponding ones for U and Ψ. The jump relation is obtained from the jump relation for U and the one for Ψ along with (1.5.8):

$$\begin{aligned} U_+\Psi_+ &= U_-\left(I_2 - 2\pi i\theta \frac{w}{2\pi i}\Psi_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \Psi_+^{-1}\right)\Psi_+ \\ &= U_-\Psi_-\left(\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} - w\theta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right). \end{aligned}$$

The singular behaviour near $\mathbf{v}$ is obtained in a similar manner: the second column of $\Psi^\theta_{|\mathbf{v}}(z-v)^{-\sigma_3}$ is obviously $\mathcal{O}(1)$ since the second column of U is $\mathcal{O}((z-v)^{-1})$ as $z \to v \in \mathbf{v}$, while for the first column we notice that for each $v \in \mathbf{v}$, by (1.5.8),

$$\begin{aligned} \lim_{z\notin\Lambda\to v} \Psi^\theta_{|\mathbf{v}}(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \lim_{z\in\Lambda\to v} \Psi^\theta_{|\mathbf{v}\pm}(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = Y_\pm(v) \lim_{z\in\Lambda\to v} R(z)^{-1}\Psi_\pm(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= Y_\pm(v)\frac{2\pi i}{w(v)} \lim_{z\in\Lambda\to v} R(z)^{-1}\frac{w(z)}{2\pi i}\Psi_\pm(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= Y_\pm(v)\frac{2\pi i}{w(v)}\mathbf{f}_\mathbf{v}(v) = 0. \end{aligned}$$

Moreover, we have by (1.5.4) that the kernel of the conditional ensemble is given by

$$\begin{aligned} K^\theta_{|\mathbf{v}}(x,y) &= \frac{1}{x-y}\begin{pmatrix} 0 & 1 \end{pmatrix}\Psi_\pm(y)^{-1}U_\pm(y)^{-1}U_\pm(x)\Psi_\pm(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{w(x)}{2\pi i} \\ &= \frac{w(x)}{2\pi i(x-y)}\left(\Psi^\theta_{|\mathbf{v}}(y)^{-1}\Psi^\theta_{|\mathbf{v}}(x)\right)_{21}. \end{aligned} \tag{1.5.10}$$

Let us illustrate the above procedure in some examples.

Example 1.5.1. *Let p_k be the normalized degree k orthogonal polynomial with respect to a weight function w on $\Lambda = \mathbb{R}$, with leading coefficient $\kappa_k > 0$. Write*

$$\Psi(z) := \begin{pmatrix} \frac{1}{\kappa_N} p_N(z) & \frac{1}{2\pi i \kappa_N} \int_{-\infty}^{+\infty} \frac{p_N(s)w(s)ds}{s-z} \\ -2\pi i \kappa_{N-1} p_{N-1}(z) & -\kappa_{N-1} \int_{-\infty}^{+\infty} \frac{p_{N-1}(s)w(s)ds}{s-z} \end{pmatrix}. \tag{1.5.11}$$

This is the solution of the Fokas-Its-Kitaev RH problem [79], which is the above RH problem for Ψ with

$$\Lambda = \mathbb{R}, \qquad \Psi_\infty(z) = z^{N\sigma_3}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With $\mathbf{f}, \mathbf{g}$ as in (1.5.8), the kernel $K_N(x, y)$ is then the Christoffel-Darboux kernel (note the factor $w(x)$ which was not present in (1.4.3); this is due to the different reference measures $\mathrm{d}x$ here and $w(x)\mathrm{d}x$ in (1.4.3))

$$K_N(x, y) = \frac{\kappa_{N-1} w(x)}{\kappa_N} \frac{p_N(x)p_{N-1}(y) - p_N(y)p_{N-1}(x)}{x - y}.$$

The RH problem for $\Psi^\theta_{|\emptyset}$ is then the Fokas-Its-Kitaev RH problem, but with a deformed weight function $(1-\theta)w$; for non-empty $\mathbf{v}$, the regularised function $\Psi^\theta_{|\mathbf{v}}(z) \prod_{v\in\mathbf{v}} (z-v)^{-\sigma_3}$ then satisfies the Fokas-Its-Kitaev RH problem with weight function $(1-\theta(z))w(z) \prod_{v\in\mathbf{v}} (z-v)^2$ and with N replaced by N minus the cardinality of $\mathbf{v}$, which is in perfect agreement with Proposition 1.4.1. This RH problem has been an object of intensive study in the past decades and large N asymptotics for its solution have been obtained for a large class of weight functions, see e.g. [68, 109, 110].

Example 1.5.2. *Write*

$$\Psi(z) = \begin{cases} \begin{pmatrix} e^{\pi i z} & e^{\pi i z} \\ -e^{-\pi i z} & 0 \end{pmatrix}, & \text{for } \operatorname{Im} z > 0, \\ \begin{pmatrix} e^{\pi i z} & 0 \\ -e^{-\pi i z} & e^{-\pi i z} \end{pmatrix}, & \text{for } \operatorname{Im} z < 0. \end{cases}$$

This matrix satisfies the RH problem for Ψ with

$$\Lambda = \mathbb{R}, \qquad w(x) = 1, \qquad \Psi_\infty = \Psi.$$

With $\mathbf{f}, \mathbf{g}$ as in (1.5.8), the kernel $K(x, y)$ is then the sine kernel (1.4.5). The associated RH problem for $\Psi^\theta_{|\emptyset}$ for $\theta = 1_B$ the indicator function of a union of intervals was the RH problem studied originally in [70], and was also analysed succesfully in [36] for $\theta = \gamma 1_B$ with $\gamma \in (0, 1)$.

Example 1.5.3. *Write*

$$\Psi(z) := \sqrt{2\pi} e^{\frac{\pi i}{6}} \times \begin{cases} \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z) \\ -i\mathrm{Ai}'(z) & -i\omega^2 \mathrm{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \text{for } \operatorname{Im} z > 0, \\ \begin{pmatrix} \mathrm{Ai}(z) & -\omega^2 \mathrm{Ai}(\omega z) \\ -i\mathrm{Ai}'(z) & i\mathrm{Ai}'(\omega z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \text{for } \operatorname{Im} z < 0, \end{cases} \tag{1.5.12}$$

with $\omega = e^{\frac{2\pi i}{3}}$ *and* Ai *the Airy function. Using the relation* $\mathrm{Ai}(z) + \omega\mathrm{Ai}(\omega z) + \omega^2\mathrm{Ai}(\omega^2 z) = 0$*, one verifies, using the asymptotic behaviour of the Airy function, that this matrix satisfies the RH problem for* Ψ *with* $\Lambda = \mathbb{R}$*,* $w(x) = 1$ *and* $\Psi_\infty(z)$ *equal to*

$$\begin{cases} \frac{1}{\sqrt{2}} z^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-\frac{2}{3}z^{3/2}\sigma_3} & |\arg z| < \pi - \delta, \\ \frac{1}{\sqrt{2}} z^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-\frac{2}{3}z^{3/2}\sigma_3} \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} & |\arg z| < \pi - \delta,\ \pm \mathrm{Im}\, z > 0, \end{cases}$$

for any sufficiently small $\delta > 0$*, with principal branches of the root functions. With* $\mathbf{f}, \mathbf{g}$ *as in* (1.5.8)*, the kernel* $K(x,y)$ *is then the Airy kernel*

$$K(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y}.$$

The RH problem for $\Psi^\theta_{|\emptyset}$*, or an equivalent RH problem obtained after opening of the lenses, was then studied in [54, 47, 48] for a rather large class of functions* θ*, in order to derive differential equations and asymptotics for Airy kernel Fredholm determinants of the form* $\det(1 - \mathrm{M}_{\sqrt{\theta}}\mathrm{K}\mathrm{M}_{\sqrt{\theta}})$*. In particular, these determinants are important in the study of the narrow wedge solution of the Kardar-Parisi-Zhang equation and in the study of finite temperature free fermions, and they have a remarkably rich integrable structure: they are connected to the Korteweg-de Vries equation and to an integro-differential version of the second Painlevé equation. The asymptotics resulting from this RH analysis allow also to derive asymptotics for the conditional kernels* $K^\theta_{|\emptyset}$*. Moreover, the density of the pushed Coulomb gas from [64, 108] can be interpreted as an approximation of the one-point function* $K^\theta_{|\emptyset}(x,x)$*.*

The conclusion of this section is two-fold. First, we just showed that the IIKS RH problem allows one to characterize the conditional kernels $K^\theta_{|\emptyset}$ and $K^\theta_{|\mathbf{v}}$ in terms of a RH problem, which can potentially be analysed asymptotically. Secondly, the conditional ensembles $\mathbb{P}^\theta_{|\emptyset}$ enable us to give a natural probabilistic interpretation to the IIKS method, as we explain next.

The starting point of the IIKS method to study Fredholm determinants of the form $\det(1-\mathrm{M}_\theta\mathrm{K})$, is the Jacobi identity: if $\theta(x) = \theta_t(x)$ depends smoothly on a deformation parameter t, we have

$$\begin{aligned} \partial_t \log\det(1 - \mathrm{M}_{\sqrt{\theta_t}}\mathrm{K}\mathrm{M}_{\sqrt{\theta_t}}) &= -\operatorname{tr}\left[\partial_t \mathrm{M}_{\theta_t}\mathrm{K}(1 - \mathrm{M}_{\theta_t}\mathrm{K})^{-1}\right] \\ &= -\int_\Lambda \partial_t\theta_t(x) K^{\theta_t}_{|\emptyset}(x,x)\mathrm{d}\mu(x). \end{aligned}$$

In analytic terms, this implies that one can compute the Fredholm determinant $\det(1 - \mathrm{M}_{\theta_t}\mathrm{K})$, or at least its logarithmic derivative, provided that one has sufficiently accurate knowledge of the conditional kernel $K^{\theta_t}_{|\emptyset}(x,x)$.

In probabilistic terms, if $1 - \theta_t$ does not vanish, this identity reads

$$\partial_t \log \mathbb{E} \prod_{x \in \operatorname{supp}\xi} (1 - \theta_t(x)) = \mathbb{E}^{\theta_t}_{|\emptyset} \int_\Lambda \partial_t \log(1 - \theta_t(x))\mathrm{d}\xi(x). \tag{1.5.13}$$

The logarithmic derivative of an average multiplicative statistic in $\mathbb{P}$ is thus equal to an average linear statistic in the conditional ensemble $\mathbb{P}^{\theta}_{|\emptyset}$. Moreover, if the function $t \mapsto \theta_t(x)$ is a smooth probability distribution function, then the function

$$h^{\theta}_{x}(t) = -\partial_t \log(1 - \theta_t(x)) = \frac{\partial_t \theta_t(x)}{1 - \theta_t(x)} = \mathrm{Prob}\left(t_x = t \mid t_x \geq t\right), \qquad (1.5.14)$$

has the natural interpretation of a hazard rate likelihood of the random variable t_x with distribution $t \mapsto \theta_t(x)$. We can interpret t_x for instance as the detection time of point x, and then $h^{\theta}_{x}(t)$ is the likelihood to detect the particle at position $x \in \operatorname{supp} \xi$ at time t, given that it was not detected before time t.

Chapter 2

Jánossy Densities of the Airy Kernel Determinantal Point Process

This chapter retakes my third paper [60] in collaboration with Tom Claeys, Giulio Ruzza and Sofia Tarricone. My contribution lies in the foundations of the article: I identified that the Jánossy densities were the appropriate generalisation of the Fredholm determinant in this context, obtained prototypes of relations expressing them in terms of the residue at infinity of the Riemann-Hilbert problem considered at the end of the previous chapter as well as a trace-formula, established factorization identities allowing me to show that the conditioning of the previous chapter is always possible for the Airy point process and our collaborators to perform asymptotic analysis of various related quantities.

Abstract

We study Jánossy densities of a randomly thinned Airy kernel determinantal point process. We prove that they can be expressed in terms of solutions to the Stark and cylindrical Korteweg–de Vries equations; these solutions are Darboux transformations of the simpler ones related to the gap probability of the same thinned Airy point process. Moreover, we prove that the associated wave functions satisfy a variation of Amir–Corwin–Quastel's integro-differential Painlevé II equation. Finally, we derive tail asymptotics for the relevant solutions to the cylindrical Korteweg–de Vries equation and show that they decompose asymptotically into a superposition of simpler solutions.

2.1 Introduction

The cylindrical Korteweg–de Vries equation admits a family of solutions which are expressed in terms of Fredholm determinants involving the Airy kernel operator [128, 129, 48]. These solutions have an interesting probabilistic interpretation, as they are gap probabilities for random thinnings of the Airy point process [48], and they are therefore connected to important problems in integrable probability, such as the extreme value statistics of finite temperature free fermions [103, 120, 125, 117], the distribution of the narrow wedge solution of the Kardar–Parisi–Zhang equation [116, 130, 9], the edge eigenvalue statistics in the complex elliptic Ginibre Ensemble at weak non-Hermiticity [38], and multiplicative statistics of Hermitian random matrices [85].

In this work, we show that Darboux transformations of such solutions also enjoy a probabilistic interpretation: they are Jánossy densities of random thinnings of the Airy point process. We investigate the integrable structure of these solutions, and show how they are connected to the Stark equation and to an integro-differential Painlevé II equation. In this way, we reveal a remarkable connection between the Airy point process and scattering theory for solutions of the cylindrical Korteweg–de Vries equation. Moreover, we show that their tail asymptotics can be described as a superposition of simpler solutions. This soliton-like behaviour finds its origin in the fact that certain conditional ensembles related to the Airy point process decorrelate asymptotically.

To set the ground and give motivation for this work, we start by recalling results about the Fredholm determinant

$$j_\sigma(s) := 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \rho_n^{\mathrm{Ai}}(\lambda_1 + s, \dots, \lambda_n + s) \prod_{i=1}^{n} \sigma(\lambda_i) \mathrm{d}\lambda_i, \tag{2.1.1}$$

where $\sigma : \mathbb{R} \to [0,1]$ is a function satisfying Assumption A below, $s \in \mathbb{R}$, and

$$\rho_n^{\mathrm{Ai}}(\lambda_1, \dots, \lambda_n) := \det\left(K^{\mathrm{Ai}}(\lambda_i, \lambda_j)\right)_{1 \le i,j \le n} \tag{2.1.2}$$

with K^{Ai} the *Airy kernel*

$$K^{\mathrm{Ai}}(\lambda, \mu) := \int_0^{+\infty} \mathrm{Ai}(\lambda + \eta)\mathrm{Ai}(\mu + \eta)\, \mathrm{d}\eta = \frac{\mathrm{Ai}(\lambda)\mathrm{Ai}'(\mu) - \mathrm{Ai}'(\lambda)\mathrm{Ai}(\mu)}{\lambda - \mu}, \tag{2.1.3}$$

Ai and Ai$'$ being the Airy function and its derivative, respectively. We will consider functions σ satisfying the following properties.

Assumption A. *The function $\sigma : \mathbb{R} \to [0,1]$ is smooth and there exists $\kappa > 0$ such that $\sigma(\lambda) = O\big(|\lambda|^{-\frac{3}{2}-\kappa}\big)$ as $\lambda \to -\infty$.*

Remark 2.1.1. *For $j_\sigma(s)$ to be well defined, we need the integrability condition $\int_{\mathbb{R}} \sigma(\lambda) K^{\mathrm{Ai}}(\lambda, \lambda)\mathrm{d}\lambda < \infty$. The decay in Assumption A is slightly stronger than this requirement. This will allow us to control the $s \to +\infty$ behaviour of certain objects, see in particular Section 2.3.3.*

As we shall prove in Lemma 2.2.1, it follows from Assumption A that $0 < j_\sigma(s) \le 1$ for all $s \in \mathbb{R}$. Moreover, as proved in [2, 33, 35, 48], introducing

$$v_\sigma(s) := \partial_s^2 \log j_\sigma(s), \tag{2.1.4}$$

one has

$$v_\sigma(s) = -\int_{\mathbb{R}} \varphi_\sigma(\lambda;s)^2\sigma'(\lambda)\mathrm{d}\lambda, \qquad \sigma'(\lambda) := \frac{\mathrm{d}\sigma(\lambda)}{\mathrm{d}\lambda}, \tag{2.1.5}$$

where $\varphi_\sigma(\lambda;s)$ solves the *Stark equation* $\left(\partial_s^2 + 2v_\sigma(s) - s\right)\varphi(\lambda;s) = \lambda\,\varphi(\lambda;s)$[1]. More precisely, $\varphi_\sigma(\lambda;s)$ is the unique solution to the *Stark boundary value problem*

$$\big(\partial_s^2 + 2v_\sigma(s) - s\big)\varphi_\sigma(\lambda;s) = \lambda\,\varphi_\sigma(\lambda;s), \qquad \varphi_\sigma(\lambda;s) \sim \mathrm{Ai}(\lambda+s), \quad s\to+\infty. \tag{2.1.6}$$

(See Proposition 2.3.10 for the proof of the boundary values under our current assumptions on σ). In particular, it follows by combining (2.1.5) and (2.1.6) that φ_σ solves the *integro-differential Painlevé II equation* of Amir, Corwin, and Quastel [2]

$$\partial_s^2\varphi_\sigma(\lambda;s) = \left(\lambda + s + 2\int_{\mathbb{R}}\varphi_\sigma(\mu;s)^2\sigma'(\mu)\mathrm{d}\mu\right)\varphi_\sigma(\lambda;s). \tag{2.1.7}$$

It is worth observing that the relation (2.1.5) is the analogue, for potentials with linear background, of the classical *Trace Formula* obtained by Deift and Trubowitz [71, Equation $(1)_R$, page 183] in scattering theory for the Schrödinger equation for potentials with zero background.

A one-parameter family of isospectral deformations of (2.1.6) can be constructed as follows. Let $T>0$ and introduce

$$J_\sigma(X,T) := 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int_{\mathbb{R}^n}\rho^{\mathrm{Ai}}_{X,T;n}(\lambda_1,\dots,\lambda_n)\prod_{i=1}^{n}\sigma(\lambda_i)\mathrm{d}\lambda_i, \tag{2.1.8}$$

where

$$\rho^{\mathrm{Ai}}_{X,T;n}(\lambda_1,\dots,\lambda_n) := \det\left(K^{\mathrm{Ai}}_{X,T}(\lambda_i,\lambda_j)\right)_{1\le i,j\le n} \tag{2.1.9}$$

for a shifted and dilated Airy kernel

$$K^{\mathrm{Ai}}_{X,T}(\lambda,\mu) := T^{-\frac{1}{3}}K^{\mathrm{Ai}}\big(T^{-\frac{1}{3}}(\lambda+X), T^{-\frac{1}{3}}(\mu+X)\big). \tag{2.1.10}$$

It is readily checked that $J_\sigma(X,T) = j_{\widetilde{\sigma}}(XT^{-\frac{1}{3}})$, where $\widetilde{\sigma}(\lambda) := \sigma(T^{\frac{1}{3}}\lambda)$, implying that

$$V_\sigma(X,T) := \partial_X^2\log J_\sigma(X,T) = T^{-\frac{2}{3}}v_{\widetilde{\sigma}}(XT^{-\frac{1}{3}}) \tag{2.1.11}$$

satisfies

$$V_\sigma(X,T) = -T^{-\frac{1}{2}}\int_{\mathbb{R}}\widehat{\varphi}_\sigma(\lambda;X,T)^2\sigma'(\lambda)\mathrm{d}\lambda \tag{2.1.12}$$

in terms of the function $\widehat{\varphi}_\sigma(\lambda;X,T) := T^{-\frac{1}{12}}\varphi_{\widetilde{\sigma}}(\lambda T^{-\frac{1}{3}};XT^{-\frac{1}{3}})$. The latter is also equivalently characterized as the unique solution to the boundary value problem

$$\begin{cases}\mathscr{L}\,\widehat{\varphi}_\sigma(\lambda;X,T) = \lambda\,\widehat{\varphi}_\sigma(\lambda;X,T)\,,\\ \widehat{\varphi}_\sigma(\lambda;X,T) \sim T^{-\frac{1}{12}}\,\mathrm{Ai}\big(T^{-\frac{1}{3}}(\lambda+X)\big)\,, \quad X\to+\infty,\end{cases} \tag{2.1.13}$$

[1] The Stark equation is nothing else than the Schrödinger equation $\left(\partial_s^2 + 2u(s)\right)\varphi(\lambda;s) = \lambda\varphi(\lambda;s)$, for a potential $u(s) = v_\sigma(s) - s/2$ with linear background $-s/2$.

where $\mathscr{L} := T\,\partial_X^2 + 2\,T\,V_\sigma(X,T) - X$. Moreover, it is proved in [48, Theorem 1.3] that $V = V_\sigma(X,T)$ solves the *cylindrical Korteweg–de Vries* (cKdV) equation

$$\partial_T V + \frac{1}{12}\,\partial_X^3 V + V\,\partial_X V + \frac{1}{2T}\,V \;=\; 0. \tag{2.1.14}$$

More precisely, the variables $x \in \mathbb{R}, t > 0$ of loc. cit. are related to $X \in \mathbb{R}, T > 0$ of this paper as $x = -XT^{-\frac{1}{2}}$, $t = T^{-\frac{1}{2}}$, such that the KdV equation for $u = u(x,t)$

$$\partial_t u + \frac{1}{6}\,\partial_x^3 u + 2\,u\,\partial_x u \;=\; 0 \tag{2.1.15}$$

[48, equation (1.7)] is equivalent to the cKdV equation (2.1.14) for the function

$$V(X,T) := T^{-1}\,u\big(x = -XT^{-\frac{1}{2}}, t = T^{-\frac{1}{2}}\big) + \frac{1}{2}XT^{-1}. \tag{2.1.16}$$

In the language of integrable PDEs, this implies that the Fredholm determinant $J_\sigma(X,T)$ is a *tau function* of the cKdV equation. In particular, $J = J_\sigma(X,T)$ solves the *bilinear form* of the cKdV equation

$$\partial_X J\,\partial_T J - J\,\partial_X\partial_T J - \frac{1}{4}\left(\partial_X^2 J\right)^2 + \frac{1}{3}\,\partial_X J\,\partial_X^3 J - \frac{1}{12}\,J\,\partial_X^4 J - \frac{1}{2T}\,J\,\partial_X J \;=\; 0. \tag{2.1.17}$$

The direct and inverse scattering transform for the cKdV equation has been established in [132, 133, 100, 101] for smooth and decaying initial data.

Example 2.1.1. *The simplest situation occurs when $\sigma = 0$. Then,*

$$J_0(X,T) = 1, \qquad V_0(X,T) = 0, \qquad \varphi_0(\lambda; X,T) = \mathrm{Ai}\big(T^{-\frac{1}{3}}(\lambda + X)\big). \tag{2.1.18}$$

Example 2.1.2. *The function $\sigma = 1_{(0,+\infty)}$ does not satisfy Assumption A, but it is nevertheless an instructive degenerate situation. (The present setting could be extended to include such case, cf. Remark 2.1.4 and Section 2.3.8.) The integro-differential Painlevé II equation reduces to the Painlevé II equation, and (2.1.5) is the celebrated Tracy–Widom formula. The cKdV tau function and solution are given by*

$$J_{1_{(0,+\infty)}}(X,T) = F_{\mathrm{TW}}(XT^{-\frac{1}{3}}), \qquad V_{1_{(0,+\infty)}}(X,T) = -T^{-\frac{2}{3}}y_{\mathrm{HM}}(XT^{-\frac{1}{3}})^2, \tag{2.1.19}$$

where F_{TW} is the Tracy–Widom distribution [137] and y_{HM} is the Hastings–McLeod solution to Painlevé II [92]. The asymptotics of the Hastings–McLeod solution imply that (see Figure 2.1)

$$V_{1_{(0,+\infty)}}(X,T) \sim \begin{cases} \frac{1}{2}XT^{-1}, & XT^{-\frac{1}{3}} \to -\infty, \\ -T^{-\frac{2}{3}}\mathrm{Ai}\big(XT^{-\frac{1}{3}}\big), & XT^{-\frac{1}{3}} \to +\infty. \end{cases} \tag{2.1.20}$$

Remark 2.1.2. *The KdV and cKdV equations, (2.1.15) and (2.1.14), respectively, are completely equivalent from an algebraic point of view, since the transformation (2.1.16) defines a one-to-one correspondence of solutions. On the other hand, this correspondence drastically changes the analytic properties of*

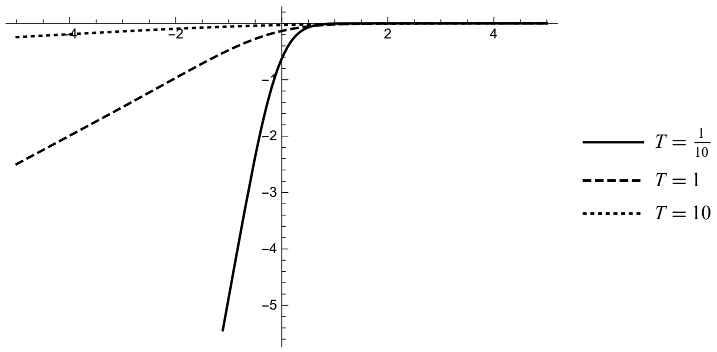

Figure 2.1: The solution $V_{1_{(0,+\infty)}}(X,T)$ as a function of X for some values of T.

solutions; e.g., if V is bounded then u is not, and vice versa. In view of the analytic properties of the solutions under consideration, we find it more natural to work with the cKdV equation; moreover, the relevant Riemann–Hilbert problem in our analysis formally matches with the one for the inverse scattering theory of the cKdV of Its and Sukhanov [100, 101], cf. Section 2.3.5.

To explain the probabilistic meaning of these cKdV solutions, let the shifted and dilated Airy point process be the determinantal point process [135] on the real line induced by the correlation kernel $K^{\mathrm{Ai}}_{X,T}$ given in (2.1.10). Equivalently, it is the probability distribution (parametrically depending on $X \in \mathbb{R}, T > 0$) on the space of locally finite configurations of points on the real line characterized by its m-point correlation functions $\rho^{\mathrm{Ai}}_{X,T;m}$ defined in (2.1.9). More explicitly, this means that for all disjoint Borel sets $B_1, \ldots, B_\ell \subseteq \mathbb{R}$ and for all integers $k_1, \ldots, k_\ell \geq 1$ summing up to $\sum_{j=1}^{\ell} k_j = m$, the expected number of m-tuples of points in a random configuration of which k_1 lie in B_1, k_2 in B_2, ..., is

$$\frac{1}{k_1! \cdots k_\ell!} \int_{B_1^{k_1} \times \cdots \times B_l^{k_\ell}} \rho^{\mathrm{Ai}}_{X,T;m}(\lambda_1, \ldots, \lambda_m)\, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_m. \tag{2.1.21}$$

Then, the relation to the cKdV tau functions $J_\sigma(X,T)$ is expressed by the identity

$$J_\sigma(X,T) = \mathbb{E}\left[\prod_{j \geq 1} \big(1 - \sigma(\lambda_j)\big)\right] \tag{2.1.22}$$

where the expectation on the right involves the particles $\lambda_1 > \lambda_2 > \cdots$ of the shifted and dilated Airy process. (It is well-known that the Airy process has almost surely infinitely many particles and a largest particle, see Remark 2.2.2.) The identity (2.1.22) follows from the general theory of determinantal point processes, cf. [27, eq. (11.2.4)].

The σ-thinned shifted and dilated Airy point process is obtained from the shifted and dilated Airy point process by removing each particle in a configuration independently with (position-dependent) probability $1 - \sigma$, thus retaining it with probability σ. This point process is also determinantal, with a correlation kernel given by $\sqrt{\sigma(\lambda)}K^{\mathrm{Ai}}_{X,T}(\lambda,\mu)\sqrt{\sigma(\mu)}$, cf. [114, eq (2.5)]; it follows that $J_\sigma(X,T)$ is the *gap probability* of the σ-thinned process, i.e., the probability that a random point configuration in this process is empty. Moreover, if $\sigma:\mathbb{R}\to[0,1]$ decays sufficiently fast at $-\infty$ (for instance, if it satisfies Assumption A) the σ-thinned shifted and dilated Airy point process has almost surely a finite number of particles (Remark 2.2.2). In such a case, we can define the global *Jánossy density* of order $m\geq 0$, denoted $J_\sigma(X,T|\nu_1,\dots,\nu_m)$. These quantities are characterized by the property that for all disjoint Borel subsets $B_1,\dots,B_\ell\subseteq\mathbb{R}$ such that $\sqcup_{j=1}^{\ell}B_j=\mathbb{R}$ and all integers $k_1,\dots,k_\ell\geq 1$ summing up to $\sum_{j=1}^{\ell}k_j=m$, the probability that a configuration in the σ-thinned shifted and dilated Airy point process contains exactly m particles, of which k_1 lie in B_1, k_2 in B_2, …, is

$$\frac{1}{k_1!\cdots k_\ell!}\int_{B_1^{k_1}\times\cdots\times B_l^{k_\ell}} J_\sigma(X,T|\nu_1,\dots,\nu_m)\prod_{j=1}^{m}\sigma(\nu_j)\mathrm{d}\nu_j. \tag{2.1.23}$$

It is known [135, eq. (1.38)] that Jánossy densities can be expressed as

$$\begin{aligned}&J_\sigma(X,T|\nu_1,\dots,\nu_m)=\\&\qquad\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{\mathbb{R}^n}\rho^{\mathrm{Ai}}_{X,T;n+m}(\lambda_1,\dots,\lambda_n,\nu_1,\dots,\nu_m)\prod_{i=1}^{n}\sigma(\lambda_i)\mathrm{d}\lambda_i.\end{aligned} \tag{2.1.24}$$

For brevity, we will collect the *distinct* real numbers ν_i into a vector (or set) $\underline{\nu}:=(\nu_1,\dots,\nu_m)$ and denote $J_\sigma(X,T|\underline{\nu}):=J_\sigma(X,T|\nu_1,\dots,\nu_m)$. To be consistent with the notation previously introduced, we have $J_\sigma(X,T)=J_\sigma(X,T|\emptyset)$.

As in the case $m=0$, it is interesting to study the situation when T is kept constant first. This is sufficient to disclose the relation to Stark boundary value problems and to a generalised integro-differential Painlevé II equation. In such case, T can be set to 1 without loss of generality because

$$J_\sigma(X,T|\underline{\nu})=T^{-\frac{m}{3}}J_{\widetilde{\sigma}}(XT^{-\frac{1}{3}},1|T^{-\frac{1}{3}}\underline{\nu}),\qquad \widetilde{\sigma}(\lambda):=\sigma(T^{\frac{1}{3}}\lambda). \tag{2.1.25}$$

Accordingly, we will formulate our results in which T is constant more concisely in terms of

$$j_\sigma(s|\underline{\nu}):=J_\sigma(s,1|\underline{\nu}),\qquad s\in\mathbb{R}. \tag{2.1.26}$$

Our first result on the integrable structure of the Jánossy densities is their expression in terms solely of the eigenfunctions of the Stark operator.

Theorem I. *Let σ satisfy Assumption A and let $\varphi_\sigma(\lambda;s)$ be the unique solution to the Stark boundary value problem (2.1.6). For all $s\in\mathbb{R}$ and all $\underline{\nu}=(\nu_1,\dots,\nu_m)$ with $\nu_i\neq\nu_j$ for all $i\neq j$, we have*

$$j_\sigma(s|\underline{\nu})=\det\left(L_s^\sigma(\nu_i,\nu_j)\right)_{i,j=1}^{m}j_\sigma(s|\emptyset), \tag{2.1.27}$$

where

$$\begin{aligned}L_s^\sigma(\lambda,\mu)&=\int_s^{+\infty}\varphi_\sigma(\lambda;r)\varphi_\sigma(\mu;r)\,\mathrm{d}r\\&=\frac{\varphi_\sigma(\lambda;s)\partial_s\varphi_\sigma(\mu;s)-\partial_s\varphi_\sigma(\lambda;s)\varphi_\sigma(\mu;s)}{\lambda-\mu},\end{aligned} \tag{2.1.28}$$

and

$$j_\sigma(s|\emptyset) = \exp\Big(-\int_s^{+\infty}(r-s)\Big(\int_{\mathbb{R}}\varphi_\sigma(\lambda;r)^2\sigma'(\lambda)\mathrm{d}\lambda\Big)\mathrm{d}r\Big). \tag{2.1.29}$$

The proof is given in Section 2.3.4.

Remark 2.1.3. *The kernel $L_s^\sigma(\cdot,\cdot)$ induces a determinantal point process (depending parametrically on $s \in \mathbb{R}$) which is obtained via a conditioning of the shifted Airy point process, in the following sense: assign independently to each point λ in a random configuration mark 1 with probability $\sigma(\lambda)$ and mark 0 otherwise, then condition the resulting marked shifted Airy point process on the event that no points have mark 1. The conditional point process obtained in this manner is determinantal, and has correlation kernel $L_s^\sigma(\cdot,\cdot)$, see Section 2.2.2 and [58] and [41, 42, 40, 85] for details. The factorization (2.1.27) receives an interesting probabilistic interpretation: it is the product of an m-point correlation function in this conditional determinantal point process and of the gap probability of the σ-thinned shifted Airy point process.*

Example 2.1.3. *It is instructive to consider again the case $\sigma = 0$, in which case $\varphi_0(\lambda; s) = \mathrm{Ai}(\lambda + s)$, and so (2.1.28) reduces to the shifted Airy kernel $L_s^0(\lambda,\mu;s) = K^{\mathrm{Ai}}(\lambda+s,\mu+s)$, cf. (2.1.3). In this sense we can regard φ_σ as a generalisation of the Airy function, and L_s^σ as a generalisation of the shifted Airy kernel; it is interesting to check that several properties of the Airy function and the shifted Airy kernel are preserved by this generalisation. See for instance Proposition 2.3.10 and Corollary 2.3.11.*

Example 2.1.4. *The choice $\sigma = 1_{(0,+\infty)}$ is again not admissible in view of Assumptions A, but with $v_\sigma(s) = \varphi(0;s)^2 = -y_{\mathrm{HM}}(s)^2$, the degenerate case of (2.1.5), the Stark boundary value problem (2.1.6) still makes sense, and the integral kernel $L_s^{1_{(0,+\infty)}}(\lambda,\mu)$ is defined. Moreover, this kernel has appeared in the soft-to-hard edge transition in random matrix theory [63, Theorem 1.3]. In terms of the notations used in [63, Theorem 1.3], we have*

$$\begin{aligned}\varphi_{1_{(0,+\infty)}}(\lambda;s) &= -\frac{1}{\sqrt{2\pi}}f_0(-\lambda;s),\\ \partial_s\varphi_{1_{(0,+\infty)}}(\lambda;s) &= -\frac{1}{\sqrt{2\pi}}\big(g_0(-\lambda;s)+p_0(s)f_0(-\lambda;s)\big),\end{aligned}$$

and

$$L_s^{1_{(0,+\infty)}}(\lambda;\mu) = \mathbb{K}_0^{\mathrm{soft/hard}}(-\lambda,-\mu;s). \tag{2.1.30}$$

Then we can identify the Stark boundary problem (2.1.6) with [63, Equations (1.12)–(1.15)]. Furthermore, $L_s^{1_{(0,+\infty)}}$ is the kernel of the determinantal point process obtained by conditioning the shifted Airy point process on absence of particles on $(0,\infty)$. Recall also that $j_{1_{(0,+\infty)}}(s|\emptyset) = j_{1_{(0,+\infty)}}(s) = F_{\mathrm{TW}}(s)$ is the Tracy–Widom distribution in this case.

Next we give a second expression for the Jánossy densities which is more directly parallel to equations (2.1.5), (2.1.6), and (2.1.7) for the case $m = 0$. This expression involves eigenfunctions of the Stark operator with a modified potential.

Theorem II. *Let σ satisfy Assumption A. For all $s \in \mathbb{R}$ and all $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for all $i \neq j$, we have*

$$\partial_s^2 \log j_\sigma(s|\underline{\nu}) = \int_{\mathbb{R}} \varphi_\sigma(\lambda; s|\underline{\nu})^2 \left(-\sigma'(\lambda) + \sum_{i=1}^m \frac{2\big(1-\sigma(\lambda)\big)}{\lambda - \nu_i} \right) \mathrm{d}\lambda \tag{2.1.31}$$

where $\varphi_\sigma(\lambda; s|\underline{\nu})$ solves the Stark equation

$$\big(\partial_s^2 + 2v_\sigma(s|\underline{\nu}) - s\big)\varphi_\sigma(\lambda; s|\underline{\nu}) = \lambda \varphi_\sigma(\lambda; s|\underline{\nu}) \tag{2.1.32}$$

with potential

$$v_\sigma(s|\underline{\nu}) := \partial_s^2 \log j_\sigma(s|\underline{\nu}). \tag{2.1.33}$$

Moreover, $\varphi_\sigma(\lambda; s|\underline{\nu})$ can be expressed in terms of $\varphi_\sigma(\lambda; s|\emptyset)$ and $\partial_s \varphi_\sigma(\lambda; s|\emptyset)$ as

$$\varphi_\sigma(\lambda; s|\underline{\nu}) = \frac{\det \begin{pmatrix} \varphi_\sigma(\lambda; s|\emptyset) & L_s^\sigma(\lambda, \nu_1) & \cdots & L_s^\sigma(\lambda, \nu_m) \\ \varphi_\sigma(\nu_1, s|\emptyset) & L_s^\sigma(\nu_1, \nu_1) & \cdots & L_s^\sigma(\nu_1, \nu_m) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_\sigma(\nu_m, s|\emptyset) & L_s^\sigma(\nu_m, \nu_1) & \cdots & L_s^\sigma(\nu_m, \nu_m) \end{pmatrix}}{\det \begin{pmatrix} L_s^\sigma(\nu_1, \nu_1) & \cdots & L_s^\sigma(\nu_1, \nu_m) \\ \vdots & \ddots & \vdots \\ L_s^\sigma(\nu_m, \nu_1) & \cdots & L_s^\sigma(\nu_m, \nu_m) \end{pmatrix}}, \tag{2.1.34}$$

so that, in particular, $\varphi_\sigma(\lambda, s|\underline{\nu})$ has zeros at $\lambda = \nu_1, \dots, \nu_m$.

The proof is given in Section 2.3.4. In the case $m = 0$, the eigenfunctions $\varphi_\sigma(\lambda; s|\emptyset)$ of this theorem reduce to what we denoted just $\varphi_\sigma(\lambda; s)$ up to this point; for the sake of clarity, we will from now on use the notation $\varphi_\sigma(\lambda; s|\emptyset)$. Similarly said for the notation $v_\sigma(s) = v_\sigma(s|\emptyset)$.

In other words, Theorem II states that the Stark equation with potential $\partial_s^2 \log j_\sigma(s|\underline{\nu})$ is obtained by *Darboux transformations* (in the original spirit of Darboux [67]) of the Stark equation with potential $\partial_s^2 \log j_\sigma(s|\emptyset)$.

Moreover, by the asymptotics $\varphi_\sigma(\lambda; s|\emptyset) \sim \mathrm{Ai}(\lambda + s) = \varphi_0(\lambda; s|\emptyset)$ as $s \to +\infty$, one obtains that the appropriate boundary condition for the solution to the Stark equation (2.1.32) is $\varphi_\sigma(\lambda; s|\underline{\nu}) \sim \varphi_0(\lambda; s|\underline{\nu})$ as $s \to +\infty$. The function $\varphi_0(\lambda; s|\underline{\nu})$ is explicit in terms of the Ai and Ai$'$ functions, by (2.1.34) and $L_s^0 = K_s^{\mathrm{Ai}}$.

It follows by combining (2.1.31), (2.1.32), and (2.1.33), that $\varphi_\sigma(\lambda; s|\underline{\nu})$ satisfies a deformation of the integro-differential Painlevé II equation:

$$\partial_s^2 \varphi_\sigma(\lambda; s|\underline{\nu}) = \left(\lambda + s + 2\int_{\mathbb{R}} \varphi_\sigma(\mu; s|\underline{\nu})^2 \left(\sigma'(\mu) - \sum_{i=1}^m \frac{2\big(1-\sigma(\mu)\big)}{\mu - \nu_i} \right) \mathrm{d}\mu \right) \varphi_\sigma(\lambda; s|\underline{\nu}). \tag{2.1.35}$$

A general framework for studying integro-differential equations related to a class of Fredholm determinants was developed in [107].

Remark 2.1.4. *Theorems I and II hold true more generally for all σ satisfying the decay condition of Assumption A even if they are only piecewise smooth with a finite number of jump singularities. In this case, one has to add to $\sigma'(\lambda)\,\mathrm{d}\lambda$ a discrete measure supported at the singularities of σ. The extension to this case can be done following [48], as we will discuss in Section 2.3.8.*

Remark 2.1.5. *Jánossy densities generally carry important information about a point process, but the global Jánossy densities of a determinantal point process are only defined if the associated kernel defines a trace class operator. This is not the case for the Airy kernel operator. To remedy this, one commonly considers Jánossy densities of determinantal point processes restricted to bounded sets B [29]. It is less customary to consider Jánossy densities of thinned determinantal point processes, as we do, but this has the advantage that, for a suitable class of thinning functions σ, global Jánossy densities exist. In the degenerate case $\sigma = 1_B$, we recover the Jánossy density of the determinantal point process restricted to the set B.*

The description of Jánossy densities of determinantal point processes on $\mathbb{R}$, restricted to bounded intervals B, in terms of solutions to certain differential equations has been recently developed in [127]. The author there proved that for determinantal point processes defined through kernels satisfying the Tracy–Widom criteria [138], the Tracy–Widom method allows to express not only the gap probability (as proved in [138]) but also the Jánossy densities of the process restricted to a bounded interval B, in terms of solutions to a system of differential equations in the endpoints of B. For kernels also enjoying the integrable structure of Its–Izergin–Korepin–Slavnov [98] (e.g., Airy, Bessel, sine kernels), the gap probability can be characterized by a Riemann–Hilbert (RH) problem. This provides an alternative approach to study underlying integrable differential equations, and a powerful tool to tackle their asymptotics. In this work, we extend the RH approach to study Jánossy densities of the thinned shifted Airy point process. We are confident that our method can also be applied to other determinantal point processes with integrable structure, like the ones associated to Bessel and sine kernels.

To construct a family of solutions to the cKdV equation in terms of the Jánossy densities, we restore the full dependence on X, T and we introduce

$$V_\sigma(X,T|\underline{\nu}) := \partial_X^2 \log J_\sigma(X,T|\underline{\nu}), \qquad X \in \mathbb{R},\ T > 0. \tag{2.1.36}$$

Theorem III. *For all σ satisfying Assumption A and all $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for all $i \neq j$, the function $V = V_\sigma(X,T|\underline{\nu})$ solves the cKdV equation,*

$$\partial_T V + \frac{1}{12}\partial_X^3 V + V\,\partial_X V + \frac{1}{2T} V = 0\,, \quad \textit{for all } X \in \mathbb{R}, T > 0. \tag{2.1.37}$$

The proof is in Section 2.3.7, see Corollary 2.3.13.

Example 2.1.5. *When $\sigma = 0$ we have $j_0(s|\emptyset) = 1$ and $L_s^0(\lambda,\mu) = K^{\mathrm{Ai}}(\lambda + s, \mu + s)$, such that, according to (2.1.27),*

$$j_0(s|\underline{\nu}) = \rho_m^{\mathrm{Ai}}(\nu_1 + s, \dots, \nu_m + s) \tag{2.1.38}$$

is the m-point correlation function in the shifted Airy ensemble (2.1.2)*. The corresponding cKdV tau function is*

$$J_0(X,T|\underline{\nu}) = \det\big(K^{\mathrm{Ai}}_{X,T}(\nu_i,\nu_j)\big)_{i,j=1}^m. \tag{2.1.39}$$

The associated cKdV solution $V_0(X,T|\underline{\nu}) = \partial_X^2 \log\det\big(K^{\mathrm{Ai}}_{X,T}(\nu_i,\nu_j)\big)_{i,j=1}^m$ is a special case of soliton-type solution [126], cf. Figure 2.2, exhibiting right tail decay, and left tail rapid oscillations with decaying amplitude. It is straightforward to verify, using the asymptotics for the Airy function and its derivative, that for any $T_0 > 0$,

$$V_0(X,T|\underline{\nu}) \sim -\frac{m}{\sqrt{XT}}, \qquad \text{as } XT^{-\frac{1}{3}} \to +\infty, \text{ uniformly for } T \geq T_0. \tag{2.1.40}$$

Indeed, as $XT^{-\frac{1}{3}} \to +\infty$, $T \geq T_0$

$$K^{\mathrm{Ai}}_{X,T}(v,w) \sim \frac{1}{8\pi X}\mathrm{e}^{-\frac{2}{3}T^{-\frac{1}{2}}(X+v)^{\frac{3}{2}}}\mathrm{e}^{-\frac{2}{3}T^{-\frac{1}{2}}(X+w)^{\frac{3}{2}}} \tag{2.1.41}$$

and this allows to prove by induction on m that

$$\det\big(K^{\mathrm{Ai}}_{X,T}(\nu_i,\nu_j)\big)_{i,j=1}^m \sim \frac{1}{(8\pi X)^m}\mathrm{e}^{-\frac{4m}{3}T^{-\frac{1}{2}}X^{\frac{3}{2}}}. \tag{2.1.42}$$

Since these asymptotics are moreover valid uniformly for complex numbers X with $|\arg X| < \delta$ and $\delta > 0$ small, we can differentiate the logarithm of this expression twice which yields (2.1.40)*.*

Similarly, after straightforward computations involving the asymptotic behaviour of the Airy function and its derivative, we obtain for the left tail if $m = 1$, as $XT^{-\frac{1}{3}} \to -\infty$, $T \geq T_0$,

$$J_0(X,T|\nu) = \frac{1}{\pi}\sqrt{\frac{|X|-\nu}{T}}\times \left(1 - \frac{\sqrt{T}}{4}(|X|-\nu)^{-\frac{3}{2}}\cos\left[\frac{4}{3\sqrt{T}}(|X|-\nu)^{\frac{3}{2}}\right] + O\left(X^{-3}T\right)\right),$$

$$V_0(X,T|\nu) = \frac{1}{\sqrt{T|X|}}\cos\left[\frac{4}{3\sqrt{T}}(|X|-\nu)^{\frac{3}{2}}\right] + O(|X|^{-\frac{3}{2}}T^{-\frac{1}{2}}).$$

Remark 2.1.6. *The function σ and the parameters $\underline{\nu}$ can be understood as scattering data for the cKdV solution under consideration. In analogy with [90, 91], we could interpret σ as a function describing a gas of solitons.*

Our final result concerns the asymptotic behaviour of the Jánossy densities and associated cKdV solutions when $T \to +\infty$, uniformly in $X \in \mathbb{R}$. To this end, we formulate some stronger conditions on the function σ, cf. [48, 56].

Assumption B. *The function $F := \frac{1}{1-\sigma}$ extends to an entire function. Moreover:*

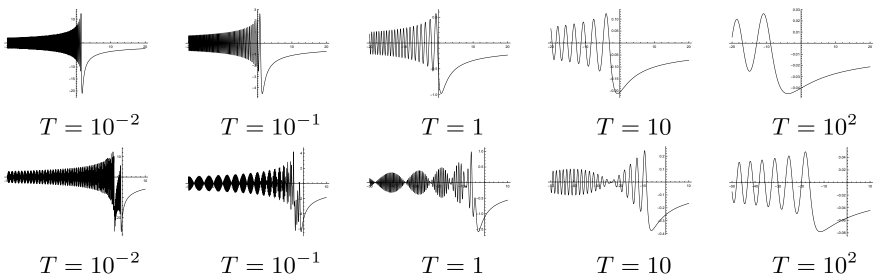

Figure 2.2: First line: 1-soliton cKdV solution $V_0(X,T|\underline{\nu})$ with $\underline{\nu}=(-2)$ as a function of X for various values of T. Second line: 2-soliton cKdV solution $V_0(X,T|\underline{\nu})$ with $\underline{\nu}=(0,3)$ as a function of X for various values of T.

- *$F'\geq 0$ and $(\log F)''\geq 0$ on the real line;*
- *$F(\lambda)=1+c'_-\mathrm{e}^{-c_-|\lambda|}(1+o(1))$ as $\lambda\to-\infty$, $F(\lambda)=c'_+\mathrm{e}^{c_+\lambda}(1+O(\mathrm{e}^{-\epsilon\lambda}))$ as $\lambda\to+\infty$, for some $c_\pm, c'_\pm,\epsilon>0$;*
- *$F(\lambda)=O(\mathrm{e}^{c_+\operatorname{Re}\lambda})$ as $\operatorname{Re}\lambda\to+\infty$.*

In particular, the second assumption implies the strong decay $\sigma(\lambda)=c'_-\mathrm{e}^{-c_-|\lambda|}\times\big(1+o(1)\big)$ as $\lambda\to-\infty$ and $\sigma(\lambda)=1-\frac{1}{c'_+}\mathrm{e}^{-c_+\lambda}\big(1+O(\mathrm{e}^{-\epsilon\lambda})\big)$ as $\lambda\to+\infty$ (for the same constants $c_\pm,c'_\pm,\epsilon>0$).

The reader may want to keep in mind the prototype example of an admissible function σ given by $\sigma(\lambda)=\frac{1}{1+\mathrm{e}^{-\lambda}}$, such that $F(\lambda)=1+\mathrm{e}^{\lambda}$.

Theorem IV. *Let $\underline{\nu}=(\nu_1,\dots,\nu_m)$ with $\nu_i\neq\nu_j$ for all $i\neq j$.*
(i) *Let σ satisfy Assumption A. For any $T_0>0$, there exists $c>0$ such that*

$$J_\sigma(X,T|\underline{\nu})=\det\big(K^{\mathrm{Ai}}_{X,T}(\nu_i,\nu_j)\big)_{i,j=1}^m\left(1+O\big(\mathrm{e}^{-cXT^{-\frac{1}{3}}}\big)\right) \tag{2.1.43}$$

$$\sim(8\pi X)^{-m}\mathrm{e}^{-\frac{4m}{3}X^{\frac{3}{2}}T^{-\frac{1}{2}}}, \tag{2.1.44}$$

$$V_\sigma(X,T|\underline{\nu})=V_0(X,T|\underline{\nu})+O\big(\mathrm{e}^{-cXT^{-\frac{1}{3}}}\big)\sim-\frac{m}{\sqrt{XT}}, \tag{2.1.45}$$

uniformly in $T\geq T_0$ as $XT^{-\frac{1}{3}}\to+\infty$.
(ii) *Let σ satisfy Assumption B. For any $T_0>0$, we have*

$$J_\sigma(X,T|\underline{\nu})\sim\frac{|X|^{\frac{m}{2}}}{\pi^mT^{\frac{m}{2}}}J_\sigma(X,T|\emptyset)\prod_{j=1}^m\frac{1}{1-\sigma(\nu_j)}, \tag{2.1.46}$$

$$\begin{aligned}V_\sigma(X,T|\underline{\nu})&=V_\sigma(X,T|\emptyset)\\&\quad+\frac{1}{\sqrt{|X|T}}\sum_{j=1}^m\cos\!\left(\frac{4|X|^{\frac{3}{2}}}{3T^{\frac{1}{2}}}(1+A_{X,T})-\frac{2|X|^{\frac{1}{2}}}{T^{\frac{1}{2}}}\nu_j(1+B_{X,T}(\nu_j))\right)\\&\quad+O(|X|^{-1}),\end{aligned} \tag{2.1.47}$$

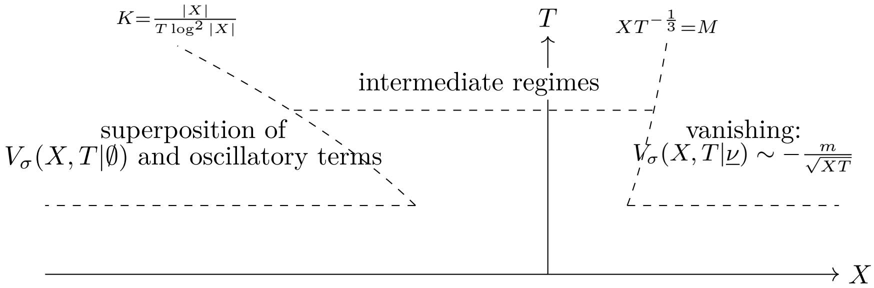

Figure 2.3: Phase diagram showing the different tail asymptotics for $V_\sigma(X,T|\underline{\nu})$, uniform in the indicated regions for fixed $M, K > 0$.

uniformly for $T \geq T_0$ as $\frac{X}{T\log^2|X|} \to -\infty$, where $A_{X,T}, B_{X,T}(\nu)$ converge to 0 as $\frac{X}{T\log^2|X|} \to -\infty$. Moreover, in the same limit,

$$\log J_\sigma(X,T|\emptyset) = \rho^3 T^2 \left(-\frac{4}{15}(1-\xi)^{\frac{5}{2}} + \frac{4}{15} - \frac{2}{3}\xi + \frac{1}{2}\xi^2\right) + O\left(|X|^{\frac{3}{2}} T^{-\frac{1}{2}}\right),$$
$$V_\sigma(X,T|\emptyset) = \rho\left(1 - \sqrt{1-\xi}\right) + O\left(|X|^{-\frac{1}{2}} T^{-\frac{1}{2}}\right),$$

where $\rho := c_+^2/\pi^2$ and $\xi := X/(\rho T)$.

Observe the specific structure of the ν_j-dependence in the cKdV solution $V_\sigma(X,T|\underline{\nu})$ and the Jánossy density $J_\sigma(X,T|\underline{\nu})$. For $XT^{-\frac{1}{3}} \to +\infty$, the leading order behaviours of $V_\sigma(X,T|\underline{\nu})$ and $J_\sigma(X,T|\underline{\nu})$ depend on the number of points m but not explicitly on the positions $\nu_1, \ldots, \nu_m$. On the other hand, for $\frac{X}{T\log^2|X|} \to -\infty$, the effect of $\underline{\nu}$ is more prominent. For $J_\sigma(X,T|\underline{\nu})$, it results in a product of factors depending on ν_j, while for $V_\sigma(X,T|\underline{\nu})$, the presence of $\nu_1, \ldots, \nu_m$ results in a superposition of rapidly oscillating terms depending on ν_j.

It is remarkable that both in the left and right tail asymptotics of $V_\sigma(X,T|\underline{\nu})$, we recognize (up to a sub-leading phase shift in the oscillatory terms) a superposition of m 1-soliton solutions whose tail asymptotics are described in Example 2.1.5, in addition to the leading order and $\underline{\nu}$-independent contribution coming from $V_\sigma(X,T|\emptyset)$.

For $-KT\log^2|X| \leq X \leq MT^{\frac{1}{3}}$, the ν_j-dependence is more involved and less explicit, as we will explain in Section 2.4.4.

Methodology and outline

In Section 2.2, we will gather several properties and identities for the Jánossy densities on which we will rely later, and we will give a probabilistic interpretation to the kernel L_s^σ. Two different factorizations of the Jánossy densities will be of particular importance.

In Section 2.3, we will characterize the Jánossy densities and other relevant quantities in terms of a 2×2 matrix-valued Riemann–Hilbert (RH) problem, by relying on the Its–Izergin–Korepin–Slavnov method [98]. This RH characterization shows strong similarities with the one from [21] and we give a detailed comparison with the general methods of op. cit. in Section 2.3.6. Moreover, we establish a connection between the RH problem, the Stark boundary value problem (2.1.6), and the cKdV equation (2.1.14). This will enable us to prove Theorem I, Theorem II, and Theorem III.

Section 2.4 will be devoted to the asymptotic analysis of the RH problem from Section 2.3. We will distinguish several regions in the (X, T)-plane which will require a different type of asymptotic analysis, and which lead to the results presented in Theorem IV. In this section, we also use previous asymptotic results [48, 56] for the case $m = 0$ in which Jánossy densities reduce to gap probabilities.

2.2 Preliminaries on Jánossy densities

In this section, we study in more detail the Jánossy densities $j_\sigma(s|\underline{\nu})$ introduced in (2.1.26). The results could be easily translated into parallel results for $J_\sigma(X, T|\underline{\nu})$ by (2.1.25) and the observation that $\widetilde{\sigma}$ satisfies Assumption A if σ does, but we will omit the details for the sake of brevity.

2.2.1 Operator preliminaries

For a given $g \in L^\infty(\mathbb{R})$, let $\mathcal{M}_g$ be the multiplication operator on $L^2(\mathbb{R})$ defined by $\mathcal{M}_g f = gf$ for all $f \in L^2(\mathbb{R})$, and let $\mathcal{K}_s^{\mathrm{Ai}}$ be the operator acting on $L^2(\mathbb{R})$ through the shifted Airy kernel,

$$(\mathcal{K}_s^{\mathrm{Ai}} f)(\lambda) = \int_{\mathbb{R}} K_s^{\mathrm{Ai}}(\lambda, \mu) f(\mu) \mathrm{d}\mu, \quad K_s^{\mathrm{Ai}}(\lambda, \mu) := K^{\mathrm{Ai}}(\lambda + s, \mu + s), \ f \in L^2(\mathbb{R}), \tag{2.2.1}$$

with K^{Ai} defined in (2.1.3). It is worth recalling that $\mathcal{K}_s^{\mathrm{Ai}}$ is an orthogonal projector which can be represented as $\mathcal{A}_s \mathcal{M}_{1_{(0,+\infty)}} \mathcal{A}_s$ where $\mathcal{A}_s$ is the unitary involution of $L^2(\mathbb{R})$ defined by

$$(\mathcal{A}_s f)(\lambda) = \int_{-\infty}^{+\infty} \mathrm{Ai}(\lambda + \mu + s) f(\mu) \, \mathrm{d}\mu, \qquad f \in L^2(\mathbb{R}), \tag{2.2.2}$$

where the integral in the right-hand side is taken as an L^2-limit of $\int_{-\Lambda}^{+\infty}$ as $\Lambda \to +\infty$.

Lemma 2.2.1. *Let σ satisfy Assumption A. The operator defined as $\mathcal{K}_s^\sigma := \mathcal{M}_{\sqrt{\sigma}} \mathcal{K}_s^{\mathrm{Ai}} \mathcal{M}_{\sqrt{\sigma}}$ is trace class on $L^2(\mathbb{R})$ and*

$$j_\sigma(s|\emptyset) = \det_{L^2(\mathbb{R})} \left(1 - \mathcal{K}_s^\sigma\right). \tag{2.2.3}$$

Moreover, $0 < j_\sigma(s|\emptyset) \le 1$ for all $s \in \mathbb{R}$.

We denote the Fredholm determinant of a trace class perturbation of the identity by $\det\limits_{L^2(\mathbb{R})}$.

Proof. We have $\mathcal{K}_s^\sigma = \mathcal{H}\mathcal{H}^\dagger$ where $\mathcal{H} := \mathcal{M}_{\sqrt{\sigma}}\mathcal{A}_s 1_{(0,+\infty)}$. It follows by the asymptotic properties of the Airy function at $+\infty$ that $\mathcal{H}$ is Hilbert–Schmidt provided σ satisfies Assumption A. Therefore, $\mathcal{K}_s^\sigma$ is the composition of two Hilbert–Schmidt operators, hence it is trace class on $L^2(\mathbb{R})$. Then, (2.2.3) follows by the classical formula for Fredholm determinants of operators with an integral kernel. Next, since $\mathcal{K}_s^{\mathrm{Ai}}$ is an orthogonal projector and $0 \le \sigma \le 1$, we have $(\mathcal{K}_s^\sigma)^2 \le \mathcal{K}_s^\sigma$ because

$$(\mathcal{K}_s^\sigma)^2 = \mathcal{M}_{\sqrt{\sigma}}\mathcal{K}_s^{\mathrm{Ai}}\mathcal{M}_\sigma\mathcal{K}_s^{\mathrm{Ai}}\mathcal{M}_{\sqrt{\sigma}} \le \mathcal{M}_{\sqrt{\sigma}}(\mathcal{K}_s^{\mathrm{Ai}})^2\mathcal{M}_{\sqrt{\sigma}} = \mathcal{M}_{\sqrt{\sigma}}\mathcal{K}_s^{\mathrm{Ai}}\mathcal{M}_{\sqrt{\sigma}} = \mathcal{K}_s^\sigma, \tag{2.2.4}$$

such that $0 \le \mathcal{K}_s^\sigma \le 1$ and so $0 \le j_\sigma(s|\emptyset) \le 1$. It remains to show that $j_\sigma(s|\emptyset) \ne 0$, or, equivalently, that 1 is not an eigenvalue of $\mathcal{K}_s^\sigma$. For, assume $f \in L^2(\mathbb{R})$ is such that $\mathcal{K}_s^\sigma f = f$. Setting $g := \mathcal{K}_s^{\mathrm{Ai}}(\sqrt{\sigma}f)$, we have $\sqrt{\sigma}g = f$ and so, since $\mathcal{K}_s^{\mathrm{Ai}}$ is an orthogonal projector,

$$||g||_2 = \left|\left|\mathcal{K}_s^{\mathrm{Ai}}(\sqrt{\sigma}f)\right|\right|_2 \le \left|\left|\sqrt{\sigma}f\right|\right|_2 = ||\sigma g||_2 \tag{2.2.5}$$

Therefore, since $0 \le \sigma \le 1$, we have $(\sigma - 1)g = 0$ almost everywhere on $\mathbb{R}$. Since $\sigma \to 0$ at $+\infty$ (cf. Assumption A), g has to vanish on some open set of $\mathbb{R}$. On the other hand, g is the restriction to the real line of an entire function, as it follows from the fact that $\mathcal{A}_s h$ is entire for all h with support bounded below by standard properties of the Airy function. Therefore, g is identically zero, so is f, and 1 is not an eigenvalue of $\mathcal{K}_s^\sigma$. ☐

Remark 2.2.2.
$\mathcal{M}_{\sqrt{\sigma}}\mathcal{K}_s^{\mathrm{Ai}}\mathcal{M}_{\sqrt{\sigma}}$ acts on $L^2(\mathbb{R})$ through the kernel $\sqrt{\sigma(\lambda)\sigma(\mu)}K^{\mathrm{Ai}}(\lambda+s,\mu+s)$, which is a correlation kernel for the σ-thinned shifted Airy point process. It follows from Lemma 2.2.1 and from the general theory of determinantal point processes [135, Theorem 4] that the σ-thinned shifted and dilated Airy point process has almost surely a finite number of particles. On the other hand, since $\mathcal{K}^{\mathrm{Ai}}$ is not trace class, the Airy point process has almost surely an infinite number of particles; it is however trace class once restricted to half-lines $(t,+\infty)$ so that the Airy point process has almost surely a largest particle.

2.2.2 Conditional ensembles

According to Lemma 2.2.1, the operator $1-\mathcal{K}_s^\sigma$ is invertible, and, therefore, so is $1-\mathcal{M}_\sigma\mathcal{K}_s^{\mathrm{Ai}}$. Thus, it makes sense to introduce

$$\mathcal{L}_s^\sigma := \mathcal{K}_s^{\mathrm{Ai}}(1-\mathcal{M}_\sigma\mathcal{K}_s^{\mathrm{Ai}})^{-1} = \mathcal{K}_s^{\mathrm{Ai}} + \mathcal{K}_s^{\mathrm{Ai}}\mathcal{M}_{\sqrt{\sigma}}(1-\mathcal{K}_s^\sigma)^{-1}\mathcal{M}_{\sqrt{\sigma}}\mathcal{K}_s^{\mathrm{Ai}}. \tag{2.2.6}$$

As we shall review below following the Its–Izergin–Korepin–Slavnov method [98], $\mathcal{L}_s^\sigma$ is an integral kernel operator, whose kernel we denote by $L_s^\sigma(\cdot,\cdot)$. It has been proved by the first two authors of this paper [58], building on [41, 42, 40], that this kernel induces a determinantal point process defined as follows. Consider the shifted Airy process and construct a σ-marked point process by assigning to each point λ in a random configuration, independently, a mark 1 with probability $\sigma(\lambda)$ or a mark 0 with probability $1-\sigma(\lambda)$. Conditioning the marked point process on the event that there are no points with mark 1, it is shown in op. cit. that the resulting conditional ensemble is determinantal, with correlation kernel with respect to the *deformed* reference measure $\big(1-\sigma(\lambda)\big)\mathrm{d}\lambda$ given precisely by $L_s^\sigma(\cdot,\cdot)$.

Let us introduce the following notation. Given vectors $\underline{u}=(u_1,\dots,u_m)\in\mathbb{R}^m$ and $\underline{w}=(w_1,\dots,w_n)\in\mathbb{R}^n$, introduce the $m\times n$ matrix $K_s^{\mathrm{Ai}}(\underline{u},\underline{w})\in\mathbb{R}^{m\times n}$ with entries

$$\big(K_s^{\mathrm{Ai}}(\underline{u},\underline{w})\big)_{i,j} := K_s^{\mathrm{Ai}}(u_i,w_j), \qquad 1\le i\le m,\ 1\le j\le n. \tag{2.2.7}$$

Lemma 2.2.3. *For any vector $\underline{\nu}=(\nu_1,\dots,\nu_m)$, with $\nu_i\neq\nu_j$ for all $i\neq j$, $K_s^{\mathrm{Ai}}(\underline{\nu},\underline{\nu})$ is positive-definite.*

Proof. According to (2.1.3) and to (2.2.7), we can rewrite $K_s^{\mathrm{Ai}}(\underline{\nu},\underline{\nu})$ as a Gram matrix

$$\begin{aligned}\big(K_s^{\mathrm{Ai}}(\underline{\nu},\underline{\nu})\big)_{i,j} &= \int_0^{+\infty}\mathrm{Ai}(\nu_i+\eta+s)\mathrm{Ai}(\nu_j+\eta+s)\,\mathrm{d}\eta\\ &= \big\langle \mathrm{Ai}(\nu_i+\cdot),\mathrm{Ai}(\nu_j+\cdot)\big\rangle_{L^2(s,+\infty)}.\end{aligned} \tag{2.2.8}$$

Hence, it suffices to show that the m vectors $\mathrm{Ai}(\nu_i+\cdot)\in L^2(s,+\infty)$, for $1\le i\le m$, are linearly independent when the points $\nu_1,\dots,\nu_m$ are distinct. In order to obtain a contradiction, let us assume that the linear span of these m vectors is k-dimensional with $k<m$ and, without loss of generality, that $\mathrm{Ai}(\nu_i+\cdot)$ for $i=1,\dots,k$ form a basis. Then, there exists $c_1,\dots,c_k\in\mathbb{C}$ such that $\mathrm{Ai}(\nu_m+t)=\sum_{i=1}^k c_i\mathrm{Ai}(\nu_i+t)$ identically in t. Subtract $(t+\nu_m)$ times this relation from the second derivative of this relation in t to get, using the Airy equation, $0=\sum_{i=1}^k(\nu_i-\nu_m)c_i\mathrm{Ai}(\nu_i+t)$. Hence $c_i=0$ for all $1\le i\le k$ because $\nu_m\neq\nu_i$ for all $i\neq m$, and so $\mathrm{Ai}(\nu_m+t)=0$ identically in t, a contradiction. ☐

According to Lemma 2.2.3, for distinct points $\nu_1,\dots,\nu_m\in\mathbb{R}$, collected into a vector $\underline{\nu}:=(\nu_1,\dots,\nu_m)$, we can introduce the integral kernel operator $\mathcal{H}_s^{\underline{\nu}}$ acting on $L^2(\mathbb{R})$ through the kernel

$$\begin{aligned}H_s^{\underline{\nu}}(\lambda,\mu) &:= \frac{\det K_s^{\mathrm{Ai}}\big((\lambda,\underline{\nu}),(\mu,\underline{\nu})\big)}{\det K_s^{\mathrm{Ai}}(\underline{\nu},\underline{\nu})}\\ &= K_s^{\mathrm{Ai}}(\lambda,\mu)-K_s^{\mathrm{Ai}}(\lambda,\underline{\nu})K_s^{\mathrm{Ai}}(\underline{\nu},\underline{\nu})^{-1}K_s^{\mathrm{Ai}}(\underline{\nu},\mu),\end{aligned} \tag{2.2.9}$$

where the second equality stems from the well-known formula

$$\det\left(\begin{array}{c|c} A & B\\ \hline C & D\end{array}\right)=\det(D)\det\left(A-BD^{-1}C\right), \qquad \text{if } \det D\neq 0, \tag{2.2.10}$$

for the determinant of a block matrix with lower-right corner invertible. It follows from the results in [131] that $H_s^{\underline{\nu}}(\lambda,\mu)$ is the kernel of the *reduced Palm measure* of the shifted Airy process at (distinct) points $\nu_1,\dots,\nu_m$, which can be interpreted as the shifted Airy point process conditioned on configurations containing points at $\nu_1,\dots,\nu_m$ and then removing the points $\nu_1,\dots,\nu_m$ from the configuration.

2.2.3 Factorizations of Jánossy densities

We can factorize the Jánossy densities $j_\sigma(s|\underline{\nu})$ in two different ways: the first one utilizes the Palm kernels $H_s^{\underline{\nu}}$, the second one involves the kernels L_s^σ of the conditional ensembles. It is convenient to introduce notations similar to (2.2.7) for these kernels, namely, given vectors $\underline{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$ and $\underline{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$, we introduce matrices $L_s^\sigma(\underline{u}, \underline{w}) \in \mathbb{R}^{m\times n}$ and $H_s^{\underline{\nu}}(\underline{u}, \underline{w}) \in \mathbb{R}^{m\times n}$ with entries

$$\left(L_s^\sigma(\underline{u}, \underline{w})\right)_{i,j} = L_s^\sigma(u_i, w_j), \qquad \left(H_s^{\underline{\nu}}(\underline{u}, \underline{w})\right)_{i,j} = H_s^{\underline{\nu}}(u_i, w_j). \tag{2.2.11}$$

Proposition 2.2.4. *For all σ satisfying Assumption A and all $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for $i \neq j$, we have the identities*

$$j_\sigma(s|\underline{\nu}) = \det\left(K_s^{\mathrm{Ai}}(\underline{\nu}, \underline{\nu})\right) \det_{L^2(\mathbb{R})}\left(1 - \mathcal{M}_{\sqrt{\sigma}} \mathcal{H}_s^{\underline{\nu}} \mathcal{M}_{\sqrt{\sigma}}\right), \tag{2.2.12}$$

$$j_\sigma(s|\underline{\nu}) = \det\left(L_s^\sigma(\underline{\nu}, \underline{\nu})\right) \det_{L^2(\mathbb{R})}\left(1 - \mathcal{M}_{\sqrt{\sigma}} \mathcal{K}_s^{\mathrm{Ai}} \mathcal{M}_{\sqrt{\sigma}}\right) = \det\left(L_s^\sigma(\underline{\nu}, \underline{\nu})\right) j_\sigma(s|\emptyset). \tag{2.2.13}$$

Proof. We start by rewriting (2.1.24):

$$\begin{aligned} j_\sigma(s|\underline{\nu}) &= \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \det\left(K_s^{\mathrm{Ai}}\left((\underline{\lambda}, \underline{\nu}), (\underline{\lambda}, \underline{\nu})\right)\right) \prod_{i=1}^n \sigma(\lambda_i) \mathrm{d}\lambda_i \\ &= \det\left(K_s^{\mathrm{Ai}}(\underline{\nu}, \underline{\nu})\right) \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \det\left(H_s^{\underline{\nu}}(\underline{\lambda}, \underline{\lambda})\right) \prod_{i=1}^n \sigma(\lambda_i) \mathrm{d}\lambda_i, \end{aligned} \tag{2.2.14}$$

where we denote $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$, $(\underline{\lambda}, \underline{\nu}) = (\lambda_1, \dots, \lambda_n, \nu_1, \dots, \nu_m)$, and we manipulate the determinant of block matrices using (2.2.10) and Lemma 2.2.3 as

$$\begin{aligned} \det\left(K_s^{\mathrm{Ai}}\left((\underline{\lambda}, \underline{\nu}), (\underline{\lambda}, \underline{\nu})\right)\right) &= \det\left(\begin{array}{c|c} K_s^{\mathrm{Ai}}(\underline{\lambda}, \underline{\lambda}) & K_s^{\mathrm{Ai}}(\underline{\lambda}, \underline{\nu}) \\ \hline K_s^{\mathrm{Ai}}(\underline{\nu}, \underline{\lambda}) & K_s^{\mathrm{Ai}}(\underline{\nu}, \underline{\nu}) \end{array}\right) \\ &= \det\left(K_s^{\mathrm{Ai}}(\underline{\nu}, \underline{\nu})\right) \times \\ &\quad \det\left(K_s^{\mathrm{Ai}}(\underline{\lambda}, \underline{\lambda}) - K_s^{\mathrm{Ai}}(\underline{\lambda}, \underline{\nu}) K_s^{\mathrm{Ai}}(\underline{\nu}, \underline{\nu})^{-1} K_s^{\mathrm{Ai}}(\underline{\nu}, \lambda)\right) \\ &= \det\left(K_s^{\mathrm{Ai}}(\underline{\nu}, \underline{\nu})\right) \det\left(H_s^{\underline{\nu}}(\underline{\lambda}, \underline{\lambda})\right). \end{aligned} \tag{2.2.15}$$

Hence, (2.2.12) is established. Next, let us introduce the operator $\mathcal{N} := \mathcal{M}_{\sqrt{\sigma}}(\mathcal{K}_s^{\mathrm{Ai}} - \mathcal{H}_s^{\underline{\nu}})\mathcal{M}_{\sqrt{\sigma}}$ such that

$$\det_{L^2(\mathbb{R})}\left(1 - \mathcal{M}_{\sqrt{\sigma}} \mathcal{H}_s^{\underline{\nu}} \mathcal{M}_{\sqrt{\sigma}}\right) = \det_{L^2(\mathbb{R})}\left(1 - \mathcal{K}_s^\sigma\right) \det_{L^2(\mathbb{R})}\left(1 + (1 - \mathcal{K}_s^\sigma)^{-1} \mathcal{N}\right). \tag{2.2.16}$$

From (2.2.9) we know that the kernel of $\mathcal{N}$ is

$$N(\lambda, \mu) = \sqrt{\sigma(\lambda)\sigma(\mu)} K_s^{\mathrm{Ai}}(\lambda, \underline{\nu}) K_s^{\mathrm{Ai}}(\underline{\nu}, \underline{\nu})^{-1} K_s^{\mathrm{Ai}}(\underline{\nu}, \mu),$$

such that the kernel of $(I - \mathcal{K}_s^\sigma)^{-1}\mathcal{N}$ is

$$\widetilde{L}_s^\sigma(\lambda, \underline{\nu}) \left(K_s^{\mathrm{Ai}}(\underline{\nu}, \underline{\nu})\right)^{-1} K_s^{\mathrm{Ai}}(\underline{\nu}, \mu) \sqrt{\sigma(\mu)} \tag{2.2.17}$$

where $\widetilde{L}_s^\sigma(\cdot,\cdot)$ is the kernel of $(1-\mathcal{K}_s^\sigma)^{-1}\mathcal{M}_{\sqrt{\sigma}}\mathcal{K}_s^{\mathrm{Ai}}$. By the general formula for the Fredholm determinant of a finite-rank perturbation of the identity, cf. [89, Theorem 3.2], we obtain (I_m denotes the $m\times m$ identity matrix)

$$\begin{aligned}
&\det_{L^2(\mathbb{R})}\big(1+(1-\mathcal{K}_s^\sigma)^{-1}\mathcal{N}\big)\\
&=\det\left(I_m+K_s^{\mathrm{Ai}}(\underline{\nu},\underline{\nu})^{-1}\int_{\mathbb{R}}K_s^{\mathrm{Ai}}(\underline{\nu},\lambda)\sqrt{\sigma(\lambda)}\widetilde{L}_s^\sigma(\lambda,\underline{\nu})\mathrm{d}\lambda\right) \qquad (2.2.18)\\
&=\frac{\det\big(K_s^{\mathrm{Ai}}(\underline{\nu},\underline{\nu})+\int_{\mathbb{R}}K_s^{\mathrm{Ai}}(\underline{\nu},\lambda)\sqrt{\sigma(\lambda)}\widetilde{L}_s^\sigma(\lambda,\underline{\nu})\mathrm{d}\lambda\big)}{\det\big(K_s^{\mathrm{Ai}}(\underline{\nu},\underline{\nu})\big)}\\
&=\frac{\det\big(L_s^\sigma(\underline{\nu},\underline{\nu})\big)}{\det\big(K_s^{\mathrm{Ai}}(\underline{\nu},\underline{\nu})\big)} \qquad (2.2.19)
\end{aligned}$$

where we use the second identity in (2.2.6). Finally, equation (2.2.13) follows from (2.2.12). □

Remark 2.2.5. *Both factorizations (2.2.12) and (2.2.13) have a natural probabilistic interpretation as products of an m-point correlation function with a gap probability. In the first factorization, we have the m-point correlation function in the shifted and rescaled Airy point process, multiplied with the gap probability in the σ-thinning of the Palm measure at points $\nu_1,\dots,\nu_m$ associated to the shifted and rescaled Airy point process. In the second factorization, we have the m-point correlation function in the conditional ensemble associated to the shifted and rescaled Airy point process introduced above, multiplied with the gap probability in the σ-thinning of the thinned shifted and rescaled Airy point process. In the first factorization, the correlation function is simpler, but the gap probability is on the other hand simpler in the second factorization.*

Using the above result, it is now easy to show that Jánossy densities are strictly positive for all distinct $\nu_1,\dots,\nu_m$.

Proposition 2.2.6. *For all σ satisfying Assumption A and all $\underline{\nu}=(\nu_1,\dots,\nu_m)$ with $\nu_i\neq\nu_j$ for $i\neq j$, we have $\det L_s^\sigma(\underline{\nu},\underline{\nu})>0$ and $j_\sigma(s|\underline{\nu})>0$.*

Proof. The operator $\mathcal{N}:=\mathcal{M}_{\sqrt{\sigma}}(\mathcal{K}_s^{\mathrm{Ai}}-\mathcal{H}_s^{\underline{\nu}})\mathcal{M}_{\sqrt{\sigma}}$ is non-negative-definite. Indeed, for all $\phi\in L^2(\mathbb{R})$,

$$\langle\mathcal{N}\phi,\phi\rangle=\underline{h}^\dagger\big(K_s^{\mathrm{Ai}}(\underline{\nu},\underline{\nu})\big)^{-1}\underline{h}\geq 0\,, \qquad (2.2.20)$$

where $\underline{h}=\int_{\mathbb{R}}\sqrt{\sigma}(\lambda)\phi(\lambda)K_s^{\mathrm{Ai}}(\underline{\nu},\lambda)\mathrm{d}\lambda$. Hence, we have proved that

$$1-\mathcal{M}_{\sqrt{\sigma}}\mathcal{H}_s^{\underline{\nu}}\mathcal{M}_{\sqrt{\sigma}}\geq 1-\mathcal{K}_s^\sigma. \qquad (2.2.21)$$

Since $1-\mathcal{K}_s^\sigma$ is (strictly) positive-definite by Lemma 2.2.1, the operator $1-\mathcal{M}_{\sqrt{\sigma}}\mathcal{H}_s^{\underline{\nu}}\mathcal{M}_{\sqrt{\sigma}}$ is also positive-definite, hence invertible. Therefore, $j_\sigma(s|\underline{\nu})>0$ by Lemma 2.2.3 and the first factorization of Jánossy densities (2.2.12), and therefore $\det\big(L_s^\sigma(\underline{\nu},\underline{\nu})\big)>0$ by the second one (2.2.13). □

2.3 RH characterization of Jánossy densities

The aim of the section is to give RH characterizations of Jánossy densities, in order to prove Theorems I, II, and III.

2.3.1 RH problems

The operator $\mathcal{M}_\sigma\mathcal{K}_s^{\mathrm{Ai}}$ is *integrable* in the sense of Its–Izergin–Korepin–Slavnov (IIKS) [98], namely it is a kernel operator whose kernel can be expressed as

$$\frac{\mathbf{f}(\lambda;s)\mathbf{h}(\mu;s)}{\lambda-\mu}, \quad \mathbf{f}(\lambda;s) := \sigma(\lambda)\begin{pmatrix} -i\,\mathrm{Ai}'(\lambda+s) \\ \mathrm{Ai}(\lambda+s)\end{pmatrix}, \; \mathbf{h}(\mu;s) := \begin{pmatrix} -i\,\mathrm{Ai}(\mu+s) \\ \mathrm{Ai}'(\mu+s)\end{pmatrix}. \tag{2.3.1}$$

Therefore, according to op. cit., the resolvent operator $(1-\mathcal{M}_\sigma\mathcal{K}_s^{\mathrm{Ai}})^{-1}-1$ can be characterized in terms of the following RH problem (see proof of Proposition 2.3.1 below).

RH problem for Y_σ

(a) $Y_\sigma(\cdot;s) : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic for all $s\in\mathbb{R}$.

(b) The boundary values of $Y_\sigma(\cdot;s)$ are continuous on $\mathbb{R}$ and are related by

$$Y_{\sigma,+}(\lambda;s) = Y_{\sigma,-}(\lambda;s)\big(I - 2\pi i\,\mathbf{f}(\lambda;s)\mathbf{h}^\top(\lambda;s)\big), \qquad \lambda\in\mathbb{R}, \tag{2.3.2}$$

where the subscript $+$ (respectively, $-$) indicates the boundary value from above (respectively, below) the real axis.

(c) As $\lambda\to\infty$, we have

$$Y_\sigma(\lambda;s) = I + \frac{1}{\lambda}\begin{pmatrix} \beta_\sigma(s) & i\eta_\sigma(s) \\ i\alpha_\sigma(s) & -\beta_\sigma(s)\end{pmatrix} + O(\lambda^{-2}), \tag{2.3.3}$$

for some $\alpha_\sigma(s)$, $\beta_\sigma(s)$, and $\eta_\sigma(s)$.

The following result has been proven in [48]. For the reader's convenience, we offer a direct proof based on the IIKS method.

Proposition 2.3.1. *The RH problem for Y_σ has a unique solution for all $s\in\mathbb{R}$ and we have*

$$\partial_s \log j_\sigma(s|\emptyset) = -\alpha_\sigma(s), \tag{2.3.4}$$

where α_σ is given in (2.3.3).

Proof. The upshot of IIKS theory [98] is that the RH problem for Y_σ is uniquely solvable if and only if $1-\mathcal{M}_s^\sigma\mathcal{K}_s^{\mathrm{Ai}}$ is invertible. The latter condition holds true by Lemma 2.2.1. Moreover, in this case, the resolvent operator $(1-\mathcal{M}_\sigma\mathcal{K}_s^{\mathrm{Ai}})^{-1}-1$ is also an integral operator with kernel

$$\frac{\mathbf{f}^\top(\lambda;s)Y_\sigma^\top(\lambda;s)Y_\sigma^{-\top}(\mu;s)\mathbf{h}(\mu;s)}{\lambda-\mu}. \tag{2.3.5}$$

Therefore, using Jacobi variational formula and the identity

$$\partial_s K_s^{\mathrm{Ai}}(\lambda,\mu) = -\mathrm{Ai}(\lambda+s)\mathrm{Ai}(\mu+s), \tag{2.3.6}$$

which follows directly from (2.1.3), we compute $\frac{\partial}{\partial s}\log j_\sigma(s|\emptyset)$ as

$$\begin{aligned}
&-\operatorname{tr}\left((1-\mathcal{M}_\sigma\mathcal{K}_s^{\mathrm{Ai}})^{-1}\mathcal{M}_\sigma\partial_s\mathcal{K}_s^{\mathrm{Ai}}\right)\\
&= -\operatorname{tr}\left(\left((1-\mathcal{M}_\sigma\mathcal{K}_s^{\mathrm{Ai}})^{-1}-1\right)\mathcal{M}_\sigma\partial_s\mathcal{K}_s^{\mathrm{Ai}}\right) - \operatorname{tr}\left(\mathcal{M}_\sigma\partial_s\mathcal{K}_s^{\mathrm{Ai}}\right)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\mathbf{f}^\top(\lambda;s)Y_\sigma^\top(\lambda;s)Y_\sigma^{-\top}(\mu;s)\mathbf{h}(\mu;s)}{\lambda-\mu}\sigma(\mu)\mathrm{Ai}(\mu+s)\mathrm{Ai}(\lambda+s)\mathrm{d}\lambda\mathrm{d}\mu\\
&\qquad\qquad + \int_{\mathbb{R}}\sigma(\mu)\mathrm{Ai}(\mu+s)^2\mathrm{d}\mu\\
&= \int_{\mathbb{R}}\left[(i,0)\int_{\mathbb{R}}\frac{\mathbf{h}(\lambda;s)\mathbf{f}^\top(\lambda;s)Y_\sigma^\top(\lambda;s)}{\lambda-\mu}\mathrm{d}\lambda\right]Y_\sigma^{-\top}(\mu;s)\mathbf{h}(\mu;s)\sigma(\mu)\mathrm{Ai}(\mu+s)\mathrm{d}\mu\\
&\qquad\qquad + \int_{\mathbb{R}}\sigma(\mu)\mathrm{Ai}(\mu+s)^2\mathrm{d}\mu\\
&\overset{(*)}{=} \int_{\mathbb{R}}(i,0)\left(I-Y_\sigma^\top(\mu;s)\right)Y_\sigma^{-\top}(\mu;s)\mathbf{h}(\mu;s)\sigma(\mu)\mathrm{Ai}(\mu+s)\mathrm{d}\mu\\
&\quad + \int_{\mathbb{R}}\sigma(\mu)\mathrm{Ai}(\mu+s)^2\mathrm{d}\mu \qquad (2.3.7)\\
&= \int_{\mathbb{R}}(i,0)Y_\sigma^{-\top}(\mu;s)\mathbf{h}(\mu;s)\mathbf{f}^\top(\mu;s)\begin{pmatrix}0\\1\end{pmatrix}\mathrm{d}\mu, \qquad (2.3.8)
\end{aligned}$$

where we use the expressions of $\mathbf{f},\mathbf{h}$ given in (2.3.1), and in the equality $(*)$ we use the identity

$$Y_\sigma^\top(\mu) = I - \int_{\mathbb{R}}\frac{\mathbf{h}(\lambda;s)\mathbf{f}^\top(\lambda;s)Y_\sigma^\top(\lambda;s)}{\lambda-\mu}\mathrm{d}\lambda, \tag{2.3.9}$$

which follows from the RH problem satisfied by Y_σ and the Sokhotski–Plemelj formula. Finally, (2.3.8) can be simplified by a residue computation:

$$\begin{aligned}
&\int_{\mathbb{R}}(i,0)Y_\sigma^{-\top}(\mu;s)\mathbf{h}(\mu;s)\mathbf{f}^\top(\mu;s)\begin{pmatrix}0\\1\end{pmatrix}\mathrm{d}\mu\\
&= i\frac{1}{2\pi i}\int_{\mathbb{R}}\left(Y_{\sigma,+}^{-\top}(\mu;s)-Y_{\sigma,-}^{-\top}(\mu;s)\right)_{1,2}\mathrm{d}\mu \qquad (2.3.10)\\
&= i\left(\operatorname*{res}_{\mu=\infty}Y_\sigma^{-\top}(\mu;s)\right)_{1,2} = -\alpha_\sigma(s), \qquad (2.3.11)
\end{aligned}$$

using (2.3.3). □

Next we consider the following RH problem which depends on $s\in\mathbb{R}$ and on a finite number of distinct points $\nu_1,\ldots,\nu_m$, $\nu_i\neq\nu_j$ for $i\neq j$, as usual collected into the vector $\underline{\nu} := (\nu_1,\ldots,\nu_m)$. When $m=0$, condition (c) below is empty.

RH problem for Ψ_σ

(a) $\Psi_\sigma(\cdot; s|\underline{\nu}) : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic for all $s \in \mathbb{R}$ and all $\underline{\nu}$.

(b) The boundary values of $\Psi_\sigma(\cdot; s|\underline{\nu})$ are continuous on $\mathbb{R} \setminus \{\nu_1, \ldots, \nu_m\}$ and are related by

$$\Psi_{\sigma,+}(\lambda; s|\underline{\nu}) = \Psi_{\sigma,-}(\lambda; s|\underline{\nu}) \begin{pmatrix} 1 & 1-\sigma(\lambda) \\ 0 & 1 \end{pmatrix}, \qquad \lambda \in \mathbb{R},\ \lambda \neq \nu_1, \ldots, \nu_m. \tag{2.3.12}$$

(c) For all $i = 1, \ldots, m$, as $\lambda \to \nu_i$ from either side of the real axis we have

$$\Psi_\sigma(\lambda; s|\underline{\nu})(\lambda - \nu_i)^{-\sigma_3} = O(1). \tag{2.3.13}$$

(d) As $\lambda \to \infty$, we have

$$\Psi_\sigma(\lambda; s|\underline{\nu}) = \left(I + \frac{1}{\lambda}\Psi_\sigma^1(s|\underline{\nu}) + O(\frac{1}{\lambda^2})\right)\lambda^{\frac{\sigma_3}{4}} G e^{\left(-\frac{2}{3}\lambda^{\frac{3}{2}} - s\lambda^{\frac{1}{2}}\right)\sigma_3} C_\delta \tag{2.3.14}$$

for any $\delta \in (0, \frac{\pi}{2})$. Here we take the principal branches of $\lambda^{\frac{1}{4}\sigma_3}$ and $\lambda^{\frac{1}{2}}$, analytic in $\mathbb{C} \setminus (-\infty, 0]$ and positive for $\lambda > 0$, and

$$\Psi_\sigma^1(s|\underline{\nu}) := \begin{pmatrix} q_\sigma(s|\underline{\nu}) & ir_\sigma(s|\underline{\nu}) \\ ip_\sigma(s|\underline{\nu}) & -q_\sigma(s|\underline{\nu}) \end{pmatrix},$$

$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad G := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \tag{2.3.15}$$

$$C_\delta := \begin{cases} I, & |\arg\lambda| < \pi - \delta, \\ \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}, & \pi - \delta < \pm\arg\lambda < \pi. \end{cases}$$

Remark 2.3.2. *We shall explain in detail in Section 2.3.5 the relation of this RH problem with the one in [101] related to the inverse scattering for the cKdV equation.*

Remark 2.3.3. *The solution to this RH problem is unique by a standard argument in RH problems based on Liouville and Morera theorems. Moreover, as we will show, the solution exists and can be constructed in terms of the solution to the RH problem for Y_σ (by an Airy dressing) and of a suitable matrix-valued rational function (by a Schlesinger transformation [21]).*

We first recall the case $m = 0$, which has already been considered in [48]. To this end we introduce the Airy model RH problem in the following form.

RH problem for Φ^{Ai}

(a) Φ^{Ai} is analytic in $\mathbb{C}\setminus\mathbb{R}$.

(b) The boundary values of Φ^{Ai} are continuous on $\mathbb{R}$ and are related by

$$\Phi^{\mathrm{Ai}}_+(\lambda) = \Phi^{\mathrm{Ai}}_-(\lambda)\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \qquad \lambda\in\mathbb{R}. \tag{2.3.16}$$

(c) As $\lambda\to\infty$, Φ^{Ai} has the asymptotic behaviour

$$\Phi^{\mathrm{Ai}}(\lambda) = \left(I+\lambda^{-1}\begin{pmatrix}0 & \frac{7i}{48}\\ 0 & 0\end{pmatrix}+\mathcal{O}(\lambda^{-2})\right)\lambda^{\frac14\sigma_3}G\mathrm{e}^{-\frac23\lambda^{\frac32}\sigma_3}C_\delta, \tag{2.3.17}$$

for any $0<\delta<\pi/2$ where G, C_δ are given in (2.3.15) and the branches of $\lambda^{\frac14\sigma_3}$ and $\lambda^{\frac12}$ are as in (2.3.14).

The (unique) solution can be expressed in terms of the Airy function as

$$\Phi^{\mathrm{Ai}}(\lambda) := \begin{cases} -\sqrt{2\pi}\begin{pmatrix}\mathrm{Ai}'(\lambda) & -\mathrm{e}^{\frac{2i\pi}{3}}\mathrm{Ai}'(\mathrm{e}^{\frac{-2i\pi}{3}}\lambda)\\ i\,\mathrm{Ai}(\lambda) & -i\mathrm{e}^{\frac{-2i\pi}{3}}\mathrm{Ai}(\mathrm{e}^{\frac{-2i\pi}{3}}\lambda)\end{pmatrix}, & \text{if } \operatorname{Im}\lambda>0,\\[2ex] -\sqrt{2\pi}\begin{pmatrix}\mathrm{Ai}'(\lambda) & \mathrm{e}^{\frac{-2i\pi}{3}}\mathrm{Ai}'(\mathrm{e}^{\frac{2i\pi}{3}}\lambda)\\ i\,\mathrm{Ai}(\lambda) & i\mathrm{e}^{\frac{2i\pi}{3}}\mathrm{Ai}(\mathrm{e}^{\frac{2i\pi}{3}}\lambda)\end{pmatrix}, & \text{if } \operatorname{Im}\lambda<0.\end{cases} \tag{2.3.18}$$

Proposition 2.3.4. *When $m=0$, the RH problem for Ψ_σ has a unique solution for all $s\in\mathbb{R}$ which can written as*

$$\Psi_\sigma(\lambda;s|\emptyset) = \begin{pmatrix}1 & \frac{is^2}{4}\\ 0 & 1\end{pmatrix} Y_\sigma(\lambda;s)\Phi^{\mathrm{Ai}}_s(\lambda) \tag{2.3.19}$$

where $\Phi^{\mathrm{Ai}}_s(\lambda) := \Phi^{\mathrm{Ai}}(\lambda+s)$. Moreover,

$$p_\sigma(s|\emptyset) = \alpha_\sigma(s)+\frac{s^2}{4}, \tag{2.3.20}$$

and the kernel $L^\sigma_s(\lambda,\mu)$ of the operator $\mathcal{L}^\sigma_s := \mathcal{K}^{\mathrm{Ai}}_s(1-\mathcal{M}_\sigma\mathcal{K}^{\mathrm{Ai}}_s)^{-1}$ can be written as

$$L^\sigma_s(\lambda,\mu) = \frac{\left(\Psi_\sigma(\mu;s|\emptyset)^{-1}\Psi_\sigma(\lambda;s|\emptyset)\right)_{2,1}}{2\pi i(\lambda-\mu)}. \tag{2.3.21}$$

Proof. As explained in Remark 2.3.3, uniqueness of the solution follows from standard arguments, so it suffices to verify that (2.3.19) solves the RH problem. Condition (a) is easily checked, while for condition (b) we use the identity

$$I-2\pi i\,\mathbf{f}(\lambda;s)\mathbf{h}^\top(\lambda;s) = \Phi^{\mathrm{Ai}}_{s,-}(\lambda)\begin{pmatrix}1 & 1-\sigma(\lambda)\\ 0 & 1\end{pmatrix}\Phi^{\mathrm{Ai}}_{s,+}(\lambda)^{-1}, \tag{2.3.22}$$

which follows directly from the identities

$$\mathbf{f}(\lambda;s) = \frac{i\,\sigma(\lambda)}{\sqrt{2\pi}}\Phi^{\mathrm{Ai}}_s(\lambda)\begin{pmatrix}1\\0\end{pmatrix}, \qquad \mathbf{h}(\lambda;s) = -\frac{1}{\sqrt{2\pi}}\Phi^{\mathrm{Ai}}_s(\lambda)^{-\top}\begin{pmatrix}0\\1\end{pmatrix}, \tag{2.3.23}$$

and from condition (b) in the RH problem for Φ^{Ai}. Finally, combining conditions (c) in the RH problems for Y and Φ^{Ai}, we obtain that as $\lambda \to \infty$ we have

$$\begin{aligned} Y_\sigma(\lambda; s)\Phi_s^{\mathrm{Ai}}(\lambda) = \left(I + \lambda^{-1}\begin{pmatrix} \beta(s) & i\big(\eta(s)+\frac{7}{48}\big) \\ i\alpha(s) & -\beta(s)\end{pmatrix} + O(\lambda^{-2})\right) \times \\ (\lambda+s)^{\frac{1}{4}\sigma_3} G \mathrm{e}^{-\frac{2}{3}(\lambda+s)^{\frac{3}{2}}\sigma_3} C_\delta \end{aligned} \tag{2.3.24}$$

and expanding for λ large and s fixed we verify condition (d) in the RH problem for Ψ_σ along with the claimed relation (2.3.20). Finally, (2.3.21) follows directly from the expression (2.3.5) for the kernel of $(1-\mathcal{M}_\sigma\mathcal{K}_s^{\mathrm{Ai}})^{-1}-1 = \mathcal{M}_\sigma\mathcal{L}_s^\sigma$, along with the identities (2.3.19) and (2.3.23) ☐

Proposition 2.3.5. *The RH problem for Ψ_σ has a unique solution for all $s \in \mathbb{R}$ and all $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for $i \neq j$, which can be expressed as*

$$\Psi_\sigma(\lambda; s|\underline{\nu}) = M(\lambda; s|\underline{\nu})\Psi_\sigma(\lambda; s|\emptyset), \tag{2.3.25}$$

where M is a rational function of λ, with poles at $\lambda = \nu_1, \dots, \nu_m$ only, given by

$$M(\lambda; s|\underline{\nu}) = I - \frac{1}{2\pi i}\sum_{i,j=1}^m \frac{\big(L_s^\sigma(\underline{\nu},\underline{\nu})^{-1}\big)_{j,i}}{\lambda-\nu_j}\Psi_\sigma(\nu_i; s|\emptyset)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\Psi_\sigma^{-1}(\nu_j; s|\emptyset), \tag{2.3.26}$$

where we use the notation (2.2.11).

Proof. By the conditions in the RH problem for Ψ_σ it is straightforward to verify that $M(\lambda; s|\underline{\nu}) := \Psi_\sigma(\lambda; s|\underline{\nu})\Psi_\sigma(\lambda; s|\emptyset)^{-1}$ is a rational matrix with simple poles at $\nu_1, \dots, \nu_m$ only and $M(\lambda; s|\underline{\nu}) \to I$ as $\lambda \to \infty$. Hence we write

$$M(\lambda; s|\underline{\nu}) = I + \sum_{j=1}^m \frac{M_j(s|\underline{\nu})}{\lambda - \nu_j}. \tag{2.3.27}$$

Condition (c) in the RH problem for Ψ_σ then translates to the condition

$$M(\lambda; s|\underline{\nu})\Psi_\sigma(\lambda; s|\emptyset)(\lambda-\nu_j)^{-\sigma_3} = O(1), \quad \text{as } \lambda \to \nu_j, \qquad j = 1, \dots, m, \tag{2.3.28}$$

and we claim that this condition uniquely determines the coefficients $M_j(s|\underline{\nu})$. Indeed, the expansion at $\lambda \to \nu_j$ of the left-hand side of (2.3.28) gives

$$\begin{aligned} \left(I + \frac{M_j(s|\underline{\nu})}{\lambda-\nu_j} + \sum_{1\le i\le m,\, i\neq j}\frac{M_i(s|\underline{\nu})}{\nu_j-\nu_i} + O(\lambda-\nu_j)\right) \\ \times\Big(\Psi_\sigma(\nu_j; s|\emptyset) + \Psi_\sigma'(\nu_j; s|\emptyset)(\lambda-\nu_j) + O((\lambda-\nu_j)^2)\Big)(\lambda-\nu_j)^{-\sigma_3} \end{aligned} \tag{2.3.29}$$

where $\Psi_\sigma'(\lambda; s|\underline{\nu}) := \partial_\lambda\Psi_\sigma(\lambda; s|\underline{\nu})$. Vanishing of singular terms in this Laurent series yields

$$M_j(s|\underline{\nu})\Psi_\sigma(\nu_j;s|\emptyset)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\tag{2.3.30}$$

$$\left(\left(I+\sum_{\substack{1\le i\le m\\ i\ne j}}\frac{M_i(s|\underline{\nu})}{\nu_j-\nu_i}\right)\Psi_\sigma(\nu_j;s|\emptyset)+M_j(s|\underline{\nu})\Psi_\sigma'(\nu_j;s|\emptyset)\right)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.\tag{2.3.31}$$

Equation (2.3.30) implies existence of a column vector $\mathbf{a}_j=\mathbf{a}_j(s|\underline{\nu})\in\mathbb{C}^2$, for every $j=1,\dots,m$, such that

$$M_j(s|\underline{\nu})=\mathbf{a}_j(0,1)\Psi_\sigma^{-1}(\nu_j;s|\emptyset).\tag{2.3.32}$$

Plugging (2.3.32) into (2.3.31) using (2.3.21) we get

$$\begin{aligned}\Psi_\sigma(\nu_j;s|\emptyset)\begin{pmatrix}1\\0\end{pmatrix}+\sum_{\substack{1\le i\le m\\ i\ne j}}\mathbf{a}_i\underbrace{\frac{(0,1)\Psi_\sigma^{-1}(\nu_i;s|\emptyset)\Psi_\sigma(\nu_j;s|\emptyset)\begin{pmatrix}1\\0\end{pmatrix}}{\nu_j-\nu_i}}_{=2\pi i\left(L_s^\sigma(\underline{\nu},\underline{\nu})\right)_{j,i}}\\
+\mathbf{a}_j\underbrace{(0,1)\Psi_\sigma^{-1}(\nu_j;s|\emptyset)\Psi_\sigma'(\nu_j;s|\emptyset)\begin{pmatrix}1\\0\end{pmatrix}}_{=2\pi i\left(L_s^\sigma(\underline{\nu},\underline{\nu})\right)_{j,j}}=\begin{pmatrix}0\\0\end{pmatrix}\end{aligned}\tag{2.3.33}$$

and so, cf. Proposition 2.2.6,

$$\begin{aligned}&\Psi_\sigma(\nu_j;s|\emptyset)\begin{pmatrix}1\\0\end{pmatrix}+2\pi i\sum_{i=1}^m\mathbf{a}_i\left(L_s^\sigma(\underline{\nu},\underline{\nu})\right)_{j,i}=\begin{pmatrix}0\\0\end{pmatrix}\\
\Rightarrow\ &\mathbf{a}_j=-\frac{1}{2\pi i}\sum_{i=1}^m\left(L_s^\sigma(\underline{\nu},\underline{\nu})^{-1}\right)_{j,i}\Psi_\sigma(\nu_i;s|\emptyset)\begin{pmatrix}1\\0\end{pmatrix},\end{aligned}\tag{2.3.34}$$

and by (2.3.32), we finally get (3.4.6). □

2.3.2 Stark equation

It is convenient to introduce the following variant of Ψ_σ, namely

$$\Theta_\sigma(\lambda;s|\underline{\nu}):=\begin{pmatrix}1&p_\sigma(s|\underline{\nu})\\0&1\end{pmatrix}\mathrm{e}^{\frac{i\pi}{4}\sigma_3}\Psi_\sigma(\lambda;s|\underline{\nu})\mathrm{e}^{-\frac{i\pi}{4}\sigma_3}.\tag{2.3.35}$$

The RH conditions on Ψ_σ imply that Θ_σ is the unique solution to the following RH problem.

RH problem for Θ_s

(a) $\Theta_\sigma(\cdot;s|\underline{\nu}) : \mathbb{C}\setminus\mathbb{R}\to\mathbb{C}^{2\times 2}$ is analytic for all $s\in\mathbb{R}$ and all finite $\underline{\nu}\subset\mathbb{R}$.

(b) The boundary values of $\Theta_\sigma(\cdot;s|\underline{\nu})$ are continuous on $\mathbb{R}\setminus\underline{\nu}$ and are related by

$$\Theta_{\sigma,+}(\lambda;s|\underline{\nu}) = \Theta_{\sigma,-}(\lambda;s|\underline{\nu})\begin{pmatrix}1 & i(1-\sigma(\lambda))\\ 0 & 1\end{pmatrix},\ \lambda\in\mathbb{R},\ \lambda\neq\nu_1,\dots,\nu_m. \tag{2.3.36}$$

(c) For all $i=1,\dots,m$, as $\lambda\to\nu_i$ from either side of the real axis we have

$$\Theta_\sigma(\lambda;s|\underline{\nu})(\lambda-\nu_i)^{-\sigma_3} = O(1). \tag{2.3.37}$$

(d) As $\lambda\to\infty$, we have

$$\Theta_\sigma(\lambda;s|\underline{\nu}) = \begin{pmatrix}1 & p\\ 0 & 1\end{pmatrix}\left(I+\lambda^{-1}\begin{pmatrix}q & -r\\ p & -q\end{pmatrix}+O(\lambda^{-2})\right)\times \lambda^{\frac14\sigma_3}\frac{\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}}{\sqrt{2}}\mathrm{e}^{\left(-\frac23\lambda^{\frac32}-s\lambda^{\frac12}\right)\sigma_3}C_\delta \tag{2.3.38}$$

for any $\delta\in(0,\frac{\pi}{2})$; here $p=p_\sigma(s|\underline{\nu})$, $q=q_\sigma(s|\underline{\nu})$, and $r=r_\sigma(s|\underline{\nu})$ are the same as in (2.3.14), C_δ is in (2.3.15), and the branches of $\lambda^{\frac14\sigma_3}$ and $\lambda^{\frac12}$ are taken as in (2.3.14).

The formula (2.3.21) is equivalent to

$$L_s^\sigma(\lambda,\mu) = \frac{\left(\Theta_\sigma(\mu;s|\emptyset)^{-1}\Theta_\sigma(\lambda;s|\emptyset)\right)_{2,1}}{2\pi(\lambda-\mu)}. \tag{2.3.39}$$

Proposition 2.3.6. *For any $\lambda\in\mathbb{C}\setminus\mathbb{R}$ and for any $\underline{\nu}=(\nu_1,\dots,\nu_m)$ with $\nu_i\neq\nu_j$ for $i\neq j$, $\Theta_\sigma(\lambda;s|\underline{\nu})$ is differentiable in s, and*

$$\partial_s\Theta_\sigma(\lambda;s|\underline{\nu}) = \begin{pmatrix}0 & \lambda+2\partial_s p_\sigma(s|\underline{\nu})\\ 1 & 0\end{pmatrix}\Theta_\sigma(\lambda;s|\underline{\nu}), \tag{2.3.40}$$

where $p_\sigma(s|\underline{\nu})$ appears in (2.3.14).

Proof. The differentiability of Θ_σ in s, and the fact (2.3.38) continues to hold after differentiating formally in s, can be proved using standard techniques from RH theory, and we refer the reader to [48, Section 3] for details. The matrix function $A(\lambda;s|\underline{\nu}) := \partial_s\Theta_\sigma(\lambda;s|\underline{\nu})\Theta_\sigma(\lambda;s|\underline{\nu})^{-1}$ is entire in λ; indeed it has no jump across the real axis and no singularities at $\underline{\nu}$ because of the RH conditions (b) and (c) for Θ_σ. Moreover, condition (d) in the RH problem for Θ_σ implies that as $\lambda\to\infty$

$$A(\lambda;s|\underline{\nu}) = \begin{pmatrix}0 & \lambda+p^2+2q+\partial_s p\\ 1 & 0\end{pmatrix}+\lambda^{-1}\begin{pmatrix}\star & \star\\ -p^2-2q+\partial_s p & \star\end{pmatrix}+O(\lambda^{-2}), \tag{2.3.41}$$

where $p = p_\sigma(s|\underline{\nu})$ and $q = q_\sigma(s|\underline{\nu})$ are as in (2.3.38) and $\star$ denote expressions which are not relevant to us now. Since $A(\lambda; s|\underline{\nu})$ is entire, Liouville's theorem implies that $A(\lambda; s|\underline{\nu})$ coincides with the linear and constant terms in the Laurent series (2.3.41) and that higher order terms vanish. This yields

$$p_\sigma(s|\underline{\nu})^2 + 2q_\sigma(s|\underline{\nu}) = \partial_s p_\sigma(s|\underline{\nu}), \tag{2.3.42}$$

and the proof is complete. □

From equation (2.3.40) it follows that

$$\Theta_\sigma(\lambda; s|\underline{\nu}) = -\sqrt{2\pi} \begin{pmatrix} \partial_s \varphi_\sigma(\lambda; s|\underline{\nu}) & \partial_s \chi_\sigma(\lambda; s|\underline{\nu}) \\ \varphi_\sigma(\lambda; s|\underline{\nu}) & \chi_\sigma(\lambda; s|\underline{\nu}) \end{pmatrix}, \tag{2.3.43}$$

where either $f = \varphi_\sigma(\lambda; s|\underline{\nu})$ or $f = \chi_\sigma(\lambda; s|\underline{\nu})$ solves

$$\left(\partial_s^2 - 2\big(\partial_s p_\sigma(s|\underline{\nu})\big)\right) f = \lambda f. \tag{2.3.44}$$

Proposition 2.3.7. *We have*

$$\partial_s L_s^\sigma(\lambda, \mu) = -\varphi_\sigma(\lambda; s|\emptyset)\varphi_\sigma(\mu; s|\emptyset) \tag{2.3.45}$$

Proof. We use (2.3.39) to compute

$$\begin{aligned} \partial_s L_s^\sigma(\lambda, \mu) &= \operatorname{tr} \partial_s \left(\frac{\Theta_\sigma(\lambda; s|\emptyset) \mathrm{E}_{12} \Theta_\sigma(\mu; s|\emptyset)^{-1}}{2\pi(\lambda - \mu)} \right) \\ &= \operatorname{tr} \frac{\big(A(\lambda; s|\emptyset) - A(\mu; s|\emptyset)\big)\Theta_\sigma(\lambda; s|\emptyset)\mathrm{E}_{12}\Theta_\sigma(\mu; s|\emptyset)^{-1}}{2\pi(\lambda - \mu)} \\ &= \operatorname{tr} \frac{\mathrm{E}_{12}\Theta_\sigma(\lambda; s|\emptyset)\mathrm{E}_{12}\Theta_\sigma(\mu; s|\emptyset)^{-1}}{2\pi}, \end{aligned} \tag{2.3.46}$$

where we used the cyclic property of the trace and Proposition 2.3.6 and we denoted

$$\mathrm{E}_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A(\lambda; s|\emptyset) := \begin{pmatrix} 0 & \lambda + 2\partial_s p_\sigma(s|\emptyset) \\ 1 & 0 \end{pmatrix}. \tag{2.3.47}$$

Finally, it suffices to insert (2.3.43) into (2.3.46). □

We can finally characterize the Jánossy densities in terms of the RH problem for Ψ_σ.

Proposition 2.3.8. *For all $s \in \mathbb{R}$ and all finite sets $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for all $i \neq j$, we have*

$$\partial_s \log j_\sigma(s|\underline{\nu}) = \frac{s^2}{4} - p_\sigma(s|\underline{\nu}) \tag{2.3.48}$$

where $p_\sigma(s|\underline{\nu})$ appears in (2.3.14).

Proof. Using Proposition 2.3.5 we get

$$ip_\sigma(s|\underline{\nu}) - ip_\sigma(s|\emptyset) = \left(M^1_\infty(s|\underline{\nu})\right)_{2,1} \tag{2.3.49}$$

where $M(\lambda; s|\underline{\nu}) = I + \lambda^{-1}M^1_\infty(s|\underline{\nu}) + O(\lambda^{-2})$ as $\lambda \to \infty$. Using (3.4.6) we compute

$$M^1_\infty(s|\underline{\nu}) = -\frac{1}{2\pi i}\sum_{i,j=1}^{m}\left(L^\sigma_s(\underline{\nu},\underline{\nu})^{-1}\right)_{j,i}\Psi_\sigma(\nu_i; s|\emptyset)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\Psi^{-1}_\sigma(\nu_j; s|\emptyset) \tag{2.3.50}$$

and so, using (2.3.35) and (2.3.43), we obtain

$$\left(M^1_\infty(s|\underline{\nu})\right)_{2,1} = i\sum_{i,j=1}^{m}\left(L^\sigma_s(\underline{\nu},\underline{\nu})^{-1}\right)_{j,i}\varphi_\sigma(\nu_i; s|\emptyset)\varphi_\sigma(\nu_j; s|\emptyset) \tag{2.3.51}$$

On the other hand, by (2.3.45) we have

$$\begin{aligned}\partial_s \log\det L^\sigma_s(\underline{\nu},\underline{\nu}) &= \sum_{i,j=1}^{m}\left(L^\sigma_s(\underline{\nu},\underline{\nu})^{-1}\right)_{j,i}\frac{\partial\left(L^\sigma_s(\underline{\nu},\underline{\nu})\right)_{i,j}}{\partial s}\\ &= -\sum_{i,j=1}^{m}\left(L^\sigma_s(\underline{\nu},\underline{\nu})^{-1}\right)_{j,i}\varphi_\sigma(\nu_i; s|\emptyset)\varphi_\sigma(\nu_j; s|\emptyset)\end{aligned} \tag{2.3.52}$$

and the proof now follows from (2.3.49) because $\log j_\sigma(s|\underline{\nu}) = \log\det L^\sigma_s(\underline{\nu},\underline{\nu}) + \log j_\sigma(s|\emptyset)$ by (2.2.13) and because

$$\frac{\partial}{\partial s}\log j_\sigma(s|\emptyset) = \frac{s^2}{4} - p_\sigma(s|\emptyset) \tag{2.3.53}$$

by (2.3.4) and (2.3.20). $\square$

2.3.3 Asymptotics as $s \to +\infty$

The jump matrix of condition (b) in the RH problem for Y can be rewritten, thanks to (2.3.23), as

$$I - 2\pi i\,\mathbf{f}(\lambda; s)\mathbf{h}^\top(\lambda; s) = I + \Phi^{\mathrm{Ai}}_s(\lambda)\begin{pmatrix}0 & \sigma(\lambda)\\ 0 & 0\end{pmatrix}\Phi^{\mathrm{Ai}}_s(\lambda)^{-1}. \tag{2.3.54}$$

We now show that this jump matrix is close to the identity in the appropriate norms in order to apply the standard *small-norm* RH theory [97]. To this end, we introduce the following notation for a measurable matrix-valued function $X : \mathbb{R} \to \mathbb{C}^{m\times n}$:

$$||X||_p := \begin{cases}\max\limits_{1\le i\le m,\ 1\le j\le n}\left\{\left(\int_{\mathbb{R}}|X_{i,j}(\mu)|^p \mathrm{d}\mu\right)^{1/p}\right\}, & p\in[1,+\infty),\\[2ex] \max\limits_{1\le i\le m,\ 1\le j\le n}\left\{\operatorname*{ess\,sup}_{\mu\in\mathbb{R}}|X_{i,j}(\mu)|\right\}, & p = \infty.\end{cases} \tag{2.3.55}$$

Lemma 2.3.9. *Let σ satisfy Assumption A. Then, with the same $\kappa > 0$ as in Assumption A, we have*

$$\left|\left|\Phi_s^{\mathrm{Ai}}\begin{pmatrix}0 & \sigma\\ 0 & 0\end{pmatrix}(\Phi_s^{\mathrm{Ai}})^{-1}\right|\right|_p = O(s^{-\kappa}), \quad \textit{as } s \to +\infty, \qquad p = 1, 2, \infty. \tag{2.3.56}$$

Proof. The entries in $\Phi_s^{\mathrm{Ai}}(\lambda)\begin{pmatrix}0 & \sigma(\lambda)\\ 0 & 0\end{pmatrix}\Phi_s^{\mathrm{Ai}}(\lambda)^{-1}$ are (possibly, up to an appropriate sign) $\sigma(\lambda)\mathcal{B}(\lambda+s)$ where $\mathcal{B}(\lambda)$ is one of the functions $\mathrm{Ai}(\lambda)^2$ or $\mathrm{Ai}(\lambda)\mathrm{Ai}'(\lambda)$ or $\mathrm{Ai}'(\lambda)^2$. By Assumption A, there are $\Lambda, C_1 > 0$ such that $\lambda < -\Lambda$ implies $\sigma(\lambda) < C_1|\lambda|^{-\frac{3}{2}-\kappa}$. Assuming $s \geq 2\Lambda$, we have, using standard asymptotic properties of Ai and Ai$'$:

- for $\lambda \geq -s/2$, $\sigma(\lambda) \leq 1$ and $|\mathcal{B}(\lambda+s)| \leq C_2\exp(-\lambda - s)$ for some $C_2 > 0$,
- for $\lambda \leq -s/2$, $\sigma(\lambda) \leq C_1|\lambda|^{-\frac{3}{2}-\kappa}$ and $|\mathcal{B}(\lambda+s)| \leq C_3|\lambda|^{\frac{1}{2}}$ for some $C_3 > 0$.

Therefore, as $s \to +\infty$ we get:

$$||\sigma(\lambda)\mathcal{B}(\lambda+s)||_1 \leq \int_{-\infty}^{-s/2} C_1C_3|\lambda|^{-\kappa-1}\mathrm{d}\lambda + \int_{-s/2}^{+\infty} C_2\mathrm{e}^{-\lambda-s}\mathrm{d}\lambda = O(s^{-\kappa}), \tag{2.3.57}$$

$$||\sigma(\lambda)\mathcal{B}(\lambda+s)||_2^2 \leq \int_{-\infty}^{-s/2}(C_1C_3)^2|\lambda|^{-2\kappa-2}\mathrm{d}\lambda + \int_{-s/2}^{+\infty} C_2^2\mathrm{e}^{-\lambda-s}\mathrm{d}\lambda \tag{2.3.58}$$

$$= O(s^{-2\kappa-1}), \tag{2.3.59}$$

$$||\sigma(\lambda)\mathcal{B}(\lambda+s)||_\infty \leq \max\left\{C_1C_3\left(\frac{s}{2}\right)^{-\kappa-1}, C_2\mathrm{e}^{-s/2}\right\} = O(s^{-\kappa-1}), \tag{2.3.60}$$

and the lemma is proved. □

Proposition 2.3.10. *If σ satisfies Assumption A, we have*

$$\varphi_\sigma(\lambda; s|\emptyset) = \mathrm{Ai}(\lambda+s)\big(1 + O(s^{-\frac{1}{2}-\kappa})\big), \qquad s \to +\infty, \tag{2.3.61}$$

for all $\lambda \in \mathbb{R}$.

Proof. Rewriting condition (b) in the RH problem for Y as

$$Y_{\sigma,+}(\lambda; s) - Y_{\sigma,-}(\lambda; s) = \Phi_s^{\mathrm{Ai}}(\lambda)\begin{pmatrix}0 & \sigma(\lambda)\\ 0 & 0\end{pmatrix}\Phi_s^{\mathrm{Ai}}(\lambda)^{-1}Y_\sigma(\lambda; s), \qquad \lambda \in \mathbb{R}, \tag{2.3.62}$$

standard RH theory implies that we can write

$$Y_\sigma(\lambda; s) = I + \frac{1}{2\pi i}\int_{\mathbb{R}}(Y_{\sigma,-}(\mu; s) - I)\Phi_s^{\mathrm{Ai}}(\mu)\begin{pmatrix}0 & \sigma(\mu)\\ 0 & 0\end{pmatrix}\Phi_s^{\mathrm{Ai}}(\mu)^{-1}\frac{\mathrm{d}\mu}{\mu-\lambda}$$
$$+ \frac{1}{2\pi i}\int_{\mathbb{R}}\Phi_s^{\mathrm{Ai}}(\mu)\begin{pmatrix}0 & \sigma(\mu)\\ 0 & 0\end{pmatrix}\Phi_s^{\mathrm{Ai}}(\mu)^{-1}\frac{\mathrm{d}\mu}{\mu-\lambda}. \tag{2.3.63}$$

Moreover, by Lemma 2.3.9 and standard small-norm RH theory [?], $Y_-(\cdot;s)-I$ is in L^2 (entry-wise) for all $s\in\mathbb{R}$ and satisfies

$$||Y_{\sigma,-}(\cdot;s)-I||_2 = O(s^{-\kappa}), \qquad s\to+\infty. \tag{2.3.64}$$

By (2.3.19), (2.3.35), (2.3.43) we have $\sqrt{2\pi}\varphi_\sigma(\lambda;s|\emptyset) = i\left(Y_\sigma(\lambda;s)\Phi_s^{\mathrm{Ai}}(\lambda)\right)_{2,1}$, such that multiplying (2.3.63) by $\Phi_s^{\mathrm{Ai}}(\lambda)$ and extracting the (2,1)-entry we obtain

$$\begin{aligned}\varphi_\sigma&(\lambda;s|\emptyset)\\ &= \mathrm{Ai}(\lambda+s) + \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(0,1)(Y_{\sigma,-}(\mu)-I)\Phi_s^{\mathrm{Ai}}(\mu)\begin{pmatrix}1\\0\end{pmatrix}\sigma(\mu)K_s^{\mathrm{Ai}}(\lambda,\mu)\mathrm{d}\mu \qquad (2.3.65)\\ &\qquad + \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(0,1)\Phi_s^{\mathrm{Ai}}(\mu)\begin{pmatrix}1\\0\end{pmatrix}\sigma(\mu)K_s^{\mathrm{Ai}}(\lambda,\mu)\mathrm{d}\mu. \qquad (2.3.66)\end{aligned}$$

Hence, for some $C>0$,

$$\begin{aligned}&\left|\frac{\varphi_\sigma(\lambda;s|\emptyset)}{\mathrm{Ai}(\lambda+s)}-1\right| \le \frac{C}{\mathrm{Ai}(\lambda+s)}\times\\ &\left(||Y_{\sigma,-}-I||_2\left|\left|\Phi_s^{\mathrm{Ai}}\begin{pmatrix}1\\0\end{pmatrix}\sigma K_s^{\mathrm{Ai}}(\cdot,\lambda)\right|\right|_2 + \left|\left|\Phi_s^{\mathrm{Ai}}\begin{pmatrix}1\\0\end{pmatrix}\sigma K_s^{\mathrm{Ai}}(\cdot,\lambda)\right|\right|_1\right).\end{aligned} \tag{2.3.67}$$

Therefore, denoting $\mathcal{A}$ either Ai or Ai′, we need to estimate the $L^p(\mathbb{R},\mathrm{d}\mu)$-norm (for $p=1,2$) of

$$a(\mu) := \mathcal{A}(\mu+s)\sigma(\mu)\frac{K_s^{\mathrm{Ai}}(\lambda,\mu)}{\mathrm{Ai}(\lambda+s)} \tag{2.3.68}$$

as $s\to+\infty$, for fixed λ. We can assume s is sufficiently large such that $s>2|\lambda|$ and $\mathrm{Ai}(\lambda+s)\le|\mathrm{Ai}'(\lambda+s)|$.

- When $\mu\le -s/2$, we have $|\mathcal{A}(\mu+s)| = O(|\mu|^{\frac14})$, $\sigma(\mu)=O(|\mu|^{-\frac32-\kappa})$, and

  $$\left|\frac{K_s^{\mathrm{Ai}}(\lambda,\mu)}{\mathrm{Ai}(\lambda+s)}\right| \le \frac{|\mathrm{Ai}'(\lambda+s)|}{\mathrm{Ai}(\lambda+s)}\frac{(|\mathrm{Ai}(\mu+s)|+|\mathrm{Ai}'(\mu+s)|)}{|\lambda-\mu|} = O(s^{-\frac12}|\mu|^{\frac14}) \tag{2.3.69}$$

  hence

  $$a(\mu) = O(|\mu|^{-1-\kappa}s^{-\frac12}). \tag{2.3.70}$$

- When $\mu\ge -s/2$, we have $|\mathcal{A}(\mu+s)| = O(\mathrm{e}^{-\mu-s})$, $\sigma(\mu)=O(1)$, and

  $$\begin{aligned}\left|\frac{K_s^{\mathrm{Ai}}(\lambda,\mu)}{\mathrm{Ai}(\lambda+s)}\right| &\le \int_s^{+\infty}\frac{\mathrm{Ai}(\lambda+\eta)}{\mathrm{Ai}(\lambda+s)}\mathrm{Ai}(\mu+\eta)\mathrm{d}\eta\\ &\le \int_s^{+\infty}\mathrm{Ai}(\mu+\eta)\mathrm{d}\eta = O(\mathrm{e}^{-\mu-s}).\end{aligned} \tag{2.3.71}$$

Hence $||a||_{L^1(\mathbb{R})} = O(s^{-\frac{1}{2}-\kappa})$ and $||a||_{L^2(\mathbb{R})} = O(s^{-1-\kappa})$, so that resuming from (2.3.67) we get

$$\left| \frac{\varphi_\sigma(\lambda; s|\emptyset)}{\mathrm{Ai}(\lambda+s)} - 1 \right| = O(s^{-\frac{1}{2}-\kappa}), \tag{2.3.72}$$

and the proof is complete. □

Corollary 2.3.11. *We have*

$$L_s^\sigma(\lambda,\mu) = \int_s^{+\infty} \varphi_\sigma(\lambda; r|\emptyset)\varphi_\sigma(\mu; r|\emptyset)\mathrm{d}r. \tag{2.3.73}$$

Proof. Follows directly by integrating (2.3.45) from s to $+\infty$ thanks to (2.3.61). □

2.3.4 Proofs of Theorems I and II

Proof of Theorem I. The first relation (2.1.27) is nothing else than (2.2.13).

The first equality in (2.1.28) is (2.3.73) while the second one is a rewriting of (2.3.39) using (2.3.43).

That φ_σ solves the Stark boundary value problem (2.1.6) with potential $v_\sigma(s|\emptyset) := \partial_s^2 \log j_\sigma(s|\emptyset)$ follows from (2.3.44), (2.3.53), and (2.3.61).

Finally, in order to prove (2.1.29), we first consider the following chain of equalities, where we use (2.3.53) and the asymptotics as $s \to +\infty$ of Section 2.3.3:

$$\begin{aligned} \log j_\sigma(s|\emptyset) = -\int_s^{+\infty} \partial_r \log j_\sigma(r|\emptyset)\mathrm{d}r &= \int_s^{+\infty} (r-s)\partial_r^2 \log j_\sigma(r|\emptyset)\mathrm{d}r \\ &= \int_s^{+\infty} (r-s) v_\sigma(r|\emptyset)\mathrm{d}r. \end{aligned}$$

In the first step we use $\lim_{r\to+\infty} j_\sigma(r|\emptyset) = \lim_{r\to+\infty} \det_{L^2(\mathbb{R})}(1-\mathcal{K}_s^\sigma) = 1$ because $\mathcal{K}_s^\sigma$ converges to the zero operator in trace-norm when $s \to +\infty$. Indeed, $\mathcal{K}_s^\sigma$ is a non-negative trace-class operator with (jointly) continuous integral kernel so that its trace-norm is

$$\begin{aligned} \int_{\mathbb{R}} \sigma(\lambda)K_s^{\mathrm{Ai}}(\lambda,\lambda)\mathrm{d}\lambda &= \int_{-\infty}^{-s/2} \underbrace{\sigma(\lambda)K_s^{\mathrm{Ai}}(\lambda,\lambda)}_{O(|\lambda|^{-1-\kappa})}\mathrm{d}\lambda + \int_{-s/2}^{+\infty} \underbrace{\sigma(\lambda)K_s^{\mathrm{Ai}}(\lambda,\lambda)}_{O(\exp(-\lambda-s))} \\ &= O(s^{-\kappa}) \end{aligned} \tag{2.3.74}$$

as $s \to +\infty$; here we use that as $s \to +\infty$ we have $\sigma(\lambda) = O(\lambda^{-\frac{3}{2}-\kappa})$ and $K_s^{\mathrm{Ai}}(\lambda,\lambda) = O(|\lambda|^{\frac{1}{2}})$ for $\lambda < -s/2$, and $\sigma(\lambda) = O(1)$ and $K_s^{\mathrm{Ai}}(\lambda,\lambda) = O(\exp(-\lambda-s))$ for $\lambda > -s/2$ (cf. Assumption A).

The identity $\int \varphi^2(\lambda; s|\emptyset)\mathrm{d}\sigma(\lambda) = -v_\sigma(s|\emptyset)$, proved in [48, Proposition 4.1], completes the proof. This identity also follows by setting $\underline{\nu} = \emptyset$ in the more general identity (2.3.80) below, which will be shown in the proof of Theorem II by an adaptation of the argument in loc. cit. (and not relying on the case $\underline{\nu} = \emptyset$). □

Proof of Theorem II. Let us introduce

$$\Xi(\lambda; s|\underline{\nu}) := \Theta_\sigma(\lambda; s|\underline{\nu})\xi(\lambda|\underline{\nu})^{-\sigma_3}, \qquad \xi(\lambda|\underline{\nu}) := \prod_{i=1}^{m}(\lambda - \nu_i). \tag{2.3.75}$$

As it follows from conditions (b) and (c) in the RH problem for Θ_σ, $\Xi(\lambda; s|\underline{\nu})$ is a sectionally analytic matrix-valued function of λ satisfying a jump condition across the real axis of the form

$$\Xi_+(\lambda; s|\underline{\nu}) = \Xi_-(\lambda; s|\underline{\nu}) \begin{pmatrix} 1 & i(1-\sigma(\lambda))\xi(\lambda|\underline{\nu})^2 \\ 0 & 1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}. \tag{2.3.76}$$

It follows that $C(\lambda; s|\underline{\nu}) := \big(\partial_\lambda \Xi(\lambda; s|\underline{\nu})\big)\Xi(\lambda; s|\underline{\nu})^{-1}$ is also a sectionally analytic matrix-valued function of λ satisfying a jump condition across the real axis of the form

$$C_+(\lambda; s|\underline{\nu}) - C_-(\lambda; s|\underline{\nu}) = \Xi(\lambda; s|\underline{\nu}) \times \\ \begin{pmatrix} 0 & i\xi(\lambda|\underline{\nu})^2\big((1-\sigma(\lambda))2\partial_\lambda \log \xi(\lambda|\underline{\nu}) - \sigma'(\lambda)\big) \\ 0 & 0 \end{pmatrix} \Xi(\lambda; s|\underline{\nu})^{-1},$$

for all $\lambda \in \mathbb{R}$. In the right-hand side of this equation we omit the choice of boundary values for Ξ as the expression is independent from this choice, as it can be shown by (2.3.76). It therefore follows from a contour deformation argument that

$$\int_{\mathbb{R}} \Xi(\lambda; s|\underline{\nu}) \begin{pmatrix} 0 & i\xi(\lambda|\underline{\nu})^2\big((1-\sigma(\lambda))2\partial_\lambda \log \xi(\lambda|\underline{\nu}) - \sigma'(\lambda)\big) \\ 0 & 0 \end{pmatrix} \Xi(\lambda; s|\underline{\nu})^{-1} \frac{\mathrm{d}\lambda}{2\pi i} \\ = \lim_{R\to+\infty} \oint_{c_R} C(\lambda; s|\underline{\nu}) \frac{\mathrm{d}\lambda}{2\pi i}, \tag{2.3.77}$$

where c_R is the *clock-wise* oriented circle $|\lambda| = R$. By the identity

$$C := \big(\partial_\lambda \Xi\big)\Xi^{-1} = \big(\partial_\lambda \Theta_\sigma\big)\Theta_\sigma^{-1} - (\partial_\lambda \log \xi)\Theta_\sigma \sigma_3 \Theta_\sigma^{-1} \tag{2.3.78}$$

and the asymptotic relation (2.3.38) we obtain that, as $\lambda \to \infty$ uniformly in the complex plane, we have

$$\big(C(\lambda; s|\underline{\nu})\big)_{2,1} = 1 + \lambda^{-1}\left(\frac{s}{2} - \partial_s p_\sigma(s|\underline{\nu})\right) + O(\lambda^{-\frac{3}{2}}). \tag{2.3.79}$$

Here we also use (2.3.42). Taking the $(2,1)$-entry of (2.3.77), we obtain, also using Proposition 2.3.8,

$$-\int_{\mathbb{R}} \varphi_\sigma(\lambda; s|\underline{\nu})^2 \left((1-\sigma(\lambda))2\partial_\lambda \log \xi(\lambda|\underline{\nu}) - \sigma'(\lambda)\right) \mathrm{d}\lambda = \partial_s p_\sigma(s|\underline{\nu}) - \frac{s}{2} \\ = -\partial_s^2 \log j_\sigma(s|\underline{\nu}). \tag{2.3.80}$$

Taking into account that $\xi(\lambda|\underline{\nu}) = \prod_{i=1}^m(\lambda - \nu_i)$, the identity (2.3.80) we just proved is (2.1.31).
Next, the Stark equation (2.1.32) follows from (2.3.44) and Proposition 2.3.8.

Finally, extracting the $(2,1)$-entry in (2.3.25), using (3.4.6) and (2.3.43),

$$\begin{aligned}\varphi_\sigma(\lambda;s|\underline{\nu}) &= \left(1-\sum_{i,j=1}^{m}\frac{\left(L_s^\sigma(\underline{\nu},\underline{\nu})^{-1}\right)_{j,i}}{\lambda-\nu_j}\varphi_\sigma(\nu_i;s|\emptyset)\partial_s\varphi_\sigma(\nu_j;s|\emptyset)\right)\varphi_\sigma(\lambda;s|\emptyset)\\ &\quad+\left(\sum_{i,j=1}^{m}\frac{\left(L_s^\sigma(\underline{\nu},\underline{\nu})^{-1}\right)_{j,i}}{\lambda-\nu_j}\varphi_\sigma(\nu_i;s|\emptyset)\varphi_\sigma(\nu_j;s|\emptyset)\right)\partial_s\varphi_\sigma(\lambda;s|\emptyset)\\ &=\varphi_\sigma(\lambda;s|\emptyset)-\sum_{i,j=1}^{m}\left(L_s^\sigma(\underline{\nu},\underline{\nu})^{-1}\right)_{j,i}\varphi_\sigma(\nu_i;s|\emptyset)L_s^\sigma(\lambda,\nu_i)\\ &=\frac{1}{\det L_s^\sigma(\underline{\nu},\underline{\nu})}\det\begin{pmatrix}\varphi_\sigma(\lambda;s|\emptyset) & L_s^\sigma(\lambda,\nu_1) & \cdots & L_s^\sigma(\lambda,\nu_m)\\ \varphi_\sigma(\nu_1,s|\emptyset) & L_s^\sigma(\nu_1,\nu_1) & \cdots & L_s^\sigma(\nu_1,\nu_m)\\ \vdots & \vdots & \ddots & \vdots\\ \varphi_\sigma(\nu_m,s|\emptyset) & L_s^\sigma(\nu_m,\nu_1) & \cdots & L_s^\sigma(\nu_m,\nu_m)\end{pmatrix}.\end{aligned} \tag{2.3.81}$$

where in the second step we use (2.1.28) and in the third a standard manipulation of the determinant of a block matrix. Therefore (2.1.34) holds true and the proof is complete. $\square$

2.3.5 Comparison with inverse scattering for the Stark operator

We now comment on the connection between our probabilistic construction based on the σ-thinned (shifted) Airy process and the classical inverse scattering problem for the Stark operator, as described in [101], see also [132, 133]. The latter can be formulated through the following RH problem, cf. [101, Definition 2.3].

RH problem for M

(a) $M(\cdot;\xi):\mathbb{C}\setminus\mathbb{R}\to\mathbb{C}^{2\times 2}$ is analytic for all $\xi\in\mathbb{R}$.

(b) The boundary values of $M(\cdot;\xi)$ are continuous on $\mathbb{R}$ and are related by

$$M_+(\mu;\xi)=\begin{pmatrix}0 & -s(\mu)\\ \overline{s(\mu)} & 1\end{pmatrix}M_-(\mu;\xi),\qquad \mu\in\mathbb{R}. \tag{2.3.82}$$

(c) As $\mu\to\infty$, we have

$$\begin{aligned}&M(\mu;\xi)=M_\infty(\mu;\xi)\big(I+o(1)\big),\\ &M_\infty(\xi;\mu):=\begin{cases}\begin{pmatrix}-w_0(\xi-\mu) & -w_0'(\xi-\mu)\\ w_1(\xi-\mu) & w_1'(\xi-\mu)\end{pmatrix}, & \operatorname{Im}\mu>0,\\ \begin{pmatrix}w_2(\xi-\mu) & w_2'(\xi-\mu)\\ w_0(\xi-\mu) & w_0'(\xi-\mu)\end{pmatrix}, & \operatorname{Im}\mu<0,\end{cases}\end{aligned} \tag{2.3.83}$$

where

$$
\begin{aligned}
w_0(k) &:= 2i\sqrt{\pi}\mathrm{Ai}(k),\\
w_1(k) &:= 2\sqrt{\pi}\mathrm{e}^{\frac{\pi i}{6}}\mathrm{Ai}(\mathrm{e}^{\frac{2\pi i}{3}}k),\\
w_2(k) &:= 2\sqrt{\pi}\mathrm{e}^{-\frac{\pi i}{6}}\mathrm{Ai}(\mathrm{e}^{-\frac{2\pi i}{3}}k).
\end{aligned}
\tag{2.3.84}
$$

In (2.3.82), $s(\mu) = a(\mu)\big(\overline{a(\mu)}\big)^{-1}$ (for $\mu \in \mathbb{R}$), and $a(\mu)$ is part of the scattering data for the Stark operator. In particular, cf. [101, Theorem 2.2], $a(\mu)$ is analytic and non-zero in the half-plane $\operatorname{Im}\mu < 0$ and $a(\mu) = 1+o(|\mu|^{-\frac{1}{2}})$ as $\mu \to \infty$ within $\operatorname{Im}\mu \leq 0$. Then, the matrix

$$
\Psi(\lambda; s) := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} \times \begin{cases} M^\top(-\lambda; s)\begin{pmatrix} 0 & a(-\lambda)^{-1} \\ -a(-\lambda) & 0 \end{pmatrix}, & \operatorname{Im}\lambda > 0, \\ M^\top(-\lambda; s)\begin{pmatrix} \overline{a(-\overline{\lambda})} & 0 \\ 0 & \big(\overline{a(-\overline{\lambda})}\big)^{-1} \end{pmatrix}, & \operatorname{Im}\lambda < 0, \end{cases}
\tag{2.3.85}
$$

essentially solves the RH problem for Ψ_σ, for $\sigma(\lambda) = 1 - |a(-\lambda)|^{-2}$ and $m = 0$, with the caveat that it only satisfies a slightly weaker normalization at $\lambda = \infty$ in which the sub-leading term is just $o(1)$ rather than $O(\lambda^{-1})$.

Indeed, the expression in the right-hand side of (2.3.85) is analytic for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ by the above mentioned properties of a, and a direct computation suffices to ascertain that

$$
\Phi^{\mathrm{Ai}}(\lambda + s) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} M_\infty^\top(-\lambda; s) \times \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \operatorname{Im}\lambda > 0, \\ I, & \operatorname{Im}\lambda < 0, \end{cases}
\tag{2.3.86}
$$

such that the normalization at ∞ of the two RH problems match (up to the order of the sub-leading contribution, as we already mentioned). Moreover, a direct computation shows that the jump condition of the RH problem for Ψ_σ is satisfied by the right-hand side of (2.3.85).

There is however an essential difference between our assumptions on σ, and the assumptions in [101]. Whereas we consider functions σ converging to 0 at $-\infty$, but not necessarily converging to 0 at $+\infty$, cf. Assumption A, it is required in [101, Theorem 2.2(c)] that $\sigma(\lambda) = 1 - |a(-\lambda)|^{-2} \to 0$ as $\lambda \to \pm\infty$. Hence, the class of functions σ that we consider, is not included in the class of scattering data considered by classical inverse scattering theory for the Stark operator.

2.3.6 Connection with the theory of Schlesinger transformations

It is also worth to make a comparison of our setting with the general theory of Schlesinger transformations (see [102, 20, 21]).For a general RH problem depending on parameters, one can define the *Malgrange–Bertola differential* on the space of parameters [20]. The general definition, applied to the RH problem for $\Gamma(\lambda;s) := \Psi_\sigma(\lambda - s; s|\emptyset)$, specializes to the following one-form in s:

$$\Omega = \omega(s)\mathrm{d}s, \quad \omega(s) := \int_{\mathbb{R}} \operatorname{tr}\left[\Gamma_-^{-1}(\lambda;s)\frac{\mathrm{d}\Gamma_-(\lambda;s)}{\mathrm{d}\lambda}\frac{\mathrm{d}J_\Gamma(\lambda;s)}{\mathrm{d}s}J_\Gamma^{-1}(\lambda;s)\right]\frac{\mathrm{d}\lambda}{2\pi i}, \tag{2.3.87}$$

where $J_\Gamma(\lambda;s) = \begin{pmatrix} 1 & 1-\sigma(\lambda-s) \\ 0 & 1 \end{pmatrix}$. Using the form of J_Γ and the jump condition $\Gamma_+(\lambda;s) = \Gamma_-(\lambda;s)J_\Gamma(\lambda;s)$, the integrand can be rewritten as

$$\begin{aligned} \operatorname{tr}\left[\Gamma_-^{-1}(\lambda;s)\frac{\mathrm{d}\Gamma_-(\lambda;s)}{\mathrm{d}\lambda}\begin{pmatrix} 0 & \sigma'(\lambda-s) \\ 0 & 0 \end{pmatrix}\right] &= \\ \frac{1}{2}\operatorname{tr}\left(\Gamma_-^{-1}(\lambda;s)\frac{\mathrm{d}^2\Gamma_-(\lambda;s)}{\mathrm{d}\lambda^2} - \Gamma_+^{-1}(\lambda;s)\frac{\mathrm{d}^2\Gamma_+(\lambda;s)}{\mathrm{d}\lambda^2}\right) \end{aligned} \tag{2.3.88}$$

and hence a residue computation gives

$$\omega(s) = -\frac{1}{2}\operatorname*{res}_{\lambda=\infty}\operatorname{tr}\left(\Gamma^{-1}(\lambda;s)\frac{\mathrm{d}^2\Gamma(\lambda;s)}{\mathrm{d}\lambda^2}\right) = \frac{s^2}{4} - p_\sigma(s|\emptyset) = \partial_s \log j_\sigma(s|\emptyset). \tag{2.3.89}$$

The logarithmic potential j_σ of Ω is then termed tau function of the RH problem [20]. Note that a tau function in this sense is defined only up to a multiplicative (integration) constant. Accordingly, we can say that the Fredholm determinant $j_\sigma(s|\emptyset)$ is the tau function associated with the RH problem for $\Gamma(\lambda;s)$.

Pole insertion in a RH problem (*Schlesinger transformation*) and its effect on Ω have been studied in depth in [21] (expanding on [102, 20]) for RH problems with identity normalization at infinity and insertion of poles off the jump contour. The general results of op. cit. formally match with our setting. Namely, in our setting we consider the RH problem for $\Gamma(\lambda;s|\underline{\nu}) := \Psi(\lambda - s; s|\underline{\nu})$ which is obtained from the one for Γ by inserting poles at $\nu_i + s$ such that $\Gamma(\lambda;s|\underline{\nu}) = (\lambda - \nu_i - s)^{-\sigma_3}O(1)$ for $i = 1, \ldots, m$. The *characteristic matrix* of [21, Definition 2.2], such that the logarithmic differential of its determinant expresses the variation between the Malgrange–Bertola differential after pole insertion [21, Theorem 2.2 part (3)], would reduce in the present setting to

$$\begin{aligned} &\operatorname*{res}_{\lambda=\nu_i+s}\operatorname*{res}_{\mu=\nu_j+s}\frac{\left(\Gamma^{-1}(\lambda;s)\Gamma(\mu;s)\right)_{2,1}}{(\lambda-\mu)(\lambda-\nu_i-s)(\mu-\nu_j-s)}\mathrm{d}\lambda \\ &= -2\pi i\, L_s^\sigma(\nu_i,\nu_j), \qquad\qquad i,j = 1,\ldots,m. \end{aligned} \tag{2.3.90}$$

Hence, the above-mentioned [21, Theorem 2.2 part (3)] predicts that the tau function associated with the RH problem for $\Gamma(\lambda;s|\underline{\nu})$ is (within an absolute multiplicative constant) $j_\sigma(s|\emptyset)$ times the determinant of $L_s^\sigma(\underline{\nu},\underline{\nu})$, i.e. $j_\sigma(s|\underline{\nu})$ by (2.2.12), as we showed in Proposition 2.3.8.

2.3.7 Isospectral deformation and cKdV: proof of Theorem III

As explained in Section 2.1, the connection with the cKdV equation is made by studying the shifted and dilated Airy kernel

$$K_{X,T}^{\mathrm{Ai}}(\lambda,\mu) := T^{-\frac{1}{3}} K^{\mathrm{Ai}}\Big(T^{-\frac{1}{3}}(\lambda+X), T^{-\frac{1}{3}}(\mu+X)\Big), \tag{2.3.91}$$

where $X \in \mathbb{R}$, $T \geq 0$ are parameters. The Jánossy density $J_\sigma(X,T|\underline{\nu})$, defined in (2.1.24), is recovered from $j_\sigma(s|\underline{\nu})$ by (2.1.25). In view of Proposition 2.3.8, we have

$$\begin{aligned} \partial_X \log J_\sigma(X,T|\underline{\nu}) &= T^{-\frac{1}{3}}\ \partial_s \log j_{\widetilde{\sigma}}(s|T^{-\frac{1}{3}}\underline{\nu})\Big|_{s=XT^{-\frac{1}{3}}} \\ &= \frac{X^2}{4T} - T^{-\frac{1}{3}} p_{\widetilde{\sigma}}(XT^{-\frac{1}{3}};T^{-\frac{1}{3}}\underline{\nu}). \end{aligned} \tag{2.3.92}$$

Throughout this section we use the notation $\widetilde{\sigma}(\lambda) = \sigma(T^{\frac{1}{3}}\lambda)$, as in (2.1.25). It is straightforward to verify by the RH problem for Ψ_σ that the matrix

$$\widehat{\Psi}_\sigma(\lambda;X,T|\underline{\nu}) := T^{\frac{1}{12}\sigma_3}\Psi_{\widetilde{\sigma}}(\lambda T^{-\frac{1}{3}};XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}) \tag{2.3.93}$$

is the (unique) solution to the following RH problem.

RH problem for $\widehat{\Psi}_\sigma$

(a) $\widehat{\Psi}_\sigma(\cdot;X,T|\underline{\nu}) : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic for all $X \in \mathbb{R}$, $T > 0$, and all $\underline{\nu}$.

(b) The boundary values of $\widehat{\Psi}_\sigma(\cdot;X,T|\underline{\nu})$ are continuous on $\mathbb{R}\setminus\{\nu_1,\dots,\nu_m\}$ and are related by

$$\widehat{\Psi}_{\sigma,+}(\lambda;X,T|\underline{\nu}) = \widehat{\Psi}_{\sigma,-}(\lambda;X,T|\underline{\nu})\begin{pmatrix}1 & 1-\sigma(\lambda)\\ 0 & 1\end{pmatrix}, \qquad \lambda\in\mathbb{R},\ \lambda\neq\nu_i. \tag{2.3.94}$$

(c) For all $i = 1,\dots,m$, as $\lambda\to\nu_i$ from either side of the real axis we have

$$\widehat{\Psi}_\sigma(\lambda;X,T|\underline{\nu})(\lambda-\nu_i)^{-\sigma_3} = O(1). \tag{2.3.95}$$

(d) As $\lambda\to\infty$, we have

$$\begin{aligned}\widehat{\Psi}_\sigma(\lambda;X,T|\underline{\nu}) = \left(I + \frac{1}{\lambda}\begin{pmatrix}\widehat{q}_\sigma(X,T|\underline{\nu}) & i\widehat{r}_\sigma(X,T|\underline{\nu})\\ i\widehat{p}_\sigma(X,T|\underline{\nu}) & -\widehat{q}_\sigma(X,T|\underline{\nu})\end{pmatrix} + O(\lambda^{-2})\right)\times \\ \lambda^{\frac{1}{4}\sigma_3} G e^{-T^{-\frac{1}{2}}\left(\frac{2}{3}\lambda^{\frac{3}{2}} + X\lambda^{\frac{1}{2}}\right)\sigma_3} C_\delta \end{aligned} \tag{2.3.96}$$

for any $\delta \in (0,\frac{\pi}{2})$. Here we take principal branches of the roots of λ as explained after (2.3.14), and G, C_δ are as in (2.3.15). Moreover, the coefficients in the sub-leading term are related to the ones in (2.3.14) by

$$\widehat{q}_\sigma(X,T|\underline{\nu}) = T^{\frac{1}{3}} q_{\widetilde{\sigma}}(XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}),$$

$$\widehat{r}_\sigma(X,T|\underline{\nu}) = T^{\frac{1}{2}} r_{\widetilde{\sigma}}(XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}), \tag{2.3.97}$$

$$\widehat{p}_\sigma(X,T|\underline{\nu}) = T^{\frac{1}{6}} p_{\widetilde{\sigma}}(XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}). \tag{2.3.98}$$

It is convenient to reformulate (2.3.92) using (2.3.98), as

$$\partial_X \log J_\sigma(X,T|\underline{\nu}) = \frac{X^2}{4T} - T^{-\frac{1}{2}}\widehat{p}_\sigma(X,T|\underline{\nu}). \tag{2.3.99}$$

Introduce now, cf. (2.3.35) and (2.3.43),

$$\begin{aligned}\widehat{\Theta}_\sigma(\lambda;X,T|\underline{\nu}) &:= T^{\frac{1}{12}\sigma_3}\Theta_{\widetilde{\sigma}}(\lambda T^{-\frac{1}{3}};XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}) \\ &= \begin{pmatrix}1 & \widehat{p}_\sigma(X,T|\underline{\nu}) \\ 0 & 1\end{pmatrix} \mathrm{e}^{\frac{i\pi}{4}\sigma_3}\widehat{\Psi}_\sigma(\lambda;X,T|\underline{\nu})\mathrm{e}^{-\frac{i\pi}{4}\sigma_3} \\ &= -\sqrt{2\pi}\begin{pmatrix}T^{\frac{1}{2}}\partial_X\widehat{\varphi}_\sigma(\lambda;X,T|\underline{\nu}) & T^{\frac{1}{2}}\partial_X\widehat{\chi}_\sigma(\lambda;X,T|\underline{\nu}) \\ \widehat{\varphi}_\sigma(\lambda;X,T|\underline{\nu}) & \widehat{\chi}_\sigma(\lambda;X,T|\underline{\nu})\end{pmatrix}\end{aligned} \tag{2.3.100}$$

where we define, cf. (2.3.43),

$$\begin{aligned}\widehat{\varphi}_\sigma(\lambda;X,T|\underline{\nu}) &= T^{-\frac{1}{12}}\varphi_{\widetilde{\sigma}}(\lambda T^{-\frac{1}{3}};XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}), \\ \widehat{\chi}_\sigma(\lambda;X,T|\underline{\nu}) &= T^{-\frac{1}{12}}\chi_{\widetilde{\sigma}}(\lambda T^{-\frac{1}{3}};XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}).\end{aligned} \tag{2.3.101}$$

Proposition 2.3.12. *Let $V_\sigma(X,T|\underline{\nu}) := \partial_X^2 \log J_\sigma(X,T|\underline{\nu})$ and f be in the linear span of $\widehat{\varphi}_\sigma(\lambda;X,T|\underline{\nu})$ and $\widehat{\chi}_\sigma(\lambda;X,T|\underline{\nu})$. We have the "Lax pair"*

$$\mathscr{L}f = \lambda f, \qquad \mathscr{A}f = \partial_T f, \tag{2.3.102}$$

where

$$\begin{aligned}\mathscr{L} &:= T\partial_X^2 + 2TV_\sigma(X,T|\underline{\nu}) - X, \\ \mathscr{A} &:= -\frac{1}{3}\partial_X^3 - V_\sigma(X,T|\underline{\nu})\partial_X - \frac{1}{2}\partial_X V_\sigma(X,T|\underline{\nu}) + \frac{1}{3}T^{-1} - \frac{1}{12}T^{-\frac{3}{2}}.\end{aligned} \tag{2.3.103}$$

Proof. Although the first equation $\mathscr{L}f = \lambda f$ follows directly from (2.3.44) by using (2.3.101), it is convenient to deduce it again; doing so will provide us with additional information useful in the derivation of the second equation.

We start by noting that the matrix function $A(\lambda;X,T|\underline{\nu}) := \big(\partial_X\widehat{\Theta}_\sigma(\lambda;X,T|\underline{\nu})\big)\widehat{\Theta}_\sigma(\lambda;X,T|\underline{\nu})^{-1}$ has no jump across the real axis because the jump condition for $\widehat{\Theta}_\sigma$ across the real axis does not depend on X as if follows from (2.3.100) along with condition (b) in the RH problem for $\widehat{\Psi}_\sigma$. Once more, we refer the reader to [48, Section 3] for the rigorous justification of the differentiability of the RH solution.

Moreover, it is readily checked, cf. (2.3.38), that as $\lambda \to \infty$ we have

$$\begin{aligned}\Theta_\sigma(\lambda;s|\underline{\nu}) = \begin{pmatrix}1 & \widehat{p}_\sigma(X,T|\underline{\nu}) \\ 0 & 1\end{pmatrix}\Bigg(I + \frac{1}{\lambda}\begin{pmatrix}\widehat{q}_\sigma(X,T|\underline{\nu}) & -\widehat{r}_\sigma(X,T|\underline{\nu}) \\ \widehat{p}_\sigma(X,T|\underline{\nu}) & -\widehat{q}_\sigma(X,T|\underline{\nu})\end{pmatrix} \\ + \frac{1}{\lambda^2}\begin{pmatrix}\star & \star \\ \widehat{n}_\sigma(X,T|\underline{\nu}) & \star\end{pmatrix} + O(\lambda^{-3})\Bigg)\lambda^{\sigma_3/4}\frac{\begin{pmatrix}1 & 1 \\ -1 & 1\end{pmatrix}}{\sqrt{2}}\mathrm{e}^{-T^{-\frac{1}{2}}\left(\frac{2}{3}\lambda^{\frac{3}{2}}+X\lambda^{\frac{1}{2}}\right)\sigma_3}C_\delta\end{aligned} \tag{2.3.104}$$

for any $\delta \in (0, \frac{\pi}{2})$. The notations are as in (2.3.38) but now we also need to explicitly record the $(2,1)$-entry of the second sub-leading term in the asymptotic series, denoted $\widehat{n}_\sigma$.

Next, from (2.3.104), we deduce that A has an expansion for large λ of the form

$$A = T^{-\frac{1}{2}} \begin{pmatrix} 0 & \lambda + \widehat{p}_\sigma^2 + 2\widehat{q}_\sigma + T^{\frac{1}{2}}\partial_X \widehat{p}_\sigma \\ 1 & 0 \end{pmatrix} + \lambda^{-1} T^{-\frac{1}{2}} \times$$
$$\begin{pmatrix} \star & \star \\ T^{\frac{1}{2}}\partial_X \widehat{p}_\sigma - \widehat{p}_\sigma^2 - 2\widehat{q}_\sigma & \widehat{n}_\sigma + \widehat{r}_\sigma + \widehat{p}_\sigma^3 + 3\widehat{p}_\sigma \widehat{q}_\sigma - \frac{1}{2}T^{\frac{1}{2}}\partial_X\left(\widehat{p}_\sigma^2 + 2\widehat{q}_\sigma\right) \end{pmatrix}$$
$$+ O(\lambda^{-2}). \tag{2.3.105}$$

Liouville theorem guarantees then that A is a polynomial in λ. Consequently, the higher-order Laurent coefficient in this expansion must vanish; we do not need the information coming from the first row (and we have accordingly omitted these terms), while from the second row at order λ^{-1} we obtain

$$\widehat{p}_\sigma^2 + 2\widehat{q}_\sigma = T^{\frac{1}{2}}\partial_X \widehat{p}_\sigma, \qquad \widehat{n}_\sigma + \widehat{r}_\sigma + \widehat{p}_\sigma^3 + 3\widehat{p}_\sigma \widehat{q}_\sigma = \frac{1}{2} T \partial_X^2 \widehat{p}_\sigma. \tag{2.3.106}$$

Summarizing, also thanks to (2.3.99), we have

$$A = T^{-\frac{1}{2}} \begin{pmatrix} 0 & \lambda + 2T^{\frac{1}{2}}\partial_X \widehat{p}_\sigma \\ 1 & 0 \end{pmatrix} = T^{-\frac{1}{2}} \begin{pmatrix} 0 & \lambda - 2V_\sigma + X \\ 1 & 0 \end{pmatrix}. \tag{2.3.107}$$

Comparing with (2.3.100) we obtain $\mathscr{L} f = \lambda f$ whenever f is in the linear span of $\widehat{\varphi}_\sigma, \widehat{\chi}_\sigma$.
Next, the matrix function $B(\lambda; X, T|\underline{\nu}) := \left(\partial_T \widehat{\Theta}_\sigma(\lambda; X, T|\underline{\nu})\right)\widehat{\Theta}_\sigma(\lambda; X, T|\underline{\nu})^{-1}$ has no jump across the real axis because the jump condition for $\widehat{\Theta}_\sigma$ across the real axis does not depend on T, and so B is entire in λ. It then follows from an application of Liouville theorem that B is a polynomial in λ. In particular, from (2.3.104), the $(2,1)$-entry of B is expressed as

$$3T^{\frac{3}{2}} B_{2,1} = -\lambda + \left(\widehat{p}_\sigma^2 + 2\widehat{q}_\sigma - \frac{3X}{2}\right) = -\lambda + T^{\frac{1}{2}}\partial_X \widehat{p}_\sigma - \frac{3}{2}X = -\lambda - TV_\sigma - X, \tag{2.3.108}$$

and, similarly, the $(2,2)$-entry of B as

$$3T^{\frac{3}{2}} B_{2,2} = -\left(\widehat{n}_\sigma + \widehat{r}_\sigma + \widehat{p}_\sigma^3 + 3\widehat{p}_\sigma \widehat{q}_\sigma\right) = -\frac{1}{2} T \partial_X^2 \widehat{p}_\sigma = \frac{1}{2} T^{\frac{3}{2}} \partial_X V_\sigma - \frac{1}{4}. \tag{2.3.109}$$

We have used (2.3.106) and (2.3.99) to simplify these expressions. Comparing with (2.3.100), we must have

$$\partial_T f = -\frac{\lambda + TV_\sigma + X}{3T}\partial_X f + \frac{2\partial_X V_\sigma - T^{-\frac{3}{2}}}{12} f, \tag{2.3.110}$$

for f equal to either $\widehat{\varphi}_\sigma$ or $\widehat{\chi}_\sigma$, and hence for any f in their linear span. By the relation $\mathscr{L} f = \lambda f$ obtained above we can rewrite the last relation by using $\lambda \partial_X f = \partial_X(\mathscr{L} f)$ which finally yields $\partial_T f = \mathscr{A} f$. $\square$

Corollary 2.3.13. *The function $V_\sigma(X,T|\underline{\nu}) := \partial_X^2 \log J_\sigma(X,T|\underline{\nu})$ satisfies the cKdV equation* (2.1.14)*.*

Proof. This is a classical argument [115]. From the compatibility condition of (2.3.102) we obtain

$$\big(\partial_T \mathscr{L} + [\mathscr{L}, \mathscr{A}]\big) f = 0. \tag{2.3.111}$$

A direct computation gives that $\partial_T \mathscr{L} + [\mathscr{L}, \mathscr{A}]$ is the operator of multiplication with the function

$$V_\sigma(X,T|\underline{\nu}) + 2T\partial_T V_\sigma(X,T|\underline{\nu}) + 2TV_\sigma(X,T|\underline{\nu})\partial_X V_\sigma(X,T|\underline{\nu}) + \frac{1}{6} T \partial_X^3 V_\sigma(X,T|\underline{\nu}). \tag{2.3.112}$$

The equation (2.3.111) must be true for any f in the linear span of $\widehat{\varphi}_\sigma, \widehat{\chi}_\sigma$. Since $\det \widehat{\Theta}_\sigma = 1$ identically in all variables λ, X, T, the functions $\widehat{\varphi}_\sigma$ and $\widehat{\chi}_\sigma$ never vanish simultaneously, hence (2.3.112) must vanish identically. □

This proves Theorem III.

2.3.8 Generalisation to discontinuous σ's

In this section we briefly explain how to extend the results to a broader class of functions σ, including in particular $\sigma = 1_{(0,+\infty)}$.

Assumption C. *The function $\sigma : \mathbb{R} \to [0,1]$ can be written as $\sigma = \sigma_0 + \sum_{j=1}^{f} w_j 1_{(\xi_j,+\infty)}$ for some (finite) integer $f \geq 0$, some $w_1, \dots, w_f > 0$ and some $\xi_1, \dots, \xi_f \in \mathbb{R}$, and a smooth function σ_0 such that $\sigma_0(\lambda) = O(|\lambda|^{-\frac{3}{2}-\kappa})$ as $\lambda \to -\infty$ for some $\kappa > 0$.*

These are the assumptions made in [48], to which we refer for more details, and they include the setting of [57] which corresponds to the case $\sigma_0 = 0$. Under these more general assumptions, the RH problems for Y_σ and Ψ_σ have to be complemented with the condition that Y_σ and Ψ_σ have, at worst, logarithmic singularities at ξ_j.

Theorem I holds true verbatim except for (2.1.29), which is to be replaced by

$$j_\sigma(s|\emptyset) = \exp\left[-\int_s^{+\infty} (r-s)\left[\int_{\mathbb{R}} \varphi_\sigma(\lambda; r|\emptyset)^2 \sigma_0'(\lambda) \mathrm{d}\lambda + \sum_{j=1}^{f} \Delta_j \varphi_\sigma(\xi_j; r|\emptyset)^2\right]\right] \tag{2.3.113}$$

where $\Delta_j := w_j - w_{j-1}$ for $2 \leq j \leq f$ and $\Delta_1 := w_1$. This follows directly from [48, equations (1.8) and (1.26)].

Moreover, Theorem II holds true verbatim except for (2.1.31), which is to be replaced by

$$\partial_s^2 \log j_\sigma(s|\underline{\nu}) = \int_{\mathbb{R}} \varphi_\sigma(\lambda; s|\underline{\nu})^2 \left(-\sigma_0'(\lambda) + \sum_{i=1}^{m} \frac{2\big(1-\sigma(\lambda)\big)}{\lambda - \nu_i}\right) \mathrm{d}\lambda - \sum_{j=1}^{f} \Delta_j \varphi_\sigma(\xi_j; s|\underline{\nu})^2, \tag{2.3.114}$$

and Theorem III holds true verbatim. These two generalisations are obtained by studying the local behaviour of Ψ_σ near the logarithmic singularities at the points ξ_j, as is done in the end of the proof of [48, Proposition 4.1], cf. equations (4.5) and (4.6) there.

2.4 Asymptotics

2.4.1 Outline

The goal of this section is to prove Theorem IV.

The proof of part (i) of Theorem IV will rely on elementary operator estimates, starting from the analogue of the factorization (2.2.12) in the cKdV variables,

$$J_\sigma(X,T|\underline{\nu}) = \det\left(K^{\mathrm{Ai}}_{X,T}(\underline{\nu},\underline{\nu})\right) \det_{L^2(\mathbb{R})}\left(1 - \mathcal{M}_{\sqrt{\sigma}}\widehat{\mathcal{H}}^{\underline{\nu}}_{X,T}\mathcal{M}_{\sqrt{\sigma}}\right), \tag{2.4.1}$$

where $\widehat{\mathcal{H}}^{\mathrm{Ai}}_{X,T}$ is the integral operator with kernel, similarly to (2.2.9),

$$\begin{aligned}\widehat{H}^{\underline{\nu}}_{X,T}(\lambda,\mu) &:= \frac{\det K^{\mathrm{Ai}}_{X,T}\big((\lambda,\underline{\nu}),(\mu,\underline{\nu})\big)}{\det K^{\mathrm{Ai}}_{X,T}(\underline{\nu},\underline{\nu})} \\ &= K^{\mathrm{Ai}}_{X,T}(\lambda,\mu) - K^{\mathrm{Ai}}_{X,T}(\lambda,\underline{\nu})K^{\mathrm{Ai}}_{X,T}(\underline{\nu},\underline{\nu})^{-1}K^{\mathrm{Ai}}_{X,T}(\underline{\nu},\mu).\end{aligned} \tag{2.4.2}$$

Consequently, by (2.1.11), we also have

$$V_\sigma(X,T|\underline{\nu}) = V_{\sigma=0}(X,T|\underline{\nu}) + \partial_X^2 \log \det_{L^2(\mathbb{R})}\left(1 - \mathcal{M}_{\sqrt{\sigma}}\widehat{\mathcal{H}}^{\underline{\nu}}_{X,T}\mathcal{M}_{\sqrt{\sigma}}\right). \tag{2.4.3}$$

We will prove in Section 2.4.2 that the second factor in (2.4.1) is close to 1 and that the second term in (2.4.3) is close to 0, and this will result in part (i) of Theorem IV.

For part (ii) of Theorem IV, we will instead use the analogue of (2.2.13) in the cKdV variables X, T. Using (2.1.25), (2.1.27), and (2.1.28), we obtain the identity

$$J_\sigma(X,T|\underline{\nu}) = \det\left(\widehat{L}^\sigma_{X,T}(\nu_i,\nu_j)\right)_{i,j=1}^m J_\sigma(X,T|\emptyset) \tag{2.4.4}$$

where

$$\widehat{L}^\sigma_{X,T}(\lambda,\mu) := T^{-\frac{1}{3}} L^{\widetilde{\sigma}}_{XT^{-\frac{1}{3}}}(T^{-\frac{1}{3}}\lambda, T^{-\frac{1}{3}}\mu), \tag{2.4.5}$$

with $\widetilde{\sigma}(\lambda) = \sigma(T^{\frac{1}{3}}\lambda)$ as in (2.1.25), which can be rewritten by (2.1.28) and (2.3.100)–(2.3.101) as

$$\begin{aligned}&\widehat{L}^\sigma_{X,T}(\lambda,\mu) \\ &\qquad = T^{\frac{1}{2}}\frac{\widehat{\varphi}_\sigma(\lambda;X,T|\emptyset)\partial_X\widehat{\varphi}_\sigma(\mu;X,T|\emptyset) - \widehat{\varphi}_\sigma(\mu;X,T|\emptyset)\partial_X\widehat{\varphi}_\sigma(\lambda;X,T|\emptyset)}{\lambda-\mu}\end{aligned} \tag{2.4.6}$$

$$= \frac{\left(\widehat{\Theta}_\sigma(\mu;X,T|\emptyset)^{-1}\widehat{\Theta}_\sigma(\lambda;X,T|\emptyset)\right)_{2,1}}{2\pi(\lambda-\mu)}. \tag{2.4.7}$$

It also follows that

$$V_\sigma(X,T|\underline{\nu}) = V_\sigma(X,T|\emptyset) + \partial_X^2 \log\det\big(\widehat{L}^\sigma_{X,T}(\nu_i,\nu_j)\big)_{i,j=1}^m. \tag{2.4.8}$$

The asymptotic behaviour of the second factor in (2.4.4) and of the first term in (2.4.8) has been established in [48, 56], and can be summarized as follows in the cases where $XT^{-\frac{1}{2}} \to -\infty$.

Theorem 2.4.1. *Let σ satisfy Assumption B. For any $T_0 > 0$ there exists $K > 0$ such that*

$$\log J_\sigma(X,T|\emptyset) = \rho^3 T^2 \left(-\frac{4}{15}(1-\xi)^{\frac{5}{2}} + \frac{4}{15} - \frac{2}{3}\xi + \frac{1}{2}\xi^2\right) + O\left(|X|^{\frac{3}{2}} T^{-\frac{1}{2}}\right), \tag{2.4.9}$$

$$V_\sigma(X,T|\emptyset) = \rho\left(1 - \sqrt{1-\xi}\right) + O\left(|X|^{-\frac{1}{2}} T^{-\frac{1}{2}}\right), \tag{2.4.10}$$

where $\rho := c_+^2/\pi^2$ and $\xi := X/(\rho T)$, uniformly for $X \le -KT^{\frac{1}{2}}$ ant $T \ge T_0$.

This result is contained in [56, Theorem 1.3].

Therefore, in order to prove part (ii) of Theorem IV we only need to study, in Section 2.4.3, the additional contributions to (2.4.4) and (2.4.8) coming from $\det\big(\widehat{L}^\sigma_{X,T}(\nu_i,\nu_j)\big)_{i,j=1}^m$.

2.4.2 Right tail: $XT^{-\frac{1}{3}} \to \infty$

We start with a Fredholm determinant estimate for the trace-class operator $\mathcal{M}_{\sqrt{\sigma}} \widehat{\mathcal{H}}^{\mathrm{Ai}}_{X,T} \mathcal{M}_{\sqrt{\sigma}}$, not only valid for large positive X, but also for complex large X with $\arg X$ sufficiently small.

Lemma 2.4.2. *Let σ satisfy Assumption A and let $\underline{\nu} = (\nu_1,\dots,\nu_m)$ with $\nu_i \ne \nu_j$ for all $i \ne j$. For any $T_0 > 0$ there exist $M, c, \delta > 0$ such that*

$$\det_{L^2(\mathbb{R})}\left(1 - \mathcal{M}_{\sqrt{\sigma}} \widehat{\mathcal{H}}^{\underline{\nu}}_{X,T} \mathcal{M}_{\sqrt{\sigma}}\right) = 1 + O\big(\mathrm{e}^{-cXT^{-\frac{1}{3}}}\big), \tag{2.4.11}$$

uniformly for $|X| \ge MT^{\frac{1}{3}}$, $|\arg X| < \delta$, and $T \ge T_0$.

Proof. Using the integral representation for the Airy kernel in (2.1.3), (2.1.10), and the asymptotic behaviour for the Airy function, it is straightforward to verify that

$$\left|K^{\mathrm{Ai}}_{X,T}(\lambda,\mu)\right| = O\left(|\lambda\mu|^{\frac{1}{4}} \mathrm{e}^{-cT^{-\frac{1}{3}}\mathrm{Re}\,(\lambda+X)_+} \mathrm{e}^{-cT^{-\frac{1}{3}}\mathrm{Re}\,(\mu+X)_+}\right), \tag{2.4.12}$$

uniformly for $|X| \ge MT^{\frac{1}{3}}$, $|\arg X| < \delta$, and $T \ge T_0$, where $R_+ = \max\{R, 0\}$, for any $c > 0$, and uniformly for $\lambda, \mu \in \mathbb{R}$. Hence, by (2.4.2),

$$\left|\widehat{H}^{\underline{\nu}}_{X,T}(\lambda,\mu)\right| = O\left(|\lambda\mu|^{\frac{1}{4}} \mathrm{e}^{-cT^{-\frac{1}{3}}\mathrm{Re}\,(\lambda+X)_+} \mathrm{e}^{-cT^{-\frac{1}{3}}\mathrm{Re}\,(\mu+X)_+}\right), \tag{2.4.13}$$

uniformly for the same values of X, T, λ, μ. We can now use the triangular inequality in the Fredholm series

$$\det_{L^2(\mathbb{R})}\left(1-\mathcal{M}_{\sqrt{\sigma}}\widehat{\mathcal{H}}^{\underline{\nu}}_{X,T}\mathcal{M}_{\sqrt{\sigma}}\right)-1 = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int_{\mathbb{R}^n}\det_{i,j=1:n}\left(\widehat{H}^{\underline{\nu}}_{X,T}(\lambda_i,\lambda_j)\right)\prod_{j=1}^{n}\sigma(\lambda_j)\mathrm{d}\lambda_j, \tag{2.4.14}$$

in order to obtain

$$\begin{aligned}&\left|\det_{L^2(\mathbb{R})}\left(1-\mathcal{M}_{\sqrt{\sigma}}\widehat{\mathcal{H}}^{\underline{\nu}}_{X,T}\mathcal{M}_{\sqrt{\sigma}}\right)-1\right|\\ &\qquad\le\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\mathbb{R}^n}\det\big(O(1)\big)_{i,j=1}^{n}\prod_{j=1}^{n}\mathrm{e}^{-2cT^{-\frac{1}{3}}\mathrm{Re}\,(\lambda_j+X)_+}|\lambda_j|^{\frac{1}{2}}\sigma(\lambda_j)\mathrm{d}\lambda_j\\ &\qquad=O\left(\sum_{n=1}^{\infty}\frac{n^{\frac{n}{2}}\mathrm{e}^{-ncT^{-\frac{1}{3}}\mathrm{Re}\,X}}{n!}\left(\int_{\mathbb{R}}\mathrm{e}^{-cT^{-\frac{1}{3}}\mathrm{Re}\,(\lambda+X)_+}|\lambda|^{\frac{1}{2}}\sigma(\lambda)\mathrm{d}\lambda\right)^n\right)\\ &=O\left(\sum_{n=1}^{\infty}\frac{n^{\frac{n}{2}}}{n!}\xi^n\right)\end{aligned}$$

where Hadamard's inequality guarantees that $\det\big(O(1)\big)_{i,j=1}^{n}=O(n^{\frac{n}{2}})$, and in the last step we set $\xi:=\mathrm{e}^{-cT^{-\frac{1}{3}}\mathrm{Re}\,X}\int_{\mathbb{R}}\mathrm{e}^{-cT^{-\frac{1}{3}}\mathrm{Re}\,(\lambda+X)_+}|\lambda|^{\frac{1}{2}}\sigma(\lambda)\mathrm{d}\lambda$. Finally, the power series $\sum_{n=1}^{\infty}\frac{n^{\frac{n}{2}}}{n!}\xi^n$ in ξ has infinite radius of convergence, hence $\sum_{n=1}^{\infty}\frac{n^{\frac{n}{2}}}{n!}\xi^n=O(\xi)$ when $\xi\to 0$; since

$$|\xi|\le\mathrm{e}^{-cT^{-\frac{1}{3}}\mathrm{Re}\,X}\int_{\mathbb{R}}|\lambda|^{\frac{1}{2}}\sigma(\lambda)\mathrm{d}\lambda, \tag{2.4.15}$$

and since the integral on the right-hand side is finite by Assumption A, we have $\xi=O(\mathrm{e}^{-cT^{-\frac{1}{3}}\mathrm{Re}\,X})$ and the proof is complete. $\square$

Taking logarithms on both sides in (2.4.1), we obtain

$$\log J_\sigma(X,T|\underline{\nu})=\log\det\left(K^{\mathrm{Ai}}_{X,T}(\underline{\nu},\underline{\nu})\right)+\log\det_{L^2(\mathbb{R})}\left(1-\mathcal{M}_{\sqrt{\sigma}}\widehat{\mathcal{H}}^{\mathrm{Ai}}_{X,T}\mathcal{M}_{\sqrt{\sigma}}\right), \tag{2.4.16}$$

and it follows from Lemma 2.4.2 (for real X) that

$$\det_{L^2(\mathbb{R})}\left(1-\mathcal{M}_{\sqrt{\sigma}}\widehat{\mathcal{H}}^{\mathrm{Ai}}_{X,T}\mathcal{M}_{\sqrt{\sigma}}\right)=1+O(\mathrm{e}^{-cXT^{-\frac{1}{3}}}) \tag{2.4.17}$$

as $X,T\to\infty$ uniformly for $X\ge MT^{\frac{1}{3}}$ and $T\ge T_0$. This implies (2.1.44).

Taking the second logarithmic X-derivative in (2.1.44), we obtain

$$V_\sigma(X,T|\underline{\nu})=V_0(X,T|\underline{\nu})+\partial_X^2\log\det_{L^2(\mathbb{R})}\left(1-\mathcal{M}_{\sqrt{\sigma}}\widehat{\mathcal{H}}^{\mathrm{Ai}}_{X,T}\mathcal{M}_{\sqrt{\sigma}}\right). \tag{2.4.18}$$

Since the estimate from Lemma 2.4.2 holds uniformly for $|X| \geq MT^{\frac{1}{3}}$, $|\arg X| < \delta$, and $T \geq T_0$, we can use Cauchy's integral formula for the second derivative to obtain

$$\partial_X^2 \log \det_{L^2(\mathbb{R})} \left(1 - \mathcal{M}_{\sqrt{\sigma}} \widehat{\mathcal{H}}^{\mathrm{Ai}}_{X,T} \mathcal{M}_{\sqrt{\sigma}}\right) = O(\mathrm{e}^{-cXT^{-\frac{1}{3}}}), \tag{2.4.19}$$

and thus we prove (2.1.45), so the proof of part (i) of Theorem IV is concluded.

2.4.3 Left tail: $X/T \to -\infty$

In this section we use the results of [56]; the latter rely on Assumption B, which we assume throughout this section. We recall the transformation $x = -XT^{-\frac{1}{2}}$ and $t = T^{-\frac{1}{2}}$ between the cKdV variables of the present paper and the KdV variables of [56]. For the ease of notations, we will denote $\widehat{\Theta}(\lambda) := \widehat{\Theta}_\sigma(\lambda; X, T|\emptyset)$ throughout this section for the function defined in (2.3.100). Let us now assume that, for an arbitrary $T_0 > 0$ and for a sufficiently large $K > 0$, we have $X \leq -KT$ and $T \geq T_0$. In this regime (in fact, in the larger regime $X \leq -KT^{\frac{1}{2}}$), the relevant asymptotics have been studied in [56] via a RH analysis involving a series of transformations which we can condense in the relation

$$\begin{aligned} \widehat{\Theta}_\pm(|X|w) &= \begin{pmatrix} 1 & \widehat{p} \\ 0 & 1 \end{pmatrix} \mathrm{e}^{\frac{i\pi}{4}\sigma_3} |X|^{\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & -i|X|^{\frac{3}{2}} T^{-\frac{1}{2}} g_1 \\ 0 & 1 \end{pmatrix} R(w) \\ &\times (w-a)^{\frac{1}{4}\sigma_3} G \begin{pmatrix} 1 & 0 \\ \pm \mathrm{e}^{-|X|^{\frac{3}{2}} T^{-\frac{1}{2}} \phi_\pm(w)} & 1 \end{pmatrix} \mathrm{e}^{|X|^{\frac{3}{2}} T^{-\frac{1}{2}} (g_\pm(w) - g_0)\sigma_3} \mathrm{e}^{-\frac{i\pi}{4}\sigma_3}, \end{aligned} \tag{2.4.20}$$

for w sufficiently close to 0. Here, c. [56, equations (4.15), (4.17)], with principal branches for the roots,

$$\begin{aligned} g(w) &= \int_a^w g'(s)\mathrm{d}s, \\ g'(w) &= -(w-a)^{\frac{1}{2}} \left(1 + \frac{T^{\frac{1}{2}}}{2\pi|X|^{\frac{1}{2}}} \int_{-\infty}^a \frac{\sigma'(|X|s)}{1-\sigma(|X|s)} \frac{1}{\sqrt{a-s}} \frac{\mathrm{d}s}{s-w}\right), \\ g_0 &= \frac{T^{\frac{1}{2}} \log\big(1-\sigma(|X|a)\big)}{2|X|^{\frac{3}{2}}}, \\ \phi(w) &= 2\big(g(w) - g_0\big) + \frac{T^{\frac{1}{2}} \log\big(1-\sigma(|X|w)\big)}{|X|^{\frac{3}{2}}}. \end{aligned} \tag{2.4.21}$$

Moreover, G is given in (2.3.15) and $\widehat{p} = \widehat{p}_\sigma(X, T|\emptyset)$. The value of $a = a(X,T)$ is implicitly defined by the *endpoint condition*, namely $g'_+(w) - g'_-(w) = O\big((w-a)^{\frac{1}{2}}\big)$ as $w \to a$ with $w < a$, cf. [56, equation (4.3)]. For $X \leq -KT$ and $T \geq T_0$, a is bounded away from zero and infinity, and by [56, Lemma 4.5],

$$R(w) = I + O(|X|^{-\frac{3}{2}} T^{\frac{1}{2}}), \qquad \partial_w R(w) = O(|X|^{-\frac{3}{2}} T^{\frac{1}{2}}), \tag{2.4.22}$$

uniformly in w, $T \geq T_0$, $X \leq -KT$. Finally, the value of g_1 is given explicitly in [56, equation (4.16)] but it is not needed for our current purposes.

As explained in Section 2.4.1, in order to describe the behaviour of $J_\sigma(X,T|\underline{\nu})$, $V_\sigma(X,T|\underline{\nu})$ in this regime we will use equations (2.4.4) and (2.4.8). Thus what is fundamental to understand is the behavior of $\widehat{L}^\sigma_{X,T}(\nu_i,\nu_j)$. The following two lemmas will show how the kernel behaves on and off the diagonal.

We are interested in values of $\widehat{\Theta}$ at $\nu = |X|w$, hence we assume throughout this section that w is real and small. Therefore, (2.4.20) implies

$$\widehat{\Theta}_\pm(|X|w) = \begin{pmatrix} 1 & \widehat{p} + \frac{X^2}{T^{\frac{1}{2}}} g_1 \\ 0 & 1 \end{pmatrix} |X|^{\frac{1}{4}\sigma_3} O(1) \times \begin{pmatrix} \mathrm{e}^{|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\pm(w)-g_0)} & 0 \\ \mp i \mathrm{e}^{|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\mp(w)-g_0)} & \mathrm{e}^{-|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\pm(w)-g_0)} \end{pmatrix} \tag{2.4.23}$$

where we also use the identity $\phi_\pm = g_\pm - g_\mp$ [56, equation below (4.17)]. From [56, Proposition 4.7 and equation (1.32)] we have

$$\widehat{p} + X^2 T^{-\frac{1}{2}} g_1 = O(|X|^{-1}T^{\frac{1}{2}}), \tag{2.4.24}$$

such that the previous relation implies

$$\widehat{\Theta}_\pm(|X|w) = \begin{pmatrix} 1 & O(|X|^{-1}T^{\frac{1}{2}}) \\ 0 & 1 \end{pmatrix} |X|^{\frac{1}{4}\sigma_3} O(1) \times \begin{pmatrix} \mathrm{e}^{|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\pm(w)-g_0)} & 0 \\ \mp i \mathrm{e}^{|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\mp(w)-g_0)} & \mathrm{e}^{-|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\pm(w)-g_0)} \end{pmatrix}. \tag{2.4.25}$$

Finally, we study the last factor, involving g, g_0. By (2.4.21) and the Sokhotski–Plemelj formula, we have

$$g'_\pm(w) = \mp i\sqrt{a-w} \times \left[1 + \frac{T^{\frac{1}{2}}}{2\pi |X|^{\frac{1}{2}}} \mathrm{p.v.}\int_{-\infty}^{a} \frac{\sigma'(|X|s)}{1-\sigma(|X|s)} \frac{1}{\sqrt{a-s}} \frac{\mathrm{d}s}{s-w} \pm i \frac{T^{\frac{1}{2}}|X|^{-\frac{1}{2}}}{2\sqrt{a-w}} \frac{\sigma'(|X|w)}{1-\sigma(|X|w)} \right] \tag{2.4.26}$$

where p.v.$\int$ is the principal value integral. It follows that

$$\operatorname{Re} g'_\pm(w) = \frac{T^{\frac{1}{2}}}{2|X|^{\frac{1}{2}}} \frac{\sigma'(|X|w)}{1-\sigma(|X|w)} \tag{2.4.27}$$

and so

$$\operatorname{Re}\big(g_\pm(\zeta) - g_0\big) = \int_a^w \operatorname{Re} g'_\pm(s) \mathrm{d}s - g_0 = -\frac{T^{\frac{1}{2}}}{2|X|^{\frac{3}{2}}} \log\big(1-\sigma(|X|w)\big). \tag{2.4.28}$$

Finally, let us set $w = |X|^{-1}\nu$, for a fixed ν and sufficiently large $|X|$. It follows from the last estimates and (2.4.25) that

$$\widehat{\Theta}_\pm(\nu) = \begin{pmatrix} 1 & O(|X|^{-1}T^{\frac{1}{2}}) \\ 0 & 1 \end{pmatrix} |X|^{\frac{1}{4}\sigma_3} O(1) \tag{2.4.29}$$

because from (2.4.28) we have

$$\left|\mathrm{e}^{|X|^{\frac{3}{2}}T^{-\frac{1}{2}}\left(g_\pm(\nu|X|^{-1})-g_0\right)}\right| = \frac{1}{1-\sigma(\nu)} \tag{2.4.30}$$

which is bounded away from 0 and ∞, uniformly in the regime under consideration. In particular, by (2.3.100),

$$\widehat{\varphi}_\sigma(\nu;X,T|\emptyset) = O(|X|^{-\frac{1}{4}}), \qquad T^{\frac{1}{2}}\partial_X\widehat{\varphi}_\sigma(\nu;X,T|\emptyset) = O(|X|^{\frac{1}{4}}). \tag{2.4.31}$$

From (2.4.7) and (2.4.31), we immediately obtain boundedness of the kernel $\widehat{L}^\sigma_{X,T}(\nu_1,\nu_2)$ for $\nu_1 \neq \nu_2$. Namely, we have proved the following lemma.

Lemma 2.4.3. *Let $T_0 > 0, \nu_1 \neq \nu_2 \in \mathbb{R}$ be fixed. There exists $K > 0$ such that*

$$\widehat{L}^\sigma_{X,T}(\nu_1,\nu_2) = O(1) \tag{2.4.32}$$

uniformly for $X \leq -KT$ and $T \geq T_0$.

On the other hand, we now show that on the diagonal, the kernel $\widehat{L}^\sigma_{X,T}(\nu,\nu)$ grows.

Lemma 2.4.4. *Let $T_0 > 0, \nu \in \mathbb{R}$ be fixed. We have*

$$\widehat{L}^\sigma_{X,T}(\nu,\nu) \sim \frac{|X|^{\frac{1}{2}}}{\pi T^{\frac{1}{2}}}\frac{1}{1-\sigma(\nu)}, \quad \text{as } \frac{X}{T\log^2|X|} \to -\infty, \tag{2.4.33}$$

uniformly for $T \geq T_0$. In particular, $\widehat{L}^\sigma_{X,T}(\nu,\nu)^{-1} = O(|X|^{-\frac{1}{2}}T^{\frac{1}{2}})$.

Proof. We combine (2.4.7) (in the confluent limit $\mu,\lambda \to \nu$) with (2.4.20) to get

$$\begin{aligned}
\widehat{L}^\sigma_{X,T}(\nu,\nu) &= \frac{1}{2\pi}\left(\widehat{\Theta}_+^{-1}(\nu)\partial_\nu\widehat{\Theta}_+(\nu)\right)_{2,1} \\
&= \frac{1}{2\pi i}\Bigg\{\mathrm{e}^{-\chi(g_+(\frac{\nu}{|X|})-g_0)\sigma_3}\begin{pmatrix}1 & 0\\ -\mathrm{e}^{-\chi\phi_+(\frac{\nu}{|X|})} & 1\end{pmatrix}G^{-1}\left(-\tfrac{\nu}{|X|}-a\right)^{-\frac{1}{4}\sigma_3}R^{-1}(\tfrac{\nu}{|X|})\times \\
&\qquad \partial_\nu\left[R(\tfrac{\nu}{|X|})\left(-\tfrac{\nu}{|X|}-a\right)^{\frac{1}{4}\sigma_3}G\begin{pmatrix}1 & 0\\ \mathrm{e}^{-\chi\phi_+(\frac{\nu}{|X|})} & 1\end{pmatrix}\mathrm{e}^{\chi(g_+(\frac{\nu}{|X|})-g_0)\sigma_3}\right]\Bigg\}_{2,1} \\
&= \frac{1}{2\pi i}\Bigg\{\begin{pmatrix}\mathrm{e}^{-\chi(g_+(\frac{\nu}{|X|})-g_0)} & 0\\ -\mathrm{e}^{\chi(g_-(\frac{\nu}{|X|})-g_0)} & \mathrm{e}^{\chi(g_+(\frac{\nu}{|X|})-g_0)}\end{pmatrix}G^{-1}\left(-\tfrac{\nu}{|X|}-a\right)^{-\frac{1}{4}\sigma_3}R^{-1}(\tfrac{\nu}{|X|})\times \\
&\qquad \partial_\nu\left[R(\tfrac{\nu}{|X|})\left(-\tfrac{\nu}{|X|}-a\right)^{\frac{1}{4}\sigma_3}G\begin{pmatrix}\mathrm{e}^{\chi(g_+(\frac{\nu}{|X|})-g_0)} & 0\\ \mathrm{e}^{\chi(g_-(\frac{\nu}{|X|})-g_0)} & \mathrm{e}^{-\chi(g_+(\frac{\nu}{|X|})-g_0)}\end{pmatrix}\right]\Bigg\}_{2,1}
\end{aligned} \tag{2.4.34}$$

where we denote $\chi := |X|^{\frac{3}{2}}T^{-\frac{1}{2}}$ and in the last step we use the relation $\phi_+ = g_+ - g_-$, cf. [56, equation below (4.17)]. As we proved in (2.4.30), $\mathrm{e}^{\chi(g_\pm(\frac{\nu}{|X|})-g_0)}$ is bounded away from $0,\infty$ and so the triangular matrices appearing in (2.4.34) are $O(1)$. Therefore, when the derivative in ν acts in (2.4.34) it produces terms of order $O(|X|^{-1})$ when it acts on the first two factors, see also (2.4.22), and another term when it acts on the triangular matrix, which provides the leading asymptotic contribution, yielding

$$\widehat{L}^{\sigma}_{X,T}(\nu,\nu) = \frac{\chi}{2\pi i\,|X|} \mathrm{e}^{\chi\left(g_+(\frac{\nu}{|X|})+g_-(\frac{\nu}{|X|})-2g_0\right)} \left(g_-'(\tfrac{\nu}{|X|}) - g_+'(\tfrac{\nu}{|X|})\right) + O(|X|^{-1}). \tag{2.4.35}$$

By the construction of g, cf. [56, Section 4.2], we have (recall that $F = 1/(1-\sigma)$)

$$\chi\big(g_+(w) + g_-(w) - 2g_0\big) = \big(\log F\big)(|X|w), \tag{2.4.36}$$

hence we can rewrite the last expression as

$$\widehat{L}^{\sigma}_{X,T}(\nu,\nu) = \frac{|X|^{\frac12}}{2\pi i\, T^{\frac12}} F(\nu)\big(g_-'(\tfrac{\nu}{|X|}) - g_+'(\tfrac{\nu}{|X|})\big) + O(|X|^{-1}). \tag{2.4.37}$$

Next, we use (2.4.26), a change of integration variable, and an integration by parts in order to get

$$\begin{aligned} g_-'(w) - g_+'(w) &= \\ &\qquad 2i\sqrt{a-w}\left(1 + \frac{T^{\frac12}}{2\pi|X|^{\frac12}} \mathrm{p.v.}\int_{-\infty}^{a|X|} (\log F)'(s) \frac{1}{\sqrt{a - s|X|^{-1}}} \frac{\mathrm{d}s}{s - |X|w}\right) \\ &= 2i\left(\sqrt{a-w} + \frac{T^{\frac12}}{2\pi|X|^{\frac12}} \int_{-\infty}^{a|X|} (\log F)''(s) \log\left|\frac{\sqrt{a-w} + \sqrt{a-s|X|^{-1}}}{\sqrt{a-w} - \sqrt{a-s|X|^{-1}}}\right| \mathrm{d}s\right). \end{aligned} \tag{2.4.38}$$

Now, it is useful to recall the following asymptotic properties for $a = a(X,T)$ from [56, Proposition 4.1]:

$$a = \left.\frac{\big(\sqrt{1+y}-1\big)^2}{y}\right|_{y=\frac{\pi^2}{c_+^2}|X|/T} + O(|X|^{-\frac32}T^{\frac12}) \tag{2.4.39}$$

uniformly in $-X/T \le K$, $T \ge T_0$ (for any $K, T_0 > 0$). Hence,

$$a = 1 - \frac{2c_+T^{\frac12}}{\pi|X|^{\frac12}} + O(|X|^{-1}T), \qquad \sqrt{a - \frac{\nu}{|X|}} = 1 - \frac{c_+T^{\frac12}}{\pi|X|^{\frac12}} + O(|X|^{-1}T), \tag{2.4.40}$$

as $X/T \to -\infty$, $T \ge T_0$. Let us now show that the second term in (2.4.38) is sub-dominant. To start with, notice that, in the same limit, and uniformly for $s \in (-\infty, a|X|)$,

$$\log\left|\frac{\sqrt{a-\nu|X|^{-1}} + \sqrt{a-s|X|^{-1}}}{\sqrt{a-\nu|X|^{-1}} - \sqrt{a-s|X|^{-1}}}\right| = \log\frac{4a|X|}{|s-\nu|} + O(1) + O\left(\log(s|X|^{-1})\right). \tag{2.4.41}$$

It follows that as $XT^{-1}\log^{-2}|X| \to -\infty$

$$\widehat{L}^{\sigma}_{X,T}(\nu,\nu) = \frac{|X|^{\frac12}}{\pi\, T^{\frac12}} F(\nu)\left[1 - \frac{c_+T^{\frac12}}{\pi|X|^{\frac12}} + O\left(|X|^{-\frac12}T^{\frac12}\log|X|\right)\right]. \tag{2.4.42}$$

To achieve this bound, we used

$$\int_{-\infty}^{a|X|} (\log F)''(s)\mathrm{d}s = \int_{-\infty}^{+\infty} (\log F)''(s)\mathrm{d}s + o(1) = c_+ + o(1) = O(1) \quad (2.4.43)$$

(note that, by Assumption B,

$$\int_{-\infty}^{+\infty} (\log F)''(s)\mathrm{d}s = \lim_{s\to+\infty} (\log F)'(s) - \lim_{s\to-\infty} (\log F)'(s) = c_+$$

) and

$$\int_{-\infty}^{a|X|} (\log F)''(s) \log \frac{4}{|s-\nu|}\mathrm{d}s = \int_{-\infty}^{+\infty} (\log F)''(s) \log \frac{4}{|s-\nu|}\mathrm{d}s + o(1) = O(1), \quad (2.4.44)$$

which, in turn, follow by the Lebesgue dominated convergence theorem, and a similar bound for $\int_{-\infty}^{a|X|} (\log F)''(s) \log |s|\mathrm{d}s$. Thus we obtain

$$\widehat{L}^{\sigma}_{X,T}(\nu,\nu) = \frac{|X|^{\frac{1}{2}}}{\pi T^{\frac{1}{2}}} F(\nu)\Big[1 + O\left(T^{\frac{1}{2}} |X|^{-\frac{1}{2}} \log |X|\right)\Big],$$

and the thesis follows. □

Remark 2.4.5. *It is straightforward to adapt the above proof in order to obtain asymptotics in the full region $X/T \to -\infty$, slightly larger than the region $\frac{X}{T\log^2|X|} \to -\infty$. Note however that the error term will then no longer be small, and the asymptotic expression contains several terms, see* (2.4.43) *and* (2.4.44)*. For the sake of simplicity, we present the results only as $\frac{X}{T\log^2|X|} \to -\infty$.*

Using the two previous results, we can prove an important decorrelation property: since the matrix $\widehat{L}^{\sigma}_{X,T}(\underline{\nu},\underline{\nu})$ is dominated by its diagonal, the m-point correlation function $\det \widehat{L}^{\sigma}_{X,T}(\underline{\nu},\underline{\nu})$ decomposes at leading order into a product of one-point correlation functions. Similarly, its second logarithmic derivative decomposes at leading order into a sum of rapidly oscillating terms.

Proposition 2.4.6. *Let $T_0 > 0$, $\underline{\nu} \in \mathbb{R}^m$. We have*

$$\det \widehat{L}^{\sigma}_{X,T}(\underline{\nu},\underline{\nu}) = \left(1 + O(|X|^{-\frac{1}{2}}T^{\frac{1}{2}})\right) \prod_{i=1}^{m} \widehat{L}^{\sigma}_{X,T}(\nu_i,\nu_i) \quad (2.4.45)$$

$$\partial_X^2 \log\det \widehat{L}^{\sigma}_{X,T}(\underline{\nu},\underline{\nu}) = \quad (2.4.46)$$

$$\frac{1}{\sqrt{|X|T}} \sum_{i=1}^{m} \cos\left(\frac{4|X|^{\frac{3}{2}}}{3T^{\frac{1}{2}}}(1 + A_{X,T}) - \frac{2|X|^{\frac{1}{2}}}{T^{\frac{1}{2}}}\nu_i(1 + B_{X,T}(\nu_i))\right) + O(|X|^{-1})$$

uniformly for $T \geq T_0$ as $\frac{X}{T\log^2|X|} \to -\infty$, where $A_{X,T}, B_{X,T}(\nu)$ converge to 0 *as $\frac{X}{T\log^2|X|} \to -\infty$.*

Proof. For the ease of notation, let us denote $\widehat{L} = (\widehat{L}_{ij})$ for the $m \times m$ matrix with entries $\widehat{L}_{ij} := \widehat{L}^{\sigma}_{X,T}(\nu_i, \nu_j)$. By Lemma 2.4.3 and Lemma 2.4.4 we have

$$\widehat{L}_{ij} = \widehat{L}_{ii}\big(\delta_{ij} + O(|X|^{-\frac{1}{2}}T^{\frac{1}{2}})\big), \qquad 1 \le i,j \le m. \tag{2.4.47}$$

Taking determinants we get (2.4.45). Moreover, (2.4.47) also implies

$$(\widehat{L}^{-1})_{ij} = \frac{1}{\widehat{L}_{ii}}\big(\delta_{ij} + O(|X|^{-\frac{1}{2}}T^{\frac{1}{2}})\big) = \frac{\delta_{ij}}{\widehat{L}_{ii}} + O(|X|^{-1}T), \qquad 1 \le i,j \le m. \tag{2.4.48}$$

(In the second equality we use again Lemma 2.4.4.) By a direct computation, we have

$$\partial_X^2 \log\det\widehat{L} = \sum_{i,j=1}^{m} (\partial_X^2 \widehat{L}_{ij})(\widehat{L}^{-1})_{ji} - \sum_{i,j,k,l=1}^{m} (\partial_X \widehat{L}_{ij})(\widehat{L}^{-1})_{jk}(\partial_X \widehat{L}_{kl})(\widehat{L}^{-1})_{li}. \tag{2.4.49}$$

We have the relation $T^{\frac{1}{2}}\partial_X \widehat{L}_{ij} = -\widehat{\varphi}_i\widehat{\varphi}_j$, where we denote $\widehat{\varphi}_i := \widehat{\varphi}_\sigma(\nu_i; X, T|\emptyset)$. This is the analogue, for the full set of cKdV variables X, T, of the relation (2.3.45). As a consequence, $T^{\frac{1}{2}}\partial_X^2 \widehat{L}_{ij} = -(\partial_X\widehat{\varphi}_i)\widehat{\varphi}_j - \widehat{\varphi}_i(\partial_X\widehat{\varphi}_j)$. Hence, by (2.4.31), we have

$$\partial_X \widehat{L}_{ij} = O(|X|^{-\frac{1}{2}}T^{-\frac{1}{2}}), \qquad \partial_X^2 \widehat{L}_{ij} = O(T^{-1}). \tag{2.4.50}$$

Therefore, by Lemma 2.4.4, (2.4.48), and (2.4.50), we have the estimates

$$\begin{aligned} \sum_{i,j=1}^{m} (\partial_X^2 \widehat{L}_{ij})(\widehat{L}^{-1})_{ji} &= \sum_{i=1}^{m} \frac{\partial_X^2 \widehat{L}_{ii}}{\widehat{L}_{ii}} + O(|X|^{-1}), \\ \sum_{i,j,k,l=1}^{m} (\partial_X \widehat{L}_{ij})(\widehat{L}^{-1})_{jk}(\partial_X \widehat{L}_{kl})(\widehat{L}^{-1})_{li} &= O(|X|^{-2}). \end{aligned} \tag{2.4.51}$$

In the last one we combined (2.4.48) and Lemma 2.4.4 to get $(\widehat{L}^{-1})_{ij} = O(|X|^{-\frac{1}{2}}T^{\frac{1}{2}})$. Substituting these estimates into (2.4.49), we obtain

$$\begin{aligned} \partial_X^2 \log\det\widehat{L} &= -\frac{2}{\sqrt{T}}\sum_{i=1}^{m}\frac{1}{\widehat{L}_{ii}}\widehat{\varphi}_i\partial_X\widehat{\varphi}_i + O(|X|^{-1}) \\ &= -\frac{1}{\pi T}\sum_{i=1}^{m}\frac{1}{\widehat{L}_{ii}}\left(\widehat{\Theta}(\nu_i)\mathrm{E}_{12}\widehat{\Theta}(\nu_i)^{-1}\right)_{2,2} + O(|X|^{-1}), \end{aligned} \tag{2.4.52}$$

where the elementary unit matrix E_{12} is defined in (2.3.47), and where we used (2.3.100).

Using (2.4.20), we can write after straightforward computations

$$\left(\widehat{\Theta}(\nu)\mathrm{E}_{12}\widehat{\Theta}(\nu)^{-1}\right)_{2,2} = \mathbf{v}B\mathbf{w}, \tag{2.4.53}$$

where, writing $\chi := |X|^{\frac{3}{2}}T^{-\frac{1}{2}}$ as before,

$$\mathbf{v} = \big(0, 1\big)\, R(\tfrac{\nu}{|X|}), \qquad \mathbf{w} = R(\tfrac{\nu}{|X|})^{-1}\begin{pmatrix} i|X|^{-\frac{1}{2}}\widehat{p} + i\chi g_1 \\ 1 \end{pmatrix}, \tag{2.4.54}$$

and,

$$B = \frac{\mathrm{e}^{2\chi(g_+(\frac{\nu}{|X|})-g_0)}}{i}(\tfrac{\nu}{|X|}-a)^{\frac{\sigma_3}{4}} G \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-\chi\phi_+(\frac{\nu}{|X|})} & 1 \end{pmatrix} \times \\ E_{12} \begin{pmatrix} 1 & 0 \\ -\mathrm{e}^{-\chi\phi_+(\frac{\nu}{|X|})} & 1 \end{pmatrix} G^{-1}(\tfrac{\nu}{|X|}-a)^{-\frac{\sigma_3}{4}}. \tag{2.4.55}$$

Using (2.4.22) and the asymptotic estimate (2.4.24), we obtain

$$\mathbf{v} = \left(O(\chi^{-1}), 1+O(\chi^{-1})\right), \qquad \mathbf{w} = \begin{pmatrix} O(\chi^{-1}) \\ 1+O(\chi^{-1}) \end{pmatrix}, \qquad \text{as } X/T \to -\infty, \tag{2.4.56}$$

and in particular as $\frac{X}{T\log^2|X|} \to -\infty$. By (2.4.21), we can simplify the expression for B and obtain

$$\begin{aligned} B &= -iF(\nu)(\tfrac{\nu}{|X|}-a)^{\frac{\sigma_3}{4}} G \begin{pmatrix} -1 & \mathrm{e}^{\chi\phi_+(\frac{\nu}{|X|})} \\ -\mathrm{e}^{-\chi\phi_+(\nu/|X|)} & 1 \end{pmatrix} G^{-1}(\tfrac{\nu}{|X|}-a)^{-\frac{\sigma_3}{4}} \\ &= \frac{F(\nu)}{2}(\tfrac{\nu}{|X|}-a)^{\frac{\sigma_3}{4}} \times \\ &\begin{pmatrix} \mathrm{e}^{\chi\phi_+(\frac{\nu}{|X|})}+\mathrm{e}^{-\chi\phi_+(\frac{\nu}{|X|})} & -2-i\mathrm{e}^{\chi\phi_+(\frac{\nu}{|X|})}+i\mathrm{e}^{-\chi\phi_+(\frac{\nu}{|X|})} \\ 2-i\mathrm{e}^{\chi\phi_+(\frac{\nu}{|X|})}+i\mathrm{e}^{-\chi\phi_+(\frac{\nu}{|X|})} & -\mathrm{e}^{\chi\phi_+(\frac{\nu}{|X|})}-\mathrm{e}^{-\chi\phi_+(\frac{\nu}{|X|})} \end{pmatrix} \\ &\times (\tfrac{\nu}{|X|}-a)^{-\frac{\sigma_3}{4}}. \end{aligned} \tag{2.4.57}$$

Since $\phi_+(\frac{\nu}{|X|})$ is purely imaginary and B is bounded and bounded away from 0, we have

$$B = \frac{1}{2}F(\nu)\begin{pmatrix} O(1) & O(1) \\ O(1) & -\mathrm{e}^{\chi\phi_+(\frac{\nu}{|X|})}-\mathrm{e}^{-\chi\phi_+(\frac{\nu}{|X|})} \end{pmatrix}. \tag{2.4.58}$$

Hence

$$\left(\widehat{\Theta}(\nu)\mathrm{E}_{12}\widehat{\Theta}(\nu)^{-1}\right)_{2,2} = \mathbf{v}B\mathbf{w} = -F(\nu)\cos\left(\chi|\phi_+(\tfrac{\nu}{|X|})|\right) + O(\chi^{-1}). \tag{2.4.59}$$

We finally obtain

$$\begin{aligned} \partial_X^2 \log\det\widehat{L} &= \frac{1}{\pi T}\sum_{i=1}^m \frac{F(\nu_i)}{\hat{L}_{ii}}\cos\left(\chi|\phi_+(\tfrac{\nu_i}{|X|})|\right) + O(|X|^{-1}) \\ &= \frac{1}{\sqrt{|X|T}}\sum_{i=1}^m \cos\left(|X|^{\frac{3}{2}}T^{-\frac{1}{2}}|\phi_+(\tfrac{\nu_i}{|X|})|\right) + O(|X|^{-1}), \end{aligned} \tag{2.4.60}$$

where we used Lemma 2.4.4. It remains to compute

$$|\phi_+(\tfrac{\nu_i}{|X|})| = \int_{\nu_i/|X|}^a |(g_+-g_-)'(s)|\mathrm{d}s. \tag{2.4.61}$$

For this, we recall (2.4.38) and the estimates below that equation; in the same way as in the proof of Lemma 2.4.4, we then obtain

$$|\phi_+(0)| \to \frac{4}{3}, \qquad |\phi_+(\tfrac{\nu_i}{|X|})| - |\phi_+(0)| \sim -2\frac{\nu_i}{|X|}, \tag{2.4.62}$$

as $\frac{X}{T\log^2|X|} \to -\infty$. The argument of the cosine is thus equal to

$$\frac{4|X|^{\frac{3}{2}}}{3\sqrt{T}}(1+A_{X,T}) - \frac{2\sqrt{|X|}}{\sqrt{T}}\nu_i(1+B_{X,T}(\nu_i)), \tag{2.4.63}$$

with $A_{X,T} \to 0$, $B_{X,T}(\nu_i) \to 0$ as $\frac{X}{T\log^2|X|} \to -\infty$, and the result follows. $\square$

Combining the above result with (2.4.4), and then substituting the asymptotics from Lemma 2.4.4, we complete the proof of part (ii) of Theorem IV.

With some more effort, we could obtain asymptotics in the slightly bigger asymptotic region where $X/T \to -\infty$, as already mentioned in Remark 2.4.5.

2.4.4 Intermediate regimes: $-KT \leq X \leq MT^{\frac{1}{3}}$

We will now discuss the asymptotic behaviour as $T \to \infty$ of various relevant quantities in the intermediate regimes where $-KT \leq X \leq MT^{\frac{1}{3}}$ for sufficiently large constants $K, M > 0$. The asymptotics for the Jánossy density $J_\sigma(X,T|\underline{\nu})$ and the cKdV solution $V_\sigma(X,T|\underline{\nu})$ become, unfortunately, rather involved and implicit. In order to understand the mechanisms behind these asymptotics, an interesting and relevant object to consider, is the kernel $\widehat{L}^\sigma_{X,T}(\nu_1,\nu_2)$. Indeed, in view of the factorization (2.4.4), determinants of this kernel describe the effect of the points $\nu_1,\dots,\nu_m$ on the Jánossy densities $J_\sigma(X,T|\underline{\nu})$, and the second logarithmic X-derivative of such determinants describe the effect of the points $\nu_1,\dots,\nu_m$ on the cKdV solutions $V_\sigma(X,T|\underline{\nu})$. Recall that the kernel $\widehat{L}^\sigma_{X,T}(\nu_1,\nu_2)$ is expressed in terms of the RH solution $\widehat{\Theta}$ through (2.4.7). We distinguish three further asymptotic regimes.

Left-intermediate regime: $-KT \leq X \leq -K'T^{\frac{1}{2}}$ for any $K,K' > 0$. The asymptotic analysis of the RH problem for $\widehat{\Theta}$ has been carried through in [56] and is very similar to the one utilized for the left tail. However, there is an important difference in that the decorrelation property from Proposition 2.4.6 no longer holds. For that reason, even if we could obtain asymptotics for $\widehat{L}^\sigma_{X,T}(\nu_1,\nu_2)$, the explicit asymptotic behaviour of the determinants $\det \widehat{L}^\sigma_{X,T}(\underline{\nu},\underline{\nu})$ and their logarithmic derivatives becomes cumbersome for $m > 1$.

Right-intermediate regime: $-MT^{\frac{1}{3}} \leq X \leq MT^{\frac{1}{3}}$ for any $M > 0$. In this case, it was proved in [48, Theorem 1.15] that there exists a (sufficiently large) $T_0 > 0$ such that for all $M > 0$ we have

$$\log J_\sigma(X,T|\emptyset) = \log F_{\mathrm{TW}}(XT^{-\frac{1}{3}}) + O(T^{-\frac{1}{6}}), \tag{2.4.64}$$

$$V_\sigma(X,T|\emptyset) = -T^{-\frac{2}{3}} y_{\mathrm{HM}}\big(XT^{-\frac{1}{3}}\big)^2 + O(T^{-1}), \tag{2.4.65}$$

uniformly for $|X| \le MT^{\frac{1}{3}}$ and $T \ge T_0$, where y_{HM} is the Hastings–McLeod solution of the Painlevé II equation, and F_{TW} is the Tracy-Widom distribution (see also Example 2.1.2).

The asymptotic analysis of $\widehat{\Theta}$ has also been obtained in [48], and it implies that the leading order asymptotics of $\widehat{L}^{\sigma}_{X,T}(\lambda,\mu)$ are determined by the soft-to-hard edge transition kernel $L_s^{1_{(0,\infty)}}$ from Example 2.1.4, as we prove next.

Proposition 2.4.7. *Let $M > 0$. As $T \to \infty$, we have uniformly for $-MT^{\frac{1}{3}} \le X \le MT^{\frac{1}{3}}$, and uniformly for ν_1, ν_2 in compact subsets of the real line that*

$$\widehat{L}^{\sigma}_{X,T}(\nu_1,\nu_2) = T^{-\frac{1}{3}} L^{1_{(0,+\infty)}}_{s=XT^{-\frac{1}{3}}}(T^{-\frac{1}{3}}\nu_1, T^{-\frac{1}{3}}\nu_2) + O(T^{-\frac{2}{3}}), \tag{2.4.66}$$

and this error estimate continues to hold upon differentiating an arbitrary number of times with respect to ν_1 and ν_2.

Proof. The proof relies on the RH analysis performed in [48, Section 6]: the result is that, for every fixed $\nu \in \mathbb{C}$, we have the factorization

$$\widehat{\Theta}_\sigma(\nu; X, T|\emptyset) = \begin{pmatrix} 1 & \widehat{p}_\sigma \\ 0 & 1 \end{pmatrix} \mathrm{e}^{\frac{i\pi}{4}\sigma_3} T^{\frac{1}{12}\sigma_3} S(\nu T^{-\frac{1}{3}}) \times \\ \Psi_{1_{(0,+\infty)}}(\nu T^{-\frac{1}{3}}; XT^{-\frac{1}{3}}|\emptyset) \begin{pmatrix} 1 & a(\nu T^{-\frac{1}{3}}) \\ 0 & 1 \end{pmatrix} \mathrm{e}^{-\frac{i\pi}{4}\sigma_3}, \tag{2.4.67}$$

where $\widehat{p}_\sigma = \widehat{p}_\sigma(X,T|\emptyset)$, a has an explicit expression which is not needed for our purposes (cf. [48, equation (6.10)]). The matrix S should be interpreted as an error term: it satisfies a *small-norm* RH problem, which means that, provided $|w| < 1$, $S(w) = I + O(T^{-\frac{1}{3}})$ and $\partial_w S(w) = O(T^{-\frac{1}{3}})$, uniformly for $-MT^{\frac{1}{3}} \le X \le MT^{\frac{1}{3}}$ (for any $M > 0$). In particular, for every fixed $\nu_1, \nu_2 \in \mathbb{R}$ and T sufficiently large we also have

$$S(\nu_2 T^{-\frac{1}{3}})^{-1} S(\nu_1 T^{-\frac{1}{3}}) = I + O(T^{-\frac{2}{3}}(\nu_1 - \nu_2)). \tag{2.4.68}$$

Combining this with (2.4.7) and (2.4.67), we get

$$\begin{aligned} &\widehat{L}^{\sigma}_{X,T}(\nu_1,\nu_2) = \\ &= T^{-\frac{1}{3}} L^{1_{(0,+\infty)}}_{s=XT^{-\frac{1}{3}}}(T^{-\frac{1}{3}}\nu_1, T^{-\frac{1}{3}}\nu_2) \\ &+ \frac{\left(\Psi_{1_{(0,+\infty)}}(\nu_2 T^{-\frac{1}{3}}; XT^{-\frac{1}{3}}|\emptyset)^{-1} O(T^{-\frac{2}{3}}) \Psi_{1_{(0,+\infty)}}(\nu_1 T^{-\frac{1}{3}}; XT^{-\frac{1}{3}}|\emptyset)\right)_{2,1}}{2\pi i}, \end{aligned} \tag{2.4.69}$$

where we also use the identity

$$L_s^{1_{(0,+\infty)}}\left(T^{-\frac{1}{3}}\nu_1, T^{-\frac{1}{3}}\nu_2\right) = \\ \frac{\left(\Psi_{1_{(0,+\infty)}}(\nu_2 T^{-\frac{1}{3}}; s|\emptyset)^{-1} \Psi_{1_{(0,+\infty)}}(\nu_1 T^{-\frac{1}{3}}; s|\emptyset)\right)_{2,1}}{2\pi i T^{-\frac{1}{3}}(\nu_1 - \nu_2)}, \tag{2.4.70}$$

which is a special case of (2.3.21). The last term is $O(T^{-\frac{2}{3}})$ since it is a combination of the entries of the first column of $\Psi_{1_{(0,+\infty)}}(\nu_i T^{-\frac{1}{3}})$, $i = 1, 2$, which are both entire. The above identities extend to ν_1, ν_2 in compact subsets of the complex plane, hence we can apply Cauchy's formula to differentiate, without affecting the error term. $\square$

A first, crucial, obstruction for obtaining explicit asymptotics for the Jánossy densities $J_\sigma(X,T|\underline{\nu})$ lies in the fact that the kernel $L^{1_{(0,+\infty)}}_{s=XT^{-\frac{1}{3}}}$ is itself a transcendental object, which we cannot evaluate explicitly. However, we can proceed in the hope of describing $J_\sigma(X,T|\underline{\nu})$ asymptotically in terms of the σ-independent quantity $L^{1_{(0,+\infty)}}_{s=XT^{-\frac{1}{3}}}$. For $m = 1$, we immediately find by (2.4.4) that

$$J_\sigma(X,T|\nu) \sim T^{-\frac{1}{3}} L^{1_{(0,+\infty)}}_{s=XT^{-\frac{1}{3}}}(0,0) J_\sigma(X,T|\emptyset), \tag{2.4.71}$$

where the asymptotics for $J_\sigma(X,T|\emptyset)$ are given by (2.4.64). For $m > 1$, we can estimate $J_\sigma(X,T|\underline{\nu})$ as follows.

Proposition 2.4.8. *Let $M > 0$, $\underline{\nu} = (\nu_1, \dots, \nu_m) \in \mathbb{R}^m$, and $\nu_i \neq \nu_j$ for $i \neq j$. As $T \to \infty$, we have uniformly for $-MT^{\frac{1}{3}} \le X \le MT^{\frac{1}{3}}$ that*

$$J_\sigma(X,T|\underline{\nu}) \sim C_m T^{-\frac{m^2}{3}} \prod_{1 \le j < k \le m} (\nu_k - \nu_j)^2 J_{1_{(0,\infty)}}(X,T|\emptyset), \tag{2.4.72}$$

for some constant $C_m > 0$ possibly depending on m but not on $\sigma, \underline{\nu}, X, T$.

Proof. Let us abbreviate $L = \widehat{L}^\sigma_{X,T}$ and $\widetilde{L} = L^{1_{(0,+\infty)}}_{s=XT^{-\frac{1}{3}}}$. By (2.4.4), we have

$$J_\sigma(X,T|\underline{\nu}) = \det\left(L(\nu_i,\nu_j)\right)_{i,j=1}^m J_\sigma(X,T|\emptyset), \tag{2.4.73}$$

hence by (2.4.64), it remains to prove that

$$\det\left(L(\nu_i,\nu_j)\right)_{i,j=1}^m \sim C_m T^{-\frac{m^2}{3}} \prod_{1 \le j < k \le m} (\nu_k - \nu_j)^2, \tag{2.4.74}$$

in the relevant limit. Since $L(\cdot,\cdot)$ is entire in its variables, we have

$$L(\nu_i,\nu_j) = \sum_{a,b \ge 0} L^{(a,b)}(0,0) \frac{\nu_i^a \nu_j^b}{a!b!} \tag{2.4.75}$$

hence

$$
\begin{pmatrix}
L(\nu_1,\nu_1) & L(\nu_1,\nu_2) & \cdots & L(\nu_1,\nu_m) \\
L(\nu_2,\nu_1) & L(\nu_2,\nu_2) & \cdots & L(\nu_2,\nu_m) \\
\vdots & \vdots & \ddots & \vdots \\
L(\nu_m,\nu_1) & L(\nu_m,\nu_2) & \cdots & L(\nu_m,\nu_m)
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1 & \nu_1 & \frac{\nu_1^2}{2!} & \cdots \\
1 & \nu_2 & \frac{\nu_2^2}{2!} & \cdots \\
\vdots & \vdots & \cdots & \cdots \\
1 & \nu_m & \frac{\nu_m^2}{2!} & \cdots
\end{pmatrix}
\begin{pmatrix}
L^{(0,0)}(0,0) & L^{(1,0)}(0,0) & L^{(2,0)}(0,0) & \cdots \\
L^{(0,1)}(0,0) & L^{(1,1)}(0,0) & L^{(2,1)}(0,0) & \cdots \\
L^{(0,2)}(0,0) & L^{(1,2)}(0,0) & L^{(2,2)}(0,0) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \times
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\nu_1 & \nu_2 & \cdots & \nu_m \\
\frac{\nu_1^2}{2!} & \frac{\nu_2^2}{2!} & \cdots & \frac{\nu_m^2}{2!} \\
\vdots & \vdots & \cdots & \cdots
\end{pmatrix}. \tag{2.4.76}
$$

Then we use Proposition 2.4.7 to obtain

$$
L^{(i-1,j-1)}(0,0) \sim T^{-\frac{i+j-1}{3}} \widetilde{L}^{(i-1,j-1)}(0,0), \quad \text{as } T \to \infty,\ |XT^{-\frac{1}{3}}| \leq M. \tag{2.4.77}
$$

Expanding (2.4.76) by the Binet–Cauchy identity, we immediately see that the leading order as $T \to \infty$ is given by

$$
\det_{i,j=1}^{m} \left(\frac{\nu_i^{j-1}}{(j-1)!} \right) \det_{i,j=1}^{m} \left(T^{-\frac{i+j-1}{3}} L^{(i-1,j-1)}(0,0) \right) \det_{i,j=1}^{m} \left(\frac{\nu_i^{j-1}}{(j-1)!} \right)
$$

$$
= T^{-\frac{m^2}{3}} \prod_{k=1}^{m-1} \frac{1}{k!^2} \det_{i,j=1}^{m} \left(\widetilde{L}^{(i-1,j-1)}(0,0) \right) \det_{i,j=1}^{m} \left(\nu_i^{j-1} \right)^2. \tag{2.4.78}
$$

In the latter, we recognize the Vandermonde determinant, and the result follows. □

Middle-intermediate regime: $-K'T^{\frac{1}{2}} \leq X \leq -MT^{\frac{1}{3}}$ for some $K', M > 0$. Here, the asymptotic analysis of the RH problem for $\widehat{\Theta}$ has also been completed in [48], but the asymptotics are implicit and described in terms of the solution of an integro-differential generalisation of the fifth Painlevé equation.

Chapter 3

Asymptotics for Averages over Classical Orthogonal Ensembles

This chapter retakes my first article [59] in collaboration with Tom Claeys, Alexander Minakov and Meng Yang. This is a stand-alone part as it came before the main topic of this thesis. I obtained a generalisation of known factorization identities expressing Toeplitz determinants in terms of Toeplitz+Hankel determinants, but involving in addition the associated orthogonal polynomials on the unit circle (OPUC). I was then able to write averages over the Classical Orthogonal Ensembles in terms of Toeplitz determinants and OPUC. This is especially powerful as Toeplitz determinants are well-studied, mainly using the famous Fokas-Its-Kitaev Riemann-Hilbert problem, and the latter naturally involve the appropriate OPUC. This allowed my collaborators to compute asymptotics for interesting symbols, and I then used those results to derive asymptotics for gap probabilities, moment generating functions of occupancy number and global concentration inequalities.

Abstract

We study averages of multiplicative eigenvalue statistics in ensembles of orthogonal Haar distributed matrices, which can alternatively be written as Toeplitz+Hankel determinants. We obtain new asymptotics for symbols with Fisher-Hartwig singularities in cases where some of the singularities merge together, and for symbols with a gap or an emerging gap. We obtain these asymptotics by relying on known analogous results in the unitary group and on asymptotics for associated orthogonal polynomials on the unit circle. As consequences of our results, we derive asymptotics for gap probabilities in the Circular Orthogonal and Symplectic Ensembles, and an upper bound for the global eigenvalue rigidity in the orthogonal ensembles.

3.1 Introduction

Consider the classical orthogonal group $\mathbb{O}_N$ of $N \times N$ orthogonal matrices equipped with the Haar measure, and its components $\mathbb{O}_N^{\pm}$ of $N \times N$ orthogonal matrices with determinant equal to ± 1. If N is even, the eigenvalues of a matrix $M \in \mathbb{O}_N^+ = \mathbb{O}_{2n}^+$ come in complex conjugate pairs $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_n}$ with $\theta_1, \dots, \theta_n \in [0, \pi]$, while a matrix $M \in \mathbb{O}_N^- = \mathbb{O}_{2n+2}^-$ has complex conjugate pairs of eigenvalues $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_n}$ with $\theta_1, \dots, \theta_n \in [0, \pi]$, and fixed eigenvalues -1 and $+1$. If $N = 2n+1$ is odd, a matrix $M \in \mathbb{O}_N^{\pm}$ has complex conjugate pairs of eigenvalues $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_n}$ with $\theta_1, \dots, \theta_n \in [0, \pi]$ complemented by the fixed eigenvalue ± 1. Due to Weyl's integration formula, the joint probability distributions of the free eigenangles $\theta_1, \dots, \theta_n \in [0, \pi]$ are given by (see e.g. [81, p71–72], [123, p76] and [106])

$$
\begin{aligned}
\mathbb{O}_{2n}^+ :& \quad \frac{2}{n!(2\pi)^n} \prod_{1 \le j < k \le n} (2\cos\theta_k - 2\cos\theta_j)^2 \prod_{j=1}^n d\theta_j, \\
\mathbb{O}_{2n+2}^- :& \quad \frac{1}{n!(2\pi)^n} \prod_{j=1}^n (2\sin\theta_j)^2 \prod_{1 \le j < k \le n} (2\cos\theta_k - 2\cos\theta_j)^2 \prod_{j=1}^n d\theta_j, \\
\mathbb{O}_{2n+1}^{\pm} :& \quad \frac{1}{n!(2\pi)^n} \prod_{j=1}^n 2(1 \mp \cos\theta_j) \prod_{1 \le j < k \le n} (2\cos\theta_k - 2\cos\theta_j)^2 \prod_{j=1}^n d\theta_j.
\end{aligned} \tag{3.1.1}
$$

We also mention that the joint probability distribution of the free eigenangles of a symplectic matrix $U \in \mathbb{S}\mathrm{p}_{2n}$ distributed with respect to Haar measure is the same as $\mathbb{O}_{2n+2}^-$ [123, Theorem 3.5]. Our results thus cover all the cases of the classical groups $\mathbb{SO}_n$ and $\mathbb{S}\mathrm{p}_{2n}$ equipped with Haar measure. In all three above cases, there are n *free variables* $\theta_1, \dots, \theta_n$. We are interested in large n asymptotics for multiplicative averages of the form

$$
\mathbb{E}_n^{(j,\pm)}[f] := \mathbb{E}_{\mathbb{O}_{2n+j}^{\pm}} \prod_{k=1}^n f(e^{i\theta_k}) f(e^{-i\theta_k}), \tag{3.1.2}
$$

where f is an integrable function on the unit circle which we will call the *symbol*, and $\mathbb{E}_{\mathbb{O}_N^{\pm}}$ denotes the average with respect to (3.1.1). In the notation at the left hand side, j is the number of fixed eigenvalues, n the number of free eigenangles, and ± 1 the determinant of the random matrix M. The 4 admissible values for the pair $(j, \pm)$ are $(0, +), (2, -), (1, +)$, and $(1, -)$.

It is well understood that such averages can be written as determinants of matrices of Toeplitz+Hankel type [10, 81]. These determinants can in turn be expressed either in terms of Hankel determinants with Jacobi-type weights depending on f, or in terms of Toeplitz determinants and orthogonal polynomials on the unit circle with symbols depending on f, see [69].

Identities relating orthogonal and unitary ensembles. Our approach will rely on a variant of such existing identities, which is particularly convenient for asymptotic analysis, and which allows us to write averages over orthogonal ensembles of a symbol f in terms of averages over the unitary group $\mathbb{U}_N$ of Haar distributed $N \times N$ unitary matrices for the symbol

$$g(e^{it}) := f(e^{it})f(e^{-it}) \tag{3.1.3}$$

and related orthogonal polynomials on the unit circle evaluated at ± 1. Before stating these identities, let us recall that the eigenvalues $e^{i\varphi_1}, \dots, e^{i\varphi_N}$, with $\varphi_1, \dots, \varphi_N \in [0, 2\pi)$, of a Haar-distributed matrix U from the unitary group $\mathbb{U}_N$ of $N \times N$ unitary matrices, often referred to as the Circular Unitary Ensemble (CUE), have the joint probability distribution

$$\mathbb{U}_N: \quad \frac{1}{(2\pi)^N N!} \prod_{1 \le k < j \le N} |e^{i\varphi_j} - e^{i\varphi_k}|^2 \prod_{j=1}^{N} d\varphi_j. \tag{3.1.4}$$

Moreover, averages

$$\mathbb{E}_N^{\mathbb{U}}[g] := \mathbb{E}_{\mathbb{U}_N} \det g(U) = \mathbb{E}_{\mathbb{U}_N} \prod_{j=1}^{N} g(e^{i\varphi_j}), \tag{3.1.5}$$

where g is a non-negative integrable function on the unit circle and where $\mathbb{E}_{\mathbb{U}_N}$ denotes the average over the unitary group $\mathbb{U}_N$, can be written via Heine's identity as Toeplitz determinants: we have

$$\mathbb{E}_N^{\mathbb{U}}[g] = \det \left(g_{j-k}\right)_{j,k=0}^{N-1}, \tag{3.1.6}$$

where g_m is the m-th Fourier coefficient of g,

$$g_m = \frac{1}{2\pi} \int_0^{2\pi} g(e^{it}) e^{-imt} dt. \tag{3.1.7}$$

We also need the monic orthogonal polynomials Φ_N of degree N on the unit circle with respect to an integrable weight function $g(e^{it}) \ge 0$, characterized by the conditions

$$\int_0^{2\pi} \Phi_N(e^{it}) e^{-ikt} g(e^{it}) dt = 0 \quad \text{for any integer } 0 \le k < N. \tag{3.1.8}$$

These polynomials can also be written as determinants

$$\Phi_N(z) = \frac{\det \begin{pmatrix} g_{j-k} & z^j \end{pmatrix}_{j,k=0}^{N,N-1}}{\det(g_{j-k})_{j,k=0}^{N-1}} = \frac{\det \begin{pmatrix} g_0 & g_{-1} & \cdots & g_{-N+1} & 1 \\ g_1 & g_0 & \cdots & g_{-N+2} & z \\ g_2 & g_1 & \cdots & g_{-N+3} & z^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_N & g_{N-1} & \cdots & g_1 & z^N \end{pmatrix}}{\det(g_{j-k})_{j,k=0}^{N-1}}. \tag{3.1.9}$$

In the next result, we express averages over the orthogonal ensembles in terms of averages over the unitary group and orthogonal polynomials, and this will be the starting point of our asymptotic analysis later on.

Proposition 3.1.1. *Let f be a function on the unit circle which is such that g defined by (3.1.3) is non-negative and integrable on $[0,2\pi]$. Let Φ_k be the degree k monic orthogonal polynomial on the unit circle with respect to the weight $g(e^{it})$. Then for all positive integers n,*

$$\begin{aligned} \mathbb{E}_n^{(0,+)}[f] &= \left[\frac{\mathbb{E}_{2n}^{\mathbb{U}}[g]}{-\Phi_{2n-1}(1)\Phi_{2n-1}(-1)}\right]^{1/2}, \\ \mathbb{E}_n^{(2,-)}[f] &= \left[\Phi_{2n}(1)\Phi_{2n}(-1)\mathbb{E}_{2n}^{\mathbb{U}}[g]\right]^{1/2}, \\ \mathbb{E}_n^{(1,\pm)}[f] &= \left[\frac{\Phi_{2n}(\pm 1)}{\Phi_{2n}(\mp 1)}\mathbb{E}_{2n}^{\mathbb{U}}[g]\right]^{1/2}. \end{aligned} \tag{3.1.10}$$

Asymptotics for averages in orthogonal ensembles. There is a vast literature on asymptotics for Toeplitz determinants, and large N asymptotics for (3.1.5)–(3.1.6) are well understood for large classes of complex-valued symbols g. The most classical result in this context is Szegő's strong limit theorem, which states that [136, 96, 103] with $g(e^{it}) = e^{V(e^{it})}$ and V sufficiently smooth on the unit circle, as $N \to \infty$,

$$\det\left(g_{j-k}\right)_{j,k=0}^{N-1} = e^{NV_0} e^{\sum_{k=1}^{\infty} kV_kV_{-k}}(1+o(1)) \text{ with } V_k = \frac{1}{2\pi}\int_0^{2\pi} V(e^{it})e^{-ikt}dt. \tag{3.1.11}$$

More precisely, this holds for any V such that $\sum_{k=1}^{\infty} k|V_k|^2 < \infty$. More general results allow for symbols which vanish on an arc of the unit circle [140] or for the presence of Fisher-Hartwig singularities, which are combinations of root-type singularities with jump discontinuities. Such symbols have a long history [12, 13, 17, 39, 76, 78, 118, 140], and asymptotics for the associated Toeplitz determinants are now completely understood in the large N limit, as long as the symbol does not depend on N [69]. In cases where the symbol depends on N, various interesting transitions in the large N asymptotics can take place, such as the emergence of a Fisher-Hartwig singularity [142, 61], the emergence of an arc of vanishing [53, 54], or the merging of Fisher-Hartwig singularities [62, 77].

Large N asymptotics for the analogues in the orthogonal ensembles $\mathbb{O}_N^{\pm}$, namely (3.1.2), are also known for fixed symbols (i.e. independent of N) with Fisher-Hartwig singularities, see [69, Theorem 1.25] for the most complete result in this respect and [10, 14, 15, 16] for earlier developments. However, the picture for averages in $\mathbb{O}_N^{\pm}$ is incomplete because, as far as we know, asymptotics are not known for symbols vanishing on an arc, and no results are available about transition asymptotics in situations where either several singularities approach each other in the large N limit (except for the results from [80] obtained simultaneously with ours, see Remark 3.2.4 below), or parameters are tuned in such a way that a gap in the support emerges as $N \to \infty$. The objective in this paper is to complete this task. In order to avoid technical and notational complications, we restrict ourselves to non-negative real-valued symbols g, although some of the results could be generalised to complex-valued symbols.

Outline for the rest of the paper. After stating our main results in Section 3.2, we will prove Proposition 3.1.1 in Section 3.3. In Section 3.4, we will analyse orthogonal polynomials on the unit circle for symbols with Fisher-Hartwig singularities, which possibly merge in the large degree limit, and this will allow us to prove Theorem 3.2.1 and Theorem 3.2.2 below. In Section 3.5, we will analyse the case of symbols with a gap or an emerging gap, and this will lead us to the proof of Theorem 3.2.5. In Section 3.6, we will study gap probabilities and global rigidity of eigenvalues in $\mathbb{O}_n^{(j,\pm)}$ and prove Theorem 3.2.12.

3.2 Statement of results

3.2.1 Symbols with Fisher-Hartwig singularities

Let V be an analytic function in a neighbourhood of the unit circle which is real-valued on the unit circle and such that $V(e^{it}) = V(e^{-it})$, and let $0 < t_1 < \ldots < t_m < \pi$, with $m \in \mathbb{N}$. For any $j = 0, 1, \ldots, m, m+1$, we have parameters $\alpha_j \geq 0$ and for any $j = 1, \ldots, m$ we have parameters $\beta_j \in i\mathbb{R}$. We will consider symbols f such that g given by (3.1.3) is of the form

$$
\begin{aligned}
g(e^{it}) &= e^{V(e^{it})}|e^{it}-1|^{2\alpha_0}|e^{it}+1|^{2\alpha_{m+1}} \\
&\times \prod_{j=1}^{m} \left(\frac{e^{it}}{e^{i(\pi+t_j)}}\right)^{\beta_j} \left(\frac{e^{-it}}{e^{i(\pi+t_j)}}\right)^{\beta_j} \left|e^{it}-e^{it_j}\right|^{2\alpha_j} \left|e^{it}-e^{-it_j}\right|^{2\alpha_j}, \quad (3.2.1)
\end{aligned}
$$

where $z^{\beta} = |z|^{\beta} e^{i\beta \arg z}$ with $-\pi < \arg z \leq \pi$. This is one of the standard forms of a positive symbol with Fisher-Hartwig singularities, symmetric with respect to the real line and having singularities at the points $e^{\pm it_j}$, $j = 1, \ldots, m$, and at the points ± 1. These singularities are combinations of jump and root singularities whose nature depends on the parameters α_j, β_j. For instance, if we set $m = 1$, $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $V \equiv 0$, then g is piece-wise constant: $g(e^{it}) = e^{-2it_1\beta_1}$ for $|t| > t_1$ and $g(e^{it}) = e^{-2it_1\beta_1} e^{2i\pi\beta_1}$ for $|t| < t_1$. Note that the symmetry with respect to the real line excludes the possibility of having jump singularities (with non-zero parameters β_0, β_{m+1}) at ± 1.

If $V, m, t_j, \alpha_j, \beta_j$ are independent of N, large N asymptotics for $\mathbb{E}_N^{\mathbb{U}}[g] = \det (g_{j-k})_{j,k=0}^{n-1}$ were obtained in [69] (in more general situations where the symbol is complex and not necessarily symmetric with respect to the real line, where V is not necessarily analytic, and where $\alpha_j > -1/2$ is allowed to be negative). Translating the results from [76] (see also [69] for a more general result) to our setting, we have

$$
\mathbb{E}_{2n}^{\mathbb{U}}[g] = E^2 e^{2nV_0} (2n)^{\alpha_0^2+\alpha_{m+1}^2+2\sum_{j=1}^{m}(\alpha_j^2-\beta_j^2)} (1+o(1)), \qquad (3.2.2)
$$

as $n \to \infty$, with E given by

$$
\begin{aligned}
E = {} & e^{\frac{1}{2}\sum_{k=1}^{+\infty} kV_k^2} e^{2i\sum_{j=1}^{m}\alpha_j \sum_{k=1}^{m} t_k\beta_k} e^{-2\pi i \sum_{1\le j<k\le m}\alpha_j\beta_k} e^{-\pi i \sum_{j=1}^{m}\alpha_j\beta_j} \\
& \times \prod_{j=1}^{m} \frac{|G(1+\alpha_j+\beta_j)|^2}{G(1+2\alpha_j)} \frac{e^{2i\beta_j \sum_{k=1}^{+\infty} V_k \sin kt_j}}{e^{\alpha_j(V(z_j)-V_0)}} |2\sin t_j|^{-(\alpha_j^2+\beta_j^2)} \\
& \times \prod_{1\le j<k\le m} \left|2\sin\frac{t_j-t_k}{2}\right|^{2(\beta_j\beta_k-\alpha_j\alpha_k)} \left|2\sin\frac{t_j+t_k}{2}\right|^{-2(\beta_j\beta_k+\alpha_j\alpha_k)} \\
& \times 2^{-\alpha_0\alpha_m} e^{i(\alpha_0+\alpha_{m+1})\sum_{j=1}^{m} t_j\beta_j} e^{-\pi i\alpha_0 \sum_{j=1}^{m}\beta_j} \\
& \times \frac{G(1+\alpha_0)}{G(1+2\alpha_0)^{\frac{1}{2}}} e^{-\frac{1}{2}\alpha_0(V(1)-V_0)} \frac{G(1+\alpha_{m+1})}{G(1+2\alpha_{m+1})^{\frac{1}{2}}} e^{-\frac{1}{2}\alpha_{m+1}(V(-1)-V_0)} \\
& \times \prod_{j=1}^{m} \left|2\sin\frac{t_j}{2}\right|^{-2\alpha_0\alpha_j} \left|2\cos\frac{t_j}{2}\right|^{-2\alpha_{m+1}\alpha_j},
\end{aligned}
\tag{3.2.3}
$$

where G is Barnes's G function. It follows from the techniques used in [69] that these asymptotics are valid uniformly for α in compact subsets of $(-1/2, \infty)$, β in compact subsets of $i\mathbb{R}$, and as long as the distance between the singularities $e^{\pm it_j}$ remains bounded from below.

One possible choice of f leading through (3.1.3) to (3.2.1) is the positive square root of g, namely

$$
\begin{aligned}
f(e^{it}) = {} & e^{\frac{1}{2}V(e^{it})}|e^{it}-1|^{\alpha_0}|e^{it}+1|^{\alpha_{m+1}} \\
& \times \prod_{j=1}^{m}\left(\frac{e^{it}}{e^{i(\pi+t_j)}}\right)^{\beta_j/2}\left(\frac{e^{-it}}{e^{i(\pi+t_j)}}\right)^{\beta_j/2} \left|e^{it}-e^{it_j}\right|^{\alpha_j}\left|e^{it}-e^{-it_j}\right|^{\alpha_j}.
\end{aligned}
\tag{3.2.4}
$$

The following result, which we will prove in Section 3.4, describes the large n asymptotics of (3.1.2), in terms of (3.1.5), in the case of a symbol with Fisher-Hartwig singularities, and holds uniformly in the position of the singularities, as long as they do not approach ± 1 too fast as $n \to \infty$.

Theorem 3.2.1. *Let $m \in \mathbb{N}$, $0 < t_1 < \ldots < t_m < \pi$, $\alpha_j \geq 0$ for $j = 0, \ldots, m+1$, $\beta_j \in i\mathbb{R}$ for $j = 1, \ldots, m$, and let V be analytic in a neighbourhood of the unit circle, real-valued on the unit circle and such that $V(e^{it}) = V(e^{-it})$, with Laurent series $V(z) = \sum_{k=-\infty}^{\infty} V_k z^k$ and $V_k = V_{-k} \in \mathbb{R}$. Let f be such that g is of the form (3.2.1). There exists $M > 0$ such that as $n \to \infty$, uniformly in the region $\frac{M}{n} < t_1 < \ldots < t_m < \pi - \frac{M}{n}$, we have*

$$
\begin{aligned}
\mathbb{E}_n^{(0,+)}[f] &= C_n \left(\mathbb{E}_{2n}^{\mathbb{U}}[g]\right)^{1/2}\left(1+\mathcal{O}\left(\frac{1}{n\min\{t_1,\pi-t_m\}}\right)\right), \\
\mathbb{E}_n^{(2,-)}[f] &= C_n^{-1} \left(\mathbb{E}_{2n}^{\mathbb{U}}[g]\right)^{1/2}\left(1+\mathcal{O}\left(\frac{1}{n\min\{t_1,\pi-t_m\}}\right)\right), \\
\mathbb{E}_n^{(1,\pm)}[f] &= \widetilde{C}_n^{\pm 1} \left(\mathbb{E}_{2n}^{\mathbb{U}}[g]\right)^{1/2}\left(1+\mathcal{O}\left(\frac{1}{n\min\{t_1,\pi-t_m\}}\right)\right),
\end{aligned}
\tag{3.2.5}
$$

where

$$C_n = \frac{2^{\alpha_0+\alpha_{m+1}}}{n^{\frac{\alpha_0+\alpha_{m+1}}{2}}\sqrt{\pi}}\Gamma\left(\frac{1}{2}+\alpha_0\right)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+\alpha_{m+1}\right)^{\frac{1}{2}} e^{\frac{1}{4}(V(1)+V(-1)-2V_0)} \times \prod_{j=1}^{m}\left[(2\sin t_j)^{\alpha_j} e^{-i\beta_j t_j} e^{\frac{i\pi}{2}\beta_j}\right],$$

$$\widetilde{C}_n = n^{\frac{\alpha_0-\alpha_{m+1}}{2}}\frac{\Gamma\left(\frac{1}{2}+\alpha_{m+1}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}+\alpha_0\right)^{\frac{1}{2}}} e^{\frac{1}{4}(V(-1)-V(1))}\prod_{j=1}^{m}\left[\left(\tan\frac{t_j}{2}\right)^{-\alpha_j} e^{-\frac{i\pi}{2}\beta_j}\right]. \tag{3.2.6}$$

In the case where m, the positions of the singularities t_j, and the values of the parameters α_j,β_j are independent of n, we can write the above results in a more explicit form by substituting (3.2.2)–(3.2.3). This yields

$$\begin{aligned}
\mathbb{E}_n^{(0,+)}[f] &= C_n E e^{nV_0}(2n)^{(\alpha_0^2+\alpha_{m+1}^2)/2+\sum_{j=1}^m(\alpha_j^2-\beta_j^2)}(1+o(1)),\\
\mathbb{E}_n^{(2,-)}[f] &= C_n^{-1} E e^{nV_0}(2n)^{(\alpha_0^2+\alpha_{m+1}^2)/2+\sum_{j=1}^m(\alpha_j^2-\beta_j^2)}(1+o(1)),\\
\mathbb{E}_n^{(1,\pm)}[f] &= \widetilde{C}_n^{\pm 1} E e^{nV_0}(2n)^{(\alpha_0^2+\alpha_{m+1}^2)/2+\sum_{j=1}^m(\alpha_j^2-\beta_j^2)}(1+o(1)).
\end{aligned} \tag{3.2.7}$$

Here, we recover [69, Theorem 1.25] in the case of a positive symbol f (to see this, one needs to use the doubling formula for Barnes' G-function, see [69, formula (2.39)]).

Let us now consider in more detail the situation where the positions of the Fisher-Hartwig singularities are allowed to vary with n. This includes in particular situations where singularities merge in the large n limit or converge to ±1. For notational convenience, we now set $\alpha_0 = \alpha_{m+1} = 0$ in (3.1.3), but one should note that we can do this without loss of generality because we will now allow $t_1 = 0$ and $t_m = \pi$. Although we expect (3.2.7) to hold whenever the distance between singularities decays slower than $1/n$, the main obstacle to prove this, is that strong asymptotics (including the value of the multiplicative constant) for $\mathbb{E}_{2n}^{\mathbb{U}}[g]$ have not been established, except for $m = 1$ [62], when they are related to the Painlevé V equation. Weak asymptotics, without explicit value for the multiplicative constant, have been obtained in general [77]. The result of [77] translated to our setting is

$$\mathbb{E}_{2n}^{\mathbb{U}}[g] = F^2 e^{2nV_0}(2n)^{\sum_{j=1}^m(2\alpha_j^2-2\beta_j^2)}\prod_{j=1}^{m}\left(\sin t_j+\frac{1}{n}\right)^{-2\alpha_j^2-2\beta_j^2} e^{\mathcal{O}(1)} \tag{3.2.8}$$

as $n\to\infty$, uniformly for $0 < t_1 < \ldots < t_m < \pi$, with

$$F = \prod_{1\le j<k\le m}\left(\frac{1}{\sin\frac{t_k-t_j}{2}+\frac{1}{n}}\right)^{2(\alpha_j\alpha_k-\beta_j\beta_k)}\left(\frac{1}{\sin\frac{t_j+t_k}{2}+\frac{1}{n}}\right)^{2(\alpha_j\alpha_k+\beta_j\beta_k)}. \tag{3.2.9}$$

We can substitute this in (3.2.5) to obtain weak large n asymptotics for the averages $\mathbb{E}_n^{(j,\pm)}[f]$, uniformly for $\frac{M}{n} < t_1 < \ldots < t_m < \pi - \frac{M}{n}$, but we can moreover extend this to cases where $t_1 \leq \frac{M}{n}$ or $t_m \geq \pi - \frac{M}{n}$. This is the content of our next result, which we will also prove in Section 3.4.

Theorem 3.2.2. *Let $m \in \mathbb{N}$, $0 \leq t_1 < \ldots < t_m \leq \pi$, $\alpha_0 = \alpha_{m+1} = 0$, $\alpha_j \geq 0$, $\beta_j \in i\mathbb{R}$ for $j = 1, \ldots, m$, and let V be analytic in a neighbourhood of the unit circle, real-valued on the unit circle and such that $V(e^{it}) = V(e^{-it})$, with Laurent series $V(z) = \sum_{k=-\infty}^{\infty} V_k z^k$ and $V_k = V_{-k} \in \mathbb{R}$. Let f be such that g is of the form (3.2.1). Then we have uniformly over the entire region $0 < t_1 < \ldots < t_m < \pi$, as $n \to \infty$,*

$$\begin{aligned}
\mathbb{E}_n^{(0,+)}[f] &= F e^{nV_0} \prod_{j=1}^{m} n^{\alpha_j^2 - \beta_j^2} \left(\sin t_j + \frac{1}{n} \right)^{\alpha_j - \alpha_j^2 - \beta_j^2} \times e^{\mathcal{O}(1)}, \\
\mathbb{E}_n^{(2,-)}[f] &= F e^{nV_0} \prod_{j=1}^{m} n^{\alpha_j^2 - \beta_j^2} \left(\sin t_j + \frac{1}{n} \right)^{-\alpha_j - \alpha_j^2 - \beta_j^2} \times e^{\mathcal{O}(1)}, \\
\mathbb{E}_n^{(1,\pm)}[f] &= F e^{nV_0} \prod_{j=1}^{m} n^{\alpha_j^2 - \beta_j^2} \left(\sin \frac{t_j}{2} + \frac{1}{n} \right)^{\mp\alpha_j - \alpha_j^2 - \beta_j^2} \left(\cos \frac{t_j}{2} + \frac{1}{n} \right)^{\pm\alpha_j - \alpha_j^2 - \beta_j^2} \\
&\qquad \times e^{\mathcal{O}(1)},
\end{aligned} \tag{3.2.10}$$

with F given by (3.2.9). Here $e^{\mathcal{O}(1)}$ denotes a function which is uniformly bounded and bounded away from 0 as $n \to \infty$. These results are also uniform for α_j and β_j in compact subsets of $[0, +\infty)$ and $i\mathbb{R}$ respectively.

Remark 3.2.3. *The factors $\sin t_j + \frac{1}{n}$ have to be interpreted as follows: whenever t_j does not converge too rapidly to 0 or π as $n \to \infty$, the sine is the dominant term; if $t_j \to 0$ or $t_j \to \pi$ as $n \to \infty$ with speed of convergence faster than $\frac{1}{n}$, the term $\frac{1}{n}$ will be dominant. Similarly for the factors $\sin \frac{t_j}{2} + \frac{1}{n}$ as $t_j \to 0$ and $\cos \frac{t_j}{2} + \frac{1}{n}$ as $t_j \to \pi$.*

Remark 3.2.4. *As mentioned before, one of the problems in determining the explicit value of the $e^{\mathcal{O}(1)}$ factor lies in the asymptotics for $\mathbb{E}_n^{\mathbb{U}}[g]$, which are known only up to a multiplicative constant as $n \to \infty$. In the case $m = 1$ where we have only two singularities, this multiplicative constant can be evaluated explicitly in terms of quantities related to a solution of the fifth Painlevé equation [62]. Simultaneously with this work, Forkel and Keating [80] evaluated the $e^{\mathcal{O}(1)}$ factor in (3.2.10) explicitly in terms of the same Painlevé V solution when $m = 2$, as long as the singularities $e^{\pm it_1}, e^{\pm it_2}$ do not approach ± 1. When there are more than two singularities approaching each other, one might expect a multiplicative constant connected to a generalisation of the fifth Painlevé equation, but the problem of evaluating the constant remains open.*

3.2.2 Symbols with a gap or an emerging gap

Next, we take $s \geq 0$ and $t_0 \in (0, \pi)$. We consider symbols f such that g, defined by (3.1.3), is of the form

$$g(e^{it}) = e^{V(e^{it})} \times \begin{cases} 1 & \text{for } 0 \leq |t| \leq t_0, \\ s & \text{for } t_0 < |t| \leq \pi, \end{cases} \tag{3.2.11}$$

and suppose that V is, as before, real on the unit circle, and analytic in a neighbourhood of the unit circle. Note that in view of (3.1.3), we have $V(e^{it}) = V(e^{-it})$. For $s > 0$ fixed, this is (up to a multiplicative constant) a special case of a symbol with two Fisher-Hartwig singularities ($m = 1$, $\alpha_1 = 0$). However, the limit $s \to 0$ corresponds to $\beta_1 \to -i\infty$, and the results stated before do not remain valid in this limit. To state our results, we need the Fourier coefficients $\widetilde{V}_k$ of the function

$$\widetilde{V}(e^{it}) := V(e^{2i \arcsin(\sin \frac{t_0}{2} \sin \frac{t}{2})}). \tag{3.2.12}$$

In the cases where either $s = 0$, or s depends on N and $s \to 0$ sufficiently fast as $N \to \infty$, such that $s \leq \left(\tan \frac{t_0}{4}\right)^{2N}$, asymptotics for $\mathbb{E}_N^{\mathbb{U}}[g]$ were obtained in [55, Theorem 1.1]:

$$\begin{aligned}\mathbb{E}_N^{\mathbb{U}}[g] = N^{-1/4} \left(\sin \frac{t_0}{2}\right)^{N^2} e^{N\widetilde{V}_0 + \sum_{k=1}^{\infty} k\widetilde{V}_k \widetilde{V}_{-k}} \left(\cos \frac{t_0}{2}\right)^{-1/4} e^{\frac{1}{12}\log 2 + 3\zeta'(-1)} \\ \times (1 + o(1)),\end{aligned}$$

as $N \to \infty$. Setting $N = 2n$, we have

$$\begin{aligned}\mathbb{E}_{2n}^{\mathbb{U}}[g] = (2n)^{-1/4} \left(\sin \frac{t_0}{2}\right)^{4n^2} e^{2n\widetilde{V}_0 + \sum_{k=1}^{\infty} k\widetilde{V}_k \widetilde{V}_{-k}} \left(\cos \frac{t_0}{2}\right)^{-1/4} e^{\frac{1}{12}\log 2 + 3\zeta'(-1)} \\ \times (1 + o(1)),\end{aligned} \tag{3.2.13}$$

as $n \to \infty$ with either $s = 0$ or $s \to 0$ sufficiently fast such that $s \leq \left(\tan \frac{t_0}{4}\right)^{4n}$. This result is moreover valid uniformly as $t_0 \to \pi$, as long as $n(\pi - t_0) \to \infty$.

We also need the function

$$\delta(z) = \exp\left\{\frac{h(z)}{2\pi i} \int_{\gamma} \frac{V(\zeta)\mathrm{d}\zeta}{(\zeta - z)h(\zeta)}\right\}, \quad \text{where } h(\zeta) = \left((\zeta - e^{it_0})(\zeta - e^{-it_0})\right)^{1/2}, \tag{3.2.14}$$

where γ denotes the counter-clockwise oriented circular arc going from e^{-it_0} to e^{it_0} and passing through 1, and where h is determined by the conditions that it has a branch cut along the complementary circular arc going from e^{it_0} to e^{-it_0} and passing through -1, and that it is asymptotic to ζ for large ζ. We will need in particular the values

$$\begin{aligned}\delta_-(-1) &:= \lim_{z \to (-1)_-} \delta(z) = \exp\left(\frac{-\cos \frac{t_0}{2}}{\pi i} \int_{\gamma} \frac{V(\zeta)\mathrm{d}\zeta}{(\zeta + 1)h(\zeta)}\right), \\ \delta(\infty) &= \exp\left\{\frac{-1}{2\pi i} \int_{\gamma} \frac{V(\zeta)\mathrm{d}\zeta}{h(\zeta)}\right\},\end{aligned}$$

which are both positive. We will prove the following in Section 3.5.

Theorem 3.2.5. *Let $t_0 \in (0,\pi)$, let V be real-valued on the unit circle, analytic in a neighbourhood of the unit circle and such that $V(e^{it}) = V(e^{-it})$. Let f be such that g is of the form (3.2.11). Then, as $n \to \infty$, uniformly with respect to $0 \le s \le \left(\tan \frac{t_0}{4}\right)^{4n}$, we have*

$$\begin{aligned} \mathbb{E}_n^{(0,+)}[f] &= C_{2n-1}^{-1} \left(\mathbb{E}_{2n}^{\mathbb{U}}[g]\right)^{1/2} (1+o(1)),\\ \mathbb{E}_n^{(2,-)}[f] &= C_{2n} \ \left(\mathbb{E}_{2n}^{\mathbb{U}}[g]\right)^{1/2} (1+o(1)),\\ \mathbb{E}_n^{(1,\pm)}[f] &= \widetilde{C}_{2n}^{\pm 1} \left(\mathbb{E}_{2n}^{\mathbb{U}}[g]\right)^{1/2} (1+o(1)), \end{aligned} \tag{3.2.15}$$

where

$$\begin{aligned} C_{2n-1} &= 2^{2n-\frac{3}{4}} \left(\sin \frac{t_0}{4}\right)^n \left(\cos \frac{t_0}{4}\right)^{3n-1} e^{-\frac{1}{4}V(1)} \frac{\delta_-(-1)^{1/2}}{\delta(\infty)},\\ C_{2n} &= 2^{2n+\frac{1}{4}} \left(\sin \frac{t_0}{4}\right)^n \left(\cos \frac{t_0}{4}\right)^{3n+1} e^{-\frac{1}{4}V(1)} \frac{\delta_-(-1)^{1/2}}{\delta(\infty)},\\ \widetilde{C}_{2n} &= 2^{\frac{1}{4}} \left(\frac{\sin \frac{t_0}{4}}{\cos \frac{t_0}{4}}\right)^{n+\frac{1}{2}} \frac{e^{-\frac{1}{4}V(1)}}{\delta_-(-1)^{1/2}}. \end{aligned} \tag{3.2.16}$$

These asymptotics are also valid as $t_0 \to \pi$ in such a way that $n(\pi - t_0) \to \infty$. The $o(1)$ terms can be written as $\mathcal{O}((n(\pi-t_0))^{-1} + (n(\pi-t_0))^{-1/2} s(\tan \frac{t_0}{4})^{-4n})$.

Using the known asymptotics for $\mathbb{E}_{2n}^{\mathbb{U}}[g]$ given by (3.2.13), we can write the above results in a more explicit form:

$$\mathbb{E}_n^{(0,+)}[f] = \frac{2^{\frac{1}{6}} e^{\frac{1}{4}V(1)} \ \delta(\infty) \ e^{\frac{3}{2}\zeta'(-1)} \ e^{\frac{1}{2}\sum_{k=1}^\infty k\widetilde{V}_k\widetilde{V}_{-k}}}{\sqrt{\delta_-(-1)} \ (\sin \frac{t_0}{2})^{\frac{1}{2}} \ \left(\cos \frac{t_0}{2}\right)^{1/8}} \times \frac{n^{-\frac{1}{8}} \left(\sin \frac{t_0}{2}\right)^{2n^2-n+\frac{1}{2}} \ e^{n\widetilde{V}_0}}{\left(1+\cos \frac{t_0}{2}\right)^{n-\frac{1}{2}}} (1+o(1)),$$

$$\mathbb{E}_n^{(2,-)}[f] = \frac{2^{\frac{1}{6}} e^{\frac{3}{2}\zeta'(-1)} \ \cos \frac{t_0}{4} \ \sqrt{\delta_-(-1)} \ e^{\frac{1}{2}\sum_{k=1}^\infty k\widetilde{V}_k\widetilde{V}_{-k}}}{e^{\frac{1}{4}V(1)} \ \delta(\infty) \ \left(\cos \frac{t_0}{2}\right)^{1/8}} \times n^{-\frac{1}{8}} \left(\sin \frac{t_0}{2}\right)^{2n^2+n} \left(1+\cos \frac{t_0}{2}\right)^n e^{n\widetilde{V}_0} (1+o(1)),$$

$$\mathbb{E}_n^{(1,\pm)}[f] = \frac{2^{\pm\frac{1}{4}} \ e^{\frac{3}{2}\zeta'(-1)} e^{\mp\frac{1}{4}V(1)} e^{\frac{1}{2}\sum_{k=1}^\infty k\widetilde{V}_k\widetilde{V}_{-k}} \delta_-(-1)^{\mp 1/2}}{2^{\frac{1}{12}} \left(\cos \frac{t_0}{2}\right)^{1/8}} \cdot n^{-\frac{1}{8}} \left(\sin \frac{t_0}{2}\right)^{2n^2 \pm n} \left(1+\cos \frac{t_0}{2}\right)^{\mp n} e^{n\widetilde{V}_0} (1+o(1)),$$

where ζ is Riemann's zeta function.

3.2.3 Gap probabilities and global rigidity

The above results can be used to compute asymptotics for gap probabilities and generating functions in $\mathbb{O}_n^{(j,\pm)}$ and also in the Circular Orthogonal Ensemble (COE) and in the Circular Symplectic Ensemble (CSE). These have the joint probability distributions [81, Proposition 2.8.7]

$$\mathrm{C}\beta\mathrm{E}_N : \quad \frac{1}{Z_N^{[\beta]}} \prod_{1\le k<j\le N} |e^{i\phi_j} - e^{i\phi_k}|^\beta \prod_{j=1}^N d\phi_j, \tag{3.2.17}$$

with the normalization constant

$$Z_N^{[\beta]} = (2\pi)^N \frac{\Gamma(\beta N/2+1)}{\Gamma(\beta/2+1)}, \tag{3.2.18}$$

where $\beta = 1$ for the COE, and $\beta = 4$ for the CSE. Recall from (3.1.4) that $\beta = 2$ corresponds to the CUE. Define the piecewise constant symbol

$$g_{t_0,s}(e^{it}) = \begin{cases} s, & 0 \le |t| \le t_0, \\ 1, & t_0 < |t| \le \pi. \end{cases} \tag{3.2.19}$$

The average

$$E_N^{[\beta]}(t_0; s) := \mathbb{E}_{\mathrm{C}\beta\mathrm{E}_N} \prod_{j=1}^N g_{t_0,s}(e^{i\phi_j})$$

over (3.2.17) is the generating function for *occupancy numbers* of the arc between e^{it_0} and e^{-it_0} passing through 1, in the sense that

$$E_N^{[\beta]}(t_0; s) = \sum_{m=0}^N s^m \mathbb{P}_{\mathrm{C}\beta\mathrm{E}_N}\left(\text{there are exactly } m \text{ eigenangles in } [-t_0, t_0]\right).$$

Equivalently, for $s \in (0,1)$, $E_N^{[\beta]}(t_0; s)$ is the probability that the thinned $\mathrm{C}\beta\mathrm{E}_N$, obtained by removing each eigenvalue independently with probability s, has no eigenangles in $[-t_0, t_0]$. Similarly, if we define

$$f_{t_0,s}(e^{it}) = \begin{cases} 1, & t_0 < t < 2\pi, \\ s, & 0 \le t \le t_0, \end{cases} \tag{3.2.20}$$

we have the following identity in the orthogonal ensembles,

$$\begin{aligned} E_n^{(j,\pm)}(t_0; s) := \mathbb{E}_n^{(j,\pm)}[f_{t_0,s}] = \sum_{m=0}^n s^m \mathbb{P}_{\mathbb{O}_{2n+j}^\pm} \\ \left(\text{there are exactly } m \text{ eigenangles in } (0, t_0)\right). \end{aligned} \tag{3.2.21}$$

Equivalently, for $s \in (0,1)$, $E_n^{(j,\pm)}(t_0, s)$ is the probability that the thinned orthogonal ensemble $\mathbb{O}_n^{(j,\pm)}$, obtained by removing each free eigenangle θ_k, $k = 1, \ldots, n$ independently with probability s, has no eigenvalues in $(0, t_0)$. The following identities relate the COE and CSE generating functions to those of the orthogonal ensembles $\mathbb{O}_N^{\pm}$, see [25, 26]:

$$\begin{aligned} E^{[1]}_{2n+1}(t_0;s) &= \frac{sE_n^{(2,-)}(t_0;s^2)+E_{n+1}^{(0,+)}(t_0;s^2)}{1+s}, \\ E^{[1]}_{2n}(t_0;s) &= \frac{sE_n^{(1,+)}(t_0;s^2)+E_{n}^{(1,-)}(t_0;s^2)}{1+s}, \\ E^{[4]}_{n}(t_0;s) &= \frac{1}{2}\left(E_n^{(1,+)}(t_0;s)+E_n^{(1,-)}(t_0;s)\right). \end{aligned} \tag{3.2.22}$$

To compute asymptotics for the right hand sides of the expressions in the case where $s=0$, we can apply Theorem 3.2.5 in the case $V=0$ and $s=0$.

In Section 3.6.1, we will show that this yields asymptotics for the $\mathbb{O}_N^{\pm}$, COE and CSE gap probabilities which correspond to $s=0$. Asymptotics for similar averages were established in [26] in the microscopic regime where t_0 is of the order of $1/n$. For $t_0\to 0$ at a slower rate, these asymptotics are new to the best of our knowledge.

Corollary 3.2.6. *Let $t_0\in(0,\pi)$. As $n\to\infty$, with fixed t_0 or with $t_0\to 0$ in such a way that $nt_0\to+\infty$,*

$$E_n^{(j,\pm)}(t_0;0) = 2^{-\frac{1}{12}}e^{\frac{3}{2}\zeta'(-1)}\left(\frac{\left(1+\sin\frac{t_0}{2}\right)^{\tilde{n}}}{2^{\frac{1}{4}}\left(\cos\frac{t_0}{2}\right)^{\tilde{n}}}\right)^{\epsilon_j^{\pm}}\frac{\left(\cos\frac{t_0}{2}\right)^{2\tilde{n}^2}}{\left(\tilde{n}\sin\frac{t_0}{2}\right)^{\frac{1}{8}}}(1+o(1)), \tag{3.2.23}$$

where $\tilde{n}=n+\frac{j-1}{2}$ and $\epsilon_j^{\pm}=1$ if $+1$ is a fixed eigenvalue of $\mathbb{O}_{2n+j}^{\pm}$ and $\epsilon_j^{\pm}=-1$ otherwise. In other words, $\epsilon_0^+=-1$, $\epsilon_1^+=1$, $\epsilon_1^-=-1$, $\epsilon_2^-=1$.
Moreover, as $N\to\infty$ and $t_0\in(0,\pi)$ is either fixed or tends to 0 in such a way that $Nt_0\to\infty$, we have

$$\begin{aligned} E_N^{[1]}(t_0;0) &= 2^{\frac{7}{24}}e^{\frac{3}{2}\zeta'(-1)}\left(\frac{\cos\frac{t_0}{2}}{1+\sin\frac{t_0}{2}}\right)^{\frac{N}{2}}\frac{\left(\cos\frac{t_0}{2}\right)^{\frac{N^2}{2}}}{\left(N\sin\frac{t_0}{2}\right)^{\frac{1}{8}}}(1+o(1)), \\ E_N^{[4]}(t_0;0) &= 2^{-\frac{4}{3}}e^{\frac{3}{2}\zeta'(-1)}\left(\frac{1+\sin\frac{t_0}{2}}{\cos\frac{t_0}{2}}\right)^{N}\frac{\left(\cos\frac{t_0}{2}\right)^{2N^2}}{\left(N\sin\frac{t_0}{2}\right)^{\frac{1}{8}}}(1+o(1)). \end{aligned}$$

Remark 3.2.7. *We can compare these results with the corresponding result in the CUE, which reads [140]*

$$E_N^{[2]}(t_0;0) = 2^{\frac{1}{12}}e^{3\zeta'(-1)}\frac{\left(\cos\frac{t_0}{2}\right)^{N^2}}{\left(N\sin\frac{t_0}{2}\right)^{\frac{1}{4}}}(1+o(1)).$$

To compute asymptotics for the right hand sides of (3.2.22) in the case where $s>0$ is fixed, we can apply Theorem 3.2.1 and [62]. Through (3.2.22), this yields asymptotics for the generating functions/gap probabilities in the thinned COE and thinned CSE, which we prove in Section 3.6.2.

Corollary 3.2.8. *As $\tilde{n} = n + \frac{j-1}{2} \to \infty$, with $t_0 \in (0,\pi)$ fixed or such that $nt_0 \to \infty$, and with $\epsilon_j^{\pm}$ as above,*

$$E_n^{(j,\pm)}(t_0; s) = s^{-\frac{1}{4}\epsilon_j^{\pm}} \left| G\left(1 + \frac{\log s}{2\pi i}\right)\right|^2 (4\tilde{n}\sin t_0)^{\frac{\log^2 s}{4\pi^2}} s^{\frac{\tilde{n}t_0}{\pi}} (1 + o(1)).$$

Moreover, as $N \to +\infty$ with $t_0 \in (0,\pi)$ fixed or such that $Nt_0 \to \infty$, we have

$$E_N^{[1]}(t_0; s) = \frac{2s^{\frac{1}{2}}}{1+s} \left| G\left(1 + \frac{\log s}{\pi i}\right)\right|^2 (2N\sin t_0)^{\frac{\log^2 s}{\pi^2}} s^{\frac{Nt_0}{\pi}} (1 + o(1)),$$

$$E_N^{[4]}(t_0; s) = \frac{1+s^{\frac{1}{2}}}{2s^{\frac{1}{4}}} \left| G\left(1 + \frac{\log s}{2\pi i}\right)\right|^2 (4N\sin t_0)^{\frac{\log^2 s}{4\pi^2}} s^{\frac{Nt_0}{\pi}} (1 + o(1)).$$

Remark 3.2.9. *The above results should be compared to the CUE analogue (see Section 3.6.2)*

$$E_N^{[2]}(t_0; s) = \left| G\left(1 + \frac{\log s}{2\pi i}\right)\right|^4 (2N\sin t_0)^{\frac{\log^2 s}{2\pi^2}} s^{\frac{Nt_0}{\pi}} (1 + o(1)).$$

Finally we can use Theorem 3.2.2 to obtain weak uniform asymptotics for the generating functions when $s > 0$, see Section 3.6.2 for the proof of this result.

Corollary 3.2.10. *Uniformly for $t_0 \in [0,\pi]$, s in compact sets of $(0,+\infty)$, as $n \to \infty$:*

$$E_n^{(j,\pm)}(t_0; s) = (n\sin t_0 + 1)^{\frac{\log^2 s}{4\pi^2}} s^{\frac{nt_0}{\pi}} e^{\mathcal{O}(1)},$$

hence for $\beta = 1, 4$, as $N \to \infty$,

$$E_N^{[\beta]}(t_0; s) = (N\sin t_0 + 1)^{\frac{\log^2 s}{\beta\pi^2}} s^{\frac{Nt_0}{\pi}} e^{\mathcal{O}(1)}.$$

Remark 3.2.11. *The above result also holds for $\beta = 2$, see [62] for an expression of the multiplicative constant.*

The previous corollary allows us to derive global rigidity estimates for the ordered eigenangles $0 < \theta_1 \leq \ldots \leq \theta_n < \pi$ in the orthogonal ensembles $\mathbb{O}_{2n}^{+}, \mathbb{O}_{2n+2}^{-}, \mathbb{O}_{2n+1}^{\pm}$. Given the joint probability distribution of the eigenvalues (3.1.1) which implies that the eigenvalues repel each other, we can expect that in a typical situation, the eigenangles are distributed in a rather regular way, in other words we can expect that θ_j will typically lie not too far from the deterministic value $\frac{j\pi}{n}$. We can also expect that the counting function $N_{(0,t)}$, counting the number of eigenangles in $(0,t)$ for $t \leq \pi$, will behave to leading order typically like $\frac{nt}{\pi}$. We prove the following in Section 3.6.3.

Theorem 3.2.12. *In the ensembles $\mathbb{O}_{2n}^{+}, \mathbb{O}_{2n+2}^{-}, \mathbb{O}_{2n+1}^{\pm}$, we have for any $\epsilon > 0$*

$$\lim_{n\to+\infty} \mathbb{P}\left(\max_{k=1,\ldots,n} \left|\theta_k - \frac{\pi k}{n}\right| < (1+\epsilon)\frac{\log n}{n}\right) = 1,$$

and

$$\lim_{n\to+\infty}\mathbb{P}\left(\sup_{t\in(0,\pi)}\left|N_{(0,t)}-\frac{nt}{\pi}\right|<\left(\frac{1}{\pi}+\epsilon\right)\log n\right)=1.$$

Remark 3.2.13. *These results should be compared to concentration inequalities in [123, Section 5.4], which yield probabilistic bounds for* $\left|N_{(0,t)}-\frac{nt}{\pi}\right|$ *rather than for its supremum, and to global rigidity results in the CβE [3, 49, 113] (see in particular Corollary 1.3 of [113]) and the sine β process [93]. The method that we use to prove this result is based on a bound for the first exponential moment of the eigenvalue counting function, and this does not allow to get a complementary lower bound for the maximum and supremum. The question of sharpness of the upper bound is closely related to the theory of Gaussian multiplicative chaos, see e.g. [3, 18, 139] in general and [80] in this specific situation.*

3.2.4 Possible generalisations

Apart from the positive symbols with Fisher-Hartwig singularities and the symbols with a gap or emerging gap, there are other types of symbols for which Toeplitz determinant asymptotics are known, and for which one could use Proposition 3.1.1 in order to generalise them to the orthogonal ensembles. One could for instance consider complex-valued symbols or non-analytic symbols with Fisher-Hartwig singularities and apply the results from [69]. Another example consists of a situation where a symbol is smooth but depends on n and develops a Fisher-Hartwig singularity in the limit $n\to\infty$, as considered in [61]. In this case, like in the case $m=1$ of Theorem 3.2.2, it is also possible to evaluate the multiplicative constant in the asymptotic expansion in terms of solutions to the Painlevé V equation. Yet another example consists of symbols with a gap, but with an additional Fisher-Hartwig singularity inside the gap, as considered in [143]. This situation is related to a system of coupled Painlevé V equations.

In principle, the results from Theorem 3.2.2 can also be applied to derive asymptotics for moments of moments of characteristic polynomials in the orthogonal ensembles, which can be written as multiple integrals of the multiplicative averages we are considering in Theorem 3.2.2, in the special case where all β_j's vanish and where all α_j's are equal. The moments of moments are of interest because they reveal some of the statistics of the extrema of characteristic polynomials. In the case of the unitary group, their asymptotics were conjectured in [83] and later proved in [62] in the case of two singularities, and in [77] in general. Both for unitary and orthogonal ensembles, these moments of moments have been evaluated exactly in terms of symmetric functions in [11, 5] for α_j integer. It would be interesting to see if Theorem 3.2.2 can be used to generalise the asymptotics to any $\alpha_j\geq 0$.

3.3 Proof of Proposition 3.1.1

Given a real-valued integrable function f on the unit circle, define the symbol $g(e^{it}) = f(e^{it})f(e^{-it})$, symmetric with respect to complex conjugation of the variable, and define its Fourier coefficients as in (3.1.7). It is known that the averages (3.1.2) can be written as determinants of Toeplitz+Hankel matrices. More precisely, we have (see e.g. [10, theorem 2.2], [81, p212], or [103]) for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}_n^{(0,+)}[f] &= \frac{1}{2}\det\left(g_{j-k} + g_{j+k}\right)_{j,k=0}^{n-1}, \\
\mathbb{E}_n^{(2,-)}[f] &= \det\left(g_{j-k} - g_{j+k+2}\right)_{j,k=0}^{n-1}, \\
\mathbb{E}_n^{(1,\pm)}[f] &= \det\left(g_{j-k} \mp g_{j+k+1}\right)_{j,k=0}^{n-1}.
\end{aligned}
\tag{3.3.1}
$$

Moreover, there exist identities expressing products of two Toeplitz+Hankel determinants as a Toeplitz determinant ([141], [10, Corollary 2.4], or [81, p211])

$$
\begin{aligned}
\mathbb{E}_{2n}^{\mathbb{U}}[g] &= \det\left(g_{j-k}\right)_{j,k=0}^{2n-1} \\
&= \det\left(g_{j-k} - g_{j+k+1}\right)_{j,k=0}^{n-1}\det\left(g_{j-k} + g_{j+k+1}\right)_{j,k=0}^{n-1}, \\
\mathbb{E}_{2n+1}^{\mathbb{U}}[g] &= \det\left(g_{j-k}\right)_{j,k=0}^{2n} \\
&= \frac{1}{2}\det\left(g_{j-k} + g_{j+k}\right)_{j,k=0}^{n}\det\left(g_{j-k} - g_{j+k+2}\right)_{j,k=0}^{n-1}.
\end{aligned}
\tag{3.3.2}
$$

We would like to invert such factorizations, and write a single Toeplitz+Hankel determinant in terms of a Toeplitz determinant. To that end, we need in addition analogues of the above identities, but with slightly different products of Toeplitz+Hankel determinants at the right. As above, let Φ_N be the degree N monic orthogonal polynomial associated with the symbol g.

Proposition 3.3.1.

$$
\begin{aligned}
&\Phi_{2n}(\pm 1)\det\left(g_{j-k}\right)_{j,k=0}^{2n-1} = \\
&\qquad \det\left(g_{j-k} - g_{j+k+2}\right)_{j,k=0}^{n-1}\det\left(g_{j-k} \mp g_{j+k+1}\right)_{j,k=0}^{n-1}, \\
&\Phi_{2n+1}(\pm 1)\det\left(g_{j-k}\right)_{j,k=0}^{2n} = \\
&\qquad \pm\det\left(g_{j-k} \mp g_{j+k+1}\right)_{j,k=0}^{n}\det\left(g_{j-k} - g_{j+k+2}\right)_{j,k=0}^{n-1}.
\end{aligned}
\tag{3.3.3}
$$

Proof. The representation of the monic orthogonal polynomials in terms of the determinant (3.1.9) yields

$$
\Phi_N(\pm 1)\det(g_{j-k})_{j,k=0}^{N-1} = \det\left(g_{j-k} \quad (\pm 1)^j\right)_{j,k=0}^{N,N-1}.
$$

Setting $N = 2n$ and subtracting the $(2n-j)$-th row from the j-th row of the matrix at the right hand side for $j = 0, \ldots, n-1$, we obtain

$$
\Phi_{2n}(\pm 1)\det(g_{j-k})_{j,k=0}^{2n-1} = \det\begin{pmatrix} \left(g_{j-k} - g_{2n-j-k}\right)_{j,k=0}^{n-1,2n-1} & (0)_{j,k=0}^{n-1,0} \\ \left(g_{n+j-k}\right)_{j,k=0}^{n,2n-1} & \left((\pm 1)^{n+j}\right)_{j,k=0}^{n,0} \end{pmatrix}.
$$

Then, adding the $(2n-k)$-th column to the k-th column for $k=n,...,2n-1$ and dividing by two to take into account the case $k=n$, we get (g being symmetric, we have $g_m = g_{-m}$)

$$\begin{aligned}&\Phi_{2n}(\pm 1)\det(g_{j-k})_{j,k=0}^{2n-1}\\&=\frac{1}{2}\det\begin{pmatrix}(g_{j-k}-g_{2n-j-k})_{j,k=0}^{n-1,n-1} & (0)_{j,k=0}^{n-1,n-1} & (0)_{j,k=0}^{n-1,0}\\(g_{n+j-k})_{j,k=0}^{n,n-1} & (g_{j-k}+g_{j+k})_{j,k=0}^{n,n-1} & \left((\pm 1)^{n+j}\right)_{j,k=0}^{n,0}\end{pmatrix}.\end{aligned}$$

This yields

$$\begin{aligned}&\Phi_{2n}(\pm 1)\det(g_{j-k})_{j,k=0}^{2n-1}\\&\qquad=\frac{1}{2}\det(g_{j-k}-g_{j+k+2})_{j,k=0}^{n-1}\det\left((g_{j-k}+g_{j+k})_{j,k=0}^{n,n-1}\quad \left((\pm 1)^{n+j}\right)_{j,k=0}^{n,0}\right).\end{aligned}$$

Adding or subtracting the $(j+1)$-th row from the j-th for $j=0,\dots,n-1$, and then expanding with respect to the last column, we end up with

$$\begin{aligned}&\det\left((g_{j-k}+g_{j+k})_{j,k=0}^{n,n-1}\quad \left((\pm 1)^{n+j}\right)_{j,k=0}^{n,0}\right)\\&\qquad=\det\left(g_{j-k}+g_{j+k}\mp(g_{j-k+1}+g_{j+k+1})\right)_{j,k=0}^{n-1}.\end{aligned}$$

For the first identity in (3.3.3), it remains to prove that the second determinant at the right hand side in the above formula is equal to $\det(g_{j-k}\mp g_{j+k+1})_{j,k=0}^{n-1}$. To see this, it suffices in the latter matrix to subtract or add the $(k-1)$-th column from the k-th for $k=1,\dots,n-1$, and to multiply the first column by 2. This indeed gives

$$\det(g_{j-k}\mp g_{j+k+1})_{j,k=0}^{n-1}=\frac{1}{2}\det(g_{j-k}\mp g_{j+k+1}\mp(g_{j-k+1}\mp g_{j+k}))_{j,k=0}^{n-1},$$

thus proving the first identity. For the second, one proceeds similarly by subtracting or adding the $(2n+1-j)$-th row from the j-th for $j=0,...,n$, and then adding or subtracting the $(2n+1-k)$-th column from the k-th for $k=n+1,\dots,2n$, leading to

$$\begin{aligned}&\Phi_{2n+1}(\pm 1)\det(g_{j-k})_{j,k=0}^{2n}\\&\qquad=\det(g_{j-k}\mp g_{j+k+1})_{j,k=0}^{n}\det\left((g_{j-k}\pm g_{j+k+1})_{j,k=0}^{n,n-1}\ \left((\pm 1)^{n+1+j}\right)_{j=0}^{n}\right).\end{aligned}$$

As before, subtracting or adding the next row to each row except the last one, and then expanding with respect to the last column, we get

$$\begin{aligned}&\det\left((g_{j-k}\pm g_{j+k+1})_{j,k=0}^{n,n-1}\ \left((\pm 1)^{n+1+j}\right)_{j=0}^{n}\right)\\&\qquad=\pm\det(g_{j-k}\pm g_{j+k+1}\mp(g_{j-k+1}\pm g_{j+k+2}))_{j,k=0}^{n-1}.\end{aligned}$$

Also similarly as before, we have

$$\det(g_{j-k}-g_{j+k+2})_{j,k=0}^{n-1}=\det(g_{j-k}-g_{j+k+2}\mp(g_{j-k+1}-g_{j+k+1}))_{j,k=0}^{n-1},$$

and the two above equations allow us to conclude the proof. $\square$

We can now combine the factorizations (3.3.2) and (3.3.3) to obtain

$$\begin{aligned}
\left(\frac{1}{2}\det\left(g_{j-k}+g_{j+k}\right)_{j,k=0}^{n-1}\right)^2 &= \frac{1}{-\Phi_{2n-1}(-1)\Phi_{2n-1}(1)}\det(g_{j-k})_{j,k=0}^{2n-1},\\
\left(\det\left(g_{j-k}-g_{j+k+2}\right)_{j,k=0}^{n-1}\right)^2 &= \Phi_{2n}(1)\Phi_{2n}(-1)\det(g_{j-k})_{j,k=0}^{2n-1},\\
\left(\det\left(g_{j-k}\pm g_{j+k+1}\right)_{j,k=0}^{n-1}\right)^2 &= \left(\frac{\Phi_{2n}(-1)}{\Phi_{2n}(1)}\right)^{\pm 1}\det(g_{j-k})_{j,k=0}^{2n-1}.
\end{aligned} \tag{3.3.4}$$

To prove Theorem 3.1.1, it then suffices to use (3.3.1) and to note that since g is non-negative, the zeros of the orthogonal polynomials are symmetric with respect to the real line and lie inside the unit disk, hence the right hand sides of (3.3.4) are positive.

3.4 Symbols with Fisher-Hartwig singularities

In this section, we let, as in Theorem 3.2.1 and Theorem 3.2.2, V be an analytic function in a neighbourhood of the unit circle, real-valued on the unit circle and such that $V(e^{it}) = V(e^{-it})$, with Fourier coefficients $V_k = V_{-k} \in \mathbb{R}$, and we let $m \in \mathbb{N}$, $0 < t_1 < \ldots < t_m < \pi$, and for any $j = 1, \ldots, m$, $\alpha_j \geq 0$ and $\beta_j \in i\mathbb{R}$. Then we let g be of the form (3.2.1). This is a positive symbol with $2m+2$ Fisher-Hartwig singularities $e^{\pm it_j}$ and ± 1.

In order to prove Theorem 3.2.1 and Theorem 3.2.2, by Proposition 3.1.1, we need to obtain asymptotics for the orthogonal polynomials $\Phi_N(\pm 1)$, with $N = 2n$ and $N = 2n-1$.

3.4.1 Asymptotics for $\Phi_N(\pm 1)$

The large N asymptotics for $\Phi_N(\pm 1)$ are not readily available in the literature, but can be computed using the RH analysis from [77], which was inspired by the analysis of [69]. Both those RH methods are based on an asymptotic analysis of the function

$$Y(z) = \begin{pmatrix} \Phi_N(z) & \frac{1}{2\pi i}\int\limits_{\mathcal{C}} \frac{\Phi_N(\zeta)g(\zeta)d\zeta}{\zeta^N(\zeta-z)} \\ -\chi_{N-1}^2 z^{N-1}\Phi_{N-1}(z^{-1}) & \frac{-\chi_{N-1}^2}{2\pi i}\int\limits_{\mathcal{C}} \frac{\Phi_{N-1}(\zeta^{-1})g(\zeta)d\zeta}{\zeta(\zeta-z)} \end{pmatrix}, \tag{3.4.1}$$

where $\chi_{N-1}^{-2} = \frac{1}{2\pi}\int_0^{2\pi}\left|\Phi_{N-1}(e^{it})\right|^2 g(e^{it})dt$ and $\mathcal{C}$ is the unit circle. This is the standard solution of the following RH problem for orthogonal polynomials on the unit circle [79].

RH problem for Y

(a) Y is analytic in $\mathbb{C}\backslash\mathcal{C}$, where the unit circle $\mathcal{C}$ is oriented counter-clockwise.

(b) Y has continuous boundary values $Y_\pm$ as $z \in \mathcal{C} \setminus \{\pm 1, e^{\pm it_1}, \ldots, e^{\pm it_m}\}$ is approached from inside $(+)$ or outside $(-)$ the unit circle, and they are related by $Y_+(z) = Y_-(z)\begin{pmatrix} 1 & z^{-N}g(z) \\ 0 & 1 \end{pmatrix}$.

(c) $Y(z) = (I + \mathcal{O}(z^{-1}))z^{N\sigma_3}$ as $z \to \infty$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

If one imposes moreover suitable conditions near the points $\pm 1, e^{\pm it_j}$, the solution to the above RH problem is unique, and one can derive asymptotics for it as $N \to \infty$ using the Deift/Zhou steepest descent method [72].

In the following result, we restrict ourselves to symbols g of the form (3.1.3) with $\alpha_0 = \alpha_{m+1} = 0$, i.e. the case where there are no Fisher-Hartwig singularities at the points ± 1, in the region needed for Theorem 3.2.1.

Proposition 3.4.1. *Let g be of the form (3.2.1) with $\alpha_0 = \alpha_{m+1} = 0$. Define $u_+ = t_1$ and $u_- = \pi - t_m$. We have*

$$\begin{aligned} \Phi_N(1) &= e^{\frac{1}{2}(V_0 - V(1))} \prod_{j=1}^{m} \left(2\sin\frac{t_j}{2}\right)^{-2\alpha_j} e^{it_j\beta_j} e^{-i\pi\beta_j} \left(1 + \mathcal{O}\left(\frac{1}{Nu_+}\right)\right) \\ \Phi_N(-1) &= (-1)^N e^{\frac{1}{2}(V_0 - V(-1))} \prod_{j=1}^{m} \left(2\cos\frac{t_j}{2}\right)^{-2\alpha_j} e^{it_j\beta_j} \left(1 + \mathcal{O}\left(\frac{1}{Nu_-}\right)\right) \end{aligned} \tag{3.4.2}$$

as $N \to \infty$, uniformly over the region $M/N < t_1 < \ldots < t_m < \pi - M/N$ with $M > 0$ sufficiently large, and uniformly for α_j and β_j in compact subsets of $[0, +\infty)$ and $i\mathbb{R}$ respectively.

Proof. The analysis in [77] is based on partitioning the $2m$ singularities $e^{\pm it_j}$ in different clusters. To do this, let us define for any $0 < M_1 < M_2$ the *clustering condition* (M_1, M_2, N) as follows. We say that clustering condition (M_1, M_2, N) is satisfied if the set $A = \{t_1, \ldots, t_m, -t_1, \ldots, -t_m\}$ can be partitioned into $\ell \le 2m$ clusters $A_1, \ldots, A_\ell$ such that the following holds:

(a) for any two values $x, y \in A$ belonging to the same cluster A_k, we have $|x - y| \le M_1/N$ or $||x - y| - 2\pi| \le M_1/N$, which means that singularities corresponding to the same cluster approach each other fast enough as $N \to \infty$,

(b) for any two values $x, y \in A$ belonging to a different cluster, we have $|x - y| > M_2/N$ and $||x - y| - 2\pi| > M_2/N$, which means that singularities corresponding to different clusters do not approach each other too fast as $N \to \infty$.

Note that any clustering condition is trivially satisfied if $m = 0$. Observe also that different values of M_1 may lead to a different number of clusters ℓ. Indeed, one cluster corresponding to a bigger value of M_1 may consist of the union of several clusters corresponding to a smaller value of M_1. Given $M_1 > 0$, partition the $\pm t_j$'s in $\ell = \ell(M_1)$ clusters A_j as above, and define

$$\mu(M_1, M_2, N) := \min_{x \in A_k, y \in A_j, j \neq k} |x - y|, \tag{3.4.3}$$

i.e. $\mu(M_1, M_2, N)$ is the minimal distance between arguments belonging to different clusters. Next, define arguments $\hat{t}_1, \ldots, \hat{t}_\ell$, also depending on M_1 and

N, where $\hat{t}_j$ is the average of the arguments t_k belonging to the cluster A_j. Under the clustering condition (M_1, M_2, N) and if in addition $M > M_2/2$ and $M_2 \geq 3M_1$, we have

$$\frac{3M_1}{N} \leq \frac{M_2}{N} \leq \mu(M_1, M_2, N) \leq 2u_\pm. \tag{3.4.4}$$

The RH analysis in [77] (which is inspired by the one from [69]) consists of explicit transformations

$$Y \mapsto T \mapsto S \mapsto R,$$

where Y is given by (3.4.1), such that in particular we have $Y_{11}(z) = \Phi_N(z)$. The transformations $Y \mapsto T$ and $T \mapsto S$ are similar as in [69] and are fairly standard; the transformation $S \mapsto R$ consists of constructing local parametrices P in disks $\mathcal{U}_j$ of radius $\mu(M_1, M_2, N)/3$ around each of the points $e^{i\hat{t}_j}$ for $j = 1, \ldots, \ell$, and a global parametrix P^∞ elsewhere in the complex plane. By (3.4.4), every singularity $e^{\pm it_k}$ is contained in one of the disjoint disks $\mathcal{U}_j$, and the points ± 1 are not contained in such a disk. Therefore, we do not need the precise form of the local parametrices.

Let us list more details about each of these transformations.

Step 1. Define

$$T(z) = \begin{cases} Y(z), & |z| < 1, \\ Y(z) z^{-N\sigma_3}, & |z| > 1. \end{cases}$$

Step 2. Define

$$S(z) = \begin{cases} T(z) \begin{pmatrix} 1 & 0 \\ z^{-N} g(z)^{-1} & 1 \end{pmatrix}, & \text{when } |z| > 1 \text{ is inside the lenses,} \\ T(z) \begin{pmatrix} 1 & 0 \\ -z^{N} g(z)^{-1} & 1 \end{pmatrix}, & \text{when } |z| < 1 \text{ is inside the lenses,} \\ T(z), & \text{when } z \text{ outside the lenses,} \end{cases}$$

where g is the analytic extension of g defined in (3.2.1) to the interior parts of the lenses, see Figure 3.1 for the shape of the lenses and [69, Section 4] or [77, Section 6] for an explicit expression of this analytic continuation.

Step 3. Define

$$R(z) = \begin{cases} S(z) P^\infty(z)^{-1}, & z \in \mathbb{C} \setminus (\bigcup_j \mathcal{U}_j), \\ S(z) P_j(z)^{-1}, & z \in \mathcal{U}_j. \end{cases}$$

Here P^∞ is the *global parametrix* and P_j's are *local parametrices*. We will not need their general expressions, but we will need the value of the global parametrix evaluated at $z = \pm 1_\pm$, which is defined in (3.4.6) below.

For $z \to 1_+$ or $z \to -1_-$, the transformations $Y \mapsto T \mapsto S \mapsto R$ imply (see [77, formulas (70), (75), (78), (83)])

$$\begin{aligned} Y_{11}(\pm 1) = T_{11}(\pm 1_\pm)(\pm 1_\pm)^N &= (\pm 1_\pm)^N S_{11}(\pm 1_\pm) - f(\pm 1_\pm)^{-1} S_{12}(\pm 1_\pm) \\ &= (\pm 1_\pm)^N (RP^\infty)_{11}(\pm 1_\pm) - f(\pm 1)^{-1} (RP^\infty)_{12}(\pm 1_\pm), \end{aligned} \tag{3.4.5}$$

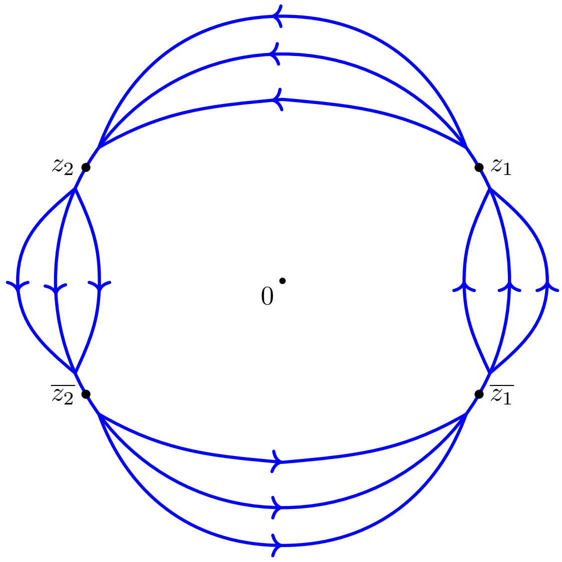

Figure 3.1: Opening of lenses in the case of 4 singularities $z_1, \overline{z_1}, z_2, \overline{z_2}$ partitioned into four clusters.

where (see [77, formulas (79) and (72)])

$$P^{\infty}(\pm 1_{\pm}) = e^{-\sum_{k=-\infty}^{-1} V_k \cdot (\pm 1)^k \sigma_3} \prod_{j=1}^{m} \left(1 \mp e^{it_j}\right)^{(\beta_j - \alpha_j)\sigma_3} \left(1 \mp e^{-it_j}\right)^{(-\beta_j - \alpha_j)\sigma_3}, \tag{3.4.6}$$

with the principal branch of the roots.

The final conclusion of the RH analysis in [77] is the following: for any $M_1 > 0$, then for large enough $M_2 > 0$, we have $R(z) = I + \mathcal{O}\left(\frac{1}{N\mu(M_1, M_2, N)}\right)$, uniformly in z as $N \to \infty$, and uniformly under clustering condition (M_1, M_2, N). Let us now choose any value of $M_1 > 0$, and let $M_2 \geq 3M_1$ be a constant induced by the above statement, i.e. let $M_2 = M_2(M_1)$ be such that $R(z) = I + \mathcal{O}\left(\frac{1}{N\mu(M_1, M_2, N)}\right)$, uniformly in z as $N \to \infty$ under clustering condition (M_1, M_2, N).

Next, we iterate by defining $M_3 = M_3(M_2) \geq 3M_2$ as some value such that (noting that $\mu(M_2, N) \geq \mu(M_1, N)$)

$$R(z) = I + \mathcal{O}\left(\frac{1}{N\mu(M_2, M_3, N)}\right) = I + \mathcal{O}\left(\frac{1}{N\mu(M_1, M_2, N)}\right),$$

uniformly in z and under clustering condition (M_2, M_3, N) as $N \to \infty$. We iterate this procedure, which allows us to deduce that $R(z) = I + \mathcal{O}\left(\frac{1}{N\mu(M_1, M_2, N)}\right)$ uniformly as $N \to \infty$ under the m disjoint clustering conditions

$$(M_1, M_2, N), (M_2, M_3, N), \ldots, (M_m, M_{m+1}),$$

for some increasing sequence $M_1 < \ldots < M_{m+1}$.

We now take $M > M_{m+1}/2$ and claim that for any configuration of t_j's such that $M/N < t_1 < \ldots < t_m < \pi - M/N$ and for sufficiently large N, at least one of the m above clustering conditions hold. By contraposition, if this were false, there would be for any $k = 1, \ldots, m$ a different value $j_k \in \{1, \ldots, m-1\}$ such that

$$M_k/N \leq t_{j_k+1} - t_{j_k} < M_{k+1}/N,$$

since $t_0, \pi - t_m \geq M_{m+1}/(2N)$. This yields a contradiction by the pigeonhole principle. We can conclude that we have the uniform bound $R(z) = I + \mathcal{O}\left(\frac{1}{N\mu(M_1,M_2,N)}\right)$ as $N \to \infty$, where we recall that the constant $M_1 > 0$ was arbitrary, but its value has an influence on how large M needs to be.

This estimate is weaker than the one needed for (3.4.2), but we know in addition from [77, formulas (84)–(85), (86), and (89)] that

$$||R(z) - I|| \leq \frac{1}{2\pi}\left|\left|\int_{\Sigma}\frac{R_-(s)(\Delta(s) - I)}{z - s}ds\right|\right|, \tag{3.4.7}$$

where $\Delta(s)$ is a matrix-valued function (the jump matrix), and Σ is the jump contour consisting of the circles $\partial\mathcal{U}_1, \ldots, \partial\mathcal{U}_\ell$, and 2ℓ arcs connecting neighbouring circles by one arc inside and one arc outside the unit circle. On $\partial\mathcal{U}_j$, we have the uniform bound $\Delta(s) - I = \mathcal{O}(\frac{1}{N\mu(M_1,M_2,N)})$ as $N \to \infty$, on the arcs inside $(+)$ or outside $(-)$ we have $\Delta(s) - I = \mathcal{O}(|s|^{\pm N})$ as $N \to \infty$. Substituting this in (3.4.7) and setting $z = \pm 1$, we obtain after straightforward estimates the uniform bound

$$||R(\pm 1) - I|| = \mathcal{O}\left(\frac{1}{Nu_\pm}\right), \qquad N \to \infty.$$

Finally, after all these preparations, the result (3.4.2) follows upon substituting the asymptotics for $R(\pm 1)$ and (3.4.6) in (3.4.5). □

We will now extend the above result to $\alpha_0, \alpha_{m+1} > 0$ in (3.2.1).

Proposition 3.4.2. *Writing $u_+ = t_1$ and $u_- = \pi - t_m$, we have*

$$\begin{aligned}\Phi_N(1) &= e^{\frac{1}{2}(V_0 - V(1))}\frac{\sqrt{\pi}N^{\alpha_0}}{2^{2\alpha_0+\alpha_{m+1}}\Gamma(\alpha_0 + \frac{1}{2})}\prod_{j=1}^{m}\left(2\sin\frac{t_j}{2}\right)^{-2\alpha_j}e^{it_j\beta_j}e^{-i\pi\beta_j} \\ &\qquad \times\left(1 + \mathcal{O}\left(\frac{1}{Nu_+}\right)\right), \\ \Phi_N(-1) &= e^{\frac{1}{2}(V_0 - V(-1))}\frac{(-1)^N\sqrt{\pi}N^{\alpha_{m+1}}}{2^{2\alpha_{m+1}+\alpha_0}\Gamma(\alpha_{m+1} + \frac{1}{2})}\prod_{j=1}^{m}\left(2\cos\frac{t_j}{2}\right)^{-2\alpha_j}e^{it_j\beta_j} \\ &\qquad \times\left(1 + \mathcal{O}\left(\frac{1}{Nu_-}\right)\right),\end{aligned} \tag{3.4.8}$$

as $N \to \infty$, uniformly over the region $M/N < t_1 < \ldots < t_m < \pi - M/N$ with $M > 0$ sufficiently large, and uniformly for α_j and β_j in compact subsets of $[0, +\infty)$ and $i\mathbb{R}$ respectively.

Proof. We again follow the RH analysis from [77] to prove this, the main difference with the proof of Proposition 3.4.1 being that the RH solution at the points ± 1 is now approximated in terms of a local parametrix instead of the global parametrix.

Let $\mathcal{U}_\pm$ be a disk with radius $\frac{u_\pm}{3}$, centred at ± 1. The RH analysis from [77] requires to construct a local parametrix in $\mathcal{U}_\pm$. We now have, because of the explicit transformations $Y \mapsto T \mapsto S \mapsto R$ in [77], the identities

$$\begin{aligned} Y_{11}(\pm 1) = T_{11}(\pm 1_\pm)(\pm 1_\pm)^N &= (\pm 1_\pm)^N S_{11}(\pm 1_\pm) - g(\pm 1_\pm)^{-1} S_{12}(\pm 1_\pm) \\ &= (\pm 1_\pm)^N (RP^\pm)_{11}(\pm 1_\pm) - g(\pm 1)^{-1} (RP^\pm)_{12}(\pm 1_\pm), \end{aligned} \tag{3.4.9}$$

where $g(\pm 1)$ is the boundary value of g when coming from the region inside the lenses in the upper half plane, where $P^\pm$ is the local parametrix defined in $\mathcal{U}_\pm$, and where R is uniformly close to I as $N \to \infty$. In order to obtain large N asymptotics for $\Phi_N(\pm 1)$, we need to substitute the exact formula for $P^\pm$ and the large N asymptotics for R. These computations have been done in [69, Section 7] (see in particular equations (7.23)–(7.26) in that paper, and note the different notations $\beta_j \mapsto -\beta_j$ and $\alpha_0 \mapsto 2\alpha_0 + \frac{1}{2}$, $\alpha_{m+1} \mapsto 2\alpha_{r+1} + \frac{1}{2}$), for the convenience of the reader we sketch these computations here, restricting ourselves to the situation in $\mathcal{U}_+$, as the case $\mathcal{U}_-$ is similar, and also restricting ourselves for simplicity to $\alpha_0 \notin \mathbb{Z}$. The local parametrix P^+ then takes the form

$$P^+(z) = E(z)\Psi(\zeta(z)) g(z)^{-\frac{\sigma_3}{2}} z^{-N\sigma_3/2}, \tag{3.4.10}$$

where $\zeta(z) = N \log z$, where $\Psi(\zeta)$ is the solution to a model RH problem (depending on α_0, see [69, Section 4.1]) whose solution can be constructed out of confluent hypergeometric functions which in the case $\beta_0 = 0$ at hand degenerate to Bessel functions, and where E is a function analytic at ± 1. $E(z)$ and $\Psi(\zeta)$ can be found explicitly in formulas (4.25), (4.32) and (4.50) of [69]. We have

$$E(z) = P^\infty(z) g(z)^{\frac{\sigma_3}{2}} z_j^{N\sigma_3/2} \begin{pmatrix} 0 & e^{2\pi i \alpha_j} \\ -e^{-\pi i \alpha_j} & 0 \end{pmatrix},$$

where P^∞ behaves close to 1, outside the unit circle, in the following way (see [77, formulas (79) and (72)]):

$$\begin{aligned} P^\infty(z) \sim 2^{-\alpha_{m+1}} (z-1)^{-\alpha_0 \sigma_3} e^{-\sum_{k=-\infty}^{-1} V_k \sigma_3} \times \\ \prod_{j=1}^{m} \left(1 - e^{it_j}\right)^{(\beta_j - \alpha_j)\sigma_3} \left(1 - e^{-it_j}\right)^{(-\beta_j - \alpha_j)\sigma_3} \end{aligned} \tag{3.4.11}$$

as $z \to 1$ from outside the unit circle, where all the roots correspond to arguments in $(-\pi, \pi)$.

After a straightforward calculation we obtain, for $z \in \mathcal{U}_+$ in the region outside the unit circle and outside the lens,

$$\begin{aligned} Y_{11}(z) &= z^N (RP_j)_{11}(z) - g(z)^{-1} (RP_j)_{12}(z) \\ &= z^N P_{11}^\infty(z) e^{2\pi i \alpha_j} \left(\Psi_{22}(\zeta(z)) + e^{2\pi i \alpha_0} \Psi_{21}(\zeta(z))\right) \left(1 + \mathcal{O}\left(\frac{1}{N u_+}\right)\right) \end{aligned}$$

as $N \to \infty$, where $\Psi_{21}(\zeta)$ and $\Psi_{22}(\zeta)$ are entries of $\Psi(\zeta)$ in a certain sector of the complex plane, given by

$$\Psi_{21}(\zeta) = -\zeta^{-\alpha_0} e^{-3\pi i \alpha_0} e^{-\zeta/2} \psi(1-\alpha_0, 1-2\alpha_0, \zeta) \frac{\Gamma(1+\alpha_0)}{\Gamma(\alpha_0)}$$

and

$$\Psi_{22}(\zeta) = \zeta^{-\alpha_0} e^{-\pi i \alpha_0} e^{\zeta/2} \psi(-\alpha_0, 1-2\alpha_0, e^{-\pi i}\zeta),$$

where $\psi(a, c; z)$ is the confluent hypergeometric function of the second kind with, in the case where $\alpha_0 \notin \mathbb{Z}$, the standard expansion of $\psi(a, c; z)$ as $z \to 0$,

$$\psi(a, c; z) = \frac{\Gamma(1-c)}{\Gamma(1+a-c)}(1 + \mathcal{O}(z)) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c}(1 + \mathcal{O}(z)).$$

Substituting these asymptotics, we obtain after a straightforward computation

$$Y_{11}(z) = z^N P_{11}^{\infty}(z) e^{\pi i \alpha_0} \zeta(z)^{\alpha_0} \frac{\Gamma(-2\alpha_0)}{\Gamma(-\alpha_0)} \times \\ \left(e^{\zeta(z)/2} e^{-2\pi i \alpha_0} + e^{-\zeta(z)/2} \right) \left(1 + \mathcal{O}\left(\frac{1}{N u_+} \right) \right).$$

Substituting the above asymptotics for P^{∞} and $\zeta(z) = N \log z$, we obtain

$$\Phi_N(1) = e^{\frac{1}{2}(V_0 - V(1))} N^{\alpha_0} \prod_{j=1}^{m} \left(2 \sin \frac{t_j}{2} \right)^{-2\alpha_j} e^{i t_j \beta_j} e^{-i\pi \beta_j} \times \\ \frac{\cos \pi \alpha_0}{2^{\alpha_{m+1}-1}} \frac{\Gamma(-2\alpha_0)}{\Gamma(-\alpha_0)} \left(1 + \mathcal{O}\left(\frac{1}{N u_+} \right) \right) \tag{3.4.12}$$

as $N \to \infty$. Using the reflection formula and the doubling formula for the Gamma function, as well as the relation $\Gamma(1+z) = z\Gamma(z)$, we obtain the statement of the proposition. The other cases, namely the asymptotics for $\Phi_N(1)$ for $\alpha_0 \in \mathbb{Z}$ and the asymptotics for $\Phi_N(-1)$ can be obtained in a similar way, we refer the reader to [69, Section 7] for details. □

Proposition 3.4.3. *We have*

$$\begin{aligned} \Phi_N(1) &= \prod_{j=1}^{m} \left(\sin \frac{t_j}{2} + \frac{1}{n} \right)^{-2\alpha_j} \times e^{\mathcal{O}(1)}, \\ \Phi_N(-1) &= \prod_{j=1}^{m} \left(\cos \frac{t_j}{2} + \frac{1}{n} \right)^{-2\alpha_j} \times e^{\mathcal{O}(1)}, \end{aligned} \tag{3.4.13}$$

as $N \to \infty$, uniformly over the entire region $0 < t_1 < \ldots < t_m < \pi$, and uniformly for α_j and β_j in compact subsets of $[0, +\infty)$ and $i\mathbb{R}$ respectively.

Proof. We again follow the RH analysis from [77] to prove this. We restrict to the computation of $\Phi_N(+1)$, as the computation of $\Phi_N(-1)$ is similar, or can be derived from $\log \Phi_N(+1)$ after transforming the symbol by a rotation. Also,

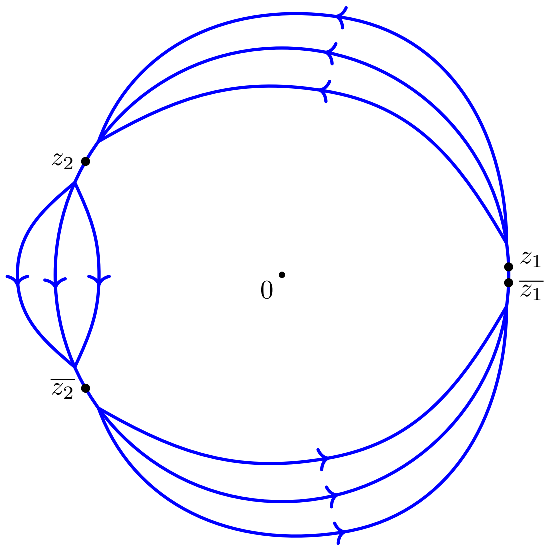

Figure 3.2: Opening of lenses in the case of 4 singularities $z_1, \overline{z_1}, z_2, \overline{z_2}$ partitioned into three clusters.

we can restrict to the case $t_1 \leq M/N$ for some large $M > 0$, since the case $t_1 > M/N$ was handled in Proposition 3.4.1 and this implies the weaker result (3.4.2).

Let us take $M_1 > 2M$, such that $2t_1 \leq M_1/N$, and define the clusters $A_1, \ldots, A_\ell$, depending on M_1 and on N, as before. The points $\pm t_1$ will then belong to the same cluster, which we label as A_1. By restricting to a subsequence of the positive integers N, we can assume that the numbers of points in each cluster are independent of N. We write $2\mu_1$ for the number of points in A_1, such that $A_1 = \{e^{\pm it_k}\}_{k=1}^{\mu_1}$, and we observe that the average of the points in A_1 is equal to $\hat{t}_1 = 0$. Next, we write $\mathcal{U}_1$ for the disk with radius $\mu(M_1, M_2, N)/3$ centred at 1, with $\mu(M_1, M_2, N)$ given by (3.4.3), and we use the local transformation $\zeta(z) = N \log z$ for $z \in \mathcal{U}_1$. We have $\zeta(1) = 0$ and we define $w_{k,N} = -i\zeta(e^{it_k}) = Nt_k$ for $1 \leq k \leq \mu_1$. Note that $w_{k,N} \leq M_1/2$ for all $k \leq \mu_1$ because of the clustering condition.

The RH analysis from [77] requires us to construct a local parametrix in $\mathcal{U}_1$. We now have, because of the explicit transformations $Y \mapsto T \mapsto S \mapsto R$ in [77] (see Figure 3.2 for the shape of the jump contour for S in this case), the identities

$$Y_{11}(1) = (1_+)^N T_{11}(1_+) = S_{11}(1_+) = (RP)_{11}(1_+),$$

where $R(1)$ is bounded as $N \to \infty$, uniformly under clustering condition (M_1, M_2, N) for sufficiently large M_2 (it is in fact close to I, but we will not need this). The corresponding lenses are described in Figure 3.2. Moreover, P is the local parametrix defined in $\mathcal{U}_1$. The construction of this local parametrix is explained in detail in [77, Section 6.3]. We omit the technical details of this construction, and restrict ourselves to the elements from it that we need for our purposes. As $z \to 1_+$, we have

$$P(1_+) = E(1)\Phi(0; w_{1,N}, \dots, w_{\mu_1,N}) \begin{pmatrix} 0 & g(1_+)^{\frac{1}{2}} \\ g(1_+)^{-\frac{1}{2}} & 0 \end{pmatrix}, \tag{3.4.14}$$

where $\Phi(\zeta; w_1, \dots, w_{\mu_1})$ is the solution to a model RH problem depending on parameters $w_1, \dots, w_{\mu_1}$, $E(1)$ is given by

$$E(1) = P^{\infty}(1_+) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \prod_{\nu=1}^{\mu_1} (-iw_{\nu,N})^{\beta_\nu \sigma_3} \exp[\pi i(\alpha_\nu - \beta_\nu)\sigma_3] g(1_+)^{-\frac{1}{2}\sigma_3}, \tag{3.4.15}$$

with $P^{\infty}(1_+)$ the global parametrix given by (3.4.6). It is easily seen from this expression that $E(1)$ is bounded as $N \to \infty$, uniformly in the parameters $t_1, \dots, t_m$.

Since $P^{\infty}(1_+)$ is diagonal, $E(1)$ is off-diagonal and after a straightforward calculation we obtain

$$\begin{aligned} Y_{11}(1) &= (RP)_{11}(1_+) \\ &= E_{12}(1)\Phi_{22}(0; w_{1,N}, \dots, w_{\mu_1,N}) g(1_+)^{-\frac{1}{2}} \\ &\qquad + \mathcal{O}\left(\frac{g(1_+)^{-\frac{1}{2}} \Phi(0; w_{1,N}, \dots, w_{\mu_1,N})}{N\mu(M_1, M_2, N)} \right), \end{aligned} \tag{3.4.16}$$

as $N \to \infty$, uniformly under clustering condition (M_1, M_2, N) for M_2 large enough.

The matrix $\Phi(0; w_1, \dots, w_{\mu_1})$ is continuous as a function of $w_1, \dots, w_{\mu_1} > \epsilon$ for any ϵ, see [77, Section 5.3], and this implies that

$$Y_{11}(1) = \mathcal{O}(g(1_+)^{-1/2}), \qquad N \to \infty,$$

uniformly under clustering condition (M_1, M_2, N) for M_2 large enough and with $w_{1,N}, \dots, w_{\mu_1,N} > \epsilon$ for some $\epsilon > 0$, which implies the result in this case by (3.2.1).

In order to evaluate the asymptotics of $\Phi(0; w_1, \dots, w_{\mu_1})$ when some of the w_j's, say $w_1, \dots, w_k$, tend to 0 as $N \to \infty$, we need to follow the construction of another local parametrix Q in [77, Section 5.3]. We again omit the details of this construction and refer the interested reader to [77]. The result from this construction is that

$$\Phi(0; w_1, \dots, w_{\mu_1}) = F_N \prod_{\nu=1}^{k} (-iw_\nu)^{2\alpha_\nu \sigma_3} U_N \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} D,$$

where F_N is uniformly bounded as $N \to \infty$, U_N is upper-triangular, and D is a diagonal matrix independent of N, and the determinants of F_N, D, U_N are all equal to 1. It follows that

$$\log \Phi_{22}(0; w_1, \dots, w_{\mu_1}) = \sum_{\nu=1}^{k} 2\alpha_\nu \log |w_\nu| + \mathcal{O}(1).$$

Substituting this in (**??**) and recalling that $E(1)$ is uniformly bounded as $N \to \infty$, we get

$$\begin{aligned}\log Y_{11}(1) &= -\frac{1}{2}\log g(1_+) + \sum_{\nu=1}^{k} 2\alpha_\nu \log|w_{\nu,N}| + \mathcal{O}(1)\\ &= -2\sum_{j=1}^{m}\alpha_j \log|1-e^{it_j}| + 2\sum_{\nu=1}^{k}\alpha_\nu \log(Nt_j) + \mathcal{O}(1).\end{aligned}$$

It is straightforward to derive the result from this estimate. □

3.4.2 Proofs of Theorem 3.2.1 and Theorem 3.2.2

Under the assumptions of Theorem 3.2.1, we have by Proposition 3.1.1 and Propositions 3.4.1–3.4.2 that

$$\mathbb{E}_n^{(0,+)}[f] = \left[\frac{\mathbb{E}_{2n}^{\mathbb{U}}[g]}{-\Phi_{2n-1}(1)\Phi_{2n-1}(-1)}\right]^{1/2} = C_n\left[\mathbb{E}_{2n}^{\mathbb{U}}[g]\right]^{1/2}(1+o(1)),$$

as $n \to \infty$, where C_n is as in (3.2.6). The asymptotics for $\mathbb{E}_n^{(2,-)}[f]$ and $\mathbb{E}_n^{(1,\pm)}[f]$ follow in a similar fashion. This ends the proof of Theorem 3.2.1.

Under the assumptions of Theorem 3.2.2, we use Proposition 3.1.1 and Proposition 3.4.3 to obtain the uniform large N asymptotics

$$\begin{aligned}\mathbb{E}_n^{(0,+)}[f] &= \left(\mathbb{E}_{2n}^{\mathbb{U}}[g]\right)^{1/2}\left[-\Phi_{2n-1}(1)\Phi_{2n-1}(-1)\right]^{-\frac{1}{2}}\\ &= \left(\mathbb{E}_{2n}^{\mathbb{U}}[g]\right)^{\frac{1}{2}}\prod_{j=1}^{m}\left(\sin\frac{t_j}{2}+\frac{1}{n}\right)^{\alpha_j}\left(\cos\frac{t_j}{2}+\frac{1}{n}\right)^{\alpha_j}\times e^{\mathcal{O}(1)}\\ &= \left(\mathbb{E}_{2n}^{\mathbb{U}}[g]\right)^{\frac{1}{2}}\prod_{j=1}^{m}\left(\sin t_j+\frac{1}{n}\right)^{\alpha_j}\times e^{\mathcal{O}(1)}.\end{aligned}$$

By similar computations, we obtain the required asymptotics for $\mathbb{E}_n^{(2,-)}[f]$ and $\mathbb{E}_n^{(1,\pm)}[f]$. This ends the proof of Theorem 3.2.2.

3.5 Symbols with a gap or an emerging gap

In this section, we assume that $g(e^{it})$ defined by (3.1.3) is of the form (3.2.11), i.e.

$$g(e^{it}) = e^{V(e^{it})} \times \begin{cases}1 & \text{for } 0 \le |t| \le t_0,\\ s & \text{for } t_0 < |t| \le \pi,\end{cases}$$

for some real-valued function V analytic in a neighbourhood of the unit circle, and with $s \in [0,1]$.

3.5.1 Asymptotics for $\Phi_N(\pm 1)$

Let Φ_N be the monic polynomial of degree N, orthogonal with the weight g on the unit circle, characterized by the orthogonality conditions (3.1.8). The proof of the following result is based on the RH representation for $\Phi_N(z)$, see Section 3.4.1, and on the large N asymptotic analysis of the RH problem in spirit of the analysis performed in [55]. We do not follow exactly the steps of transformations from [55], but introduce a slightly different sequence of transformations. The most significant differences of our analysis from the one done in [55] is that, first, during the step $Y \mapsto T$ we make a cosmetic transformation inside the unit disk, $|z| < 1$, and second, the function ϕ used in Step 3 is different from the one in [55]: they coincide up to a constant for $|z| > 1$ but have opposite signs for $|z| < 1$.

Proposition 3.5.1. *Let V be as in Theorem 3.2.5. As $N \to \infty$ with $s = 0$, or as $N \to \infty$ and at the same time $s \to 0$ in such a way that $s \le \left(\tan \frac{t_0}{4}\right)^{2N}$, we have the large N asymptotics*

$$\Phi_N(1) = \sqrt{2}\cos\frac{t_0 - \pi + (-1)^N \pi}{4}\left(\sin\frac{t_0}{2}\right)^N e^{-\frac{V(1)}{2}}\delta(\infty)^{-1}(1 + o(1)),$$
$$\Phi_N(-1) = (-1)^N \cos\frac{t_0}{4}\left(1 + \cos\frac{t_0}{2}\right)^N \delta_-(-1)\delta(\infty)^{-1}(1 + o(1)).$$

These asymptotics are also valid as $t_0 \to \pi$, as long as $N(\pi - t_0) \to \infty$. The $o(1)$ terms can be written as $\mathcal{O}\left(\frac{1}{N(\pi - t_0)} + s\,(\tan\frac{t_0}{4})^{-2N}\frac{1}{\sqrt{N(\pi - t_0)}}\right)$.

Remark 3.5.2. *Note that when $s = 0$ or $s\lambda(\tan\frac{t_0}{4})^{2N}$, the first error term $\frac{1}{N(\pi - t_0)}$ dominates the second. On the other hand, when s is close to $(\tan\frac{t_0}{4})^{2N}$, the second error term becomes dominant, and is $\mathcal{O}(\frac{1}{\sqrt{N(\pi - t_0)}})$. Furthermore, when t_0 is not approaching π, the factor $(\pi - t_0)$ in the error terms can be omitted, but as $t_0 \to \pi$, the error term becomes larger due to it.*

Denote
$$\gamma = \{z : |z| = 1, \arg z \in (-t_0, t_0)\},$$
$$\gamma^c = \{z : |z| = 1, \arg z \in (t_0, \pi) \cup (-\pi, -t_0)\},$$
both oriented in the counter-clockwise direction.

Proof. The asymptotic analysis of the RH problem from Section 3.4.1 can be done using the following steps of transformations,

$$Y \mapsto T \mapsto \widehat{T} \mapsto \widetilde{T} \mapsto S \mapsto R.$$

Here the transformation $Y \mapsto T$ normalizes the asymptotics at infinity, while the transformations $T \mapsto \widehat{T} \mapsto \widetilde{T}$ are preparatory transformations before opening of the lenses. Then, $\widetilde{T} \mapsto S$ consists of opening of the lenses, and $S \mapsto R$ is the final transformation to pass to a small-norm RH problem; this step involves construction of parametrices. We start by giving some more details about each of these transformations.

Step 1. Define

$$T(z) = \begin{cases} Y(z)z^{-N\sigma_3}, & |z| > 1, \\ Y(z)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & |z| < 1. \end{cases}$$

Here the transformation for $|z| > 1$ aims at improving the large z asymptotics of Y, while the transformation for $|z| < 1$ is a cosmetic one, which makes factorizations at further steps more transparent. $T(z)$ has the asymptotics $T(z) \to I$ as $z \to \infty$ and satisfies the jump $T_+(z) = T_-(z)\begin{pmatrix} g(z) & -z^N \\ z^{-N} & 0 \end{pmatrix}$ for z on the unit circle $\mathcal{C}$.

Step 2. The jump for $T(z)$ is highly oscillating for $z \in \mathcal{C}$, and the next step is to factorize it into product of two matrix functions, which can then be moved respectively inside or outside the unit disk where they would be exponentially small. This is done differently for $z \in \gamma$ and for $z \in \gamma^c$, and we start with γ. The idea is to exchange the term $g(z)$ in the $(1, 1)$ entry of the jump for the T with 1; an appropriate factorization will then easily follow. This is achieved with the help of the following function $\delta(z)$,

$$\delta(z) = \exp\left\{ \frac{h(z)}{2\pi i} \int\limits_{\gamma} \frac{V(\zeta)d\zeta}{(\zeta - z)h(\zeta)} \right\},$$

where the function $h(\zeta) = ((\zeta - z_0)(\zeta - \overline{z_0}))^{1/2}$

is analytic in $\zeta \in \mathbb{C} \setminus \gamma^c$ and asymptotic to ζ as $\zeta \to \infty$. The function δ is analytic in $\mathbb{C} \setminus \mathcal{C}$, has a finite non-zero limit as $\zeta \to \infty$, and its boundary values satisfy the following conjugation conditions on the circle $\mathcal{C}$:

$$\delta_+(z)\delta_-(z) = 1, \ z \in \gamma^c, \qquad \frac{\delta_+(z)}{\delta_-(z)} = \mathrm{e}^{V(z)}, \ z \in \gamma.$$

Using the properties $V(z) = V(z^{-1})$ for $|z| = 1$ and $h(\zeta) = \zeta h(\zeta^{-1})$, one can check that for all z we have $\delta(z)\delta(z^{-1}) = 1$ and $\overline{\delta(\overline{z})} = \delta(z)$. Let

$$\widehat{T}(z) = \delta(\infty)^{\sigma_3} T(z)\delta(z)^{-\sigma_3}.$$

$\widehat{T}$ tends to I as $z \to \infty$ and satisfies the following jumps:

$$\widehat{T}_+(z) = \widehat{T}_-(z)\begin{pmatrix} 1 & \frac{-z^N\delta_+(z)^2}{\mathrm{e}^{V(z)}} \\ \frac{1}{z^N\delta_-(z)^2\mathrm{e}^{V(z)}} & 0 \end{pmatrix}, z \in \gamma,$$

$$\widehat{T}_+(z) = \widehat{T}_-(z)\begin{pmatrix} s\mathrm{e}^{V(z)}\frac{\delta_-(z)}{\delta_+(z)} & -z^N \\ z^{-N} & 0 \end{pmatrix}, z \in \gamma^c,$$

We see that the jump matrix on γ can be factorized into a product of a lower-triangular and an upper-triangular matrix with ones on the diagonals, and this allows to “open lenses” around γ, in other words allows to get rid of oscillating entries on γ by transforming them into exponentially small ones on lenses. However, we still have oscillating entries on γ^c, and we cannot follow the same strategy as for γ (i.e., to transform the $(1,1)$ entry in the jump matrix to 1). Instead, we transform off-diagonal entries into constant ones, by introducing the following function $\phi(z)$, which is to replace the function $\log z$ in $z^N = \mathrm{e}^{N\log z}$, and thus to transform the entries $z^{\pm N}$ into 1.

Step 3. Define

$$\phi(z) = \int_{z_0}^{z} \frac{(\zeta+1)\mathrm{d}\zeta}{\zeta h(\zeta)} + \pi i, \qquad \ell = -2\log\sin\frac{t_0}{2} > 0,$$

where the path of integration should not cross $(-\infty, 0] \cup \gamma^c$. Then one can check that $\phi(z) - \log z = \ell + \mathcal{O}(z^{-1})$ as $z \to \infty$, and $\overline{\phi(\overline{z})} = \phi(z)$ for all z, and $\phi(z) - \log z$ is analytic in $\mathbb{C} \setminus \gamma^c$, where the principal branch of the logarithm is taken. The function $\phi_-(z) - \phi_+(z)$ is continuous and real-valued on γ^c, and its maximum over γ^c is attained at the point -1, with $\phi_-(-1) - \phi_+(-1) = -4\log\tan\frac{t_0}{4} > 0$. Let

$$\widetilde{T}(z) = e^{\frac{N}{2}(\ell-\pi i)\sigma_3}\widehat{T}(z)e^{-\frac{N}{2}(\phi(z)-\pi i-\log z)\sigma_3},$$

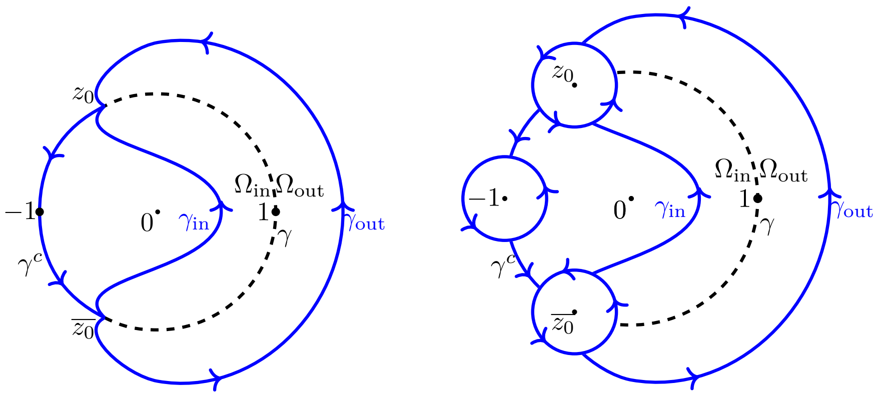

Figure 3.3: Jump contour for S (on the left), and for R (on the right).

then $\widetilde{T}(z) \to I$ as $z \to \infty$ and $\widetilde{T}$ satisfies the following jumps:

$$\begin{aligned}
\widetilde{T}_+(z) &= \widetilde{T}_-(z)\begin{pmatrix} 1 & \frac{\mathrm{e}^{N(\phi(z)-\pi i)}\delta_+(z)^2}{-\mathrm{e}^{V(z)}} \\ \frac{\mathrm{e}^{-N(\phi(z)-\pi i)}}{\delta_-(z)^2\mathrm{e}^{V(z)}} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ \frac{\mathrm{e}^{-N(\phi(z)-\pi i)}}{\delta_-(z)^2\mathrm{e}^{V(z)}} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{\mathrm{e}^{N(\phi(z)-\pi i)}\delta_+(z)^2}{-\mathrm{e}^{V(z)}} \\ 0 & 1 \end{pmatrix}, z \in \gamma, \\
\widetilde{T}_+(z) &= \widetilde{T}_-(z)\begin{pmatrix} s\mathrm{e}^{\frac{N}{2}(\phi_-(z)-\phi_+(z))}\mathrm{e}^{V(z)}\,\frac{\delta_-(z)}{\delta_+(z)} & -z^N \\ z^{-N} & 0 \end{pmatrix}, z \in \gamma^c.
\end{aligned}$$

Step 4. The next step is the opening of lenses around γ. Consider the regions as indicated in the left part of Figure 3.3, and define

$$S(z) = \begin{cases} \widetilde{T}(z) \begin{pmatrix} 1 & 0 \\ \delta(z)^{-2} e^{-V(z)} (-1)^N e^{-N(\phi(z))} & 1 \end{pmatrix}, & z \in \Omega_{\text{out}}, \\ \widetilde{T}(z) \begin{pmatrix} 1 & \delta(z)^2 e^{-V(z)} (-1)^N e^{N\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_{\text{in}}, \\ \widetilde{T}(z), & \text{elsewhere.} \end{cases}$$

Step 5a. Now, we take $r > 0$ sufficiently small (but fixed) and we define parametrices, i.e. local approximations, for S as follows. Let $U_{z_0}, U_{\overline{z_0}}, U_{-1}$ be (non-intersecting) disks centred at $z_0, \overline{z_0}, -1$, respectively, of the radius $r \cos \frac{t_0}{2}$; their boundaries $\partial U_{z_0}, \partial U_{\overline{z_0}}, \partial U_{-1}$ are oriented in the counter-clockwise direction. Define (we use the letters u(up), d(down), l(left) to distinguish between the parametrices at the points $z_0, \overline{z}_0, -1$, respectively; see also the right part of Figure 3.3)

$$P(z) = \begin{cases} P^{\infty}(z), & z \in \mathbb{C} \setminus (U_{z_0} \cup U_{\overline{z}_0} \cup U_{-1}), \\ P_u(z), & z \in U_{z_0}, \\ P_d(z), & z \in U_{\overline{z}_0}, \\ P_l(z), & z \in U_{-1}, \end{cases}$$

We see that the radius $r \cos \frac{t_0}{2}$ of the disks shrinks as t_0 approaches π. For us, the explicit expressions for the local parametrices P_u and P_d will be unimportant because we only need to evaluate Φ_N at the points ± 1; however, we will still need them in order to estimate the error term. The form of the outer parametrix P^{∞} on the other hand is more important: it is given by

$$P^{\infty}(z) = \begin{pmatrix} \frac{1}{2}(\kappa(z) + \kappa(z)^{-1}) & \frac{i}{2}(\kappa(z) - \kappa(z)^{-1}) \\ \frac{-i}{2}(\kappa(z) - \kappa(z)^{-1}) & \frac{1}{2}(\kappa(z) + \kappa(z)^{-1}) \end{pmatrix},$$

where

$$\kappa(z) = \left(\frac{z - \overline{z_0}}{z - z_0} \right)^{1/4},$$

analytic in $z \in \mathbb{C} \setminus \gamma^c$ and asymptotic to 1 at infinity. Note that $\kappa(1) = e^{i(\pi - t_0)/4}$, $\kappa_-(-1) = e^{-it_0/4}$.

Step 5b: Local parametrix at z_0.

Change of variable. First of all, the linear fractional change of variable $k = k(z) = \frac{1 - z\,\overline{z_0}}{z - \overline{z_0}}$ maps the points of the unit circle to the real line as follows:

$$z_0 \mapsto 0, \quad -1 \mapsto -1, \quad \overline{z_0} \to \infty, \quad 1 \mapsto 1,$$

and thus allows to separate the points $z_0, -1, \overline{z_0}$, which might be merging as $t_0 \to \pi$. Next, using the variable k the function $\phi(z)$ can be written as

$$\phi(z) = \pi i - 2i \cos \frac{t_0}{2} \int_0^{k(z)} \frac{(k+1)\mathrm{d}k}{(k + z_0)(k + \overline{z_0})\sqrt{k}}, \tag{3.5.1}$$

where the path of integration does not intersect $(-\infty, 0]$, and the principal branch of the square root is taken. This prompts to introduce a local variable $\zeta = \zeta(z; t_0)$ in the disk U_{z_0} as follows: $\phi(z) =: \pi i - 4i \cos\frac{t_0}{2}\sqrt{\zeta}$, so that $\zeta = k(1+\mathcal{O}(k)), k \to 0$, and the branch cut for $\sqrt{\zeta}$, i.e. the half-line $\zeta < 0$, corresponds to $z \in \gamma^c$. Introduce also the new large parameter $\tau := 2N\cos\frac{t_0}{2}$, then $N(\phi(z) - \pi i) = -2i\tau\sqrt{\zeta}$.

Bessel parametrix. Similarly as e.g. in [111, Section 6] (but note the different sign of the off-diagonal entries of the jump matrices), we construct a function which solves exactly the same jumps as S in a small neighbourhood of the point z_0. Define

$$\Psi(\zeta) = \begin{pmatrix} \sqrt{\pi}\,\mathrm{e}^{\frac{-\pi i}{4}}\sqrt{\zeta}I_1(-i\sqrt{\zeta}) & \frac{-1}{\sqrt{\pi}}\mathrm{e}^{\frac{\pi i}{4}}\sqrt{\zeta}K_1(-i\sqrt{\zeta}) \\ -\sqrt{\pi}\,\mathrm{e}^{\frac{\pi i}{4}}I_0(-i\sqrt{\zeta}) & \frac{1}{\sqrt{\pi}}\mathrm{e}^{\frac{-\pi i}{4}}K_0(-i\sqrt{\zeta}) \end{pmatrix}, \quad \arg\zeta \in (\pi-\alpha, \pi),$$
$$= \begin{pmatrix} \frac{1}{\sqrt{\pi}}\mathrm{e}^{\frac{\pi i}{4}}\sqrt{\zeta}K_1(i\sqrt{\zeta}) & \frac{-1}{\sqrt{\pi}}\mathrm{e}^{\frac{\pi i}{4}}\sqrt{\zeta}K_1(-i\sqrt{\zeta}) \\ \frac{1}{\sqrt{\pi}}\mathrm{e}^{\frac{-\pi i}{4}}K_0(i\sqrt{\zeta}) & \frac{1}{\sqrt{\pi}}\mathrm{e}^{\frac{-\pi i}{4}}K_0(-i\sqrt{\zeta}) \end{pmatrix}, \quad \arg\zeta \in (-\pi+\alpha, \pi-\alpha),$$
$$= \begin{pmatrix} \frac{1}{\sqrt{\pi}}\mathrm{e}^{\frac{\pi i}{4}}\sqrt{\zeta}K_1(i\sqrt{\zeta}) & \sqrt{\pi}\mathrm{e}^{\frac{-\pi i}{4}}\sqrt{\zeta}I_1(i\sqrt{\zeta}) \\ \frac{1}{\sqrt{\pi}}\mathrm{e}^{\frac{-\pi i}{4}}K_0(i\sqrt{\zeta}) & \sqrt{\pi}\mathrm{e}^{\frac{\pi i}{4}}I_0(i\sqrt{\zeta}) \end{pmatrix}, \quad \arg\zeta \in (-\pi, -\pi+\alpha),$$

where $\alpha \in (0,\pi)$ and $I_j, K_j, j = 0,1$ are the modified Bessel functions [1, Chapter 9.6]. The function Ψ satisfies the jump conditions $\Psi_+(\zeta) = \Psi_-(\zeta)\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$, $\zeta \in (\infty\mathrm{e}^{i(\pi-\alpha)}, 0)$; $\Psi_+(\zeta) = \Psi_-(\zeta)\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}, \zeta \in (\infty\mathrm{e}^{-i(\pi+\alpha)}, 0)$; $\Psi_+(\zeta) = \Psi_-(\zeta)\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \zeta \in (0, -\infty)$, where the orientation of the segments is from the first mentioned point to the last one, and $(\infty\mathrm{e}^{i\beta}, 0)$ denotes the ray coming from infinity to the origin at an angle $\beta \in \mathbb{R}$ (see the left part of Figure 3.4). Besides, the function Ψ satisfies the uniform in $\arg\zeta \in [-\pi, \pi]$ asymptotics

$$\Psi(\zeta) = \zeta^{\frac{\sigma_3}{4}}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\mathcal{E}(\zeta)\mathrm{e}^{-i\sqrt{\zeta}\sigma_3}, \quad \mathcal{E}(\zeta) = I + \mathcal{O}(\frac{1}{\sqrt{\zeta}}), \qquad \zeta \to \infty.$$

We will also need the function $\widehat{\Psi}(\zeta) := \Psi(\zeta)G(\zeta)$, where

$$G(\zeta) := \begin{cases} I - s\frac{1}{2\pi i}\log\zeta\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, & \arg\zeta \in (\pi-\alpha, \pi), \\ I - s\frac{1}{2\pi i}\log\zeta\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}, & \arg\zeta \in (-\pi+\alpha, \pi-\alpha), \\ I + s\frac{1}{2\pi i}\log\zeta\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, & \arg\zeta \in (-\pi, -\pi+\alpha). \end{cases}$$

The function $\widehat{\Psi}$ satisfies the jumps as in the right part of Figure 3.4.

For $z : |z - z_0| < r\cos\frac{t_0}{2}$, define

$$P_u(z) = B_u(z)\widehat{\Psi}(\tau^2\zeta)\delta(z)^{-\sigma_3}\mathrm{e}^{-\frac{1}{2}V(z)\mathrm{sgn}(\log|z|)\sigma_3}\mathrm{e}^{-\frac{N}{2}(\phi(z)-\pi i)\sigma_3},$$

where $B_u(z) = P^{(\infty)}(z)\delta(z)^{\sigma_3}\mathrm{e}^{\frac{1}{2}V(z)\mathrm{sgn}(\log|z|)\sigma_3}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}(\tau^2\zeta)^{-\frac{\sigma_3}{4}}$ and $B_u(z)$ is analytic in U_{z_0} (i.e., does not have jumps across $\mathcal{C}$). Here $\mathrm{sgn}(x) = \frac{x}{|x|}$ is the signum function, so that $\mathrm{e}^{-\frac{1}{2}V(z)\mathrm{sgn}(\log|z|)\sigma_3}$ equals $\mathrm{e}^{\frac{1}{2}V(z)\sigma_3}$ for $|z| < 1$ and equals $\mathrm{e}^{-\frac{1}{2}V(z)\sigma_3}$ for $|z| > 1$. The function $P_u(z)$ satisfies the same jumps as $S(z)$ inside U_{z_0}, and on the boundary ∂U_{z_0} we have the following matching condition:

$$\begin{aligned}P(z)P^{(\infty)}(z)^{-1} = {} & P^{(\infty)}(z)\delta(z)^{\sigma_3}\mathrm{e}^{\frac{1}{2}V(z)\mathrm{sgn}(\log|z|)\sigma_3}\times\\ & \mathcal{E}(\tau^2\zeta)\mathrm{e}^{-i\tau\sqrt{\zeta}\sigma_3}G(\tau^2\zeta)\mathrm{e}^{-\frac{N}{2}(\phi(z)-\pi)\sigma_3}\delta(z)^{-\sigma_3}\times\\ & \mathrm{e}^{-\frac{1}{2}V(z)\mathrm{sgn}(\log|z|)\sigma_3}P^{(\infty)}(z)^{-1}\\ = {} & I + \mathcal{O}(\frac{1}{\tau\sqrt{\zeta}}),\end{aligned}$$

as $\tau^2\zeta \to \infty$. Here we used that P^∞ is bounded on ∂U_z uniformly in t_0.

Step 5c: Local parametrix at $\overline{z}_0$. For z inside $U_{\overline{z}_0}$ we define $P(z) := \sigma\overline{P(\overline{z})}\sigma$, where $\sigma = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.

Step 5d: Local parametrix at -1. For $z \in U_{-1}$, define

$$P_l(z) = P^{(\infty)}(z)G_l(z),$$

where $G_l(z) = \begin{bmatrix}1 & 0\\ -sf(z) & 1\end{bmatrix}$ for $|z| < 1$ and $G_l(z) = \begin{bmatrix}1 & sf(z)\\ 0 & 1\end{bmatrix}$ for $|z| > 1$, with

$$f(z) = \frac{1}{2\pi i}\int\limits_{\gamma^c}\mathrm{e}^{\frac{N}{2}(\phi_-(\xi)-\phi_+(\xi))}\,\mathrm{e}^{V(\xi)}\,\frac{\delta_-(\xi)}{\delta_+(\xi)}\,\frac{\mathrm{d}\xi}{\xi - z}.$$

Note that $\phi_- - \phi_+$ has a double zero at the point $z = -1$, and hence large N asymptotics of $f(z)$ can be obtained by classical saddle point methods. Using (3.5.1), we see that the large parameter is $N\cos\frac{t_0}{2}$ rather than N, and for $z \in \partial D_{-1}$ we have $|f(z)| = \mathcal{O}(\frac{1}{\sqrt{N\cos\frac{t_0}{2}}})\mathrm{e}^{N\phi_-(-1)}$. The matching condition on the circle $|z+1| = r\cos\frac{t_0}{2}$ is

$$\begin{aligned}P_l(z)P^{(\infty)}(z)^{-1} &= P^{(\infty)}(z)G(z)P^{(\infty)}(z)^{-1}\\ &= I + \mathcal{O}\left(\frac{1}{\sqrt{N\cos\frac{t_0}{2}}}\,s\,(\tan\frac{t_0}{4})^{-2N}\right),\end{aligned}$$

as $N\cos\frac{t_0}{2} \to \infty$. Here we used that P^∞ is bounded on ∂U_{-1} uniformly in t_0.

Step 6. Define the error function R by the formula

$$S(z) = R(z)P(z),$$

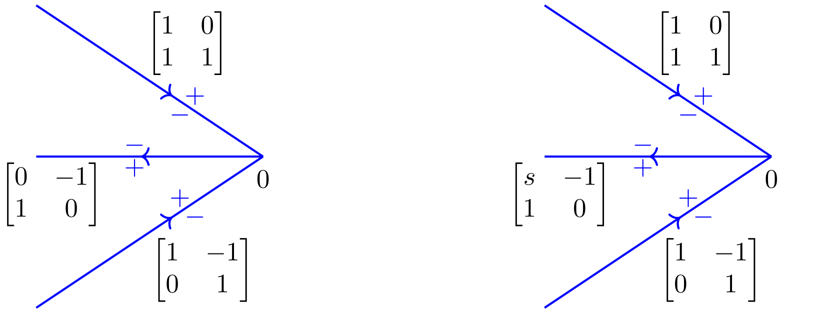

Figure 3.4: Jumps for the functions $\Psi(\zeta)$ (on the left) and $\widehat{\Psi}(\zeta)$ (on the right).

where as before, P means P_u, P_d, P_l in the relevant disks, and P means P^∞ elsewhere. The jump conditions for R on the disks $U_{z_0}, U_{\overline{z}_0}, U_{-1}$ allow to conclude that $R(z) = I + \mathcal{O}((N\cos\frac{t_0}{2})^{-1} + (N\cos\frac{t_0}{2})^{-1/2}s(\tan\frac{t_0}{4})^{-2N})$ uniformly in z, as $N\cos\frac{t_0}{2} \to \infty$, under the conditions of Theorem 3.2.5 (note that this is consistent with the results of the RH analysis from [55]). Tracing back the chain of transformations from R to Y, we find that (as $N\cos\frac{t_0}{2} \to \infty$)

$$\begin{aligned} Y_{11}(z) = \Bigg(&\frac{1}{2}(\kappa(z)+\kappa(z)^{-1})R_{11}(z) - \frac{i}{2}(\kappa(z)-\kappa(z)^{-1})R_{12}(z) \\ &- \frac{\frac{1}{2}(\kappa(z)+\kappa(z)^{-1})R_{12}(z) + \frac{i}{2}(\kappa(z)-\kappa(z)^{-1})R_{11}(z)}{\delta^2(z)e^{V(z)}e^{N(\phi(z)-\pi i)}}\Bigg) \\ &\times e^{\frac{N}{2}(\phi(z)+\log z-\ell)}\frac{\delta(z)}{\delta(\infty)}, \end{aligned}$$

for $z \in \Omega_{\mathrm{out}}$, and that

$$\begin{aligned} Y_{11}(z) = &\left(\frac{1}{2}(\kappa(z)+\kappa(z)^{-1})R_{11}(z) - i\frac{1}{2}(\kappa(z)-\kappa(z)^{-1})R_{12}(z)\right)\times \\ &e^{\frac{N}{2}(\phi(z)+\log z-\ell)}\frac{\delta(z)}{\delta(\infty)}, \end{aligned}$$

for $z \in \{z : |z| > 1\} \setminus \Omega_{\mathrm{out}}$. From here, using $\delta_-(1) = \mathrm{e}^{-\frac{1}{2}V(1)}, \phi(1) = 0$, we obtain

$$\begin{aligned} Y_{11}(-1) = &\left(\cos\frac{t_0}{4}R_{11}(-1-0) - \sin\frac{t_0}{4}R_{12}(-1-0)\right)\times \\ &\left(1+\cos\frac{t_0}{2}\right)^N(-1)^N\frac{\delta_-(-1)}{\delta(\infty)}, \end{aligned}$$

$$\begin{aligned} Y_{11}(1) = &\Bigg(\left(\cos\frac{\pi-t_0}{4} + (-1)^N\sin\frac{\pi-t_0}{4}\right)R_{11}(1) \\ &+\left(\sin\frac{\pi-t_0}{4} - (-1)^N\cos\frac{\pi-t_0}{4}\;\right)R_{12}(1)\Bigg)\times\left(\sin\frac{t_0}{2}\right)^N\frac{e^{-V(1)/2}}{\delta(\infty)}. \end{aligned}$$

Substituting the asymptotics $R(z) = I + o(1)$ for R, we obtain the result. $\square$

3.5.2 Proof of Theorem 3.2.5

From Proposition 3.5.1, we obtain

$$\Phi_N(1)\Phi_N(-1) = (-1)^N C_N^2(1+o(1)), \qquad \frac{\Phi_N(1)}{\Phi_N(-1)} = (-1)^N \widetilde{C}_N^2(1+o(1)),$$

as $N \to \infty$, where

$$C_N^2 = \sqrt{2}\cos\frac{t_0}{4}\cos\frac{t_0-\pi+(-1)^N\pi}{4}\left(\sin\frac{t_0}{2}\left(1+\cos\frac{t_0}{2}\right)\right)^N e^{-\frac{1}{2}V(1)}\frac{\delta_-(-1)}{\delta(\infty)^2},$$
$$\widetilde{C}_N^2 = \sqrt{2}\frac{\cos\frac{t_0-\pi+(-1)^N\pi}{4}}{\cos\frac{t_0}{4}}\left(\frac{\sin\frac{t_0}{2}}{1+\cos\frac{t_0}{2}}\right)^N\frac{e^{-\frac{1}{2}V(1)}}{\delta_-(-1)}.$$

Substituting this in Proposition 3.1.1, we obtain (3.2.15).

3.6 Gap probabilities and global rigidity

3.6.1 Proof of Corollary 3.2.6

The goal is to apply Theorem 3.2.5 to compute the averages in (3.2.21), but this requires certain adaptations. One needs to make the change of variables $\theta_k \mapsto \pi - \theta_k$ for $k = 1, ..., n$ in the averages (3.2.21), which given (3.1.1) yields

$$\begin{aligned} E_{2n}^{+}(t_0;0) &= \mathbb{E}_n^{(0,+)}[f], \\ E_{2n+2}^{-}(t_0;0) &= \mathbb{E}_n^{(2,-)}[f], \\ E_{2n+1}^{\pm}(t_0;0) &= \mathbb{E}_n^{(1,\mp)}[f], \end{aligned}$$

where f is related to g in (3.2.11) with $V = 0$, $s = 0$ and with the change of parameter $t_0 \mapsto \pi - t_0$. One may therefore compute the right-hand side of the above equalities using Theorem 3.2.5, and this yields with $c = 2^{\frac{1}{24}}e^{\frac{3}{2}\zeta'(-1)}$

$$\begin{aligned} E_{2n}^{+}(t_0;0) &= 2^{\frac{1}{4}}\left(\cos\frac{t_0}{2}\right)^{-n}\left(1+\sin\frac{t_0}{2}\right)^{-\frac{2n-1}{2}}\frac{\left(\cos\frac{t_0}{2}\right)^{2n^2}}{(2n\sin\frac{t_0}{2})^{\frac{1}{8}}}(1+o(1)), \\ E_{2n+2}^{-}(t_0;0) &= c2^{\frac{1}{24}}e^{\frac{3}{2}\zeta'(-1)}2^{-\frac{1}{4}}\left(\cos\frac{t_0}{2}\right)^{n}\left(1+\sin\frac{t_0}{2}\right)^{\frac{2n+1}{2}}\frac{\left(\cos\frac{t_0}{2}\right)^{2n^2}}{(2n\sin\frac{t_0}{2})^{\frac{1}{8}}} \\ &\quad\times(1+o(1)), \\ E_{2n+1}^{\pm}(t_0;0) &= c\left[\frac{(1+\sin\frac{t_0}{2})^n}{2^{\frac{1}{4}}\left(\cos\frac{t_0}{2}\right)^n}\right]^{\pm 1}\frac{\left(\cos\frac{t_0}{2}\right)^{2n^2}}{(2n\sin\frac{t_0}{2})^{\frac{1}{8}}}(1+o(1)), \end{aligned}$$

as $n \to \infty$, and this is equivalent to the desired result. One then applies the interrelation (3.2.22) to obtain the asymptotics for the CβE ensembles with $\beta = 1, 4$.

3.6.2 Proof of Corollaries 3.2.8 and 3.2.10

The symbol $f_{t_0,s}$ in (3.2.20) is associated to $g_{t_0,s}$ in (3.2.19) through equation (3.1.3). One then notices the relation

$$g_{t_0,s} = s^{\frac{t_0}{\pi}} g,$$

where g is defined by (3.2.1) with $V = 0$, $m = 1$, $t_1 = t_0$, $\alpha_0 = \alpha_1 = \alpha_{m+1} = 0$ and $\beta_1 = \frac{\log s}{2\pi i}$. Applying Theorem 3.2.1, we get

$$\begin{aligned}
E^+_{2n}(t_0; s) &= s^{\frac{nt_0}{\pi}} C \mathbb{E}^{\mathbb{U}}_{2n}[g]^{\frac{1}{2}}(1 + o(1)),\\
E^-_{2n+2}(t_0; s) &= s^{\frac{nt_0}{\pi}} C^{-1} \mathbb{E}^{\mathbb{U}}_{2n}[g]^{\frac{1}{2}}(1 + o(1)),\\
E^{\pm}_{2n+1}(t_0; s) &= s^{\frac{nt_0}{\pi}} \tilde{C}^{\pm 1} \mathbb{E}^{\mathbb{U}}_{2n}[g]^{\frac{1}{2}}(1 + o(1)),
\end{aligned}$$

where

$$C = e^{\frac{\log s}{4}} e^{-\frac{t_0 \log s}{2\pi}}, \qquad \tilde{C} = e^{-\frac{\log s}{4}}.$$

But now from [62, Theorem 1.11], for t_0 fixed or when $t_0 \to 0$ and $nt_0 \to +\infty$ one knows that

$$\mathbb{E}^{\mathbb{U}}_{2n}[g] = (4n \sin t_0)^{\frac{\log^2 s}{2\pi^2}} \left| G\left(1 + \frac{\log s}{2\pi i}\right)\right|^4 (1 + o(1)),$$

from which the result follows. One then applies the interrelation (3.2.22) to obtain the asymptotics in the CβE ensembles. In a similar fashion, to prove Corollary 3.2.10, one uses Theorem 3.2.2.

3.6.3 Proof of Theorem 3.2.12

Let n be a positive integer and consider the n free eigenangles $\theta_1 \le \ldots \le \theta_n$ in $\mathbb{O}^{\pm}_N$. Define the counting measure $N_{(0,t)} = \sum_{k=1}^n \chi_{(0,t)}(\theta_k)$ as the number of eigenangles in $(0, t)$, for $0 < t \le \pi$. For later convenience, let us also write $\theta_0 = 0$ and $\theta_{n+1} = \pi$.

We first use a discretization of the supremum of the counting function to bound the two quantities of interest in Theorem 3.2.12.

Lemma 3.6.1. *In $\mathbb{O}^+_{2n}$, $\mathbb{O}^-_{2n+2}$, and $\mathbb{O}^{\pm}_{2n+1}$, we have almost surely*

$$\begin{aligned}
\max_{k=1,\ldots,n} \left|\theta_k - \frac{\pi k}{n}\right| &\le \frac{\pi}{n}\left(1 + \max_{k=1,\ldots,n}\left|N_{(0,\frac{\pi k}{n})} - (k-1)\right|\right),\\
\sup_{t\in(0,\pi)} \left|N_{(0,t)} - \frac{nt}{\pi}\right| &\le 2 + \max_{k=1,\ldots,n}\left|N_{(0,\frac{\pi k}{n})} - (k-1)\right|.
\end{aligned}$$

Proof. Since $[0, \pi) = \sqcup_{j=0}^{n-1}[\frac{\pi j}{n}, \frac{\pi(j+1)}{n})$, for each $k = 1, ..., n$ there exists a unique $j \in \{0, ..., n-1\}$ such that $\frac{\pi j}{n} \le \theta_k < \frac{\pi(j+1)}{n}$. Given that $N_{(0,t)}$ is a non-decreasing function of t, we find the following estimates,

$$N_{(0,\frac{\pi j}{n})} - (j + 1) \le N_{(0,\theta_k)} - \frac{n\theta_k}{\pi} \le N_{(0,\frac{\pi(j+1)}{n})} - j.$$

Because of the ordering of the eigenangles, $N_{(0,\theta_k)} = k-1$, so that

$$\left(N_{(0,\frac{\pi j}{n})} - (j-1)\right) - 1 \leq \frac{n}{\pi}\left(\frac{\pi k}{n} - \theta_k\right) \leq \left(N_{(0,\frac{\pi(j+1)}{n})} - j\right) + 1,$$

and it then suffices to take the maximum or minimum over k and j to obtain the first estimate. Using a similar partitioning argument, one has

$$\sup_{t\in(0,\pi)} \left|N_{(0,t)} - \frac{nt}{\pi}\right| = \max_{k=0,\ldots,n} \sup_{t\in(\theta_k,\theta_{k+1}]} \left|N_{(0,t)} - \frac{nt}{\pi}\right|.$$

Now as a function of t, $N_{(0,t)}$ is left-continuous, has a jump of size 1 at each θ_k, is constant and equals k on $(\theta_k, \theta_{k+1}]$, therefore

$$\sup_{t\in(\theta_k,\theta_{k+1}]} N_{(0,t)} - \frac{nt}{\pi} = k - \frac{n\theta_k}{\pi} = \frac{n}{\pi}\left(\frac{\pi k}{n} - \theta_k\right),$$
$$\inf_{t\in(\theta_k,\theta_{k+1}]} N_{(0,t)} - \frac{nt}{\pi} = k - \frac{n\theta_{k+1}}{\pi} = \frac{n}{\pi}\left(\frac{\pi(k+1)}{n} - \theta_{k+1}\right) - 1.$$

This implies the upper bound

$$\sup_{t\in(0,\pi)} \left|N_{(0,t)} - \frac{nt}{\pi}\right| \leq 1 + \frac{n}{\pi}\max_{k=1,\ldots,n}\left|\theta_k - \frac{\pi k}{n}\right|,$$

and it then suffices to use the previous estimate to conclude. $\square$

Lemma 3.6.2. *In $\mathbb{O}^+_{2n}$, $\mathbb{O}^-_{2n+2}$, and $\mathbb{O}^\pm_{2n+1}$, for any $\alpha > 1, \gamma > 0$ there exists $C_\gamma > 0$ such that*

$$\mathbb{P}\left(\max_{k=1,\ldots,n}\left|N_{(0,\frac{\pi k}{n})} - (k-1)\right| > \alpha\right) \leq C_\gamma e^{-\gamma\alpha} n^{\frac{\gamma^2}{4\pi^2}+1}.$$

Proof. By definition and Boole's inequality one has

$$\mathbb{P}\left(\max_{k=1,\ldots,n}\left|N_{(0,\frac{\pi k}{n})} - (k-1)\right| > \alpha\right) \leq \mathbb{P}\left(\max_{k=1,\ldots,n} N_{(0,\frac{\pi k}{n})} - (k-1) > \alpha\right)$$
$$+ \mathbb{P}\left(\min_{k=1,\ldots,n} N_{(0,\frac{\pi k}{n})} - (k-1) < -\alpha\right),$$

as well as (the last term of the sum always vanishes)

$$\mathbb{P}\left(\max_{k=1,\ldots,n} N_{(0,\frac{\pi k}{n})} - (k-1) > \alpha\right) \leq \sum_{k=1}^{n-1} \mathbb{P}\left(N_{(0,\frac{\pi k}{n})} > k-1+\alpha\right),$$
$$\mathbb{P}\left(\min_{k=1,\ldots,n} N_{(0,\frac{\pi k}{n})} - (k-1) < -\alpha\right) \leq \sum_{k=1}^{n-1} \mathbb{P}\left(-N_{(0,\frac{\pi k}{n})} > -k+1+\alpha\right).$$

Applying Chernoff's bound for $\gamma > 0$ yields for any $t \in (0,\pi)$

$$\mathbb{P}\left(N_{(0,t)} > \frac{nt}{\pi} + \alpha\right) \leq e^{-\gamma\alpha} e^{-\frac{\gamma nt}{\pi}} \mathbb{E}_n^{(j,\pm)}\left[e^{\gamma\chi_{[-t,t]}(\arg z)}\right],$$
$$\mathbb{P}\left(-N_{(0,t)} > -\frac{nt}{\pi} + \alpha\right) \leq e^{-\gamma\alpha} e^{\frac{\gamma nt}{\pi}} \mathbb{E}_n^{(j,\pm)}\left[e^{-\gamma\chi_{[-t,t]}(\arg z)}\right],$$

Therefore, for any $\delta \in \mathbb{R} \setminus \{0\}$, $t \in [\frac{1}{n}, \pi - \frac{1}{n}]$, one may write, using Corollary 3.2.8, for some $C_\delta > 0$

$$\mathbb{E}_n^{(j,\pm)} \left[e^{\delta \chi_{[-t,t]}(\arg z)} \right] \leq C_\delta e^{\frac{\delta n t}{\pi}} (n \sin t + 1)^{\frac{\delta^2}{4\pi^2}}.$$

This leads to the following estimate for some $C_\gamma > 0$,

$$\mathbb{P}\left(\max_{k=1,\dots,n} \left| N_{(0,\frac{\pi k}{n})} - (k-1) \right| > \alpha \right) \leq C_\gamma e^{-\gamma\alpha} n^{\frac{\gamma^2}{4\pi^2}} \sum_{k=1}^{n-1} \left(\sin \frac{\pi k}{n} \right)^{\frac{\gamma^2}{4\pi^2}},$$

and since as $n \to +\infty$

$$\sum_{k=1}^{n-1} \left(\sin \frac{\pi k}{n} \right)^{\frac{\gamma^2}{4\pi^2}} \sim n \int_0^1 (\sin \pi t)^{\frac{\gamma^2}{4\pi^2}} \, dt,$$

this ends the proof. $\square$

In order to prove Theorem 3.2.12, we use on one hand Lemma 3.6.1, which implies

$$\begin{aligned}
&\mathbb{P}\left(\max_{k=1,\dots,n} \left| \theta_k - \frac{\pi k}{n} \right| > (1+\epsilon) \frac{\log n}{n} \right) \\
&+ \mathbb{P}\left(\sup_{t \in (0,\pi)} \left| N_{(0,t)} - \frac{nt}{\pi} \right| > \left(\frac{1}{\pi} + \epsilon \right) \log n \right) \\
&\leq 2\mathbb{P}\left(\max_{k=1,\dots,n} \left| N_{(0,\frac{\pi k}{n})} - (k-1) \right| > (1+\epsilon) \frac{\log n}{\pi} - 2 \right),
\end{aligned}$$

while on the other it follows from Lemma 3.6.2 that for any $\gamma > 0$ there exists $C_\gamma > 0$ such that

$$\mathbb{P}\left(\max_{k=1,\dots,n} \left| N_{(0,\frac{\pi k}{n})} - (k-1) \right| > (1+\epsilon) \frac{\log n}{\pi} - 2 \right) \leq C_\gamma n^{\frac{\gamma^2}{4\pi^2} - (1+\epsilon)\frac{\gamma}{\pi} + 1}.$$

Since the minimum of the polynomial $\frac{\gamma^2}{4\pi^2} - (1+\epsilon)\frac{\gamma}{\pi} + 1$ is attained at $\frac{\gamma}{\pi} = 2(1+\epsilon)$ and is equal to $1 - (1+\epsilon)^2 < 0$, the desired result follows by letting $n \to +\infty$.

Outlook of Further Research

Similar integrable kernels

We are confident that the results of Chapter 2 can be extended to other classical kernels of integrable form, such as the sine and Bessel kernels, which share many similar properties. In particular, both admit integral expressions involving their integrable representations as well as an associated differential system; e.g. for the sine kernel (0.0.76) and its integrable representation (0.0.77), we have

$$K^{\sin}(x,y) = \pi \int_0^1 \left(\frac{1}{\sqrt{\pi}} \cos \pi t x \frac{1}{\sqrt{\pi}} \sin \pi t y + \frac{1}{\sqrt{\pi}} \sin \pi t y \frac{1}{\sqrt{\pi}} \cos \pi t x \right) \mathrm{d}t, \tag{3.6.1}$$

and

$$\partial_x f(x) = -f(x) \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}, \qquad \partial_y g(y) = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix} g(y). \tag{3.6.2}$$

Fredholm minors

The Jánossy densities are actually a special case of Fredholm minors, which are defined as follows: let $(\mathbf{u},\mathbf{v}) := \{(u_j, v_j)\}_{j=1:m} \subset \Lambda^2$, then the Fredholm minor associated to an integral operator K on $L^2(\Lambda,\mu)$ with kernel $K : \Lambda^2 \to \mathbb{C}$, a bounded function $\theta : \Lambda \to \mathbb{C}$ and $(\mathbf{u},\mathbf{v})$ is defined via the series

$$\begin{aligned} \mathfrak{m}_K(\theta;\mathbf{u},\mathbf{v}) &:= \sum_{n\geq 0} \frac{1}{n!} \int_{\Lambda^n} \det \begin{pmatrix} K(\mathbf{u},\mathbf{v}) & K(\mathbf{u},\mathbf{w}) \\ K(\mathbf{w},\mathbf{v}) & K(\mathbf{w},\mathbf{w}) \end{pmatrix} \mathrm{d}^n \mu_\theta(\mathbf{w}) \\ &= \det_{l,k=1:m} \big(K(u_l,v_k)\big) \\ &\quad + \int_\Lambda \det \begin{pmatrix} (K(u_l,v_k))_{l,k=1}^m & (K(u_l,w))_{l=1}^m \\ (K(w,v_k)_{k=1}^m) & K(w,w) \end{pmatrix} \theta(w) \mathrm{d}\mu(w) + \dots . \end{aligned} \tag{3.6.3}$$

If for all $j = 1 : m$ we have $u_j = v_j$, and replacing θ by $-\theta$, then this reduces to a Jánossy density. We already started investigating Fredholm minors and found that many of the properties of Jánossy densities found in Chapters 1 and 2 generalise to Fredholm minors.

Derivatives of Fredholm minors

Here we assume furthermore $\Lambda \subset \mathbb{C}$, so that we can talk about operators of integrable form and (real or complex) derivatives. It is readily seen that the Fredholm minors vanish whenever two points in $\mathbf{u}$ or $\mathbf{v}$, for instance

$$\mathfrak{m}_K(0; (u_1, v_1), (u_1, v_2)) = \det \begin{pmatrix} K(u_1, v_1) & K(u_1, v_2) \\ K(u_1, v_1) & K(u_1, v_2) \end{pmatrix} = 0. \tag{3.6.4}$$

It is then natural to then regularise and take a limit; we investigated the quantities that arise in this way but with additional symmetries in the derivatives. For instance the most simplest case would be

$$\begin{aligned} \lim_{(u_2,v_2)\to(u_1,v_1)} \frac{\mathfrak{m}_K(0; (u_1, v_1), (u_2, v_2))}{(u_2 - u_1)(v_2 - v_1)} &= \\ \det \begin{pmatrix} K(u_1, v_1) & \partial^{(0,1)} K(u_1, v_1) \\ \partial^{(1,0)} K(u_1, v_1) & \partial^{(1,1)} K(u_1, v_1) \end{pmatrix}&. \end{aligned} \tag{3.6.5}$$

This construction led us to generalise the notions of discrete Hilbert and Cauchy transforms, therefore of discrete Riemann-Hilbert problems as well: instead of having simple poles, now all these have poles of higher order, for example in the case above double poles at u_1 or v_1. Now by definition, the above is a particular derivative of a Fredholm minor:

$$\begin{aligned} \lim_{(u_2,v_2)\to(u_1,v_1)} \frac{\mathfrak{m}_K(0; (u_1, v_1), (u_2, v_2))}{(u_2 - u_1)(v_2 - v_1)} &= \\ \partial_{u_2} \partial_{v_2} \mathfrak{m}_K(0; (u_1, v_1), (u_2, v_2))\big|_{u_2=u_1, v_2=v_1}&; \end{aligned} \tag{3.6.6}$$

we thus expect generalisations to exist for the derivatives defined for $(\mathbf{p}, \mathbf{q}) := \{(p_j, q_j)\}_{j=1:m}$ with $p_j, q_j \in \mathbb{N}$ by

$$\partial^{(\mathbf{p},\mathbf{q})} \mathfrak{m}_K(\theta; \mathbf{u}, \mathbf{v}) := \partial_{u_1}^{p_1} \partial_{v_1}^{q_1} ... \partial_{u_m}^{p_m} \partial_{v_m}^{q_m} \mathfrak{m}_K(\theta; \mathbf{u}, \mathbf{v}). \tag{3.6.7}$$

The essential ingredient for the generalisation of discrete Hilbert/Cauchy transform and Riemann-Hilbert problems we have obtained was that the integrable representation of $K(x, y) = f(x) \frac{1}{x-y} g(y)$, which is nothing but the simplest Fredholm minor $\mathfrak{m}_K(0; (u, v))$, leads to a more general integrable representation of the derivatives of Fredholm minors $\partial^{(\mathbf{p},\mathbf{q})} \mathfrak{m}_K(0; \mathbf{u}, \mathbf{v})$. For instance the matrix whose determinant we take in

$$\partial_{u_2} \mathfrak{m}_K(0; (u_1, v_1), (u_2, v_2)) = \det \begin{pmatrix} K(u_1, v_1) & K(u_1, v_2) \\ \partial^{(1,0)} K(u_2, v_1) & \partial^{(1,0)} K(u_2, v_2) \end{pmatrix} \tag{3.6.8}$$

can be factorized as

$$\begin{pmatrix} f(u_1) & & \\ & \partial f(u_2) & f(u_2) \end{pmatrix} \begin{pmatrix} \frac{1}{u_1 - v_1} & & \frac{1}{u_1 - v_2} \\ & \frac{1}{u_2 - v_1} & \frac{1}{u_2 - v_2} \\ & \partial_{u_2} \frac{1}{u_2 - v_1} & \partial_{u_2} \frac{1}{u_2 - v_2} \end{pmatrix} \begin{pmatrix} g(v_1) & \\ g(v_1) & \\ & g(v_2) \end{pmatrix}. \tag{3.6.9}$$

In the same way the "factorization" $K(x,y) = f(x)\frac{1}{x-y}g(y)$, by cyclic permutation (arising in the computations), will lead us to consider the quantity $g(w)f(w)\frac{1}{w-z}$, which is the building block of the Hilbert and Cauchy transforms and thus of Riemann-Hilbert problems, we expect that the above factorisation will lead to appropriate generalisations of the discrete Hilbert/Cauchy transforms and Riemann-Hilbert problems, in turn guiding to generalisation of the results obtained in Chapter 2.

From scalars to matrices to operators

Recently, the need to generalise scalar integral kernels to matrix-valued ones and classical integrable representations involving matrices to operator-valued ones has arisen. Let us explain why: in the setting of Chapter 1, i.e. given an operator K on $L^2(\Lambda,\mu)$ inducing a determinantal point process and a thinning function θ, consider the operator $\mathrm{K}^\theta_{2\times 2}$ acting on $L^2(\Lambda,\mu;\mathbb{C}^2)$ (here $\mathbb{C}^2$ is understood as a Hilbert space) whose matrix-valued kernel is defined by

$$K^\theta_{2\times 2}(x,y) := \begin{pmatrix} \sqrt{\theta(x))} \\ \sqrt{1-\theta(x)} \end{pmatrix} K(x,y) \begin{pmatrix} \sqrt{\theta(y)} & \sqrt{1-\theta(y)} \end{pmatrix}. \tag{3.6.10}$$

Because of the isometry $L^2(\Lambda,\mu;\mathbb{C}^2) \simeq L^2(\Lambda\times\{0,1\},\mu)$, we can identify $\mathrm{K}^\theta_{2\times 2} \simeq \mathrm{K}^\theta$, where K^θ is defined in (1.2.6). Now if K is of integrable form $K(x,y) = \frac{f(x)g(y)}{x-y}$ with $f(x)\in\mathbb{C}^{1\times p}$, $g(y)\in\mathbb{C}^{p\times 1}$, then K^θ cannot be since its domain is never a subset of the complex plane, however $\mathrm{K}^\theta_{2\times 2} = \frac{f^\theta(x)g^\theta(y)}{x-y}$ can be if we admit a more general definition of integrable representation with $f^\theta(x)\in\mathbb{C}^{2\times p}$, $g^\theta(y)\in\mathbb{C}^{p\times 2}$ defined by

$$f^\theta(x) = \begin{pmatrix} \sqrt{\theta(x)} \\ \sqrt{1-\theta(x)} \end{pmatrix} f(x), \qquad g^\theta(y) = g(y)\begin{pmatrix} \sqrt{\theta(x)} & \sqrt{1-\theta(x)} \end{pmatrix}. \tag{3.6.11}$$

This construction may seem a bit forced, yet matrix-valued kernels of integral form do appear more naturally in the context of matrix-valued orthogonal polynomials. Now consider a so-called finite-temperature version of the Airy kernel which depends on a function $\sigma : \mathbb{R}\to\mathbb{C}$ (assuming that the following defines a nice enough kernel)

$$K^{\mathrm{Ai}}_\sigma(x,y) = \int_{\mathbb{R}} \mathrm{Ai}(x+t)\mathrm{Ai}(y+t)\sigma(t)\mathrm{d}t. \tag{3.6.12}$$

In the physics literature, this arose in the context of fermionic systems with a non-zero temperature, i.e. a finite inverse temperature $\beta > 0$, and σ was given by

$$\sigma(t) = \begin{cases} \dfrac{1}{1+e^{-\beta t}} & \beta > 0, \\ \mathbb{1}_{(0,\infty)}(t) & \beta = \infty. \end{cases} \tag{3.6.13}$$

As we can see, the case of zero-temperature ($\beta=\infty$) yields the standard Airy kernel (0.0.92). Because $\mathrm{Ai}(x+t)\mathrm{Ai}(y+t) = -\partial_t K^{\mathrm{Ai}}(x+t,y+t)$, if we assume that σ is of bounded variation (i.e. its distributional derivative is a finite measure $\mathrm{d}\sigma$, this is the case for the explicit σ given above), we can integrate by parts revealing that

$$K_\sigma^{\mathrm{Ai}}(x,y) = \int_{\mathbb{R}} K^{\mathrm{Ai}}(x+t, y+t)\mathrm{d}\sigma(t). \tag{3.6.14}$$

Once again we are led to a more general integrable representation

$$K_\sigma^{\mathrm{Ai}}(x,y) = \frac{f_\sigma^{\mathrm{Ai}}(x) g_\sigma^{\mathrm{Ai}}(y)}{x-y} \tag{3.6.15}$$

where instead of $f(x)$ and $g(y)$ respectively being row and column vectors, they are now linear maps

$$f_\sigma^{\mathrm{Ai}}(x) \in \mathcal{L}(L^2(\mathbb{R}, \mathrm{d}\sigma; \mathbb{C}^2), \mathbb{C}), \qquad g_\sigma^{\mathrm{Ai}}(y) \in \mathcal{L}(\mathbb{C}, L^2(\mathbb{R}, \mathrm{d}\sigma; \mathbb{C}^2)), \tag{3.6.16}$$

defined for $\phi = (\phi_1, \phi_2) \in L^2(\mathbb{R}, \mathrm{d}\sigma; \mathbb{C}^2) \simeq L^2(\mathbb{R}, \mathrm{d}\sigma)^2$, $\alpha \in \mathbb{C}$ by

$$\begin{aligned} f_\sigma^{\mathrm{Ai}}(x)[\phi] &= \int_{\mathbb{R}} f^{\mathrm{Ai}}(x+t) \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix} \mathrm{d}\sigma(t), \\ g_\sigma^{\mathrm{Ai}}(y)[\alpha](t) &= g^{\mathrm{Ai}}(y+t)\alpha. \end{aligned} \tag{3.6.17}$$

Moreover, the useful properties that we relied on in Chapter 2 remain: the integral representation

$$K_\sigma^{\mathrm{Ai}}(x,y) = \int_0^\infty \left[\int_{\mathbb{R}} \mathrm{Ai}(x+t+s)\mathrm{Ai}(y+t+s)\mathrm{d}\sigma(t) \right] \mathrm{d}s, \tag{3.6.18}$$

as well as the differential system

$$\partial_x f(x) = -f(x) \begin{pmatrix} 0 & 1 \\ x+\hat{\mathrm{t}} & 0 \end{pmatrix}, \qquad \partial_y g(y) = \begin{pmatrix} 0 & 1 \\ y+\hat{\mathrm{t}} & 0 \end{pmatrix} g(y), \tag{3.6.19}$$

where $\hat{\mathrm{t}}$ is the position operator defined for $\psi \in L^2(\mathbb{R}, \mathrm{d}\sigma)$ by

$$\hat{\mathrm{t}}[\psi](t) = t\psi(t). \tag{3.6.20}$$

Note that unless the support of $\mathrm{d}\sigma$ is bounded, this is not a bounded operator; this is one of the "analytical" challenges which appear when considering operators; yet the algebraic computations remain the same, as such we expect that the results of Chapter 2 and their generalisations to (derivatives of) Fredholm minors generalise appropriately to this operator setting. Operator-valued Riemann-Hilbert problems already appeared in the literature, see e.g. [99], [35], and will most likely increase in popularity in the years to come.

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