Theorems of Euclidean Geometry through Calculus

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Throughout the ages, notably in ancient Greece and Rome, the Arab Empire, up to the Renaissance and the Enlightenment, geometers have discovered and proved theorems of pure, classical geometry. They used shapes, alignments, distances, angles. All of their tremendous work is known as geometry, and taught to billions of people around the world since the existence of education.

Many of those theorems are expressed as an equality, stating that in such circumstances, such quantity is equal to a function of others – the Pythagoras theorem is an emblematic illustration. They can actually be written as real functions, from a domain included in $\mathbb{R}^n$, to $\mathbb{R}$. As the intputs corresponding to lengths, magnitudes, etc. change in their domain, reflecting all possible configurations under the assumptions of the theorem, the function gives the output predicted by the theorem. But the concept of function was not known to the Ancients, at least not in this form.

The Enlightenment was precisely the time of the rise of another famous branch of mathematics, also widely taught in most colleges on the planet: calculus, which deals with functions and small deviations called infinitesimals – a point recently emphasized by Strogatz [1]. Among others, including of course Leibniz [2], Newton played a central role in the development of calculus [3]. What if History had been reversed and all this had happened earlier, much earlier...

I. THALES OF MILETUS

Imagine that Newton was born before Thales. When considering a triangle with two sides of lengths x and y, he could have fantasized about moving the third side parallel to itself and thought: "Well, I am not an Greek geometer but I am rather good in calculus and I feel there might be some connection between the way x and y vary in such circumstances."

He would have materialized his suspicion in a function

$$y = y(x) \tag{1}$$

connecting x and y whatever the position of the third side, as long as it is moved parallel to itself. In particular, after a slight displacement resulting in small deviations δx and δy, he would have had to first order

$$\delta y = y'(x)\delta x \tag{2}$$

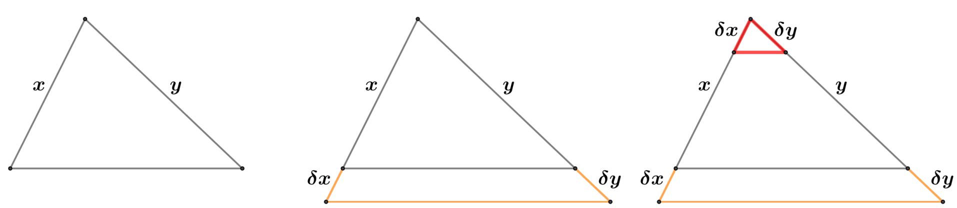

Figure 1. Small parallel displacement of a side in any triangle, resulting in small deviations of the two other sides.

But the lengths δx and δy of the small added segments must themselves obey equation (1), that is

$$\delta y = y(\delta x) \tag{3}$$

To see it, translate those segments to the (x, y) vertex. Developing the right-hand-side member of equation (3) to first order and noticing that $y(0) = 0$, we have

$$\delta y = y'(0)\delta x \tag{4}$$

which, compared to eq. (2), implies that $y'(x)$ is constant. Integrating $y' = k$, k being a positive constant since $y(x)$ is an increasing and smooth function, gives the Thales theorem [4, Book VI, Prop. II]

$$y(x) = kx \tag{5}$$

II. PYTHAGORAS OF SAMOS

If he was born before Thales, Newton was born before Pythagoras too, so that we do not have to make any further unlikely hypothesis. Imagine that driven by his success in suspecting the existence of a Greek theorem, he moved to consider a right triangle of legs of lengths x and y and of hypotenuse of length z.

He might have been tempted to speculate about the link, if any, between x, y and z in every right triangle. And again, as calculus master, he could have postulated that

$$z = z(x, y) \tag{6}$$

a relation that must be true for any x, y and z in a right triangle. In particular, after a slight increase in the length of x, leading to a small deviation δx, while $\delta y = 0$, he would have found, to first order, that

$$\delta z = \partial_x z \, \delta x \tag{7}$$

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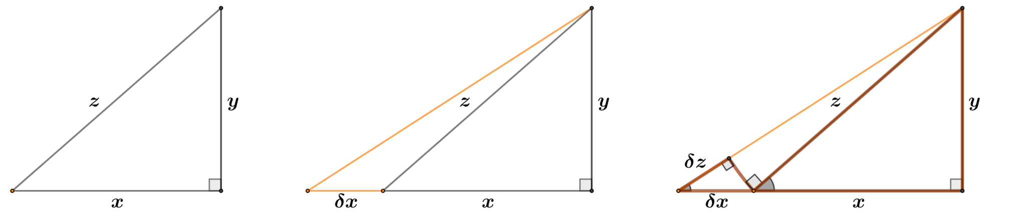


Figure 2. Small displacement of a vertex in a right triangle, resulting in small deviations of one of the legs and the hypotenuse.

Here, δx is the length of the hypotenuse of a right triangle with one leg of length δz. To first order, this small triangle is similar to the initial one. Using the Thales theorem that he has just found out,

$$\delta z = \frac{x}{z}\delta x \tag{8}$$

Substituting this result to δz in eq. (7) would have led him to the partial differential equation

$$z\partial_x z = x \tag{9}$$

whose general solution is

$$z^2(x, y) = x^2 + k(y) \tag{10}$$

where $k(y)$ is an arbitrary function of y. But the function z has to be symmetric in x and y (since he could have made the same reasoning with a non-zero δy while $\delta x = 0$) and $z(x, 0) = x$. Hence

$$z^2(x, y) = x^2 + y^2 \tag{11}$$

that is, the Pythagorean theorem, which, like Thales', was probably discovered long before – and taken up later by Euclid, in his Elements [4, Book I, Prop. XLVII]. This proof is known and is published in a slightly different form in [5, 6].

III. APOLLONIUS OF PERGA

Thales and Pythagoras theorems are not the only ones that are named before famous Greek geometers. Newton could have gone a step further – eastwards, a few centuries later – and assumed that in any triangle of side-lengths x, y and z, the length d of the median relative to the z-length side is a smooth function of x, y and z, i.e.

$$d = d(x, y, z) \tag{12}$$

After an infinitesimal rotation of the y-length side around the (y, z) vertex resulting in a small deviation δx, with $\delta y = 0$ and $\delta z = 0$, to first order:

$$\delta d = \partial_x d\, \delta x \tag{13}$$

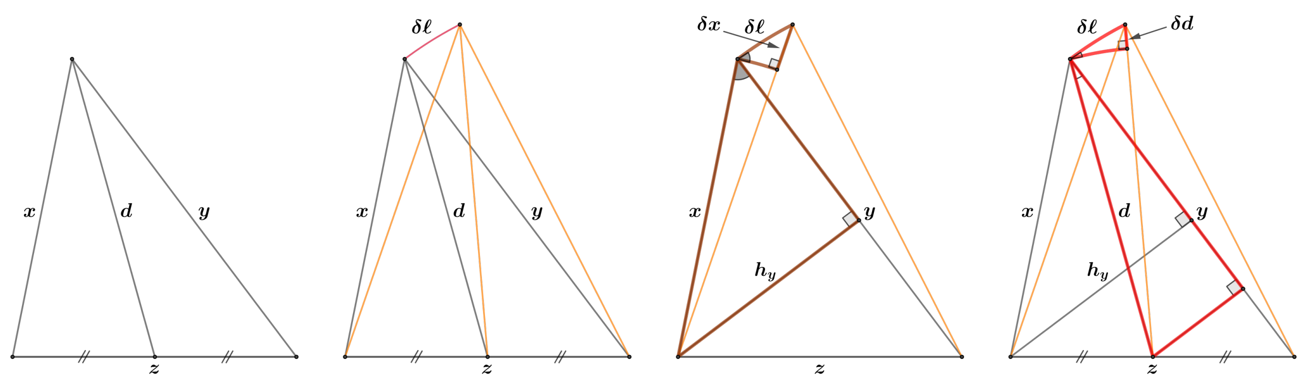


Figure 3. In any triangle, small rotation of a vertex around another one, resulting in small deviations of the sides connected by the rotated vertex, and of the corresponding median.

In order to get an expression for δx and δd and then a differential equation leading to the would-be theorem, consider the infinitesimal arc travelled by the moved vertex, of length $\delta\ell$.

To first order, it can be seen as the hypotenuse of a small right triangle with one leg of length δx, which is similar to a larger right triangle whose corresponding leg is the h_y-length height relative to the y-length side, and the hypotenuse is the x-length side, so that

$$\delta x = \frac{h_y}{x}\delta\ell \tag{14}$$

The $\delta\ell$-length arc is also the first order hypotenuse of another small right triangle with one leg of length δd, which is similar to the triangle with a d-length hypotenuse and whose corresponding leg is a segment starting from the foot of the median and parallel to – and thus half of the length of – the h_y-length height, so that

$$\delta d = \frac{h_y}{2d}\delta\ell \tag{15}$$

Inserting those deviations in eq. (13) leads to the partial differential equation

$$d\partial_x d = \frac{x}{2} \tag{16}$$

whose general solution is

$$d^2(x, y, z) = \frac{x^2}{2} + k(y, z) \tag{17}$$

where $k(y, z)$ is a function of y and z. But $d(x, y, z)$ has to be symmetric in x and y (since we can make the same reasoning with a non-zero δy while $\delta x = 0$). Hence

$$d^2(x, y, z) = \frac{x^2 + y^2}{2} + c(z) \tag{18}$$

with $c(z)$ a function of z. Furthermore, if $x = 0$ (or $y = 0$), $y = z$ (or $x = z$) and $d = z/2$. This yields $c(z) = -z^2/4$, which can be obtained alternatively by invoking the Pythagorean theorem for $y = x$. The particular solution reads

$$d^2(x, y, z) = \frac{x^2 + y^2 - 2(z/2)^2}{2} \tag{19}$$

that is, Apollonius's theorem, to be found in a slightly more elaborate form in [7].

IV. MATTHEW STEWART

Suppose Newton was born before Stewart, an 18th-century Scottish mathematician (and reverend). Well, he was. Perhaps he was not interested, or did not have the time, otherwise he could have used this tool to generalise Apollonius's theorem to any cevian.

In a triangle of sidelengths x, y and z, assume that the length d of a cevian dividing the side of length z in two segments of lengths m and n, is a smooth function of x, y, m and n, that is

$$d = d(x, y, m, n) \tag{20}$$

After an infinitesimal rotation of the y-length side around the (y, z) vertex resulting in a small deviation δx, with $\delta y = 0$, $\delta m = 0$ and $\delta n = 0$, to first order:

$$\delta d = \partial_x d\, \delta x \tag{21}$$

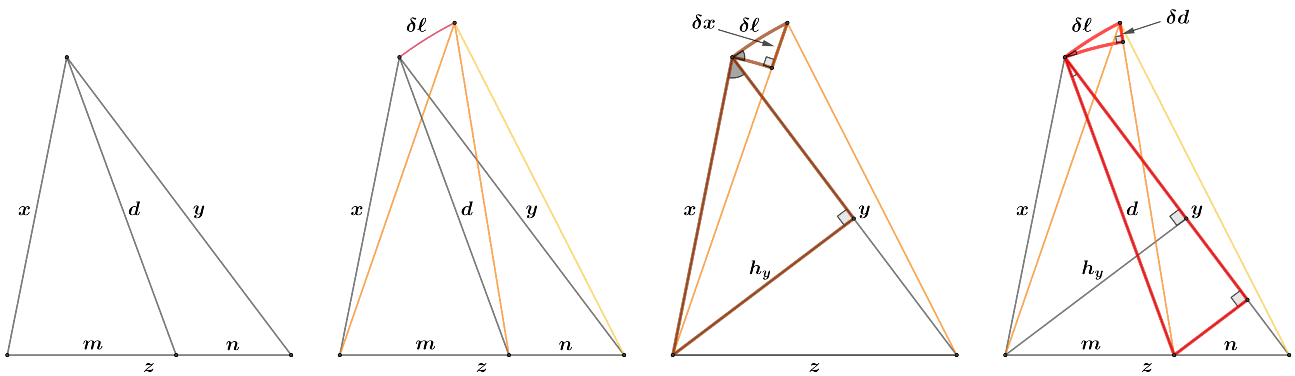


Figure 4. In any triangle, small rotation of a vertex around another one, resulting in small deviations of the sides connected by the rotated vertex, and of the corresponding cevian.

Using the same similarities as for the Apollonius's theorem, with the unique difference that the foot of the cevian is not necessarily the middle of the $(m + n)$-length side but falls at a distance n from its right vertex, we find

$$\delta x = \frac{h_y}{x}\delta\ell \qquad \delta d = \frac{n h_y}{(m+n)d}\delta\ell \tag{22}$$

h_y being the length of the height relative to y, we have the partial differential equation

$$d\partial_x d = \frac{n}{m+n}x \tag{23}$$

whose general solution is

$$d^2(x, y, m, n) = \frac{n}{m+n}x^2 + k(y, m, n) \tag{24}$$

where $k(y, m, n)$ is a function of y, m and n. But $d(x, y, m, n)$ must be symmetric in (x, m) and (y, n) (since we can make the same reasoning with a non-zero δy while $\delta x = 0$). Hence

$$d^2(x, y, m, n) = \frac{nx^2 + my^2}{m+n} + c(m, n) \tag{25}$$

with $c(m, n)$ a symmetric function of m and n. Furthermore, if $x = 0$ (or $y = 0$), $y = m + n$ (or $x = m + n$) and $d = m$ (or $d = n$). This yields $k(m, n) = -mn$. The particular solution reads

$$d^2(x, y, m, n) = \frac{n(x^2 - m^2) + m(y^2 - n^2)}{m+n} \tag{26}$$

that is, Stewart's theorem [8].

V. HERON OF ALEXANDRIA

Intoxicated by his findings, Newton could have switched to a more elaborate, though older, challenge – as probably did an Ancient Greek Roman Egyptian mathematician... What if, for any triangle, the area A could be a smooth function of the sides lengths x, y and z? He would have assumed

$$A = A(x, y, z) \tag{27}$$

After an infinitesimal rotation of the y-length side around the (y, z) vertex resulting in a small deviation δx, while $\delta y = 0$ and $\delta z = 0$, to first order:

$$\delta A = \partial_x A\, \delta x \tag{28}$$

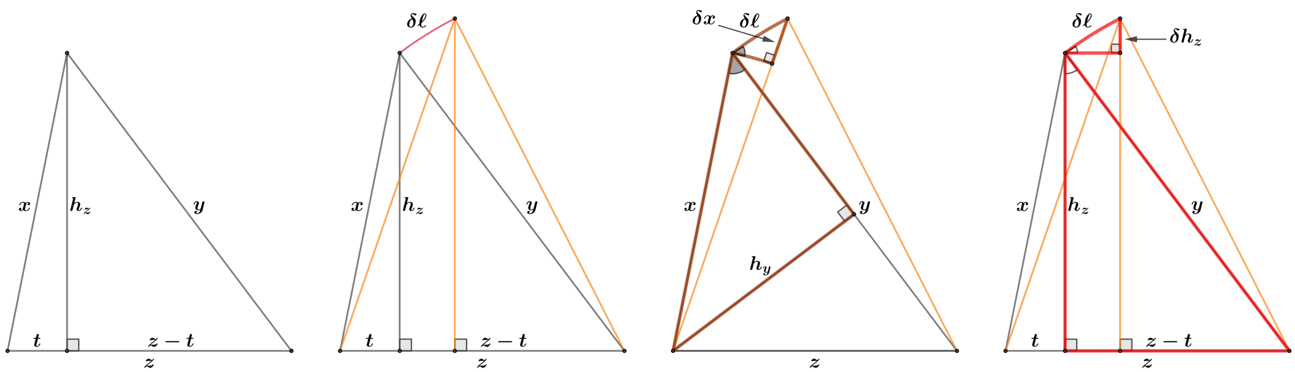


Figure 5. In any triangle, small rotation of a vertex around another one, resulting in small deviations of the sides connected by the rotated vertex, and of the corresponding height.

First note that the h_z-length height relative to the z-length side divides the initial triangle in two right triangles of horizontal legs of lengths t and $z - t$ respectively. One can express h_z as a result of the Pythagorean theorem in both right triangles.
Equating those expressions yields $x^2 - t^2 = y^2 - (z - t)^2$ and hence

$$t = \frac{x^2 - y^2 + z^2}{2z} \qquad z - t = \frac{y^2 - x^2 + z^2}{2z} \tag{29}$$

Again, $\delta\ell$ is the length of the infinitesimal arc travelled by the moved vertex. Like in the two last sections, it can be considered as the first-order hypotenuse of a small triangle whose similarity with a larger one allows to find δx. But it is also, to first order, the hypotenuse of another small triangle with one leg of length δh_z, which is similar to the large right triangle whose corresponding leg is the $(z - t)$-length segment, and the hypotenuse the y-length side. Since $h_y = 2A/y$ and $\delta h_z = 2\delta A/z$, we have

$$\delta x = \frac{2A}{xy}\delta\ell \qquad \delta A = \frac{y^2 - x^2 + z^2}{4y}\delta\ell \tag{30}$$

Plugging in results (30) into equation (28), gives the partial differential equation

$$A\partial_x A = \frac{1}{8}[x(y^2 + z^2) - x^3] \tag{31}$$

which can be integrated out to give the general solution

$$A^2(x, y, z) = \frac{1}{16}[2x^2(y^2 + z^2) - x^4 + k(y, z)] \tag{32}$$

where $k(y, z)$ is an homogeneous function of y and z. Since $A(x, y, z)$ must be symmetric in x, y and z (since we can make the same reasoning with a non-zero δy or δz), $k(y, z) = 2y^2z^2 - y^4 - z^4$. Hence

$$A(x, y, z) = \frac{1}{4}\sqrt{2(x^2y^2 + x^2z^2 + y^2z^2) - (x^4 + y^4 + z^4)} \tag{33}$$

which can be factorized into the Heron theorem [9]

$$A(x, y, z) = \sqrt{\frac{x + y + z}{2}\frac{-x + y + z}{2}\frac{x - y + z}{2}\frac{x + y - z}{2}} \tag{34}$$

whose discovery could actually be Archimedes' [10].

VI. JAMSHID AL-KASHI

Newton could have chosen to deal with angles – besides calculus, he knew a bit about trigonometry. Let us send him to Persia, a few centuries before his birth, and wonder wether in any triangle of sidelengths x, y and z, the angle $\gamma = \widehat{(x, y)}$ could be a smooth function of x, y and z, that is

$$\gamma = \gamma(x, y, z) \tag{35}$$

After an infinitesimal rotation of the y-length side around the (y, x) vertex resulting in a small deviation δz, with $\delta x = 0$ and $\delta y = 0$, to first order:

$$\delta\gamma = \partial_z \gamma\, \delta z \tag{36}$$

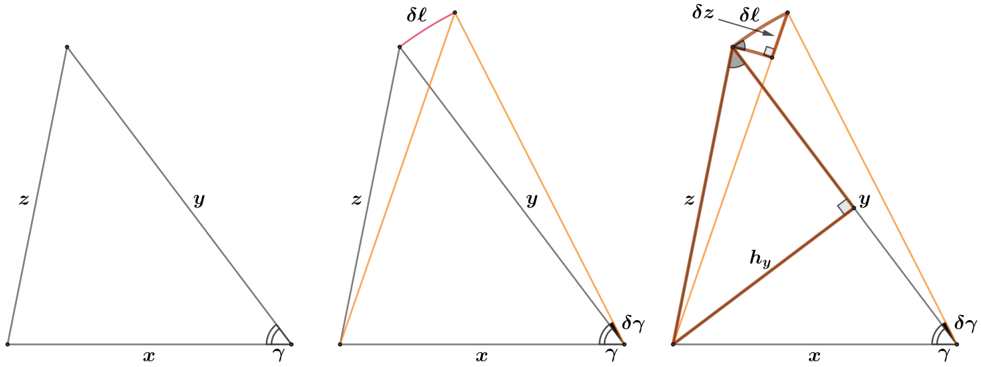


Figure 6. In any triangle, small rotation of a vertex around another one, resulting in small deviations of the sides connected by the rotated vertex, and (of one) of its angles.

While $\delta\gamma$ is easy to connect to $\delta\ell$, the length of the arc travelled by the moved vertex (in the illustrative figures, x and z have been swapped for aesthetic reasons), δz can be determined thanks to the same similiarity as in the three previous sections. We have thus

$$\delta\gamma = \frac{\delta\ell}{y} \qquad \delta z = \frac{h_y}{z}\delta\ell \quad \text{with} \quad h_y = x\sin\gamma \tag{37}$$

Inserting those deviations in eq. (36) yields the partial differential equation

$$\sin\gamma\,\partial_z\gamma = \frac{z}{xy} \tag{38}$$

whose general solution is

$$\cos[\gamma(x,y,z)] = -\frac{z^2 + k(x,y)}{2xy} \tag{39}$$

where $k(x,y)$ is a symmetric, homogeneous function of x and y. According to Pythagoras, when $\gamma = \pi/2$, $z^2 = x^2 + y^2$, i.e. $k(x,y) = -x^2 - y^2$. Hence

$$\cos[\gamma(x,y,z)] = \frac{-z^2 + x^2 + y^2}{2xy} \tag{40}$$

that is, al-Kashi's theorem [11] – also known as the law of cosines or generalized Pythagorean theorem, and already familiar to Euclid [4, Book II, Prop. XII & XIII].

VII. OLRY TERQUEM

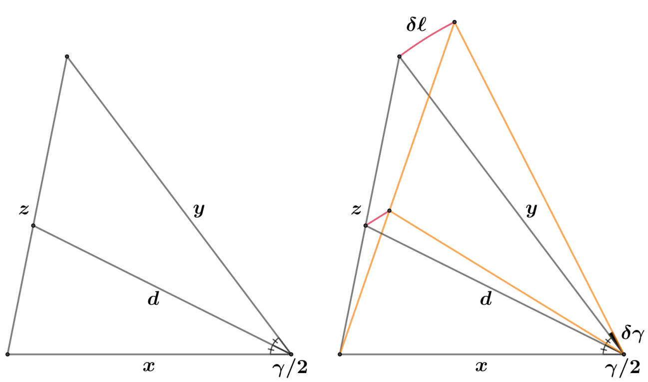
Completely exhilarated, Newton could have taken on a bigger piece and assumed that in any triangle of side-lengths x, y and z, the length d of the $\gamma = \widehat{(x,y)}$ angle bisector is a smooth function of x, y and z, i.e.

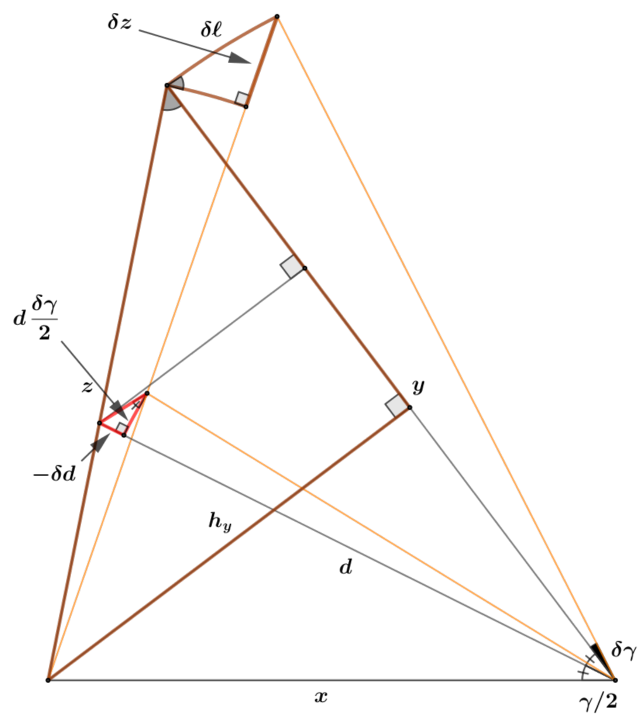
$$d = d(x,y,z) \tag{41}$$

After an infinitesimal rotation of the y-length side around the (x,y) vertex resulting in a small deviation δz, with $\delta x = 0$ and $\delta y = 0$, to first order:

$$\delta d = \partial_z d\,\delta z \tag{42}$$

Figure 7. In any triangle, small rotation of a vertex around another one, resulting in small deviations of the sides connected by the rotated vertex, and of the bisectors.





Again, thanks to the same similiarity as in the four previous sections, δz can easily be linked to $\delta\ell$, the length of the arc travelled by the moved vertex.

It is a little more complicated for δd. First note that in the illustrative figure, $\delta d < 0$, so that we will consider the positive length $-\delta d$. Then observe that when the y-length side infinitesimally rotates around the (x, y) vertex, the foot of the $\gamma = \widehat{(x, y)}$ angle bisector moves along a perpendicular to the y-length side, just like the (z, y) vertex. But the angle between this perpendicular and the angle bisector is the complementary of $\gamma/2$. Thus in the small right triangle of legs of lengths $-\delta d$ and $d\,\delta\gamma/2$, the opposite angle to the $-\delta d$-length leg is, to first order, equal to $\gamma/2$, implying that $\tan(\gamma/2) = -\delta d/(d\,\delta\gamma/2)$. Hence

$$\delta z = \frac{h_y}{z}\delta\ell \qquad \delta d = -\tan\frac{\gamma}{2}\frac{d}{2}\delta\gamma \quad \text{with} \quad \delta\ell = y\delta\gamma \tag{43}$$

h_y being the length of the height relative to y. Using

$$\tan\frac{\gamma}{2} = \frac{\sin\gamma}{1+\cos\gamma} \tag{44}$$

with

$$\sin\gamma = \frac{h_y}{x} \quad \text{and} \quad \cos\gamma = \frac{-z^2+x^2+y^2}{2xy} \tag{45}$$

we have the partial differential equation

$$\frac{\partial_z d}{d} = \frac{-z}{-z^2+(x+y)^2} \tag{46}$$

whose general solution is

$$d(x, y, z) = k(x, y)\sqrt{(x+y)^2 - z^2} \tag{47}$$

where $k(x, y)$ is a symmetric function of x and y. To determine it, note that in the particular case of a right triangle with hypotenuse of length z, the angle bisector is the diagonal of the inscribed square of sidelength $xy/(x+y)$ – as can be deduced from similarities between the right triangles generated by the square in the initial triangle. We find $k(x, y) = \sqrt{xy}/(x+y)$. Hence

$$d(x, y, z) = \sqrt{xy\left(1 - \frac{z^2}{(x+y)^2}\right)} \tag{48}$$

that is, the length of the angle bisector, as Terquem computed in the 19th century [12].

VIII. JEAN-PAUL DE GUA DE MALVES

Armed with this powerful theorem-finding tool, Newton could have moved on to even bolder challenges, like leaving the plane for the real space, and imagining, say, a generalization of the Pythagorean theorem in three dimensions! Let him consider a trirectangular tetrahedron, that is a tetrahedron with a right angle corner, like the corner of a cube: what if, for any of them, the area of the face opposite to the right angle was a function of the areas of the other faces?

A convenient way to parametrize the problem is to give arbitrary lengths to the three edges from the right angle vertex, say x, y and z. The areas of the three right triangle faces are $xy/2$, $xz/2$ and $yz/2$. For the area of the last face, opposite to the right angle, say A, we can have an expression by choosing a base, say the edge of length $\sqrt{y^2+z^2}$ (thanks Pythagoras) and the relative height of length h. We have

$$A = \frac{1}{2}\sqrt{y^2+z^2}\,h \tag{49}$$

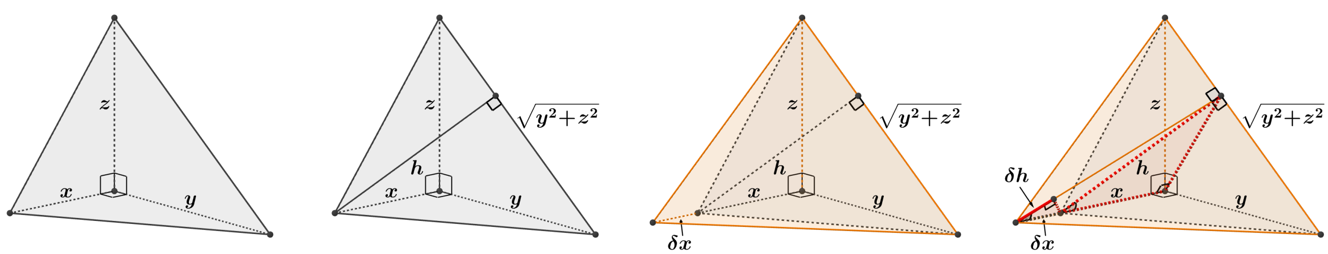


Figure 8. In a trirectangular tetrahedron, small displacement of a vertex, resulting in small deviations of one of the three orthogonal edges and of two of the other edges connected to the moved vertex.

Let us go back to Newton and his obsession. He could have stated that A is a smooth function of x and y:

$$A = A(x, y, z) \tag{50}$$

Choosing to slightly increase x, while leaving y and z invariants, that is, an infinitesimal deviation δx, with $\delta y = 0$ and $\delta z = 0$, we find

$$\delta A = \partial_x A\,\delta x \tag{51}$$

Eq. (49) implies that

$$\delta A = \frac{1}{2}\sqrt{y^2+z^2}\,\delta h \tag{52}$$

But what do we know of δh? First note that the foot of the h-length height is not affected by the deviation δx since this h-length height and the x-length edge are in a plane orthogonal to the base of the A-area face. In this plane, we can check that to first order, the right triangle with h-length hypotenuse and x-length leg is similar to the one with δx-length hypotenuse and δh-length leg, so that

$$\delta h = \frac{x}{h}\delta x \tag{53}$$

Combining this equation with result (52), itself plugged in into eq. (51) with $\delta y = 0$, we have

$$\frac{1}{2}\sqrt{y^2+z^2}\,\frac{x}{h}\delta x = \partial_x A\,\delta x \tag{54}$$

Simplifying by δx and using eq. (49) to get rid of h, we find a partial differential equation

$$A\partial_x A = \frac{1}{4}(y^2+z^2)x \tag{55}$$

It can be integrated out to give the general solution

$$A^2(x,y,z) = \frac{1}{4}[(y^2+z^2)x^2 + k(y,z)] \tag{56}$$

with $k(y,z)$ an homogeneous and symmetric function of y and z. Since $A(x,y,z)$ must itself be symmetric in x, y and z (i.e. we can make the same reasoning with a non-zero δy or δz), $k(y,z) = y^2z^2$. Hence

$$A^2(x,y,z) = \left(\frac{xy}{2}\right)^2 + \left(\frac{xz}{2}\right)^2 + \left(\frac{yz}{2}\right)^2 \tag{57}$$

known as de Gua's theorem [14], first formulated by Descartes [15], which states that in any trirectangular tetrahedron, the square of the area of the face opposite to the right corner is equal to the sum of the squares of the areas of the other faces – a three-dimensional generalization of the Pythagorean theorem.

CONCLUSION

This way to derive classical theorems in Euclidean geometry can be used for many more of them, including Ptolemy's, Brahmagupta's, Euler's and the law of sines (by examining infinitesimal rotations and scale transformations along/of the circumcircle). It also allows to find quantities like the circumradius and the inradius of a triangle expressed as functions of the sidelengths.

It cannot be used for all theorems, of course. It does not work for theorems in discrete geometry or involving number theory, for theorems stating that this or that line cuts another at this or that point, is perpendicular or tangent to this or that circle, etc. It has to be a theorem involving an equation that defines a function, which will be seen as a particular solution of a (system of) differential equation(s). The proofs that we propose are not necessarily simpler than others. They do not evade the geometric difficulties at stake. We displace the argument of the proof into the game of infinitesimals, but it remains as geometric.

The main advantage of this method is that the theorem does not need to be known. We start with an unknown function and observe the way it behaves, to first order, under small deviations of some quantities. In the best case, it gives us a (system of) differential equation(s) that we can solve and, therefore, discover the theorem. We hope this method may be used to discover new theorems, perhaps in other fields of mathematics.

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