On the optimal combination of naive and mean-variance portfolio strategies

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Outline

Motivation

Contributions

Optimal vs. constrained combination

Mixed strategy & empirical results

Takeaways

Motivation

Efficient mean-variance portfolios

- Markowitz (1952) defines efficient mean-variance portfolios:

$$\boldsymbol{w}^\star = \operatorname*{argmax}_{\boldsymbol{w}} \underbrace{\boldsymbol{w}^\top \boldsymbol{\mu}}_{\text{portfolio mean}} - \frac{\gamma}{2} \overbrace{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}^{\text{portfolio risk}} = \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$$

  - $\boldsymbol{w}$: vector of weights on risky assets ($1 - \mathbf{1}'\boldsymbol{w}$ is invested in the risk-free asset)
  - $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$: vector of mean returns and covariance matrix of returns
  - γ: risk-aversion coefficient

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- Challenge: Investors must estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ and rely on

$$\widehat{\boldsymbol{w}}^\star = \frac{1}{\gamma} \widehat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\mu}}$$

Estimation risk and poor performance

- Problem: $\widehat{\boldsymbol{w}}^\star$ performs poorly out-of-sample. The naive equally weighted portfolio ($\boldsymbol{w}_{ew} = \mathbf{1}/N$) often outperforms it.
- See DeMiguel et al. (2009, RFS).
- This fact questions the added value of portfolio optimization.

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- Plug-in: relying on $\widehat{\boldsymbol{w}}^{\star}$. It is theoretically optimal, but subject to heavy estimation risk
  - Approaches exist to limit estimation errors on parameters (e.g. Ledoit & Wolf shrinkage)
- Naive: relying on $\boldsymbol{w}_{ew}$. It is insensitive to estimation risk but it is also suboptimal!
- Let's try to find the middle ground ...

Combination of naive and mean-variance portfolio strategies

- Tu and Zhou (2011, JFE) introduce a portfolio combination:

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \kappa_1 \hat{\boldsymbol{w}}^{\star} + \kappa_2 \boldsymbol{w}_{ew} \quad \text{with} \quad \boldsymbol{c} := \kappa_1 + \kappa_2 = 1.$$

- $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ are the combination coefficients.

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- As in Kan And Zhou (2007, JFQA), they are found by maximizing the expected out-of-sample utility (***EU***)

$$\boldsymbol{\kappa}^{tz} = \underset{\boldsymbol{\kappa}}{\text{argmax}} \ \mathbb{E} \underbrace{\left[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})' \boldsymbol{\mu} - \frac{\gamma}{2} \hat{\boldsymbol{w}}(\boldsymbol{\kappa})' \boldsymbol{\Sigma} \hat{\boldsymbol{w}}(\boldsymbol{\kappa}) \right]}_{\text{out-of-sample utility}} \quad \text{s.t.} \quad \boldsymbol{c}$$

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$$= \underset{\boldsymbol{\kappa}}{\operatorname{argmax}} \; \boldsymbol{EU} \quad \text{s.t.} \quad \boldsymbol{c}$$

- This combined rule significantly outperforms $\widehat{\boldsymbol{w}}^{\star}$ and may outperform $\boldsymbol{w}_{ew}$.

This seems all well and good, but ...

- Tu and Zhou (2011) force combination coefficients to sum to one:

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- Problem: $\hat{\boldsymbol{w}}^\star$ is not a fully invested portfolio: $\mathbf{1}'\widehat{\boldsymbol{w}}^\star \neq 1$.
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- We actually invest in three portfolios: the sample tangent, the risk-free asset, and the equally-weighted portfolio...

- ... But under $\boldsymbol{c}$, we only have one coefficient to control the weight allocated to each one of them !

This paper

- We propose to relax the constraint on combination coefficients:

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \kappa_1 \hat{\boldsymbol{w}}^\star + \kappa_2 \boldsymbol{w}_{ew} \quad \text{~~with~~} \quad \cancel{\boldsymbol{c}}$$

- We find the optimal combination coefficients that maximize the expected out-of-sample utility of the portfolio combination:

$$\boldsymbol{\kappa}^{opt} = \underset{\boldsymbol{\kappa}}{\text{argmax}} \quad \boldsymbol{EU} \quad \text{~~s.t.~~} \quad \cancel{\boldsymbol{c}}$$

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Optimal vs. constrained combination

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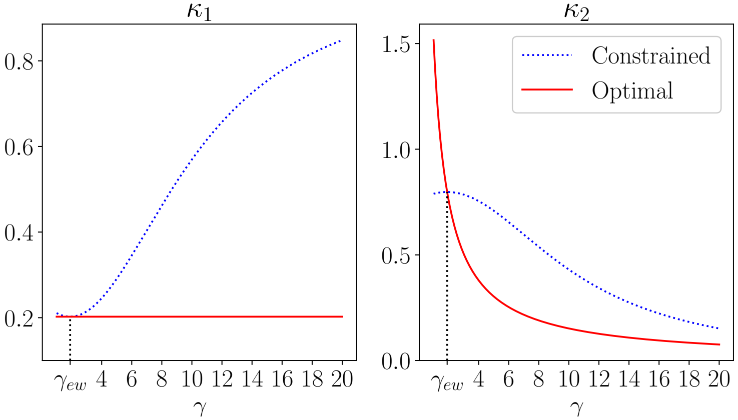
2. Outperformance relative to the risk-free asset
3. Less extreme portfolio weights
4. Outperformance in expected out-of-sample utility (EU)
5. Outperformance relative to the two-fund rule
6. Outperformance in expected out-of-sample Sharpe Ratio

Combination coefficients

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \kappa_1 \hat{\boldsymbol{w}}^\star + \kappa_2 \boldsymbol{w}_{ew}$$



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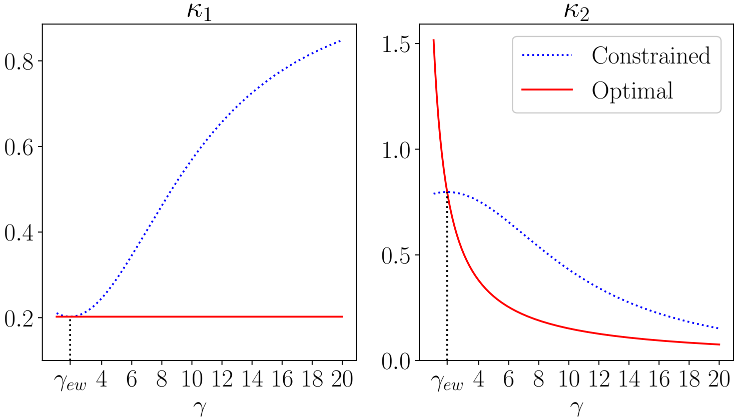
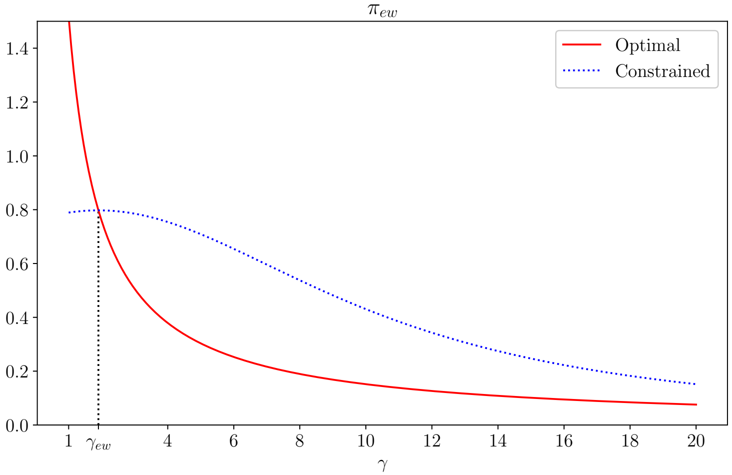


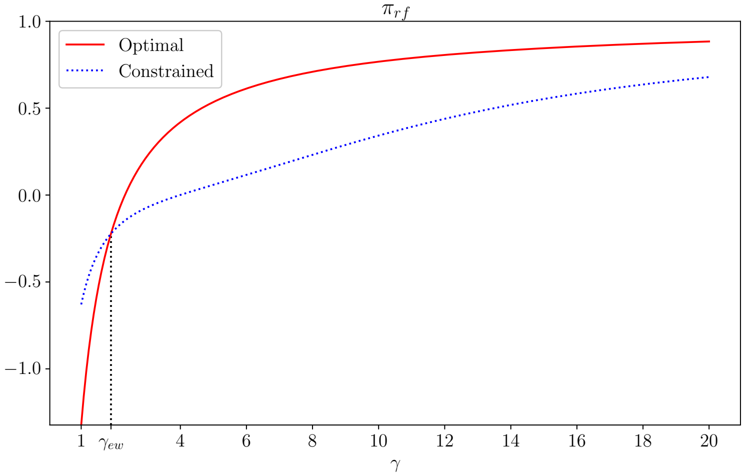
- Key facts: $\kappa_1^{tz} \geq \kappa_1^{opt}$ and equality iff $\gamma = \gamma_{ew} = \mu_{ew}/\sigma_{ew}^2$
- We use the 25 portfolios of stocks sorted on size and book-to-market (25SBTM) dataset spanning 07/26 to 12/21.

Benefit 1: Optimal exposure to the equally-weighted portfolio



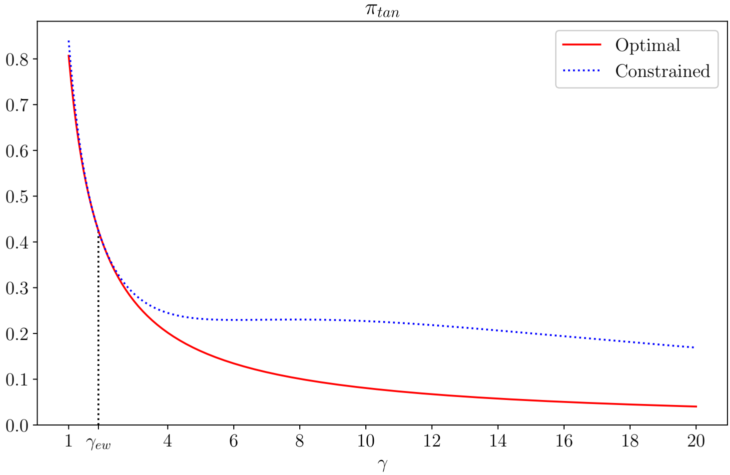
When $\gamma = 10$, $T = 120$, and $N = 25$, $\pi_{ew}^{tz} = 43\%$ and $\pi_{ew}^{opt} = 15\%$

Benefit 1: Optimal exposure to the risk-free asset



When $\gamma = 10$, $T = 120$, and $N = 25$, $\pi_{rf}^{tz} = 34\%$ and $\pi_{rf}^{opt} = 77\%$

Benefit 1: Optimal exposure to the tangent portfolio



When $\gamma = 10$, $T = 120$, and $N = 25$, $\pi_{tan}^{tz} = 23\%$ and $\pi_{tan}^{opt} = 8\%$

Benefit 2: Outperformance relative to the risk-free asset

Problem: Under mild conditions, the constrained combination can underperform the risk-free asset for investors with a risk-aversion higher than a given threshold: γ_{neg}[1]

[1] $\gamma_{neg} = 5.41$ with the 25SBTM dataset.

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Problem: Under mild conditions, the constrained combination can underperform the risk-free asset for investors with a risk-aversion higher than a given threshold: γ_{neg}[1]

Intuition: The combination should outperform each of its components, but it might not be possible because of the constraint.

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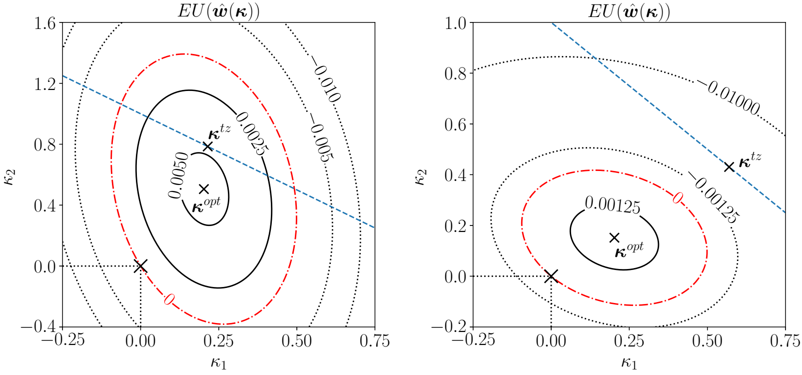
Benefit 2: Outperformance relative to the risk-free asset



(a) $\gamma = 3 < \gamma_{neg}$ (b) $\gamma = 10 > \gamma_{neg}$

Benefit 3: Less extreme weight allocation

- Problem: The weights allocated to the risky assets by the constrained strategy are extreme.
- Why it matters: practical implementation and transaction costs

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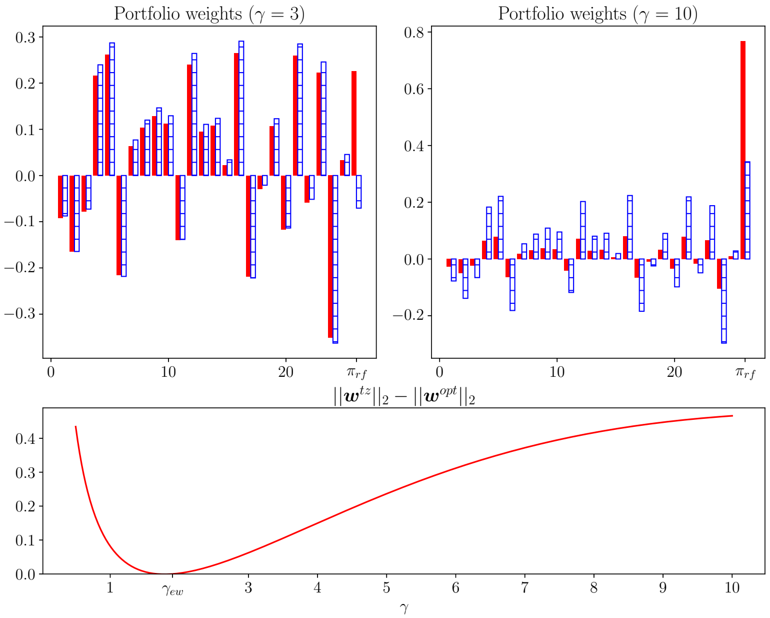

Mixed strategy & empirical results

Mixing $\hat{w}^{opt}$ and $\hat{w}^{tz}$

- Practical concern: $\hat{\kappa}_2^{opt}$ is unbounded and very sensitive to estimation errors in $\hat{\boldsymbol{\mu}}$ while $\hat{\boldsymbol{\kappa}}^{tz}$ is bounded in $[0, 1]$

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- Practical concern: $\hat{\kappa}_2^{opt}$ is unbounded and very sensitive to estimation errors in $\hat{\boldsymbol{\mu}}$ while $\hat{\boldsymbol{\kappa}}^{tz}$ is bounded in $[0, 1]$
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- When theoretical gain is small, *the wrong constraint helps* (cf. Jagannathan and Ma (2003)).

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- Estimation errors have a substantial impact on out-of-sample performance
- When theoretical gain is small, *the wrong constraint helps* (cf. Jagannathan and Ma (2003)).
- We define a symmetric interval around γ_{ew}: $[\gamma_{ew} \pm \epsilon]$. In this interval, $\hat{\boldsymbol{w}}^{tz}$ is preferred to $\hat{\boldsymbol{w}}^{opt}$:

$$\hat{\boldsymbol{w}}^{opt,tz} = \begin{cases} \hat{\boldsymbol{w}}^{tz} & \text{if } \ \gamma \in [\gamma_{ew} \pm \epsilon], \\ \hat{\boldsymbol{w}}^{opt} & \text{otherwise.} \end{cases}$$

Empirical results

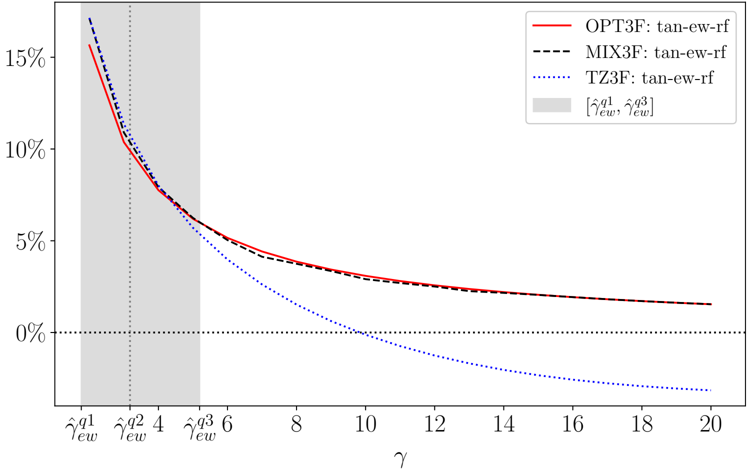
Dataset	N	Time period	Abbr.
25 portfolios formed on size and book-to-market	25	07/1926 - 12/2021	25SBTM
10 momentum portfolios	10	01/1927 - 12/2021	10MOM
25 portfolios formed on OP[2] and investment	25	07/1963 - 12/2021	25OPINV
48 industry portfolios	48	07/1969 - 12/2021	48IND

Table 1: List of datasets used in the empirical analysis (monthly frequency)

- Different rolling window sizes
- Various estimators for $\hat{\boldsymbol{\Sigma}}$: sample, linear shrinkage, nonlinear shrinkage (Ledoit & Wolf (2004, 2017))
- Monthly rebalancing and proportional transaction costs of 10bps

[2] Operating profitability.

Empirical results



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3. *The wrong constraint helps*: the $\boldsymbol{c}$ constraint is helpful in some cases
4. We introduce a mixed strategy combining both strategies
5. The optimal strategy always delivers a positive net utility
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Thank you!

Please send comments to

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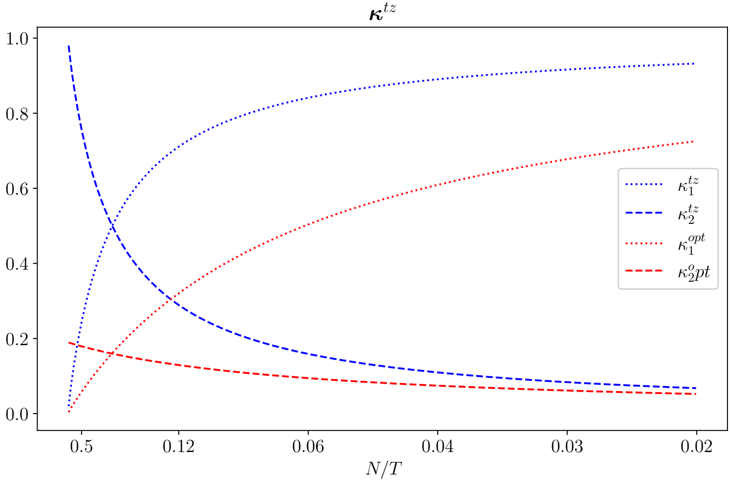
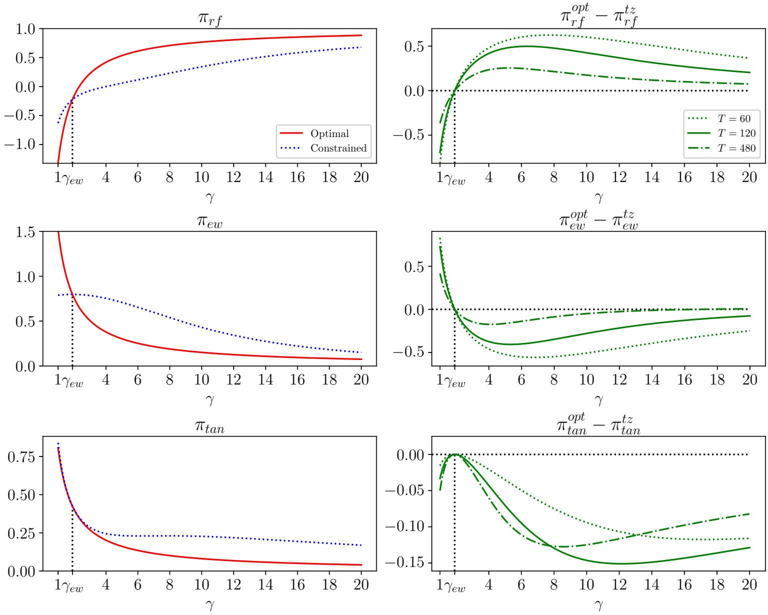
Impact of estimation risk



Optimal exposure to the three funds + estimation risk



Combination coefficients

The constrained combination coefficients are

$$\kappa_1^{tz} = \frac{\psi^2 + \sigma_{ew}^2(\gamma - \gamma_{ew})^2}{\psi^2 + \sigma_{ew}^2(\gamma - \gamma_{ew})^2 + d} \in [0,1] \quad \text{and} \quad \kappa_2^{tz} = 1 - \kappa_1^{tz} \in [0,1].$$

The optimal combination coefficients are

$$\kappa_1^{opt} = \frac{\psi^2}{\psi^2 + d} \in [0,1] \quad \text{and} \quad \kappa_2^{opt} = \frac{\gamma_{ew}}{\gamma}(1 - \kappa_1^{opt}) \in \mathbb{R},$$

d is a parameter that increases with estimation risk N/T

γ_{neg} threshold definition

$$\gamma > \gamma_{ew}\left(1 + \sqrt{1 + \frac{\theta^4/\theta_{ew}^2}{d - \theta^2}}\right) = \gamma_{neg}.$$

Notation

- Sample mean-variance portfolio (SMV) (from sample of size T)

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- We define $\theta^2 = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ as the maximum squared Sharpe ratio, $\theta_{ew}^2 = \mu_{ew}^2/\sigma_{ew}^2$ as the squared Sharpe ratio of the EW portfolio, and ψ^2 as the difference between the two:

$$\psi^2 = \theta^2 - \theta_{ew}^2 \geq 0.$$

Sample estimators

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{R_t}, \quad \widehat{\boldsymbol{\Sigma}} = \frac{1}{T-N-2}\sum_{t=1}^{T}(\boldsymbol{R_t} - \hat{\mu})(\boldsymbol{R_t} - \hat{\mu})'.$$

Efficient frontier - Short reminder