

Stability of Flapping Flight Dynamics of Large Birds

Gianmarco Ducci

PhD thesis

Dissertation committee:

Prof. Renaud Ronsse (UCLouvain, advisor) Prof. Philippe Chatelain (UCLouvain, advisor) Prof. Pierre-Antoine Absil (UCLouvain, Belgium) Prof. Julien Hendrickx (UCLouvain, Belgium) Prof. Emily Shepard (University of Swansea, UK) Prof. Mark Lowenberg (University of Bristol, UK) Prof. Sandra Soares-Frazao (UCLouvain, chairperson)

November, 2022

Acknowlegdments

This thesis marks the final achievement of my last four years of work at UCLouvain. First, I would like to express my sincere gratitude to the people who made possible this Belgian adventure, and guided me through this new journey. I would like to thank my two supervisors: Renaud Ronsse and Philippe Chatelain, for giving me the opportunity of working on such an interesting topic, and constantly supporting my work. I am very thankful for their advices, the scientific meetings we had, and for putting me in the best condition to work and to keep learning.

Very important parts of this work have been discussed and conceptualized in company of Victor and Gennaro. Thank you both for the time you dedicated and for the availability you always have shown.

I would also like to thank all the people composing my PhD committee. Pierre-Antoine Absil, thank you very much for your availability during these years, and for helping me in the initial part of the project. Thank you also to Julien Hendrickx, for all the interesting interactions and insightful conversations during these years of Reveal Flight meetings. I would also like to thank Mark Lowenberg, and Emily Shepard, for accepting of being part of my committee, and the careful reading of this manuscript. The suggestions and the interesting discussions emerged during the private defense, definitely improved its quality. Moreover, thanks to Sandra Soares-Frazao for chairing the jury.

My gratitude goes also to Nathalie Sergoigne, for all the administrative support you provide, your prompt availability, and last but not less important your joviality.

Thanks Umbe, my Glaswegian buddy. I cannot imagine my time there without our reciprocal support.

Gennaro (again), you deserve to be mentioned twice. Sharing this Belgian experience with you was a real pleasure. Thanks for all the time spent together, the precious scientific (and less scientific) discussions, and for teaching me the invaluable secrets of *patate*, *riso e cozze*.

I also wish to express my gratitude to the people from the lab. Raül, Aurélia, Henri, Ali, Edu. The working environment was so great with having you around.

A very big thanks also to Jeanne, Sara, and Anouk. Your kindness is special. I will miss a lot the "climbing after-work", as well as the "pizza after-climbing".

Things are easier when you have a friend like Branzi. We have been lucky enough to share quite a lot of time these years, and basically being housemates again. Well Meli, thank you too in this regard. You both have been so helpful and you have made me feel home always when needed the most.

Thanks to Marta, for being that inspiring, supportive, and for making these last years quite unique. *Que suerte*, what a quality time, and what a lucky summer school, indeed!

To conclude, thanks to all my family as a whole, for supporting me during these years, and for their constant encouragements. I would like to explicitly mention my brothers, Titto and Fede: you are such a good example.

This work was supported by Fédération Wallonie-Bruxelles (FWB) under the Action de recherche concertée (ARC) RevealFlight (grant number **17/22-080, REVEALFLIGHT** – The reverse- engineering of flight: a bottomup reproduction of bird biomechanics and of self-organization into a flock).

Abstract

Birds have inspired human innovation and technology for centuries. The first documented human flying attempt is in the late 800, from Abbas ibn Firnas who created what is considered the first aviation experiment. Later in the years, around the 15th century, Leonardo da Vinci dedicated long time of his research to formally understand bird's flight, leading him to the famous ornithopter invention. Biological fliers are still nowadays a source of scientific inspiration for unsolved research questions, both invoking a deeper understanding of sophisticated flight mechanisms, and sparking new engineering ideas.

This Thesis aims at contributing in the field of flight dynamics of migratory birds, trying to understand the role of biological and kinematic parameters on flapping flight stability. From a physical point of view, the flapping represents a forcing term in the equations of motion of bird flight. Generally speaking, due to this action, these equations do not display fixed points of equilibrium. The problem of studying stability of bird flight is therefore re-formulated via a limit cycle analysis of such equations of motion.

We leverage such a formalism to provide evidence that the morphological and kinematic parameters responsible for the generation of the pitching moment, are the most impacting the longitudinal stability. Our numerical results suggest that passive stability cannot be achieved in absence of the tail surface. However we show a trade-off between passively stable flights, and power expenditure, suggesting explanations for the fields observation that birds flap with furled tail during long flights.

We conclude by validating a multi-body approach for modeling the bird dynamics, that can be of direct help in understanding the role of bioinspired compliant elements on flight stability. A preliminary investigation modeling the shoulder joint compliance of the wing is proposed.

Contents

	Con	tents		iii
1	Intr	roduction		
	1.1	Locon	notion and gait	3
	1.2	Bird wing and tail anatomy		5
		1.2.1	Bird wing	6
		1.2.2	Bird tail	8
	1.3	Stability of bird flight		
	1.4	Estima	ating power in biological fliers	10
	1.5	The in	nportance of modeling bird flight	11
	1.6	Resear	rch questions	11
	1.7	Struct	ure of the dissertation	13
	1.8	Relate	d scientific publications	14
2	Lite	rature l	Review	15
	2.1	Biome	chanics of flying birds	16
		2.1.1	Dimensional scaling	16
		2.1.2	Experimental work on flapping kinematics	18
	2.2	2 Aerodynamic models for flapping wings		19
		2.2.1	Momentum jet theory	23
		2.2.2	Blade element model	23
		2.2.3	Lifting line theory	25
	2.3	Flight dynamics stability		28
		2.3.1	Averaging theory	29
		2.3.2	Limit cycle approach and Floquet theory	30
	2.4	Outco	me of the literature review	32

3	Dyr	namics of Flapping Flight		35
	3.1	Introduction		35
	3.2	Dynamical model of a flying bird		37
		3.2.1 Equations of longitudinal	motion	38
		3.2.2 Wing kinematics		39
		3.2.3 Aerodynamic model of the	e wing	41
		3.2.4 Aerodynamic model of the	e tail	44
	3.3	Coupling aerodynamic and flight	dynamics	46
		3.3.1 Reference wing kinematics	3	46
		3.3.2 Computation of the aerody	ynamic forces and moments	47
	3.4	Limitations of the current framew	ork	51
	3.5	Conclusion		52
4	Det	tection of limit cycles and multifla	ар	55
	4.1	Introduction		55
	4.2	Background on dynamical system	IS	56
	4.3	Stability of limit cycles		58
		4.3.1 The Jacobian matrix		60
		4.3.2 A graphical example		64
	4.4	The multiple-shooting algorithm		66
		4.4.1 Multiple-shooting scheme	for unknown period of the	
		orbit		66
		4.4.2 Multiple-shooting scheme	for known period of the	
		orbit		68
		4.4.3 Solving the multiple-shoot	ing scheme	70
	4.5	Computation of the Jacobian mat	ix	71
		4.5.1 Analytical computation of	the Jacobian matrix	71
		4.5.2 Numerical computation of	the Jacobian matrix	72
	4.6	multiflap package		72
		4.6.1 Architecture of the code .		73
		4.6.2 Current limitations		74
	4.7	Conclusion		75
5	Analysis of a representative flapping flight regime			
	5.1	Introduction		
	5.2	Numerical parameters and wingtip trajectory		
5.3 Results		Results		81
		5.3.1 Experiment 1: representati	ve limit cycle and stability	
		analysis		81

		5.3.2	Experiment 2: Sensitivity analysis of the shoulder amplitude	86
		5.3.3	Experiment 3: Sensitivity Analysis on the Wing In- sertion Point	89
	5.4	Comp	arison between two methods for computing the Jaco-	
		bian n	natrix	91
	5.5	A pos	teriori verification of the aerodynamic model	91
	5.6	Discus	ssions and conclusion	92
6 On the role of tail in stability and energetic cost of bird fla			e of tail in stability and energetic cost of bird flapping	
	flight			95
	6.1	Introd	uction	95
	6.2	Drag J	production by body and wing	97
	6.3	Param	netric space	98
	6.4	Power	Consumption and Cost of Transport	100
	6.5	Nume	erical settings	101
	6.6	Result	ts	102
		6.6.1	Manifold of the solutions	102
		6.6.2	Comparison between stable and unstable limit cycles	103
		6.6.3	Trade-off between stability and energetic consumption	107
	6.7	Discus	ssions and Conclusion	108
7	A m	ulti-bo	dy approach for bird flapping flight	111
	7.1	Introd	luction	111
	7.2	The m	ulti-body model	113
		7.2.1	Bird topology	114
		7.2.2	Robotran workflow	116
		7.2.3	Interfacing multiflap and Robotran	117
	7.3	Valida	ition method	119
	7.4	Result	s of the validation	119
	7.5	Concl	usion	121
8	Moo	deling j	oint compliance	123
	8.1	Introd	uction	123
	8.2	Comp	liance of the shoulder articulation	125
	8.3 Modification on the multi-body topology		ication on the multi-body topology	126
	8.4	Result	s	127
		8.4.1	Effect of the spring element	127
		842	Effect of the damping element	- <i></i> 129
		0.1.4	Encer of the dulliping clement	/

	8.5 Current limitations			
	8.6 Conclusion			
0				
9	Conclusions and future perspectives		135	
	9.1 Overview of the manuscript		135	
	9.2 Discussions of the key findings		136	
		9.2.1 Wing model and aerodynamics	136	
		9.2.2 Development of the multiple-shooting algorithm	138	
		9.2.3 Limit cycle approach for flapping flight dynamics	139	
		9.2.4 Compliance of wing articulations	139	
	9.3	Review of the research questions	140	
	9.4	Future perspectives	142	
		9.4.1 Body compliance	142	
		9.4.2 A supporting tool for in-vivo experiments	145	
	9.5	Final remarks	145	
	Bibl	iography	147	
	DIU	logruphy	11/	
Aŗ	Appendices			
Α	Vali	dation of multiflap on Rössler's system	165	
B	multiflap tutorial		171	
C	Comparison of different integrator schemes 1			
D	Modification of main.py file for Robotran 1			
E	Estimation of the largest Lyapunov exponent 17			

Chapter 1 Introduction

Flight dynamics is a broad subject that aims at understanding and predicting the motion of flying apparatus. Classical theories of flight mechanics were developed during the last century, and pushed by a driving need of knowledge to advance with aerospace technology. We are here interested in the branch of flight mechanics applied to motions in the atmosphere. Studying the dynamics of flight in the atmosphere requires an interplay of cross-disciplines such as classical mechanics, fluid dynamics and mathematics; whose foundational contributions were provided by the greatest mathematicians and physicists of the 18th and 19th century, such as Newton, Bernoulli, Euler and Laplace [1]. This PhD Thesis is primarily focused on the dynamics of birds, and among all the flying species, the ones of interest are migratory birds of large scale (e.g. ibis). In particular the work is aimed at shedding light on questions regarding flight stability of fast forward flight in flapping regime. In this Chapter, the reader will discover fundamental concepts of bird flight and animal locomotion which will be largely used in the rest of the manuscript. The objective is to introduce necessary glossary for ultimately building a mathematical model. Key terms such as *gait*, *flapping phases*, *passive stability* will thus be defined.

Birds have inspired human innovation and technology for centuries. The first documented human flying attempt dates back from the late 800, from Abbas ibn Firnas who created what is considered to be the first aviation experiment. Later in the 15th and 16th centuries, Leonardo da Vinci dedicated long time of his research to formally understand birds' flight, leading him to the realization of a human powered flying machine with flapping wings, that nowadays we call ornithopter. He also put great ef-

forts on understanding complex flight maneuvers, now called dynamic soaring [2], documented for the first time in his manuscript *E* and in his *Codice sul volo degli uccelli* where he stated: "The kite and other birds do not flap their wings very much, but seek the current of the wind. [...] When the wind is not blowing, the Kite flaps its wings in such a way as to fly upward freely. Then it begins to descend losing lots of altitude, but gaining speed." His descriptions predated the first generally accepted physical explanations of this phenomenon, proposed by Lord Rayleigh in 1883 [3].



Figure 1.1: Flock of four bird exploiting upwind currents. First documented maneuver study by Leonardo da Vinci, Manusctipt *E*, folio 40 verso.

Nowadays, biomimicry challenges new technological limits, and among others examples, biological fliers are a source of scientific inspiration for unsolved research questions. Their capability of flying over long distances during migrations, responding to environmental perturbations and handling maneuvers, is unique. As an example, barn swallows show accelerations up to 14G, and roll rates up to $5000\frac{deg}{sec}$ [4]. Advances in ornithology have proven evidence of non-stop migratory flight of over more than 5000 km [5, 6], and a recent observation by Gill et al. [7] captured a non-stop migration from Alaska to New Zealand of over 11 000 km which arguably represents the longest flyway measured so far. The ability of fliers of mitigating environmental perturbations (such as gusts, wind etc.) is stimulating the interest of biologist and engineers to investigate the morphological and physical mechanisms that govern bird flight stability. These performance are attracting a broad scientific interest, for both having a pure understanding of the physics governing such delicate phenomena, and for sparking new ideas in engineering design and advances [8].

1.1 Locomotion and gait

Migrations are fascinating phenomena happening in nature. It is difficult to agree on a general definition, but what characterizes migration in animals, is the movement from one habitat to another [9]. Many animals and species migrate as an adaptive response to seasonal changes, or geographic variation of available resources [10]. Changes in light duration, light intensity, and temperature are critical factors that trigger this event.

This phenomenon often highlights astonishing endurance performance. The bar-tailed godwit flies across the Pacific Ocean covering about 11 000 km in non-stop flight over nine days [7, 11]. Humpback whales swim from the cold polar water where they feed, to breeding waters of subtropical or tropical regions, covering about 16 000 km each year [12, 13]. This variety of method that animals (or humans) exploit to accomplish any movement from one place to another, is generally called **locomotion**.



Figure 1.2: Scheme of the forces acting on a bird.

Bird locomotion is governed by four main fundamental forces. Considering Figure 1.2, these are:

- Weight: It is a downward force caused by gravity action.
- Lift: It is the main force for sustaining the bird in air. It acts upward and it is mostly generated by the wings. Contributions of lift can also be generated by the tail surface.

- Drag: It is the force that acts in the opposite direction of bird motion. Drag forces are generated by pressure distributions around the body, and to friction effects between the surface and the flow.
- Thrust: It is the propelling force produced by the flapping of the wings. It determines the movement of the bird in the preferred direction of motion.

Importantly, bird locomotion can be split in two main categories: **unpowered** flight and **powered** flight. Unpowered flight has the peculiarity that birds do not flap their wings to sustain their weight, and the wings produce lift forces exploiting the environment, but do not produce thrust. This happens during *gliding*, where the wings are fully extended and can produce enough lift while minimizing drag, pending a sufficient initial forward velocity (Figure 1.3). It also happens in *soaring* where the flier exploits upward thermal currents or wind profiles to gain altitude, perform maneuvers, or travel long distances [14, 15] (Figure 1.4). Soaring is a typical flight regime observed in large seabirds, mostly due to the windy conditions of the environments where they adapted [16].



Figure 1.3: Red-beaked seagull in gliding regime. Photo Credits Max Gray, usage under Unsplash Licence conditions.



Figure 1.4: Schematic diagram of wandering albatross in soaring regime. Figure from [17] under CC-BY license.

Conversely, powered flight is characterized by the flapping motion of the wings as illustrated in Figure 1.5. This allows the flier to simultaneously produce lift and thrust that vary over time. This locomotion regime is the one of interest in the present manuscript. An important definition when talking about flapping, is gait:



Figure 1.5: Flapping regime. Photo Credit Gary Bendig, usage under Unsplash Licence conditions.

Gait (adapted from [18])

In bird locomotion, the gait is the temporal pattern described by the movements of the wing, over one flapping period.

All flapping gaits build upon two main phases. A *downstroke* phase, that provides the majority of the locomotion forces (lift and thrust), and an *upstroke* phase which represents the recovery part of the cycle. During the upstroke, the wing is re-positioned to repeat a next downstroke [19, 20]. A sequence of a complete flapping cycle is pictured in Figure 1.6.

It is the goal of the next Section to give an overview of the two main aerodynamic surfaces of the bird — namely the wing and the tail — and how their anatomy allow to generate complex gaits and develop forces.

1.2 Bird wing and tail anatomy

In order to model the dynamics of bird flight and mathematically reconstruct a gait, it is important to describe the anatomy of the main lifting



Figure 1.6: Sequence of a flapping cycle. (a) Full upstroke. (b) Middle downstroke. (c) Full downstroke. (d) Middle upstroke. (e) Full upstroke. Original illustration: Pau Olivares.

surfaces, namely the wing and the tail.

1.2.1 Bird wing

Bird wings are flexible surfaces that are capable of developing aerodynamic forces over a variety of flight velocities and environmental circumstances [21]. From an anatomic perspective, the bird wing is composed of three main joints, namely the shoulder joint, the elbow joint and the wrist joint, as illustrated in Figure 1.7. Shoulder and elbow are reciprocally connected via the humerus, elbow and wrist between ulna and radius, and the metacarpals is attached to the wrist [22].

Thanks to this poly-articulation, bird wings are therefore capable of flapping, sweeping, twisting, and changing its extension (folding). The motion of such wing joints can be actively actuated by muscles, i.e. through the so-called **active wing morphing**, or passively powered by external loads exploiting the joint compliance, i.e. through **passive wing morphing** [24].

Active morphing regulates the locomotion gait, by modulating the force production in both gliding and flapping regimes, and acts on an aerodynamic level to vary the lift and drag coefficients, as well as the wing



Figure 1.7: The skeletal system of bird wings. Adaptation from[23], under licence Creative Commons Attribution.

area [24, 25]. Most birds, during the downstroke, tend to fully extend the wing for maximizing the production of lift forces [26]. Conversely, during the upstroke, the wing is folded via the elbow joint, in the so-called sweep movement. This geometry change aims at reducing the wing surface and minimizing the generation of drag forces in the recovery phase.

The mechanical work needed to produce the locomotion gait comes from muscles. Biewener et al. [27] in 1992 performed an experiment to measure muscle activities in starlings. They showed that the downstroke phase was mostly regulated by pectoralis muscles (see Figure 1.7). This muscle connects the humerus with the keeled sternum and it is the largest in birds. In contrast, the principal muscle that governs the upstroke is the supracoracoideus. It also plays a role to decelerate the end of the downstroke, to respond to inertial loads due to the flapping. Supracoracoideus muscle is also attached to sternum and the upper part of the humerus (see Figure 1.7). On the other hand, passive morphing is a complex phenomenon that has recently been put in the equations of bird modeling, in order to study the influence of compliant elements of the wing on external environmental perturbations [28].

The lifting surface of birds wing is mainly composed of feathers. Feathers are very complex structures that have the function of changing the wing surface via their spreading and folding, and transmitting the aerodynamic loads to the wing bones. In order to support the generated forces, these elements combine properties of stiffness and flexibility [29]. The feather surface is composed by *primary* and *secondary* feathers as illustrated in Figure 1.8. Primary feathers are rigidly attached to the metacarpals. Sec-

ondary feathers are attached to the ulna and can rotate about the spanwise axis of the wing [19].

Interestingly, feathered surfaces evolved showing colored patterns. This aesthetic feature, play also a role in signaling functions, allowing birds to better track the position of nearby individuals in order to avoid collisions in fast flight regimes [30, 31].



Secondary feathers

Figure 1.8: Planar form of a bird wing. Primary feathers — pictured in blue — are rigidly attached to the metacarpals. Secondary feathers — pictured in green — are attached to the ulna, and can rotate about the spanwise axis of the wing.

1.2.2 Bird tail

In birds, the tail is a flexible surface that has the capability of dynamic morphing. Like the wing, this surface is also feathered and can be controlled during the flight in three main degrees of freedom, namely its incidence, its rotation and its opening angle [22, 32]. Incidence and rotation are illustrated in Figure 1.9.



Figure 1.9: Tail incidence and rotation. **(a)**: Lateral view of the bird, where the tail degree of freedom controlling the incidence is pictured. **(b)**: Back view of the bird, where the tail degree of freedom controlling the rotation is pictured.

A planar view of a tail surface is illustrated in Figure 1.10 where the opening angle is better highlighted. Indeed birds have the authority to

Figure 1.10: Planar view of the tail surface. This surface is composed of feathers which are anchored to the bird main body. Two important configurations are illustrated, namely an open configuration (pictured in red), and a furled configuration (pictured in grey).



open and close the tail depending on circumstances, setting *open tail* configurations or *furled tail* configurations respectively. The tail shape and the number of covering feathers vary across species. The whole tail apparatus is controlled by six major muscles and the tail is functionally decoupled from the rest of the trunk [22, 32, 33]. Harvey et al. suggested that this decoupling allows an independent control of this surface, and thus the tail could be specialized in the context of flight control [22]. However the active role of the tail in forward flight stabilization remains an open question. There are some intuitions suggesting that evolution traded-off the tail as a stabilizer, in favor of sensor-driven closed-loop mechanism to optimize flight efficiency. To support this hypothesis many birds have been observed to successfully fly with a furled tail [25, 34] and spread it up to perform energetically costly maneuvers. In the next Section this important concept of flight stabilization, will be formally defined and clarified.

1.3 Stability of bird flight

Birds fly in a highly perturbed environment, where wind, thermal currents and gusts constantly interact with the animal. In order to fly birds not only have to develop the required forces for sustaining their weight and modulating their velocity, but it is crucial for them to control their flight trajectory and deal with such disturbances [35, 36].

A key concept for studying this capacity to fly in perturbed environments is the one of *stability*. In our context, stability can be defined as the capacity of birds of rejecting external perturbations in order to restore an original flight regime. This flight regime can be restored by inherent morphological properties of the flier (passive stability), by continuously actively adjusting the gait kinematics as function of external stimuli (active stability), or finally a combination of both strategies. A central topic of the present Thesis is the one of passive stability. It is defined as follows:

Passive stability

Passive stability is the inherent capacity of a biological flier of responding to external perturbations, restoring an original flight regime, without any active feedback control of its gait.

A foundational study by Smith [35] developed a theory of *evolution of instability*, establishing how inherently unstable flight regimes might have provided a selective advantage for fliers through evolution. Indeed, passively unstable systems are more responsive to changes in command, and this might have facilitated maneuverability for birds. This had to come in parallel with the development of sensory-driven neural circuitries to actively control the flight in order to display stable closed-loop behavior. But is passive stability completely lost through the evolution of bird species? This has been recently shown to be false in gliding [22]. In this regime, some specific elbow movements are used to transition between unstable and stable regimes. Studying similar phenomenon in flapping is one of the driving questions that motivated the present work.

1.4 Estimating power in biological fliers

An important feature in locomotion is the one of power consumption. This aspect in fact leads animals to adopt different locomotion regimes, in order to maximize or minimize certain biologically-relevant functions [37]. As an example, a bird can select an energetic costly flight regime to escape a predator via maximizing the forward velocity through gait adaptation. Conversely, during long migrations birds tend to minimize the cost of transport, explaining the emergence of important collective behaviors that usually take the name of "V-formation" flights [38, 39]. Birds within V-flocks, position themselves in optimal group configurations in order to exploit the leader wake, and showing a determined wingtip path phase that serve for energy saving purpose [40].

In the vision of studying passive stability, characterizing the flight by estimating the power consumption is relevant. In particular, it serves as a criterion in the understanding which — under the limitations of mathematical models — can be the preferred locomotion regimes a flier may choose, and thus helping in identifying the ideal kinematics and morphological

configurations that could be adopted in order to reduce the energetic expenditure of the flight.

1.5 The importance of modeling bird flight

The approach that will be presented in this manuscript relies on numerical modeling. There are several reasons that justify the need of building insilico tools.

First, reproducing the biomechanics of bird flight allows to study the influence of different gait parameters on related flight regimes. As a direct consequence, this modeling approach can be exploited to explain the reasons that might govern specific kinematics adopted by birds and observed in experiments. Numerical approaches can also be extended to replicate dynamical responses to broad environmental conditions that a bird may encounter during its flight (gust rejection, ground effect, thermal currents, etc.) and that would otherwise require specific and very expensive experimental setups, as well as dedicated wind tunnels.

Moreover, it can complement some inevitable limitations of in-vivo experiments. As an example, electromyography is often used in birds to measure muscle work. Respirometers are used to estimate metabolic power, and consist in tubing the bird during the entire duration of the experiment. Such experiments require intrusive instrumentation, which may alter the natural way of flying [41, 42]. Numerical simulations, in this regard, can help in making tailored predictions that might guide experimental research.

Ultimately, the in-silico approach is the most versatile tool for helping the emerging engineering interest in unmanned air vehicles (UAVs) and bird-inspired robots [43, 44]. This offers the grounds for numerical optimizations of specific cost functions, helping in the design and development of this technology, for example offering bio-inspired control schemes to stabilize the flight, or implementing dedicated wing kinematics to improve flight efficiency.

1.6 Research questions

The main research questions raised in the context of the Thesis are listed below.

1. How can we accurately build a framework accounting for active wing morphing to assess flapping flight stability?

Flapping flight, due to the continuous variation of the aerodynamic forces induced by the wing motion requires a dedicated framework. The first building block is composed by an accurate model aiming at capturing realistic gaits and movements of the wing skeleton, and estimate the aerodynamic forces accordingly. The second building block is about describing the dynamics of the bird. Finally the last building block aims at assessing the stability of this dynamical system. A literature review is carried out in order to choose the most appropriate models composing each part of this global framework.

2. What is the role of the kinematics parameters of the wing and the function of the tail in longitudinal stability?

Leveraging the aforementioned framework, many flight regimes can be analyzed and assessed in terms of stability. We envisage the importance of studying different flapping gaits, and assessing their stability properties. In particular, seeking whether for some set of parameters passively stable flight configurations could be achieved.

3. Can our numerical investigations give insight on the evolutionary process of bird flight?

Biological observation shows fast forward flight with furled tails. This indeed suggests that developing an active sensor-driven control scheme improves the efficiency of the flight with respect of having passively stability mechanism, such as a tail surface. We want to challenge this hypothesis, via energetically quantifying the flight regimes identified, and look for trade-offs between passive stability and energetic expenditure.

- 4. How does the passive morphing of the wing influence flight stability?
- Compliance mechanisms are omnipresent in nature. Among birds, the wing flexibility was observed to be the principal passive mechanism of gust rejection [28]. To address this task, we present an extended framework of the multiple-shooting algorithm, that is coupled with a multi-body generator of equations, Robotran [45]. Thanks to such an extended framework, the interplay of active and passive morphing can be studied to help the understanding of the role of wing compliance in flight stability.

1.7 Structure of the dissertation

Chapter 2 presents a literature review of bird flight aimed at covering the relevant aspects needed for building a numerical model. It will review the fundamental scaling laws that apply to birds, in order to relate their morphology to their mass. Then it will report quasi-steady aerodynamic models of flapping wings. As a last point, it will describe the two main approaches to study the flight stability of flapping flight, namely averaging theory and Floquet theory.

Chapter 3 presents the assumptions and the equations of motion used in our computations. It will describe the choice of the aerodynamic model highlighting the range of validity, and it will show the coupling between aerodynamic and flight dynamics.

Chapter 4 explains the kernel of the computations, namely the multipleshooting algorithm. It will present the numerical scheme for both autonomous and nonautonomous systems and reports a description of the architecture of multiflap, the Python package to identify limit cycles and assess their stability.

Chapter 5 reports the first investigation carried out with this framework. It will show a representative limit cycle corresponding to a steady fast forward flight regime, and the quantification of its stability. It presents the first attempt to also level the flight, and a sensitivity analysis via varying the wing insertion point.

Chapter 6 presents the extension the framework to a parametric study. It involves computations of sources of drag, i.e. parasitic and profile drag, and employs a tail-like surface. The trend of the stability and the corresponding performance are quantified as a function of different kinematics and morphological configurations.

Chapter 7 shows the implementation of the multiple-shooting framework coupled with Robotran, and presents a validation of the new framework with previous obtained results.

Chapter 8 opens a new investigation leveraging Robotran environment in order to generate the equations of motion accounting for the compliance of the shoulder joint. It describes how the joint flexibility is modeled, and shows the preliminary results obtained.

Chapter 9 concludes the dissertation summarizing the findings and commenting on the limitations of our assumptions. It illustrates future perspective that remained uncovered in the present work.

1.8 Related scientific publications

The following chapters are related to published work. For the sake of readability, their content is adapted compared to the published articles.

Peer-reviewed journal papers

- Ducci G., Colognesi V., Vitucci G., Chatelain P., and Ronsse R. "Stability and sensitivity analysis of bird flapping flight", *Journal of Nonlinear Science*, 2021
- Ducci G., Vitucci G., Chatelain P., and Ronsse R. "On the role of tail in stability and energetic cost of bird flapping flight", *manuscript in preparation*, 2022

Conferences and dissemination

- Ducci G., Colognesi V., Chatelain P., and Ronsse R. "Simulation of flapping bird flight, part 2: Gait parametrization, limit cycle, and dynamic stability", Talk. *APS Division of Fluid Dynamics*, Seattle, December 2019.
- Ducci G., Chatelain P., and Ronsse R. "Stability and Gait Analysis of Bird Flight", Talk. *Interdisciplinary Seminar Series on Biolocomotion*, Happening online, September 2020.
- Ducci G., Chatelain P., and Ronsse R.. "The Application of Floquet Theory to the Stability of Birds Flight", Talk, *SIAM*, *Conference on Dynamical Systems*, Happening virtually, May 2021.
- Ducci G., Chatelain P., and Ronsse R. "The Application of Floquet Theory to the Stability of Birds Flight", Poster session, *EMBL*, *oscillator meeting*, Heidelberg, March 2022.

Summer school

• Summer School on Nonlinear Dynamics in Life Science, *Fields Institute*, Toronto (Canada), July 2019 - Travel grant award.

Chapter 2

Literature Review

Bird flight is a multi-disciplinary problem which invokes the development of a dedicated framework to ultimately tackle the topic of flight stability. Breaking down the problem in fundamental units, building such a framework comprises: (i) a mathematical model aimed at describing the flapping gait; (ii) an aerodynamic model aimed at estimating the aerodynamic forces; (iii) a dynamical model describing the equations of motion of the flier, and the related mathematical tools to study their stability. These three fundamental blocks are illustrated in Figure 2.1.

This Chapter first reviews the literature about biomechanics and wing morphology. This part of review reports the information about bird morphology and wing kinematics, aimed at reconstructing an in-silico flapping wing model. More in detail, we give an overview of the most relevant scaling laws that apply for birds, via the so-called allometric formulas. Moreover, we report the most relevant experimental work aimed at understanding kinematics features of the musculoskeletal apparatus of bird wings.

We next present an overview of relevant aerodynamic models that are appropriate to study flight dynamics. We mainly focus of quasi-steady models that permit to compute the aerodynamic loads on the wing, although they fail in capturing the unsteadiness and the topology of the wake.

As the aerodynamic models advanced in the course of the years, numerical methods tailored at studying the flight dynamics and the stability emerged at the same pace. We also provide a review of the two main approaches employed for such a purpose, namely the averaging theory and the Floquet theory, highlighting the most relevant works carried out in the



Figure 2.1: Taxonomy of the reviewed topics. Biomechanics and morphology contains the work done in order to retrieve allometric formulas and morphological parameters needed to model a bird wing. The aerodynamics section revisit the most relevant aerodynamic models, aimed at computing the forces generated by flapping wings. Flight dynamics section reviews the recent tools aimed at studying stability in flight dynamics of flapping wings.

field.

2.1 Biomechanics of flying birds

In order to build a realistic model of the wing, it is important to account for its morphology, size, and dimensions. This part of the review reports the mathematical scaling laws that apply to birds, relating their mass with their principal morphological parameters. This provides the modeler a quick estimation of paramount variables of interest that will be also further employed in our framework.

2.1.1 Dimensional scaling

When modeling biological phenomena, it is helpful to understand the effect that parameters play on different scales problem. A biological challenge when studying bird locomotion, is to correlate variables of interest so that to obtain generalized scaling laws.

The null hypothesis of scaling laws postulates that all the animals of a particular set are *geometrically similar* [19]. Given a characteristic length *l*, the following scaling would apply for the surface *S*

$$S \propto l^2$$

and under the assumption of constant density, the mass would scale as

$$m \propto l^3$$

Since in ornithology the body mass of the bird m_b is usually the easiest variable to measure [19], normally these relationships read

$$l = m_{h}^{1/3}$$

and

$$S = m_b^{2/3}$$

y has the important feature

Geometrical similarity has the important feature of *isometry*. The rejection of this null hypothesis leads to what is called *allometry*. Allometric formulas therefore look for deviation from the isometric scaling law, in order to correlate biologically relevant variables. These equations take the general form of

$$Y = am_h^{\gamma} \tag{2.1}$$

where γ is the allometric exponent and it is normally calculated as the slope of this linear regression in a log – log plot, of data collected from field observations. An example is shown in Figure 2.2 where the wingspans of different birds are related with the body mass.

Relevant work to estimate allometric relationships has been done over the years. Notably, Greenwalt [46, 47] calculated the trend of wing area and wing span of different bird species and insects as a function of the mass. Rayner [48], found a significant insight about flight efficiency. He suggested that body dimensions deviate from isometric scaling, in such a way that larger birds have lower cost of transport. Important correlations between different variables and the bird body mass are summarized in Table 2.1.

Allometric formulas are an important tool which can provide a quick estimate of relevant kinematics and morphological parameters. Indeed, in order to build a realistic model of a wing, quantities such as the wingspan,



Figure 2.2: Allometric plot of the wingspan with respect to the body mass of different birds species. Adopted from [20].

the aspect ratio, and the wingbeat frequency are fundamental, and can be rapidly estimated via algebraic relationships. However, for other quantities of interest the estimation of the correlation coefficient might be insufficient. Aerodynamic forces, or intrinsic phenomena that are inherently dependent on the gait are not captured with a sufficient level of accuracy by this approach, requiring the need of different mathematical methods.

2.1.2 Experimental work on flapping kinematics

An important contribution to the development of realistic aerodynamic models for bird flight, came with observations from Tobalske and Dial [50] in 1996. They recorded the wingbeat kinematics of pigeons in different flight regimes. They tried to understand the flight velocities at which a gait transition would occur, and the body orientation in different conditions. Interesting gait reconstructions of wing tip and wing root trajectories are reported in Figure 2.3.

This work bridged the gap between observation and modeling, furnishing detailed and quantified information on flapping gaits, in order to reconstruct them computationally. First numerical models of realistic bird wings started emerging and growing with accuracy thanks to these results, thus also improving the fidelity of the gait-dependent forces esti-

	Isometric exponent	Allometric formula [48, 49]
Wingspan [m]	0.33	$1.17m_b^{0.39}$
Wing area m^2	0.67	$0.16m_b^{0.72}$
Wing loading $[Nm^1]$	0.33	$62.2m_b^{0.28}$
Aspect ratio	0.0	$8.56m_b^{0.06}$
Minimum power speed $[ms^{-1}]$	0.17	$5.70m_b^{0.16}$
Maximum range speed ms^{-1}	0.17	$15.4m_b^{1.10}$
Minimum power [W]	1.17	$10.9m_b^{0.19}$
Minimum cost of transport	0.0	$0.21 m_b^{-0.07}$
Wingbeat frequency [Hz]	-0.33	$3.87m_{h}^{-0.33}$

Table 2.1: Isometric scaling and allometric deviations for birds species.

Figure 2.3: Wing tip and wing root trajectory reconstruction for a pigeon flying at different speeds. Adopted from [50].



mation [51].

2.2 Aerodynamic models for flapping wings

In order to study the flight dynamics of a bird, the output of interest from an aerodynamic model is the estimation of the aerodynamic forces. This Section provides a review of the main predictive methods proposed in literature to estimate such a variable of interest. It aims at explaining the main physical principle behind the model analyzed, and describes their main applications documented in literature.

Importantly, the proposed review is restricted to those methods that

are considered relevant for the topic of this Thesis. It thus omits: (i) Insect scale aerodynamics; (ii) Wake topology of flapping wings and related CFD methods. We rather focus on the so-called *low-fidelity methods* which are suitable for parametric analyses and investigations.

Before entering the details, it is important to introduce the nomenclature adopted, and the main dimensionless quantities that will be often recalled through the manuscript.

Aerodynamic airfoils are slender bodies whose boundary layer remain attached to the their whole profile. Referring to Figure 2.4, the fore extremity of the airfoil is called *leading edge*, the aft extremity of the airfoil is called *trailing edge*. The straight line connecting these extremities is termed *chord* and indicated with *c* [52]. Importantly, very often the airfoil chord is the parameter defining the scale of the problem. The line positioned midway between the upper and lower profile of the airfoil is called *mean line*. In symmetric profiles — such as the one reported in Figure 2.4 — the length of the mean line coincides with the length of the aerodynamic chord. When profiles are not symmetric, the mean line necessarily presents a curvature, and the airfoil is said to be *cambered*. The angle α between the direction of the flow field *U* and the chord, defines the *angle of attack*.



Figure 2.4: 2D airfoil profile. The extreme points of the airfoil are called leading edge and trailing edge. The straight line connecting these two points identifies the chord length. The angle α between the velocity vector U and the chord identifies the angle of attack.

Reynolds number

An important similarity parameter which is often used to characterize the flow is the so-called *Reynolds number*. Reynolds number is a dimensionless number expressing the ratio between inertial forces and viscous forces, and it is defined as

$$Re = \frac{\rho UL}{\mu} \tag{2.2}$$

where ρ is the density of the fluid, *U* a reference velocity, *L* is a reference length (often taken as a reference length in the direction of the fluid), and μ

is the dynamic viscosity. Indeed, the flow pattern around a body depends on the ratio between these two forces. Reynolds number allows comparing problems across scales. Aircraft operate at high Reynolds number, in the order of $10^7 - 10^8$, where inertial forces dominate the viscous ones. In most birds and bats the Reynolds number varies between $10^4 - 10^5$, whereas in insect scales can drop until 10^2 , where viscous effects prevail.

Quasi-steady assumption and reduced frequency

In quasi-steady analyses it is assumed that the instantaneous force of a flapping wing (or airfoil) is the one corresponding to an equivalent steadystate motion at the same instantaneous velocities and attitudes [20]. As a rule of thumb, the validity of quasi-steady assumption is determined by the so-called *reduced frequency*. The reduced frequency is another important dimensionless quantity. Adopting the definition of [19], it physically represents the distance that the wing tip moves up and down for each unit distance moved horizontally. If the excursion of the wing is large compared to the advance ratio, then sharp transition between upstroke and downstroke cause vortex shedding and related unsteady aerodynamic effects. Vice-versa, the motion can be considered quasi-steady. The reduced frequency is normally estimated as

$$k = \frac{(2\pi f)c}{2U} \tag{2.3}$$

with *f* being the wingbeat frequency, *c* the airfoil chord, and U the forward flight velocity. Quasi-steady assumptions generally apply if k < 0.2 [19].

Strouhal number

The last dimensionless parameter that we introduce, is the *Strouhal number* (St). It usually governs phenomena that present vortex growth and shedding, such as the a bird wake due to the flapping motion of the wings. Practically, in animal locomotion this number represents the ratio between the vertical velocity of the wing tip, and the forward flight velocity of the body motion. It is expressed as [53]

$$St = \frac{f\Psi}{U} \tag{2.4}$$

21

where f is the wingbeat frequency, Ψ the vertical distance of the wingtip, and U the bird forward flight velocity.

Aerodynamic coefficients

For a 2D profile a generic aerodynamic force per unit span can be expressed in the form

$$F_a = \frac{1}{2}\rho U^2 c C_f \tag{2.5}$$

where the quantity $\frac{1}{2}\rho U^2$ is the dynamic pressure, *c* is the airfoil chord, and C_f is a so-called aerodynamic coefficient. Aerodynamic coefficients are dimensionless quantities, that are particularly useful when comparing performance of wings at different scales. Under the assumptions of incompressible flow and steady conditions, for a flapping airfoil these aerodynamic parameters are functions of

$$C_f = f(\text{geometry}, \alpha, Re, k) \tag{2.6}$$

where the geometry embeds all the characteristics of the wing and its crosssections, α is the angle of attack, and *Re* and *k* the dimensionless quantities previously defined, namely the Reynolds number and the reduced frequency. The lift coefficient *C*_l, is defined as

$$C_l = \frac{L}{\frac{1}{2}\rho|U|^2c} \tag{2.7}$$

analogously the drag coefficient C_d is defined as

$$C_d = \frac{D}{\frac{1}{2}\rho|U|^2c} \tag{2.8}$$

and finally

$$C_m = \frac{M}{\frac{1}{2}\rho|U|^2 c^2}$$
(2.9)

where *L*, *D*, and *M* are the lift, drag and moment per unit span respectively, ρ the air density, and *U* the inflow velocity. The aerodynamic coefficients are extremely important because they remain constant under geometric similarities of aerodynamic profiles, and under the same flow conditions (angle of attack and Reynolds number), thus allowing to perform experiments (or numerical simulations) on different problem scales.

2.2.1 Momentum jet theory

The momentum jet theory (or actuator disk theory), is a steady-state model that represents one of the first attempts to mathematically describe the aerodynamics of flapping flight. Momentum jet theory, represented a readaptation of previous theories proposed by Shapiro to study helicopter flight [54].

This model replaces the bird wings with a circular disc of diameter equal to the wingspan. This disc represents a pressure discontinuity in the flow field, which accelerates the fluid downward.

From conservation of mass flow rate, actuator disk theory allows to estimate the induced velocity w_d via the algebraic relationship [54]

$$w_d = \frac{W}{2\rho U S_d} \tag{2.10}$$

where *W* is the weight of the bird, and S_d is the area of the actuator disk, having the diameter of the bird wingspan.

The first documented work applying momentum jet theory to model bio-inspired aerodynamics was proposed by Pennicuick [55]. In his work he was interested in calculating the power required for a pigeon to fly at various speeds. Indeed, this was the first effort to derive different contribution for power requirements in bird flight, using a rigorous framework.

However this method has major limitations. Considering Equation (2.10), the induced velocity only depends on two morphological parameters of the bird, namely the wingspan and its mass. This model therefore overlooks the flapping nature of the flier. It cannot capture the wingbeat, and all the kinematics changes that the wing may have.

2.2.2 Blade element model

Blade element model is still a steady-state model that consists in splitting the wing in chordwise elements. Each of these strips is then considered acting as a two-dimensional airfoil, and represents the fundamental units of bird propulsion, as pictured in Figure 2.5.

In this model, the local velocity that each profile sees, is given by the contribution of the free stream velocity \mathbf{U}_{∞} which incorporates the velocity of the bird, the induced velocity w_d , and the kinematic velocity due to motion of the of the wing \mathbf{v}_{kin} , relative to the bird's body.



Figure 2.5: Schematic of the blade element model. The wing is discretized in fundamental units named blade elements, and each blade is treated as a 2D profile embedded in a wing portion of planar area $S_j = c_j ds_j$. The velocity component \mathbf{U}_j accounts for the induced velocity and the kinematic velocity of the wing due to the flapping motion. The resulting lift and drag on each discretized element are l_i and d_j respectively.

$$\mathbf{U} = \mathbf{U}_{\infty} - w_d \hat{\mathbf{e}}_n - \mathbf{v}_{kin} \tag{2.11}$$

where the unit vector $\hat{\mathbf{e}}_n$ is the normal direction of the planar area S_j of each wing element, and the kinematic velocity is known from prescribed laws. Blade element model does not provide a tailored computation of the induced velocity. Therefore, commonly this quantity is estimated using momentum jet theory, and solving Equation (2.10).

Once **U** is calculated, then the lift and drag component at each wing element j can be computed as

$$l_j = \frac{1}{2}\rho |\mathbf{U}_j|^2 S_j C_{l,j}$$

$$d_j = \frac{1}{2}\rho |\mathbf{U}_j|^2 S_j C_{d,j}$$
(2.12)

where $S_j = c_j ds_j$ is the element reference area, and $C_{l,j}$ and $C_{d,j}$ are the lift and drag coefficients respectively of each element *j*. Thus at each time step, the total lift and drag developed by the wing are given by

$$L(t) = \sum_{j=1}^{N} \frac{1}{2} \rho |\mathbf{U}_{j}(t)|^{2} S_{j} c_{l,j}(t)$$

$$D(t) = \sum_{j=1}^{N} \frac{1}{2} \rho |\mathbf{U}_{j}(t)|^{2} S_{j} c_{d,j}(t)$$
(2.13)

The first documented adaptation of blade element theory on flapping flight, is from Weis-Fogh, in 1956 [56, 57] in order to study the flight performance of the desert locusta *Schistocerca gregaria*. Ellington in 1984, in a series of six papers, re-examined Weis-Fogh's results. He firstly reached to the conclusion that blade element theory could be used in insect aerodynamics in forward flight. However, he claimed that this method would become increasingly unreliable as the hovering state is approached, due to the unsteadiness of the wake [58].

More recently, this method has been employed by Taylor and Thomas, to study the stability of flapping flight in forward motion. They leveraged this formulation to derive the condition of static stability in flapping regime [59].

Wu and Popovic [51] simulated birds flight with a high fidelity model of avian geometry and kinematics. Based on Tobalske's observations, they modeled the wing with all of its joints and degrees of freedom, accounting for the feathers and tail. Interestingly, they coupled this complex bioinspired model with a blade element solver for the aerodynamic forces. It thus represented the first computational attempt, able to produce high fidelity wing gait, with an aerodynamic model capable to predict the forces.

In a recent study, Parslew used this model to study the flight performance of a pigeon subject to different wing kinematics [60]. The advantage of the blade element theory is the computational speed, and despite momentum jet theory represents a leap forward to capture the motion of the wing. However, its weakest point is the computation of the induced velocity, which may lead to non negligible errors in the force computation.

2.2.3 Lifting line theory

Lifting line theory was first proposed by Prandtl in 1918, and still nowadays represents a powerful tool to estimate aerodynamic forces acting on finite slender bodies, such as high aspect ratio wings [61]. The model assumes the the lifting surface is reduced to a single line, that sheds downstream vortex filaments as pictured in Figure 2.6.



Figure 2.6: Schematic representation of the lifting line. The wing is reduced to a line, where the aerodynamic forces are computed. The wake is modeled as a series of vortex filaments shed behind the wing.

Each element along the line is assumed to behave as a 2D airfoil, neglecting any transverse flow component, and subject to a modified angle of attack from the velocity induced by the wake w_d as

$$\alpha_r = \alpha - \alpha_i \simeq \alpha - \frac{w_d}{|\mathbf{U}|} \tag{2.14}$$

The elementary contribution of the induced velocity in a point y_0 along the lifting line, by a single vortex filament in y is calculated via Biot-Savart law

$$\delta w_d(y_0) = -\frac{\left(\frac{d\Gamma}{dy}\right)dy}{4\pi(y_0 - y)} \tag{2.15}$$

and the global value of the downwash velocity in the point y_0 is given by the integral of Equation (2.15) as

$$w_d(y_0) = -\frac{1}{4\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\left(\frac{d1}{dy}\right)}{(y_0 - y)} dy$$
(2.16)

We may now express the aerodynamic force per unit span acting on a generic cross section *y* via the chain of relationships

$$F(y) = \frac{1}{2}\rho |\mathbf{U}_r(y)|^2 c(y) C_{l,\alpha} \alpha_r = \rho |\mathbf{U}_r(y)| \Gamma(y)$$
(2.17)

26
where the right-most part comes from Kutta-Joukowski theorem. From Equation (2.17), the circulation thus read

$$\Gamma(y) = \frac{1}{2} |\mathbf{U}_r(y)| c(y) C_{l,\alpha}(\alpha - \alpha_i)$$
(2.18)

Plugging Equation (2.16) in (2.14), and substituting α_r in Equation (2.18), it is obtained the following

$$\Gamma(y) = \frac{1}{2}c(y)C_{l,\alpha}\Big[|\mathbf{U}|\alpha - \frac{1}{4\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\left(\frac{d1}{dy}\right)}{(y_0 - y)}dy\Big]$$
(2.19)

which represented the so-called Prandtl's integro-differential equation of the lifting line. Once Equation (2.19) is solved for $\Gamma(y)$, the total lift is computed as

$$L = \rho |\mathbf{U}| \int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma(y) dy$$
(2.20)

15

Note that in deriving these Equations, the geometrical parameters characterizing the wing and the aerodynamic profiles, namely the chord c(y), the angle of attack $\alpha(y)$, and the lift coefficient $C_{l,\alpha}$ are assumed to be known.

Lifting line theory was initially developed for fixed wings, but gradually extended also to flapping. The first documented case of the application of this model to flapping wing was proposed by Betteridge and Archer [62], where they used quasi-steady assumptions in order to calculate the induced velocity from the wake. This model was tested to calculate the lift distribution simulating a bird flight scenario. However, this wing model only presented a single degree of freedom allowing the flapping motion, but was unable to capture a realistic wing morphing.

In 1981, Philips et al. [63], extended the model from Betteridge and Archer, accounting unsteady effects from the wake. They modeled a pair of flapping wings flying at a constant velocity. The motion was prescribed to flap in a plane perpendicular to the hinge axis with one degree of freedom only as described likewise in [62]. A schematic is shown in Figure 2.7. The wake was modeled as a vortex sheet in a near field, and as a series of closed loop elements in the far field. Although this was a novel application of embedding lifting line theory with unsteady effects, this work was still far from capturing the aerodynamics of birds. It did not account for realistic kinematics and complex wing morphing. A similar approach was proposed by Izraelevitz et al. [64] where they also applied a lifting line



Figure 2.7: Flapping wing scheme employed by Philips [63] for the development of a flapping lifting line.

model to capture the aerodynamic of a flapping wing, and tried to account for some local unsteadiness of the wake while avoiding to remember the full wake history. This model was proposed to be real-time, however no applications with the body dynamics were tested. As in [63] the wing was modeled as having a single rotational degree of freedom at the level of the shoulder.

A recent work developed by Colognesi et al. [65], proposed a lifting line model based on a continuous extraction from a bird wing capable to morph in all of its degrees of freedom. This model, described in [65, 66], could be used both via modeling the wake as quasi-steady, or coupled with a vortex particle mesh solver to characterize the full wake topology.

2.3 Flight dynamics stability

As a consequence of the advances in the aerodynamic models for flapping fliers, a great effort has been put in developing models to study the flight stability of flapping animals. The flight dynamics of flapping fliers constitutes a nonlinear time dependent system of ordinary differential equations in the compact form

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t, \nu) \tag{2.21}$$

where **x** represents the dynamical state variables of interest, **v** the velocity field constituting the equations of motion, *t* the time dependency which embeds the time-varying motion of the wings, and v a set of parameters.

As research advanced, two main concurrent approaches to address the

problem of flight stability affirmed, namely *averaging theory* and *Floquet theory* [43]. This Section reviews these two methods, highlighting both advantages and disadvantages, and reviewing the significant contribution in the field.

2.3.1 Averaging theory

The averaging theorem converts a nonlinear time dependent system, into a nonlinear time independent system. It applies to systems in the form

$$\dot{\mathbf{x}} = \epsilon \mathbf{v}(\mathbf{x}, t, \epsilon) \tag{2.22}$$

where ϵ is a small positive parameter, **v** is a periodic vector field with period T > 0. Averaging method approximates the solution of system (2.22) by studying the dynamics of the averaged vector field **v**(**x**, *t*, ϵ) at $\epsilon = 0$, such that the resulting system is nonlinear time independent in the form

$$\dot{\overline{\mathbf{x}}} = \epsilon \overline{\mathbf{v}}(\overline{\mathbf{x}}) \tag{2.23}$$

with the averaged vector field $\overline{\mathbf{v}} = \frac{1}{T} \int_0^T \mathbf{v}(\mathbf{x}, t, 0)$ [67]. Averaging theorem then states that [43, 67, 68]:

- 1. If $\mathbf{x}(0) \overline{\mathbf{x}}(0) = O(\epsilon)$, there exists a pair (b, ϵ^*) such that $\mathbf{x}(t) \overline{\mathbf{x}}(t) = O(\epsilon)$, for all $t \in [0, b/\epsilon]$ and $\forall \epsilon \in [0, \epsilon^*]$
- 2. If the origin of the solution of the averaged system $\overline{\mathbf{x}}(0) = 0$ is an exponentially stable fixed point of equilibrium of Equation (2.23), and if $\mathbf{x}(0) \overline{\mathbf{x}}(0) = O(\epsilon)$, then there exists ϵ^* such that $\mathbf{x}(t) \overline{\mathbf{x}}(t) = O(\epsilon)$, $\forall t > 0$ and $\forall \epsilon \in [0, \epsilon^*]$. Moreover, Equation (2.23) has an unique and exponentially stable *T*-periodic solution $\mathbf{x}_T(t)$ such that $||\mathbf{x}_T(t)|| \le k\epsilon$ for some *k*.

In flapping flight, commonly the parameter ϵ corresponds to the wingbeat period *T*. It thus implies that the faster the wingbeat frequency, and the lower is the error associated to the averaging approximation.

Taylor and Thomas [59] provided the first quantitative analysis of a flying animal leveraging averaging theory. In this study, they modeled the longitudinal flight dynamics of the desert locust *Schistocerca gregaria* using the equations derived for aircraft. These equations then were linearized and the aerodynamic coefficients were estimated experimentally. Likewise, this same approach was also employed by Xiong and Sun [69].

Although this method represents the first effort to formally study longitudinal flight dynamics, from its very beginning the limitations emerged. Its main drawback is that it fails if the wingbeat is close to the natural frequency of the body motion, such as for large birds in slow forward flight, or in transitions between two different flight regimes, such as from fast forward flight to hovering [36, 70]. Under these situations, the errors due to this approximation are not negligible.

Moreover, recent studies showed that averaging may also fail for larger time scales separation, requiring higher-order averaging methods [71, 72].

An important advance, was marked by Taylor and Żbikowski [73]. For the first time, they tackled the problem of flight dynamics via a novel approach, based on the concept of limit cycle stability. It was for the first time proposed a new definition of *stability of flapping flight* as the asymptotic orbital stability in phase space. This called the need of a new framework tailored at quantifying the stability of limit cycles, described by the equations of motion of the flier. This work represented the landmark that inspired further studies based on the so-called Floquet theory.

2.3.2 Limit cycle approach and Floquet theory

Limit cycle approach differs from averaging theory, because it looks at the evolution of the system without further assumptions or approximations.

Let assume the flight dynamics of a flier are described by a nonlinear time periodic system in the form of Equation (2.21), then the limit cycle is a particular solution such that

$$\mathbf{x}(t+T) = \mathbf{x}(t)$$

for a certain period T > 0. Such a periodic solution defines a steady-state flight regime. This approach addresses the stability problem by looking at the stability of such limit cycle in the phase space: if a perturbed trajectory converges back to the orbit, then it is stable, and vice-versa. This problem is addressed by Floquet theory. The stability of the set of equations is governed by the eigenvalues of the so-called Floquet matrix J, also known as Floquet multipliers, Λ_i . This Floquet matrix maps perturbations within an infinitesimal sphere around a point of the limit cycle (\mathbf{x}_0, t_0) into an ellipsoid after a time *T* equal to the period of the orbit. Stretching or contracting ratios of the principal axes of this first order transformation are governed by the Floquet multipliers. Floquet multipliers have the property of being invariant along the limit cycle, whereas the Floquet matrix and its eigenvectors depend on it. Concretely, the Floquet matrix J can be calculated as the solution of the variational Equation:

$$\frac{d\mathbb{J}}{dt}(\mathbf{x}_{0})\Big|_{t_{0}}^{t} = \mathbb{A}(\mathbf{x}, t)\mathbb{J}(\mathbf{x}_{0})\Big|_{t_{0}}^{t}$$

$$\mathbb{J}(\mathbf{x}_{0})\Big|_{t_{0}}^{t_{0}} = \mathbb{I}$$
(2.24)

where the matrix

$$\mathbb{A}(\mathbf{x},t) = \nabla \mathbf{v}(\mathbf{x},t)|_{\mathbf{x}=\mathbf{x}^*}$$
(2.25)

is called the stability matrix [74] and is *T*-periodic on the limit cycle. If the absolute values of all Floquet multipliers Λ_i are smaller than one, the corresponding periodic orbit is stable. If the absolute value of at least one multiplier is larger than one, the corresponding orbit is unstable and the perturbation spirals out of the limit cycle along the corresponding eigendirection(s). Conversely to the averaging method, Floquet theory is independent by the time scale of the problem, allowing thus to perform stability analyses in conditions where the flapping frequency and the natural body frequency are on the same order of magnitude. We refer to Chapter 4 for details about limit cycles and Floquet theory, and the physical meaning of the Jacobian matrix.

Following this approach, Dietl and Garcia [75] build the first Floquetbased framework to study the longitudinal stability of an ornithopter. They defined the trim condition as the limit cycle described by the state-space variables of the equations of motion with the same period as the flapping wingbeat, and used Floquet theory to determine its stability. They studied the longitudinal flight dynamics of an ornithopter treated as a rigid body, with imposed joint kinematic trajectories, and developed a limit cycle detection method based on a multiple-shooting algorithm to concomitantly identify the limit cycle, and assess its stability. Importantly, they restricted the kinematic analysis to two degrees of freedom only, namely the plunging angle, and the wing twist, defining the wingbeat amplitude and the angle of attack, respectively. This approach has also been found suitable to shed light on flapping flight dynamics at a level of insect scales [76, 77].

2.4 Outcome of the literature review

Modeling bird flight dynamics requires a strong interplay between three main identified elements: (i) biomechanics of the wing apparatus; (ii) aerodynamic model; (iii) a mathematical tool to study the stability.

Modeling the wing with good level of accuracy requires capturing the innate wing morphing motion that birds can achieve via their skeleton. Capturing the morphing is paramount, because it permits to study all the phenomena of wing sweep and folding upon which stability and energetic performance may depend. This task is achievable by modeling the wing in all of its joints, and scale it with realistic dimensions and parameters. Allometric relationships are a powerful tool in order to provide rapid estimation of wing and body morphology, such as masses, wingspan, or wingbeat frequency.

In order to estimate the aerodynamic loads acting on the wings, there are various available models. Since the main focus of this manuscript is on flight dynamics stability, it is important to have a fast aerodynamic solver, which ideally does not depend on the unsteadiness and history of the wake. Quasi-steady aerodynamic models suit these needs, however a dedicated review is carried out in order to maintain the quasi-steady assumption while preserving the accuracy of the solver. We conclude that momentum jet theory is suited only for a rapid estimation of the downwash velocity. It can be coupled with blade elements, but fails in capturing the wing motion. On the contrary, the lifting line model is found to be the most accurate method to compute the induced velocities of the wake, while preserving accuracy in describing a realistic gait.

The aerodynamic model constitutes the input of the flight dynamics equations. This makes this system nonlinear time dependent. Two main approaches are discussed, namely averaging theory and Floquet theory. Although averaging theory can work for large time scale separation (such as insect flight), it fails at giving accurate results when the wingbeat frequency is close to the natural body frequencies. Conversely, Floquet theory looks at the stability of the limit cycle described by the equations of motion, and the calculation of the Floquet multipliers is independent of the time scales of the problem. For such a reason, we consider it appropriate for our case, and we further develop this method for our particular applications.

In light of what discussed, such a complete framework comprising a

high fidelity wing morphing, a quasi-steady lifting line solver, and a limit cycle formalism to capture steady state bird flight regimes in flapping has not been found in literature. Combining these existing elements, would thus allow us to contribute to the state-of-the-art with a novel and unique framework, tailored for large scale birds in flapping regimes.

Chapter 3

Dynamics of Flapping Flight

3.1 Introduction

In this Chapter we describe how the dynamics of a flapping flier is modeled. This represents the foundation in order to ultimately assess the flight stability. It also describes the wing and aerodynamic model, and presents the coupling between the aerodynamics and the equations of motion of the flier.

It is important to introduce the adopted notation. Sticking to the fixed wing formalism [1] we introduce the three principal planes and rotations about the body-axes of the bird. The body frame is assumed to be centered in *G*, the center of mass of the bird, and oriented by taking the x'-axis aligned with the longitudinal axis of the flier and pointing forward, the z'-axis pointing downward, and y'-axis to define a right-handed frame as pictured in Figure 3.1.

The sagittal or longitudinal plane is identified by the axes (x', z'). When the flight is restricted to this plane, the degree of freedom in rotation is about the y'-axis as shown in Figure 3.1(a). The moment $M_{y'}$ associated with this rotation is called *pitching moment*. Positive pitching moment is said to be *nose-up moment*, conversely negative pitching moment is said to be *nose-down moment*.

The transverse plane is identified by the axes (x', y'). When the flight is restricted to this plane, the degree of freedom in rotation is about the z'-axis as shown in Figure 3.1(b). The moment $M_{z'}$ associated with this rotation is called *yawing moment*.

The frontal plane is identified by the axes (y', z'). When the flight is restricted to this plane, the degree of freedom in rotation is about the x'-axis as shown in Figure 3.1(c). The moment $M_{x'}$ associated with this rotation is called *rolling moment*.



Figure 3.1: Principal planes adopted for describing the dynamics of a flapping bird. (a): Longitudinal plane defined by x', z'. $M_{y'}$ is the pitching moment about y'-axis. (b): Transversal plane defined by x', y'. $M_{z'}$ is the yawing moment about z'-axis. (c): Frontal plane defined by y', z'. $M_{x'}$ is the rolling moment about x'-axis.

The bird is considered to be symmetric with respect to the longitudinal plane. This allows to decouple longitudinal dynamics (pitch) from lateral dynamics (roll and yaw) [78]. The equations of motion that will be presented in this Chapter refer to the longitudinal plane only.

To describe the flight of the bird, a bio-mechanical model of the flapping wing is introduced. In our model, the wing is composed of poly-articulated rigid bodies representing the wing skeleton. This wing skeleton is then actuated by prescribing the kinematics of the shoulder, elbow, and wrist joints. This model, inspired by previous works [51], aims at representing the wing anatomy described in Chapter 1. Based on this skeleton actuation, our model is able to trace out the full wing envelope at every time step of the simulation.

This Chapter also introduces the aerodynamic solver, namely the quasisteady morphing lifting line. The importance of the aerodynamic solver in our context is to calculate the forces acting on the wing. As previously introduced in Chapter 2, lifting line builds upon Prandtl's theory to model the lift on finite wings of high aspect ratio [79]. It thus assumes that each profile of a finite wing behaves as a 2D wing at a modified angle of attack [52, 79]. This modified angle of attack is calculated accounting the induced velocity of the wake, which is itself modeled as straight semi-infinite vortex filaments shed behind the bird. The classical lifting line theory for a fixed wing is here re-adapted to account for the flapping motion of the bird. This lifting line is therefore continuously extracted from the timevarying wing envelope and consequently the forces are calculated under a quasi-steady assumption.

The Chapter follows describing the tail model. For this particular surface where the aerodynamic chord is of the same order of magnitude as its span, the lifting line model fails to predict accurate forces. A different model, i.e. the slender theory, is used to model such surface [80].

We conclude the Chapter with a comparison of the aerodynamic forces between two cases: (i) a steady and constant inflow where the bird body does not move, aimed at reproducing an in-silico wind tunnel situation; (ii) a case in which the body is free to move on the longitudinal plane and the dynamics and aerodynamic model are coupled. These examples are aimed at highlighting the unsteady nature of flapping phenomena, and at explaining the need of a rigorous framework to investigate the problem of flight stability.

3.2 Dynamical model of a flying bird

In this section, the equations of motion of a flying bird are developed. We build this model upon two main assumptions:

- The flight is restricted to the longitudinal plane, so that the bird main body has only three degrees of freedom: two in translation and one in rotation. The system is symmetric with respect to this plane. As a consequence, lateral forces, rolling moments and yawing moments are identically equal to zero at every time and therefore do not have to be considered in the equations of motion. Stability of these degrees of freedom is thus not discussed in the present manuscript.
- 2. The inertial effect of the wings on the main body can be neglected. The model therefore does not account of the effect of flapping on the motion of the center of gravity. This is guided by the fact that for large and fast migratory birds, the wing mass is much lower than the body mass, about 5% according to [81]. This assumption has been extensively used for both ornithopter and insect scale models [73, 75, 82, 83].

The main morphological parameters used to model the main body and the

wing kinematics are introduced in the following sections.

3.2.1 Equations of longitudinal motion

The body is modeled with a mass m_b and a rotational inertia I_{yy} about its center of mass. The equations of motion are expressed in the body frame G(x', y', z') with unit vectors $(\hat{\mathbf{e}}_{x'}, \hat{\mathbf{e}}_{y'}, \hat{\mathbf{e}}_{z'})$, and an origin located at the center of mass, as pictured in Figure 3.2. The state space vector is thus

$$\mathbf{x} = \{u, w, q, \theta\}$$

where *u* and *w* are the body velocities along the x' – and z' –axis and θ and *q* are the pitch angle and its time derivative about the y' –axis, respectively. Consequently, the equations of motion read [75, 84]



Figure 3.2: Reference frames describing flight dynamics in the longitudinal plane. The origin of the moving body-frame is taken at the bird's center of mass G.

$$\begin{split} \dot{u} &= -qw - g\sin\theta + \frac{1}{m_b} \left(F_{x'}(\mathbf{x}(t), t) + F_{x',t}(\mathbf{x}(t), t) \right) \\ \dot{w} &= qu + g\cos\theta + \frac{1}{m_b} \left(F_{z'}(\mathbf{x}(t), t) + F_{z',t}(\mathbf{x}(t), t) \right) \\ \dot{q} &= \frac{1}{I_{yy}} \left(M_{y'}(\mathbf{x}(t), t) + M_{y',t}(\mathbf{x}(t), t) \right) \\ \dot{\theta} &= q \end{split}$$
(3.1)

The forcing terms in Equation (3.1) are the aerodynamic forces and moments applied to the wing (namely $F_{x'}$, $F_{z'}$, and $M_{y'}$) and to the tail ($F_{x',t}$, $F_{z',t}$, and $M_{y',t}$). Consequently, the bird model is actually a 4-states *non-autonomous* system, where the aerodynamic terms at the generic time t > 0 depends not only on the instantaneous state vector, but also on the instantaneous configuration of the wing in the flapping cycle. This is obviously the main difference with respect to an equivalent fixed-wing aircraft model.

Formally, the forcing terms depend on the whole past state history. Therefore, these forcing aerodynamic terms can be expressed in the form of Equation (3.1) only if a quasi-steady-state approximation is used.

3.2.2 Wing kinematics

The bird has two wings. Each wing is a rigid poly-articulated body, comprising the bird arm, forearm and hand, as pictured in Figure 3.3. Each segment is actuated by a joint to induce wing morphing. The wing segment representing the humerus is anchored to the bird main body via the shoulder joint (**s**), which has three rotational degrees of freedom. The wing segment representing the ulna and radius is anchored to the humerus via the elbow joint (**e**), which has one rotational degree of freedom. The metacarparls is anchored to the ulna and radius via the wrist joint (**w**), which has two rotational degrees of freedom.

We do not solve the wing dynamics in the state space equations of the flier, but we rather assume that their kinematics are imposed. Consequently, the internal torques in the wing joints do not have to be computed for solving the body equations of motion. The description of the right and the left wing kinematics are assumed to be mirror, since movements are imposed to be symmetric. For the sake of simplicity, each joint angle *i* is considered to follow a harmonic trajectory $q_i(t)$, with respective amplitude



Figure 3.3: Top view of the bird model. The left wing emphasizes a cartoon model of the skeleton. The shoulder joint **s** connects the wing to the body via three rotational degrees of freedom (RDoF), the elbow joint **e** connects the arm with the forearm via one RDoF and the wrist joint **w** connects the forearm to the hand via two RDoF. Each feather is attached to a bone via two additional RDoF, except the most distal one "1" which is rigidly aligned with the hand. The right wing further emphasizes the lifting line (red) which is computed as a function of the wing morphing. The aerodynamic forces generated on the wing are computed on the discretized elements P_i . The tail is modeled as a triangular shape with fixed chord c_t and maximum width b_t that can be morphed as a function of its opening angle β .

 A_i , offset $q_{0,i}$ and phase $\phi_{0,i}$. We do not exclude that some joints might follow more complex variations during the flapping cycle, but we decided to model the articulations as harmonic functions in order to minimize the number of parameters describing the gait. A generic equation of a wing joint reads

$$q_i(t) = q_{0,i} + A_i \sin(\omega t + \phi_{0,i})$$
(3.2)

with $\omega = 2\pi f$ and f is the flapping frequency, identical for each joint. For the six rotational joints of the model in Figure 3.3, this makes a total of 19 gait parameters (including the wingbeat frequency) prescribing a particular set of wing kinematics.

Feather movements are governed by a simplified version of the model developed in [65], which is itself inspired from [85]. Indeed, feathers are

similarly attached to the wing bodies via two rotational degrees of freedom (allowing spreading and pitching in a bone-relative frame of reference) but the motion of these degrees of freedom follows here predefined trajectories, while they feature some dynamic compliance in bending and twist [65, 85]. More precisely, we constrained feather movements via kinematic relationships depending on the angles between the wing segments in order to make them spreading and folding smoothly with the wing.

3.2.3 Aerodynamic model of the wing

In this section, the model used to compute the aerodynamic forces acting on the wing is developed. The model assumes that all aerodynamic forces act on the wings, and none on the main body. We use a quasi-steady lifting line approach, where the wake is shed backward in the form of straight and infinitely long vortex filaments at each time-step of the simulation.

The wing motion and its position are defined in a wing-bone frame (x_w, y_w, z_w) shown in Figure 3.2. The respective unit vectors along these axes are $(\hat{\mathbf{e}}_{x'_{uv}}, \hat{\mathbf{e}}_{y'_{uv}}, \hat{\mathbf{e}}_{z'_{uv}})$. This frame is taken to follow the orientation of the body frame, while the translation of its origin fixes the position of the bird shoulder with respect to its center of mass through the flapping cycle. The projection of O_w on the x' axis, identifies the O'_w point. From this wing position, the lifting line is then consequently extracted. It is defined as the line passing through the quarter of the chord, which is itself defined as the segment orthogonal to the lifting line, going from the leading to the trailing edge of the wing, as pictured in Figure 3.3. From a given wing configuration, the leading and trailing edges are defined as follows. The former goes from the shoulder to the wrist in a straight line, then to the tip of the outermost feather. The latter connects the tip of each feather from the innermost to the outermost. The lifting line is then obtained through an iterative process guaranteeing that it is located at the quarter of the chord distance and that it is orthogonal to the chord at each points.

In order to compute the aerodynamic forces, it is required to know the wing angle of attack. A generic wing cross section is shown in Figure 3.4, where $c(\mathbf{y})$ represents the aerodynamic chord length. Each wing element is identified by a plane containing the lifting line. The unit vector orthogonal to such a plane is denoted by $\hat{\mathbf{e}}_n$, the unit vector tangent to the lifting line $\hat{\mathbf{e}}_t$ and the binormal one $\hat{\mathbf{e}}_b = \hat{\mathbf{e}}_t \times \hat{\mathbf{e}}_n$.

According to this notation, $\mathbf{v}_i = -w_d \hat{\mathbf{e}}_n$ is the induced velocity (down-

wash) and

$$\mathbf{U} = \mathbf{U}_{\infty} - \mathbf{U}_{kin} - \mathbf{v}_q$$

is the relative velocity seen by a wing profile, which accounts for the flight speed \mathbf{U}_{∞} and the wing motion \mathbf{U}_{kin} while its component along $\hat{\mathbf{e}}_t$ is previously eliminated, and \mathbf{v}_q is the contribution induced from the angular velocity of the body q computed as $\mathbf{v}_q(\mathbf{y}) = q\hat{\mathbf{j}} \times (\mathbf{P}_i - \mathbf{G})$. Hence, the effective angle of attack is given by

$$\alpha_r = \alpha - \alpha_i \simeq \alpha - \frac{w_d}{|\mathbf{U}|} \tag{3.3}$$



Figure 3.4: Left: Wing element between two wing profiles, and identifying a plane Σ containing the lifting line. Right: Cross section containing the chord point \mathbf{P}_i where the velocities are applied.

The wake is considered to be composed of semi-infinite vortex tubes aligned with the x-axis, as shown in Figure 3.5. In theory, because the wing is not straight, the bound vortex (i.e. the circulation of the lifting line itself) also induces velocities on the line itself. However in the presented model, these induced velocities are neglected since their magnitude is much lower as compared to the flight velocity. Therefore the only induced velocity accounted for is a vertical component due to the wake. This velocity at a point $\mathbf{y} \equiv (y'_w, z'_w)$ along the wingspan and induced by a set of semiinfinite vortex tubes of circulation $d\Gamma_i$ is computed via Biot-Savart law [52], i.e.

$$w_d(\mathbf{y}) = -\frac{1}{4\pi} \sum_{i} \left(-d\Gamma_i \frac{(\mathbf{y} - \mathbf{y}_i) \times \hat{\mathbf{e}}_{\mathbf{x}'_w}}{|\mathbf{y} - \mathbf{y}_i|^2} \right) \cdot \hat{\mathbf{e}}_n$$
(3.4)

where *i* are the discretized elements of the lifting line. Considering the theorem of Kutta-Joukowski, the local circulation Γ is computed as

$$\Gamma(\mathbf{y}) = \frac{1}{2} |\mathbf{U}_r(\mathbf{y})| c(\mathbf{y}) C_{l\alpha}(\alpha - \alpha_i)$$
(3.5)

where $\mathbf{U}_{\mathbf{r}}$ is the norm of the local relative velocity vector, *c* is the local chord, and $C_{l\alpha}$ is equal to 2π as a result of thin airfoil theory. We now

assume the downwash velocity to be sufficiently small to approximate $|\mathbf{U}_r(\mathbf{y})| \simeq |\mathbf{U}|$. Considering Equation 3.5 and injecting Equation 3.4 in Equation 3.3, we get the lifting line Equation

$$\Gamma(\mathbf{y}) = \frac{1}{2}c(\mathbf{y})C_{l\alpha}\left[\left|\mathbf{U}\right|\alpha(\mathbf{y}) - \frac{1}{4\pi}\sum_{i}\left(-d\Gamma_{i}\frac{(\mathbf{y}-\mathbf{y}_{i})\times\hat{\mathbf{e}}_{x'_{w}}}{\left|\mathbf{y}-\mathbf{y}_{i}\right|^{2}}\right)\cdot\hat{\mathbf{e}}_{n}\right]$$
(3.6)

To satisfy the solenoidal property of the vorticity field, the circulation Γ



Figure 3.5: The vortex wake of the bird is considered straight and infinite at each time-step of the flapping period. The variations in the line circulation Γ induce vortex tubes of circulation $d\Gamma_{i+1} = \Gamma_{i+1} - \Gamma_i$, which in turn induce velocities in the wake.

must form closed loops. This means that the circulation of the vortex tubes can be computed from the variations of Γ along the lifting line. For a given vortex tube i + 1 between the points i and i + 1 of the lifting line, the circulation of a shed tube $d\Gamma_{i+1}$ is equal to

$$d\Gamma_{i+1} = \Gamma_{i+1} - \Gamma_i \tag{3.7}$$

where Γ_i is the local circulation at the *i*th point of the lifting line.

The circulation of the lifting line is computed iteratively. Starting from an initial guess, the induced velocities are computed at each point of the lifting line, with the contribution of each vortex tube from Biot-Savart law. The angle of attack is then modified with the new local flow conditions and new values are obtained from Equation (3.3). The circulation of the vortex tubes are then computed from Equation (3.7), thus closing the loop. Once all the circulations are computed at every time step, the aerodynamic force acting on each discretized point of the lifting line is computed as

$$d\mathbf{F}_i = \rho \Gamma_i (\mathbf{U}_{\mathbf{r}} \times \hat{\mathbf{e}}_t) dl_i \tag{3.8}$$

Finally, the global forces acting on the wing, can be evaluated in order to close the system of Equation (3.1), by summing each contribution and computing the corresponding pitching moment, i.e.

$$F_{x'} = \sum_{i=1}^{N} (d\mathbf{F}_i) \cdot \hat{\mathbf{e}}_{x'}$$

$$F_{z'} = \sum_{i=1}^{N} (d\mathbf{F}_i) \cdot \hat{\mathbf{e}}_{z'}$$

$$M_{y'} = \sum_{i=1}^{N} (\mathbf{GP}_i) \times (d\mathbf{F}_i) \cdot \hat{\mathbf{e}}_{y'}$$
(3.9)

3.2.4 Aerodynamic model of the tail

Since the wingspan of bird tails is of the same magnitude as its aerodynamic chord, here the lifting line approach cannot be used [52, 86, 87]. Therefore, the tail is modeled using the so-called *slender theory*. Let us consider this lifting surface as a thin flat plate , with a triangular planform [80], as pictured in Figure 3.6. The morphology is defined via the tail opening angle β , and the chord c_t . This latter parameter is kept constant, thus the tail span is controlled via β from the trigonometrical relationship

$$b_t = 2c_t \tan\left(\frac{\beta}{2}\right) \tag{3.10}$$

This framework remains valid for low angles of attack of the tail ($\alpha_{tail} < 5^{\circ}$) within which it provides accurate results [88]. This limitation is valid in our context of fast forward flight, where the bird flight is straight, horizontal and the forward velocity *u* is much larger than the vertical one *w*.

Figure 3.6: Tail scheme modeled as a delta wing. The tail opening is identified by β , the tail chord by c_t and the tail span by b_t . The aerodynamic forces are applied in the point **N**.



The velocity component acting on the tail-like surface is

$$\mathbf{U}_t(t) = \mathbf{U}_{\infty} + \mathbf{v}_i^{w \to t} + \mathbf{v}_{ind,b}$$
(3.11)

where the term $\mathbf{v}_{ind}^{w \to t}$ is the velocity acting on the tail, induced by vortex filament shed by the wing calculated according to Biot-Savart law [52], and

$$\mathbf{v}_{ind,b} = q\hat{\mathbf{e}}_{u'} \wedge (\mathbf{G} - \mathbf{N})$$

is the velocity induced by the body angular velocity q, with $(\mathbf{G} - \mathbf{N})$ the vector between the center of mass of the body (**G**) and the point of application of the forces on the tail (**N**) taken at two third along the tail chord, as illustrated in Figure 3.3. The forces generated by the tail are computed as a result of potential flow theory [80, 89]

$$F_{x',t} = \left(\frac{\pi}{4}\rho\alpha_t |\mathbf{U}_t|^2 b_t^2\right) \cdot \hat{\mathbf{e}}_{x'}$$

$$F_{z',t} = \left(\frac{\pi}{4}\rho\alpha_t |\mathbf{U}_t|^2 b_t^2\right) \cdot \hat{\mathbf{e}}_{z'}$$
(3.12)

These forces are applied at the point **N**.

In addition, adding this tail-like surface introduces another source of drag that needs to be accounted for. This parasitic drag contribution is, according to [80]

$$D_{p,t} = \frac{1}{2}\rho |\mathbf{U}_t|^2 S_t C_{D,f}$$
(3.13)

where S_t is the tail planar surface and $C_{D,f}$ the dimensionless friction coefficient. This coefficient is estimated leveraging the results from thin plates [52] according to the relationship

$$C_{D,f} = \frac{1.328}{\sqrt{Re_T}} \tag{3.14}$$

with Re_T being the Reynolds number with the characteristic length of the tail chord c_t .

3.3 Coupling aerodynamic and flight dynamics

In order to integrate Equation (3.1), at each time step the aerodynamic forces and moment have to be computed, since they depend not only on the time, but also on the dynamic state of the system \mathbf{x} .

Considering Equation (3.1), the aerodynamic model at the generic time step *i* is initialized by the state space vector $\mathbf{x}^i = [u^i, w^i, q^i, \theta^i]$, and by the position of the wing φ^i imposed by prescribing the kinematics. Once the aerodynamic forces are computed, the dynamic state \mathbf{x}^{i+1} can be evaluated via numerical integration, and this process repeated for each time step. A schematic of the coupling is pictured in Figure 3.7.



Figure 3.7: Schematic of the coupling between the aerodynamics and the flight mechanics. At each time step *i*, the state variables \mathbf{x}^i and the wing position φ^i are the input for the aerodynamic model. The aerodynamic model computes the aerodynamic forces \mathbf{F}_a^i and moment M_a^i . At this stage, the numerical integrator can march in time to compute the state variables \mathbf{x}^{i+1} at the step i + 1.

3.3.1 Reference wing kinematics

The aerodynamic model can be adapted to large scale flapping fliers, and we implemented lengths of the bones and feathers to match those of the northern bald ibis (*Geronticus eremita*). This particular bird has been chosen because it has a high aspect ratio wing [90] – which is well suited for the lifting line approach used – and uses non-stop flapping flight.

The parameters governing the wing kinematics described by Equation (3.2) are constrained to follow the wing kinematics of real birds. No accurate data about the kinematics of ibises and other large birds are available in the literature. We thus exploit available observations for fast forward flight regimes of pigeons reported in [50], in order to produce a consistent qualitative kinematics, checked via the behavior of the wingtip trajectory. The typical wingbeat frequency is retrieved from [81, 91] and we tuned the wingbeat amplitude accordingly, in order to keep the angle of attack in a realistic range.

Based on the aforementioned observations, all simulations reported in the rest of the manuscript have been computed with the morphological parameters gathered in Table 3.1a, except for a dedicated study on the tail opening, where the value of this parameter will change. The wingbeat frequency for the investigations presented in the manuscript is taken equal to f = 4Hz. This has two main reasons: from a mathematical perspective the frequency does not alter the qualitative behavior of the forces and moments. It affect these values quantitatively via changing the angle of attack of the wing profile, due to a change in the kinematic velocity of the wing. This same effect is instead obtained via fixing the frequency, and varying the wingbeat amplitude. Moreover, from a biological point of view birds tend to select a preferred frequency for a cruise flight, and even though for changing flight regimes they can change it, the available range for each species is limited [92].

The resulting wing kinematics over one flapping period, obtained with the parameters of Table 3.1b, is also reported in Figure 3.8. The tip trajectory of the wing kinematics over one flapping period is further reported in Figure 3.9.

3.3.2 Computation of the aerodynamic forces and moments

The reference kinematics and wing morphology described in Table 3.1 are here employed to compute the aerodynamic forces and moment from the lifting line model.

In order to highlight the unsteady nature emerging in flapping flight we extend the application of this aerodynamic model to two further cases: the first one is an in-silico wind tunnel situation in which the bird flaps its wings but its body position is fixed and constrained. Under these conditions, the body dynamics is not modeled and the bird is impacted by a constant horizontal inflow of fixed velocity U_{∞} as pictured in Figure 3.10(a).



Figure 3.8: Snapshots of the wing kinematics within one flapping period, taken every T/4.

The second case reports the aerodynamic forces in the case of coupling with the body dynamics. Under these conditions the bird is free to fly on the longitudinal plane while flapping its wings with the same prescribed kinematics as before, Figure 3.10(b). The ODE system (3.1) is therefore solved for one flapping period with the initial condition $\mathbf{x}_0 = [U_{\infty}, 0, 0, 0]$, with $U_{\infty} = 14ms^{-1}$.

The resulting aerodynamic forces and the pitching moment for these two scenarios are reported in Figure 3.11. For the in-silico wind tunnel case the periodicity of forces and pitching moment is maintained over one flapping period (red solid line). This is because the inflow U_{∞} is kept constant, and the flapping motion is periodic. Conversely, when the bird is free to move on the longitudinal plane, the periodicity of the aerodynamic forces is lost (blue solid line). In this dynamical situation, for such initial values the inflow continuously changes over time, showing a non-periodic behavior of $\mathbf{x} = [u(t), w(t), q(t), \theta(t)]$. The evolution of these state variables in one flapping period is pictured in Figure 3.12.

This case highlights an important fact: although the wing kinematics is prescribed by periodic sinusoidal functions, the resulting evolution of the aerodynamic forces and the body dynamics are not necessarily periodic. This makes flapping problems unsteady phenomena. To study stability,



Figure 3.9: Wing tip trajectory. **(a)**: 3D evolution of the wingtip over one flapping period (black solid line). The wing is reported in three different flapping phases, namely the middle downstroke, full downstroke, full upstroke. **(b)**: Top view of the wing tip trajectory (black solid line) over one flapping period. The wing is reported at middle downstroke only.

we thus need to first define the steady-state of this system.

For a given prescribed kinematics, infinite solutions of the body dynamics can be found, depending on the initial conditions $\mathbf{x}(0)$. For some particular combination of such initial conditions, the body dynamics might show a periodic behavior. When this happens, the system is said to be *steady-state*. Indeed, when all the state variables are periodic the average acceleration over one period is zero. This case where all the state variables are periodic is called *limit cycle*. To study flapping flight stability, we therefore need to: (i) find the associated limit cycle for a prescribed kinematics; (ii) study the stability of such associated limit cycle.

The next Chapter is fully dedicated to introduce a rigorous limit cycle definition, with a particular focus on the tool developed in order to detect this particular solution, and quantify its stability.



Figure 3.10: Comparison between two different cases. (a) In-silico wind tunnel scenario. The bird body is constrained and the flapping kinematics imposed. In this situation the inflow is constant and aligned horizontally with the bird body. (b) Body dynamics scenario. The bird body is free to move on the longitudinal plane according to Equation (3.1), with imposed flapping kinematics. The dynamics is initialized at time t = 0 with the state space values $\mathbf{x}(0)$.



Figure 3.11: Comparison of the aerodynamic forces and pitching moment between a steady inflow condition (red) and a coupling with the body dynamics (blue).



Figure 3.12: Evolution of the 4 state space variables describing the longitudinal dynamics of bird flight.

3.4 Limitations of the current framework

It is important to illustrate what we cannot achieve with the current framework.

One limitation which is important to discuss is about the aerodynamic model. Lifting line theory builds on Prandtl's hypothesis of high aspect ratio wings [52]. This restricts the field of application of our model to birds that respect such condition, i.e. migratory birds with large wing span. Indeed, when the wing span reduces, then other mechanisms of lift generation come into the equations, for example leading edge vortex phenomena, which cannot be captured in the current framework [4, 43].

Assuming the wake being quasi-steady, implies that the forward flight velocity is sufficiently high not to consider the unsteady effect of the wake on the computation of the angle of attack. As previously introduced, the quasi-steady assumption can be checked by computing the reduced frequency of Equation (2.3) and checking that k < 0.2 [4, 43].

The assumption of ignoring the wing dynamics, but rather imposing its kinematics, leads to the next limitation of our model: it does not account for the complex phenomena of joints compliance, and ultimately for closed-loop control and adaptation in a broader perspective. Real articulations could in fact significantly react to external loads [93, 94], influencing therefore the entire dynamics of the wing, and consequently the production of the aerodynamic forces and moments. To include this effect, the current system of Equation (3.1) necessarily needs to be modified accounting for the dynamics of the wing. A proposed approach based on multibody dynamics will be described in Chapter 7.

3.5 Conclusion

The work depicted in this Chapter highlighted the different elements that interplay in bird flight dynamics.

We first introduced the equations of motion on the longitudinal plane. Our bird is modeled as a rigid body with poly-articulated wings. The inertial effect of the wings on the main body is neglected, given the small wing mass compared with the body mass of migratory birds [81]. Under these assumptions, the resulting system of Equation (3.1) is nonlinear and time dependent. Flapping regimes, in fact, introduce a forcing term due to the wing motion in the form of aerodynamic forces and moments. These forcing terms, are the substantial difference between flapping wing dynamics and fixed wing dynamics.

The wing model allows the morphing during the flapping cycle and thus permits to reproduce realistic flight gaits. This biomechanics is made possible through the articulations at the level of shoulder, elbow and wrist, and actuated by a prescribed kinematics governed by sinusoidal functions. This model can further be exploited for investigating the impact that each degree of freedom has on the problem.

The aerodynamic forces during the flapping cycle are calculated using a quasi-steady lifting line model. This lifting line is extracted at each timestep according to the position of the wing. The wake is modeled as semiinfinite vortex tubes shed behind the bird and parallel to the stream flow velocity, as pictured in Figure 3.5. The main advantage of using a quasisteady model to study the flight dynamics is for its computational speed, while preserving good accuracy of forces estimation.

We concluded the Chapter showing the aerodynamic forces and moment in two cases of bird flight. The first one reproduced a wind tunnel scenario, where the bird body does not move, and the wings are flapping according to a prescribed kinematics. The second case solves Equation (3.1) via the coupling of the aerodynamics with the body dynamics. The evolution of this system, starting from a generic initial condition, is non necessarily periodic. This makes flapping flight an unsteady phenomenon. In order to evaluate the stability of the flight dynamics, we first need to define a steady state. Given the forcing aerodynamic terms in the equations of motion, the only steady-state toward which flapping flight can evolve is the asymptotic limit cycle, i.e. the flight regime where all the state variables are simultaneously periodic. In the next Chapter we present the tool we implemented in order to find steady state flapping flight regimes, namely the multiple-shooting algorithm.
 Table 3.1:
 Morphological and kinematics parameters for the computation of bird flight.

Bird body					
Mass (m_b) , $[kg]$	1.2				
Moment of Inertia (I_y) , $[kg \cdot m^2]$	0.1				
Bird wing					
Wingspan (b) , $[m]$	1.35				
Mean aerodynamic chord (\overline{c}) , $[m]$	0.15				
Arm bone length (l_a) , $[m]$	0.134				
Forearm bone length (l_f) , $[m]$	0.162				
Hand bone length (l_h) , $[m]$	0.084				
Bird feathers					
Primary feather 1 (l_{k1}) , $[m]$	0.25				
Primary feather 2 (l_{k2}) , $[m]$	0.275				
Primary feather 3 (l_{k3}) , $[m]$	0.25				
Secondary feather 1 (l_{k4}) , $[m]$	0.225				
Secondary feather 2 (l_{k5}) , $[m]$	0.2				
Secondary feather 3 (l_{k6}) , $[m]$	0.175				
Secondary feather 4 (l_{k7}) , $[m]$	0.15				
Bird tail					
Tail opening (β) , [deg]	0				
Tail chord $(c_t), [m]$	0.25				

(a) Morphological parameters of bird body, wing and feathers.

(b) Joint parameters to describe the wing kinematics.							
	Joint	$q_0[deg]$	A[deg]	ϕ [deg]			

Joint	$q_0[deg]$	A[deg]	$\phi[deg]$
Shoulder y	11.5	0.8	-90
Shoulder <i>x</i>	0	42	180
Shoulder z	19	20	90
Elbow z	30	30	-90
Wrist y	0	30	-90
Wrist z	-30	30	90

Chapter 4

Detection of limit cycles and multiflap

4.1 Introduction

In Chapter 3, we stated the two necessary steps for studying flapping flight stability, namely detecting the limit cycle corresponding to a prescribed kinematics, and then assessing the stability of such limit cycle.

This Chapter thus aims at describing the mathematical theory and the numerical model used in multiflap, the Python package developed in the framework of this thesis, in order to detect limit cycles and assess their stability according to Floquet theory.

We start by defining dynamical systems and some fundamental notions, such as *phase space* and *trajectory*. Then a characterization of dynamical systems as a function of their long term behavior is provided, with a particular focus on the definition of limit cycles, and by explaining the stable or unstable behavior they may display.

We then explain what drives the stability (or instability) of such limit cycles. In Section 4.3 is reported the mathematical formalism of the *lilnear stability analysis* of periodic orbits, where it is explained the importance of the Jacobian (or Monodromy) matrix. According to Floquet theory, the eigenvalues of this Jacobian matrix — the so-called *Floquet multipliers* — are invariant along the orbit, and characterize its local stability properties [74, 95, 96].

In Section 4.4 is described the multiple-shooting algorithm. This is the

numerical scheme that allows to detect limit cycles (independently from their stability), and automatically compute the Floquet multipliers of the orbit. Two versions of this algorithm are illustrated, a first one when the period of the orbit is unknown, and a second one when the period of the orbit is known a priori.

Then, we describe both methods implemented in our code in order to calculate this Jacobian matrix, namely an analytical method, and a numerical method based on finite differentiation of the trajectory.

The terminology that follows is a re-adaptation from Cvitanović et al. [74], the reference textbook used to develop this part of work.

4.2 Background on dynamical systems

Dynamical systems describe the evolution of an event with time. Considering a bird flight, if we were able to know at every time the position of the center of mass and its velocity, we would have a complete description of its motion. The variables describing positions and velocities are called *state variables* and are indicated with the state vector $\mathbf{x} = (x_1, x_2, ..., x_n)$. All their possible values form the so-called *phase space*, which is here indicated with \mathcal{M} [74]. At a given time, the state of a dynamical system can be fully represented by a single point in the state space \mathcal{M} of dimension n.

Without losing generalities, let us consider a three dimensional dynamical system $\mathbf{x} = (x_1, x_2, x_3)$, forming a phase space $\mathcal{M} \in \mathbb{R}^3$ as pictured in Figure 4.1. Let us assume this system has a definite law that captures how the points move in the phase space. A generic point $\mathbf{x}_0(t_0)$ subject to this law is mapped after a time τ to $\mathbf{x}_0(t_0 + \tau)$. The continuous path described during the evolution, is called a *trajectory*.

In general, instead of considering points in the phase space, we can track the evolution of entire regions. Considering $\mathcal{M}_i \in \mathcal{M}$, all the infinite points belonging to this subset can be mapped into $\mathcal{M}_f \in \mathcal{M}$ as indicated by the dashed lines in Figure 4.1. The derivative with respect to time of all the possible trajectories describe a so-called *velocity field* in \mathcal{M} .

Dynamical systems can often be expressed by this velocity field, by expressing evolution laws in the form of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t) \tag{4.1}$$

If the velocity field \mathbf{v} does not depends explicitly on time, the system is said to be *autonomous* or *self-sustained*, conversely it is *non autonomous* or



Figure 4.1: Schematic of a phase space $\mathcal{M} \in \mathbb{R}^3$. The evolution of $\mathbf{x}_0(t_0)$ into $\mathbf{x}(t_0 + \tau)$ traces out a solid black trajectory in the phase space. The derivative of the trajectory with respect to time, identifies a velocity vector (red). This schematic also reports the evolution of an entire region $\mathcal{M}_i \in \mathcal{M}$ into $\mathcal{M}_f \in \mathcal{M}$.

forced [74, 97].

Characterization of dynamical systems

Considering the dynamical system of Equation (4.1), we define the mapping operator

$$f(\mathbf{x}_0)\Big|_{t_0}^{t_0+\tau} = \int_{t_0}^{t_0+\tau} \mathbf{v}(\mathbf{x}, t) dt + \mathbf{x}_0$$
(4.2)

and using this notation, we express the evolution of a point in the phase space in a compact form

$$\mathbf{x}(t_0 + \tau) = f(\mathbf{x}_0) \Big|_{t_0}^{t_0 + \tau}.$$
(4.3)

Three main types of trajectories can be identified

- Stationary (or fixed points): $\mathbf{x}_0(t_0) = f(\mathbf{x}_0)\Big|_{t_0}^{t_0+\tau} \forall \tau > 0$
- Periodic: $\mathbf{x}_0(t_0) = f(\mathbf{x}_0)\Big|_{t_0}^{t_0+T_p}$ for a given period $T_p > 0$

• Aperiodic:
$$f(\mathbf{x}_0)\Big|_{t_0}^{t_0+\tau} \neq f(\mathbf{x}_0)\Big|_{t_0}^{t_0+\tau'} \forall \tau \neq \tau'$$

Among these three families of solutions, we are interested in periodic trajectories (or periodic orbits). An important subset of periodic orbits are *limit cycles*. We refer the definition of the book by Strogatz [95]:

```
Limit cycle (adapted from [95])
```

A limit cycle is an **isolated** closed trajectory in the phase space. Isolated means that neighboring trajectories are not closed.

These particular solutions exclusively emerge from nonlinear phenomena; linear systems cannot exhibit such behavior [95].

4.3 Stability of limit cycles

Limit cycles can be stable or unstable. If all neighboring perturbed trajectories tend to be dynamically attracted by the limit cycle, we say it is *stable*. Otherwise, the limit cycle is *unstable* (Figure 4.2). Exceptional cases of *half stable* manifolds are not discussed in this manuscript, and we refer to the book from Strogatz for a thorough explanation [95].



Figure 4.2: Stability behavior of limit cycles. **(a)**: Qualitative example of a stable limit cycle. Perturbed solutions are dynamically attracted by the orbit (green trajectories). **(b)**: Qualitative example of an unstable limit cycle. Perturbed solutions are repelled by the orbit, spiraling away from it (red trajectories).

Stable limit cycles are very present in nature: examples of self-sustained oscillations happen in heart beating, circadian rhythms, tidal rhythms,





etc. [95, 98, 99]. An important characteristic is their capability of restoring oscillatory conditions after perturbations. An example of stable limit cycles is here reported for the Poincaré oscillator [100, 101, 102]. Poincaré oscillator is a phenomenological model that can describe the dynamics of any sinusoidal rhythmic system via only two state space variables, namely the radius *r* and the phase θ as pictured in Figure 4.3. The radius, and the coordinates in the phase space (*x*, *y*) are expressed in arbitrary units (a.u.), the azimuthal coordinate is expressed in radians, and the temporal evolution is tracked in seconds. The equations in polar coordinates read

$$\dot{r} = \lambda r (A - r)$$

$$\dot{\theta} = \omega$$
(4.4)

where λ , A, and ω , are three parameters that regulate the relaxation rate, the oscillation amplitude, and the oscillation period respectively. Setting the parameters A = 1[a.u.], $\lambda = 0.01s^{-1}$ and $\omega = 2\pi/24[rad/s]$, we can integrate the system starting from an arbitrary initial value $(r, \theta) = [2, 0]$. Having looked at Figure 4.3, and considering the change of coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{4.5}$$

the evolution of the dynamical system is pictured in Figure 4.4. The green trajectory shown in Figure 4.4(a), describes the evolution of x_0 and is eventually attracted by the black limit cycle. The duration of the transient is about 4 oscillations, and visible in Figure 4.4(b).



Figure 4.4: Poincaré oscillator. (a): Phase space. The green trajectory starts out of the limit cycle, and evolves towards it guided by the velocity field (gray arrows). (b): Time series of the x variable. After an initial transient, the variable settles on stable oscillations.

As revealed by this example, stable limit cycles can be directly found by integrating the equations of motion of the dynamical system from an initial condition located close to it. The trajectory will naturally be attracted by the stable orbit after a certain transient. The duration of this transient in the particular case just analyzed is regulated by the relaxation rate λ . On the contrary, this direct numerical integration approach does not work if the nature of the limit cycle is unstable, since the trajectory will be constantly repelled away by the orbit. In these situations, a different tailored approach is thus required.

4.3.1 The Jacobian matrix

The example of Figure 4.4 shows that the initial value, after about 4 oscillations settles on the limit cycle. We are now interested in quantifying this transient time. More generally, we would like to quantify information about the stability (or instability) properties of limit cycles.

Having looked at Figure 4.5, let $\mathbf{x}^*(t_0)$ be a point belonging to the limit cycle at time t_0 , and $\mathbf{x}^*(t_0) + \delta \mathbf{x}(t_0)$ be its neighbor, where $\delta \mathbf{x}(t_0)$ thus captures the initial perturbation. After a time equal to the period *T*, this perturbed point is mapped to

$$\mathbf{x}^{*}(t_{0}) + \delta \mathbf{x}(t_{0} + T) = f(\mathbf{x}^{*}(t_{0}) + \delta \mathbf{x}(t_{0}))\Big|_{t_{0}}^{t_{0} + T}.$$
(4.6)



Figure 4.5: Illustration of a periodic orbit (red) and a first integration over one period of the neighboring trajectory (blue). The point belonging to the limit cycle is identified with $\mathbf{x}^*(t_0)$, and its tangent to the trajectory is identified with $\mathbf{v}(\mathbf{x}^*(t_0))$. The perturbation vector $\delta \mathbf{x}(t_0)$ identifies the initial condition of the neighboring trajectory. After one-period integration, the mapped point is identified by the evolution of the perturbed vector $\delta \mathbf{x}(t_0 + T)$.

By expanding the right hand side of Equation (4.6) to the first order, and expressing the terms for the generic *i*-th coordinate of the state space

$$x_{i}^{*}(t_{0}) + \delta x_{i}(t_{0} + T) = f_{i}(\mathbf{x}^{*}(t_{0}))\Big|_{t_{0}}^{t_{0}+T} + \sum_{j=1}^{N} \frac{\partial f_{i}(\mathbf{x}^{*}(t_{0}))\Big|_{t_{0}}^{t_{0}+T}}{\partial x_{j}} \delta x_{j}(t_{0}) \quad (4.7)$$

and considering the limit cycle condition $x_i^*(t_0) = f_i(\mathbf{x}^*(t_0))\Big|_t^{t_0+T}$, Equation (4.7) reduces to

$$\delta x_i(t_0+T) = \sum_{j=1}^N \frac{\partial f_i(\mathbf{x}^*(t_0))\Big|_{t_0}^{t_0+T}}{\partial x_j} \delta x_j(t_0).$$
(4.8)

61

Equation (4.8) describes the dynamic of the perturbed neighboring state, and the matrix

$$\mathbb{J}_{i,j}(\mathbf{x}^*)\Big|_{t_0}^{t_0+T} = \frac{\partial f_i(\mathbf{x}^*(t_0))\Big|_{t_0}^{t_0+T}}{\partial x_i}$$

is called the Jacobian matrix or Monodromy matrix. Stability of limit cycles is governed by Floquet theory [96], and assessed by the eigenvalues Λ_i of the Jacobian matrix that quantifies how much a small perturbation out of the limit cycle is deformed by the flow, after a period *T* [74, 95, 103]. These eigenvalues are also called *Floquet multipliers*.

Calling \mathbf{x}_0 a generic initial condition, this Jacobian matrix is thus the result of the integration of the following system up to time t = T [74, 103]

$$\frac{d\mathbf{J}}{dt}(\mathbf{x}_0)\Big|_{t_0}^t = \mathbf{A}(\mathbf{x}, t)\mathbf{J}(\mathbf{x}_0)\Big|_{t_0}^t$$

$$\mathbf{J}(\mathbf{x}_0)\Big|_{t_0}^{t_0} = \mathbf{I}$$
(4.9)

where the matrix

$$\mathbb{A}(\mathbf{x},t) = \nabla \mathbf{v}(\mathbf{x},t)|_{\mathbf{x}=\mathbf{x}^*}$$
(4.10)

is called the stability matrix [74] and is *T*-periodic on the limit cycle.

In our particular case, the matrix A(t) accordingly to Equation (3.1) is $A(\mathbf{x}(t), t) = \begin{pmatrix} \frac{1}{m} \frac{\partial F_{x'}}{\partial u} & \frac{1}{m} \frac{\partial F_{x'}}{\partial w} - q & -w + \frac{1}{m} \frac{\partial F_{x'}}{\partial q} & -g \cos \theta + \frac{1}{m} \frac{\partial F_{x'}}{\partial \theta} \\ q + \frac{1}{m} \frac{\partial F_{z'}}{\partial u} & \frac{1}{m} \frac{\partial F_{z'}}{\partial w} & u + \frac{1}{m} \frac{\partial F_{z'}}{\partial q} & g \sin \theta + \frac{1}{m} \frac{\partial F_{z'}}{\partial \theta} \\ \frac{1}{l_{yy}} \frac{\partial M_{y'}}{\partial u} & \frac{1}{l_{yy}} \frac{\partial M_{y'}}{\partial w} & \frac{1}{l_{yy}} \frac{\partial M_{y'}}{\partial q} & \frac{1}{l_{yy}} \frac{\partial M_{y'}}{\partial \theta} \\ 0 & 0 & 1 & 0 \end{pmatrix}$ (4.11)

Floquet multipliers have the property to be independent of the choice of x^* on the limit cycle, while the Jacobian matrix and its eigenvectors depend on it [104].

If the system was autonomous, i.e. of the form $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$, one of the Floquet multipliers would systematically be equal to one and its eigenvector would be tangent to the limit cycle at \mathbf{x}^* . The linear transformation of Equation (4.8) also holds for the vector tangent to the limit cycle, i.e. the
velocity vector $\mathbf{v}(\mathbf{x}^*)$

$$\mathbf{v}(f(\mathbf{x}^*)\big|_{t_0}^{t_0+T}) = \mathbb{J}(\mathbf{x}^*)\big|_{t_0}^{t_0+T} \mathbf{v}(\mathbf{x}^*)$$
(4.12)

where for the limit cycle condition the vector field satisfies

$$\mathbf{v}(f(\mathbf{x}^*)\big|_{t_0}^{t_0+T}) = \mathbf{v}(\mathbf{x}^*).$$
(4.13)

Since the system is autonomous, i.e. the velocity field is independent from time, Equation (4.12) is true for all the \mathbf{x}^* belonging to the limit cycle, at any initial time t_0 . Excluding the trivial solution of Equation (4.12) corresponding to stationary points ($\mathbf{v}(\mathbf{x}) = 0$), the vector $\mathbf{v}(\mathbf{x}^*)$ is necessarily eigenvector for the Jacobian matrix $\mathbb{J}(\mathbf{x}^*)\Big|_{t_0}^{t_0+T}$, with the associated unitary eigenvalue. In the literature, this eigenvalue is often called the *marginal* multiplier.

Theorem 1 (Local asymptotical stability of limit cycles of autonomous systems). Let us consider a nonlinear autonomous system which admits at least one limit cycle solution, and let $\mathbf{J}(\mathbf{x}^*)\Big|_{t_0}^{t_0+T}$ be the Jacobian matrix along this limit cycle. Let $\Lambda_1 = 1$ denote the marginal Floquet multiplier corresponding to the eigendirection $\mathbf{v}(\mathbf{x})$. Let Λ_i with i = 2...n, be the other (n-1) Floquet multipliers of the Jacobian matrix. If $|\Lambda_i| < 1$ for i = 2...n, then this limit cycle solution is said to be asymptotically stable. It follows as a corollary, that if at least one Floquet multiplier $|\Lambda_i| > 1$ then the system is unstable.

We refer to the book from Leine [105] for a complete proof. An example of these two scenarios is pictured in Figure 4.6.

Theorem 2 (Local asymptotical stability of limit cycles of non-autono mous systems). Let us consider a nonlinear non-autonomous system which admits at least one limit cycle solution, and let $\mathbb{J}(\mathbf{x}^*)\Big|_{t_0}^{t_0+T}$ be the Jacobian matrix associated this limit cycle. Let Λ_i with $i = 1 \dots n$, be the Floquet multipliers of the Jacobian matrix. If $|\Lambda_i| < 1$ for $i = 1 \dots n$, then the periodic solution is said to be asymptotically stable. It follows as a corollary, that if at least one Floquet multiplier $|\Lambda_i| > 1$ then the system is unstable.

We refer to the book from Leine [105] for a complete proof.



Figure 4.6: Example of the locus of the Floquet multipliers for an autonomous system. (a): Stable situation. All the multipliers are strictly smaller than one in absolute value (black scatter points) except the marginal one (blue scatter point). This marginal multiplier is a feature of autonomous systems only. (b): Unstable situation. One Floquet multiplier is larger than one in absolute value (red scatter point), driving the system to be unstable along the corresponding eigendirection.

4.3.2 A graphical example

The Jacobian matrix maps infinitesimal perturbations embedded within a sphere around a specific point of the limit cycle at a given time (\mathbf{x}^*, t), to a stretched ellipsoid after a time t + T [74]. This stretching ratio is governed by the Floquet multipliers, and the stretching directions by the eigenvectors.

As a graphical example, let us consider the limit cycle of Figure 4.7, and assume that we know its period *T*, and at least one point belonging to it, i.e. \mathbf{x}^* . Let $\mathcal{M}_s \in \mathcal{M}$ be a spherical region around \mathbf{x}^* as pictured in Figure 4.7(b).

We now distribute a set of randomized points $\mathbf{x}_{s,i}$ on the surface of this infinitesimal sphere (green scatter points), and $\forall \mathbf{x}_{s,i} \in \mathcal{M}_s$ we compute their evolution over one cycle period $f(\mathbf{x}_{s,i})\Big|_{t_0}^{t_0+T}$. The result is pictured in Figure 4.8. The evolution of these points describes an ellipsoid. The principal axes of this ellipsoid are the eigenvector of the Jacobian matrix. The ratio between these principal axes and the canonical base of the sphere are the eigenvalues of this linear transformation, i.e. the Floquet multipliers.

This framework provides another important feature, namely the stretching/contracting rate per unit of time, or Floquet exponent, λ_i . Given the



Figure 4.7: Example of a limit cycle in three dimensions. (a): Shows the limit cycle with a point belonging to it (red scatter point). (b): Pictures a zoom-in highlighting the neighboring spherical region \mathcal{M}_s , and the set of randomized points distributed on it (green scatter points).



Figure 4.8: Map of the the neighboring points $\mathbf{x}_{i,s}$ after one period time integration. The spherical region is mapped into an ellipsoid whose principal axes are the Floquet eigenvectors of the matrix $\mathbf{J}(\mathbf{x}^*)\Big|_{t_0}^{t_0+T}$. The ratio between the principal axes of the ellipsoid, $(\hat{e'}_i, \hat{e'}_j)$, and the canonical base of the sphere, (\hat{i}, \hat{j}) , are the eigenvalues of this linear transformation, i.e. the Floquet multipliers.

Floquet multiplier Λ_i , it is defined as [74]

$$\lambda_i = \frac{1}{T} |\Lambda_i|. \tag{4.14}$$

Knowing these Floquet exponents, allows to compute important characteristic times of the dynamics of perturbations, such as the *doubling time* t_{doub} for unstable systems, i.e. the time it takes a perturbation to double its value with respect to the limit cycle, or the *half time* t_{half} for stable systems, i.e. the time it takes for a perturbation to halve its value with respect to the limit cycle.

$$t_{doub} = \frac{\ln(2)}{\lambda_i}$$

$$t_{half} = \frac{\ln(0.5)}{\lambda_i}$$
(4.15)

where in the first equation *i* corresponds to the expanding multipliers, while in the second equation it corresponds to the contracting multipliers.

4.4 The multiple-shooting algorithm

As previously discussed in Section 4.3, detecting unstable limit cycle requires a dedicated framework, since direct numerical integration cannot work. We thus illustrate a generalization for limit cycle detection, based on a so-called multiple-shooting algorithm. This multiple-shooting algorithm, simultaneously detects periodic orbits, and computes the corresponding Floquet multiplier, in order to characterize them based on stability properties.

It was first proposed by Keller [106], and successively modified by Lust [104], and we present two dedicated solvers, depending if the cycle period is known or if unknown. Both schemes are illustrated in the next Sections.

4.4.1 Multiple-shooting scheme for unknown period of the orbit

The scheme here presented, is suitable for all the cases of limit cycle problems where the period is **unknown** a-priori. This is mostly useful for autonomous system, in presence of self-sustained oscillations that are not driven by any external periodic terms.



Figure 4.9: Generic computational scheme describing the multiple-shooting method of a phase-space $\mathcal{M} \in \mathbb{R}^3$ (without loss of generalities). The black trajectory is traced out by integrating the equations from the guessed points (blue dots). The asymptotic limit cycle is represented in red, and the points belonging to it in green.

The multiple-shooting algorithm splits the limit cycle in *M* discrete points, as shown in Figure 4.9, for an example where n = 3 and M = 7. We indicate the unknown points belonging to the limit cycles as \mathbf{x}_i^* , and the guess points as \mathbf{x}_i .

The point \mathbf{x}_i^* is mapped to the point \mathbf{x}_{i+1}^* by

$$\mathbf{x}_{i+1}^{*} = f(\mathbf{x}_{i}^{*})\big|_{t_{i}}^{t_{i}+\tau} = f(\mathbf{x}_{i} + \Delta \mathbf{x}_{i})\big|_{t_{i}}^{t_{i}+\tau}$$
(4.16)

where $\tau = T_p / M - 1$ is the partition of the unknown limit cycle period T_p .

Computing the Taylor first order expansion of the right hand side of Equation (4.16), the point \mathbf{x}_{i+1}^* can be expressed as function of the guessed points only

$$\mathbf{x}_{i+1} + \Delta \mathbf{x}_{i+1} = f(\mathbf{x}_i) \Big|_{t_i}^{t_i + \tau} + \mathbb{J}(\mathbf{x}_i) \Big|_{t_i}^{t_i + \tau} \cdot \Delta \mathbf{x}_i$$
(4.17)

where $\mathbb{J}|_{t_i}^{t_i+\tau}(\mathbf{x}_i)$ is the Jacobian matrix defined in Equation (4.9). Since τ in this case is unknown, we introduce the guessed period in Equation (4.17),

67

via the relationship

$$f(\mathbf{x}_i)\big|_{t_i}^{t_i+\tau} = f(\mathbf{x}_i)\big|_{t_i}^{t_i+\overline{\tau}} + \mathbf{v}\big(f(\mathbf{x}_i)\big|_{t_i}^{t_i+\overline{\tau}}\big)\Delta\tau$$
(4.18)

where $\overline{\tau} = T_g/M - 1$ with T_g being the guessed period of the orbit and $\Delta \tau$ a new unknown of the problem. Furthermore we approximate

$$\mathbf{J}|_{t_i}^{t_i+\tau}(\mathbf{x}_i) \approx \mathbf{J}|_{t_i}^{t_i+\overline{\tau}}(\mathbf{x}_i)$$
(4.19)

Plugging Equations (4.18) and (4.19) into Equation (4.17), and re-arranging the terms, we obtain the following relationship between two consecutive points

$$\mathbb{J}(\mathbf{x}_{i})\Big|_{t_{i}}^{t_{i}+\overline{\tau}}\Delta\mathbf{x}_{i}-\Delta\mathbf{x}_{i+1}+\mathbf{v}\big(f(\mathbf{x}_{i})\big|_{t_{i}}^{t_{i}+\overline{\tau}}\big)\Delta\tau=\underbrace{-\big(f(\mathbf{x}_{i})\big|_{t_{i}}^{t_{i}+\overline{\tau}}-\mathbf{x}_{i+1}\big)}_{Error}$$
(4.20)

The multiple-shooting scheme for unknown period therefore can be put in the form

$$\underbrace{\begin{pmatrix} \mathbb{J}(\mathbf{x}_{0}) \Big|_{0}^{\overline{\tau}} & -\mathbb{I} & 0 & \dots & 0 & \mathbf{v}(f(\mathbf{x}_{0}) \Big|_{0}^{\overline{\tau}} \\ 0 & \mathbb{J}(\mathbf{x}_{1}) \Big|_{t_{1}}^{t_{1}+\overline{\tau}} & -\mathbb{I} & \dots & 0 & \mathbf{v}(f(\mathbf{x}_{1}) \Big|_{t_{1}}^{t_{1}+\overline{\tau}} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbb{J}(\mathbf{x}_{m-1}) \Big|_{t_{m-1}}^{\overline{T}} & -\mathbb{I} & \mathbf{v}(f(\mathbf{x}_{m-1}) \Big|_{t_{m-1}}^{\overline{T}} \\ -\mathbb{I} & 0 & \dots & 0 & \mathbb{I} & 0 \end{pmatrix} \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}} \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}}}_{\Delta \mathbf{x}_{n} \times M+1} = \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}} \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1} = \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}} \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1} = \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}} \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1} = \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1} = \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1} = \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1} = \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1} = \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1} = \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1} = \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1} = \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1} = \underbrace{\mathsf{M}_{[n \times M, n \times M+1]}^{\overline{T}} = \mathsf{M}_{\mathbf{x}_{n} \times M+1}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1}^{\overline{T}}}_{\mathbf{x}_{n} \times M+1}^{\overline{T}}_{\mathbf{x}_{n} \times M+1}^{\overline{T}}_{\mathbf{x}$$

Calling *n* the number of states of the dynamical system and *M* the amount of points employed in the multiple-shooting, **M** is the multiple-shooting matrix of dimension $[n \times M, n \times M + 1]$ and is non-square, $\Delta \mathbf{x}$ the unknown vector of dimension $[n \times M + 1]$ and **E** the error vector of dimension $[n \times M]$.

4.4.2 Multiple-shooting scheme for known period of the orbit

The multiple-shooting scheme can be re-adapted in the case the period of the orbit is known a priori. This can be the case when the system is forced with a periodic term, i.e. in flapping flight where the forcing terms determine the wingbeat period.

Looking to Figure 4.9, and analogously to the previous case, we start the derivation of this scheme by expressing the evolution of \mathbf{x}_i^* to the point \mathbf{x}_{i+1}^* by

$$\mathbf{x}_{i+1}^{*} = f(\mathbf{x}_{i}^{*})\big|_{t_{i}}^{t_{i}+\tau} = f(\mathbf{x}_{i} + \Delta \mathbf{x}_{i})\big|_{t_{i}}^{t_{i}+\tau}$$
(4.22)

where in this particular case $\tau = T_p/M - 1$ is known. Computing the Taylor first order expansion of the right hand side of Equation (4.22), the point \mathbf{x}_{i+1}^* can be expressed as function of the guessed points only

$$\mathbf{x}_{i+1} + \Delta \mathbf{x}_{i+1} = f(\mathbf{x}_i) \Big|_{t_i}^{t_i + \tau} + \mathbb{J}(\mathbf{x}_i) \Big|_{t_i}^{t_i + \tau} \cdot \Delta \mathbf{x}_i$$
(4.23)

where $\mathbb{J}|_{t_i}^{t_i+\tau}(\mathbf{x}_i)$ is the Jacobian matrix defined in Equation (4.9). Re-arranging Equation (4.23) as

$$\mathbb{J}(\mathbf{x}_{i})\Big|_{t_{i}}^{t_{i}+\tau} \cdot \Delta \mathbf{x}_{i} - \Delta \mathbf{x}_{i+1} = \underbrace{-\left(f(\mathbf{x}_{i})\Big|_{t_{i}}^{t_{i}+\tau} - \mathbf{x}_{i+1}\right)}_{Error}$$
(4.24)

the multiple-shooting scheme can be derived as

$$\underbrace{\begin{pmatrix} \mathbb{J}(\mathbf{x}_{0}) \Big|_{0}^{\tau} & -\mathbb{I} & 0 & \dots & 0 \\ 0 & \mathbb{J}(\mathbf{x}_{1}) \Big|_{t_{1}}^{t_{1}+\tau} & -\mathbb{I} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbb{J}(\mathbf{x}_{m-1}) \Big|_{t_{m-1}}^{T} & -\mathbb{I} \\ -\mathbb{I} & 0 & \dots & 0 & \mathbb{I} \end{pmatrix}}_{\mathbf{M} [n \times M, n \times M]} \underbrace{\begin{pmatrix} \Delta \mathbf{x}_{0} \\ \Delta \mathbf{x}_{1} \\ \vdots \\ \vdots \\ \vdots \\ \Delta \mathbf{x}_{m} \\ \mathbf{x}$$

Calling *n* the number of states of the dynamical system and *M* the amount of points employed in the multiple-shooting, **M** is the multiple-shooting matrix of dimension $[n \times M, n \times M]$, $\Delta \mathbf{x}$ the unknown vector of dimension $[n \times M]$ and **E** the error vector of dimension $[n \times M]$.

4.4.3 Solving the multiple-shooting scheme

We now want to solve the multiple-shooting scheme for a limit cycle solution. Equations (4.21) and (4.25), can both be re-expressed in a compact form

$$\mathbf{M}(\mathbf{x}_i) \Delta \mathbf{x} = \mathbf{E}(\mathbf{x}_i) \tag{4.26}$$

finding the solution of Equation (4.26), consists in finding $\mathbf{x}_i^* \in \mathbb{R}^n$ such that $\mathbf{E}(\mathbf{x}_i^*) = \mathbf{0}$ and this can be solved with an iterative scheme.

We implemented in our code a modified Newton's method, namely the Levenberg–Marquardt algorithm [107] (LMA). This choice is made for two reasons. First LMA is a solver that can deal with systems in the form of Equation (4.26) in both cases, where the multiple-shooting matrix **M** is square or non-square. Secondly, the main drawback of classic Newton's method is the high sensitivity on the choice of the initial guess $\mathbf{x}_i^{(0)}$. It can be mathematically proved that Newton's method quadratically converges only if the choice of the initial conditions is sufficiently close to the solution (for a proof of this Theorem, please refer to [108], chap. 7), while LMA shows more robustness with respect to guessed values. Such an implementation of LMA in a multi-shooting code was already adopted by Dednam and Botha [109] and the code was validated with both autonomous and non-autonomous systems.

Leveraging on LMA, the estimation of a new unknown vector δx that solves for Equation (4.27), and used to update the state variables at the generic iteration step *k*, is computed as follows

$$\left[\mathbf{M}^{T}\mathbf{M} + \lambda \operatorname{diag}(\mathbf{M}^{T}\mathbf{M})\right]\delta\mathbf{x} = \mathbf{M}^{T}\mathbf{E}$$
(4.27)

where λ is a non-negative, adaptive damping parameter. Introducing the relaxation factor ν for λ , a candidate algorithm is presented in Algorithm 1. When the trajectory eventually converges to the limit cycle, the Jacobian matrix of the whole limit cycle obeys the semigroup property and can be expressed as the product of the submatrices of Equation (4.25), i.e.

$$\mathbb{J}(\mathbf{x}_0)\Big|_0^T = \mathbb{J}(\mathbf{x}_{m-1})\Big|_{t_{m-1}}^T \cdots \mathbb{J}(\mathbf{x}_1)\Big|_{t_1}^{t_1+\tau} \cdot \mathbb{J}(\mathbf{x}_0)\Big|_0^\tau$$
(4.28)

It is important to point out that also stationary trajectories, i.e. fixed points of equilibrium, identically set to zero the right hand side of Equations (4.26) and (4.27). This implies that if the guessed points are too closed

```
Algorithm 1 Levenberg–Marquardt
```

```
1: \mathbf{x}_i^{(0)} \leftarrow \texttt{guessed_points}
  2: k \leftarrow 0
  3: \epsilon \leftarrow \texttt{tolerance}
  4: function LMA(\mathbf{x}_{i}^{(0)}, \epsilon)
                 \lambda \leftarrow set
\mathbf{M}(\mathbf{x}_{i}^{(o)}), \mathbf{E}(\mathbf{x}_{i}^{(o)}) \leftarrow Compute
  5.
  6:
                 while |\mathbf{E}(\mathbf{x}_i^{(k)})| > \epsilon do
  7:
                           \mathbf{M}(\mathbf{x}_{i}^{(k)}), \mathbf{E}(\mathbf{x}_{i}^{(k)}) \leftarrow Compute
  8:
                           [\mathbf{M}^T\mathbf{M} + \lambda \operatorname{diag}(\mathbf{M}^T\mathbf{M})] \,\delta \mathbf{x} = \mathbf{M}^T\mathbf{E} \leftarrow Solve
  g.
                          \mathbf{x}_{i}^{(k+1)} \leftarrow \mathbf{x}_{i}^{(k)} + \delta \mathbf{x}_{i}^{(k)}
10:
                          \mathbf{E}(\mathbf{x}_{i}^{(k+1)}) \leftarrow Compute
11:
                          if \min |\mathbf{E}(\mathbf{x}_{i}^{(k+1)})| < \min |\mathbf{E}(\mathbf{x}_{i}^{(k)})| then
12:
                                    \lambda = \lambda / \nu
13:
14:
                          else
                 \lambda = \lambda * \nu
return \mathbf{x}_i^{(k+1)}
15:
16:
```

to fixed points of equilibrium, the iterative scheme could be attracted by this latter solution, which does not represent a limit cycle condition, while satisfying the convergence criteria.

4.5 Computation of the Jacobian matrix

Two concurrent approaches can be used to evaluate the Jacobian matrix and build the diagonal blocks of the multiple-shooting matrix \mathbf{M} in Equations (4.21) and (4.25): the first one relies on an analytical approach, while the second one relies on numerical computations only. Both methods are implemented in our code.

4.5.1 Analytical computation of the Jacobian matrix

The analytical approach is the one we used in the following simulations, and it is here described in detail. The Jacobian matrix is obtained by solving the variational Equation (4.9). However, to compute the matrix $\mathbb{A}(\mathbf{x}, t)$, it is necessary to know the state of the system \mathbf{x} at every time t. Solving

Equation (4.9), thus also require to integrate the state Equation (4.1), i.e.

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{J}} \end{pmatrix} = \begin{pmatrix} \mathbf{v}(\mathbf{x}, t) \\ \mathbb{A}(\mathbf{x}, t) \ \mathbb{J} \end{pmatrix}$$
(4.29)

with the initial condition

$$\begin{pmatrix} \mathbf{x}(t_0) \\ \mathbb{J}^0 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbb{I} \end{pmatrix}$$
(4.30)

This approach simultaneously solves $(n + n^2)$ ordinary differential equations [103]. The solution of this system corresponds to the Jacobian matrix of a generic trajectory at time t_f , obtained from an initial condition at time t_0 .

4.5.2 Numerical computation of the Jacobian matrix

The second method that can be used to compute the Jacobian matrix J rellies on numerical differentiation. In this case, the component $J_{i,j}$ of the Jacobian matrix is evaluated by a finite difference of the perturbed trajectory along each state variable, i.e.

$$\mathbb{J}_{i,j}(\mathbf{x}_0)\Big|_t^{t+T} = \frac{f_i(\mathbf{x}_0 + \varepsilon \hat{\mathbf{e}}_j)\Big|_t^{t+T} - f_i(\mathbf{x}_0)\Big|_t^{t+T}}{\varepsilon}$$
(4.31)

where ϵ is the absolute value of the perturbation.

4.6 multiflap package

The multiple-shooting algorithm just described, has been developed in the context of this Thesis, and released in a Python package called multiflap. It is a toolbox that handles multiple-shooting schemes in the form of Equation (4.21) and (4.25), and solves them leveraging the Levenberg–Marquardt algorithm. In multiflap both analytical and numerical models of Jacobian computation are implemented. This package has the flexibility to be used with large types of ordinary differential equations, and recently employed also to detect limit cycles for different purposes [110]. Concurrent packages however are available. To the best of our knowledge, this numerical

algorithm is implemented in AUTO-07P, a software for numerical continuation [111], PyDSTool [112], and a recent package written in Julia called BifurcationKit.jl [113]. However, the main advantage of having an *in-house* software is the adaptability to the required interfaces such as the flight dynamics model and our aerodynamic model.

4.6.1 Architecture of the code

We present here the architecture of multiflap. It is an object-oriented toolbox that comes with three main packages: (i) aerodynamic package, which stores the wing model and lifting line; (ii) the multiple-shooting package, which contains the classes and methods to build and solve the multipleshooting scheme; (iii) the equation package, in which the user encodes the set of ordinary differential equations that is the subject of the study. The tree structure of the code is reported in Figure 4.10



Figure 4.10: Tree structure of multiflap. The current version comes with three main packages, namely the aerodynamic package, the multiple-shooting package, and the equations package.

The file multiple_shooting.py contains the MultipleShooting class that takes as input the set of equations defined in multiflap/odes.

This MultipleShooting class has the following methods:

• get_mappedpoint: Returns the last element of the time integration.

This solves numerically Equation (4.3);

- get_jacobian_analytical: Returns the Jacobian (or Monodromy) matrix, via solving Equation (4.29);
- get_jacobian_numerical: Returns the Jacobian (or Monodromy) matrix, via solving Equation (4.31);
- get_ms_scheme: Returns the multiple-shooting matrix **M** and the error vector **E** in order to get Equation (4.26).

The file lma_solver.py contains the Solver class which relies on LMA to solve Equation (4.27).

We report the validation of multiflap in Appendix A. Moreover, a full example of a case study is described in Appendix B, where it is shown how to hard code the equations, and how to build the main.py file to run a complete simulation.

4.6.2 Current limitations

As previously mentioned, two concurrent approaches are implemented in multiflap in order to compute the Jacobian matrix, i.e. the analytical and the numerical approach. In order to compute the Jacobian analytically, the user has to hard code the stability matrix of Equation (4.10) (see Appendix B). This is a significant limitation, because for high dimensional systems it requires a big effort and constitutes a potential errorprone process. Thus, in all of these cases, it is preferred to compute the Jacobian numerically. This problem could be solved by providing multiflap with automatic differentiation packages, that read the equations of motions and build the stability matrix automatically. A candidate software which relies on symbolic framework to compute algorithmic differentiation is CasADi [114]. No work has been done so far towards this direction, but interfacing multiflap to CasADi could be the first step to overcome this current limitation.

Finally, multiflap is not parallelized. This feature would drastically speed up the computational efficiency of the package. The nature of the algorithm could allow a parallelization via assigning one process for each point of the multiple-shooting algorithm. Previous works successfully attempted in realizing this parallelization, showing a drastic improvement of computational speed [115].

4.7 Conclusion

The work illustrated in this Chapter developed the formalism to address limit cycle problems, and described the numerical algorithm implemented in our framework in order to detect limit cycles and assess their stability. This algorithm has been implemented in a Python package called multiflap, and hosted at https://github.com/vortexlab-uclouvain/multiflap/.

We first described the notation adopted, by defining the phase space, the trajectory and ultimately the limit cycle. Limit cycle solutions can be either stable or unstable. The former can be detected via direct numerical integration of the ODE system, starting from an initial condition located closed to it. By doing that, the trajectory will be naturally attracted by the orbit after a transient time. This approach does not apply to unstable limit cycles, as the trajectory will be constantly repelled by the unstable orbit. For these cases, a tailored approach is thus required.

The stability of these limit cycles is governed by Floquet theory [96] and assessed by the eigenvalues of the Jacobian or (or Monodromy) matrix, the so-called Floquet multipliers. These eigenvalues have the important property to be invariant of the orbit. If all the Floquet multipliers (except the trivial one) are smaller than one in absolute value, the limit cycle is stable. Conversely, if at least one Floquet multiplier is larger than one in absolute value, the system is unstable.

The detection of limit cycles relies on the multiple-shooting algorithm. This numerical scheme, inspired from a previous version of Lust [104], iteratively detects periodic orbits, and evaluates the corresponding Floquet multipliers by computing the Jacobian matrix. In the current version of our code, this Jacobian matrix can either be computed analytically or numerically.

We conclude the Chapter by presenting our package multiflap and its architecture. It is an object-oriented toolbox, which leverages the Levenberg-Marquardt algorithm to solve the multiple-shooting schemes. multiflap handles both autonomous and non-autonomous systems of ODEs, and embeds two ways of computing the Jacobian matrix, namely the analytical or numerical computation. We finally illustrate the main classes and functions.

This work represents the last piece of the computational puzzle in order to ultimately study steady-state flight regimes, and evaluate their stability. A first case study is reported in the next Chapter.

Chapter 5

Analysis of a representative flapping flight regime

5.1 Introduction

Among others, one of the most impressive capabilities of birds flight, is their reaction to environmental perturbations, such as wind and gusts. It is important to get biological insights on the mechanisms that govern such performance for different reasons. From a purely zoological perspective, this could give precious information to understand the evolutionary path, that led bird to have nowadays the observed morphology. On the other hand, engineering could largely benefit from bio-inspired findings. Biomimetics is an important topic of inspiration for emerging flapping vehicles, and building a computational framework could lead to progress.

As discussed in Chapter 3, due to time-varying aerodynamic forces, flapping flight does not display fixed points of equilibrium. The problem is therefore approached via a limit cycle analysis based on Floquet theory. In this Chapter, we merge together the work previously developed, namely using the multiple-shooting algorithm in order to find limit cycles of Equation (3.1). This allows us to conduct a rigorous longitudinal stability analysis of fast forward flight regimes.

The global architecture of the framework is presented in Figure 5.1. In the present study, no effect of the tail surface is accounted for. The biomechanical model of the wing, is actuated by prescribing the kinematics at the level of its articulations: shoulder, elbow, and wrist. This constitutes the input of our aerodynamic model. We base our calculations on the quasi-steady lifting line theory, modeling the wake as semi-infinite vortex filaments. The aerodynamic model is then coupled with the equations of motion in the longitudinal plane. The scheme is initialized by guessed initial values, which are successively refined by the multiple shooting algorithm. Once the limit cycle is found, the Floquet multipliers are computed.



Figure 5.1: Global architecture of the numerical framework for studying flapping flight stability. The kinematics is imposed, and it is passed to the aerodynamic model. This lifting line model is coupled with the equations of motion (double arrow). This system is integrated starting from guessed initial values. The multiple-shooting algorithm refines such initial values until the limit cycle condition is satisfied. The Floquet multipliers comes as an output of the system too.

We further report three in-silico experiments. The first experiment aims at validating the capacity of the developed framework to identify steady state flight regimes and assessing their stability with a particular reference kinematics. We found that for this prescribed kinematics, the solution had one unstable mode.

The second experiment, aims at finding particular kinematics that guarantee a certain flight regime. In the case presented, we modulate the wingbeat amplitude at the level of the shoulder, in order to find steady-state and level flight. This particular parameter, is found to affect marginally the dynamical properties of the system.

The last experiment quantifies the sensitivity of the stability with re-

spect to the position of the wing insertion point. This scenario changes the distribution of nose-up and nose-down pitching moment, without altering the prescribed kinematics. Interestingly, the relative position between the wing and the center of mass is found to significantly affect the values of Floquet multipliers, suggesting that this distribution of pitching moment plays a very important role in flapping flight stability.

The Chapter follows by showing a comparison between the two methods of computing the Jacobian matrix, namely the analytical method and the numerical method. We perform this test in order to further validate our multiple-shooting algorithm, and to benchmark these two approaches.

The aerodynamic model is then a posteriori verified, to assure that the limit cycles detected were compatible with the aerodynamics assumptions. It was verified that the reduced frequency assured the validity of the quasisteady model of the wake, and we then verified that the angle of attack remains within the validity of the thin airfoil theory in order to calculate the lift coefficient.

5.2 Numerical parameters and wingtip trajectory

The morphological and kinematics parameters for describing the wing geometry, are reported in Table 3.1. The aerodynamic effects due to the tail are not explicitly modeled in this case study. Therefore the corresponding aerodynamic terms $F_{x',t}$, $F_{z',t}$, $M_{y',t}$ of Equation (3.1) are identically set to zero. The proposed study compensates for this missing tail by sweeping the wing around the body center of mass, thus providing the possibility to generate both nose-up and nose-down pitching moments. This constitutes a necessary condition for the existence of limit cycles, in order to achieve rotational equilibrium of the bird body over a period. Moreover, the parasitic and profile sources of drag have not been accounted for in the equations of motion. The resulting tip trajectory is shown in Figure 5.2. Considering a counter-clockwise motion, the blue trajectory represents the region where the lifting line lies ahead of the center of mass G. Assuming a positive lift for the whole wingbeat period, this region provides a positive pitching moment due to the lift (nose-up). The tip positions corresponding to the orange segment of the trajectory correspond to the region where the lifting line lies behind the center of mass, generating a negative pitching moment due to the lift effect, under the aforementioned assumption.

We look for limit cycles having the period of the flapping wingbeat.



Figure 5.2: Tip path trajectory over a wingbeat cycle. Blue line: position of the lift ahead of the center of mass, contributing to a nose-up (positive) pitching moment; orange line: position of the lift behind the center of mass, contributing to a nose-down (negative) pitching moment. Scatter points represent the position of the wing profile every T/20.

This period is known, because of the imposed periodicity of the kinematics. We therefore rely on the multiple-shooting scheme of Equation (4.25). The numerical parameters employed in multiple-shooting algorithm are reported in Table 5.1 and the numerical integration has been performed using a fourth-order Runge–Kutta scheme. A sensitivity analysis of the integrator scheme is further illustrated in Appendix C.

We leverage on a semi-analytical computation of the Jacobian matrix in the form of Equation (4.29). The challenge of using this approach, is the computation of the derivatives of the aerodynamic forces to compute the stability matrix of Equation (4.11). This approach is called semi-analytical, because while it relied on an analytical formulation of the Jacobian matrix J, it still required a numerical estimation of the first partial derivatives of the aerodynamic forces to compute the stability matrix A.

The initial value at time t = 0 to start the multiple-shooting scheme was chosen to resemble a reasonable cruise flight condition of large birds

$$\mathbf{x}_0(t=0) = [16.5, 0.5, 0.5, 0]$$

Multiple-Shooting settings		
Integrator order	4	
Time steps over a period	150	
Amount of points (M)	5	
Iteration error	10^{-5}	

Table 5.1: Numerical parameters used in the Multiple shooting algorithm.

and the other M - 1 points were spread by straightforward numerical integration

$$\mathbf{x}_{n+1}(t_{n+1}) = f(\mathbf{x}_n(t_n))\Big|_{t_n}^{t_n+\tau}$$

in order to have a column of zeros in the RHS of Equation (4.25), except for the last element.

5.3 Results

This section reports three experiments that were conducted to validate the capacity of the developed framework to identify limit cycles, assess their stability, and quantify the sensitivity of flight regime and stability with respect to kinematic and morphological parameters.

5.3.1 Experiment 1: representative limit cycle and stability analysis

A representative limit cycle solution is reported here, as the result of a multiple-shooting computation. This solution corresponds to the reference case in which the kinematics is described by the governing parameters of Table 3.1a, with $A_{s,x} = 42$ deg.

Convergence analysis and consequently stability results are shown in Figure 5.3. In particular, rapid convergence is obtained, after 7 iterations. The error is evaluated as the max{|E(x)|} of Equation (4.26). Such configuration presents one expanding eigenvalue, which leads the system to be unstable.

The numerical values of the Floquet multipliers, are also reported in Table 5.2

The expanding eigenvalue has an absolute value of $\Lambda_1 = 1.40$. The



Figure 5.3: Convergence error (left) and Floquet multipliers (right) for the reference case with the imposed kinematics of Table 3.1a.

Multiplier	Value
Λ_1	1.40 + 0j
Λ_2	0.829 + 0.260j
Λ_3	0.829 - 0.260j
Λ_4	0.212 + 0j

Table 5.2: Floquet multipliers obtained for the reference case with the imposed kinematics of Table 3.1a.

related eigenvector is $e_1 = [-0.69, 0.68, 0.12, 0.16]^T$ evaluated at the point $\mathbf{x}(t_0 = 0) = [18.24, -1.93, -0.10, -0.115]^T$. This therefore excites the perturbation along each eigenbase directions with the same order of magnitude. Considering a flapping period of T = 0.25s, the expanding Floquet exponent is

$$\lambda_1 = \frac{1}{T}\ln(\Lambda_1) = 1.34s^{-1}$$

and therefore the time needed for the perturbation to double its value is approximately

$$t_{doub} = \frac{\ln(2)}{\lambda_1} \approx 0.51s$$

This corresponds to approximatively two flapping periods and is thus larger than the one reported in previous studies focusing on smaller scale animals [75, 116]. The eigenvalue spectrum was found qualitatively similar to the one presented by [75], and interestingly a similar pattern was also observed for insect scales in previous research efforts [69, 83, 117].

Figure 5.4 pictures the periodic solution of the state variables describing this limit cycle in the phase space. This solution is steady state, for the prescribed kinematics and morphology. The states are plotted with respect to one cycle in the moving body frame. At time zero the wing position corresponds to the middle of the downstroke.

On the right side of Figure 5.4 is pictured the 4-state limit cycle in the phase portrait. The periodicity in θ is plotted as a color map on the trajectory described by the three others state space variables.



Figure 5.4: Reference limit cycle solution. Trajectory of each state variable (left) and phase portrait (right).

To better highlight what leads to trajectory separation, we ran another simulation with a perturbation of the limit cycle solution along the pure unstable direction e_1 and let the system evolve over five flapping periods. The separation is driven from the pitch angle θ which quickly tilts down, and is subsequently followed by an increase of the *u*-component of velocity, likely resulting from acceleration due the larger action of the gravity along the local *x*-axis, as shown in Figure 5.5.



Figure 5.5: Separation of the perturbed solution along the unstable eigenvector (black) over five flapping periods, from the periodic orbit (red). Trajectory of each state variable (left) and phase portrait of the variables [u, w, q] (right).

Similarly we perturbed the solution purely along the stable eigendirection as shown in Figure 5.6. Indeed, when there is no component acting along the the unstable direction, the system tends to be dynamically attracted to the orbit.

The aerodynamic forces and moment of the limit cycle solution are plotted in Figure 5.7 and normalized with respect to $m_b g$ (forces) and $m_b g \overline{c}$ (moment). The global action of the forces and pitching moment over one period is zero, confirming the state of trimmed flight, and limit cycle condition (zero acceleration over one period).

The validity of this aerodynamic model was assessed by studying the resulting reduced frequency and angle of attack.

Results from the simulation settles on values that are very compatible with biological observations for species of similar mass and aspect ratio [118, 119, 120]. The corresponding Strouhal number for this flight condition is

$$St = rac{f\Psi}{U_{\infty}} pprox 0.18$$



Figure 5.6: Evolution of the perturbed solution along the stable eigenvector (black) over five flapping periods. The trajectory is attracted by the periodic orbit (red). Trajectory of each state variable (left) and phase portrait of the variables [u, w, q] (right).



Figure 5.7: Dimensionless forces and pitching moment developed by the flier, expressed in the fixed frame O(X, Z).

where *f* is the flapping frequency, Ψ the vertical length identified by the tip-to-tip excursion, as pictured in Figure 5.8, and U_{∞} the forward flight velocity. It lies in the lower margin identified by Taylor [53] for maximizing power efficiency for birds in forward flight.



Figure 5.8: Position of the lifting line over one flapping period. (1): Middle downstroke. (2): Full downstroke. (3) Middle upstroke. (4): Full upstroke. The time evolution of the lifting line is color plotted. The tip excursion identifies the reference length Ψ to calculate the Strouhal number.

5.3.2 Experiment 2: Sensitivity analysis of the shoulder amplitude

The multiple-shooting method has been applied to address the question of the gait sensitivity, since the framework can handle the analysis of several gait configurations. In particular, we exploited this to achieve a specific limit cycle solution corresponding to level flight. Indeed, the solution reported in Experiment 1 corresponds to trimmed flight, but not necessarily to level flight: trimmed flight might correspond to a flight regime with a non-zero averaged vertical velocity.

In order to achieve level flight, the mean vertical velocity with respect to the fixed frame has to be zero over the flapping period. Considering Figure 3.2, the velocity components in the fixed frame are

$$U_{ff} = \dot{X} = u \cos \theta + w \sin \theta$$

$$W_{ff} = -\dot{Z} = u \sin \theta - w \cos \theta$$
(5.1)

In particular, we report here a sensitivity analysis of the flapping gait as a function of one of the most important kinematic parameters, namely the wingbeat amplitude of the shoulder $A_{s,x}$. Consequently, seeking for a level-flight configuration reduces to a single parametric study consisting in finding the shoulder amplitude $A_{s,x}^*$ that corresponds to a limit cycle whose mean vertical velocity is equal to zero, i.e.

$$\overline{W}_{ff}(A_{s,x}^*) = 0 \tag{5.2}$$

Since W_{ff} is a non-linear function of $A_{s,x}$, we rely on a Newton-Raphson method to find its root. Finally, the climbing or descending ratio is identified by the trajectory angle, defined as

$$\gamma = \tan^{-1} \frac{W_{ff}}{\overline{U}_{ff}}$$

By applying the frame transformation of Equation (5.1) and subsequently integrating the velocities, three different trajectories of the bird corresponding to three different flight conditions, are illustrated in Figure 5.9. The relationships between the shoulder amplitude, and the corresponding flight velocities and trajectory angles have also been investigated for an interval of [40 deg $< A_{s,x} < 46$ deg], and results are plotted in Figure 5.10. A quasilinear relationship is found with high sensitivity response of the amplitude parameter to the flight condition.

Descending trajectories are observed for amplitudes smaller than 43.47 deg corresponding to a negative vertical velocity, while climbing trajectories correspond to higher values of amplitudes. Also concerning the forward flight velocity, an increase in amplitude determines a linear increment of the flight speed, suggesting an active role of this parameter on the production of the thrust. For each of the shoulder amplitude tested in this experiment, the corresponding Floquet multipliers are reported in Figure 5.11.



Figure 5.9: Different flight trajectories over one flapping period. Red path: descending behavior with a shoulder amplitude of 42 deg; green path: level flight solution obtained with a shoulder amplitude of 43.47 deg; blue path: climbing behavior for a wingbeat amplitude of 44 deg.



Figure 5.10: (a) Averaged horizontal velocity, (b) averaged vertical velocity, (c) averaged norm of the velocities, (d) trajectory angle. Descending regions are plotted in red. Climbing regions are plotted in blue. The level flight condition is plotted with the green point.



Figure 5.11: Eigenvalues corresponding to several shoulder amplitudes and zoom on the unstable branch.

5.3.3 Experiment 3: Sensitivity Analysis on the Wing Insertion Point

We now conduct a sensitivity analysis of stability as a function of the relative position between the center of mass and the insertion point of the wing in the body frame. This wing instertion point coincides with the rotational shoulder joint. Since the wing is free to sweep around the body center of mass, we need to revisit the classical definition of stability margin from the literature in flight mechanics. Considering Figure 3.2, we introduce a new morphological parameter that we call longitudinal margin *LM*, which defines the position of the center of mass with respect to the wing root projection along x', normalized by the mean aerodynamic chord, i.e.

$$LM = \frac{(O'_w - G)}{\overline{c}} \tag{5.3}$$

When LM = 0%, the center of mass longitudinally coincides with the origin of the wing frame. The interval of variation of the longitudinal margin, has been explored in the range 17.5% < LM < 32%.

Results are illustrated in Figure 5.12 where the locus of the eigenvalues is reported for different longitudinal margins. It shows that the eigenval-



Figure 5.12: Sensitivity of the eigenvalues with respect to the reciprocal position between wing-bone frame and the body frame.

ues locus (and therefore stability) is governed by the wing position, relative to the center of mass, and thus from the generation of an adequate pitching moment. Looking at Figure 5.12, the expanding eigenvalue is smaller as the wing tends to get closer to the center of mass. This means that the capability of the wing to generate negative (nose-down) moment, is beneficial for the global stability behavior. In contrast, if the position of the wing is too much ahead of the center of mass, and consequently the capability of generating nose-down moment is reduced, the absolute value of the expanding eigenvalue, drastically increases, leading the system to be more and more unstable. Thus the characteristic doubling time is putatively modulated by the wing kinematics and position, and is not unique for a given specie as suggested by [75].

Importantly, none of the tested configurations corresponds to a stable solution. The distance between the wing insertion point and the center of mass could be made smaller to continue bringing the largest eigenvalue close to the unit circle, but there is a threshold above which the wing is not capable of generating enough pitching up moment, for guaranteeing the existence of a limit cycle solution. Indeed, zero average pitching moment over one period is a necessary condition of existence of a limit cycle. In sum, we did not manage to find such limit cycles for LM < 17.5%, and thus no stable limit cycle could be found.

5.4 Comparison between two methods for computing the Jacobian matrix

The same reference case of Table 3.1 has been used for comparing the two concurrent methods for computing the Jacobian matrix, i.e., solving Equation (4.29) and (4.31), respectively. The step size of the numerical integration was set constant and equal to dt = T/100.

The computational time per iteration with the semi-analytical approach was found to be about 40s while the numerical approach took more than the double, around 95s per iteration. A comparison of the resulting errors and Floquet multipliers is provided in Figure 5.13. In conclusion, both approaches provided the same eigenvalues with a very similar convergence rate, although the semi-analytical approach took less than half of the computational time of the fully numerical one.



Figure 5.13: Left: Error evolution of the reference limit cycle solution, computed with a semi-analytical approach (black) and a numerical approach (red). Right: Floquet multipliers of the reference limit cycle computed with a semi-analytical approach (black) and a numerical approach (red).

5.5 A posteriori verification of the aerodynamic model

The aerodynamic model considers the lifting line being quasi-steady. To verify its validity, we computed the reduced frequency, which is found to be about

$$\frac{(2\pi f)\overline{c}}{2U_{\infty}} = 0.1$$

where f and \bar{c} are the frequency and the mean aerodynamic chord respectively, reported in Table 3.1, and U_{∞} is the average forward flight velocity of the reference limit cycle which is about $U_{\infty} = 18.2ms^{-1}$.

We moreover report the evolution of the angle of attack at three wing sections. The lift coefficient is calculated in the linear range of the $C_l - \alpha$ curves. The angle of attack through the flapping period is shown in Figure 5.14.



Figure 5.14: Angle of attack corresponding to the kinematic parameters reported in Experiment 1. Left: Change of angle of attack α as a function of the cycle phase, evaluated in the arm (blue), forearm (green), and hand (red). Right: 2D top view of the wing envelope at the middle of downstroke, highlighting the three positions where the angles of attack are computed.

The maximum angle of attack is reached at the tip with a value which stays below 15°. Moreover considering the short chord length and the low contribution of the lift generation of this wing region, we consider that our assumption of a linear relationship between the lift coefficient and the angle of attack is satisfied [52].

5.6 Discussions and conclusion

Flapping flight stability is a central concept for understanding how complex a control scheme is, or needs to be, in animal fliers such as birds, or ornithopters. Experimental, theoretical, and numerical studies on such "flapping systems" have provided valuable insights on their dynamics and stability ([36, 59, 69]). With this contribution we have made a step forward by using a new model which is more accurate than existing ones in a couple of ways: (i) our wing is morphing during the cycle and we have enriched the bird motion by introducing critical degrees of freedom of the wing, especially the shoulder sweep angle; and (ii) we have considered the wake effect on aerodynamic force production via a quasi-steady lifting line approach.

Several steady state flight trajectories have been retrieved by varying the gait parameters. In particular, the wingbeat amplitude relates quasilinearly with the climbing angle and with the forward flight velocity in the explored range. In this study this parameter played a central role for achieving level flight, however, these results suggest that it can also be fundamental for selecting a preferred forward flight speed, thus regulating the thrust.

Results from the simulations agree with experimental observations of large bird species with a comparable mass and wingspan. In terms of forward flight velocity the limit cycle converges to values that are effectively adopted by birds during migrations [118, 119, 120], and consequently the related Strouhal number lies in the lower range identified by Taylor [53] in order to maximize power efficiency.

Furthermore, Floquet theory combined with multiple-shooting algorithms is confirmed as an elegant and powerful framework for analyzing the solutions of such flapping gaits. It turns out that the relative position between the wing and bird center of mass clearly affects pitching moments and global stability. Since our wings are massless, this effect has been investigated by moving the relative position between the center of mass and shoulder joint, whereas in reality this can be obtained by sweeping movements. This choice was made in order to unveil the main physical relationship between the generation of pitching moment and stability.

Although the expanding eigenvalue gets closer to the unit circle when the center of mass approaches the wing root, stable configurations have not been found. In sum, within the accuracy of our model, and within the variations permitted by the parameters investigated in the present study, the wing only cannot generate a fully stabilizing effect in pitch. If so, birds would need to continuously rely on sensory feedback to adapt their gait via active control. Moreover, we clearly showed how the instability doubling time depends not only on the species, but also on the wing kinematics, its morphology, and how they relate with the pitching moment.

Nevertheless, at least two important complementary aspects have not been investigated here. Stabilizing benefits could indeed arise either from wing compliance or from the aerodynamic contribution of the tail. Although the former mechanism requires an extension of the current framework, analyzing the role of the tail can be handled with the current model.

This invokes a more extended and dedicated parametric study of key kinematics degrees of freedom (in particular, regarding sweeping motion),

combined with the tail opening. It is hypothesized that a tail-like surface could have stabilizing effects, but at a cost of introducing additional drag components, thus penalizing the flight efficiency [35]. This more extended parametric study, will aim at answering questions as: *Can passive flight stability be achieved via introducing a tail-like surface?* and if so, *Is there a trade-off between power consumption and flight stability?*.

Other questions regarding inertial contributions of bird wings and head to dynamics, also remain open. Comprehensive answers within this framework would require substantial modifications of the equations of motion adopted here.

Chapter 6

On the role of tail in stability and energetic cost of bird flapping flight

6.1 Introduction

Chapter 5 introduced new research scenarios, invoking the continuation of the study towards a broader parametric space involving the tail surface. One of the questions we opened at this stage, was the understanding whether there is a trade-off between passively stable flight, and flight efficiency. A foundational study by Smith in 1952 [35], developed a theory of *evolution of instability*, establishing how inherently unstable flight regimes might have provided a selective advantage for fliers through evolution. Indeed, passively unstable systems are more responsive to changes in command [1], and this might have facilitated maneuverability for birds. This had to come in parallel with the development of sensory-driven neural circuitries to actively control the flight in order to display stable closed-loop behavior.

Over the last couple of decades, several studies have investigated how such stability might be achieved, with a specific focus on the gliding regime. Thomas and Taylor [121] studied gliding flight and showed that birds use a combination of passive stability — alleviating perturbations without active control — governed by their morphology, and active stabilization from

neural pathways to regulate their flight. For example, in gliding gulls, static longitudinal stability is achievable by modulating the opening of the elbow joint over a large range [22, 122, 123]. Cheney et al. [28] investigated the role of wing compliance and tail actuation in order to alleviate perturbations. Ajanic et al. [124] conducted a dedicated study on wing morphing and the mechanism of wing sweep on a propelled fixed wing robot. For each morphological configuration, the authors estimated the required power to fly. They showed that sweeping the wing backwards and increasing the tail surface was beneficial for longitudinal passive stability, although at the cost of increasing parasitic drag and thus the energetic performance.

Moreover, in flapping flight, the wing kinematics has an important impact on power consumption. In an analysis on pigeons, Parslew [125] suggested that particular kinematics modes might be selected in specific flight regimes for energy saving purposes. Colognesi et al. [65] also showed a dependency between power requirement and key parameters of the wing kinematics, specifically the wingbeat amplitude.

Although the role of the tail has been studied in gliding regimes, and the influence of wing kinematics has been studied to assess the performance in flapping regimes, no study to date combined both in a whole body characterization of flapping gaits.

In pursuit of having a better understanding on the biological insights governing stability of bird flight, we extend the investigation presented in Chapter (5) to a broader range of parameters and morphological configurations.

The objective of this Chapter, is thus to provide such a complete modeling frameworks that can simultaneously assess the influence of wing kinematics and tail morphology in stability and energetic performance. Based on observations with real birds [35], we hypothesize that opening the tail should inherently lead to passively stable flight regimes, at the price of an increased energetic cost.

The rest of the Chapter is structured as follows: we first introduce the additional drag contributes that have to be accounted to estimate flight performance. These contributions are the parasitic drag of the body, and the profile drag of the wing.

We next describe the parametric space. This is composed of the tail opening, the sweep offset of the shoulder joint, the flapping amplitude of the shoulder and the mean rotation angle of the wing profiles from the wrist joint. This latter parameter is here used to level the flight, while the wingbeat amplitude is explored in order to modulate the thrust production.

Finally we present the results of this parametric investigation. First we show the locus of the solutions detected by our algorithm. These solutions are first characterized by the Cost of Transport, and then by their stability. Interestingly, a change in stability properties is observed as the tail opening increases. We then analyze three reference solutions, corresponding to the same forward flight velocity, in order to highlight the beneficial role of the tail on the stability properties. These solutions are also perturbed via a Gaussian-like gust, and the response of the system is presented. We finish our investigation showing a trade-off between Cost of Transport and stability.

6.2 Drag production by body and wing

The main body and the wings induce drag that should be accounted for in Equation (3.1), for a model aiming at characterizing energetic performance. Body-induced drag is named parasitic because the body itself does not contribute to lift generation, and only induces skin friction and pressure drags around its envelope [126]. The total body drag is

$$D_b = \frac{1}{2}\rho C_{d,b} S_b |\mathbf{U}_{\infty}|^2 \tag{6.1}$$

where ρ is the air density. We implemented the model described by Maybury [126] to compute the body drag coefficient $C_{d,b}$. This depends on the morphology of the bird and the Reynolds number *Re* according to

$$C_{d,b} = 66.6m_b^{-0.511} F R_t^{0.9015} S_b^{1.063} R e^{-0.197}$$
(6.2)

with S_b and FR_t are respectively the frontal area of the body and the fitness ratio of the bird, defined as the length of the body divided by its maximum cross-sectional size. Both of them can be estimated from other allometric formulas i.e. [126, 127]

$$S_b = 0.00813 m_b^{2/3} \tag{6.3}$$

$$FR_t = 6.0799 m_h^{0.1523} \tag{6.4}$$

97

The Reynolds number introduced in Equation (2.2) is calculated with the reference length of the mean aerodynamic chord \bar{c} . This model is found to be suitable for Reynolds number in the range $10^4 - 10^5$ [126] which applies for our case. Another source of drag is the profile drag due to friction between the air and the feathers on the wings. It is the sum of the profile drag at each section along the wingspan, i.e.

$$D_{p,w} = \frac{1}{2}\rho C_{d,pro} \sum_{i=1}^{n} c_i |\mathbf{U}_{r,i}|^2 ds_i$$
(6.5)

with c_i the chord length, ds_i the length of the lifting line element along the wingspan, and $\mathbf{U}_{r,i}$ the velocity at the wing section *i* accounting for the body velocity, the kinematics velocity of the wing and the downwash velocity as pictured in Figure 3.4. We used a value of profile drag of $C_{d,pro} = 0.02$ and this is assumed to be constant over the wingspan and throughout the flapping cycle [26]. In reality, due to the wing motion, this value should be gait dependent. However, the aforementioned assumption has been largely used and accepted in previous works [128, 129].

6.3 Parametric space

In the present study, we leverage the aforementioned multiple-shooting algorithm, for seeking steady flight regimes within the following parametric space

$$\nu = (\beta, \psi_{s,z}, A_{s,x}, q_{w,y}) \tag{6.6}$$

where β is the tail opening angle, $\psi_{s,z}$ is the shoulder sweep offset, $A_{s,x}$ is the wingbeat amplitude, and $q_{w,y}$ is the mean rotation angle of the wing profiles about the axis y, see Figure 6.1. The other parameters defining the wing kinematics are kept fixed and reported in Table 6.1.

Previous studies have shown that these four parameters decisively govern the flight regime in bird flapping and gliding modes. The tail opening β and shoulder sweep offset $\psi_{s,z}$ influence flight stability, since these are the parameters having a paramount influence on the generation of pitching moment. Then, the shoulder wingbeat amplitude $A_{s,x}$ has a direct impact on thrust production and therefore on airspeed and power consumption. The last parameter $q_{w,y}$ modulates the generation of lift [65].

On top of seeking steady flight regimes, it is important to identify those corresponding to level flight, i.e. with the bird flying at a constant altitude.


Figure 6.1: (a): Front view of the bird model. The wingbeat amplitude $A_{s,x}$ about the x'-axis is the main kinematic parameter governing the wing amplitude of movement. **(b)**: Top view of the bird model. The shoulder sweep offset $\psi_{s,z}$ captures the average angle of the arm bone with respect to x'_w w over the period. The angle β captures the magnitude of tail opening. **(c)**: Lateral view of the bird wing. The wing profile rotation about y'-axis is indicated with $q_{w,y}$.

Table 6.1: Numerical parameters of the wing used in the simulations. Parameters highlighted with an asterisk * are those being varied in the parametric study. The additional parameter to those reported in the table that varies in the case study, is the tail opening β .

Joint	$q_0[deg]$	A[deg]	ϕ [deg]
Shoulder y	11.5	0.8	-90
Shoulder <i>x</i>	0	$A_{s,x}^*$	180
Shoulder z	$\psi^*_{s,z}$	14	90
Elbow z	10	10	-90
Wrist y	$q_{w,y}^*$	30	-90
Wrist z	-30	30	90

This level flight condition thus corresponds to an average mean vertical velocity being equal to zero over the period, in a fixed reference frame according to Equation (5.1). Concretely, satisfying the level flight condition isolates a three-dimensional manifold within the four-dimensional parametric space of Equation (6.6). Finding this manifold is done by searching

the value of the parameter $q_{w,y}$ that corresponds to level flight for all possible combinations of the three other parameters β , $\psi_{s,z}$ and $A_{s,x}$. In other words, we report here only the limit cycles that belong to the manifold satisfying

$$\left|\overline{W_{ff}}(\beta,\psi_{s,z},A_{s,x},q_{w,y})\right| < \epsilon \tag{6.7}$$

with $\epsilon = 5 \cdot 10^{-3} m s^{-1}$, which corresponds to a maximum vertical deviation of 1mm per flapping cycle.

6.4 Power Consumption and Cost of Transport

Each limit cycle corresponds to a particular flapping gait with its own mechanical power consumption and the corresponding cost of transport.

The global power input, i.e. the metabolic rate, is diverted part in living activities (such as respiration, circulation, etc.), and part in actuation power to provide the input to the bird muscles [131]. Then, the power output of these bird muscles actuates the wing unit. In particular, part of this power counter-acts the inertial effect of the wing motion, and the remaining is transferred to the fluid. In the present analysis, we only focus on the actuation power, and we omit all the possible sources of power losses in the form of heat, that occur through inefficiencies in muscle contraction and joint frictions.

Since inertial power for accelerating and decelerating a wing is neglected, the actuation power produced by each joint is exactly equal to the power transferred by the wing to the environment, i.e.

$$P_{act}(t) = \sum_{i=1}^{n} \mathbf{F}_{aero,i}(t) \cdot (-\mathbf{v}_{kin,i}(t))$$
(6.8)

where $\mathbf{v}_{kin,i}(t)$ is the velocity of the lifting line computed at the discretized point P_i and time t, and $F_{aero,i}(t)$ is the corresponding aerodynamic load on the wing element i, computed by the quasi-steady lifting line model.

Instantaneously, $P_{act}(t)$ can assume both positive or negative values. This depends if the wing is injecting energy into the fluid, or receiving energy from it. The mean power consumption over one flapping period is thus

$$\overline{P}_{act} = \frac{1}{T} \int_0^T P_{act}(t) dt$$
(6.9)

Equation (6.9) implicitly assumes that the actuators can store the full amount

of energy received by the fluid.

Another important metric to assess locomotion performance is the socalled Cost of Transport (CoT), i.e. a dimensionless ratio equal to the mechanical work produced by the actuators to transport a unit of body weight across a unit of distance [132]. Here, it is thus defined as

$$CoT = \frac{\overline{P}_{act}}{mg|\mathbf{U}_{\infty}|} \tag{6.10}$$

with $\overline{|U_{\infty}|}$ being the magnitude of the flight speed of the corresponding limit cycle, averaged over one period.

6.5 Numerical settings

This study is performed within the following parametric space: tail opening $\beta \in [0^\circ, 45^\circ]$, wingbeat amplitude $A_{s,x} \in [29^\circ, 45^\circ]$ and sweep offset $\psi_{s,z} \in [9^{\circ}, 15^{\circ}]$. The interval for the tail opening has been chosen to have a resulting tail span morphologically compatible with the size of the bird. The lower value for the wingbeat amplitude has been selected to yield forward flight velocities high enough to respect the quasi-steady assumption. The upper value of the wingbeat amplitude is based on previous works and observations [50]. Finally, the interval of the sweep offset is tuned to correctly distribute the pitching moment ahead/behind the center of mass, in order to satisfy the limit cycle condition. The parametric space is meshed with an uniform grid spaced along $\psi_{s,z}$ and $A_{s,x}$ with a step size of 0.5°, and a step size of 1° along β . This resulted in 19734 possible flight configurations. In the results, we report all solutions satisfying Equation (6.7), with the addition of two exclusion criteria. First, we excluded limit cycles that do not correspond to fast forward flight. In [60], Parslew identified two specific flight modes corresponding to such condition of fast forward flight (mode 4 and mode 5). Both require the body pitch angle to stay close to the horizontal configuration. Concretely, we excluded from the results limit cycles corresponding to a mean body pitch angle larger than 6° in absolute value. Second, we excluded limit cycles corresponding to biologically incompatible kinematics. This was implemented by excluding limit cycles with a mean rotation angle of the wing $q_{w,y}$ larger than 12° in absolute value. Indeed, remembering that the related amplitude of this joint was fixed to 30° (Table 3.1a), this criterion excluded solutions corresponding to geometrical rotation of the forearm larger than $\pm 42^{\circ}$ in absolute value, which we considered to be not physiologically consistent.

6.6 Results

In this section, we report the results of the systematic exploration of the gait parametric space. First, the locus of the solutions is reported, i.e. the set of parametric values for which a limit cycle has been identified. Next, three representative limit cycles are analyzed in detail: the first one with a completely furled tail ($\beta = 0$), and the both other ones with an open tail ($\beta = 40^{\circ}$). Finally, these solutions are assessed in terms of energetic expenditure, quantified by the CoT.

6.6.1 Manifold of the solutions

Among the 19734 possible parametric configurations, our algorithm detected 5604 steady leveled limit cycles. The locus of these identified solutions is pictured in Figure 6.2. Figure 6.2(a) shows the CoT at equally-



Figure 6.2: Locus of the steady and leveled solutions in the gait parameter space. Colored surfaces are representing slices of this locus, at every 5° of tail opening. (a): Cost of Transport. (b): Stability indicator captured via the largest Floquet multiplier. Unstable limit cycles are represented with different shades of yellow-to-red, and stables ones are in green; the transition appears around $\beta = 25^{\circ}$.

spaced planes of tail opening angle β , projected in the parameter space. The CoT progressively increases as the tail spreads out. It further displays a higher gradient with respect to both other parameters for a given opening angle. This reveals that this cost of transport is sensitive to the kine-

	(a)	(b)	(c)	
$\beta[^{\circ}]$	0	40	40	
$\psi_{s,z}[^\circ]$	10.5	14.5	13.5	
$A_{s,x}[^{\circ}]$	39.5	42	44	
$q_{w,y}[^{\circ}]$	2.3	-5.5	-9.67	

Table 6.2: Parameters for the three representative limit cycles studied in more detail, corresponding to one unstable and two stable flight regimes, respectively, at a forward flight velocity of $14ms^{-1}$.

matic parameters governing wing movements. Moreover, with lower tail opening angles, the CoT gradient is lower and less sensitive to changes in kinematics. Figure 6.2(b) illustrates the stability transition that occurs as a function of the tail opening. The bifurcation point happens for a value around $\beta = 25^{\circ}$. The largest Floquet multipliers in the stable region are however never smaller than about 0.96, corresponding to a largest stable Floquet exponent being equal to (see Equation (4.15))

$$\lambda_{max} = -0.16s^{-1} \tag{6.11}$$

Solving Equation (4.15) for *t*, the time taken for halving a perturbation is therefore

$$t_{half} = 4.2s \tag{6.12}$$

which corresponds to about 17 flapping periods.

As revealed by the quasi-horizontal stripes of uniform colors in Figure 6.2(b), the shoulder amplitude has a marginal effect on stability — because of its marginal role on the distribution of nose up/down pitching moment — in contrast to the tail opening and sweep angle.

6.6.2 Comparison between stable and unstable limit cycles

In this section, three representative limit cycles are further investigated: one corresponding to a tail completely furled ($\beta = 0$) and the other ones to a tail opening of $\beta = 40^{\circ}$. These reference limit cycles are selected to have the same resulting forward flight velocity, i.e. $14ms^{-1}$. The whole set of corresponding parameters is reported in Table 6.2.

The free-body diagram of these configurations is illustrated in Figure 6.3top panel. The actual pitching moment characterizing the limit cycle solu-

Chapter 6. On the role of tail in stability and energetic cost of bird flapping flight



Figure 6.3: Characterization of three representative limit cycles: one with furled tail (a), and two with a tail opened with an angle $eta\,=\,40^\circ$ (b, c). The upper panel represents the free-body diagram of the three different flight configurations. **Case (a):** The pitching moment (black solid line) is only due to the wing movement, and averages at zero. This flight regime is characterized by an unstable mode, highlighted by an eigenvalue larger than 1 (inset middle panel). The bottom panel gives the CoT for this flight configuration. Case (b): The average pitching moment M_w due to the wing lift (L_w) is negative (nose down) and the average moment due to the tail lift, M_t , is positive (nose up). The time variation of M_w is illustrated with the black solid line, and the time variation of M_t is illustrated with the red solid line. This solution is stable as all the eigenvalues are smaller than 1 in absolute value (inset middle panel). The CoT is quantified in the bottom panel, and the tail dissipation is highlighted in red. Case (c): The average pitching moment due to the wing lift (L_w) is positive (nose up) and the average moment due to the tail lift is negative (nose down). The time variation of M_w is illustrated with the black solid line, and the time variation of M_t is illustrated with the red solid line. This solution is stable as all the eigenvalues are smaller than 1 in absolute value (inset middle panel). The CoT is quantified in the bottom panel, and the tail dissipation is highlighted in red.

tions is reported in Figure 6.3-middle panel. In case (a) (furled tail), the wing must generate an equal amount of nose-up and nose-down moment to guarantee the existence of a limit cycle. Both open-tail configurations, corresponds to different configurations of momentum equilibrium. In case (b), the wing contributes for nose-down moment (on average) balanced by the nose-up moment (on average) of the tail. Conversely, in the case (c), the wing contributes for nose-up moment (on average) balanced by the nose-down moment (on average) of the tail. All these configurations exhibit a similar limit cycle regarding the phase space trajectory, although the former has one unstable mode in pitch stability governed by a Floquet multiplier equal to $\Lambda = 1.33$. The open tail cases are both stable since their largest multiplier has a magnitude equal to about $\Lambda = 0.96$. All multipliers are pictured in the inset plot of the middle panel of Figure 6.3.

The power required to achieve level flight in the furled tail case (a), averaged over one wingbeat cycle, is equal to 15.4W. In case (b), it is equal to 16.7W with a contribution to to the tail-parasitic drag of about 0.4W, while in case (c) it is equal to around 17.9W with a power dissipated by drag-induced forces in the tail of about 0.4W. This power assessment is pictured adimensionally in the bottom panel of Figure 6.3, where the red stripes corresponds to the power dissipation from the tail. There is thus a trade-off between robustness to perturbations — characterized by passive stability — and performance — characterized by the required mechanical power.

These three representative limit cycles have been perturbed by an upward gust along the local z'-axis. The gust is modeled as a Gaussian signal w_g in the form:

$$w_g = -w_0 \exp\left(-\frac{1}{2} \left(\frac{t-t_o}{\sigma}\right)^2\right) \hat{\mathbf{e}}_{z'} \quad \forall t > 0 \tag{6.13}$$

with $t_0 = 0.25s$ and $\sigma = 0.05s$. The intensity of the gust varies, in order to observe comparable effects in phase space. In the unstable case (a), $w_0 = 0.1ms^{-1}$, whereas in the stable cases (b) and (c) $w_0 = 1ms^{-1}$. The dynamic response of these three configurations is captured by the black solid lines in Figure 6.4. Figure 6.4(a) shows a quick separation from the limit cycle condition (red curves), driven by the unstable Floquet multiplier. Figure 6.4(b) and (c) show passively stable responses to the perturbation as all the Floquet multipliers are smaller than 1 in both cases. This attraction is dominated by two characteristic times, depending on the absolute value



Chapter 6. On the role of tail in stability and energetic cost of bird flapping flight

106

of the Floquet multipliers. A rapid response happens for w and q, while a slower response resembling a phugoidal mode [1, 84], with period of about 8*s* characterizes the trends of u and θ .

6.6.3 Trade-off between stability and energetic consumption

Figure 6.5 further illustrates a trade-off between stability and CoT. Figure 6.5(a), illustrates the lowest achievable CoT in the explored parametric space as a function of the forward flight velocity. This is represented at four different values of tail opening $\beta = [0, 15, 30, 45]$ deg. The minimum of the four curves is around 0.085 and corresponds to forward flight velocities of approximately $11ms^{-1}$. The steepness of the curve at increasing velocities monotonically increases with the tail opening. For the same values of β , Figure 6.5(b) reports the Pareto front of the largest Floquet multiplier Λ and CoT. This front captures the optimal solutions for which one of these features could not be more favorable without negatively affecting the other. The transition between stable and unstable flight regimes is highlighted by the vertical purple line.



Figure 6.5: Trade-off between CoT and stability. (a): Lower envelope of the CoT as a function of the forward flight velocity of four equally-spaced β -planes. We report all the possible solutions, from which the lower envelope is extracted, with transparent points, colored accordingly with the respective tail opening. (b): Pareto front of the CoT as a function of the largest Floquet multiplier of four equally-spaced β -planes. We report all the possible solutions, from which the Pareto front is extracted, with transparent points, colored accordingly with the respective tail opening. The stability transition is highlighted with the purple vertical line.

6.7 Discussions and Conclusion

We performed a four-dimensional bifurcation study in the parametric space of flapping gaits. Our numerical analysis highlights the existence of two sets of solutions as a function of the tail opening, with their respective stability properties. Figure 6.2 shows that steady leveled flight can be achieved for a large set of parameter combinations. Such combinations have to balance the pitching moment generated by the wing and the tail. This condition is mainly driven by the sweep offset of the wing and the angle of tail opening. Both of these parameters, indeed modulate the distribution of nose-up and nose-down moment, and further play a fundamental role in the limit cycle stability. The shoulder amplitude only marginally affects stability, confirming the results of Chapter 5. This is due to the fact that it does not have an effect in moving the aerodynamic forces forward or backward — on average — with respect to the center of mass, and thus in altering the pitching moment distribution.

Two profiles of pitching moment that guarantees a passively stable limit cycle have been found. One configuration is similar to those guaranteeing static pitch stability in aircraft and in bird gliding [84, 122], i.e. with the wings generating nose-down moment on average, and the tail generating nose-up moment on average (Figure 6.3(b)). The second configuration guaranteeing stable limit cycles produces a nose-up moment on average with the wing, and nose-down moment with the tail. These two stable configurations previously discovered for gliding [124], are shown here to also apply to flapping of medium to large-size birds. In [35], Smith stated a biological intuition that birds lost the capacity to rely on passively stable configurations while developing sensory-driven neural circuitries to actively control their flight over the course of evolution. However, this has been recently challenged for gliding flight [22, 122]. It was shown that in gliding regimes, birds can modulate the elbow sweep to achieve passive stability. Here, we extend this promising thesis also for flapping regimes, showing that passive stability can also be achieved with appropriate wing kinematics, and tail opening. However opening the tail comes with an important additional energetic cost. A power analysis revealed that this additional energetic cost is due both to overcoming the extra drag produced by the tail, but also to the intrinsic efficiency of the adopted wing kinematics leading to the same flight velocity. Figure 6.3 indeed shows that the extra power required to operate with the open tail conditions is not only due to

drag forces produced by the tail itself, but also to a more costly kinematics adaptation of the gait to level the flight.

Stability of these three conditions has been analyzed under a Gaussianlike gust perturbation. The unstable case shows a quick separation from the limit cycle condition. Interestingly, in the stable solutions, two characteristic times appear: a fast mode of response, affecting the variables w, q, and a slow phugoid-like mode that affects u and θ .

The trade-off between stability and energetic performance is highlighted in Figure 6.5. The lower envelope of the CoT shows a monotonic increase of steepness with the tail opening. For forward flight velocities of $14ms^{-1}$ the saving in terms of CoT between a furled tail configuration and a full open tail is of about 10%. This is comparable to energetic advantages that are estimated to make a difference between solo and formation flight, according to [39]. Indeed, having low steepness in the CoT curves is crucial for long range flights, as it allows to modulate the velocity at a lower energetic price. Figure 6.5(b) shows the Pareto front of the CoT with respect to the largest Floquet multiplier. The Pareto front of the stable solutions (black and green) is very steep, suggesting that little advantages of stability gain comes with a disproportionate energetic cost. We infer the existence of a close interplay between stability and energetic cost of flapping for medium to large size birds in steady flight. Indeed, whereas the absolute minimum CoT only marginally changes as a function of the tail opening, the steepness of its variation with respect to the forward velocity does vary as a function of this angle. Said differently, the typical U-shaped curve characterizing the CoT as function of the forward flight velocity is found at each tail opening angle, but their asymptote is lower for smaller angles.

We conducted our study with a rigorous framework to study steady flight stability, i.e. Floquet theory combined with a multiple shooting algorithm. We concluded that in spite of the gain in stability, having a tail-like surface determines an increase of steepness of the CoT with respect to the forward flight velocity, that limits the authority of the bird to modulate the flight speed. This suggests an explanation for the field observation that birds flap with furled tail in long flights [89], i.e. that a loss of dynamic stability might be traded-off in exchange for freedom of modulating velocity at lower energetic expenditure. This might be a crucial factor, for instance, in seasonal migrations, where the time of arrival at foraging, breeding and wintering sites is naturally constrained by environmental factors such as daylight duration, food availability or social reasons. However, our results show that birds still have the authority to select passively stable modes — i.e. with an open tail — that may prevail in certain circumstances, such as flying while sleeping [133].

To increase our model fidelity, it will be necessary to account for other morphological and biological elements that may contribute to stabilize the flight. These would need a substantial adaptation of the equations of motion used in the current framework. In particular, the current version of our model assumes rigid, imposed, and fixed kinematic actuation of the wings, and a rigid tail. This should be relaxed in a more bio-compatible version of the model, that should account for the intrinsic joint compliance due to actuation by muscle-tendon units. A proposed extension of our framework is introduced in the next Chapter, where the fidelity of the model is augmented via a multi-body approach.

Chapter 7

A multi-body approach for bird flapping flight

7.1 Introduction

The framework described until now, fully relies on the equations of motion in the form of (3.1). These equations have two driving assumptions: (i) the wing kinematics is imposed; (ii) the wing inertial effects are neglected. Although these assumptions are justified with previous works [43, 75, 82], having a model that embeds the dynamics of the wing can give the possibility to raise new relevant research questions. This could open the perspective of having a more realistic description of joints actuation, and quantifying the effect of the wing inertia in flight stability, which up to now still represents an open question [43].

Analytical derivations of the equations of motion accounting for the wing dynamics, are available for insect scales[76, 134]. Given their anatomy, each insect wing can be modeled by adding a single rigid segment connected to the main body, to accurately describe the biomechanics. However, bird wings are more articulated. To represent their dynamics, three additional rigid bodies have to be added — corresponding to each wing bone (Figure 1.7) — together with the related joints.

Our wing motion is composed of three rotational joints at the level of the shoulder, one rotational joint at the level of the elbow, and two rotational joints at the level of the wrist. From a dynamical perspective, each wing thus consists of six (6) degrees of freedom, to be added to the three (3)

degrees of freedom of the body motion on the longitudinal plane. Deriving by hand the equations of motion of this system is intractable, and it is meaningful to automatize this process. Within this context, symbolic software packages can be used to workaround the problem. To improve the fidelity of our model, accounting for the wing dynamics, we leverage on Robotran. Robotran is a generator of kinematics and dynamics equations for multi-body systems[45, 135]. It takes as input the topology of polyarticulated bodies, connected via different types of joints (rotational, prismatic), and generates compact sets of Newton-Euler equations of motion in symbolic formalism. These equations can be written in different programming languages (C, Python, or Matlab). It finally permits to simulate the multi-body system, i.e. predicting the time evolution of each state variable, via numerical integration. Several works addressing different locomotion problems have already been tackled with Robotran, among others it was previously employed to study human locomotion, fish locomotion, and recently also a parallel work on bird flight [65, 136, 137].

Previous efforts of multi-body approaches tailored at studying bird flight are also present in literature. An early attempt was proposed by Grauer and Hubbard to study the flight of an ornithopter [138]. Although this model was able to capture wing inertia and tail effects, it failed at describing a realistic flapping gait. A recent study published by Shen et al. [139], derived multi-body equations for a bird-scale flapping flier. The model was composed of a poly-articulated wing comprising the shoulder and elbow joints, a tail and a rudder. The stability of this system was studied via Floquet theory. Although this represented a step forward in the field, the flapping gait remained imposed. Importantly, Colognesi et al. proposed a high fidelity multi-body approach for flapping birds [65]. This work aimed at the development of a longitudinal controller to stabilize the flight, and to characterize the wake topology via a state-of-the-art vortex particle mesh method [140].

The main objective of this Chapter is to drafting up the framework for the design of a whole wing-body dynamical model of bird flight. To accomplish this task, we thus describe the interface of multiflap with Robotran. This would allow to add levels of complexity in the biomechanical model of bird flight, while maintaining the careful limit cycle formalism to study its stability. The Robotran architecture is thus presented, explaining the required classes to perform the numerical integration of the multibody system. It follows the modification of the workflow to let Robotran communicate with multiflap. We therefore explain in more details the modules that have to be added in order to solve the equations of motion exploiting the multiple-shooting algorithm.

To conclude the Chapter, we further report the first validation of a new multi-body model to describe the biomechanics of the bird. We compare the results obtained within the new Robotran framework, against a reference limit cycle previously found with the Equations (3.1).

7.2 The multi-body model

In the current model, the wings move according to a prescribed kinematics. Such an approach — although allows to reproduce various flapping gaits with good accuracy — has several limitations. Among others, it cannot permit to implement the building block of a bio-inspired actuated wing, and does not allow to study the so-called morphological adaptations for responding to environmental disturbances. These morphological adaptations, together with compliant phenomena, are observed to be ubiquitous among living systems, and we envisage that they could play a role in stability of bird flight, and being of direct interest within the context of this research.

In order to add this layer in the current framework, it is inevitable to re-derive the equations of motion of the flier, accounting for the polyarticulated elements composing the anatomy of the wing. However, the derivation of this new set of equations, given the complexity of the wing apparatus, can be very challenging by hand, and a very risky error-prone process. In this work, we present a multi-body approach to study the dynamics of flapping flight, based on Robotran environment [45]. Robotran is a multi-body generator and solver of equations of motion. In a first step, the multi-body system is topologically defined (bodies, connections, masses, etc.), and then in a second step, the equations of motion are symbolically generated.

Having a multi-body model offers several advantages. It has high versatility, which permits to implement auxiliary models, such as constitutive equations for spring-mass systems, and to be coupled with our lifting line solver. Moreover, modifications in the multi-body model, such as adding new bodies, or changing natures of the joint, can be made with no additional computational time and effort for a new derivation of the equations of motion.

Several formalisms can be adopted to derive equations of motion for a

multi-body system. The preferred one within Robotran environment is the semi-explicit formulation based on Newton/Euler formalism, also called *direct dynamics*. Given a set on independent coordinates *q*, the direct dynamics computes the joint acceleration \ddot{q} for a given pair (q, \dot{q}) . The equations of motion read in the form

$$M(q,\delta)\ddot{q} + c(q,\dot{q},\delta,frc,trq,g) = Q(q,\dot{q})$$
(7.1)

where $M[n \times n]$ is the symmetric mass matrix of the system; $c[n \times 1]$ is the nonlinear dynamic vector, that contains the gyroscopic, centripetal, Coriolis, and gravity terms, and the contribution of external forces (*frc*) and torques (*trq*); δ [10 $n \times 1$] contains the dynamic parameters of the system (body masses, centers of mass, and the inertia matrices); $Q[n \times 1]$ represents the generalized joint force or torque.

Robotran not only handles the generation of the equations of motion, but also provides the solver for integrating them. However, it is important to re-call that in our work we look for steady state flapping flights, which correspond to limit cycle solutions of the equations of motion. In order to stick to the careful limit cycle formalism, our development effort is to adapt the communication between Robotran and multiflap environments.

It is the goal of this Section, to provide the reader with a description of the new multi-body model, detailing its topology. We then report the Robotran workflow aimed at integrating the generated equations of motion, and finally we propose a modification of such workflow, in order to treat these equations with the limit cycle formalism developed to study bird stability.

7.2.1 Bird topology

This section describes the skeletal topology of bird model. This is created via the MBsysPad environment of Robotran, and illustrated in Figure 7.1.

The topology consists in a bird main body, and two wings. Each wing is composed of three rigid bodies to biomimic the bird anatomy, and the entire system is symmetric with respect to the body longitudinal plane. Exploiting this symmetry, the right part only of the bird topology is here described.

In order to reproduce the system of Equation (3.1), the Robotran model is initially constrained to fly only on the longitudinal plane. Thus, the bird main body has three degrees of freedom. The two translations in x and z





directions, are identified by the prismatic joints $T_{1,x}$ and $T_{2,z}$ respectively. The rotation about the *y*-axis, i.e. the body pitch, is identified by the rotational joint $R_{3,y}$.

The segment of the wing representing the humerus, is anchored to the bird main body through the shoulder. This joint consists of three rotational degrees of freedom, about x, y, and z-axes respectively. These rotations are identified by $R_{4,x}$, $R_{5,y}$, and $R_{6,z}$.

The segment of the wing representing the ulna and radius unit, is anchored to the humerus via the elbow. This joint consists of one rotational degree of freedom about *z*-axis. This rotation is identified by $R_{7,z}$.

The segment of the wing representing the metacarpals, is anchored to the ulna and radius unit via the wrist. This joint composes of two rotational degrees of freedom about x and y-axes. These rotations are identified by $R_{8,x}$ and $R_{9,y}$.

On each wing is applied the system of external forces and moment due to the aerodynamics. Each external force is identified with dF_i and computed with the quasi-steady lifting line model described in Chapter 3.

7.2.2 Robotran workflow

In order to run a direct dynamics simulation of a multi-body system with Robotran, three steps are required (Figure 7.2(a)).

The topology of the system represents the first step of the workflow. Once defined, all of its information are automatically stored in a .mbs file.

The second step is the generation of the equations of motion. The symbolic generator reads the .mbs file, and outputs the equations of motion of the system in a symbolic formalism. In the current case, it is explicitly asked to output these equations in Python language. Simultaneously, the working tree of the case study is also automatically created by the software. It organizes the different modules in dedicated sub-folders as illustrated in Figure 7.2(b).

The third step is to simulate the physics of the system by integrating the equations of motion. This is done by running the main.py file automatically generated by the software, inside the workR directory. We here omit the description of all the other modules and functionalities, redirecting the reader to the Robotran documentation for a thorough explanation [141].

Two classes are called inside the main.py file, in order to perform the integration of the equations of motion.



Figure 7.2: (a): Workflow of a Robotran case study. The multi-body topology — stored in a .mbs file — is read by the symbolic kernel, and the symbolic equations of motion are generated. These equations of motion and the system data are loaded by the main.py file, which can perform the time integration via the Robotran class MBsysPy.MbsDirdyn. (b): Working tree structure automatically created by Robotran.

- MBsysPy.MbsData: This class reads the multi-body topology and converts it into a Python instance object, mbs_data. The system data previously defined are accessible as attribute of this instance object. Importantly, the initial conditions of each state variable can be also accessed here.
- 2. MBsysPy.MbsDirdyn: This class takes as argument mbs_data in order to build the direct dynamics object. At this point the numerical integrator can be called as a method.

The steps just described, are also presented in the form of a candidate main.py file in Appendix D (Listing D.1).

7.2.3 Interfacing multiflap and Robotran

multiflap acts as an external module, i.e. none of its functions are encoded in the current release of Robotran. However, the architecture previously presented is slightly re-adapted for the communication of the two software (Figure 7.3(a)).



Figure 7.3: (a): Workflow of a Robotran case study. The multi-body topology, stored in a .mbs file is read by the symbolic kernel, and the symbolic equations of motion are generated. The main.py file calls the multiple-shooting module, which performs all the needed numerical integration to set up the multiple-shooting scheme of Equation (4.21) or (4.25). This scheme is then solved by LM algorithm, which also iterates over the initial values. (b): Working tree structure automatically created by Robotran.

The first two steps of the workflow, i.e. the topology definition and the symbolic generation of the equations, remain unchanged. In the usual working tree, the multiple-shooting files are manually added inside workR directory, as illustrated in Figure 7.3(b). More specifically, the required files are:

- multiple_shooting.py: it builds the system of Equation (4.21) or (4.25). Therefore, this module loads the Robotran class MBsysPy.MbsDirdyn in order to perform all the required numerical integrations.
- lma_solver.py: it contains the Levenberg-Marquardt algorithm for solving Equation (4.27).

We here explain the communication of the two software. The main.py still calls MBsysPy.MbsData to transform the .mbs file into the mbs_data object instance. At this stage, mbs_data is passed as argument to the MultipleShooting class, in order to construct the multiple-shooting object. By doing so, all of the MultipleShooting functions can: (i) access the topology of the system generated by Robotran; (ii) perform the required numerical integration to build Equation (4.21) or (4.25) via the MBsysPy.MbsDirdyn module of Robotran. Finally, the multiple-shooting object is passed to the LMA module, which solves Equation (4.27), and updates the initial values if required.

A candidate modified main.py file to run a multiple-shooting simulation in Robotran is presented in Appendix D (Listing D.2).

7.3 Validation method

Before raising new research questions within the new framework, we check its consistency by verifying the results against a previous detected flapping flight regime, i.e. the limit cycle corresponding to the level flight regime identified in Chapter 5.

For the sake of validation, all the joints of the wings are set as *driven variables*. Their motion is thus forced to obey the kinematics law described by Equation (3.2), with the parameters reported in Table 3.1b, except the shoulder amplitude, which is set for the value that guarantees the level flight condition $A_{s,x} = 43.47^{\circ}$ (Figure 5.10). Moreover, the mass of each wing element is set to a $m_w = 10^{-5}kg$, in order to minimize the inertial effects.

7.4 Results of the validation

Under the aforementioned conditions, in Figure 7.4 is reported the comparison between the limit cycle detected with the multi-body dynamic approach whose equations of motion are generated by Robotran (green scatter points), and the level limit cycle found with Equation (3.1) (black solid line). We report the comparison between the initial values of these two limit cycles in Table 7.1.

The curves follow the same behavior for each state variable, even if a small discrepancy in the results is observed. We attribute the main reason of this deviation to possible inertial effects that the wings — although with a very low mass — may produce on the whole dynamics. Indeed the acceleration and deceleration within the flapping period can produce residual forces and moments that may justify the offset we observe. However, we



Figure 7.4: Comparison between the limit cycle obtained with a multi-body approach via integrating the equations of motion generated by Robotran (green scatter points), and the 4 states model of Equation (3.1) (black solid line). This limit cycle corresponds to the level flight regime of Chapter 5.

Table 7.1: Comparison of the initial values of the two limit cycles. The first one is obtained via the Equation (3.1), the second one is obtained with a multibody approach, via integrating the equations of motion automatically generated by Robotran.

Initial value	Equation (3.1)	MB approach		
u(0) [m/s]	18.511	18.516		
w(0) [m/s]	-2.110	-2.102		
q(0) [rad/s]	-0.118	-0.120		
$\theta(0)$ [rad]	-0.1164	-0.1161		

consider that the results obtained via this multi-body approach are consistent to previous results obtained via integrating Equation (3.1) in terms of limit cycle detection.

The second validation is presented in terms of the Floquet multipliers associated with these flight regimes (Figure 7.5). The eigenvalues corresponding to the multi-body system are reported in red scatter points. The eigenvalues corresponding to Equation (3.1) are reported with black crosses. Also the locus of the Floquet multipliers is very consistent with



Figure 7.5: Comparison of the Floquet multipliers obtained with the multibody approach (red scatter points) and with the 4 states system of Equation (3.1) (black crosses).

the ones detected solving Equation (3.1).

7.5 Conclusion

The work reported in this Chapter is aimed at improving the accuracy of the current framework, via introducing a multi-body approach to study the whole body-wings dynamics of the bird. In order to achieve this task, we rely on the computational work of Robotran environment, a multi-body generator of equations of motion. Moreover, in order to detect the steadystate of such equations, we coupled Robotran with multiflap to maintain the limit cycle formalism.

We first described the Robotran workflow in order to study the direct dynamics of multi-body systems. Then we showed the required modifications of this workflow, in order to solve the multi-body equations of motion using the multiple-shooting algorithm. In addition to the file generated by Robotran, two multiflap modules need to be added in the working tree. These modules allow to: (i) build the multiple-shooting scheme of Equation (4.21) or (4.25); (ii) solve the multiple-shooting scheme via the Levenberg-Marquardt algorithm.

A first multi-body case study is then presented. It represents the numerical validation of the results obtained with the new framework, against a previously detected flight regime. To this end, the multi-body topology of the bird is created. Each wing is composed of three poly-articulated rigid bodies connected via rotational joints. For validation purpose, we restricted the multi-body system to fly on the longitudinal plane, and we imposed the wing joints to follow the driven kinematics described in Table 3.1b. In addition, we set the shoulder amplitude to match the condition of level flight found in Chapter 5. To reproduce the previous assumption of neglecting the wing inertia, we drastically reduced the wing masses compared to realistic scenarios. The validation of the multi-body framework shows a very good agreement with the flight configuration detected via solving Equation (3.1), thus opening room for adding new levels of complexity in the equations of motion.

Chapter 8

Modeling joint compliance

8.1 Introduction

The biological motor of the wing is the muscle-tendon apparatus. Muscles receive a signal from the nervous system (central of peripheral), and as a response — at a macroscopic scale — they contract, thus developing forces that are transmitted by tendons in order to actuate the skeleton [142]. Vice versa, when the skeleton is excited by external stimuli, muscles react in order to control the joint [20, 143]. The muscle-tendon unit is a compliant biological element, which is also able to store energy from the environment and to release it in form of mechanical power.

We here raise the question of how a compliant joint could influence the stability properties in flapping flight. This capacity of vertebrates in passively reacting to external stimuli is well observed in nature, and among others it is also one of the driving mechanism of gust rejection in owls in gliding regime [28]. In vision of increasing the fidelity of our model, the wing motion should be therefore embodied by such a compliant element, capable to capture at different levels of complexity the muscle-tendon unit properties. Put differently, the position and angular velocity of a certain joint will not any longer be purely prescribed but will also depend on the external system of forces and moments acting on the wing.

There are two different documented approaches in order to model muscle-tendon units, a bottom-up and a top-down. The bottom-up approach consists in describing the muscle contraction through the so-called *sliding filament theory*, which captures the dynamics at the level of my-



Figure 8.1: Structure of the skeletal muscle. Muscles are composed of fascicles which are bundles of muscle fibers. Each muscle fiber is composed of organized structures called myofibrils. Myofibrils are groups of elementary contractile units called sarcomeres. This contractile unit is actioned by myofilaments, composed of two main proteins, namely actin and myosin.

ofilaments up to the generation of the resulting force of the muscle fascicle¹ [144, 145]. An illustration of the biological structure of skeletal muscles is reported in Figure 8.1.

The top-down approach — also known as Hill's phenomenological model [146] — aims at capturing the macroscopic effect of forces generation and the compliant effects combining active and passive equivalent elements (such as spring and dampers). A simplified version of this last approach is the one privileged in the context of this work, because it is much more computationally efficient in order to be coupled with locomotion phenomena [142].

In this Chapter we thus propose the implementation of a compliant joint in the bird model. In order to achieve this task, we rely on the extended multi-body framework described in Chapter 7 for capturing the dynamics of the wing, and for implementing the new compliant feature. In particular, we here act at the level of the shoulder joint, on the degree

¹Myofilaments are protein structures composed mainly of actin and myosin. These are grouped in structures called sarcomeres, that constitute the elementary contractile unit of the muscle. Bundled units of sarcomers form the so-called *myofibril*. Organized units of myofibrils give raise to the so-called *muscle fiber*. A muscle fascicle is the result of bundled muscle fibers.

of freedom responsible for the flapping amplitude, $q_{s,x}$. Importantly, in this contribution we do not aim at augmenting the fidelity of the actuation. The wing motion is still driven by governing kinematics functions. We rather introduce a passive spring-damper element for capturing an equivalent passive compliance of muscle-tendon apparatus, which will affect the resulting dynamics of the wing.

The Chapter is structured as follows. It first introduces the joint model and its constitutive equation. Then the modifications of the multi-body system are reported. Finally, we show the preliminary results of this investigation, stressing unstable case configurations, in order to seek changes in stability via modulating the joint compliance properties.

8.2 Compliance of the shoulder articulation

In birds, the two muscles responsible for flapping the wings are the pectoralis and the supracoracoideus [20]. This biological configuration, mostly governs the actuation of the shoulder joint, which for such a reason is the major driving articulation for powering the flight.

Motivated by this reason, we are interested in developing a more biocompatible version of the shoulder joint model, that aims at capturing the passive nature of the muscle-tendon unit properties. Importantly, here we do not focus on improving the joint actuation in a more biological way which is the primary function of the muscle-tendon unit — but we solely focus on the passive reaction of the articulation subject to external forces and moments.

In the present model, we account for the joint flexibility via introducing a torsional spring-damper element, between a driven actuation and the joint, as pictured in Figure 8.2. Interestingly, such a type of joint model is also largely used in robotic manipulators, in order to account for the flexibility of the driving elements constituting the transmission (gears, bearings, etc.) [147].

Considering Figure 8.2, the driving actuation is prescribed via imposing the pair (φ_k , $\dot{\varphi}_k$), namely the driven position and the driven angular velocity respectively. The time varying displacement is introduced by the spring-damper unit which aims at representing an equivalent elastic and dissipative effect of the muscle-tendon unit via the constants (k, C), namely the spring stiffness and the damping coefficient respectively. The torque $Q_{s,x}$ on the shoulder joint along the local *x*-axis, is thus governed by the



Figure 8.2: Schematic of the compliant joint. The driving motion is prescribed by imposing φ_k , $\dot{\varphi}_k$, and the resulting dynamics is solved accounting the spring-damper element.

constitutive equation [94, 148]

$$Q_{s,x} = -k(\varphi - \varphi_k) - C(\dot{\varphi} - \dot{\varphi_k}) \tag{8.1}$$

where $(\varphi, \dot{\varphi})$ represent the resulting shoulder joint position and angular velocity respectively. The Equation (8.1) for the shoulder joint torque, is accounted in the *Q*-vector in the right-hand-side of Equation (7.1). This process is automatically handled by Robotran, which via the computation of the torque $Q_{s,x}$, calculates the evolution of the new state variables φ and $\dot{\varphi}$.

8.3 Modification on the multi-body topology

In order to conduct this numerical investigation, the multi-body topology described in Chapter 7 has been modified accordingly. The first modification concerns the nature of the joint responsible for the wingbeat amplitude at the shoulder level. Referring to Figure 7.1 the joint labeled $R_{4,x}$ — and its mirror-image joint on the left side of the symmetrical plane — have been set as independent variables, responding to the constitutive Equation (8.1). The rest of the joints remain untouched, responding to the driven kinematics, and constraining the bird main body on the longitudinal plane.

The second modification regards the mass of the wing. Setting the shoulder joint $R_{4,x}$ as independent variable, requires to add realistic values of mass on each wing element, in order to avoid numerical instabilities in the computation of the joints acceleration. Each wing element has a mass of 0.01kg, so that their global mass represents the 5% of the body mass. This value has been set arbitrarily to test out the new framework, and sensitivity investigations about the role of the wing mass on the whole dynamics are left for future work. Moreover, the moment of inertia about the principal axes of each wing segment are set $I_{xx} = I_{yy} = I_{zz} = 0.01kgm^2$.

The driving elements φ_k , obey the governing joint kinematics of Equa-

tion (3.2), with the parameter listed in Table 3.1b, except for the value of the shoulder sweep offset that is increased of half degree ($\psi_{s,z} = 19.5^{\circ}$).

8.4 Results

In this Section we report a preliminary investigation of steady-state flight regimes, accounting for the compliant shoulder joint. We present two numerical experiments: (i) we show the limit cycles and the Floquet multiplier on varying the stiffness parameter k for a fixed value of damping C; (ii) we perform the complementary study by fixing the parameter k and analyzing the effect of three different values of C.

8.4.1 Effect of the spring element

We present three flight regime solutions, corresponding to three different values of spring stiffness, namely k = [10, 15, 20]Nm/rad. In the cases analyzed, the damping coefficient is fixed and taken equal to C = 0.2Nms/rad. The resulting limit cycle solutions obtained with multiflap, via integrating the multi-body equations generated by Robotran, are presented in Figure 8.3, for the variables u, w, q, and θ . These solutions are steady-state flight regimes, but in this particular case study they are not leveled.

Importantly, the flight regimes identified display higher values of forward flight velocities, compared to previous cases. This is mainly due to the increase of the global mass of the system accounting for the wing mass. Indeed this requires a higher velocity to generate higher aerodynamic loads to sustain the flight. Moreover, changes in the limit cycle solutions are observed when varying the stiffness parameter k.

For each of these three cases the Floquet multipliers have been calculated. The locus of the eigenvalues is presented in Figure 8.4. Each of the solution investigated presents one expanding eigenvalue, thus leading the system to be unstable. The value of the expanding Floquet multipliers are reported in Table 8.1. The spectrum of the Floquet multipliers is found qualitatively similar to those previously detected, and corresponding to unstable flights. In addition, here, there are two additional eigenvalues corresponding to the two new state variables (φ , $\dot{\varphi}$).

For the cases analyzed, a more compliant joint is found beneficial in terms of stability, as the largest Floquet multiplier tends to get closer to the unitary circle as the spring stiffness decreases. This can also be quantified by the time needed for a perturbation to double its value, according to



Figure 8.3: Limit cycle of the state variables u, w, q, and θ expressed in the body frame. These limit cycles are detected for three values of spring stiffness: k = 10Nm/rad (black solid line), k = 15Nm/rad (blue solid line), k = 20Nm/rad (red solid line).



Figure 8.4: Locus of the Floquet multipliers for three values of the spring stiffness: k = 10Nm/rad black cross; k = 15Nm/rad blue cross; k = 20Nm/rad red dot. Each of these solutions is governed by an unstable eigenvalue leading the system to be unstable.

Equation (4.15). In the most beneficial case (k = 10Nm/s), this doubling time is about $t_{doub} = 0.87s$, which thus corresponds to approximately 3.5 wingbeat periods. In the least favorable case encountered (k = 20Nm/s) this doubling time corresponds to about 2.2 wingbeat periods.

These flight regimes have also been characterized regarding energetic performance, by computing the average power consumption over a cycle. Since in this case the flights are not leveled, and in particular they are descending flight regimes, this actuation power has been corrected with the rate of change in potential energy, i.e.

$$p_g(t) = -mg\dot{h} \tag{8.2}$$

with \dot{h} being the rate of altitude change of the bird's center of mass. The values of power consumption are reported in Table 8.1. Moreover, this power computation does not account for the dissipation due to profile and parasitic effects, since the drag models of Equation (6.1) and (6.5) have not been implemented in this preliminary version of the extended framework. The values of the power are reported in Table 8.1, together with the comparison with the reference limit cycle of Chapter 7, in which all the joints were kinematically driven, i.e. corresponding to $k \to +\infty$.

 Table 8.1: Floquet multipliers and power consumption for the three cases of spring stiffness.

k[Nm/rad]	Λ_{max}	$t_{doub}[s]$	$\overline{P}_{act}[W]$	$\overline{P}_{g}[W]$	$\overline{P}_{corr}[W]$
10	1.22	0.87	-2.18	11.92	9.74
15	1.31	0.64	0.25	8.90	9.15
20	1.37	0.55	1.42	7.41	8.43
$+\infty$	1.40	0.51	6.58	0	6.58

8.4.2 Effect of the damping element

To complement the previous experiment, we here show three cases of limit cycle solutions, characterized by three different values of the damping constant *C*, for a fixed value of spring stiffness *k*. In particular, the spring stiffness is fixed to a value of k = 20Nm/rad, while the three values of the damping coefficient are set C = [0, 0.1, 0.2]Nms/rad. The limit cycle solution of the variables *u*, *w*, *q*, and θ is presented in Figure 8.5. These

solutions are steady-state flight regimes, but also in this case study they are again not leveled. Similarly to the observations reported in Figure 8.3, also here changing the value of the damping coefficient affects the resulting limit cycle solutions.



Figure 8.5: Limit cycle of the state variables u, w, q, and θ expressed in the body frame. These limit cycles are detected for three values of damping coefficient: C = 0Nms/rad (black solid line), C = 0.1Nms/rad (blue solid line), C = 0.2Nms/rad (red solid line).

We calculated the stability properties of these three solutions, reporting the Floquet multipliers in Figure 8.4. As it happened for the previous case, each of these solution presents one expanding eigenvalue, thus leading the system to be unstable. Although we do observe a change in the absolute value of the expanding Floquet multiplier on varying *C*, this change of Λ_{max} is very small in the two damped cases ($C \neq 0$), as reported in Table 8.2. The doubling time is also reported. The most beneficial case (C = 0.2Nms/rad) is characterized by a doubling time which corresponds of about 2.2 flapping wingbeats, whereas the least favorable case (C = 0) is characterized by a doubling time which corresponds to about 1.8 flapping wingbeats.

These flight regimes have also been characterized regarding energetic performance, by computing the power. Also in this case, the power has been corrected accounting for the rate of change in potential energy, computed according to Equation (8.2).



Figure 8.6: Locus of the Floquet multipliers for three values of the damping coefficient: C = 0Nms/rad black cross; C = 0.1Nms/rad blue cross; C = 0.2Nms/rad red dot. Each of these solutions is governed by an unstable eigenvalue leading the system to be unstable.

C[Nms/rad]	Λ_{max}	$t_{doub}[s]$	$\overline{P}_{act}[W]$	$\overline{P}_{g}[W]$	$\overline{P}_{corr}[W]$
0	1.45	0.46	-3.76	12.55	8.79
0.1	1.39	0.52	0.33	8.45	8.78
0.2	1.37	0.55	1.42	7.41	8.43

Table 8.2: Floquet multipliers and power consumption for the three cases of damping coefficient.

8.5 Current limitations

The results obtained with the extended version relying on Robotran engine to generate the equations of motion are promising, and open new avenues of research. However, the current version of the framework comes with a major limitation, namely it can only rely on the numerical computation of the Jacobian matrix, solving Equation (4.31). The main reason behind this choice is that the current release of Robotran cannot perform automatic differentiation, and thus it cannot produce a symbolic form of the stability matrix (4.10) for the resolution of Equation (4.9).

Solving the Jacobian numerically is computationally inefficient, as revealed by the benchmark test shown in Section 5.4. Moreover, as Robotran allows to unlock the degrees of freedom of the wing in order to have an augmented accuracy of the whole wing-body dynamics, this inevitably increases the number of state variables of the problem and the subsequent dimension of the Jacobian matrix to solve. As a consequence, running optimizations and parametric analysis is very time consuming, and not possible without the use of high performance computers. A direction to solve this limitation, would be to interface Robotran with automatic differentiation software, such as the previously mentioned CasADi [114], but this is not done at the current state of this research.

8.6 Conclusion

In nature, the joint movement and its control is governed by muscle-tendon units, i.e. biological apparatus that are able to transmit forces to activate the movement, and to store energy from the environment. Motivated by such biological reasons, in this Chapter we proposed a bio-inspired joint accounting for the compliance of the shoulder articulation.

We tackled such a problem using a phenomenological model composed of: a prescribed actuation; a spring-damper unit, which aims at capturing an equivalent effect of the elasticity and viscosity properties of muscles and tendons; and the actual shoulder joint, whose position and angular velocity do not depend only on the driven element, but also on the resulting forces and moments acting on the wings.

We performed two complementary experiments: (i) we fixed the damping coefficient, and we tested the system for three different values of the spring stiffness; (ii) we fixed the spring stiffness, and we tested the sys-



Figure 8.7: Schematic of an external perturbation. (a): The wing moves with a prescribed kinematics, thus its position is not affected by the system of external forces and moments. As a consequence of an upward gust, it generates an extra lift contribution ΔL due to an increase of the angle of attack. (b): The wing joint is compliant thus the actual wing position depends on the driving motion (dashed line) and the system of external forces and moments. As a reaction to the upward gust, the wing actual position moves upward, mitigating the increment of angle of attack, and thus the extra lift ΔL .

tem for three different values of damping coefficient. The former investigation, showed a clear beneficial trend in stability, observed via reducing the spring stiffness (i.e. making the joint more compliant). This trend is highlighted by the locus of the Floquet multiplier. However — similarly to what we experienced in Chapter 5 via changing the wing insert position - no stable limit cycles were detected. We indeed experienced a threshold of k = 8Nm/rad, below which no limit cycles could be found from our algorithm. The second experiment also showed a beneficial effect in adding a damping component in the equations, via a gradual reduction of the largest Floquet multiplier, although the effect is less marked. The reason why a compliant joint is beneficial for the stability, is because it dynamically responds to external perturbations in such a way to reduce the effect that these external perturbations play on the angle of attack of the wing. Let assume the wing encounters an upward gust. If the joint is completely stiff, the wing position and velocity are not affected by the external system of forces and moments, and the wing profiles experience an increase of angle of attack due to a net effect of the upward gust. This increase angle of attack turns into an additional lift component ΔL (Figure 8.7(a)). If the joint is compliant, the wing position and velocity depend on the resulting system of forces and moments. Therefore, as a consequence to the increase in angle of attack due to the gust, and so the extra lift, the wing reacts moving

133

upwards (Figure 8.7(b)). This upward motion is thus beneficial, because it is followed by a component of kinematic velocity of the wing that partially cancels the effect of the gust, and thus alleviates the consequent increase of angle of attack. Analogously, the same action on reducing the kinematic velocity of the wing — with consequent reduction on the angle of attack is also played by the damping element.

The approach proposed is found to be suitable for opening a future scenario in bird flight modeling, adding a complexity layer in the direction of the joint compliance. It could thus increase the fidelity with realistic and bio-compatible mechanism that govern the joint actuation, and could arguably play important role in the stability and control of the flight. However it is of crucial importance to state that in this preliminary investigation the cases analyzed are not sufficient to generalize these findings. A deeper investigation on the role of the stiffness and damping is thus required. Moreover, the coefficients employed at the level of the joint model do not represent realistic biological data. One of the main challenges in this respect, is thus to extract such information about the muscle stiffness and damping. An accurate estimation would likely require in-vivo investigation and experiments.
Chapter 9

Conclusions and future perspectives

9.1 Overview of the manuscript

This research aims at contributing on the topic of flight dynamics stability of flapping wings, restricting our attention to large scale birds, i.e. migratory birds.

In this work, we piece together the fundamental layers required to give rise of an accurate bird flight modeling, following a bottom-up approach. The first layer of this approach consists in a biomechanical model of bird wings. It comprises a detailed representation of poly-articulated bodies, capable to morph and fold, and to represent realistic flapping gaits.

The second layer of our framework is composed of a quasi-steady morphing lifting line solver, which captures gait dependent aerodynamics forces. The use of this model represents the foundation of our further investigations, i.e. the study of passive flight stability in flapping regime.

The third layer of our framework is composed of the dynamical model of the flier. We restricted our flight to the longitudinal plane, imposing the kinematics of the wing skeleton apparatus, and further coupling the dynamics with the aerodynamic model. These three fundamental unit of our work are reported in Chapter 3.

The ultimate layer is the development of a careful limit cycle formalism, which we described in Chapter 4. Such a formalism permits of identifying steady-state flight regimes, and assessing their stability via Floquet theory. This framework can handle parametric studies and quantify flight performance metrics, such as actuation power and cost of transport, allowing also to study trade-offs between flight stability and energetic requirements.

Our framework is then challenged in Chapter 5 and 6. In Chapter 5 we report the results corresponding to a tail-less configuration of the flier, showing that such configurations present unstable modes. We extended these results in Chapter 6, where a systematic parametric search led us to observed stability transitions associated with the opening of the tail. However an energetic analysis revealed that this gain in stability comes with a disproportionate increase of cost of transport, suggesting that birds can rely on alternative mechanism of flight control for maximizing their efficiency.

In Chapter 7 and 8 we eventually proposed an extension of such framework, leveraging the computational power of a generator of multi-body dynamic equations called Robotran in order to derive automatically the equations of motion of the flier. This extension increases the level of complexity of our model by capturing the dynamics of the wing, and thus raising new research questions that we will leave as future perspectives.

This final Chapter is structured as follows. It first summarizes the major scientific contributions observed during this investigation, and discusses the limitations and potential improvements of the current framework. Then, it reviews the scientific questions developed in Chapter 1 in light of the findings and limitations. The last part presents possible perspectives and research ideas for future work.

9.2 Discussions of the key findings

This Section reports the major contributions achieved by this Thesis, with a specific focus on discussing their current limitations.

9.2.1 Wing model and aerodynamics

The proposed model is inspired by the anatomy of bird wings, and mimics the articulations via the same degrees of freedom as those of birds skeleton. Although this model captures realistic flapping gaits, it is based on driven kinematics at the level of the wing joints. This assumption fails in describing the dynamic nature of the wing, and the related mechanisms of adaptations for responding to environmental perturbations (such as gust alleviation, wake exploitation, etc.). Possible improvements of this model are preliminarily explored in Chapter 8 and discussed in the next Section.

The quasi-steady lifting line model allows to compute the aerodynamic forces and moments while capturing the wing morphing of the bird. However, its first limitation is about the aspect ratio of the wing. Lifting-line theory requires high aspect ratio (AR>7) to accurately estimate the forces. This restricts its application to large bird species that satisfy this condition, but it unsuccessfully predicts forces for birds of smaller scales.

The quasi-steady assumption applies under the condition on the reduced frequency [19, 43]. It is therefore accurate for fast forward flight regimes, but the unsteadiness of the wake should be accounted for if a lower range of velocities is to be explored. We tackled fast forward flight scenarios of migratory birds. The velocities obtained from our simulations vary depending on the kinematics, but they are never below about $10ms^{-1}$ and never above about $18ms^{-1}$. These regimes can be addressed with quasi-steady assumption, and they are aligned with observations [119, 149]. Conversely, for higher values of reduced frequency — corresponding to lower values of forward flight velocities — the unsteady behavior of the wake is necessary to be accounted for. In such cases, effects such as the so-called leading edge vortex are main responsible of lift generation, and therefore more sophisticated methods looking at the boundary layer separation are needed [150, 151].

Moreover the feathers in the current model are rigid bodies that follow a prescribed kinematics. This feature is used to define the wing envelope (from which the lifting line is extracted), but they do not play a structural role. In reality, feathers deform under the aerodynamic loads [152]. This aeroelastic problem is not tackled in the current version of the aerodynamic model.

An important aspect is the one of the validation of the aerodynamic model. Tailored experiments might be designed for such a purpose, and two possible strategies are here discussed. The first experiment could be validating the aerodynamic model against an *in-vivo* experiment, by measuring accelerations and forces on flying birds. However, this experiment implies the stringent condition of replicating — *in-silico* — the same wing kinematics of the *in-vivo* case. A second strategy is the design and implementation of a mechanical robot with flapping wings, which have all the necessary degrees of freedom at the level of the shoulder, elbow, and wrist. This experiment would be accurate enough to produce the exact wing kinematics between the robot and our virtual environment. A wind tunnel val-

idation could then be performed, in order to purely test the aerodynamic performance, without further complications related to the body-wing dynamics. Similar experiments have not been found in literature to allow us to validate our aerodynamic model.

The quasi-steady lifting line solver was compared with the vortex particle mesh method, in order to quantify the effect of the wake unsteadiness in fast forward flight regimes, and results shown in [153] justify the quasisteady approximation for studying our dynamical problems.

Although these limitations leave room for future improvements, we consider our wing and aerodynamic model a first contribution to the current state-of-the-art, in order to study flapping flight dynamics.

9.2.2 Development of the multiple-shooting algorithm

In the pursuit of answering the research questions related to bird flight dynamics, the multiple-shooting algorithm has been developed. It allows to detect limit cycle solutions and to assess their stability according to Floquet theory. The accuracy of the algorithm in identifying limit cycles and computing the Floquet multipliers has been validated against previous works found in literature, and reported in Appendix A. The algorithm has been released as a Python toolbox, which is openly accessible, provided with documentation, and recently employed by other users to detect limit cycle for different purposes [110].

At the current development stage, multiflap has two main limitations: (i) it does not perform symbolic differentiation; (ii) it is not parallelized. The first point creates major issues when the ODE system is large, thus making complicated for the user to hard-code the stability matrix. This problem has been encountered also in this project, within the context of the Robotran framework. In all these cases, the computation of the Jacobian matrix is preferred to be numerical, paying higher computational costs.

The limitation concerning the parallelization has not been addressed in this work. This would drastically speed up the computational time when multiple points are used in the simulations. A potential solution, could be parallelizing in such a way to assign each processor to each point of the multiple-shooting scheme.

9.2.3 Limit cycle approach for flapping flight dynamics

The biomechanical model of the wing and the multiple-shooting algorithm are used to detect steady-state flapping flight regimes. Two main studies were conducted and documented in Chapter 5 and 6 respectively. These results build upon the Equations of motion in the form of (3.1).

In the study case presented in Chapter 5, the morphology of the bird captures the aerodynamics effects from the wing surface only. Under this condition, we observed only passively unstable flight regimes. However, we performed two important sensitivity cases. The first one was to level the flight with wingbeat amplitude. We observed that the Floquet multipliers were only marginally affected by this parameter.

The second experiment was to change the wing insertion point with respect to the body center of mass. We observed that the expanding Floquet multiplier gets significantly smaller as the wing approaches the center of mass, suggesting that the pitching moment clearly affects the stability behavior of the flier.

This study suggested to explore a broader range of parameters and morphological scenarios, that are consequently reported in Chapter 6. Each solution of the parameter space was then analyzed in terms of performance. Results highlighted a trade-off between passively stable solutions obtained via introducing a tail-like surface, and the energetic expenditure. This finding, potentially explains observation of long range flight with furled tail, suggesting alternative mechanisms of flight stabilization developed through evolutionary process.

Both works presented in the two mentioned Chapters, are driven by important assumptions. Equations (3.1) apply only to study longitudinal dynamics of the flight, under the assumption of massless wings. Our model-based predictions, therefore, do not capture the effect of wing dynamics and its inertia on flight stability. Including these effects could allow to investigate how a passive compliant joint could affect flight stability. In order to account for the wing dynamics, the equations of motion need to be re-formulated accordingly.

9.2.4 Compliance of wing articulations

In Chapter 7, we proposed a multi-body approach to study flapping flight dynamics. This research direction aims at augmenting the fidelity of the model, thus capturing the dynamics of the wing. This effect is indeed cru-

cial in order to provide the wing skeleton with a bio-inspired actuation, and to study its response to natural joint compliance that are present in living systems. Driven by this motivation, we leveraged Robotran in order to generate the equations of motion of this multi-body system, and we coupled it with multiflap in order to transpose the limit cycle formalism to study flapping flight dynamics. In Chapter 8 we reported the first investigation of this extended framework. We focused on the passive compliance that the muscle-tendon unit may introduce at the level of the shoulder joint. We observed that introducing a compliant element, is beneficial in terms of stability. Although all the detected solutions present an unstable mode, reducing the stiffness of the joint brings the expanding Floquet multiplier closer to the unit circle.

These results however need further confirmation. It would be interesting to test different flapping gaits with such compliant joint model, in order to see whether stability can be achieved without the help of the tail surface, and via solely relying on passive compliant elements. Moreover, a more accurate representation of this joint would require replacing the driven kinematics by adding a bio-inspired activation. A possible strategy to achieve this goal is proposed in the next Section.

9.3 Review of the research questions

After the discussion of the key findings, we review the proposed research questions listed in Chapter 1.

1. How can we accurately build a framework accounting for active wing morphing to assess flapping flight stability?

Answering this question led us to the development of three foundational blocks. We built a realistic model of a bird wing, accounting for all of its degrees of freedom, and provided with a quasi-steady lifting line solver to compute the gait-dependent aerodynamic forces. Such a wing model and aerodynamic solver, can be tuned to mimic other morphologies of large birds, thus allowing to extend studies on different species. We then built our dynamical model restricting the flight on its longitudinal plane. These equations of motion, due to the continuous variation of the aerodynamic forces induced by the wing motion, require a dedicated framework to study stability.

We adopted a limit cycle formalism to study the flight dynamics of flapping bird and identify the steady states. The stability of these limit cycles is quantified via Floquet theory. This formalism led us to the development of a dedicated tool — called multiflap — based on multiple-shooting algorithm. Importantly, this tool is found to be versatile, and adaptable to different nonlinear phenomena not necessarily related with flight mechanics.

2. What is the role of the kinematics parameters of the wing and the function of the tail in longitudinal stability?

We leveraged our framework to study the impact of different gait parameters on flight stability. We found that the parameters regulating the pitching moment generation — sweep angle of the wing, wing insertion point, and tail opening — play a fundamental role. In particular we observed bifurcation occurring via gradually opening the tail surface angle.

Parameters having a direct impact on thrust and lift — such as the wingbeat amplitude or the mean rotation of the wing profiles of the forearm — marginally affect the stability properties of the system. Moreover, our results show that increasing the wingbeat amplitude leads to increasing flight velocities. This insight is consistent with experimental observations [154, 155]. However, many other parameters have not been explored in the context of this Thesis, leaving this question still open for further investigations.

3. Can our numerical investigations give insights on the evolutionary process of bird flight?

We tackled this point trying to quantify the energetic cost of flight in direct comparison with its stability properties. We found passively stable regimes only via introducing a tail-like surface. However, these configurations are more energetically expensive than the unstable tail-less configurations detected.

These results can help in explaining why we observe birds flying for long-range with furled tail. Moreover, they suggest that evolution could have replaced the tail surface with an active sensory-motor control scheme, that stabilizes the flight at lower energetic cost.

4. *How does the passive morphing of the wing influences flight stability?* We addressed this point by extending the framework with a multibody generator of equations of motion. This allowed to model the dynamics of the wing, and therefore to study the response of the joint compliance. We modeled such compliance with a spring-damper system aimed at capturing — at a first level of complexity — the equivalent elastic and dissipative nature of a muscle-tendon unit. Results show that a more compliant joint is beneficial in terms of stability, although in the tail-less situation case analyzed, no stable limit cycles were found via changing this parameter. However, this preliminary investigation invokes further tests, especially to dig in having more accurate estimations of the equivalent joint stiffness and damping coefficients, which do not represent biological data.

9.4 Future perspectives

The developed work provides a consistent framework tailored at studying the flight stability of flapping fliers. There is however remaining work to be done. Based on the obtained insights, we draft future research directions worth being explored. Some of the perspectives actually exploit current limitations and aim at improving them, while others open new research scenarios.

9.4.1 Body compliance

Compliance properties are omnipresent in nature, and they often represent an important mean to simplify the control of dynamic locomotion [156], allowing the morphology to self-adapt to environmental disturbances. In birds, the compliant structures that we envisage playing important roles are essentially two: the muscle-tendon apparatus, and the feathered surface of the wing.

Muscle-tendon apparatus

The muscle-tendon apparatus is the foundational block for actuating body joints, and this naturally holds for the wing. This could directly address some current limitations presented in the previous Section. A phenomenological component-based approach to model muscle-tendon units, was first proposed by Hill [146], and still vastly used nowadays to predict muscle forces in human locomotion [157]. Hill muscle-tendon model builds upon three main elements (Figure 9.1). The force generator of this model is represented by a contractile element (CE), which receives an activation signal

Figure 9.1: Hill muscle model for contraction dynamics. It is composed of an active contractile element (CE) excited by a signal a(t), a series of elastic elements (SEE), and a parallel element (PE). This model would allow to actuate the wing skeleton.



a(t). The series elements (SEE) represent tendons and the intrinsic elasticity behavior of muscle myofilaments. The parallel element (PE) accounts for the passive behavior of the muscle.

This model allows to estimate the force produced by the unit, and transmitted to the skeleton. Such a model could be of direct interest within the multi-body framework. Constitutive equations of muscle-tendons forces can be implemented in Robotran, and ultimately used to actuate the wing, replacing the driven kinematics imposed in each joint. This model would then directly account for the joint compliance via the intrinsic elastic elements.

Experimental work analyzing muscle activity of European starlings via electromyography measurements, shows that pectoralis and supracoracoideus muscles are the most stressed during powered flight [27]. It thus suggests that the first muscle actuation to be carefully considered is at the level of the shoulder joint.

The main challenge of this approach is the estimation of biologically relevant parameters composing the unit. In human locomotion such estimation has been done following two different strategies: experimental work, or inverse dynamics [157, 158], and these strategies can be re-adapted also for bird anatomy.

A separate discussion is worth being dedicated to the signal activation a(t), which constitutes the input for the wing movement. The origin of this signal, in fact, represents a century-long debate between scientists, which started in the beginning of the 20th century between Graham Brown and Sherrington. Experiments on neural control locomotion performed by Graham Brown, envisioned a so-called *feed-forward* mechanism of activation. This mechanism built upon neural rhythm generator located in the central nervous system [159]. Arguably, this finding represented the seminal work for more recent discoveries of central pattern generators, i.e. oscillators



Figure 9.2: Candidate approach to implement bending feathers in multi-body system. Each feather element is anchored to the wing via a rotational degree of freedom about *y*-axis.

that can produce coordinated activities also when isolated from sensory feedback [160, 161, 162].

Conversely, Sherrington pioneered the idea of a *feedback* mechanism of activation. In this mechanism, the activation of locomotory system was mostly due to a chain of rhythmic reflexes produced by the peripheral nervous system [163, 164].

A recent work done by Thandiackal et al. [165] challenged these two theories. Importantly, they advanced the thesis that a combination of both feed-forward and feedback signals contributes favorably in locomotion, increasing redundancies that play a role against possible neural disruptions.

Feathers compliance

Feathers are the structural compliant elements of the wing, that bend under the aerodynamic forces. This bending and deformation is known to be beneficial in the lift-drag ratio production, and can definitely play a role in the mechanism of gust rejection [166]. This phenomenon could be addressed by the multi-body formalism.

The topology presented in Figure 7.1 could be improved by anchoring rigid bodies to the wing elements. A candidate scheme is proposed in Figure 9.2 where each feather element can be anchored to the corresponding wing segment via one rotational degree of freedom about the local *y*-axis. The nature of the joint can be set as independent, namely that the final position of the feather will depend on the resulting dynamics of the multi-body system. A similar approach has been used in [65] where the torque at the level of the feather joint, $Q_{f,y}$, is governed by a spring-damper equation in the form

$$Q_{f,y} = -k_f(\phi_f - \phi_{0,f}) - C_f \dot{\phi_f}$$
(9.1)

where k_f represents the joint flexibility, ϕ_f and $\dot{\phi_f}$ the actual state variables

describing the feather position and velocity, C_f the damping coefficient, and $\phi_{0,f}$ a driving input. An attempt to quantify the flexibility of the feathers, is reported in [167]. Although further experimental work is needed to evaluate the scaling law on different species, this work could represent a first starting point for the estimation of this parameter.

9.4.2 A supporting tool for in-vivo experiments

Various in-vivo experiments are reported in literature aiming at quantifying the power consumption of bird flight, and to analyze the impact of relevant parameters, such as wingbeat amplitude and frequency on the power outcome. Various strategies of in-vivo measurements are adopted to tackle this problem. Among others, three common techniques are often used: (i) respirometry analysis; (ii) muscle electromyography; (iii) body-mounted accelerometers.

One of the main challenges of in-vivo measurements is minimizing the environment alterations, in order to capture natural responses from the flier. As an example, birds can experience different behaviors depending on whether the measurements are done in laboratory or in wild conditions [155]. We envisage that our model-based prediction framework could help in this respect to have better understanding of parameter correlations in measurements. Our level of accuracy captures phenomena of power consumption at a much higher level than allometric formulas and scaling laws, being able to simulate the actual kinematics observed in experiments and reproduce in-silico estimations. It thus constitutes a powerful tool to successfully complement experimental work. Moreover, numerical simulations are versatile. They give the possibility to decide which parameter to change and how, and possibly trigger dedicated in-vivo experiments based on numerical results.

9.5 Final remarks

The work depicted in this manuscript aimed at contributing in the field of flight stability of migratory birds in flapping regime. We studied this spectacular phenomenon in a mathematical approach, i.e. via conceptualizing a deterministic model that — within some accuracy — attempts at describing what observed in nature. At the current stage of development, such a model invokes two fundamental disciplines: classical mechanics, and fluid dynamics. The former describes the motion of the bird with Newtonian

laws, while the latter estimates the aerodynamics forces developed during the flight, leveraging on an extension of Prandtl's lifting line. What makes this problem extremely fascinating is the strong interplay among the variables belonging to these two subjects. Their coupling nature, and their reciprocal dependency, give rise to complex nonlinear dynamics equations. This nonlinear nature prevents from breaking down the problem into its fundamental parts and eventually superimposing each individual effect, but it necessarily entails the entire system being studied as a whole. For this reason, each new component that will be introduced in the model aimed at increasing its fidelity — cannot be studied just separately, but it will interact with all the other elements making more complicated the entire dynamics.

Undoubtedly, nowadays computational resources allow to have more and more realistic representations of phenomena of interest. A virtual bird reproducing the delicate locomotion machinery is a reality. The equations of motion can be extended in all degrees of freedom, aerodynamic advances can solve intricate wake structures, and the muscle activation can be embodied and triggered by central pattern generator (CPG) based signals. This however comes at a cost. On the one hand computational time makes this problem very expensive. On the other hand, a complex model is normally more difficult both to reproduce, and to analyze. It thus makes more challenging to understand the effect that each element plays on the final result.

Many questions concerning bird flight are open, and some new ones emerged as perspectives of this project. However, the continuation of this work should be goal-driven. It first calls for identifying a purpose, and then tune the level of complexity of the model to a careful compromise that justifies what is achievable.

Bibliography

- B. Etkin and L. D. Reid, *Dynamics of flight*, vol. 2. Wiley New York, 1959.
- [2] P. Lissaman, "Fundamentals of energy extraction from natural winds," *Technical Soaring*, vol. 31, no. 2, pp. 36–41, 2007.
- [3] L. Rayleigh, "The soaring of birds," *Nature*, vol. 27, no. 701, pp. 534– 535, 1883.
- [4] W. Shyy, M. Berg, and D. Ljungqvist, "Flapping and flexible wings for biological and micro air vehicles," *Progress in aerospace sciences*, vol. 35, no. 5, pp. 455–505, 1999.
- [5] T. Alerstam and A. Lindström, "Optimal bird migration: the relative importance of time, energy, and safety," in *Bird migration*, pp. 331– 351, Springer, 1990.
- [6] P. F. Battley, T. Piersma, D. I. Rogers, A. Dekinga, B. Spaans, and J. A. Van Gils, "Do body condition and plumage during fuelling predict northwards departure dates of great knots calidris tenuirostris from north-west Australia?," *Ibis*, vol. 146, no. 1, pp. 46–60, 2004.
- [7] R. E. Gill Jr, T. L. Tibbitts, D. C. Douglas, C. M. Handel, D. M. Mulcahy, J. C. Gottschalck, N. Warnock, B. J. McCaffery, P. F. Battley, and T. Piersma, "Extreme endurance flights by landbirds crossing the Pacific ocean: ecological corridor rather than barrier?," *Proceedings of the Royal Society B: Biological Sciences*, vol. 276, no. 1656, pp. 447–457, 2009.
- [8] C. Harvey and D. J. Inman, "Aerodynamic efficiency of gliding birds vs comparable UAVs: a review," *Bioinspiration & Biomimetics*, vol. 16, no. 3, p. 031001, 2021.

- [9] D. J. Aidley, *Animal migration*, vol. 13. Cambridge University Press, 1981.
- [10] S. A. Gauthreaux Jr, "The ecology and evolution of avian migration systems," in *Avian biology*, pp. 93–168, Elsevier, 1982.
- [11] R. E. Gill Jr, T. Piersma, G. Hufford, R. Servranckx, and A. Riegen, "Crossing the ultimate ecological barrier: evidence for an 11 000-kmlong nonstop flight from Alaska to New Zealand and eastern Australia by bar-tailed godwits," *The Condor*, vol. 107, no. 1, pp. 1–20, 2005.
- [12] T. W. Horton, R. N. Holdaway, A. N. Zerbini, N. Hauser, C. Garrigue, A. Andriolo, and P. J. Clapham, "Straight as an arrow: humpback whales swim constant course tracks during long-distance migration," *Biology letters*, vol. 7, no. 5, pp. 674–679, 2011.
- [13] L. Rizzo and D. Schulte, "A review of humpback whales' migration patterns worldwide and their consequences to gene flow," *Journal of the Marine Biological Association of the United Kingdom*, vol. 89, no. 5, pp. 995–1002, 2009.
- [14] C. J. Pennycuick, "Gust soaring as a basis for the flight of petrels and albatrosses (procellariiformes)," Avian Science, vol. 2, pp. 1–12, 2002.
- [15] P. L. Richardson, "How do albatrosses fly around the world without flapping their wings?," *Progress in Oceanography*, vol. 88, no. 1-4, pp. 46–58, 2011.
- [16] P. L. Richardson, E. D. Wakefield, and R. A. Phillips, "Flight speed and performance of the wandering albatross with respect to wind," *Movement ecology*, vol. 6, no. 1, pp. 1–15, 2018.
- [17] P. L. Richardson, "Leonardo da Vinci's discovery of the dynamic soaring by birds in wind shear," *Notes and Records: the Royal Society journal of the history of science*, vol. 73, no. 3, pp. 285–301, 2019.
- [18] D. V. Lee and S. L. Harris, "Linking gait dynamics to mechanical cost of legged locomotion," *Frontiers in Robotics and AI*, p. 111, 2018.
- [19] C. J. Pennycuick, *Modelling the flying bird*. Elsevier, 2008.

148

- [20] U. M. Norberg, Vertebrate flight: mechanics, physiology, morphology, ecology and evolution, vol. 27. Springer Science & Business Media, 2012.
- [21] T. L. Hieronymus, "Flight feather attachment in rock pigeons (columba livia): covert feathers and smooth muscle coordinate a morphing wing," *Journal of anatomy*, vol. 229, no. 5, pp. 631–656, 2016.
- [22] C. Harvey, V. Baliga, J. Wong, D. Altshuler, and D. Inman, "Birds can transition between stable and unstable states via wing morphing," *Nature*, vol. 603, no. 7902, pp. 648–653, 2022.
- [23] "Wikimedia commons, wing muscle." L. Shyamal, https: //commons.wikimedia.org/wiki/File:Wing_Muscles,_color. svg?uselang=fr, accessed September 2022.
- [24] D. L. Altshuler, J. W. Bahlman, R. Dakin, A. H. Gaede, B. Goller, D. Lentink, P. S. Segre, and D. A. Skandalis, "The biophysics of bird flight: functional relationships integrate aerodynamics, morphology, kinematics, muscles, and sensors," *Canadian Journal of Zoology*, vol. 93, no. 12, pp. 961–975, 2015.
- [25] A. L. Thomas, "The flight of birds that have wings and a tail: variable geometry expands the envelope of flight performance," *Journal of Theoretical Biology*, vol. 183, no. 3, pp. 237–245, 1996.
- [26] B. Tobalske, T. Hedrick, K. Dial, and A. Biewener, "Comparative power curves in bird flight," *Nature*, vol. 421, no. 6921, p. 363, 2003.
- [27] A. A. Biewener, D. K. P, and G. Goslow Jr, "Pectoralis muscle force and power output during flight in the starling," *Journal of Experimental Biology*, vol. 164, no. 1, pp. 1–18, 1992.
- [28] J. A. Cheney, J. P. Stevenson, N. E. Durston, J. Song, J. R. Usherwood, R. J. Bomphrey, and S. P. Windsor, "Bird wings act as a suspension system that rejects gusts," *Proceedings of the Royal Society B*, vol. 287, no. 1937, p. 20201748, 2020.
- [29] A. Kovalev, A. E. Filippov, and S. N. Gorb, "Unzipping bird feathers," *Journal of the Royal Society Interface*, vol. 11, no. 92, p. 20130988, 2014.

- [30] W. K. Potts, "The chorus-line hypothesis of manoeuvre coordination in avian flocks," *Nature*, vol. 309, no. 5966, pp. 344–345, 1984.
- [31] K. Zheng, D. Liang, X. Wang, Y. Han, M. Griesser, Y. Liu, and P. Fan, "Contrasting coloured ventral wings are a visual collision avoidance signal in birds," *Proceedings of the Royal Society B*, vol. 289, no. 1978, p. 20220678, 2022.
- [32] S. M. Gatesy and K. P. Dial, "Locomotor modules and the evolution of avian flight," *Evolution*, vol. 50, no. 1, pp. 331–340, 1996.
- [33] J. J. Baumel, Functional morphology of the tail apparatus of the pigeon (Columba livia). Springer, 1988.
- [34] E. H. Hankin, Animal flight: a record of observation. Iliffe & Sons Limited, 1913.
- [35] J. M. Smith, "The importance of the nervous system in the evolution of animal flight," *Evolution*, vol. 6, no. 1, pp. 127–129, 1952.
- [36] G. Taylor and A. Thomas, "Animal flight dynamics ii. longitudinal stability in flapping flight," *Journal of theoretical biology*, vol. 214, no. 3, pp. 351–370, 2002.
- [37] B. W. Tobalske, "Biomechanics and physiology of gait selection in flying birds," *Physiological and Biochemical Zoology*, vol. 73, no. 6, pp. 736–750, 2000.
- [38] I. D. Couzin, J. Krause, N. R. Franks, and S. A. Levin, "Effective leadership and decision-making in animal groups on the move," *Nature*, vol. 433, no. 7025, pp. 513–516, 2005.
- [39] P. B. Lissaman and C. A. Shollenberger, "Formation flight of birds," Science, vol. 168, no. 3934, pp. 1003–1005, 1970.
- [40] S. J. Portugal, T. Y. Hubel, J. Fritz, S. Heese, D. Trobe, B. Voelkl, S. Hailes, A. M. Wilson, and J. R. Usherwood, "Upwash exploitation and downwash avoidance by flap phasing in ibis formation flight," *Nature*, vol. 505, no. 7483, pp. 399–402, 2014.
- [41] L. Hedh, C. G. Guglielmo, L. C. Johansson, J. E. Deakin, C. C. Voigt, and A. Hedenström, "Measuring power input, power output and energy conversion efficiency in un-instrumented flying birds," *Journal* of Experimental Biology, vol. 223, no. 18, 2020.

- [42] V. A. Tucker, "Metabolism during flight in the laughing gull, larus atricilla," *American Journal of Physiology-Legacy Content*, vol. 222, no. 2, pp. 237–245, 1972.
- [43] H. E. Taha, M. R. Hajj, and A. H. Nayfeh, "Flight dynamics and control of flapping-wing MAVs: a review," *Nonlinear Dynamics*, vol. 70, no. 2, pp. 907–939, 2012.
- [44] G. A. Folkertsma, W. Straatman, N. Nijenhuis, C. H. Venner, and S. Stramigioli, "Robird: a robotic bird of prey," *IEEE robotics & automation magazine*, vol. 24, no. 3, pp. 22–29, 2017.
- [45] N. Docquier, A. Poncelet, and P. Fisette, "Robotran: a powerful symbolic gnerator of multibody models," *Mechanical Sciences*, vol. 4, no. 1, pp. 199–219, 2013.
- [46] C. H. Greenewalt, "Dimensional relationships for flying animals," Smithsonian miscellaneous collections, 144, pp. 1-46, 1962.
- [47] C. H. Greenewalt, "The flight of birds: the significant dimensions, their departure from the requirements for dimensional similarity, and the effect on flight aerodynamics of that departure," *Transactions of the American Philosophical society*, vol. 65, no. 4, pp. 1–67, 1975.
- [48] J. M. Rayner, "A new approach to animal flight mechanics," *Journal* of *Experimental Biology*, vol. 80, no. 1, pp. 17–54, 1979.
- [49] J. Rayner, "Form and function in avian flight," in *Current ornithology*, pp. 1–66, Springer, 1988.
- [50] B. Tobalske and K. Dial, "Flight kinematics of black-billed magpies and pigeons over a wide range of speeds," *Journal of Experimental Biology*, vol. 199, no. 2, pp. 263–280, 1996.
- [51] J.-c. Wu and Z. Popović, "Realistic modeling of bird flight animations," ACM Transactions on Graphics (TOG), vol. 22, no. 3, pp. 888– 895, 2003.
- [52] G. Buresti, *Elements of fluid dynamics*, vol. 3. World Scientific Publishing Company, 2012.
- [53] G. K. Taylor, R. L. Nudds, and A. L. Thomas, "Flying and swimming animals cruise at a strouhal number tuned for high power efficiency," *Nature*, vol. 425, no. 6959, pp. 707–711, 2003.

- [54] J. Shapiro, Principles of helicopter engineering. McGraw Hill, 1956.
- [55] C. J. Pennycuick, "Power requirements for horizontal flight in the pigeon columba livia," *Journal of Experimental Biology*, vol. 49, no. 3, pp. 527–555, 1968.
- [56] T. Weis-Fogh and M. Jensen, "Biology and physics of locust flight. i. basic principles in insect flight. a critical review," *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences*, vol. 239, no. 667, pp. 415–458, 1956.
- [57] T. Weis-Fogh, "Biology and physics of locust flight ii. flight performance of the desert locust (schistocerca gregaria)," *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences*, vol. 239, no. 667, pp. 459–510, 1956.
- [58] C. P. Ellington, "The aerodynamics of hovering insect flight. i. the quasi-steady analysis," *Philosophical Transactions of the Royal Society* of London. B, Biological Sciences, vol. 305, no. 1122, pp. 1–15, 1984.
- [59] G. K. Taylor and A. L. Thomas, "Dynamic flight stability in the desert locust schistocerca gregaria," *Journal of Experimental Biology*, vol. 206, no. 16, pp. 2803–2829, 2003.
- [60] B. Parslew and W. J. Crowther, "Simulating avian wingbeat kinematics," *Journal of Biomechanics*, vol. 43, no. 16, pp. 3191–3198, 2010.
- [61] L. Prandtl, "Tragflugeltheorie, gottinger nachrichten," Mathematisch-Physikalische Klasse, Germany, pp. 451–477, 1918.
- [62] D. Betteridge and R. Archer, "A study of the mechanics of flapping wings," *Aeronautical Quarterly*, vol. 25, no. 2, pp. 129–142, 1974.
- [63] P. Phlips, R. East, and N. Pratt, "An unsteady lifting line theory of flapping wings with application to the forward flight of birds," *Journal of fluid mechanics*, vol. 112, pp. 97–125, 1981.
- [64] J. S. Izraelevitz, Q. Zhu, and M. S. Triantafyllou, "State-space adaptation of unsteady lifting line theory: twisting/flapping wings of finite span," AIAA Journal, vol. 55, no. 4, pp. 1279–1294, 2017.
- [65] V. Colognesi, R. Ronsse, and P. Chatelain, "A model coupling biomechanics and fluid dynamics for the simulation of controlled flapping flight," *Bioinspiration & Biomimetics*, vol. 16, no. 2, p. 026023, 2021.

- [66] G. Ducci, V. Colognesi, G. Vitucci, P. Chatelain, and R. Ronsse, "Stability and sensitivity analysis of bird flapping flight," *Journal of Nonlinear Science*, vol. 31, no. 2, pp. 1–30, 2021.
- [67] H. K. Khalil, "Nonlinear systems," Prentice Hall, vol. 115, 2002.
- [68] M. Maggia, S. A. Eisa, and H. E. Taha, "On higher-order averaging of time-periodic systems: reconciliation of two averaging techniques," *Nonlinear Dynamics*, vol. 99, no. 1, pp. 813–836, 2020.
- [69] Y. Xiong and M. Sun, "Dynamic flight stability of a bumblebee in forward flight," *Acta Mechanica Sinica*, vol. 24, no. 1, pp. 25–36, 2008.
- [70] G. Iosilevskii, "Forward flight of birds revisited. part 2: short-term dynamic stability and trim," *Royal Society Open Science*, vol. 1, no. 2, 2014. 140249.
- [71] H. E. Taha, S. Tahmasian, C. A. Woolsey, A. H. Nayfeh, and M. R. Hajj, "The need for higher-order averaging in the stability analysis of hovering, flapping-wing flight," *Bioinspiration & biomimetics*, vol. 10, no. 1, p. 016002, 2015.
- [72] M. Maggia, S. A. Eisa, and H. E. Taha, "On higher-order averaging of time-periodic systems: reconciliation of two averaging techniques," *Nonlinear Dynamics*, vol. 99, pp. 813–836, 2020.
- [73] G. K. Taylor and R. Żbikowski, "Nonlinear time-periodic models of the longitudinal flight dynamics of desert locusts schistocerca gregaria," *Journal of the Royal Society Interface*, vol. 2, no. 3, pp. 197–221, 2005.
- [74] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay, Chaos: Classical and Quantum. Niels Bohr Institute, Copenhagen 2016.
- [75] J. M. Dietl and E. Garcia, "Stability in ornithopter longitudinal flight dynamics," *Journal of Guidance, Control, and Dynamics*, vol. 31, no. 4, pp. 1157–1163, 2008.
- [76] A. M. Hassan and H. E. Taha, "Differential-geometric-control formulation of flapping flight multi-body dynamics," *Journal of Nonlinear Science*, vol. 29, no. 4, pp. 1379–1417, 2019.

- [77] J. H. Wu and M. Sun, "Floquet stability analysis of the longitudinal dynamics of two hovering model insects," *Journal of the Royal Society Interface*, vol. 9, no. 74, pp. 2033–2046, 2012.
- [78] D. Raymer, *Aircraft design: a conceptual approach*. American Institute of Aeronautics and Astronautics, Inc., 2012.
- [79] I. H. Abbott and A. E. Von Doenhoff, *Theory of wing sections: including a summary of airfoil data*. Courier Corporation, 2012.
- [80] A. Hedenström, "Aerodynamics, evolution and ecology of avian flight," *Trends in Ecology & Evolution*, vol. 17, no. 9, pp. 415–422, 2002.
- [81] C. Berg and J. Rayner, "The moment of inertia of bird wings and the inertial power requirement for flapping flight," *Journal of Experimental Biology*, vol. 198, no. 8, pp. 1655–1664, 1995.
- [82] J. Dietl, T. Herrmann, G. Reich, and E. Garcia, "Dynamic modeling, testing, and stability analysis of an ornithoptic blimp," *Journal* of *Bionic Engineering*, vol. 8, no. 4, pp. 375–386, 2011.
- [83] H. E. Taha, M. R. Hajj, and A. H. Nayfeh, "Longitudinal flight dynamics of hovering MAVs/insects," *Journal of Guidance, Control, and Dynamics*, vol. 37, no. 3, pp. 970–979, 2014.
- [84] C. Casarosa, *Meccanica del volo*. Didattica e Ricerca. Manuali, Pisa University Press, 2013.
- [85] J.-c. Wu and Z. Popović, "Realistic modeling of bird flight animations," ACM Transactions on Graphics (TOG), vol. 22, pp. 888–895, July 2003.
- [86] R. T. Jones, "Wing theory," in Wing Theory, Princeton University Press, 2014.
- [87] A. L. Thomas, "On the tails of birds," Bioscience, pp. 215–225, 1997.
- [88] W. Maybury, J. Rayner, and L. Couldrick, "Lift generation by the avian tail," *Proceedings of the Royal Society of London. Series B: Biological Sciences*, vol. 268, no. 1475, pp. 1443–1448, 2001.
- [89] A. L. Thomas, "On the aerodynamics of birds' tails," *Philosophical Transactions of the Royal Society of London. Series B: Biological Sciences*, vol. 340, no. 1294, pp. 361–380, 1993.

- [90] "Animal Diversity Web university of michigan." https: //animaldiversity.org/accounts/Geronticus_eremita/. Accessed October 2022.
- [91] H. Wehner, J. Fritz, and B. Voelkl, "Soaring and intermittent flapgliding during migratory flights of Northern Bald Ibis," *Journal of Ornithology*, pp. 1–11, 2022.
- [92] C. Pennycuick, "Wingbeat frequency of birds in steady cruising flight: new data and improved predictions," *Journal of Experimental Biology*, vol. 199, no. 7, pp. 1613–1618, 1996.
- [93] M. Spong, "Review article: Modeling and control of elastic joint robots," *Mathematical and Computer Modelling*, vol. 12, no. 7, p. 912, 1989.
- [94] M. W. Spong and M. Vidyasagar, *Robot dynamics and control*. John Wiley & Sons, 2008.
- [95] S. H. Strogatz, Nonlinear Dynamics and Chaos with Student Solutions Manual: With Applications to Physics, Biology, Chemistry, and Engineering. CRC Press, 2018.
- [96] G. Floquet, "Sur les équations différentielles linéaires à coefficients périodiques," in Annales scientifiques de l'École normale supérieure, vol. 12, pp. 47–88, 1883.
- [97] E. A. Jackson, *Perspectives of Nonlinear Dynamics: Volume 1*, vol. 1. CUP Archive, 1989.
- [98] L. Glass and M. C. Mackey, "From clocks to chaos," in *From Clocks to Chaos*, Princeton University Press, 2020.
- [99] A. Goldbeter, "Biological rhythms: clocks for all times," *Current biology*, vol. 18, no. 17, pp. R751–R753, 2008.
- [100] A. T. Winfree, *The geometry of biological time*, vol. 2. Springer, 1980.
- [101] A. E. Granada and H. Herzel, "How to achieve fast entrainment? the timescale to synchronization," *PloS one*, vol. 4, no. 9, p. e7057, 2009.
- [102] M. d. Olmo, "Searching for order in body clocks: Circadian rhythms and redox balance," 2021.

- [103] R. Seydel, Practical bifurcation and stability analysis, vol. 5. Springer Science & Business Media, 2009.
- [104] K. Lust, "Improved numerical Floquet multipliers," International Journal of Bifurcation and Chaos, vol. 11, no. 09, pp. 2389–2410, 2001.
- [105] R. I. Leine and H. Nijmeijer, Dynamics and bifurcations of non-smooth mechanical systems, vol. 18. Springer Science & Business Media, 2013.
- [106] H. Keller, Numerical Methods for Two-point Boundary-value Problems. A Blaisdell Book in Numerical Analysis and Computer Science, Blaisdel, 1968.
- [107] D. W. Marquardt, "An algorithm for least-squares estimation of nonlinear parameters," *Journal of the society for Industrial and Applied Mathematics*, vol. 11, no. 2, pp. 431–441, 1963.
- [108] A. Quarteroni, R. Sacco, and F. Saleri, *Numerical mathematics*, vol. 37. Springer Science & Business Media, 2010.
- [109] W. Dednam and A. E. Botha, "Optimized shooting method for finding periodic orbits of nonlinear dynamical systems," *Engineering with Computers*, vol. 31, no. 4, pp. 749–762, 2015.
- [110] C. H. Gabriel, M. Del Olmo, A. Zehtabian, M. Jäger, S. Reischl, H. van Dijk, C. Ulbricht, A. Rakhymzhan, T. Korte, B. Koller, *et al.*, "Livecell imaging of circadian clock protein dynamics in crispr-generated knock-in cells," *Nature Communications*, vol. 12, no. 1, pp. 1–15, 2021.
- [111] E. J. Doedel, A. R. Champneys, F. Dercole, T. F. Fairgrieve, Y. A. Kuznetsov, B. Oldeman, R. Paffenroth, B. Sandstede, X. Wang, and C. Zhang, "Auto-07p: Continuation and bifurcation software for ordinary differential equations," 2007. Software manual. Department of Computer Science, Concordia University, Montreal, QC.
- [112] R. H. Clewley, W. Sherwood, M. LaMar, and J. Guckenheimer, "Pydstool, a software environment for dynamical systems modeling," URL http://pydstool. sourceforge. net, 2007.
- [113] R. Veltz, "PseudoArcLengthContinuation.jl," 2019.
- [114] J. A. E. Andersson, J. Gillis, G. Horn, J. B. Rawlings, and M. Diehl, "CasADi – A software framework for nonlinear optimization and

optimal control," *Mathematical Programming Computation*, vol. 11, no. 1, pp. 1–36, 2019.

- [115] B. Bachmann, L. Ochel, V. Ruge, M. Gebremedhin, P. Fritzson, V. Nezhadali, L. Eriksson, and M. Sivertsson, "Parallel multipleshooting and collocation optimization with openmodelica," in *Proceedings of the 9th International Modelica Conference*, vol. 76, pp. 659– 668, Linkoping University Press Munich, Germany, 2012.
- [116] G. Taylor, R. Bomphrey, and J. 't Hoen, "Insect flight dynamics and control," in 44th AIAA Aerospace Sciences Meeting and Exhibit, p. 32, 2006.
- [117] N. Gao, H. Aono, and H. Liu, "A numerical analysis of dynamic flight stability of hawkmoth hovering," *Journal of Biomechanical Science and Engineering*, vol. 4, no. 1, pp. 105–116, 2009.
- [118] C. Sperger, A. Heller, B. Völkl, and J. Fritz, "Flight strategies of migrating northern bald ibises—analysis of gps data during humanled migration flights," AGIT 2017 – Symposium und Fachmesse Angewandte Geoinformatik, 2017.
- [119] C. Pennycuick, S. Åkesson, and A. Hedenström, "Air speeds of migrating birds observed by ornithodolite and compared with predictions from flight theory," *Journal of the Royal Society Interface*, vol. 10, no. 86, 2013. 20130419.
- [120] C. Pennycuick, "Span-ratio analysis used to estimate effective lift: drag ratio in the double-crested cormorant phalacrocorax auritus from field observations," *Journal of Experimental Biology*, vol. 142, no. 1, pp. 1–15, 1989.
- [121] A. L. Thomas and G. K. Taylor, "Animal flight dynamics i. stability in gliding flight," *Journal of Theoretical Biology*, vol. 212, no. 3, pp. 399– 424, 2001.
- [122] C. Harvey, V. Baliga, P. Lavoie, and D. Altshuler, "Wing morphing allows gulls to modulate static pitch stability during gliding," *Journal* of the Royal Society Interface, vol. 16, no. 150, 2019. 20180641.
- [123] C. Harvey, V. Baliga, C. Goates, D. Hunsaker, and D. Inman, "Gullinspired joint-driven wing morphing allows adaptive longitudinal

flight control," Journal of the Royal Society Interface, vol. 18, no. 179, 2021. 20210132.

- [124] E. Ajanic, M. Feroskhan, S. Mintchev, F. Noca, and D. Floreano, "Bioinspired wing and tail morphing extends drone flight capabilities," *Sci. Robot.*, vol. 5, 2020.
- [125] B. Parslew, "Predicting power-optimal kinematics of avian wings," *Journal of the Royal Society Interface*, vol. 12, no. 102, 2015. 20140953.
- [126] W. J. Maybury, *The aerodynamics of bird bodies*. PhD thesis, University of Bristol, 2000.
- [127] C. Pennycuick, H. H. Obrecht III, and M. R. Fuller, "Empirical estimates of body drag of large waterfowl and raptors," *Journal of Experimental Biology*, vol. 135, no. 1, pp. 253–264, 1988.
- [128] S. Zhong and W. Xu, "Power modeling and experiment study of large flapping-wing flying robot during forward flight," *Applied Sciences*, vol. 12, no. 6, pp. 31–76, 2022.
- [129] G. Sachs, "Comparison of power requirements: Flapping vs. fixed wing vehicles," *Aerospace*, vol. 3, no. 4, p. 31, 2016.
- [130] J. M. V. Rayner, "A vortex theory of animal flight. part 2. the forward flight of birds," *Journal of Fluid Mechanics*, vol. 91, no. 4, pp. 731–763, 1979.
- [131] V. A. Tucker, "The energetic cost of moving about: walking and running are extremely inefficient forms of locomotion. much greater efficiency is achieved by birds, fish—and bicyclists," *American Scientist*, vol. 63, no. 4, pp. 413–419, 1975.
- [132] U. L. Norberg, "Flight and scaling of flyers in nature," *Flow Phenom*ena in Nature, vol. 1, pp. 120–154, 2007.
- [133] N. C. Rattenborg, B. Voirin, S. M. Cruz, R. Tisdale, G. Dell'Omo, H.-P. Lipp, M. Wikelski, and A. L. Vyssotski, "Evidence that birds sleep in mid-flight," *Nature communications*, vol. 7, no. 1, pp. 1–9, 2016.
- [134] J. H. Wu, Y. L. Zhang, and M. Sun, "Hovering of model insects: simulation by coupling equations of motion with navier–stokes equations," *Journal of Experimental Biology*, vol. 212, no. 20, pp. 3313–3329, 2009.

- [135] J.-C. Samin and P. Fisette, Symbolic modeling of multibody systems, vol. 112. Springer Science & Business Media, 2003.
- [136] N. Van der Noot, A. J. Ijspeert, and R. Ronsse, "Biped gait controller for large speed variations, combining reflexes and a central pattern generator in a neuromuscular model," in 2015 IEEE international conference on robotics and automation (ICRA), pp. 6267–6274, IEEE, 2015.
- [137] C. Bernier, M. Gazzola, R. Ronsse, and P. Chatelain, "Simulations of propelling and energy harvesting articulated bodies via vortex particle-mesh methods," *Journal of Computational Physics*, vol. 392, pp. 34–55, 2019.
- [138] J. A. Grauer and J. E. Hubbard Jr, "Multibody model of an ornithopter," *Journal of Guidance, Control, and Dynamics*, vol. 32, no. 5, pp. 1675–1679, 2009.
- [139] Y. Shen, W. Ge, and P. Miao, "Multibody-dynamic modeling and stability analysis for a bird-scale flapping-wing aerial vehicle," *Journal of Intelligent & Robotic Systems*, vol. 103, no. 1, pp. 1–17, 2021.
- [140] P. Chatelain, A. Curioni, M. Bergdorf, D. Rossinelli, W. Andreoni, and P. Koumoutsakos, "Billion vortex particle direct numerical simulations of aircraft wakes," *Computer Methods in Applied Mechanics and Engineering*, vol. 197, no. 13-16, pp. 1296–1304, 2008.
- [141] "Robotran user guide, https://www.robotran.be/docs/ tutorials/."
- [142] F. E. Zajac, "Muscle and tendon: properties, models, scaling, and application to biomechanics and motor control.," *Critical reviews in Biomedical Engineering*, vol. 17, no. 4, pp. 359–411, 1989.
- [143] R. M. Alexander, Animal mechanics. Blackwell, 1983.
- [144] A. F. Huxley, "Muscle structure and theories of contraction," Prog. Biophys. Biophys. Chem, vol. 7, pp. 255–318, 1957.
- [145] W. Williams, "Huxley's model of muscle contraction with compliance," *Journal of elasticity*, vol. 105, no. 1, pp. 365–380, 2011.
- [146] A. V. Hill, "The heat of shortening and the dynamic constants of muscle," *Proceedings of the Royal Society of London. Series B-Biological Sciences*, vol. 126, no. 843, pp. 136–195, 1938.

- [147] S. K. Dwivedy and P. Eberhard, "Dynamic analysis of flexible manipulators, a literature review," *Mechanism and machine theory*, vol. 41, no. 7, pp. 749–777, 2006.
- [148] Z. Chen, N. Y. Lii, T. Wimboeck, S. Fan, and H. Liu, "Experimental evaluation of Cartesian and joint impedance control with adaptive friction compensation for the dexterous robot hand dlr-hit ii," *International Journal of Humanoid Robotics*, vol. 8, no. 04, pp. 649–671, 2011.
- [149] A. Hedenström and T. Alerstam, "How fast can birds migrate?," *Journal of Avian Biology*, pp. 424–432, 1998.
- [150] C. P. Ellington, C. Van Den Berg, A. P. Willmott, and A. L. Thomas, "Leading-edge vortices in insect flight," *Nature*, vol. 384, no. 6610, pp. 626–630, 1996.
- [151] J. D. Eldredge and A. R. Jones, "Leading-edge vortices: mechanics and modeling," Annual Review of Fluid Mechanics, pp. 75–104, 2019.
- [152] L. Y. Matloff, E. Chang, T. J. Feo, L. Jeffries, A. K. Stowers, C. Thomson, and D. Lentink, "How flight feathers stick together to form a continuous morphing wing," *Science*, vol. 367, no. 6475, pp. 293–297, 2020.
- [153] V. Colognesi, Modeling and simulation of bird flapping flight : control, wakes and formation. PhD thesis, Université catholique de Louvain, 2022.
- [154] J. R. Usherwood, M. Stavrou, J. C. Lowe, K. Roskilly, and A. M. Wilson, "Flying in a flock comes at a cost in pigeons," *Nature*, vol. 474, no. 7352, pp. 494–497, 2011.
- [155] K. Krishnan, B. Garde, A. Bennison, N. C. Cole, E.-L. Cole, J. Darby, K. H. Elliott, A. Fell, A. Gómez-Laich, S. de Grissac, *et al.*, "The role of wingbeat frequency and amplitude in flight power," *Journal of the Royal Society Interface*, 2022. 20220168.
- [156] R. Pfeifer, M. Lungarella, and F. Iida, "Self-organization, embodiment, and biologically inspired robotics," *Science*, vol. 318, no. 5853, pp. 1088–1093, 2007.
- [157] A. J. van den Bogert, K. G. Gerritsen, and G. K. Cole, "Human muscle modelling from a user's perspective," *Journal of Electromyography and Kinesiology*, vol. 8, no. 2, pp. 119–124, 1998.

- [158] T. S. Buchanan, D. G. Lloyd, K. Manal, and T. F. Besier, "Estimation of muscle forces and joint moments using a forward-inverse dynamics model," *Medicine and Science in Sports and exercise*, vol. 37, no. 11, p. 1911, 2005.
- [159] T. G. Brown, "On the nature of the fundamental activity of the nervous centres; together with an analysis of the conditioning of rhythmic activity in progression, and a theory of the evolution of function in the nervous system," *The Journal of physiology*, vol. 48, no. 1, p. 18, 1914.
- [160] A. I. Selverston, "Invertebrate central pattern generator circuits," *Philosophical Transactions of the Royal Society B: Biological Sciences*, vol. 365, no. 1551, pp. 2329–2345, 2010.
- [161] O. Kiehn, "Decoding the organization of spinal circuits that control locomotion," *Nature Reviews Neuroscience*, vol. 17, no. 4, pp. 224–238, 2016.
- [162] S. Grillner and A. El Manira, "Current principles of motor control, with special reference to vertebrate locomotion," *Physiological Reviews*, vol. 100, no. 1, pp. 271–320, 2020.
- [163] C. Sherrington, "The integrative action of the nervous system.," The Journal of Nervous and Mental Disease, vol. 34, no. 12, pp. 801–802, 1907.
- [164] C. Sherrington, "Remarks on the reflex mechanism of the step," *Brain*, vol. 33, no. 1, pp. 1–25, 1910.
- [165] R. Thandiackal, K. Melo, L. Paez, J. Herault, T. Kano, K. Akiyama, F. Boyer, D. Ryczko, A. Ishiguro, and A. J. Ijspeert, "Emergence of robust self-organized undulatory swimming based on local hydrodynamic force sensing," *Science Robotics*, vol. 6, no. 57, 2021.
- [166] Y. Shim and P. Husbands, "Feathered flyer: integrating morphological computation and sensory reflexes into a physically simulated flapping-wing robot for robust flight manoeuvre," in *European Conference on Artificial Life*, pp. 756–765, Springer, 2007.
- [167] E. Chang, L. Y. Matloff, A. K. Stowers, and D. Lentink, "Soft biohybrid morphing wings with feathers underactuated by wrist and finger motion," *Science Robotics*, vol. 5, no. 38, p. eaay1246, 2020.

- [168] O. E. Rössler, "An equation for continuous chaos," *Physics Letters A*, vol. 57, no. 5, pp. 397–398, 1976.
- [169] O. E. Rössler, "An equation for hyperchaos," *Physics Letters A*, vol. 71, no. 2-3, pp. 155–157, 1979.
- [170] T. P. das Chagas, Stabilization of periodic orbits in discrete and continuous-time systems. PhD thesis, Université Paris Sud-Paris XI; Instituto Tecnológico de Aeronáutica, 2013.
- [171] M. Del Olmo, A. Kramer, and H. Herzel, "A robust model for circadian redox oscillations," *International Journal of Molecular Sciences*, vol. 20, no. 9, p. 2368, 2019.
- [172] M. Casartelli, E. Diana, L. Galgani, and A. Scotti, "Numerical computations on a stochastic parameter related to the Klmogorov entropy," *Physical Review A*, vol. 13, no. 5, p. 1921, 1976.

Appendices

Appendix A

Validation of multiflap on Rössler's system

This Section presents the validation of multiflap. This validation is performed in terms of limit cycle detection and calculation of the largest absolute value of the non-trivial Floquet multipliers. This is done for three cases of Rössler's system.

Rössler system is a continuous system of non-linear ODEs described by the equations [168, 169]

$$\dot{x} = -y - z$$

$$\dot{y} = x + ay$$

$$\dot{z} = b + z(x - c)$$
(A.1)

This system is automonous, and depending on the values of the parameters *a*, *b*, *c* can exhibit different long time behaviors.

Case 1: *a* = 0.15, *b* = 0.2, *c* = 3.5

For this set of parameters, the system shows at least one stable orbit [74]. The trajectory starting out the limit cycle spirals towards it. The stable limit cycle is reported in the phase space in Figure A.1(a) with a period of T = 5.9, while the eigenvalues are pictured in Figure A.1(b). The projection of the orbit on 2D planes is reported in Figure A.1(c) and (d), where the orbit found by multiflap (black solid line) is compared with the orbit shown



Figure A.1: Rössler's system for a = 0.15, b = 0.2, c = 3.5. (a): Phase space. Trajectory (red dashed line) spiraling towards the stable limit cycle (black solid line). (b): Floquet multipliers identified by multiflap (black crosses) compared with the largest absolute value of the ones reported in [109] (blue circles). (c) and (d): Projection of the periodic orbit identified in the Rössler's system, on the plane x - y, and on the plane x - z. The identified periodic orbit (black solid line) is compared with the one identified in [109] (blue scatter points) for sake of validation.

in [109] (blue scatter points).

Case 2: *a* = 0.15, *b* = 0.2, *c* = 5



Figure A.2: Rössler's system for a = 0.15, b = 0.2, c = 5. (a): Phase space. Trajectory (red dashed line) spiraling towards the stable limit cycle (black solid line). (b): Floquet multipliers identified by multiflap (black crosses) compared with the largest absolute value of the ones reported in [109] (blue circles). (c) and (d): Projection of the periodic orbit identified in the Rössler's system, on the plane x - y, and on the plane x - z. The identified periodic orbit (black solid line) is compared with the one identified in [109] (blue scatter points) for sake of validation.

For this set of parameters the limit cycle is stable. The trajectory starting out the limit cycle spirals towards it. The stable limit cycle is reported in the phase space in Figure A.2(a) with a period of T = 11.9, while the eigenvalues are pictured in Figure A.2(b). The projection of the orbit on 2D planes is reported in Figure A.2(c) and (d), where the orbit found by multiflap (black solid line) is compared with the orbit shown in [109] (blue scatter points).

Case 3: *a* = 0.2, *b* = 0.2, *c* = 4.5



Figure A.3: Rössler's system for a = 0.2, b = 0.2, c = 4.5. (a): Phase space. Trajectory (red dashed line) showing a strange attractor behavior, is constantly repelled by the unstable limit cycle (black solid line). (b): Floquet multipliers identified by multiflap (black crosses) compared with the largest absolute value of the ones reported in [170] (blue circles). (c) and (d): Projection of the periodic orbit identified in the Rössler's system, on the plane x - y, and on the plane x - z. The identified periodic orbit (black solid line) is compared with the one identified in [170] (blue scatter points) for sake of validation.

For this set of parameters the limit cycle is unstable. The trajectory (red) starting out the limit cycle spirals in the phase space behaving as strange attractor. The unstable limit cycle in the phase space is reported in Figure A.3(a) with a period T = 5.8, while the eigenvalues are pictured in Figure A.3(b). The projection of the orbit on 2D planes is reported in Figure A.3(c) and (d), where the orbit found by multiflap (black solid line) is compared with the orbit shown in [170] (blue scatter points).

Case 1	a = 0.15, b = 0.2, c = 3.5	
	multiflap	Reference case [109]
Period	5.920340	5.920340
$ \Lambda_{max} $	0.812186	0.812252
Case 2	a = 0.15, b = 0.2, c = 5	
	multiflap	Reference case [109]
Period	11.904275	11.904275
$ \Lambda_{max} $	0.571964	0.572052
Case 3	a = 0.2, b = 0.2, c = 4.5	
	multiflap	Reference case [170]
Period	5.843969	5.843969
$ \Lambda_{max} $	1.917734	1.918286

 Table A.1: Summary of the validation of Rössler's system.

The summary of the validation against Rössler's system is reported in Table A.1.
Appendix **B**

multiflap tutorial

The code is hosted at https://github.com/vortexlab-uclouvain/multiflap. In order to install the package, from command line:

\$ git clone https://github.com/vortexlab-uclouvain/multiflap.git \$ cd multiflap

and run the installer

\$ python setup.py install

To run multiflap the user has to create two files. The first one will contain the equations constituting the dynamical system, while the second one is the main file which will call the solver. We illustrate an example based on the following ODE system [171]

$$\frac{dD_1}{dt} = p - aAD_1 - dD_1$$

$$\frac{dD_2}{dt} = dD_1 - eD_2$$

$$\frac{dR}{dt} = eD_2 - qR$$

$$\frac{dA}{dt} = bIR - aAD_1$$
(B.1)

whose Stability matrix

$$\mathbb{A}(\mathbf{x}(t),t) = \begin{pmatrix} -d - aA & 0 & 0 & -aD_1 \\ d & -e & 0 & 0 \\ 0 & e & -q & 0 \\ -aA & 0 & b(1-a) & -bR - aD_1 \end{pmatrix}$$
(B.2)

Equation file

The first step to set up the case is to generate the equation file. It creates the class for the system. The methods of this class will be the ODE system, and the hard coding of the Stability matrix. Note that this last step is necessary only if the Jacobian computation is performed analytically according to Equation 4.9. This file needs to be stored in multiflap/odes.

```
....
2
   filename: redox_oscillation.py
   .....
3
   class RedoxModel:
4
           def __init__(self, a=1000, b=2, c=10000, d=0.2, e=0.1, q=0.1, p
        =1):
6
                self.a = a
                self.b = b
8
                self.c = c
9
                self.d = d
10
                self.e = e
                self.q = q
                self.p = p
13
14
                self.dimension = 4
15
16 def dynamics(self, x0, t):
            """ODE system
18
           This function will be passed to the numerical integrator
19
20
21
           Inputs:
           x0: initial values
            t: time
24
           Outputs:
           x_dot: velocity vector
26
            .....
27
           D1, D2, R, A = x0
28
           dD1_dt = self.p - self.a*A*D1 - self.d*D1
29
           dD2_dt = self.d*D1 - self.e*D2
30
31
           dR_dt = self.e*D2 - self.q*R
           dA_dt = self.b*(1-A)*R - self.a*A*D1
32
           vel_array = np.array([dD1_dt, dD2_dt, dR_dt, dA_dt], float)
34
35
           return vel_array
```

```
36
37
    def get_stability_matrix(self, x0, t):
38
            .....
39
40
            Stability matrix of the ODE system
41
42
            Inputs:
            x0: initial condition
43
            Outputs:
44
            A: Stability matrix evaluated at x0. (dxd) dimension
45
            A[i, j] = dv[i]/dx[j]
46
            .....
47
            D1, D2, R, A = x0
48
            A_matrix = np.array([[-self.d - self.a*A, 0., 0.,-self.a*D1],
49
            [self.d, -self.e, 0., 0.],
50
            [0., self.e, -self.q, 0.],
51
            [-self.a*A, 0., self.b*(1-A), -self.b*R -self.a*D1]], float)
52
53
54
            return A_matrix
```

main file

The main file has to be located in multiflap directory. It contains the following.

```
1 """
2 filename: main_redox.py
3 """
4 from odes.redox_oscillation import RedoxModel
5 from ms_package.rk_integrator import rk4
6 from ms_package.multiple_shooting_period import MultipleShootingPeriod
7 from ms_package.lma_solver_period import SolverPeriod
```

Set the initial guess value, and then generate the model object.

```
8 x = [0.5, 0.5, 0.6, 0.2]
9 mymodel = RedoxModel() # empty argument, default parameters previously
set
```

Pass the object to the multiple-shooting class, and solve it

```
10 ms_obj = MultipleShootingPeriod(x, # <- initial values
11 M=2, # <- number of points
12 period_guess= 23., # <- guess peeriod
13 t_steps=50000, # <- numbers of time step
14 model=mymodel)
15
16
17 mysol = SolverPeriod(ms_obj = ms_obj).lma()
```

Once the computation is done, the results can be accessed as follows:

```
16 # Accessing the Jacobian (or Monodromy) matrix
```

```
jac = mysol[4]
18
   # Floquet multipliers and related eigenvector in x*
19
   eigenvalues, eigenvectors = np.linalg.eig(jac)
20
21
22
  # State space limit cycle solution
23
   sol_array = mysol[3].space
24
   # Time array & period
25
26
   sol_time = mysol[3].time
27
   period = sol_time[-1]
28
29
   # Plotting results
30
   plt.plot(sol_time, sol_array[:,0], label='D1')
   plt.plot(sol_time, sol_array[:,1], label='D2', color='orange')
31
   plt.plot(sol_time, sol_array[:,2, label='R', color='g')
32
   plt.plot(sol_time, sol_array[:,3], label='A', color='r')
33
   plt.show()
34
```

Finally run the main file

\$ python3 main_redox.py

It will produce the following output



Appendix C

Comparison of different integrator schemes

We present a comparison between three different schemes of numerical integration, namely a Runge-Kutta of second order, a Runge-Kutta of fourth order, and the native Python integrator odeint. We tested them on the limit cycle solution corresponding to the unstable case reported in Figure 6.3(a). The result of the comparison is illustrated in Figure C.1.



Figure C.1: Comparison between three integrator schemes. The blue solid line is traced out using the native Python integrator odeint. Red points correspond to the fourth order Runge-Kutta scheme. Black points correspond to a second order Runge-Kutta scheme.

Appendix D

Modification of main.py file for Robotran

Listing D.1 reports the series of instruction described in Chapter 7, aimed at integrating the equations of motion generated by Robotran.

Listing D.1: Candidate main.py file to run a direct dynamic integration with ROBOTRAN

```
.....
   filename: main.py
   .....
3
   # Modules loading
4
   import MBsysPy
5
7 # 1) Project loading
mbs_data = MBsysPy.MbsData('.../dataR/myproject.mbs')
9 # Set arbitrary initial conditions
10 mbs_data.reset()
mbs_data.q[1] = 0. # position joint 1
12 mbs_data.q[2] = 0. # position joint 2
13 mbs_data.qd[1] = 0. # velocity joint 1
14 mbs_data.qd[2] = 0. # velocity joint 2
15
16 # 2) Direct Dynamics
17 mbs_dirdyn = MBsysPy.MbsDirdyn(mbs_data)
18 mbs_dirdyn.set_options(dt0=1e-3, tf=1.0, save2file=1)
19 run = mbs_dirdyn.run() # Start the numerical integration of the system
20
```

The modification of the the main.py file in order to run a multipleshooting simulation, is presented in Listing D.2. **Listing D.2:** Candidate main.py file to run a multiple-shooting case with ROBOTRAN.

```
.....
1
   filename: main.py
   .....
3
   # Modules loading
4
   import MBsysPy
5
  from multiple_shooting import MultipleShooting
6
   from lma_solver import Solver
8
9
   # Project loading
10
  mbs_data = MBsysPy.MbsData('../dataR/myproject.mbs')
12
  # Setting arbitrary initial conditions
13
14 mbs_data.reset()
15 mbs_data.q[1] = 0.
   mbs_data.q[2] = 0.
16
  mbs_data.qd[1] = 0.
  mbs_data.qd[2] = 0.
18
19
   # Store the initial conditions in a single array
20
   x0 = [mbs_data.q[1], mbs_data.q[2], mbs_data.qd[1], mbs_data.qd[2]]
21
22
23 mymodel = MultipleShooting(x0, # guess value
  M = 2, # number of points dim = 4, # dimension of the phase space
24
25
  period_guess = 2., # period guess
26
   model = mbs_data) # mbs object
27
28
   results = Solver(ms_obj = mymodel).lma() # Call LMA solver
29
30
```

Appendix E

Estimation of the largest Lyapunov exponent

Another multiflap feature is the computation of the largest Lyapunov exponent. Its estimation is based on a long time integration of two nearby trajectories, in which the separation in calculated at fixed time intervals via a recursive re-scaling of the mutual distance between the two trajectories [103].



Figure E.1: Two neighboring trajectories for the estimation of the largest Lyapunov exponent. The reference trajectory is pictured in black solid line. The perturbed trajectory is pictured in blue solid line, and re-scaled at each control point.

The following formalism applies. The leading trajectory starts from an

initial value \mathbf{x}_1 . Fixed a sampling time Δt , the (i + 1)-th point of the reference trajectory is

$$\mathbf{x}_{i+1} = f(\mathbf{x}_i) \Big|_{t_i}^{t_i + \Delta t}$$
(E.1)

The neighboring trajectory is initially perturbed with a perturbation δx . At every sampling point this trajectory is re-scaled with the module of the initial perturbation at time t_0 . The exact separation of the trajectories is

$$\mathbf{d}_{i+1} = f(\mathbf{z}_i)\Big|_{t_i}^{t_i + \Delta t} - \mathbf{x}_{i+1}$$
(E.2)

the vector \mathbf{d}_{i+1} is re-scaled with the module of the initial perturbation $|\delta \mathbf{x}|$. The next integration point for the perturbed trajectory is therefore

$$\mathbf{z}_{i+1} = \mathbf{x}_{i+1} + \mathbf{d}_{i+1} \frac{|\delta \mathbf{x}|}{|\mathbf{d}_{i+1}|}$$
 (E.3)

By recursively finding the z_i points via re-scaling the distance, and then measuring the deviation of the two trajectories, the estimation of the largest Lyapunov exponent is calculated then from the series [172]

$$\lambda_t = \frac{1}{N\Delta t} \sum_{i=1}^N \ln\left(\frac{|\mathbf{d}_i|}{|\delta \mathbf{x}|}\right). \tag{E.4}$$