OPTION PRICING AND HEDGING IN ILLIQUID MARKETS IN PRESENCE OF JUMP CLUSTERING

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Option pricing and hedging in illiquid markets in presence of jump clustering

John-John Ketelbuters[*] Donatien Hainaut[†]

Abstract

The topic of this paper is the pricing and hedging of options on small capitalization stocks. Such stocks tend to exhibit two features. The first is the presence of motionless periods in their prices. It is a consequence of a lack of liquidity, since these stocks are not heavily traded. The second is the occurrence of clustered sudden moves in the price at the times the stock is traded. The model we propose is therefore a self-exciting Hawkes jump-diffusion process that is time-changed by the inverse of an alpha-stable subordinator, a process that exhibits motionless periods. This article is divided into two parts. In the first part, we prove that, when adding some information to the inverse alpha-stable subordinator, we obtain a Markov process. This result allows us to obtain the dynamic framework we need to establish a hedging strategy. In the second part, we deal with the pricing and hedging of options. To this end, we derive a fractional partial differential equation (FPDE) for the Fourier transform of the log asset price. We introduce a finite difference method to solve this FPDE. Prices of options can be obtained by a numerical inversion of the Fourier transform with a fast-Fourier transform algorithm. Changes of measures are then discussed, as well as an optimal quadratic hedging strategy for options. Finally, the last section of this paper presents some numerical experiments.

Introduction

Illiquid assets, for example in emerging markets, are characterized by periods of time without any trade, and thus without movement in their prices. Properly taking into account this feature require stochastic processes that exhibit motionless periods. In this regard, usual models that involve Brownian motions or more general Lévy processes do not seem appropriate, as such processes move all the time. This problem also occurs in physical systems exhibiting sub-diffusion. The periods without trades correspond to the trapping events in which the subdiffusive particle gets immobilized. Subdiffusion is thus a well known phenomenon in statistical physics. The usual mathematical tool for this phenomenon is fractional calculus. More precisely, the density of a subdiffusive model is characterized by a fractional partial differential equation (FPDE), as in Barkai, Metzler, and Klafter (2000), Metzler and Klafter (2004) and Metzler and Klafter (2000). That is, the partial derivative with respect to time is replaced by a fractional derivative. This particular equations are also called fractional Fokker-Planck equations. Due to the link between the subdiffusive behaviour and fractional calculus, we also refer to subdiffusive models as fractional models. Subdiffusive dynamics can also be obtained by the technique of time-change, as in e.g. Ketelbuters and Hainaut (2022). A subdiffusion is obtained when a standard diffusion process is time-changed by the inverse of an α-stable Lévy process. This approach is used in Magdziarz (2009) to obtain a subdiffusive version of the famous Black and Scholes model and to show that vanilla option prices can be computed by numerical integration. It is also established that despite the Black and Scholes market being complete, its subdiffusive counterpart is not, since more than one risk neutral measure exist. Hainaut and Leonenko (2021) propose an extension to a subdiffusive jump diffusion model, for which option prices can be computed by solving a fractional partial integro-differential equation. This equation is similar to Dupire's forward partial differential equation (PDE) for call prices.

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A feature that often characterize financial data is the clustering of certain events. Events that are typically clustered are jumps in the asset prices and defaults of companies. As an example, we can cite the 2008 economic crisis. The collapse of Lehman Brothers brought the financial system to the brink of a breakdown. The important economical consequences indicates the existence of a propagation phenomena that is channeled through the complex economic system. Self-exciting point processes, also called Hawkes processes, allow to mathematically take into account this phenomenon of contagion. This approach finds its origins in the papers Hawkes (1971a) and Hawkes (1971b), as well as in Hawkes and Oakes (1974). The arrival of events is ruled by an intensity process which represents the instantaneous probability of occurrence. In the most common and simplest setting, the intensity process suddenly increases when an event occurs. Motivated by the 2008 crisis, the Hawkes process approach was used in Errais, Giesecke, and Goldberg (2010) to model the spillover effect between defaults. Self-exciting processes were also used in Ait-Sahalia, Cacho-Diaz, and Laeven (2015) to model the clustering of jumps in the prices of financial assets or in Hainaut (2016b) and Hainaut (2016a) for modeling the term structure of interest rates. The pricing and hedging of vanilla options when the asset is driven by a self-exciting jump diffusion was studied in Moraux and Hainaut (2018). More precisely, this article shows that the Fourier transform of the log-asset price is described by a system of ordinary differential equations. After having solved this system numerically, the method of Carr and Madan (1999) is used to invert the transform with a fast Fourier transform (FFT) algorithm, leading to the option prices.

A self-exciting model that exhibits subdiffusive behaviour was introduced in Hainaut (2020) under the name of fractional Hawkes process. It is shown in Ketelbuters and Hainaut (2022) that such a process can be calibrated to credit default swaps by solving numerically a FPDE that characterizes the associated Laplace transform. The fractional Hawkes process is obtained by time-changing a Hawkes process, which is similar to the work of Magdziarz (2009) with the Black and Scholes model. More precisely, the time-change used for subdiffusions is a process $(S_t)_{t\geqslant 0}$ defined as $S_t := \inf\{\tau > 0 : U_\tau > t\}$, where $(U_t)_{t\geqslant 0}$ is an α-stable subordinator, i.e. an increasing Lévy process to be described with more details hereinafter. In other words, $(S_t)_{t\geqslant 0}$ is the inverse of the α-stable subordinator $(U_t)_{t\geqslant 0}$. $(S_t)_{t\geqslant 0}$ is of course nondecreasing, since S_t is defined as the smallest time $(U_t)_{t\geqslant 0}$ goes above the level t.

We refer to $(S_t)_{t\geqslant 0}$ as a time-change and not as a subordinator because it is not a Markov process, and thus not a Lévy process either. Related to that, the practical use in finance of such a time-change suffers from a twofold problem: nothing can be said about the distribution of S_t conditional on S_s, $s < t$. Even worse is the fact that even if the distribution of S_t given S_s could be known, the fact that $(S_t)_{t\geqslant 0}$ is not a Markov process would make this distribution irrelevant. This is of course a significant problem if, as often, one wants to re-evaluate a financial product after it is issued. A solution to the first part of this problem was proposed in Hainaut (2021). More specifically, Hainaut (2021) shows how to determine the distribution of S_t conditional on (S_s, U_{S_s}), $s < t$. That is, we have to adjunct the additional information on the last known value U_{S_s} of $(U_{S_t})_{t\geqslant 0}$, which represents the actual value reached by $(U_t)_{t\geqslant 0}$ on the first time it goes above the level s. This solution provides an insight for the second part of the problem, which becomes: *Is it relevant to use the distribution of S_t given the last known values (S_s, U_{S_s})?* As a matter of fact, it could be that the distribution S_t seen from time $s \in (0, t)$ depends on some values (S_u, U_{S_u}) with $u < s$, leading to a loss of information. The first main contribution of this paper is to show that the answer to the question in italics is *yes*. We show that even though $(S_t)_{t\geqslant 0}$ is not a Markov process, the bivariate process $(S_t, U_{S_t})_{t\geqslant 0}$ satisfies the Markov property, ensuring that the approach of Hainaut (2021) does not suffer from any loss of information.

The second main contribution of this paper is a method for pricing vanilla options when the asset model is both self-exciting and subdiffusive. We obtain such a model by applying the technique of time-change mentioned above on the self-exciting jump diffusion asset model of Moraux and Hainaut (2018), leading to what we call a fractional (or subdiffusive) self-exciting jump diffusion model. We show that the Fourier transform of the log-price is ruled by a FPDE. Inspired by the finite difference method of Alikhanov (2015), we propose a numerical method to solve the aforementioned FPDE and show that its error is (at least) of order 2. Then, we compute vanilla option prices by numerical inversion of the transform with the help of a FFT algorithm.

The third contribution of this paper is to tackle the question of the hedging of contingent claims in a fractional self-exciting jump diffusion setting. As the market is incomplete, a perfect replication of all the contingent claims is impossible. In this case, we aim at approximating the payoff of the contingent claim by a self-financing portfolio (or self-financing strategy) that minimizes the expected squared hedging error, as in e.g. Föllmer and Sondermann (1985) and Cont, Tankov, and Voltchkova (2012). Unlike approaches based on other loss functions, quadratic hedging yields linear hedging rules that are very convenient to implement, see Schweizer (2001). Relying on the predictable representation theorem for square integrable martingales of Kunita and Watanabe (1967), we derive the existence and uniqueness of an optimal quadratic hedging strategy for any square integrable contingent claim. Moreover, if the payoff of the contingent claim is not path dependent, i.e. if it depends on the final value of the asset only, we give an explicit formula for this optimal quadratic hedging strategy. In particular, we deduce the hedging strategies for call options.

This paper is organized as follows. In the first section, we describe precisely the model we propose. In the second section, we deal with the Markov property of $(S_t, U_{S_t})_{t\geqslant 0}$. To do so, we derive some properties of the processes that are involved and study the different filtrations that arise in our setting. In the third section, we derive a fractional partial differential equation for the transform of the log-asset price. In the fourth section, we describe a whole class of risk neutral measures. As a consequence, the market is arbitrage free and incomplete. In the fifth section, we propose a numerical method based on finite differences that allows to solve numerically the fractional partial differential equation derived in the third section. Furthermore, we show how the results of this method can be used to retrieve the price of a call option under a particular risk neutral measure. In Section 6, we deal with the optimal quadratic hedging of contingent claims. Finally, the last section presents our numerical results that consist of call option prices and implied volatilities computed with the method we proposed.

0.1 The Model

This first section is dedicated to the precise introduction of the model we use. For a technical reason, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is assumed to be the product of two complete probability spaces $(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{P}^{(1)})$ and $(\Omega^{(2)}, \mathcal{F}^{(2)}, \mathbb{P}^{(2)})$. The first probability space $(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{P}^{(1)})$ carries the non-fractional model which, as mentioned in the introduction, is the same as in Moraux and Hainaut (2018). The other probability space carries all the processes related to the time-change operation (the α-stable Lévy process and its inverse). The introduction of the model with two probability spaces is the reason why the different stochastic processes will be indexed by a superscript (i) indicating on which probability space they live ($i = 1, 2$). This gives a little cumbersome notations at the beginning of this article, but the product space construction is crucial to simplify the proof of one of our results (this result is Proposition 0.17).

Let $(A_t^{(1)})_{t\geqslant 0}$ be a stochastic process on $(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{P}^{(1)})$ that satisfy the following stochastic differential equation (SDE)

$$\frac{\mathrm{d}A_t^{(1)}}{A_{t-}^{(1)}} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t^{(1)} + \mathrm{d}D_t^{(1)} - \lambda_t^{(1)} \mathbb{E}[e^{\xi} - 1]\mathrm{d}t, \tag{1}$$

where $(W_t^{(1)})_{t\geqslant 0}$ is a standard Brownian motion on $(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{P}^{(1)})$. The processes $(J_t^{(1)})_{t\geqslant 0}$ and $(\lambda_t^{(1)})_{t\geqslant 0}$ are defined as we explain now. Let $(N_t^{(1)})_{t\geqslant 0}$ be a pure jump process with jumps of size 1 and intensity $(\lambda_t^{(1)})_{t\geqslant 0}$ and $\xi_1, \xi_2, \cdots =_{\text{dist}} \xi$ be independent and identically distributed random variables whose distribution is given by the probability measure ν on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. We assume that the jumps $\xi_1, \xi_2, \ldots$ are double-exponential random variables. That is, the probability measure ν is of the form

$$\nu(B) := p \int_{B\cap\mathbb{R}^+} \rho^+ e^{-\rho_+ z} \mathrm{d}z - (1-p) \int_{B\cap\mathbb{R}^-} \rho^- e^{-\rho_- z} \mathrm{d}z \tag{2}$$

where p and $(1-p)$ are respectively the probabilities to observe upward and downward jumps and $\rho_+ > 0$, $\rho_- < 0$ are the jump size parameters. The average upward jump size is $1/\rho_+ > 0$ whereas the average downward jump size is $1/\rho_- < 0$. Overall, the expected size of a jump is $\mathbb{E}[\xi] = p(1/\rho_+) + (1-p)(1/\rho_-)$. The joint moment-generating function of $(\xi, |\xi|)$ will be needed in the subsequent developments and is denoted by ψ. From Equation (2), we have

$$\begin{aligned}\psi(z_1, z_2) &:= \mathbb{E}\left[e^{z_1\xi + z_2|\xi|}\right] \\ &= p\frac{\rho_+}{\rho_+ - (z_1 + z_2)} + (1-p)\frac{\rho_-}{\rho_- - (z_1 - z_2)}\end{aligned} \tag{3}$$

provided that $(z_1 + z_2) < \rho_+$ and $(z_1 - z_2) > \rho_-$. We set

$$D_t^{(1)} := \sum_{k=1}^{N_t^{(1)}} (e^{\xi_k} - 1) \qquad J_t^{(1)} := \sum_{k=1}^{N_t^{(1)}} \xi_k \qquad L_t^{(1)} := \sum_{k=1}^{N_t^{(1)}} |\xi_k|, \tag{4}$$

and define the stochastic intensity process $(\lambda_t^{(1)})_{t\geqslant 0}$ as

$$\mathrm{d}\lambda_t^{(1)} = \kappa(\theta - \lambda_t^{(1)})\mathrm{d}t + \eta \mathrm{d}L_t^{(1)}. \tag{5}$$

Finally, we define the log-price of the stock as the process $(X_t^{(1)})_{t\geqslant 0} := (\ln A_t^{(1)})_{t\geqslant 0}$. Using Ito's Lemma, we have

$$\mathrm{d}X_t^{(1)} = \left(\mu - \frac{\sigma^2}{2} - \lambda_t^{(1)}\mathbb{E}[e^{\xi} - 1]\right)\mathrm{d}t + \sigma \mathrm{d}W_t^{(1)} + \mathrm{d}J_t^{(1)}. \tag{6}$$

This relation implies that

$$A_t^{(1)} = A_0^{(1)} \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t - \mathbb{E}[e^{\xi} - 1]\int_0^t \lambda_s^{(1)}\mathrm{d}s + \sigma W_t^{(1)} + J_t^{(1)}\right\}. \tag{7}$$

This completes the introduction of all the stochastic processes defined on $(\Omega^{(1)}, \mathcal{F}^{(1)}, \mathbb{P}^{(1)})$. Let $(U_t^{(2)})_{t\geqslant 0}$ be an α-stable subordinator on the probability space $(\Omega^{(2)}, \mathcal{F}^{(2)}, \mathbb{P}^{(2)})$. The process $(U_t^{(2)})_{t\geqslant 0}$ is thus a strictly increasing Lévy process whose Laplace transform satisfies

$$\mathbb{E}[e^{-\omega U_t^{(2)}}] = e^{-t\omega^{\alpha}}$$

for some $\alpha \in (0,1)$. We denote by $(S_t^{(2)})_{t\geqslant 0}$ the inverse of $(U_t^{(2)})_{t\geqslant 0}$, that is

$$S_t^{(2)} := \inf\{\tau > 0 : U_\tau^{(2)} > t\}$$

for each $t \geqslant 0$. The paths of $(S_t^{(2)})_{t\geqslant 0}$ are nondecreasing $\mathbb{P}^{(2)}$-a.s., but not strictly increasing. Note that $(S_t^{(2)})_{t\geqslant 0}$ has continuous paths $\mathbb{P}^{(2)}$-a.s., which is a consequence of the strictly increasing paths of $(U_t^{(2)})_{t\geqslant 0}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be defined as follows: $\Omega = \Omega^{(1)} \times \Omega^{(2)}$, $\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$ and $\mathbb{P}(B_1 \times B_2) = \mathbb{P}^{(1)}(B_1)\mathbb{P}^{(2)}(B_2)$ for any $B_1 \times B_2 \in \mathcal{F}^{(1)} \times \mathcal{F}^{(2)}$. The process $(A_t)_{t\geqslant 0}$ is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ as $A_t(\omega_1, \omega_2) = A_t^{(1)}(\omega_1)$ for all $(\omega_1, \omega_2) \in \Omega$. The process $(U_t)_{t\geqslant 0}$ is defined as $U_t(\omega_1, \omega_2) = U_t^{(2)}(\omega_2)$. The other processes $(\lambda_t)_{t\geqslant 0}$, $(J_t)_{t\geqslant 0}$, $(S_t)_{t\geqslant 0}$ and the others are defined similarly. This construction obviously implies that the processes $(A_t)_{t\geqslant 0}$ and $(U_t)_{t\geqslant 0}$ are independent. The null sets of $(\Omega, \mathcal{F}, \mathbb{P})$ are denoted by $\mathcal{N}$. By the assumption of completeness of the probability space, we obviously have $\mathcal{N} \subset \mathcal{F}$.

The stock price is represented by the time-changed process $(A_{S_t})_{t\geqslant 0}$. Figure 1 displays examples of paths for the asset price $(A_{S_t})_{t\geqslant 0}$ (or $(A_t)_{t\geqslant 0}$ in the bottom right non fractional case) for different fractional orders α. The cases $\alpha < 1$ exhibit motionless periods, corresponding to the absence of trade for the asset. This is the feature that allows to take into account the potential illiquidity of the asset. Observe that when α is smaller, the periods of illiquidity tend to last longer.

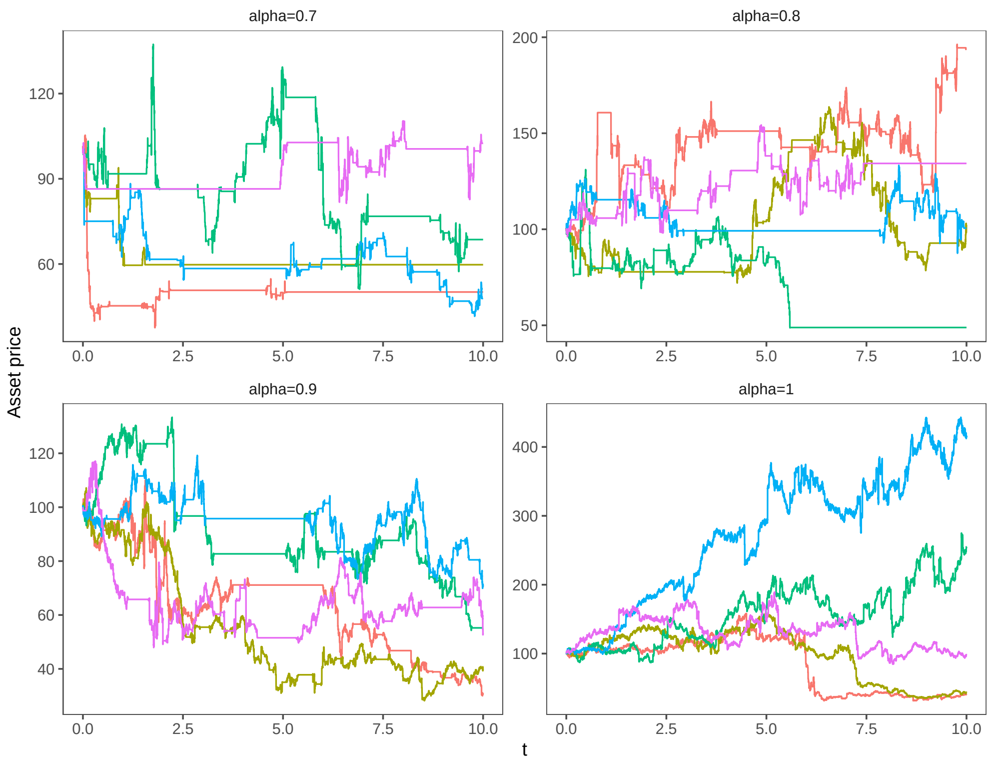

Figure 1: Five simulated paths for fractional order varying from 0.7 to 1. A fractional order equal to 1 corresponds to the non fractional model (no time change).

0.2 Markov property of the time-change process

We denote by $(\mathcal{F}_t^U)_{t\geqslant 0}$ and $(\mathcal{F}_t^S)_{t\geqslant 0}$ the natural filtrations of $(U_t)_{t\geqslant 0}$ and $(S_t)_{t\geqslant 0}$. Note that all the natural filtrations are assumed to contain the null sets $\mathcal{N}$. The α-stable subordinator $(U_t)_{t\geqslant 0}$ admits a PDF that we denote by $p_U(t,\tau) = \frac{\partial}{\partial \tau}\mathbb{P}(U_t \leqslant \tau)$. This PDF satisfies the representation

$$p_U(t,\tau) = \frac{1}{\pi}\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\Gamma(1+k\alpha)\tau^{-(1+\alpha k)}t^k\sin(\pi\alpha k)$$

for $\tau > 0$. See e.g. Proposition 3.1 in Gupta and Kumar (2022) for a proof. As a consequence, the subordinator $(U_t)_{t\geqslant 0}$ has strictly increasing paths $\mathbb{P}$-a.s.. Note that being a Lévy process, $(U_t)_{t\geqslant 0}$ is also a Feller process, for which the reader is referred to the Chapter 3 of Revuz and Yor (2004). If we consider $(U_t)_{t\geqslant 0}$ as a Feller process, any initial distribution can be considered for U_0. For a random variable X and a $\bigvee_{t\geqslant 0}\mathcal{F}_t^U$-measurable random variable Z, we write $\mathbb{E}^X[Z]$ for the expected value obtained for Z if the initial distribution of U_0 is the distribution of X. When no superscript is added on the expectation operator, we consider $U_0 = 0$ $\mathbb{P}$-a.s., i.e. $\mathbb{E}[Z] = \mathbb{E}^0[Z]$ and $(U_t)_{t\geqslant 0}$ is then a Lévy process.

The next proposition gives some further basic properties of $(U_t)_{t\geqslant 0}$ and its inverse $(S_t)_{t\geqslant 0}$.

Proposition 0.1. *(i) For any $t \geqslant 0$, $S_t \in \{\tau \geqslant 0 : U_\tau \geqslant t\}$ $\mathbb{P}$-a.s., so that $U_{S_t} \geqslant t$ $\mathbb{P}$-a.s..*

(ii) For any $t \geqslant 0$, S_t is an $\mathbb{P}$-a.s. finite stopping-time with respect to the filtration $(\mathcal{F}_t^U)_{t\geqslant 0}$. Moreover, $\sup_{u\geqslant 0} S_u < +\infty$ $\mathbb{P}$-a.s..

(iii) $(S_t)_{t\geqslant 0}$ has continuous paths $\mathbb{P}$-a.s..

Proof. (i) By definition of S_t and the definition of infimum, there is a random sequence $\tau_n \in \{\tau \geqslant 0 : U_\tau > t\}$ such that $\tau_n \downarrow S_t$. By the right-continuity of $(U_\tau)_{\tau\geqslant 0}$, we have $\lim_{n\to+\infty} U_{\tau_n} = U_{S_t}$. Since $U_{\tau_n} > t$ for all n, we also have $\lim_{n\to+\infty} U_{\tau_n} = U_{S_t} \geqslant t$.

(ii) That S_t is a stopping-time is a consequence of $\{S_t \leqslant s\} = \{U_s \geqslant t\} \in \mathcal{F}_s^U$ for all $s \geqslant 0$. Let us prove that it is finite a.s.. For any fixed $t > 0$,

$$\begin{aligned}\mathbb{P}(S_t < +\infty) &= \mathbb{P}\left(\bigcup_{n\in\mathbb{N}}\{S_t \leqslant n\}\right)\\ &= \mathbb{P}\left(\bigcup_{n\in\mathbb{N}}\{U_n \geqslant t\}\right),\end{aligned}$$

where we used that $\{S_t \leqslant n\} = \{U_n \geqslant t\}$. Moreover, since the process $(U_t)_{t\geqslant 0}$ is increasing, the sequence of events $(\{U_n \geqslant t\})_{n\in\mathbb{N}}$ is increasing. As a consequence, the continuity of probability measures implies that

$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}}\{U_n \geqslant t\}\right) = \lim_{n\to+\infty}\mathbb{P}(U_n \geqslant t).$$

Since $(U_s)_{s\geqslant 0}$ is an α-stable process, the random variables U_n and $n^{1/\alpha}U_1$ share the same probability distribution, thus

$$\lim_{n\to+\infty}\mathbb{P}(U_n \geqslant t) = \lim_{n\to+\infty}\mathbb{P}\left(\left\{U_1 \geqslant \frac{t}{n^{1/\alpha}}\right\}\right).$$

Again, the sequence of events $\left(\left\{U_1 \geqslant \frac{t}{n^{1/\alpha}}\right\}\right)_{n\in\mathbb{N}}$ is increasing so that the continuity of probability measures yields

$$\lim_{n\to+\infty} \mathbb{P}\left(\left\{U_1 \geqslant \frac{t}{n^{1/\alpha}}\right\}\right) = \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} \left\{U_1 \geqslant \frac{t}{n^{1/\alpha}}\right\}\right)$$
$$= \mathbb{P}(U_1 > 0)$$
$$= 1.$$

This proves that S_t is a.s. finite. As a consequence of the nondecreasing paths of $(S_t)_{t\geqslant 0}$, the sequence of events $(\{S_n < +\infty\})_{n\in\mathbb{N}}$ is decreasing. Therefore, the continuity of probability measures implies

$$\mathbb{P}\left(\sup_{u\geqslant 0} S_u < +\infty\right) = \mathbb{P}\left(\bigcap_{n\in\mathbb{N}} \{S_n < +\infty\}\right)$$
$$= \lim_{n\to\infty} \mathbb{P}(S_n < +\infty) = 1,$$

which concludes the proof of *(ii)*.

(iii) A proof of the right-continuity of $(S_t)_{t\geqslant 0}$ is given at Lemma 4.8 in Chapter 0 of Revuz and Yor (2004). We show now that the left-continuity can be obtained as a consequence of the strictly increasing paths of $(U_t)_{t\geqslant 0}$. Let

$$A := \{\omega \in \Omega : \text{the function } t \mapsto U_t(\omega) \text{ is strictly increasing}\}$$

and

$$B_l := \{\omega \in \Omega : \text{the function } t \mapsto S_t(\omega) \text{ is left-continuous}\}.$$

Let $\omega \in A$ and assume by contradiction that $\omega \notin B_l$. Then there would be some $t \geqslant 0$ such that for all $\varepsilon > 0$,

$$S_t(\omega) - S_{t-\varepsilon}(\omega) > \eta.$$

It implies that

$$U_{S_t(\omega)-\varepsilon+\eta}(\omega) - U_{S_t(\omega)-\varepsilon}(\omega) < \varepsilon.$$

Letting ε decrease to 0 yields

$$U_{(S_t(\omega)+\eta)-}(\omega) - U_{S_t(\omega)-}(\omega) \leqslant 0,$$

where of course U_{t-} denotes the left limit $\lim_{s\uparrow t} U_s$. As a consequence, we should have

$$U_s(\omega) = U_{(S_t(\omega)+\eta)-}(\omega) = U_{S_t(\omega)-}(\omega)$$

for all $s \in (S_t(\omega), S_t(\omega)+\eta)$, contradicting that $\omega \in A$. This proves that $A \subset B_l$ and hence the left-continuity of $(S_t)_{t\geqslant 0}$ follows from the strictly increasing paths of $(U_t)_{t\geqslant 0}$. □

Revuz and Yor (2004) also note that $S_{t-} = \inf\{s \geqslant 0 : U_s \geqslant t\}$, but since $(S_t)_{t\geqslant 0}$ is left-continuous by *(iii)* in Proposition 0.1, we have

$$S_t = \inf\{s \geqslant 0 : U_s \geqslant t\} = \inf\{s \geqslant 0 : U_s > t\}.$$

As $(S_t)_{t\geqslant 0}$ is a nondecreasing process, Proposition 0.1 *(ii)* implies that it is has bounded variation. Since it has continuous (and thus càdlàg) paths, Proposition 0.1 *(ii)* also implies that $(S_t)_{t\geqslant 0}$ is a semimartingale. Moreover, having proved that S_t is a stopping-time for any $t \geqslant 0$, we can introduce the stopped sigma-algebras

$$\mathcal{F}^U_{S_t} := \left\{A \in \mathcal{F}^U_\infty : A \cap \{S_t \leqslant u\} \in \mathcal{F}^U_u \text{ for all } u \geqslant 0\right\},$$

where $\mathcal{F}^U_\infty = \bigvee_{t\geqslant 0} \mathcal{F}^U_t$. Since for any $0 \leqslant s \leqslant t$, S_s and S_t are stopping-times that satisfy $S_s \leqslant S_t$ a.s., it is clear that the collection $(\mathcal{F}^U_{S_t})_{t\geqslant 0}$ of sigma-algebras is a filtration. We also introduce the filtrations $(\mathcal{F}^{U\circ S}_t)_{t\geqslant 0}$ and $(\mathcal{F}^{S,U\circ S}_t)_{t\geqslant 0}$ to be the natural filtrations of respectively $(U_{S_t})_{t\geqslant 0}$ and $(S_t, U_{S_t})_{t\geqslant 0}$. The next proposition shows that $(\mathcal{F}^U_{S_t})_{t\geqslant 0}$ is finer than the natural filtration of $(S_t)_{t\geqslant 0}$.

Proposition 0.2. *For all $t \geqslant 0$, $\mathcal{F}^S_t \subset \mathcal{F}^U_{S_t}$.*

Proof. Fix any $r \in \mathbb{R}$ and $\tau \leqslant t$. We start by showing that $\{S_\tau \leqslant r\} \cap \{S_t \leqslant u\} \in \mathcal{F}^U_u$ for all $u \geqslant 0$. To this end, note that $\{S_\tau \leqslant r\} \cap \{S_t \leqslant u\} = \{U_r \geqslant \tau\} \cap \{U_u \geqslant t\}$. If $r > u$, then

$$\{U_u \geqslant t\} \subset \{U_r \geqslant t\} \subset \{U_r \geqslant \tau\}$$

so that $\{U_r \geqslant \tau\} \cap \{U_u \geqslant t\} = \{U_u \geqslant t\} \in \mathcal{F}^U_u$. If $r \leqslant u$, it is clear that $\{U_r \geqslant \tau\} \cap \{U_u \geqslant t\} \in \mathcal{F}^U_u$. This allows us to conclude that $\{S_\tau \leqslant r\} \in \mathcal{F}^U_{S_t}$. Since the intervals of the form $(-\infty, r]$ generates the Borel σ-algebra $\mathcal{B}_\mathbb{R}$, it implies that for any $B \in \mathcal{B}_\mathbb{R}$ and $0 \leqslant \tau \leqslant t$, we have

$$\{S_\tau \in B\} \in \mathcal{F}^U_{S_t}.$$

The conclusion follows. □

Similarly, we can define the σ-algebra of events that happen before the stopping-time S_t as

$$\mathcal{F}^U_{S_t-} := \mathcal{F}^U_0 \vee \sigma\{A \cap \{s < S_t\} : A \in \mathcal{F}^U_s, s \in \mathbb{R}^+\}.$$

It is well known that $\mathcal{F}^U_{S_t-} \subset \mathcal{F}^U_{S_t}$, for all $t \geqslant 0$. Finally, we will also use the stopped processes $(U^{S_t}_s)_{s\geqslant 0}$ that are defined as

$$U^{S_t}_s := \begin{Bmatrix} U_{S_t} & \text{if } s \geqslant S_t \\ U_s & \text{if } s < S_t \end{Bmatrix} = U_s \mathbf{1}_{\{s<S_t\}} + U_{S_t}\mathbf{1}_{\{s\geqslant S_t\}} = U_{s\wedge S_t}.$$

We give now some intermediary results that will be useful to prove that the bivariate process $(S_t, U_{S_t})_{t\geqslant 0}$ is a Markov process with respect to its natural filtration $(\mathcal{F}^{S,U\circ S}_t)_{t\geqslant 0}$. Before stating and proving these results, let us stress their purpose. The proof of the Markov property of $(S_t, U_{S_t})_{t\geqslant 0}$ relies on the strong Markov property of the Feller process $(U_t)_{t\geqslant 0}$, which states that such processes renew themselves at stopping-times. When stopping $(U_t)_{t\geqslant 0}$ at the stopping-time S_t, the relevant σ-algebra is the stopped σ-algebra $\mathcal{F}^U_{S_t}$. However, as mentioned before, we want to prove that $(S_t, U_{S_t})_{t\geqslant 0}$ satisfies the Markov property with respect to its natural filtration $(\mathcal{F}^{S,U\circ S}_t)_{t\geqslant 0}$. The first goal of our preliminary results is thus to establish that both filtration $(\mathcal{F}^{S,U\circ S}_t)_{t\geqslant 0}$ and $(\mathcal{F}^U_{S_t})_{t\geqslant 0}$ coincide. The other goal of the preliminary results is to express some events of interest as countable unions or intersections of more simple events to prove that they are measurable.

Lemma 0.1. *Let $\mathcal{N}$ denote the collection of all null sets in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then for each $t \geqslant 0$ it holds that*

$$\mathcal{F}^U_{S_t} = \sigma(S_t, U^{S_t}) \vee \mathcal{N},$$

where $\sigma(S_t, U^{S_t})$ denote the sigma-algebra generated by the random variable S_t and the stopped process $(U^{S_t}_s)_{s\geqslant 0}$.

Proof. **First part.** The first step is to prove $\mathcal{F}^U_{S_t} \supset \sigma(S_t, U^{S_t}) \vee \mathcal{N}$. By Proposition 0.2, it is clear that $\mathcal{F}^U_{S_t} \supset \sigma(S_t)$. By the assumption of completeness of the probability space, it also holds that $\mathcal{F}^U_{S_t} \supset \mathcal{N}$. Therefore, the first step will be completed by proving that $\mathcal{F}^U_{S_t} \supset \sigma(U^{S_t})$. Fix $s \geqslant 0$ and $B \in \mathcal{B}_\mathbb{R}$. For any $u \geqslant 0$,

$$\begin{aligned}
&\{U^{S_t}_s \in B\} \cap \{S_t \leqslant u\} \\
&= \Big(\{U^{S_t}_s \in B\} \cap \{S_t \leqslant u\} \cap \{S_t \leqslant s\}\Big) \cup \Big(\{U^{S_t}_s \in B\} \cap \{S_t \leqslant u\} \cap \{S_t > s\}\Big) \\
&= \Big(\{U_{S_t} \in B\} \cap \{S_t \leqslant u \wedge s\}\Big) \cup \Big(\{U_s \in B\} \cap \{S_t \leqslant u\} \cap \{S_t > s\}\Big)
\end{aligned}$$

Theorem 6 (P.5) in Protter (2005) implies that $\{U_{S_t} \in B\} \in \mathcal{F}^U_{S_t}$ and consequently that $\{U^{S_t}_s \in B\} \cap \{S_t \leqslant u \wedge s\} \in \mathcal{F}^U_{u\wedge s} \subset \mathcal{F}^U_u$. Moreover,

$$\{U_s \in B\} \cap \{S_t \leqslant u\} \cap \{S_t > s\} = \begin{Bmatrix} \emptyset & \text{if } s \geqslant u \\ \{U_s \in B\} \cap \overline{\{U_s \geqslant t\}} \cap \{U_u \geqslant t\} & \text{if } s < u \end{Bmatrix} \in \mathcal{F}^U_u.$$

It follows that $\{U^{S_t}_s \in B\} \cap \{S_t \leqslant u\} \in \mathcal{F}^U_u$, which proves the first inclusion.

Second part. The second step is to prove $\mathcal{F}^U_{S_t} \subset \sigma(S_t, U^{S_t}) \vee \mathcal{N}$. Define

$$\mathcal{A} := \left\{ A \in \mathcal{F}^U_\infty : \mathbb{E}[\mathbf{1}_A | \mathcal{F}^U_{S_t}] = \mathbb{E}[\mathbf{1}_A | \sigma(S_t, U^{S_t}) \vee \mathcal{N}] \right\}.$$

Clearly $\mathcal{A}$ is a λ-system that contains $\mathcal{N}$. Let $\mathcal{C}$ be the collection of subsets

$$\mathcal{C} := \left\{ \bigcap_{j=1}^{n} \{U_{t_j} \in B_j\} : n \in \mathbb{N}, t_j \in \mathbb{R}^+, B_j \in \mathcal{B}_{\mathbb{R}} \right\}.$$

This collection is a π-system that satisfies $\sigma(\mathcal{C}) \vee \mathcal{N} = \mathcal{F}^U_\infty$. For $C \in \mathcal{C}$,

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_C | \mathcal{F}^U_{S_t}] &= \mathbb{E}\left[\prod_{j=1}^{n} \mathbf{1}_{\{U_{t_j} \in B_j\}} \middle| \mathcal{F}^U_{S_t} \right] \\
&= \sum_{k=1}^{n+1} \mathbb{E}\left[\mathbf{1}_{\{S_t \in [t_{k-1}, t_k)\}} \prod_{j=1}^{n} \mathbf{1}_{\{U_{t_j} \in B_j\}} \middle| \mathcal{F}^U_{S_t} \right],
\end{aligned}$$

where t_0 and t_{n+1} have to be understood as 0 and $+\infty$ respectively. Note that

$$\mathbf{1}_{\{S_t \in [t_{k-1}, t_k)\}} \mathbf{1}_{\{U_{t_j} \in B_j\}} = \begin{cases} \mathbf{1}_{\{S_t \in [t_{k-1}, t_k)\}} \mathbf{1}_{\{U_{t_j \wedge S_t} \in B_j\}} & \text{if } j < k \\ \mathbf{1}_{\{S_t \in [t_{k-1}, t_k)\}} \mathbf{1}_{\{U_{t_j \vee S_t} \in B_j\}} & \text{if } j \geqslant k. \end{cases}$$

Moreover, by the $\mathcal{F}^U_{S_t}$-measurability of the random variables S_t (Proposition 0.2) and $U_{t_j \wedge S_t} = U^{S_t}_{t_j}$ (first part of this proof), we deduce that $\mathbf{1}_{\{S_t \in [t_{k-1}, t_k)\}} \mathbf{1}_{\{U_{t_j \wedge S_t} \in B_j\}}$ is $\mathcal{F}^U_{S_t}$-measurable whenever $j < k$. As a consequence,

$$\mathbb{E}[\mathbf{1}_C | \mathcal{F}^U_{S_t}] = \sum_{k=1}^{n+1} \mathbf{1}_{\{S_t \in [t_{k-1}, t_k)\}} \prod_{j=1}^{k-1} \mathbf{1}_{\{U_{t_j \wedge S_t} \in B_j\}} \mathbb{E}\left[\prod_{j=k}^{n} \mathbf{1}_{\{U_{t_j \vee S_t} \in B_j\}} \middle| \mathcal{F}^U_{S_t} \right]$$

Let Z be the $\mathcal{F}^U_\infty$-measurable random variable $\prod_{j=k}^{n} \mathbf{1}_{\{U_{(t_j \vee S_t) - S_t} \in B_j\}}$. Then

$$\prod_{j=k}^{n} \mathbf{1}_{\{U_{t_j \vee S_t} \in B_j\}} = Z \circ \theta_{S_t},$$

where $(\theta_t)_{t \geqslant 0}$ denotes the translation operators defined as $\theta_t(U_s) = U_{s+t}$ (see Revuz and Yor (2004), Chapters 1 and 3). From Theorem 3.1 in Chapter 3 of Revuz and Yor (2004), we have

$$\mathbb{E}\left[Z \circ \theta_{S_t} | \mathcal{F}^U_{S_t} \right] = \mathbb{E}^{U_{S_t}}[Z],$$

and thus

$$\mathbb{E}[\mathbf{1}_C | \mathcal{F}^U_{S_t}] = \sum_{k=1}^{n+1} \mathbf{1}_{\{S_t \in [t_{k-1}, t_k)\}} \prod_{j=1}^{k-1} \mathbf{1}_{\{U_{t_j \wedge S_t} \in B_j\}} \mathbb{E}^{U_{S_t}}[Z].$$

Since $\mathcal{F}^U_{S_t} \supset \sigma(S_t, U^{S_t}) \vee \mathcal{N}$, the tower property for conditional expectations gives

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_C | \sigma(S_t, U^{S_t}) \vee \mathcal{N}] &= \mathbb{E}[\mathbb{E}[\mathbf{1}_C | \mathcal{F}^U_{S_t}] | \sigma(S_t, U^{S_t}) \vee \mathcal{N}] \\
&= \sum_{k=1}^{n+1} \mathbb{E}\left[\mathbf{1}_{\{S_t \in [t_{k-1}, t_k)\}} \prod_{j=1}^{k-1} \mathbf{1}_{\{U_{t_j \wedge S_t} \in B_j\}} \mathbb{E}^{U_{S_t}}[Z] \middle| \sigma(S_t, U^{S_t}) \vee \mathcal{N} \right] \\
&= \sum_{k=1}^{n+1} \mathbf{1}_{\{S_t \in [t_{k-1}, t_k)\}} \prod_{j=1}^{k-1} \mathbf{1}_{\{U_{t_j \wedge S_t} \in B_j\}} \mathbb{E}^{U_{S_t}}[Z] \\
&= \mathbb{E}[\mathbf{1}_C | \mathcal{F}^U_{S_t}],
\end{aligned}$$

where the third equality comes from the $\sigma(S_t, U^{S_t})$-measurability of the random variable in the conditional expectation. From Dynkin's π-λ theorem, we conclude that $\sigma(\mathcal{C}) \subset \mathcal{A}$. Since $\sigma(\mathcal{C}) \vee \mathcal{N} = \mathcal{F}^U_\infty$, it implies that $\mathcal{F}^U_\infty = \mathcal{A}$. As a conclusion, for any $A \in \mathcal{F}^U_{S_t}$, $A \in \mathcal{A}$ and therefore

$$\mathbf{1}_A = \mathbb{E}[\mathbf{1}_A | \mathcal{F}^U_{S_t}] = \mathbb{E}[\mathbf{1}_A | \sigma(S_t, U^{S_t}) \vee \mathcal{N}],$$

so that $A \in \sigma(S_t, U^{S_t}) \vee \mathcal{N}$, concluding the proof. □

In the following, we write $(\mathcal{F}^{U\circ S}_t)_{t\geqslant 0}$ for the natural filtration of the process $(U_{S_t})_{t\geqslant 0}$ and $(\mathcal{F}^{U\circ S,S}_t)_{t\geqslant 0}$ for the natural filtration of the bivariate process $(U_{S_t}, S_t)_{t\geqslant 0}$.

Lemma 0.2. *For all $t \geqslant 0$,*

(i) $\mathcal{F}^U_{S_t} = \sigma(\mathcal{F}^U_{S_t-}, U_{S_t})$,

(ii) $\mathcal{F}^U_{S_t-} \subset \mathcal{F}^S_t$.

Proof. (i) Recall that the $\mathcal{F}^U_{S_t}$-measurability of U_{S_t} follows from Theorem 6 (P.5) in Protter (2005). Therefore we just have to prove that $\mathcal{F}^U_{S_t} \subset \sigma(\mathcal{F}^U_{S_t-}, U_{S_t})$. By Lemma 0.1, $\mathcal{F}^U_{S_t} = \sigma(S_t, U^{S_t}) \vee \mathcal{N}$, so that we will prove that $\sigma(\mathcal{F}^U_{S_t-}, U_{S_t})$ contains both $\sigma(S_t)$ and $\sigma(U^{S_t})$. From the definition of $\mathcal{F}^U_{S_t-}$, clearly $\{S_t > s\} \in \mathcal{F}^U_{S_t-}$ for all $s \geqslant 0$, so that $\sigma(U^{S_t}) \subset \sigma(\mathcal{F}^U_{S_t-}, U_{S_t})$. For the second inclusion, note that for any $s \geqslant 0$ and $B \in \mathcal{B}_\mathbb{R}$, one has

$$\begin{aligned}\{U^{S_t}_s \in B\} &= \Big(\{U^{S_t}_s \in B\} \cap \{S_t \geqslant s\}\Big) \cup \Big(\{U^{S_t}_s \in B\} \cap \{S_t > s\}\Big)\\ &= \Big(\{U_{S_t} \in B\} \cap \{S_t \geqslant s\}\Big) \cup \Big(\{U_s \in B\} \cap \{S_t > s\}\Big).\end{aligned}$$

It is an easy task to check that the sets $\{S_t \geqslant s\}$ and $\{U_s \in B\} \cap \{S_t > s\}$ belong to $\mathcal{F}^U_{S_t-}$. We thus conclude that the second inclusion $\sigma(U^{S_t}) \subset \sigma(\mathcal{F}^U_{S_t-}, U_{S_t})$ holds.

(ii) We have to show that $A \cap \{S_t > s\} \in \mathcal{F}^S_t$ whenever $s \geqslant 0$ and $A \in \mathcal{F}^U_s$. For this, it is enough to show that $\{U_s < b\} \cap \{S_t > u\} \in \mathcal{F}^S_t$ for any $b \in \mathbb{R}$ and $s \leqslant u$. Such sets satisfy

$$\{U_s < b\} \cap \{S_t > u\} = \{S_b > s\} \cap \{S_t > u\},$$

which proves that they are in $\mathcal{F}^S_t$ if $b \leqslant t$. In the case $b > t$, the nondecreasing paths of $(S_t)_{t\geqslant 0}$ entail that $\{S_b > s\} \cap \{S_t > u\} = \{S_t > u\}$, and thus the set also belongs to $\mathcal{F}^S_t$. □

Lemma 0.3. *Let $t > 0$ and $x_1 \geqslant 0$. Then*

$$\{U_{S_t} \geqslant x_1\} = \begin{cases}\Omega & \text{if } x_1 \leqslant t\\ \{S_t = S_{x_1}\} & \text{if } x_1 > t.\end{cases}$$

Proof. The case $x_1 \leqslant t$ is obvious since $U_{S_t} \geqslant t$ (Proposition 0.1). Let $x_1 > t$ and $\omega \in \{S_{x_1} > S_t\}$. Then there exists $\xi \in (S_t(\omega), S_{x_1}(\omega))$. It follows that $U_{S_t(\omega)}(\omega) < U_\xi(\omega) \leqslant x_1$ and thus $\{U_{S_t} \geqslant x_1\} \cap \{S_{x_1} > S_t\} = \emptyset$. Since $(S_u)_{u\geqslant 0}$ is nondecreasing, we have

$$\begin{aligned}\{U_{S_t} \geqslant x_1\} &= \{U_{S_t} \geqslant x_1\} \cap \{S_{x_1} \geqslant S_t\}\\ &= (\{U_{S_t} \geqslant x_1\} \cap \{S_{x_1} > S_t\}) \cup (\{U_{S_t} \geqslant x_1\} \cap \{S_{x_1} = S_t\})\\ &= \{U_{S_t} \geqslant x_1\} \cap \{S_{x_1} = S_t\}\end{aligned}$$

so that $\{U_{S_t} \geqslant x_1\} \subset \{S_{x_1} = S_t\}$. For the converse inclusion, let $\omega \in \{S_{x_1} = S_t\}$. Proposition 0.1 implies that $x_1 \leqslant U_{S_{x_1}(\omega)}(\omega)$. Since $U_{S_{x_1}(\omega)}(\omega) = U_{S_t(\omega)}(\omega)$, we conclude that $\omega \in \{U_{S_t} \geqslant x_1\}$. □

Lemma 0.4. *Let $0 < t < x_1$. Then*

$$\{S_t = S_{x_1}\} = \bigcap_{n\in\mathbb{N}} \bigcup_{\substack{(q_1,q_2)\in\mathbb{Q}\times\mathbb{Q}\\ 0\leqslant q_1<q_2<q_1+\frac{1}{n}}} \Big[\{U_{q_1} < t\} \cap \{U_{q_2} \geqslant x_1\}\Big]. \tag{8}$$

Proof. We denote by C the set at the right-hand side of Equation (8). Let $\omega \in C$. We can find sequences $(q_{1,n})_{n\in\mathbb{N}}$, $(q_{2,n})_{n\in\mathbb{N}}$ such that for all n, $q_{1,n} < q_{2,n} < q_{1,n} + \frac{1}{n}$, $U_{q_{1,n}}(\omega) < t$ and $U_{q_{2,n}}(\omega) \geqslant x_1$. By definition of $(S_t)_{t\geqslant 0}$, we have

$$q_{1,n} \leqslant S_t(\omega) \leqslant S_{x_1}(\omega) \leqslant q_{2,n} < q_{1,n} + \frac{1}{n}.$$

It is then clear that both the sequences $(q_{1,n})_{n\in\mathbb{N}}$, $(q_{2,n})_{n\in\mathbb{N}}$ are convergent and converge towards the same limit. We get $\omega \in \{S_t = S_{x_1}\}$ as a conclusion.

Conversely, let $\omega \in \{S_t = S_{x_1}\}$. For each n, let $q_{1,n} \in (S_t(\omega) - \frac{1}{2n}, S_t(\omega)) \cap \mathbb{Q}^+$ and $q_{2,n} \in (S_{x_1}(\omega), S_{x_1}(\omega) + \frac{1}{2n}) \cap \mathbb{Q}^+$. We clearly have $U_{q_{1,n}}(\omega) < t$, $U_{q_{2,n}}(\omega) \geqslant x_1$, $q_{1,n} < q_{2,n}$ and $q_{2,n} - q_{1,n} < \frac{1}{n}$, the latter being a consequence of $S_t(\omega) = S_{x_1}(\omega)$. This proves that $\omega \in C$. □

Corollary 0.1. *For any $t \geqslant 0$, $\mathcal{F}_t^{U\circ S} \subset \mathcal{F}_\infty^U$.*

Proof. This is a consequence of Lemmas 0.3 and 0.4, as they imply that $\{U_{S_s} \geqslant x\} \in \mathcal{F}_\infty^U$ for all $s \leqslant t$. □

Lemma 0.5. *For any $t \geqslant 0$, we have $\mathcal{F}_t^{U\circ S,S} = \mathcal{F}_{S_t}^U$.*

Proof. **First part.** We begin with the proof of $\mathcal{F}_t^{U\circ S,S} \subset \mathcal{F}_{S_t}^U$. Since Proposition 0.2 states that $\mathcal{F}_t^S \subset \mathcal{F}_{S_t}^U$, we only need to prove $\mathcal{F}_t^{U\circ S} \subset \mathcal{F}_{S_t}^U$. To this end, it suffices to show that for any $x \in \mathbb{R}$ and $s \leqslant t$, $\{U_{S_s} > x\} \in \mathcal{F}_{S_t}^U$. To this end, we must show that if $u \geqslant 0$, then $\{U_{S_s} > x\} \cap \{S_t \leqslant u\} \in \mathcal{F}_u^U$. This is trivial if $x \leqslant s$ so we assume that $x > s$. We have

$$\begin{aligned}
\{U_{S_s} > x\} \cap \{S_t \leqslant u\} &= \{S_s = S_x\} \cap \{S_t \geqslant u\} \\
&= \{S_s = S_x\} \cap \{S_t \leqslant u\} \cap \{S_x \leqslant u\} \\
&= \{S_s = S_x\} \cap \{U_u \geqslant t \vee x\} \\
&= \bigcap_{n\in\mathbb{N}} \bigcup_{\substack{(q_1,q_2)\in\mathbb{Q}\times\mathbb{Q}\\ 0\leqslant q_1<q_2<q_1+\frac{1}{n}}} \Big[\{U_{q_1} < s\} \cap \{U_{q_2} \geqslant x\} \cap \{U_u \geqslant t \vee x\}\Big] \in \mathcal{F}_u^U
\end{aligned}$$

The first equality comes from Lemma 0.3, the second is a consequence of $S_x = S_s \leqslant S_t$, the third comes from the definition of $(S_t)_{t\geqslant 0}$ and the fourth is an application of Lemma 0.4. We conclude that the last set is in $\mathcal{F}_u^U$ because whenever $q_2 > u$,

$$\{U_{q_1} < s\} \cap \{U_{q_2} \geqslant x\} \cap \{U_u \geqslant t \vee x\} = \{U_{q_1} < s\} \cap \{U_u \geqslant t \vee x\}$$
$$\begin{cases} \in \mathcal{F}_u^U & \text{if } q_1 \leqslant u \\ = \emptyset \in \mathcal{F}_u^U & \text{if } q_1 > u, \end{cases}$$

which concludes the proof.

Second part. We now prove $\mathcal{F}_t^{U\circ S,S} \supset \mathcal{F}_{S_t}^U$. By Lemma 0.2 (i), $\mathcal{F}_{S_t}^U = \sigma(\mathcal{F}_{S_t-}^U, U_{S_t})$. According to (ii) of the same lemma, $\mathcal{F}_{S_t-}^U \subset \mathcal{F}_t^S$, so that $\mathcal{F}_{S_t-}^U \subset \mathcal{F}_t^{U\circ S,S}$. Since the $\mathcal{F}_t^{U\circ S,S}$-measurability of U_{S_t} is trivial, the result follows. □

We can now prove the Markov property for $(S_t, U_{S_t})_{t\geqslant 0}$.

Theorem 0.1. *The bivariate process $(S_t, U_{S_t})_{t\geqslant 0}$ is a Markov process with respect to its natural filtration.*

Proof. According to Theorem 45 in the first chapter of Protter (2005), proving that the Markov property holds amounts to showing that

$$\mathbb{E}[f(S_t, U_{S_t})|\mathcal{F}_s^{S,U\circ S}] = \mathbb{E}[f(S_t, U_{S_t})|\sigma(S_s, U_{S_s})]$$

for any $t \geqslant s \geqslant 0$ and any bounded Borel measurable function $f : \mathbb{R}^2 \to \mathbb{R}$. Let

$$\mathcal{A} = \left\{A \in \sigma(S_t, U_{S_t}) : \mathbb{E}[\mathbf{1}_A|\mathcal{F}_s^{S,U\circ S}] = \mathbb{E}[\mathbf{1}_A|\sigma(S_s, U_{S_s})]\right\}$$

and

$$\mathcal{C} = \{\{S_t \leqslant x_1\} \cap \{U_{S_t} \geqslant x_2\} : x_1, x_2 \in \mathbb{R}\}\,.$$

Note that $\mathcal{C}$ is a π-system and $\mathcal{A}$ is a λ-system. Since the collection $\{(-\infty, x_1] \times [x_2, +\infty) : x_1, x_2 \in \mathbb{R}\}$ generates the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^2}$, it is clear that $\sigma(\mathcal{C}) = \sigma(S_t, U_{S_t})$. We prove now that $\mathcal{C} \subset \mathcal{A}$. To this end, we start by observing that

$$\begin{aligned}
&\{S_t \leqslant x_1\} \cap \{U_{S_t} \geqslant x_2\} \\
&\quad= \Big(\{S_t \leqslant x_1\} \cap \{U_{S_t} \geqslant x_2\} \cap \{S_s = S_t\}\Big) \cup \Big(\{S_t \leqslant x_1\} \cap \{U_{S_t} \geqslant x_2\} \cap \{S_s < S_t\}\Big) \\
&\quad= \Big(\{S_s \leqslant x_1\} \cap \{U_{S_s} \geqslant x_2\} \cap \{S_s = S_t\}\Big) \cup \Big(\{S_t \leqslant x_1\} \cap \{U_{S_t} \geqslant x_2\} \cap \{S_s < S_t\}\Big) \\
&\quad= \Big(\{S_s \leqslant x_1\} \cap \{U_{S_s} \geqslant x_2 \vee t\}\Big) \cup \Big(\{S_t \leqslant x_1\} \cap \{U_{S_t} \geqslant x_2\} \cap \{S_s < S_t\}\Big).
\end{aligned}$$

Since $\{S_s \leqslant x_1\} \cap \{U_{S_s} \geqslant x_2 \vee t\}$ is in $\sigma(S_s, U_{S_s})$, it remains to show that

$$\mathbb{E}[\mathbf{1}_{\{S_t\leqslant x_1\}\cap\{U_{S_t}\geqslant x_2\}\cap\{S_s<S_t\}}|\mathcal{F}_s^{S,U\circ S}] = \mathbb{E}[\mathbf{1}_{\{S_t\leqslant x_1\}\cap\{U_{S_t}\geqslant x_2\}\cap\{S_s<S_t\}}|\sigma(S_s, U_{S_s})].$$

To do so, we proceed in two steps. In the first step we will assume that $x_2 \leqslant t$, so that $\{U_{S_t} \geqslant x_2\} = \Omega$. This case is the most simple. In the second step, we will work with $x_2 > t$.

Let $x_2 \leqslant t$. Then $\{S_t \leqslant x_1\} \cap \{U_{S_t} \geqslant x_2\} \cap \{S_s < S_t\} = \{S_t \leqslant x_1\} \cap \{S_s < S_t\}$. We will establish that

$$\begin{aligned}
&\{S_t \leqslant x_1\} \cap \{S_s < S_t\} \\
&\quad= \left(\bigcap_{n\in\mathbb{N}} \bigcup_{\substack{q\in\mathbb{Q}\\ q>0}} \left[\{U_{S_s+q} \geqslant t\} \cap \{S_s + q < x_1 + n^{-1}\}\right]\right) \cap \{S_s < S_t\}.
\end{aligned} \tag{9}$$

Let ω be a member of the set at the left-hand side of Equation (9). Let $n \in \mathbb{N}$ and $q \in (S_t(\omega) - S_s(\omega), x_1 + n^{-1} - S_s(\omega)) \cap \mathbb{Q}$. Such a q exists because

$$S_t(\omega) - S_s(\omega) \leqslant x_1 - S_s(\omega) < x_1 + n^{-1} - S_s(\omega).$$

Moreover, $q + S_s(\omega) > S_t(\omega)$ ensures that $U_{q+S_s(\omega)}(\omega) > t$, so that $\omega \in \{U_{S_s+q} \geqslant t\}$. Conversely, let ω be a member of the set at the right-hand side of Equation (9). For each $n \in \mathbb{N}$, let $q_n \in \mathbb{Q}, q_n > 0$ satisfy $U_{S_s(\omega)+q_n}(\omega) \geqslant t$ and $q_n < x_1 + n^{-1} - S_s(\omega)$. Since $S_s(\omega) + q_n < x_1 + n^{-1}$ for any n, we have

$$t \leqslant U_{S_s(\omega)+q_n}(\omega) < U_{x_1+n^{-1}}(\omega).$$

By the right-continuity of $(U_t)_{t\geqslant 0}$, $\lim_{n\to+\infty} U_{x_1+n^{-1}}(\omega) = U_{x_1}(\omega) \geqslant t$. By definition of $(S_t)_{t\geqslant 0}$, this is equivalent to $S_t(\omega) \leqslant x_1$. This proves that $\omega \in \{S_t \leqslant x_1\}$ and finishes the proof of Equation (9). Since the bivariate process $(U_t, t)_{t\geqslant 0}$ is a Feller process[1] with respect to its natural filtration $(\mathcal{F}_t^U)_{t\geqslant 0}$, it follows that if we define the $\mathcal{F}_\infty^U$-measurable indicator random variable Z as

$$Z := \mathbf{1}\left\{\bigcap_{n\in\mathbb{N}} \bigcup_{\substack{q\in\mathbb{Q}\\ q>0}} \left[\{U_q \geqslant t\} \cap \{q < x_1 + n^{-1}\}\right]\right\}$$

[1]this follows from the fact that it is a Lévy process.

then Equation (9) implies that

$$\mathbf{1}_{\{S_t \leqslant x_1\} \cap \{S_s < S_t\}} = \mathbf{1}_{\{S_s < S_t\}} Z \circ \theta_{S_s}.$$

By using the $\mathcal{F}^U_{S_s}$-measurablility of $\mathbf{1}_{\{S_s < S_t\}}$ and Theorem 3.1 from the Chapter 3 of Revuz and Yor (2004), we find

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{\{S_t \leqslant x_1\} \cap \{S_s < S_t\}} | \mathcal{F}^U_{S_s}] &= \mathbf{1}_{\{S_s < S_t\}} \mathbb{E}[Z \circ \theta_{S_s} | \mathcal{F}^U_{S_s}] \\ &= \mathbf{1}_{\{S_s < S_t\}} \mathbb{E}^{(U_{S_s}, S_s)}[Z]. \end{aligned} \tag{10}$$

Similarly, Lemma 0.3 entails that $\mathbf{1}_{\{S_s < S_t\}}$ is $\sigma(S_s, U_{S_s})$-measurable and thus by the tower property,

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{\{S_t \leqslant x_1\} \cap \{S_s < S_t\}} | \sigma(S_s, U_{S_s})] &= \mathbb{E}[\mathbf{1}_{\{S_s < S_t\}} \mathbb{E}[Z \circ \theta_{S_s} | \mathcal{F}^U_{S_s}] | \sigma(S_s, U_{S_s})] \\ &= \mathbf{1}_{\{S_s < S_t\}} \mathbb{E}[\mathbb{E}^{(U_{S_s}, S_s)}[Z] | \sigma(S_s, U_{S_s})] \\ &= \mathbf{1}_{\{S_s < S_t\}} \mathbb{E}^{(U_{S_s}, S_s)}[Z]. \end{aligned} \tag{11}$$

By combining Equations (10) and (11), it follows that

$$\mathbb{E}[\mathbf{1}_{\{S_t \leqslant x_1\} \cap \{S_s < S_t\}} | \mathcal{F}^U_{S_s}] = \mathbb{E}[\mathbf{1}_{\{S_t \leqslant x_1\} \cap \{S_s < S_t\}} | \sigma(S_s, U_{S_s})]$$

whenever $x_2 \leqslant t$.

We now consider the case $x_2 > t$. We will establish that when $x_2 > t$,

$$\begin{aligned} &\{S_t \leqslant x_1\} \cap \{U_{S_t} \geqslant x_2\} \cap \{S_s < S_t\} \\ &= \left(\bigcap_{n \in \mathbb{N}} \bigcup_{\substack{(q_1, q_2) \in \mathbb{Q}^2 \\ 0 < q_1 < q_2 < q_1 + \frac{1}{n}}} \left[\{U_{S_s + q_1} < t\} \cap \{U_{S_s + q_2} \geqslant x_2\} \cap \{S_s + q_2 < x_1 + n^{-1}\} \right] \right) \cap \{S_s < S_t\}. \end{aligned} \tag{12}$$

Let ω be a member of the first set of Equation (12) and $n \in \mathbb{N}$. Let

$$q_1 \in (0 \vee (S_t(\omega) - S_s(\omega) - (2n)^{-1}), S_t(\omega) - S_s(\omega)) \cap \mathbb{Q}$$

and

$$q_2 \in \big(S_t(\omega) - S_s(\omega), (S_t(\omega) - S_s(\omega) + (2n)^{-1}) \wedge (x_1 + n^{-1} - S_s(\omega))\big) \cap \mathbb{Q}.$$

The existence of q_1 follows from $S_t(\omega) > S_s(\omega)$ whereas the existence of q_2 follows from $S_t(\omega) \leqslant x_1 < x_1 + n^{-1}$. The inequality $S_s(\omega) + q_1 < S_t(\omega)$ implies that $U_{S_s(\omega)+q_1}(\omega) < t$. By Lemma 0.3, we have $S_{x_2}(\omega) = S_t(\omega)$, so that $S_s(\omega) + q_2 > S_t(\omega)$ implies that $S_s(\omega) + q_2 > S_{x_2}(\omega)$, which in turn implies that $U_{S_s(\omega)+q_2}(\omega) \geqslant x_2$. Finally, note that by construction, it also holds that $0 < q_1 < q_2 < q_1 + n^{-1}$. This proves that ω is also a member of the second set of Equation (12). We prove now the reverse inclusion: assume that ω is a member of the second set of Equation (12). For each n, let $q_{1,n}, q_{2,n}$ be numbers in $\mathbb{Q}$ that satisfy $0 < q_{1,n} < q_{2,n} < q_{1,n} + n^{-1}$, $U_{S_s(\omega)+q_{1,n}}(\omega) < t$, $U_{S_s(\omega)+q_{2,n}}(\omega) \geqslant x_2 > t$ and $q_{2,n} < x_1 + n^{-1} - S_s(\omega)$. By construction, $S_s(\omega) + q_{1,n} < S_t(\omega)$, thus

$$x_2 \leqslant U_{S_s(\omega)+q_{2,n}}(\omega) \leqslant U_{S_s(\omega)+q_{1,n}+n^{-1}}(\omega) \leqslant U_{S_t(\omega)+n^{-1}}(\omega)$$

and the right-continuity of $(U_t)_{t \geqslant 0}$ implies that $U_{S_t(\omega)}(\omega) \geqslant x_2$. Similarly, $S_s(\omega) + q_{2,n} < x_1 + n^{-1}$ implies that

$$t < x_2 < U_{S_s(\omega)+q_{2,n}}(\omega) < U_{x_1+n^{-1}}(\omega)$$

and $U_{x_1}(\omega) > t$ follows from the right-continuity of $(U_t)_{t \geqslant 0}$. We get $S_t(\omega) \leqslant x_1$ as a consequence, which ends the proof of Equation (12). As a consequence, the $\mathcal{F}^U_\infty$-measurable random variable Z defined as

$$Z := \mathbf{1} \left\{ \bigcap_{n \in \mathbb{N}} \bigcup_{\substack{(q_1, q_2) \in \mathbb{Q}^2 \\ 0 < q_1 < q_2 < q_1 + \frac{1}{n}}} \left[\{U_{q_1} < t\} \cap \{U_{q_2} \geqslant x_2\} \cap \{q_2 < x_1 + n^{-1}\} \right] \right\}$$

satisfies

$$\mathbf{1}_{\{S_t \leqslant x_1\}\cap\{U_{S_t}\geqslant x_2\}\cap\{S_s<S_t\}} = \mathbf{1}_{\{S_s<S_t\}} Z\circ\theta_{S_s}.$$

Again, by using the $\mathcal{F}^U_{S_s}$-measurability of $\mathbf{1}_{\{S_s<S_t\}}$ and Theorem 3.1 from the Chapter 3 of Revuz and Yor (2004), we find

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\{S_t \leqslant x_1\}\cap\{U_{S_t}\geqslant x_2\}\cap\{S_s<S_t\}}|\mathcal{F}^U_{S_s}] &= \mathbf{1}_{\{S_s<S_t\}}\mathbb{E}[Z\circ\theta_{S_s}|\mathcal{F}^U_{S_s}] \\ &= \mathbf{1}_{\{S_s<S_t\}}\mathbb{E}^{(U_{S_s},S_s)}[Z].\end{aligned} \tag{13}$$

Similarly, Lemma 0.3 entails that $\mathbf{1}_{\{S_s<S_t\}}$ is $\sigma(S_s, U_{S_s})$-measurable and thus by the tower property,

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\{S_t \leqslant x_1\}\cap\{U_{S_t}\geqslant x_2\}\cap\{S_s<S_t\}}|\sigma(S_s,U_{S_s})] &= \mathbb{E}[\mathbf{1}_{\{S_s<S_t\}}\mathbb{E}[Z\circ\theta_{S_s}|\mathcal{F}^U_{S_s}]|\sigma(S_s,U_{S_s})] \\ &= \mathbf{1}_{\{S_s<S_t\}}\mathbb{E}[\mathbb{E}^{(U_{S_s},S_s)}[Z]|\sigma(S_s,U_{S_s})] \\ &= \mathbf{1}_{\{S_s<S_t\}}\mathbb{E}^{(U_{S_s},S_s)}[Z].\end{aligned} \tag{14}$$

By combining Equations (13) and (14), it follows that

$$\mathbb{E}[\mathbf{1}_{\{S_t \leqslant x_1\}\cap\{U_{S_t}\geqslant x_2\}\cap\{S_s<S_t\}}|\mathcal{F}^U_{S_s}] = \mathbb{E}[\mathbf{1}_{\{S_t \leqslant x_1\}\cap\{U_{S_t}\geqslant x_2\}\cap\{S_s<S_t\}}|\sigma(S_s,U_{S_s})] \tag{15}$$

whenever $x_2 > t$. It follows that $\mathcal{C}\subset\mathcal{A}$ and thus by Dynkin's π-λ theorem, for any $A\in\sigma(\mathcal{C}) = \sigma(S_t, U_{S_t})$,

$$\mathbb{E}[\mathbf{1}_A|\mathcal{F}^{S,U\circ S}_s] = \mathbb{E}[\mathbf{1}_A|\sigma(S_s,U_{S_s})].$$

Let $f:\mathbb{R}^2\to\mathbb{R}$ be a bounded Borel measurable function. Using the decomposition of f into its positive and negative parts, i.e. $f = (f\vee 0) - ((-f)\vee 0)$, and the linearity of conditional expectations, we can assume without loss of generality that $f\geqslant 0$. Define the sequence of functions $(f_n:\mathbb{R}^2\to\mathbb{R}^+)_{n\in\mathbb{N}}$ as

$$f_n(x,y) := \sum_{k=0}^{2^{2n}} \frac{k}{2^n}\mathbf{1}_{\{f(x,y)\in[k/2^n,(k+1)/2^n)\}}.$$

We clearly have that $0\leqslant f_n\leqslant f_{n+1}\leqslant f$ for all $n\in\mathbb{N}$ and $f_n\to f$. Therefore, Equation (15) and two applications of the monotone convergence theorem yield

$$\begin{aligned}\mathbb{E}[f(S_t,U_{S_t})|\mathcal{F}^{S,U\circ S}_s] &= \mathbb{E}\left[\lim_{n\to+\infty} f_n(S_t,U_{S_t})|\mathcal{F}^{S,U\circ S}_s\right] \\ &= \lim_{n\to+\infty}\mathbb{E}[f_n(S_t,U_{S_t})|\mathcal{F}^{S,U\circ S}_s] \\ &= \lim_{n\to+\infty}\sum_{k=0}^{2^{2n}}\frac{k}{2^n}\mathbb{E}\left[\mathbf{1}_{\left\{(S_t,U_{S_t})\in f^{-1}\left[\frac{k}{2^n},\frac{k+1}{2^n}\right)\right\}}|\mathcal{F}^{S,U\circ S}_s\right] \\ &= \lim_{n\to+\infty}\sum_{k=0}^{2^{2n}}\frac{k}{2^n}\mathbb{E}\left[\mathbf{1}_{\left\{(S_t,U_{S_t})\in f^{-1}\left[\frac{k}{2^n},\frac{k+1}{2^n}\right)\right\}}|\sigma(S_s,U_{S_s})\right] \\ &= \lim_{n\to+\infty}\mathbb{E}[f_n(S_t,U_{S_t})|\sigma(S_s,U_{S_s})] \\ &= \mathbb{E}\left[\lim_{n\to+\infty} f_n(S_t,U_{S_t})|\sigma(S_s,U_{S_s})\right] \\ &= \mathbb{E}[f(S_t,U_{S_t})|\sigma(S_s,U_{S_s})],\end{aligned}$$

which is what we had to prove. □

In the next section, we study the transform of our asset process. This will be useful to compute option prices.

0.3 FPDE's for densities and Fourier transforms

This section is devoted to the derivation of FPDE's for the joint densities and transforms of the bivariate process $(X_{S_t}, \lambda_{S_t})_{t\geqslant 0}$[2]. Before diving into the main topic of this section, let us start by introducing the filtrations we consider for this process. In the non-fractional market, the filtration of the asset price $(A_t)_{t\geqslant 0}$ is denoted by $(\mathcal{F}_t^A)_{t\geqslant 0}$ and is defined as

$$\mathcal{F}_t^A = \mathcal{N} \vee \sigma\{W_s : 0 \leqslant s \leqslant t\} \vee \sigma\{J_s : 0 \leqslant s \leqslant t\} = \mathcal{F}_t^W \vee \mathcal{F}_t^J,$$

where $(\mathcal{F}_t^W)_{t\geqslant 0}$ and $(\mathcal{F}_t^J)_{t\geqslant 0}$ are the completed natural filtrations of the brownian motion $(W_t)_{t\geqslant 0}$ and the jump process $(J_t)_{t\geqslant 0}$ respectively. In other words, we assume that the impacts on the asset price of the brownian component (i.e. the noise component) and the jump component can be isolated from one another. Recall that $\mathcal{N}$ denotes the null sets of the probability space and are in $\mathcal{F}_t^W \vee \mathcal{F}_t^J$ because we assumed that the natural filtrations are completed. Note that the natural filtration $(\mathcal{F}_t^J)_{t\geqslant 0}$ carries the information about the other jump processes $(D_t)_{t\geqslant 0}$ and $(L_t)_{t\geqslant 0}$ and about the intensity $(\lambda_t)_{t\geqslant 0}$, that is

$$\sigma\{D_s : 0 \leqslant s \leqslant t\} \vee \sigma\{L_s : 0 \leqslant s \leqslant t\} \vee \sigma\{\lambda_s : 0 \leqslant s \leqslant t\} \subset \mathcal{F}_t^J,$$

for each $t \geqslant 0$. Let us now introduce the filtration for the fractional market. Let $(\mathcal{F}_t^{A,U})_{t\geqslant 0}$ be defined as $\mathcal{F}_t^{A,U} = \mathcal{F}_t^A \vee \mathcal{F}_t^U$. Since S_s is a $(\mathcal{F}_t^U)_{t\geqslant 0}$-stopping-time for all $s \geqslant 0$, it is also a $(\mathcal{F}_t^{A,U})_{t\geqslant 0}$-stopping-time. Therefore, we can introduce the filtration $(\mathcal{F}_t^{A\circ S})_{t\geqslant 0}$ defined as the time-changed filtration $\mathcal{F}_t^{A\circ S} = \mathcal{F}_{S_t}^{A,U}$. It is well known from the chapter 10 of J. Jacod (1979) that if a process $(A_t)_{t\geqslant 0}$ is a $(\mathcal{F}_t^{A,U})_{t\geqslant 0}$-semimartingale, then the time-changed process $(A_{S_t})_{t\geqslant 0}$ is a $(\mathcal{F}_{S_t}^{A,U})_{t\geqslant 0}$-semimartingale[3]. Therefore, the filtration of the asset price $(A_{S_t})_{t\geqslant 0}$ can be chosen to be $(\mathcal{F}_t^{A\circ S})_{t\geqslant 0}$.

We move now to the main topic of this section. We first need to derive PDE's for the joint densities and transform of the process $(X_t, \lambda_t)_{t\geqslant 0}$. Afterwards, we use the PDE of $(X_t, \lambda_t)_{t\geqslant 0}$ to show that the joint densities and transforms of the time-changed process $(X_{S_t}, \lambda_{S_t})_{t\geqslant 0}$ satisfy similar equations, with the difference that the partial derivative with respect to time is replaced by a fractional Caputo derivative. In the non-fractional case, we work conditionally on the last known value for $(X_t, \lambda_t)_{t\geqslant 0}$. This is justified because self-exciting Hawkes processes satisfy the Markov property provided that we keep track of their intensity. More precisely, $(J_t, \lambda_t)_{t\geqslant 0}$ is a Markov process whereas the jump process $(J_t)_{t\geqslant 0}$ alone is not.

Proposition 0.3. *Let $p(t, x_1, x_2|s, y_1, y_2)$ be the bivariate probability density function (PDF) of (X_t, λ_t) given $(X_s, \lambda_s) = (y_1, y_2)$, $s < t$, i.e.*

$$p(t, x_1, x_2|s, y_1, y_2) := \frac{\partial^2}{\partial x_1 \partial x_2} \mathbb{P}(X_t \leqslant x_1, \lambda_t \leqslant x_2 | X_s = y_1, \lambda_s = y_2).$$

Then p satisfies the PDE

$$\begin{aligned}
&\frac{\partial}{\partial t} p(t, x_1, x_2|s, y_1, y_2) = -\left(\mu - \frac{\sigma^2}{2} - x_2 \mathbb{E}[e^{\xi} - 1]\right) \frac{\partial}{\partial x_1} p(t, x_1, x_2|s, y_1, y_2) \\
&+ \frac{\sigma^2}{2} \frac{\partial^2}{\partial x_1^2} p(t, x_1, x_2|s, y_1, y_2) - \kappa(\theta - x_2) \frac{\partial}{\partial x_2} p(t, x_1, x_2|s, y_1, y_2) + \kappa p(t, x_1, x_2|s, y_1, y_2) \\
&- \eta \mathbb{E}[|\xi| p(t, x_1 - \xi, x_2 - \eta|\xi| | s, y_1, y_2)] + \mathbb{E}[p(t, x_1 - \xi, x_2 - \eta|\xi| | s, y_1, y_2) - p(t, x_1, x_2|s, y_1, y_2)],
\end{aligned}$$

with initial condition $p(s, x_1, x_2|s, y_1, y_2) = \delta_{\mathbb{R}^2}(x_1 - y_1, x_2 - y_2)$, $\delta_{\mathbb{R}^2}$ being the Dirac measure located at $(0,0) \in \mathbb{R}^2$.

Proof. This proof relies on the Cramers-Moyal expansion. It states that the bivariate PDF p satisfies

$$\begin{aligned}
&p(t + \Delta, x_1, x_2|s, y_1, y_2) - p(t, x_1, x_2|s, y_1, y_2) \\
&= \sum_{n=1}^{+\infty} \sum_{j=0}^{n} \frac{(-1)^n}{j!(n-j)!} \frac{\partial^j}{\partial x_1^j} \frac{\partial^{n-j}}{\partial x_2^{n-j}} \left(M(j, n-j, \Delta|t, x_1, x_2) p(t, x_1, x_2|s, y_1, y_2)\right)
\end{aligned} \tag{16}$$

[2]Recall that $(X_t)_{t\geqslant 0}$ is the logarithm of the asset price, i.e. $X_t = \ln A_t$.

[3]This is stated in Theorem 10.16, which can be applied to our case because our time-change $(S_t)_{t\geqslant 0}$ is both finite and continuous. We shall come back more precisely to this point in the section about changes of measures.

where $M(j, n-j, \Delta|t, x_1, x_2) = \mathbb{E}[(X_{t+\Delta} - X_t)^j (\lambda_{t+\Delta} - \lambda_t)^{n-j} | X_t = x_1, \lambda_t = x_2]$. The proof of this statement can be found in Hainaut (2020). Applying Ito's Lemma on the functions $\mathbb{R}^2 \ni (x, y) \mapsto h_{j,n}(x, y) := x^j y^{n-j}$ leads to

$$\begin{aligned}
& h_{j,n}(X_{t+\Delta} - X_t, \lambda_{t+\Delta} - \lambda_t) \\
& = j \int_{s=0}^{\Delta} (X_{t+s} - X_t)^{j-1} (\lambda_{t+s} - \lambda_t)^{n-j} \left(\left[\mu - \frac{\sigma^2}{2} - \lambda_{t+s} \mathbb{E}[e^{\xi} - 1] \right] \mathrm{d}s + \sigma \mathrm{d}W_{t+s} \right) \\
& + \frac{j(j-1)\sigma^2}{2} \int_{s=0}^{\Delta} (X_{t+s} - X_t)^{j-2} (\lambda_{t+s} - \lambda_t)^{n-j} \mathrm{d}s \\
& + (n-j) \int_{s=0}^{\Delta} (X_{t+s} - X_t)^{j} (\lambda_{t+s} - \lambda_t)^{n-j-1} \kappa(\theta - \lambda_{t+s}) \mathrm{d}s \\
& + \int_{s=0}^{\Delta} \left((X_{t+s} - X_t)^j - (X_{t+s-} - X_t)^j \right) \left((\lambda_{t+s} - \lambda_t)^j - (\lambda_{t+s-} - \lambda_t)^j \right) \mathrm{d}N_{t+s}.
\end{aligned} \tag{17}$$

Let us now examine the expectation of each term of Equation (17). We have for the first term

$$\begin{aligned}
& \mathbb{E}\left[j \int_{s=0}^{\Delta} (X_{t+s} - X_t)^{j-1} (\lambda_{t+s} - \lambda_t)^{n-j} \left(\left[\mu - \frac{\sigma^2}{2} - \lambda_{t+s} \mathbb{E}[e^{\xi} - 1] \right] \mathrm{d}s + \sigma \mathrm{d}W_{t+s} \right) \right] \\
& = \begin{cases} \left(\mu - \frac{\sigma^2}{2} - \lambda_t \mathbb{E}[e^{\xi} - 1] \right) \Delta + O(\Delta^2) & \text{if } n = j = 1 \\ O(\Delta^2) & \text{otherwise,} \end{cases}
\end{aligned} \tag{18}$$

whereas the other terms can be respectively written as

$$\begin{aligned}
& \mathbb{E}\left[\frac{j(j-1)\sigma^2}{2} \int_{s=0}^{\Delta} (X_{t+s} - X_t)^{j-2} (\lambda_{t+s} - \lambda_t)^{n-j} \mathrm{d}s \right] \\
& = \begin{cases} \sigma^2 \Delta + O(\Delta^2) & \text{if } n = j = 2 \\ O(\Delta^2) & \text{otherwise,} \end{cases}
\end{aligned} \tag{19}$$

$$\begin{aligned}
& \mathbb{E}\left[(n-j) \int_{s=0}^{\Delta} (X_{t+s} - X_t)^{j} (\lambda_{t+s} - \lambda_t)^{n-j-1} \kappa(\theta - \lambda_{t+s}) \mathrm{d}s \right] \\
& = \begin{cases} \kappa(\theta - \lambda_t)\Delta + O(\Delta^2) & \text{if } n = 1, j = 0 \\ O(\Delta^2) & \text{otherwise,} \end{cases}
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
& \mathbb{E}\left[\int_{s=0}^{\Delta} \left((X_{t+s} - X_t)^j - (X_{t+s-} - X_t)^j \right) \left((\lambda_{t+s} - \lambda_t)^{n-j} - (\lambda_{t+s-} - \lambda_t)^{n-j} \right) \mathrm{d}N_{t+s} \right] \\
& = \lambda_t \eta^{n-j} \mathbb{E}[\xi^j |\xi|^{n-j}] \Delta + O(\Delta^2).
\end{aligned} \tag{21}$$

Using Equations (16) to (21), we obtain

$$
\begin{aligned}
&\frac{p(t+\Delta,x_1,x_2|s,y_1,y_2)-p(t,x_1,x_2|s,y_1,y_2)}{\Delta}\\
&=-\frac{\partial}{\partial x_1}\left[\left(\mu-\frac{\sigma^2}{2}-x_2\mathbb{E}[e^\xi-1]\right)p(t,x_1,x_2|s,y_1,y_2)\right]\\
&+\frac{\sigma^2}{2}\frac{\partial^2}{\partial x_1^2}p(t,x_1,x_2|s,y_1,y_2)-\frac{\partial}{\partial x_2}\left[\kappa(\theta-x_2)p(t,x_1,x_2|s,y_2,y_3)\right]\\
&+\mathbb{E}\left[\sum_{n=1}^{+\infty}\sum_{j=0}^{n}\frac{(-\xi)^j(-|\eta\xi|)^{n-j}}{j!(n-j)!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\left(x_2p(t,x_1,x_2|s,y_1,y_2)\right)\right]+O(\Delta)\\
&=-\left(\mu-\frac{\sigma^2}{2}-x_2\mathbb{E}[e^\xi-1]\right)\frac{\partial}{\partial x_1}p(t,x_1,x_2|s,y_1,y_2)\\
&+\frac{\sigma^2}{2}\frac{\partial^2}{\partial x_1^2}p(t,x_1,x_2|s,y_1,y_2)-\kappa(\theta-x_2)\frac{\partial}{\partial x_2}p(t,x_1,x_2|s,y_1,y_2)+\kappa p(t,x_1,x_2|s,y_1,y_2)\\
&+\mathbb{E}\left[\sum_{n=1}^{+\infty}\sum_{j=0}^{n}\frac{(-\xi)^j(-|\eta\xi|)^{n-j}}{j!(n-j)!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\left(x_2p(t,x_1,x_2|s,y_1,y_2)\right)\right]+O(\Delta).
\end{aligned}
\tag{22}
$$

The expectation term of Equation (22) can be rewritten as

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{n=1}^{+\infty}\sum_{j=0}^{n}\frac{(-\xi)^j(-|\eta\xi|)^{n-j}}{j!(n-j)!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\left(x_2p(t,x_1,x_2|s,y_1,y_2)\right)\right]\\
&=\mathbb{E}\left[\sum_{n=1}^{+\infty}\sum_{j=0}^{n-1}\frac{(-\xi)^j(-|\eta\xi|)^{n-j}}{j!(n-j)!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\left(x_2p(t,x_1,x_2|s,y_1,y_2)\right)\right]\\
&+\mathbb{E}\left[\sum_{n=1}^{+\infty}\frac{(-\xi)^n}{n!}\frac{\partial^n}{\partial x_1^n}(x_2p(t,x_1,x_2|s,y_1,y_2))\right]\\
&=\mathbb{E}\left[-\eta|\xi|\sum_{n=1}^{+\infty}\sum_{j=0}^{n-1}\frac{(-\xi)^j(-|\eta\xi|)^{n-j-1}}{j!(n-j-1)!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j-1}}{\partial x_2^{n-j-1}}p(t,x_1,x_2|s,y_1,y_2)\right]\\
&+\mathbb{E}\left[\sum_{n=1}^{+\infty}\sum_{j=0}^{n-1}\frac{(-\xi)^j(-|\eta\xi|)^{n-j}}{j!(n-j)!}x_2\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}p(t,x_1,x_2|s,y_1,y_2)\right]\\
&+\mathbb{E}\left[\sum_{n=1}^{+\infty}\frac{(-\xi)^n}{n!}x_2\frac{\partial^n}{\partial x_1^n}p(t,x_1,x_2|s,y_1,y_2)\right]\\
&=\mathbb{E}\left[-\eta|\xi|\sum_{n=0}^{+\infty}\sum_{j=0}^{n}\frac{(-\xi)^j(-|\eta\xi|)^{n-j}}{j!(n-j)!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}p(t,x_1,x_2|s,y_1,y_2)\right]\\
&+\mathbb{E}\left[\sum_{n=1}^{+\infty}\sum_{j=0}^{n}\frac{(-\xi)^j(-|\eta\xi|)^{n-j}}{j!(n-j)!}x_2\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}p(t,x_1,x_2|s,y_1,y_2)\right]\\
&=-\eta\mathbb{E}[|\xi|p(t,x_1-\xi,x_2-\eta|\xi||s,y_1,y_2)]\\
&+x_2\mathbb{E}[p(t,x_1-\xi,x_2-\eta|\xi||s,y_1,y_2)-p(t,x_1,x_2|s,y_1,y_2)].
\end{aligned}
\tag{23}
$$

The proof is then completed by combining Equations (22) and (23). □

For $t \geqslant 0$, $z_1 \in \mathbb{C}$ and $z_2 \in \mathbb{R}$, define the conditional transform φ as

$$\varphi(t,z_1,z_2|s,y_1,y_2) := \mathbb{E}[e^{z_1X_t+\mathrm{i}z_2\lambda_t}|X_s=z_1,\lambda_s=y_2].$$

The function φ also satisfies a PDE.

Proposition 0.4. *The function φ satisfies the PDE*

$$\begin{aligned}\frac{\partial \varphi}{\partial t}(t,z_1,z_2|s,y_1,y_2) &= \left(\mathrm{i}z_1\mathbb{E}[e^{\xi}-1]-\kappa z_2+\mathrm{i}-\mathrm{i}\mathbb{E}\left[e^{z_1\xi+\mathrm{i}z_2\eta|\xi|}\right]\right)\frac{\partial \varphi}{\partial z_2}(t,z_1,z_2|s,y_1,y_2)\\ &+\left(\left(\mu-\frac{\sigma^2}{2}\right)z_1+\frac{(\sigma z_1)^2}{2}+\mathrm{i}\kappa\theta z_2\right)\varphi(t,z_1,z_2|s,y_1,y_2)\end{aligned}$$

with boundary conditions $\varphi(s,z_1,z_2|s,y_1,y_2)=e^{z_1y_1+\mathrm{i}z_2y_2}$ and $\varphi(s,0,0|s,y_1,y_2)=1$.

Proof. The partial derivative of φ satisfies

$$\begin{aligned}\frac{\partial \varphi}{\partial t}(t,z_1,z_2|s,y_1,y_2) &= \lim_{\Delta\to 0}\frac{\int_{\mathbb{R}^+}\int_{\mathbb{R}}e^{z_1x_1+\mathrm{i}z_2x_2}(p(t+\Delta,x_1,x_2|s,y_1,y_2)-p(t,x_1,x_2|s,y_1,y_2))\mathrm{d}x_1\mathrm{d}x_2}{\Delta}\\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}}e^{z_1x_1+\mathrm{i}z_2x_2}\frac{\partial p}{\partial t}(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2.\end{aligned}$$

Thanks to Proposition 0.3, we find

$$\begin{aligned}\frac{\partial \varphi}{\partial t}(t,z_1,z_2|s,y_1,y_2) &= -\int_{\mathbb{R}^+}\int_{\mathbb{R}}e^{z_1x_1+\mathrm{i}z_2x_2}\left(\mu-\frac{\sigma^2}{2}-x_2\mathbb{E}[e^{\xi}-1]\right)\frac{\partial p}{\partial x_1}(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\ &+\frac{\sigma^2}{2}\int_{\mathbb{R}^+}\int_{\mathbb{R}}e^{z_1x_1+\mathrm{i}z_2x_2}\frac{\partial^2 p}{\partial x_1^2}(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\ &-\kappa\int_{\mathbb{R}^+}\int_{\mathbb{R}}e^{z_1x_1+\mathrm{i}z_2x_2}(\theta-x_2)\frac{\partial p}{\partial x_2}(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\ &\kappa\int_{\mathbb{R}^+}\int_{\mathbb{R}}e^{z_1x_1+\mathrm{i}z_2x_2}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\ &-\eta\mathbb{E}\left[|\xi|\int_{\mathbb{R}^+}\int_{\mathbb{R}}e^{z_1x_1+\mathrm{i}z_2x_2}p(t,x_1-\xi,x_2-\eta|\xi||s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\right]\\ &\mathbb{E}\left[\int_{\mathbb{R}^+}\int_{\mathbb{R}}x_2e^{z_1x_1+\mathrm{i}z_2x_2}(p(t,x_1-\xi,x_2-\eta|\xi||s,y_1,y_2)-p(t,x_1,x_2|s,y_1,y_2))\mathrm{d}x_1\mathrm{d}x_2\right]\end{aligned}$$

We can now compute all the terms with the help of integrations by parts. To this end, note that if $\omega\in\mathbb{R}$ and $\mathbb{E}[e^{\omega X_t}]<+\infty$, then

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}}e^{\omega x_1}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2<+\infty$$

so that

$$\int_{\mathbb{R}}e^{\omega x_1}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1<+\infty \tag{24}$$

for almost all x_2. Fix any $x_2\in\mathbb{R}$ such that (24) holds. Since the function $x_1\mapsto e^{\omega x_1}p(t,x_1,x_2|s,y_1,y_2)$ is positive and continuous, Equation (24) implies that

$$\lim_{x_1\to\pm\infty}e^{\omega x_1}p(t,x_1,x_2|s,y_1,y_2)=0.$$

Also note that $p(t,x_1,x_2|s,y_1,y_2)$ vanishes whenever $x_2\to+\infty$. Moreover, since by definition of $(\lambda_t)_{t\geqslant 0}$, $\lambda_t\geqslant\theta$ for all t, it follows that $p(t,x_1,0|s,y_1,y_2)=0$ for all t,x_1,s,y_1,y_2. Bearing these facts in mind, the first term is rewritten as

$$
\begin{aligned}
&-\int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{z_1x_1+\mathrm{i}z_2x_2}\left(\mu-\frac{\sigma^2}{2}-x_2\mathbb{E}[e^{\xi}-1]\right)\frac{\partial p}{\partial x_1}(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\
&=-\left(\mu-\frac{\sigma^2}{2}\right)\int_{\mathbb{R}^+} z_1e^{\mathrm{i}z_2x_2}\left[p(t,x_1,x_2|s,y_1,y_2)\frac{e^{z_1x_1}}{z_1}-\int e^{z_1x_1}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\right]_{x_1=-\infty}^{+\infty}\mathrm{d}x_2\\
&+\mathbb{E}[e^{\xi}-1]\int_{\mathbb{R}^+} x_2e^{\mathrm{i}z_2x_2}z_1\left[p(t,x_1,x_2|s,y_1,y_2)\frac{e^{z_1x_1}}{z_1}-\int e^{z_1x_1}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\right]_{x_1=-\infty}^{+\infty}\mathrm{d}x_2\\
&=z_1\left(\mu-\frac{\sigma^2}{2}\right)\int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{z_1x_1+\mathrm{i}z_2x_2}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\
&+\mathrm{i}z_1\mathbb{E}[e^{\xi}-1]\frac{\partial}{\partial z_2}\int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{z_1x_1+\mathrm{i}z_2x_2}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\
&=z_1\left(\mu-\frac{\sigma^2}{2}\right)\varphi(t,z_1,z_2|s,y_1,y_2)+\mathrm{i}z_1\mathbb{E}[e^{\xi}-1]\frac{\partial\varphi}{\partial z_2}(t,z_1,z_2|s,y_1,y_2).
\end{aligned}
\tag{25}
$$

The second second term is computed with two integrations by parts, that is

$$
\begin{aligned}
&\frac{\sigma^2}{2}\int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{z_1x_1+\mathrm{i}z_2x_2}\frac{\partial^2 p}{\partial x_1^2}(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\
&=\frac{\sigma^2}{2}\int_{\mathbb{R}^+} e^{\mathrm{i}z_2x_2}z_1\left[\frac{e^{z_1x_1}}{z_1}\frac{\partial p}{\partial x_1}(t,x_1,x_2|s,y_1,y_2)-\int e^{z_1x_1}\frac{\partial p}{\partial x_1}(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\right]_{x_1=-\infty}^{+\infty}\mathrm{d}x_2\\
&=-\frac{\sigma^2}{2}z_1\int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{z_1x_1+\mathrm{i}z_2x_2}\frac{\partial p}{\partial x_1}(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\
&=-\frac{\sigma^2}{2}z_1\int_{\mathbb{R}^+} e^{\mathrm{i}z_2x_2}z_1\left[\frac{e^{z_1x_1}}{z_1}p(t,x_1,x_2|s,y_1,y_2)-\int e^{z_1x_1}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\right]_{x_1=-\infty}^{+\infty}\mathrm{d}x_2\\
&=\frac{\sigma^2}{2}z_1^2\int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{z_1x_1+\mathrm{i}z_2x_2}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\
&=\frac{\sigma^2}{2}z_1^2\varphi(t,z_1,z_2|s,y_1,y_2).
\end{aligned}
\tag{26}
$$

The third term is

$$
\begin{aligned}
&-\kappa\int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{z_1x_1+\mathrm{i}z_2x_2}(\theta-x_2)\frac{\partial p}{\partial x_2}(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\
&=-\kappa\theta\int_{\mathbb{R}} e^{z_1x_1}\mathrm{i}z_2\int_{\mathbb{R}^+}\frac{e^{\mathrm{i}z_2x_2}}{\mathrm{i}z_2}\frac{\partial p}{\partial x_2}(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_2\mathrm{d}x_1\\
&-\kappa\mathrm{i}\int_{\mathbb{R}} e^{z_1x_1}\int_{\mathbb{R}^+}\mathrm{i}x_2e^{\mathrm{i}z_2x_2}\frac{\partial p}{\partial x_2}(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_2\mathrm{d}x_1\\
&=-\kappa\theta\mathrm{i}z_2\int_{\mathbb{R}} e^{z_1x_1}\left[p(t,x_1,x_2|s,y_1,y_2)-\int e^{\mathrm{i}x_2z_2}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_2\right]_{x_2=0}^{+\infty}\mathrm{d}x_1\\
&-\kappa\mathrm{i}\frac{\partial}{\partial z_2}\int_{\mathbb{R}} e^{z_1x_1}\int_{\mathbb{R}^+}e^{\mathrm{i}z_2x_2}\frac{\partial p}{\partial x_2}(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_2\mathrm{d}x_1\\
&=\kappa\theta\mathrm{i}z_2\int_{\mathbb{R}^+}\int_{\mathbb{R}} e^{z_1x_1+\mathrm{i}z_2x_2}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\
&-\kappa\mathrm{i}\frac{\partial}{\partial z_2}\int_{\mathbb{R}} e^{z_1x_1}\mathrm{i}z_2\left[\frac{e^{\mathrm{i}z_2x_2}}{\mathrm{i}z_2}p(t,x_1,x_2|s,y_1,y_2)-\int e^{\mathrm{i}z_2x_2}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_2\right]_{x_2=0}^{+\infty}\mathrm{d}x_1\\
&=\kappa\theta\mathrm{i}z_2\varphi(t,z_1,z_2|s,y_1,y_2)-\kappa\frac{\partial}{\partial z_2}\left(z_2\varphi(t,z_1,z_2|s,y_1,y_2)\right)\\
&=\kappa\varphi(t,z_1,z_2|s,y_1,y_2)[\theta\mathrm{i}z_2-1]-z_2\kappa\frac{\partial\varphi}{\partial z_2}(t,z_1,z_2|s,y_1,y_2).
\end{aligned}
\tag{27}
$$

The fourth term is simply

$$\kappa \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{z_1 x_1 + \mathrm{i} z_2 x_2} p(t, x_1, x_2 | s, y_1, y_2) \mathrm{d}x_1 \mathrm{d}x_2 = \kappa \varphi(t, z_1, z_2 | s, y_1, y_2). \tag{28}$$

The fifth term is computed as

$$\begin{aligned}
&- \eta \mathbb{E}\left[|\xi| \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{z_1 x_1 + \mathrm{i} z_2 x_2} p(t, x_1 - \xi, x_2 - \eta|\xi| | s, y_1, y_2) \mathrm{d}x_1 \mathrm{d}x_2 \right] \\
&= -\eta \mathbb{E}\left[|\xi| \int_{-\eta|\xi|}^{+\infty} \int_{\mathbb{R}} e^{z_1(x_1+\xi) + \mathrm{i} z_2 (x_2 + \eta|\xi|)} p(t, x_1, x_2 | s, y_1, y_2) \mathrm{d}x_1 \mathrm{d}x_2 \right] \\
&= -\eta \mathbb{E}\left[|\xi| e^{z_1 \xi + \mathrm{i} z_2 \eta |\xi|} \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{z_1 x_1 + \mathrm{i} z_2 x_2} p(t, x_1, x_2 | s, y_1, y_2) \mathrm{d}x_1 \mathrm{d}x_2 \right] \\
&= -\eta \mathbb{E}\left[|\xi| e^{z_1 \xi + \mathrm{i} z_2 \eta |\xi|} \right] \varphi(t, z_1, z_2 | s, y_1, y_2).
\end{aligned} \tag{29}$$

Similarly, the last term is rewritten as

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^+} \int_{\mathbb{R}} x_2 e^{z_1 x_1 + \mathrm{i} z_2 x_2} (p(t, x_1 - \xi, x_2 - \eta|\xi| | s, y_1, y_2) - p(t, x_1, x_2 | s, y_1, y_2)) \mathrm{d}x_1 \mathrm{d}x_2 \right] \\
&= \varphi(t, z_1, z_2 | s, y_1, y_2) \eta \mathbb{E}\left[|\xi| e^{z_1 \xi + \mathrm{i} z_2 \eta |\xi|} \right] + \mathrm{i} \left(1 - \mathbb{E}\left[e^{z_1 \xi + \mathrm{i} z_2 \eta |\xi|} \right] \right) \frac{\varphi}{\partial z_2}(t, z_1, z_2 | s, y_1, y_2).
\end{aligned} \tag{30}$$

Adding all the terms of Equations (25) to (30) gives the announced result. □

Define $\varphi^{(0)}$ as

$$\varphi^{(0)}(t, z_1, z_2 | s, y_1, y_2) := \mathbb{E}[e^{z_1(X_t - y_1) + \mathrm{i} z_2 (\lambda_t - y_2)} | X_s = y_1, \lambda_s = y_2],$$

which is a variant of φ where the initial values of the processes $(X_t)_{t \geqslant 0}$ and $(\lambda_t)_{t \geqslant 0}$ are substracted. The PDE of Proposition 0.4 easily imply a PDE for $\varphi^{(0)}$.

Corollary 0.2. *The function $\varphi^{(0)}$ satisfies the PDE*

$$\frac{\partial \varphi^{(0)}}{\partial t}(t, z_1, z_2 | s, y_1, y_2) = \gamma(z_1, z_2) \frac{\partial \varphi^{(0)}}{\partial z_2}(t, z_1, z_2 | s, y_1, y_2) + \beta(z_1, z_2) \varphi^{(0)}(t, z_1, z_2 | s, y_1, y_2)$$

where

$$\gamma(z_1, z_2) := \mathrm{i} z_1 \mathbb{E}[e^{\xi} - 1] - \kappa z_2 + \mathrm{i} - \mathrm{i} \mathbb{E}\left[e^{z_1 \xi + \mathrm{i} z_2 \eta |\xi|} \right]$$

and

$$\beta(z_1, z_2) := \left(r - \frac{\sigma^2}{2} \right) z_1 + \frac{(\sigma z_1)^2}{2} + \mathrm{i} \kappa (\theta - y_2) z_2 - z_1 y_2 \mathbb{E}[e^{\xi} - 1] - y_2 + y_2 \mathbb{E}\left[e^{z_1 \xi + \mathrm{i} z_2 \eta |\xi|} \right].$$

Moreover the boundary conditions $\varphi(s, z_1, z_2 | s, y_1, y_2) = \varphi(s, 0, 0 | s, y_1, y_2) = 1$ hold.

Proof. It follows from

$$\varphi^{(0)}(t, z_1, z_2 | s, y_1, y_2) = \exp\{-z_1 x_1 - \mathrm{i} z_2 y_2\} \varphi(t, z_1, z_2 | s, y_1, y_2)$$

and Proposition 0.4. □

The interest of Corollary 0.2 is that the PDE of this corollary happened to be slightly easier to solve numerically than the PDE of Proposition 0.4.

We now turn to the derivation of fractional counterparts of the PDE's. In the following, for $s \leqslant t$, p_α will denote either the conditional PDF of (X_{S_t}, λ_{S_t}) given that $(X_{S_s}, \lambda_{S_s}) = (y_1, y_2)$, $S_s = v$ and $U_{S_s} = u$, in which case it is written as

$$p_\alpha(t, x_1, x_2 | s, y_1, y_2, v, u) := \frac{\partial^2}{\partial x_1 \partial x_2} \mathbb{P}\left(X_{S_t} \leqslant x_1, \lambda_{S_t} \leqslant x_2 | X_{S_s} = y_1, \lambda_{S_s} = y_2, S_s = v, U_{S_s} = u \right),$$

the conditional PDF of $(S_t, X_{S_t}, \lambda_{S_t})$ given the same information, in which case we denote it by

$$\begin{aligned} & p_\alpha(t, x_0, x_1, x_2 | s, y_1, y_2, v, u) \\ & := \frac{\partial^3}{\partial x_0 \partial x_1 \partial x_2} \mathbb{P}\left(S_t \leqslant x_0, X_{S_t} \leqslant x_1, \lambda_{S_t} \leqslant x_2 | X_{S_s} = y_1, \lambda_{S_s} = y_2, S_s = v, U_{S_s} = u\right). \end{aligned}$$

or finally the conditional PDF of (S_t, X_{S_t}). In this last case, we write

$$p_\alpha(t, x_0, x_1 | s, y_1, y_2, v, u) := \frac{\partial^2}{\partial x_0 \partial x_1} \mathbb{P}\left(S_t \leqslant x_0, X_{S_t} \leqslant x_1 | X_{S_s} = y_1, \lambda_{S_s} = y_2, S_s = v, U_{S_s} = u\right).$$

The letter p still denotes the conditional PDF of (X_t, λ_t) knowing (X_s, λ_s). The function g will denote the PDF of S_t given $S_s = v$ and $U_{S_s} = u$, i.e.

$$g(t, \tau | s, v, u) := \frac{\partial}{\partial \tau} \mathbb{P}(S_t - S_s \leqslant \tau | S_s = v, U_{S_s} = u),$$

where $t \geqslant s$. It is shown in Hainaut (2021) that the Laplace transform of g with respect to time satisfies

$$\int_u^{+\infty} e^{-\omega t} g(t, \tau | s, v, u) \mathrm{d}t = \omega^{\alpha-1} e^{-\omega u - \tau \omega^\alpha}$$

so that

$$\begin{aligned} \tilde{g}(\omega, \tau | s, v, u) &:= \int_{\mathbb{R}^+} e^{-\omega t} g(u + t, \tau | s, v, u) \mathrm{d}t \\ &= \omega^{\alpha-1} e^{-\tau \omega^\alpha} \end{aligned} \tag{31}$$

for $t \geqslant u$. Note that the conditional PDF's are related by the following identity

$$p_\alpha(t, x_1, x_2 | s, y_1, y_2, v, u) = \int_{\mathbb{R}^+} p(v + \tau, x_1, x_2 | v, y_1, y_2) g(t, \tau | s, v, u) \mathrm{d}\tau \tag{32}$$

for $t > u$. It follows follows from

$$\begin{aligned} & \mathbb{P}\left(X_{S_t} \leqslant x_1, \lambda_{S_t} \leqslant x_2 | X_{S_s} = y_1, \lambda_{S_s} = y_2, S_s = v, U_{S_s} = u\right) \\ & = \int_{\mathbb{R}^+} \mathbb{P}(X_{v+\tau} \leqslant x_1, \lambda_{v+\tau} \leqslant x_2 | X_{S_s} = y_1, \lambda_{S_s} = y_2, S_s = v, U_{S_s} = u) g(t, \tau | s, v, u) \mathrm{d}\tau \\ & = \int_{\mathbb{R}^+} \mathbb{P}(X_{v+\tau} \leqslant x_1, \lambda_{v+\tau} \leqslant x_2 | X_v = y_1, \lambda_v = y_2) g(t, \tau | s, v, u) \mathrm{d}\tau. \end{aligned}$$

The reason for imposing $t > u$ is the fact that if $s \leqslant t \leqslant u = U_{S_s}$, then we have $U_{S_t} = U_{S_s} = u$, that is U_{S_t} is not random anymore.

The next lemma comes as a consequence of the relation between p and p_α at Equation (32).

Lemma 0.6. *The Laplace transforms of p and p_α satisfy*

$$\tilde{p}_\alpha(\omega, x_1, x_2 | s, y_1, y_2, v, u) = \omega^{\alpha-1} \tilde{p}(\omega^\alpha, x_1, x_2 | v, y_1, y_2)$$

where

$$\tilde{p}_\alpha(\omega, x_1, x_2 | s, y_1, y_2, v, u) := \int_{\mathbb{R}^+} e^{-\omega t} p_\alpha(u + t, x_1, x_2 | s, y_1, y_2, v, u) \mathrm{d}t$$

and

$$\tilde{p}(\omega, x_1, x_2 | v, y_1, y_2) := \int_{\mathbb{R}^+} e^{-t\omega} p(v + t, x_1, x_2 | v, y_1, y_2) \mathrm{d}t.$$

Proof. From Equation (32) and Fubini's Theorem,

$$\begin{aligned}
\tilde{p}_\alpha(\omega, x_1, x_2|s, y_1, y_2, v, u) &:= \int_{\mathbb{R}^+} e^{-\omega t} p_\alpha(u+t, x_1, x_2|s, y_1, y_2, v, u)\mathrm{d}t \\
&= \int_{\mathbb{R}^+} e^{-\omega t} \int_{\mathbb{R}^+} p(v+\tau, x_1, x_2|v, y_1, y_2) g(u+t, \tau|s, v, u)\mathrm{d}\tau\mathrm{d}t \\
&= \int_{\mathbb{R}^+} p(v+\tau, x_1, x_2|v, y_1, y_2) \int_{\mathbb{R}^+} e^{-\omega t} g(u+t, \tau|s, v, u)\mathrm{d}t\mathrm{d}\tau \\
&= \omega^{\alpha-1} \int_{\mathbb{R}^+} e^{-\tau\omega^\alpha} p(v+\tau, x_1, x_2|v, y_1, y_2)\mathrm{d}\tau \\
&= \omega^{\alpha-1} \tilde{p}(\omega^\alpha, x_1, x_2|v, y_1, y_2),
\end{aligned}$$

as announced. □

In the next lemma, we recall another useful result.

Lemma 0.7. *For any function $h : \mathbb{R}^+ \to \mathbb{R}$, it holds that*

$$\widetilde{\frac{\mathrm{d}^\alpha h}{\mathrm{d}t^\alpha}}(\omega) = \omega^\alpha \tilde{h}(\omega) - \omega^{\alpha-1} h(0)$$

where

$$\widetilde{\frac{\mathrm{d}^\alpha h}{\mathrm{d}t^\alpha}}(\omega) := \int_{\mathbb{R}^+} \frac{\mathrm{d}^\alpha h}{\mathrm{d}t^\alpha}(t) e^{-\omega t}\mathrm{d}t \text{ and } \tilde{h}(\omega) := \int_{\mathbb{R}^+} h(t) e^{-\omega t}\mathrm{d}t.$$

The proof can be found in Podlubny (1999). Thanks to these preliminary results about fractional calculus, we can extend our PDE's to the fractional case.

Proposition 0.5. *For $t \geqslant u$, the joint conditional PDF p_α of (X_{S_t}, λ_{S_t}) satisfies the following fractional partial differential equation (FPDE)*

$$\begin{aligned}
\frac{\partial^\alpha p_\alpha}{\partial t^\alpha}(t, x_1, x_2|s, y_1, y_2, v, u) &= -\left(\mu - \frac{\sigma^2}{2} - x_2\mathbb{E}[e^\xi - 1]\right)\frac{\partial}{\partial x_1} p_\alpha(t, x_1, x_2|s, y_1, y_2, v, u) \\
&+ \frac{\sigma^2}{2}\frac{\partial^2}{\partial x_1^2} p_\alpha(t, x_1, x_2|s, y_1, y_2, v, u) - \kappa(\theta - x_2)\frac{\partial}{\partial x_2} p_\alpha(t, x_1, x_2|s, y_1, y_2, v, u) \\
&+ \kappa p_\alpha(t, x_1, x_2|s, y_1, y_2, v, u) - \eta\mathbb{E}[|\xi| p_\alpha(t, x_1 - \xi, x_2 - \eta|\xi||s, y_1, y_2, v, u)] \\
&+ \mathbb{E}[p_\alpha(t, x_1 - \xi, x_2 - \eta|\xi||s, y_1, y_2, v, u) - p_\alpha(t, x_1, x_2|s, y_1, y_2, v, u)],
\end{aligned}$$

with initial condition $p_\alpha(s, x_1, x_2|s, y_1, y_2, v, u) = \delta_{\{x_1 - y_1, x_2 - y_2\}}$.

Proof. Recall that for $s \leqslant t$, $p(t, x_1, x_2|s, y_1, y_2)$ denotes the PDF of (X_t, λ_t) given that $(X_s, \lambda_s) = (y_1, y_2)$ evaluated at $(x_1, x_2) \in \mathbb{R}^2$. The identity

$$\int_{\mathbb{R}^+} \frac{\partial}{\partial t}\left(e^{-\omega t} p(s+t, x_1, x_2|s, y_1, y_2)\right)\mathrm{d}t = -p(s, x_1, x_2|s, y_1, y_2)$$

follows from the fundamental theorem of calculus while an explicit computation of the integral gives

$$\begin{aligned}
&\int_{\mathbb{R}^+} \frac{\partial}{\partial t}\left(e^{-\omega t} p(s+t, x_1, x_2|s, y_1, y_2)\right)\mathrm{d}t \\
&\quad = -\omega\tilde{p}(\omega, x_1, x_2|s, y_1, y_2) + \int_{\mathbb{R}^+} e^{-\omega t}\frac{\partial}{\partial t} p(s+t, x_1, x_2|s, y_1, y_2)\mathrm{d}t
\end{aligned}$$

with $\tilde{p}(\omega, x_1, x_2|s, y_1, y_2) := \int_{\mathbb{R}^+} e^{-\omega t} p(s+t, x_1, x_2|s, y_1, y_2)\mathrm{d}t$. As a consequence, we have

$$\omega\tilde{p}(\omega, x_1, x_2|s, y_1, y_2) - p(s, x_1, x_2|s, y_1, y_2) = \int_{\mathbb{R}^+} e^{-\omega t}\frac{\partial}{\partial t} p(s+t, x_1, x_2|s, y_1, y_2)\mathrm{d}t. \tag{33}$$

From the PDE of p (see Proposition 0.3), the right-hand side of Equation (33) becomes

$$
\begin{aligned}
&-\int_{\mathbb{R}^+} e^{-\omega t}\left(\mu-\frac{\sigma^2}{2}-x_2\mathbb{E}[e^{\xi}-1]\right)\frac{\partial}{\partial x_1}p(s+t,x_1,x_2|s,y_1,y_2)\mathrm{d}t\\
&+\frac{\sigma^2}{2}\int_{\mathbb{R}^+} e^{-\omega t}\frac{\partial^2}{\partial x_1^2}p(s+t,x_1,x_2|s,y_1,y_2)\mathrm{d}t-\kappa(\theta-x_2)\int_{\mathbb{R}^+} e^{-\omega t}\frac{\partial}{\partial x_2}p(s+t,x_1,x_2|s,y_1,y_2)\mathrm{d}t\\
&+\kappa\int_{\mathbb{R}^+} e^{-\omega t}p(s+t,x_1,x_2|s,y_1,y_2)\mathrm{d}t-\eta\mathbb{E}\left[|\xi|\int_{\mathbb{R}^+} e^{-\omega t}p(s+t,x_1-\xi,x_2-\eta|\xi||s,y_1,y_2)\mathrm{d}t\right]\\
&+\mathbb{E}\left[\int_{\mathbb{R}^+} e^{-\omega t}[p(s+t,x_1-\xi,x_2-\eta|\xi||s,y_1,y_2)-p(s+t,x_1,x_2|s,y_1,y_2)]\mathrm{d}t\right]\\
&=-\left(\mu-\frac{\sigma^2}{2}-x_2\mathbb{E}[e^{\xi}-1]\right)\frac{\partial}{\partial x_1}\tilde{p}(\omega,x_1,x_2|s,y_1,y_2)+\frac{\sigma^2}{2}\frac{\partial^2}{\partial x_1^2}\tilde{p}(\omega,x_1,x_2|s,y_1,y_2)\\
&-\kappa(\theta-x_2)\frac{\partial}{\partial x_2}\tilde{p}(\omega,x_1,x_2|s,y_1,y_2)+\kappa\tilde{p}(\omega,x_1,x_2|s,y_1,y_2)\\
&-\eta\mathbb{E}[|\xi|\tilde{p}(\omega,x_1-\xi,x_2-\eta|\xi||s,y_1,y_2)]+\mathbb{E}[\tilde{p}(\omega,x_1-\xi,x_2-\eta|\xi||s,y_1,y_2)-\tilde{p}(\omega,x_1,x_2|s,y_1,y_2)]
\end{aligned}
$$

Since this is valid for any ω, we can replace ω by ω^α. After multiplying both sides by $\omega^{\alpha-1}$, it gives

$$
\begin{aligned}
&\omega^\alpha(\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1,x_2|s,y_1,y_2))-\omega^{\alpha-1}p(s,x_1,x_2|s,y_1,y_2)\\
&=-\left(\mu-\frac{\sigma^2}{2}-x_2\mathbb{E}[e^{\xi}-1]\right)\frac{\partial}{\partial x_1}(\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1,x_2|s,y_1,y_2))+\frac{\sigma^2}{2}\frac{\partial^2}{\partial x_1^2}(\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1,x_2|s,y_1,y_2))\\
&-\kappa(\theta-x_2)\frac{\partial}{\partial x_2}(\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1,x_2|s,y_1,y_2))+\kappa(\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1,x_2|s,y_1,y_2))\\
&-\eta\mathbb{E}[|\xi|(\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1-\xi,x_2-\eta|\xi||s,y_1,y_2))]\\
&+\mathbb{E}[(\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1-\xi,x_2-\eta|\xi||s,y_1,y_2))-(\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1,x_2|s,y_1,y_2))]
\end{aligned}
$$

Using Lemma 0.6 and the fact that $p(s,x_1,x_2|s,y_1,y_2)=p_\alpha(u,x_1,x_2|s,y_1,y_2,v,u)=\delta_{\{x_1-y_1,x_2-y_2\}}$, we get

$$
\begin{aligned}
&\omega^\alpha(\tilde{p}_\alpha(\omega,x_1,x_2|s,y_1,y_2,v,u))-\omega^{\alpha-1}p(s,x_1,x_2|s,y_1,y_2)\\
&=\widetilde{\frac{\partial^\alpha p_\alpha}{\partial t^\alpha}}(\omega,x_1,x_2|s,y_1,y_2,v,u)\\
&=-\left(\mu-\frac{\sigma^2}{2}-x_2\mathbb{E}[e^{\xi}-1]\right)\frac{\partial}{\partial x_1}\tilde{p}_\alpha(\omega,x_1,x_2|s,y_1,y_2,v,u)+\frac{\sigma^2}{2}\frac{\partial^2}{\partial x_1^2}\tilde{p}_\alpha(\omega,x_1,x_2|s,y_1,y_2,v,u)\\
&-\kappa(\theta-x_2)\frac{\partial}{\partial x_2}\tilde{p}_\alpha(\omega,x_1,x_2|s,y_1,y_2,v,u)+\kappa\tilde{p}_\alpha(\omega,x_1,x_2|s,y_1,y_2,v,u)\\
&-\eta\mathbb{E}[|\xi|\tilde{p}_\alpha(\omega,x_1-\xi,x_2-\eta|\xi||s,y_1,y_2,v,u)]\\
&+\mathbb{E}[\tilde{p}_\alpha(\omega,x_1-\xi,x_2-\eta|\xi||s,y_1,y_2,v,u)-\tilde{p}_\alpha(\omega,x_1,x_2|s,y_1,y_2,v,u)],
\end{aligned}
$$

and the result follows. □

From the definition of conditional PDF, we have

$$p_\alpha(t,x_0,x_1,x_2|s,y_1,y_2,v,u)=g(t,x_0-v|s,v,u)p(x_0,x_1,x_2|v,y_1,y_2).$$

The Laplace transform of p_α therefore satisfies

$$
\begin{aligned}
\tilde{p}_\alpha(\omega,x_0,x_1,x_2|s,y_1,y_2,v,u)&:=\int_{\mathbb{R}^+} e^{-\omega t}p_\alpha(u+t,x_0,x_1,x_2|s,y_1,y_2,v,u)\mathrm{d}t\\
&=p(x_0,x_1,x_2|s,y_1,y_2,v,u)\int_{\mathbb{R}^+} e^{-\omega t}g(u+t,x_0-v|s,v,u)\mathrm{d}t \qquad (34)\\
&=p(x_0,x_1,x_2|s,y_1,y_2,v,u)\tilde{g}(\omega,x_0-v|s,v,u)\\
&=\omega^{\alpha-1}p(x_0,x_1,x_2|v,y_1,y_2)e^{-(x_0-v)\omega^\alpha},
\end{aligned}
$$

where the last equality follows from Equation (31). For $z_0, z_2 \in \mathbb{R}$ and $z_1 \in \mathbb{C}$, define the transform φ_α as

$$\varphi_\alpha(t, z_0, z_1, z_2|s, y_1, y_2, v, u) := \mathbb{E}[e^{-z_0 S_t + z_1 X_{S_t} + \mathrm{i} z_2 \lambda_{S_t}} | X_{S_s} = y_1, \lambda_{S_s} = y_2, S_s = v, U_{S_s} = u],$$

with $t \geqslant u$. Then the function φ_α satisfies a fractional PDE.

Proposition 0.6. *The following FPDE holds*

$$\begin{aligned}\frac{\partial^\alpha \varphi_\alpha}{\partial t^\alpha}&(t, z_0, z_1, z_2|s, y_1, y_2, v, u)\\ &= \left(\mathrm{i} z_1 \mathbb{E}[e^\xi - 1] - \kappa z_2 + \mathrm{i} - \mathbb{E}\left[e^{z_1\xi + \mathrm{i} z_2 \eta|\xi|}\right]\right) \frac{\partial \varphi_\alpha}{\partial z_2}(t, z_0, z_1, z_2|s, y_1, y_2, v, u)\\ &+ \left(\left(\mu - \frac{\sigma^2}{2}\right) z_1 + \frac{(\sigma z_1)^2}{2} + \mathrm{i}\kappa\theta z_2 - z_0\right) \varphi_\alpha(t, z_0, z_1, z_2|s, y_1, y_2, v, u)\end{aligned}$$

with boundary conditions

$$\varphi_\alpha(t, z_0, 0, 0|s, y_1, y_2, v, u) = \mathrm{E}_\alpha(-z_0(t-u)^\alpha)$$
$$\varphi_\alpha(v, z_0, z_1, z_2|s, y_1, y_2, v, u) = e^{-z_0 v + z_1 y_1 + \mathrm{i} z_2 y_2}$$

and

$$\varphi_\alpha(t, 0, 0, 0|s, y_1, y_2, v, u) = 1.$$

Proof. Note that the time-Laplace transform of $\varphi^\alpha_{X,\lambda}$ satisfies

$$\begin{aligned}\tilde{\varphi}^\alpha_{X,\lambda}(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u) &:= \int_{\mathbb{R}^+} e^{-\omega t} \varphi_\alpha(u+t, z_0, z_1, z_2|s, y_1, y_2, v, u)\mathrm{d}t\\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathbb{R}^+} e^{-\omega t - z_0 x_0 + z_1 x_1 + \mathrm{i} z_2 x_2} p_\alpha(u+t, x_0, x_1, x_2|s, y_1, y_2, v, u)\mathrm{d}x_0\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}t\\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_v^{+\infty} e^{-z_0 x_0 + z_1 x_1 + \mathrm{i} z_2 x_2} \tilde{p}_\alpha(\omega, x_0, x_1, x_2|s, y_1, y_2, v, u)\mathrm{d}x_0\mathrm{d}x_1\mathrm{d}x_2 \qquad (35)\\ &= \omega^{\alpha-1} e^{v\omega^\alpha} \int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_v^{+\infty} e^{-x_0[z_0+\omega^\alpha] + z_1 x_1 + \mathrm{i} z_2 x_2} p(x_0, x_1, x_2|v, y_1, y_2)\mathrm{d}x_0\mathrm{d}x_1\mathrm{d}x_2\\ &= \omega^{\alpha-1} e^{v\omega^\alpha} \int_v^{+\infty} e^{-x_0[z_0+\omega^\alpha]} \varphi(x_0, z_1, z_2|v, y_1, y_2)\mathrm{d}x_0,\end{aligned}$$

where the penultimate equality comes as a consequence of Equation (34). Next, an integration by parts yields

$$\begin{aligned}\tilde{\varphi}^\alpha_{X,\lambda}&(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u)\\ &= \frac{-\omega^{\alpha-1} e^{v\omega^\alpha}}{z_0 + \omega^\alpha}\Bigg[\varphi(x_0, z_1, z_2|v, y_1, y_2) e^{-[z_0+\omega^\alpha]x_0}\\ &\qquad\qquad - \int e^{-[z_0+\omega^\alpha]x_0} \frac{\varphi}{\partial x_0}(x_0, z_1, z_2|v, y_1, y_2)\mathrm{d}x_0\Bigg]_{x_0=v}^{+\infty} \qquad (36)\\ &= \frac{\omega^{\alpha-1} e^{-z_0 v}}{z_0 + \omega^\alpha}\varphi(v, z_1, z_2|v, y_1, y_2)\\ &\quad + \frac{\omega^{\alpha-1} e^{v\omega^\alpha}}{z_0 + \omega^\alpha}\int_v^{+\infty} e^{-[z_0+\omega^\alpha]t} \frac{\partial\varphi}{\partial t}(t, z_1, z_2|v, y_1, y_2)\mathrm{d}t.\end{aligned}$$

From Proposition 0.4, we know that

$$
\begin{aligned}
&\omega^{\alpha-1} e^{v\omega^\alpha} \int_v^{+\infty} e^{-[z_0+\omega^\alpha]t} \frac{\partial \varphi}{\partial t}(t, z_1, z_2|v, y_1, y_2)\mathrm{d}t \\
&= \omega^{\alpha-1} e^{v\omega^\alpha} \left(\mathrm{i}z_1 \mathbb{E}[e^\xi - 1] - \kappa z_2\right) \int_v^{+\infty} e^{-[z_0+\omega^\alpha]t} \frac{\partial \varphi}{\partial z_2}(t, z_1, z_2|v, y_1, y_2)\mathrm{d}t \\
&+ \frac{\left(\mu - \frac{\sigma^2}{2}\right) z_1 + \frac{(\sigma z_1)^2}{2} + \mathrm{i}\kappa\theta z_2 + \mathbb{E}\left[(1-\eta|\xi|)e^{z_1\xi + \mathrm{i}z_2|\xi|}\right] - 1}{\omega^{1-\alpha} e^{-v\omega^\alpha}} \int_v^{+\infty} e^{-[z_0+\omega^\alpha]t} \varphi(t, z_1, z_2|v, y_1, y_2)\mathrm{d}t \\
&= \left(\mathrm{i}z_1 \mathbb{E}[e^\xi - 1] - \kappa z_2\right) \frac{\partial}{\partial z_2} \tilde{\varphi}^\alpha_{X,\lambda}(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u) \\
&+ \left(\left(\mu - \frac{\sigma^2}{2}\right) z_1 + \frac{(\sigma z_1)^2}{2} + \mathrm{i}\kappa\theta z_2 + \mathbb{E}\left[(1-\eta|\xi|)e^{z_1\xi + \mathrm{i}z_2|\xi|}\right] - 1\right) \tilde{\varphi}^\alpha_{X,\lambda}(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u)
\end{aligned}
$$

where the second equality is a consequence of Equation (35). Equation (36) is then rewritten as

$$
\begin{aligned}
&(z_0 + \omega^\alpha)\tilde{\varphi}^\alpha_{X,\lambda}(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u) - \omega^{\alpha-1} e^{-z_0 v} \varphi(v, z_1, z_2|v, y_1, y_2) \\
&= \left(\mathrm{i}z_1 \mathbb{E}[e^\xi - 1] - \kappa z_2\right) \frac{\partial}{\partial z_2} \tilde{\varphi}^\alpha_{X,\lambda}(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u) \\
&+ \left(\left(\mu - \frac{\sigma^2}{2}\right) z_1 + \frac{(\sigma z_1)^2}{2} + \mathrm{i}\kappa\theta z_2 + \mathbb{E}\left[(1-\eta|\xi|)e^{z_1\xi + \mathrm{i}z_2|\xi|}\right] - 1\right) \tilde{\varphi}^\alpha_{X,\lambda}(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u).
\end{aligned}
$$

By noting that $\varphi_\alpha(u, z_0, z_1, z_2|s, y_1, y_2, v, u) = e^{-z_0 v + z_1 y_1 + \mathrm{i}z_2 y_2} = e^{-z_0 v}\varphi(v, z_1, z_2|v, y_1, y_2)$, it follows that the last equation becomes

$$
\begin{aligned}
&\omega^\alpha \tilde{\varphi}^\alpha_{X,\lambda}(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u) - \omega^{\alpha-1}\varphi_\alpha(u, z_0, z_1, z_2|s, y_1, y_2, v, u) \\
&\quad = -z_0 \tilde{\varphi}^\alpha_{X,\lambda}(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u) + \left(\mathrm{i}z_1 \mathbb{E}[e^\xi - 1] - \kappa z_2\right) \frac{\partial}{\partial z_2} \tilde{\varphi}^\alpha_{X,\lambda}(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u) \\
&\quad + \left(\left(\mu - \frac{\sigma^2}{2}\right) z_1 + \frac{(\sigma z_1)^2}{2} + \mathrm{i}\kappa\theta z_2 + \mathbb{E}\left[(1-\eta|\xi|)e^{z_1\xi + \mathrm{i}z_2|\xi|}\right] - 1\right) \tilde{\varphi}^\alpha_{X,\lambda}(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u).
\end{aligned}
$$

Since Lemma 0.7 implies that

$$
\widetilde{\frac{\partial \varphi_\alpha}{\partial t^\alpha}}(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u) = \omega^\alpha \tilde{\varphi}^\alpha_{X,\lambda}(\omega, z_0, z_1, z_2|s, y_1, y_2, v, u) - \omega^{\alpha-1}\varphi_\alpha(0, z_0, z_1, z_2|s, y_1, y_2, v, u),
$$

the conclusion follows. □

As in the non-fractional case, we derive an equation for the variant $\varphi^{(0)}_\alpha$ of φ^α where the initial values of the processes are substracted. More precisely, if

$$
\varphi^{(0)}_\alpha(t, z_0, z_1, z_2|s, y_1, y_2, v, u) := \mathbb{E}[e^{-z_0 S_t + z_1(X_{S_t} - y_1) + \mathrm{i}z_2(\lambda_{S_t} - y_2)}|X_{S_s} = y_1, \lambda_{S_s} = y_2, S_s = v, U_{S_s} = u],
$$

then $\varphi^{(0)}_\alpha$ satisfies a FPDE.

Corollary 0.3. *The function $\varphi^{(0)}_\alpha$ satisfies the FPDE*

$$
\begin{aligned}
\frac{\partial^\alpha \varphi^{(0)}_\alpha}{\partial t^\alpha}(t, z_1, z_2|s, y_1, y_2, v, u) =& \gamma(z_1, z_2)\frac{\partial \varphi^{(0)}_\alpha}{\partial z_2}(t, z_0, z_1, z_2|s, y_1, y_2, v, u) \\
&+ \beta(z_0, z_1, z_2)\varphi^{(0)}_\alpha(t, z_0, z_1, z_2|s, y_1, y_2, v, u)
\end{aligned}
$$

where

$$
\gamma(z_1, z_2) := \mathrm{i}z_1 \mathbb{E}[e^\xi - 1] - \kappa z_2 + \mathrm{i} - \mathrm{i}\mathbb{E}\left[e^{z_1\xi + \mathrm{i}z_2\eta|\xi|}\right]
$$

and

$$\beta(z_0,z_1,z_2) := \left(r-\frac{\sigma^2}{2}\right)z_1 + \frac{(\sigma z_1)^2}{2} + \mathrm{i}\kappa(\theta-y_2)z_2 - z_1y_2\mathbb{E}[e^{\xi}-1] - y_2 + y_2\mathbb{E}\left[e^{z_1\xi+\mathrm{i}z_2\eta|\xi|}\right] - z_0.$$

Moreover the boundary conditions

$$\varphi_\alpha(t,z_0,0,0|s,y_1,y_2,v,u) = \mathrm{E}_\alpha(-z_0(t-u)^\alpha),$$

$$\varphi_\alpha(v,z_0,z_1,z_2|s,y_1,y_2,v,u) = e^{-z_0v}$$

and

$$\varphi_\alpha(t,0,0,0|s,y_1,y_2,v,u) = 1$$

hold.

The proof uses Corollary 0.2 and is essentially the same as Proposition 0.6. It is therefore omitted.

0.4 Changes of measure

In this section, we extend the results of the changes of measure section of Moraux and Hainaut (2018) to the subdiffusive model. To this end, we use a result of J. Jacod (1979) that states that under a condition called *adaptation to a time change*, sometimes also referred to as *synchronization with a time-change*, a time-changed local-martingale remains a local-martingale. We begin this section by stating this result precisely. Then, we show that it applies to our setting and thereby obtain risk neutral measures for the time-changed model.

For a probability measure $\mathbb{Q}$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ on $(\Omega,\mathcal{F})$, we define $\mathcal{M}(\mathbb{Q},\mathbb{F})$ and $\mathcal{M}_{\text{loc}}(\mathbb{Q},\mathbb{F})$ to be respectively the set of all $(\mathbb{Q},\mathbb{F})$-martingales and the set of all $(\mathbb{Q},\mathbb{F})$-local martingales. Of course, it holds that $\mathcal{M}(\mathbb{Q},\mathbb{F}) \subset \mathcal{M}_{\text{loc}}(\mathbb{Q},\mathbb{F})$. A subscript 0 is added to these notations when the collection is restricted to the (local-) martingales $(M_t)_{t\geqslant 0}$ that starts at 0, i.e. $M_0 = 0$ $\mathbb{Q}$-a.s.. A superscript c is added when the collection is restricted to the (local-) martingales that has continuous paths $\mathbb{Q}$-a.s.. For example, the set $\mathcal{M}_0^{\text{c}}(\mathbb{Q},\mathbb{F})$ (resp. $\mathcal{M}_{0,\text{loc}}(\mathbb{Q},\mathbb{F})$) thus contains the continuous martingales which start at 0 (resp. the local-martingales which start at 0 but are not necessarily continous). We give the definition of two orthogonal local-martingales.

Definition 0.1. Let $(M_t)_{t\geqslant 0}$ and $(H_t)_{t\geqslant 0}$ be two processes in $\mathcal{M}_{\text{loc}}(\mathbb{Q},\mathbb{F})$. We say that the local-martingales $(M_t)_{t\geqslant 0}$ and $(H_t)_{t\geqslant 0}$ are *orthogonal* if the process $(M_tH_t)_{t\geqslant 0}$ belongs to $\mathcal{M}_{0,\text{loc}}(\mathbb{Q},\mathbb{F})$. This is written as $(M_t)_{t\geqslant 0} \perp (H_t)_{t\geqslant 0}$.

This definition can be found in e.g. J. Jacod (1979) (Definition 2.10). It allows us to further define the collection

$$\mathcal{M}_{\text{loc}}^{\text{d}}(\mathbb{Q},\mathbb{F}) := \{(M_t)_{t\geqslant 0} \in \mathcal{M}_{\text{loc}}(\mathbb{Q},\mathbb{F}) : (M_t)_{t\geqslant 0} \perp (H_t)_{t\geqslant 0} \text{ for all } (H_t)_{t\geqslant 0} \in \mathcal{M}_{0,\text{loc}}^{\text{c}}(\mathbb{Q},\mathbb{F})\}.$$

For convenience, we will write $\mathbb{F}^U$, $\mathbb{F}^A$ and $\mathbb{F}^{A\circ S}$ instead of $(\mathcal{F}_t^U)_{t\geqslant 0}$, $(\mathcal{F}_t^A)_{t\geqslant 0}$ and $(\mathcal{F}_t^{A\circ S})_{t\geqslant 0}$, see the beginning of Section 0.3 for the definitions of these filtrations. In particular, the collection $\mathcal{M}_{\text{loc}}^{\text{d}}(\mathbb{Q},\mathbb{F})$ is known to contain the compensated jump processes, as shown in the next proposition.

Proposition 0.7. *Let Ξ be the random measure associated with the jump process $(J_t)_{t\geqslant 0}$ and $g:\mathbb{R}\to\mathbb{R}$ be a function that satisfies $\int_{\mathbb{R}}|g(z)|\nu(\mathrm{d}z) < +\infty$. Recall that $(N_t)_{t\geqslant 0}$ is the process that counts the number of jumps of $(J_t)_{t\geqslant 0}$, i.e. N_t is the number of jumps of $(J_t)_{t\geqslant 0}$ in the interval $[0,t]$. Then the compensated jump process $(\tilde{J}_t^g)_{t\geqslant 0}$ defined as*

$$\tilde{J}_t^g := \int_0^t \int_{\mathbb{R}} g(z)\left(\Xi(\mathrm{d}z)\mathrm{d}N_s - \lambda_{s-}\nu(\mathrm{d}z)\mathrm{d}s\right)$$

is in $\mathcal{M}_{\text{loc}}^{\text{d}}(\mathbb{P},\mathbb{F}^A)$.

Proof. The compensated jump process can also be expressed as

$$\tilde{J}_t^g = \sum_{k=1}^{N_t} g(\xi_k) - \mathbb{E}[g(\xi_1)]\Lambda_t,$$

where $\xi_1, \xi_2, \dots$ are the i.i.d. jump sizes of $(J_t)_{t\geqslant 0}$ distributed according to ν and $\Lambda_t = \int_0^t \lambda_{s-}\mathrm{d}s$. Let $(H_t)_{t\geqslant 0} \in \mathcal{M}^{\mathrm{c}}_{0,\mathrm{loc}}(\mathbb{P}, \mathbb{F}^A)$. We have to show that $(\tilde{J}_t^g H_t)_{t\geqslant 0}$ belongs to $\mathcal{M}_{0,\mathrm{loc}}(\mathbb{P}, \mathbb{F}^A)$. From the definition of quadratic covariation,

$$\tilde{J}_t^g H_t = \int_0^t H_{s-}\mathrm{d}\tilde{J}_s^g + \int_0^t \tilde{J}_{s-}^g \mathrm{d}H_s + [\tilde{J}^g, H]_t.$$

Since the compensated process $(\tilde{J}_t^g)_{t\geqslant 0}$ and $(H_t)_{t\geqslant 0}$ are in $\mathcal{M}_{\mathrm{loc}}(\mathbb{P}, \mathbb{F}^A)$, it is enough to show that $([\tilde{J}^g, H]_t)_{t\geqslant 0}$ is indistinguishable from the null process. Since $(\sum_{k=1}^{N_t} g(\xi_k))_{t\geqslant 0}$ is quadratic pure jump,

$$[\tilde{J}^g, H]_t = \sum_{0<s\leqslant t} g(\Delta J_s)\Delta H_s - \mathbb{E}[g(\xi_1)][\Lambda, H]_t$$

but the continuity of $(H_t)_{t\geqslant 0}$ implies that $\sum_{0<s\leqslant t} g(\Delta J_s)\Delta H_s = 0$. Moreover, if we denote by $(E_t)_{t\geqslant 0}$ the time process $(t)_{t\geqslant 0}$, then it is a continuous quadratic pure jump semimartingale[4]. As a consequence $([E, H]_t)_{t\geqslant 0}$ is indistinguishable from the null process. Then, Theorem 29 in Protter (2005) (Chapter 2) yields

$$[\Lambda, H]_t = [\lambda \cdot E, H]_t = \int_0^t \lambda_{s-}\mathrm{d}[E, H]_s = 0,$$

where $((\lambda \cdot E)_t)_{t\geqslant 0}$ denotes the stochastic integral $(\int_0^t \lambda_{s-}\mathrm{d}E_s) = (\Lambda_t)_{t\geqslant 0}$. □

The second important notion is the one of *adaptation to a time-change*.

Definition 0.2. An $\mathbb{F}$-time-change $(S_t)_{t\geqslant 0}$ is a nondecreasing process such that for any $t \geqslant 0$, S_t is an $\mathbb{F}$-stopping-time. Moreover, a stochastic process $(X_t)_{t\geqslant 0}$ is *adapted to the time-change* $(S_t)_{t\geqslant 0}$ if $(X_t)_{t\geqslant 0}$ is constant on any interval $[S_{t-}, S_t]$ for all $t \geqslant 0$, $\mathbb{Q}$-a.s..

It is clear that if $(S_t)_{t\geqslant 0}$ is as described in the previous sections, it is an $(\mathbb{F}^U \vee \mathbb{F}^A)$-time-change. Moreover, its continuity entails that any process is adapted to it, as $S_{t-} = S_t$ for all t. Adaptation to time-changes is discussed in details in the Chapter 10 of J. Jacod (1979). It is also referred to as *synchronization with a time-change*, as e.g. in Kobayashi (2011).

Recall that a *random set* is a subset of $\Omega \times \mathbb{R}^+$. A particular case of random set is given by *random intervals*. Given two random variables $S, T : \Omega \to \mathbb{R}^+ \cup \{+\infty\}$, the random interval $[\![S, T]\!]$ is the random set defined as

$$[\![S, T]\!] := \{(\omega, t) \in \Omega \times \mathbb{R}^+ : S(\omega) \leqslant t \leqslant T(\omega)\}.$$

To an $\mathbb{F}$-time-change $(S_t)_{t\geqslant 0}$, we can associate the random set J_S as

$$J_S = \{(\omega, t) \in \Omega \times \mathbb{R}^+ : S_{t-}(\omega) < +\infty\}.$$

Let $\mathcal{P}$ be a class of stochastic processes and define

$$\mathcal{P}^{J_S} = \{(X_t)_{t\geqslant 0} : (X_{t\wedge T})_{t\geqslant 0} \in \mathcal{P} \text{ for any } \mathbb{F}\text{-stopping-time } T \text{ that satisfies } [\![0, T]\!] \subset J_S\}.$$

With all these notations introduced, we can state the result that is of interest to us, namely that under some conditions, a time-changed local-martingale remains a local-martingale. This result is contained[5] in Theorem 10.16 in J. Jacod (1979).

[4]Because it is càdlàg and has finite variation on compacts, see Theorem 26 in the chapter 2 of Protter (2005).

[5]This theorem also contains the same result for other class of processes such as semimartingales for example, as noted at the beginning of Section 0.3, but we are only interested with local-martingales here.

Lemma 0.8. *Let* $(M_t)_{t\geqslant 0}$ *be a process that belongs to* $\mathcal{M}^{\mathrm{c}}_{\mathrm{loc}}(\mathbb{Q},\mathbb{F})$ *(resp. a process that belongs to* $\mathcal{M}^{\mathrm{d}}_{\mathrm{loc}}(\mathbb{Q},\mathbb{F})$*) and that is adapted to an* $\mathbb{F}$*-time-change* $(S_t)_{t\geqslant 0}$*. Then the time-changed process* $(M_{S_t})_{t\geqslant 0}$ *belongs to* $\mathcal{M}^{\mathrm{c}}_{\mathrm{loc}}(\mathbb{Q},\mathbb{G})^{J_S}$ *(resp. belongs to* $\mathcal{M}^{\mathrm{d}}_{\mathrm{loc}}(\mathbb{Q},\mathbb{G})^{J_S}$*), where* $\mathbb{G}$ *denotes the time-changed filtration* $(\mathcal{F}_{S_t})_{t\geqslant 0}$*.*

If the time-change $(S_t)_{t\geqslant 0}$ is the inverse of an α-stable subordinator, Proposition 0.1 implies that $\mathbb{P}(\{S_t < +\infty \text{ for all } t \geqslant 0\}) = 1$ so that by ignoring a null set, we can assume that $\{S_t < +\infty \text{ for all } t \geqslant 0\} = \Omega$. It follows that the associated random set J_S is simply $\Omega\times\mathbb{R}^+$, and thus that $[\![0,+\infty]\!] = \Omega\times\mathbb{R}^+ \subset J_S$. As a consequence, the inclusions $(M_{S_t})_{t\geqslant 0} \in \mathcal{M}^{\mathrm{c}}_{\mathrm{loc}}(\mathbb{Q},\mathbb{G})^{J_S}$ (resp. $(M_{S_t})_{t\geqslant 0} \in \mathcal{M}^{\mathrm{d}}_{\mathrm{loc}}(\mathbb{Q},\mathbb{G})^{J_S}$) in particular means that $(M_{S_t})_{t\geqslant 0} \in \mathcal{M}^{\mathrm{c}}_{\mathrm{loc}}(\mathbb{Q},\mathbb{G})$ (resp. $(M_{S_t})_{t\geqslant 0} \in \mathcal{M}^{\mathrm{d}}_{\mathrm{loc}}(\mathbb{Q},\mathbb{G})$). Finally, note that since the inverse of an α-stable process has continuous paths a.s., we can drop the assumption of a process adapted to the time-change in Lemma 0.8. This is summarized in the next corollary.

Corollary 0.4. *Let* $(M_t)_{t\geqslant 0}$ *be a process that belongs to* $\mathcal{M}^{\mathrm{c}}_{\mathrm{loc}}(\mathbb{Q},\mathbb{F})$ *(resp. a process that belongs to* $\mathcal{M}^{\mathrm{d}}_{\mathrm{loc}}(\mathbb{Q},\mathbb{F})$*) and an* $\mathbb{F}$*-time-change* $(S_t)_{t\geqslant 0}$ *that is the inverse of an* α*-stable subordinator Then the time-changed process* $(M_{S_t})_{t\geqslant 0}$ *belongs to* $\mathcal{M}^{\mathrm{c}}_{\mathrm{loc}}(\mathbb{Q},\mathbb{G})$ *(resp. belongs to* $\mathcal{M}^{\mathrm{d}}_{\mathrm{loc}}(\mathbb{Q},\mathbb{G})$*), where* $\mathbb{G}$ *denotes the time-changed filtration* $(\mathcal{F}_{S_t})_{t\geqslant 0}$*.*

We will also use use a change of variable formula for time-changed stochastic integral. This formula is stated at Proposition 10.21 of J. Jacod (1979). Let $(Z_t)_{t\geqslant 0}$ and $(H_t)_{t\geqslant 0}$ be two a semimartingales, with $(Z_t)_{t\geqslant 0}$ in synchronization with a time-change $(S_t)_{t\geqslant 0}$ and $(H_t)_{t\geqslant 0}$ predictable. If the stochastic integral $\int_0^t H_s \mathrm{d}Z_s$ exists, then the integral $\int_0^{S_t} H_s \mathrm{d}Z_s$ also exists and satisfies the change of variable formula

$$\int_0^{S_t} H_s \mathrm{d}Z_s = \int_0^t H_{S_{s-}} \mathrm{d}Z_{S_s}. \tag{37}$$

This formula can also be found in Kobayashi (2011) (Lemma 2.3). Again, since the inverse of an α-stable subordinator is continuous, we can drop the assumption of synchronization with the time-change.

In the following, we focus on a family of changes of measure induced by exponential martingales $(M_t(x,\zeta))_{t\geqslant 0}$ that satisfy

$$M_t(x,\zeta) = \exp\left\{ b_1(x)\lambda_t + xL_{S_t} + b_2(x)S_t - \frac{1}{2}\int_0^t \zeta_s^2 \mathrm{d}s - \int_0^t \zeta_s \mathrm{d}W_s \right\}, \tag{38}$$

where $(\zeta_t)_{t\geqslant 0}$ is an adapted process of the form $\zeta_t = \zeta_0 + \zeta_1\lambda_t$ and ζ_0, ζ_1 and x are constants. The functions b_1, b_2 correspond to the price of jump risk. The next proposition gives sufficient conditions under which $(M_{S_t}(x,\zeta))_{t\geqslant 0}$ is in $\mathcal{M}_{\mathrm{loc}}(\mathbb{P},\mathbb{F}^{A,U})$.

Proposition 0.8. *If the relations*

$$\kappa b_1(x) - (\psi(0, \eta b_1(x) + x) - 1) = 0$$

and

$$b_2(x) + \kappa\theta b_1(x) = 0$$

hold, then the process $(M_t(x,\zeta))_{t\geqslant 0}$ *is in* $\mathcal{M}_{\mathrm{loc}}(\mathbb{P},\mathbb{F}^{A,U})$*.*

Proof. Proposition 4.1 in Moraux and Hainaut (2018) states that the two conditions of the statement are sufficient for $(M_t(x,\zeta))_{t\geqslant 0}$ to be in $\mathcal{M}_{\mathrm{loc}}(\mathbb{P},\mathbb{F}^{A})$. It remains to be shown that it is also in $\mathcal{M}_{\mathrm{loc}}(\mathbb{P},\mathbb{F}^{A,U})$. By definition of local-martingale, there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of $\mathbb{F}^A$-stopping-times that satisfies $T_n \uparrow +\infty$ $\mathbb{P}$-a.s. and such that the stopped process $(M_{t\wedge T_n}(x,\zeta))_{t\geqslant 0}$ is a $(\mathbb{P},\mathbb{F}^A)$-martingale for any $n\in\mathbb{N}$. Recall that the filtrations $\mathbb{F}^A$ and $\mathbb{F}^U$ are independent. We will use the following property: for an integrable random variable X and two σ-algebras $\mathcal{H}$ and $\mathcal{G}$, if $\mathcal{H}$ is independent of $\sigma(X)\vee\mathcal{G}$, then

$$\mathbb{E}[X|\mathcal{G}\vee\mathcal{H}] = \mathbb{E}[X|\mathcal{G}]$$

$\mathbb{P}$-a.s.. This is property (k) in the chapter 9.7 of Williams (1991). As a consequence, we find for $t \geqslant s$,

$$\mathbb{E}[M_{t\wedge T_n}(x,\zeta)|\mathcal{F}_s^{A,U}] = \mathbb{E}[M_{t\wedge T_n}(x,\zeta)|\mathcal{F}_s^{A}] = M_{s\wedge T_n}(x,\zeta).$$

It follows that $(M_t(x,\zeta))_{t\geqslant 0} \in \mathcal{M}_{\text{loc}}(\mathbb{P},\mathbb{F}^{A,U})$. $\square$

We can use the local-martingales $M_t(x,\zeta)$ as densities to define equivalent measures $\mathbb{Q}^{x,\zeta}$ as

$$\mathbb{Q}^{x,\zeta}(A|\mathcal{F}_s^{A,U}) = \mathbb{E}\left[\mathbf{1}_A \frac{M_t(x,\zeta)}{M_s(x,\zeta)}|\mathcal{F}_s^{A,U}\right]. \tag{39}$$

The next proposition shows that such measures preserves the structure of the model we specified.

Proposition 0.9. *Under the measure $\mathbb{Q}^{x,\zeta}$, the intensity $(\lambda_{S_t})_{t\geqslant 0}$ satisfies the SDE*

$$\mathrm{d}\lambda_{S_t} = \kappa(\theta^{x,\zeta} - \lambda_{S_t})\mathrm{d}S_t + \eta^{x,\zeta}\mathrm{d}L_{S_t}$$

where

$$\frac{\theta^{x,\zeta}}{\theta} = \frac{\eta^{x,\zeta}}{\eta} = \psi(0,\eta b_1(x)+x).$$

The distribution of the α-stable subordinator remains unchanged under $\mathbb{Q}^{x,\zeta}$. The jumps of the processes $(D_t)_{t\geqslant 0}$, $(J_t)_{t\geqslant 0}$ and $(L_t)_{t\geqslant 0}$ defined at Equation (4) have intensity $(\lambda_t)_{t\geqslant 0}$ under $\mathbb{Q}^{x,\zeta}$. Moreover, the jump sizes $\xi_1,\xi_2,\ldots$ remain double exponential under this measure and their distribution is given by

$$\nu^{x,\zeta}(B) = p^{x,\zeta}\int_{B\cap\mathbb{R}^+}\rho_+^{x,\zeta}e^{-\rho_+^{x,\zeta}z}\mathrm{d}z + (1-p^{x,\zeta})\int_{B\cap\mathbb{R}^-}\rho_-^{x,\zeta}e^{-\rho_-^{x,\zeta}z}\mathrm{d}z,$$

with

$$\rho_+^{x,\zeta} = \rho_+ - (\eta b_1(x)+x),$$
$$\rho_-^{x,\zeta} = \rho_- + (\eta b_1(x)+x)$$

and

$$p^{x,\zeta} = \frac{p\rho_+\rho_-^{x,\zeta}}{p\rho_+\rho_-^{x,\zeta} + (1-p)\rho_+^{x,\zeta}\rho_-}.$$

Proof. Propositions 4.2 and 4.3 in Moraux and Hainaut (2018) establish that under $\mathbb{Q}^{x,\zeta}$,

$$\lambda_t = \lambda_0 + \int_0^t \kappa(\theta^{x,\zeta} - \lambda_{s-})\mathrm{d}s + \eta^{x,\zeta}L_t.$$

Thanks to the change of variable formula (37), it follows that

$$\lambda_{S_t} = \lambda_0 + \int_0^t \kappa(\theta^{x,\zeta} - \lambda_{S_{s-}})\mathrm{d}S_s + \eta^{x,\zeta}L_{S_t}.$$

The fact that the law of $(U_t)_{t\geqslant t}$ remains unchanged follows from the independence of $\mathbb{F}^A$ and $\mathbb{F}^U$. The remainder of the statement corresponds to Propositions 4.2 and 4.3 in Moraux and Hainaut (2018). $\square$

Proposition 0.10. *If the process $(\zeta_t)_{t\geqslant 0}$ satisfy*

$$\zeta_t = \frac{\mu + \lambda_t[\psi(0,\eta b_1(x)+x)(\psi^{x,\zeta}(1,0)-1) - (\psi(1,0)-1)] - r}{\sigma},$$

and if the relations of Proposition 0.8 are satisfied, then the equivalent measure $\mathbb{Q}^{x,\zeta}$ of Equation (39) is a risk-neutral measure. Under this measure, the asset price $(A_{S_t})_{t\geqslant 0}$ satisfies the SDE

$$\mathrm{d}A_{S_t} = rA_{S_t}\mathrm{d}S_t + \sigma A_{S_t}\mathrm{d}W_{S_t} + A_{S_t}\left(\mathrm{d}D_{S_t} - \mathbb{E}_{\mathbb{Q}^{x,\zeta}}[e^{\xi}-1]\lambda_{S_t}\mathrm{d}S_t\right)$$

whereas its logarithm $(X_{S_t})_{t\geqslant 0} = (\ln A_{S_t})_{t\geqslant 0}$ satisfies

$$\mathrm{d}X_{S_t} = \left(r - \frac{\sigma^2}{2} - \mathbb{E}_{\mathbb{Q}^{x,\zeta}}[e^{\xi}-1]\lambda_{S_t}\right)\mathrm{d}S_t + \sigma\mathrm{d}W_{S_t} + \mathrm{d}J_{S_t}.$$

Proof. It is proved in Moraux and Hainaut (2018) (Proposition 4.4) that under $\mathbb{Q}^{x,\zeta}$,

$$A_t = r\int_0^t A_{s-}\mathrm{d}s + \sigma\int_0^t A_{s-}\mathrm{d}W_s + \int_0^t A_{s-}\left(\mathrm{d}D_s - \mathbb{E}_{\mathbb{Q}^{x,\zeta}}[e^{\xi}-1]\lambda_s\mathrm{d}s\right) \tag{40}$$

and

$$X_t = \int_0^t \left(r - \frac{\sigma^2}{2} - \mathbb{E}_{\mathbb{Q}^{x,\zeta}}[e^{\xi}-1]\lambda_s\right)\mathrm{d}s + \sigma W_t + J_t.$$

The dynamics of $(A_{S_t})_{t\geqslant 0}$ and $(X_{S_t})_{t\geqslant 0}$ then follow from the change of variable formula (37). Moreover, note that by Equation (40) and Ito's lemma applied on the process $(\tilde{A}_t)_{t\geqslant 0} = (e^{-rt}A_t)_{t\geqslant 0}$, we have

$$\tilde{A}_t = \sigma\int_0^t \tilde{A}_s\mathrm{d}W_s + \int_0^t \tilde{A}_s\left(\mathrm{d}D_s - \mathbb{E}_{\mathbb{Q}^{x,\zeta}}[e^{\xi}-1]\lambda_s\mathrm{d}s\right)$$

or using the random measure Ξ,

$$\tilde{A}_t = \sigma\int_0^t \tilde{A}_s\mathrm{d}W_s + \int_0^t \tilde{A}_s\int_{\mathbb{R}}(e^z-1)(\Xi(\mathrm{d}z)\mathrm{d}N_s - \lambda_{s-}\nu(\mathrm{d}z)\mathrm{d}s)$$

under $\mathbb{Q}^{x,\zeta}$. Again, from the change of variable formula (37),

$$\tilde{A}_{S_t} = \sigma\int_0^t \tilde{A}_{S_{s-}}\mathrm{d}W_{S_s} + \int_0^t \tilde{A}_{S_{s-}}\int_{\mathbb{R}}(e^z-1)(\Xi(\mathrm{d}z)\mathrm{d}N_{S_s} - \lambda_{s-}\nu(\mathrm{d}z)\mathrm{d}S_s). \tag{41}$$

Since the brownian motion $(W_t)_{t\geqslant 0}$ belongs to $\mathcal{M}^{\mathrm{c}}_{\mathrm{loc}}(\mathbb{Q}^{x,\zeta},\mathbb{F}^A)$, Corollary 0.4 implies that the time-changed brownian motion $(W_{S_t})_{t\geqslant 0}$ belongs to $\mathcal{M}^{\mathrm{c}}_{\mathrm{loc}}(\mathbb{Q}^{x,\zeta},\mathbb{F}^{A\circ S})$. Moreover, Proposition 0.7 states that

$$\left(\int_0^t\int_{\mathbb{R}}(e^z-1)\left(\Xi(\mathrm{d}z)\mathrm{d}N_s - \lambda_{s-}\nu(\mathrm{d}z)\mathrm{d}s\right)\right)_{t\geqslant 0}$$

is in $\mathcal{M}^{\mathrm{d}}_{\mathrm{loc}}(\mathbb{Q}^{x,\zeta},\mathbb{F}^A)$. Therefore, Corollary 0.4 and the change of variable formula (37) entails that the time-changed compensated jump process

$$\left(\int_0^t\int_{\mathbb{R}}(e^z-1)\left(\Xi(\mathrm{d}z)\mathrm{d}N_{S_s} - \lambda_{S_{s-}}\nu(\mathrm{d}z)\mathrm{d}S_s\right)\right)_{t\geqslant 0}$$

is in $\mathcal{M}^{\mathrm{d}}_{\mathrm{loc}}(\mathbb{Q}^{x,\zeta},\mathbb{F}^{A\circ S})$. Since stochastic integrals with respect to local-martingales are local-martingales, Equation (41) implies that $(\tilde{A}_{S_t})_{t\geqslant 0}$ is in $\mathcal{M}_{\mathrm{loc}}(\mathbb{Q}^{x,\zeta},\mathbb{F}^{A\circ S})$, which ends the proof. ☐

0.5 Pricing of call options

In this section, we show how to price call and put options in our framework. We proceed by numerically inverting the Fourier transform of the option prices.

0.5.1 Numerical computation of the Fourier transform

This subsection is devoted to a numerical method that allows to solve the FPDE of Corollary 0.3. We work with a grid on $[0,T]\times[-z_{\max},z_{\max}]$ where $T, z_{\max}>0$ are fixed constants. The discretization step sizes are $\Delta_t := T/n_t$ and $\Delta_z := z_{\max}/n_z$ with $n_t, n_z \in \mathbb{N}$. The grid is defined as $\{t_0,\dots,t_{n_t}\}\times\{z_{-n_z}, z_{-n_z+1},\dots,z_{n_z}\}$ where $t_x := x\Delta_t$ and $z_y := y\Delta_z$. This grid contains $(n_t+1)\times(2n_z+1)$ points. We set $\sigma_\alpha := 1-\frac{\alpha}{2}$. Note that for any $\alpha\in(0,1)$, we have $\sigma_\alpha\in(\frac{1}{2},1)$, so that the inequalities $t_j < t_j + \frac{1}{2}\Delta_t < t_{j+\sigma_\alpha} < t_{j+1}$ always hold.

We use $\hat{\varphi}_\alpha^{(0)}$ to denote the approximated values of the function $\varphi_\alpha^{(0)}$ from Corollary 0.3. That is, for fixed s, y_1, y_2, v, u and $\omega \in \mathbb{C}$, we write

$$\beta\hat{\varphi}_\alpha^{(0)}(t,z) \approx \beta(\omega,z)\varphi_\alpha^{(0)}(t,r,\omega,z|s,y_1,y_2,v,u)$$

to express the fact that $\beta\hat{\varphi}_\alpha^{(0)}(t,z)$ is the quantity that aims to approximate

$$\beta(\omega,z_i)\varphi_\alpha^{(0)}(t,r,\omega,z|s,y_1,y_2,v,u).$$

Similarly, we also write

$$\frac{\Delta^\alpha\hat{\varphi}_\alpha^{(0)}}{\Delta t^\alpha}(t,z) \approx \frac{\partial^\alpha\varphi_\alpha^{(0)}}{\partial t^\alpha}(t,r,\omega,z|s,y_1,y_2,v,u)$$

and

$$\gamma\frac{\Delta\hat{\varphi}_\alpha^{(0)}}{\Delta z}(t,z) \approx \gamma(r,\omega,z)\frac{\partial\varphi_\alpha^{(0)}}{\partial z}(t,r,\omega,z|s,y_1,y_2,v,u).$$

The FPDE of Corollary 0.3 translates into

$$\frac{\Delta^\alpha\hat{\varphi}_\alpha^{(0)}}{\Delta t^\alpha}(t_{j+\sigma_\alpha},z_i) = \beta\hat{\varphi}_\alpha^{(0)}(t_{j+\sigma_\alpha},z_i) + \gamma\frac{\Delta\hat{\varphi}_\alpha^{(0)}}{\Delta z}(t_{j+\sigma_\alpha},z_i). \tag{42}$$

We will now give the formulas for our approximations and study their rates of convergence.

The approximation used for the Caputo fractional derivative is the one introduced in Alikhanov (2015). The Caputo derivative approximation for a function f is

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha}(t_{j+\sigma_\alpha}) \approx \frac{\Delta^\alpha f}{\Delta t^\alpha}(t_{j+\sigma_\alpha}) := \frac{\Delta_t^\alpha}{\Gamma(1-\alpha)}\sum_{s=0}^{j} c_{j-s}^{(\alpha,j)}\frac{f(t_{s+1})-f(t_s)}{\Delta_t} \tag{43}$$

where $c_0^{(\alpha,0)} := \sigma_\alpha^{1-\alpha}$ and

$$c_s^{(\alpha,j)} := \begin{cases} a_0^{(\alpha)} + b_1^{(\alpha)} & \text{if } s=0, \\ a_s^{(\alpha)} + b_{s+1}^{(\alpha)} - b_s^{(\alpha)} & \text{if } 1 \leqslant s \leqslant j-1, \\ a_j^{(\alpha)} - b_j^{(\alpha)} & \text{if } s=j, \end{cases} \tag{44}$$

when $j \in \{1,\dots,n_t\}$. The constants $a_s^{(\alpha)}$ and $b_s^{(\alpha)}$ are defined as

$$a_s^{(\alpha)} := (s+\sigma_\alpha)^{1-\alpha} - (s-1+\sigma_\alpha)^{1-\alpha} \tag{45}$$

and

$$\begin{aligned} b_s^{(\alpha)} :=& \frac{1}{2-\alpha}\left[(s+\sigma_\alpha)^{2-\alpha} - (s-1+\sigma_\alpha)^{2-\alpha}\right] \\ &- \frac{1}{2}\left[(s+\sigma_\alpha)^{1-\alpha} + (s-1+\sigma_\alpha)^{1-\alpha}\right]. \end{aligned} \tag{46}$$

The approximation formula of Equations (43)-(46) is based on a quadratic interpolation of f on the grid $\{0 = t_0, t_1, \dots, t_{n_t} = T\}$ and is derived in Alikhanov (2015). This approximation has the benefit of a high order error, that is $O(\Delta_t^{3-\alpha})$, as stated in the next proposition.

Proposition 0.11. *For any $\alpha \in (0,1)$ and $f \in \mathcal{C}^3[0,t_{j+1}]$,*

$$\left|\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha}(t_{j+\sigma_\alpha}) - \frac{\Delta^\alpha f}{\Delta t^\alpha}(t_{j+\sigma_\alpha})\right| = O(\Delta_t^{3-\alpha}).$$

Proof. See Lemma 2 in Alikhanov (2015). □

The fractional derivative of Corollary 0.3 is thus approximated by

$$\frac{\Delta^\alpha \hat{\varphi}_\alpha^{(0)}}{\Delta t^\alpha}(t_{j+\sigma_\alpha}, z_i) := \frac{\Delta_t^\alpha}{\Gamma(1-\alpha)} \sum_{s=0}^{j} c_{j-s}^{(\alpha,j)} \frac{\hat{\varphi}_\alpha^{(0)}(t_{s+1}, z_i) - \hat{\varphi}_\alpha^{(0)}(t_s, z_i)}{\Delta_t}. \tag{47}$$

We move to the approximation $\beta\hat{\varphi}_\alpha^{(0)}(t,z)$. The next proposition introduces the formula we use.

Proposition 0.12. *Let $f \in \mathcal{C}^2[0,T]$. Then,*

$$f(t_{j+\sigma_\alpha}) = (1-\sigma_\alpha)f(t_j) + \sigma_\alpha f(t_{j+1}) + O(\Delta_t^2).$$

Proof. The proof is based on Taylor's theorem. This theorem implies that

$$\begin{aligned}\sigma_\alpha f(t_{j+1}) &= \sigma_\alpha \left(f(t_{j+\sigma_\alpha}) + (t_{j+1} - t_{j+\sigma_\alpha}) f'(t_{t_{j+\sigma_\alpha}}) + \frac{(t_{j+1} - t_{j+\sigma_\alpha})^2}{2} f''(\tilde{t}_{j+1}) \right) \\ &= \sigma_\alpha \left(f(t_{j+\sigma_\alpha}) + (1-\sigma_\alpha)\Delta_t f'(t_{t_{j+\sigma_\alpha}}) + \frac{((1-\sigma_\alpha)\Delta_t)^2}{2} f''(\tilde{t}_{j+1}) \right)\end{aligned} \tag{48}$$

and

$$\begin{aligned}&(1-\sigma_\alpha)f(t_j) \\ &\quad = (1-\sigma_\alpha)\left(f(t_{j+\sigma_\alpha}) + (t_j - t_{j+\sigma_\alpha}) f'(t_{j+\sigma_\alpha}) + \frac{(t_{j+\sigma_\alpha} - t_j)^2}{2} f''(\tilde{t}_j) \right) \\ &\quad = (1-\sigma_\alpha)\left(f(t_{j+\sigma_\alpha}) - \sigma_\alpha \Delta_t f'(t_{j+\sigma_\alpha}) + \frac{(\sigma_\alpha \Delta_t)^2}{2} f''(\tilde{t}_j) \right)\end{aligned} \tag{49}$$

for some $\tilde{t}_{j+1} \in (t_{j+\sigma_\alpha}, t_{j+1})$ and $\tilde{t}_j \in (t_j, t_{j+\sigma_\alpha})$. Summing Equations (48) and (49) yields the announced result. □

The term $\beta\hat{\varphi}_\alpha^{(0)}(t,z)$ is therefore computed as

$$\beta\hat{\varphi}_\alpha^{(0)}(t_{j+\sigma_\alpha}, z_i) := \beta(r,\omega,z_i) \left[\sigma_\alpha \hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_i) + (1-\sigma_\alpha)\hat{\varphi}_\alpha^{(0)}(t_j, z_i) \right] \tag{50}$$

for all $i \in \{-n_z, \dots, n_z\}$. The approximation $\gamma \frac{\Delta \hat{\varphi}_\alpha^{(0)}}{\Delta z}(t,z)$ is more tricky. This happens for two reasons. The first is that we have to control the rates of convergence with respect to both Δ_t and Δ_z. The second is that we have to rely on 3 different approximations in order not to fall outside of the grid $\{z_{-n_z}, \dots, z_{n_z}\}$ when approximating the partial derivative with respect to z. The falling outside of the grid problem will be explained more precisely and addressed later on, in the case $i \in \{-n_z, n_z\}$. For now, we assume that $i \in \{-n_z+1, \dots, n_z-1\}$. The approximation is in this case based on the following proposition.

Proposition 0.13. *Let $f, k \in \mathcal{C}^3[-z_{\max}, z_{\max}]$. Then,*

$$k(z_i)f'(z_i) = \frac{k(z_{i+\frac{1}{2}})f(z_{i+1}) - \Delta_z k'(z_i)f(z_i) - k(z_{i-\frac{1}{2}})f(z_{i-1})}{2\Delta_z} + O(\Delta_z^2).$$

Proof. By Taylor's theorem,

$$k(z_{i+\frac{1}{2}}) = k(z_i) + \frac{\Delta_z}{2}k'(z_i) + \frac{1}{2}\left(\frac{\Delta_z}{2}\right)^2 k''(z_i) + \frac{1}{6}\left(\frac{\Delta_z}{2}\right)^3 k'''(\tilde{z}_{i+\frac{1}{2}}) \tag{51}$$

for some $\tilde{z}_{i+\frac{1}{2}} \in (z_i, z_{i+\frac{1}{2}})$ and

$$k(z_{i-\frac{1}{2}}) = k(z_i) - \frac{\Delta_z}{2}k'(z_i) + \frac{1}{2}\left(\frac{\Delta_z}{2}\right)^2 k''(z_i) - \frac{1}{6}\left(\frac{\Delta_z}{2}\right)^3 k'''(\tilde{z}_{i-\frac{1}{2}}) \tag{52}$$

for some $\tilde{z}_{i-\frac{1}{2}} \in (z_{i-\frac{1}{2}}, z_i)$. Similarly,

$$f(z_{i+1}) = f(z_i) + \Delta_z f'(z_i) + \frac{\Delta_z^2}{2} f''(z_i) + \frac{\Delta_z^3}{6} f'''(\tilde{z}_{i+1}) \tag{53}$$

for some $\tilde{z}_{i+1} \in (z_i, z_{i+1})$ and

$$f(z_{i-1}) = f(z_i) - \Delta_z f'(z_i) + \frac{\Delta_z^2}{2} f''(z_i) - \frac{\Delta_z^3}{6} f'''(\tilde{z}_{i-1}) \tag{54}$$

for some $\tilde{z}_{i-1} \in (z_{i-1}, z_i)$. Multiplying Equation (51) by Equation (53) and Equation (52) by Equation (54) leads to

$$k(z_{i+\frac{1}{2}})f(z_{i+1}) - k(z_{i-\frac{1}{2}})f(z_{i-1}) = 2\Delta_z k(z_i)f'(z_i) + \Delta_z k'(z_i)f(z_i) + O(\Delta_z^3).$$

A division by Δ_z then implies that

$$\frac{k(z_{i+\frac{1}{2}})f(x_{i+1}) - \Delta_z k'(z_i)f(x_i) - k(z_{i-\frac{1}{2}})f(x_{i-1})}{2\Delta_z} = k(z_i)f'(z_i) + O(\Delta_z^2),$$

as announced. □

Let k be a function that belongs to $\mathcal{C}^3[-z_{\max}, z_{\max}]$ and $g : [0,T] \times [-z_{\max}, z_{\max}] \to \mathbb{R} : (t,z) \mapsto g(t,z)$ be a function that is twice continuously differentiable with respect to its first argument and three times continuously differentiable with respect to its second argument. Proposition 0.12 implies that

$$k(z_i)\frac{\partial g}{\partial z}(t_{j+\sigma_\alpha}, z_i) = \sigma_\alpha k(z_i)\frac{\partial g}{\partial z}(t_{j+1}, z_i) + (1-\sigma_\alpha)k(z_i)\frac{\partial g}{\partial z}(t_j, z_i) + O(\Delta_t^2). \tag{55}$$

Moreover, Proposition 0.13 applied on each term at the right-hand side of Equation (55) provides us with

$$\begin{aligned} k(z_i)\frac{\partial g}{\partial z}(t_{j+\sigma_\alpha}, z_i) &= O(\Delta_z^2 + \Delta_t^2) \\ &+ \sigma_\alpha \frac{k(z_{i+\frac{1}{2}})g(t_{j+1}, z_{i+1}) - k'(z_i)g(t_{j+1}, z_i) - k(z_{i-\frac{1}{2}})g(t_{j+1}, z_{i-1})}{2\Delta_z} \\ &+ (1-\sigma_\alpha)\frac{k(z_{i+\frac{1}{2}})g(t_j, z_{i+1}) - k'(z_i)g(t_j, z_i) - k(z_{i-\frac{1}{2}})g(t_j, z_{i-1})}{2\Delta_z} \end{aligned} \tag{56}$$

The approximation $\gamma\frac{\Delta\hat{\varphi}_\alpha^{(0)}}{\Delta z}(t,z)$ then follows from Equation (56), that is

$$\begin{aligned} &\gamma\frac{\Delta\hat{\varphi}_\alpha^{(0)}}{\Delta z}(t_{j+\sigma_\alpha}, z_i) \\ &:= \sigma_\alpha \frac{\gamma(\omega, z_{i+\frac{1}{2}})\hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i+1}) - \Delta_z \frac{\partial\gamma}{\partial z_i}(\omega, z_i)\hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_i) - \gamma(\omega, z_{i-\frac{1}{2}})\hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i-1})}{2\Delta_z} \\ &+ (1-\sigma_\alpha)\frac{\gamma(\omega, z_{i+\frac{1}{2}})\hat{\varphi}_\alpha^{(0)}(t_j, z_{i+1}) - \Delta_z \frac{\partial\gamma}{\partial z_i}(\omega, z_i)\hat{\varphi}_\alpha^{(0)}(t_j, z_i) - \gamma(\omega, z_{i-\frac{1}{2}})\hat{\varphi}_\alpha^{(0)}(t_j, z_{i-1})}{2\Delta_z}. \end{aligned} \tag{57}$$

Unfortunately, Approximation (57) cannot be used when $i \in \{-n_z, n_z\}$. The reason is what we previously referred to as the falling outside of the grid problem. As a matter of fact, the z_{i+1} (resp. z_{i-1}) that appears in Equation (57) prevents us from using this formula for $i = n_z$ (resp. $i = -n_z$) without falling outside of the grid $\{z_{-n_z}, \ldots, z_{n_z}\}$. The next proposition introduces the approximations we use when $i \in \{-n_z, n_z\}$.

Proposition 0.14. *Let $f \in \mathcal{C}^3[-z_{\max}, z_{\max}]$. Then we have*

(i)

$$f'(-z_{\max}) = \frac{-f(-z_{\max} + 2\Delta_z) + 4f(-z_{\max} + \Delta_z) - 3f(-z_{\max})}{2\Delta_z} + O(\Delta_z^2)$$

(*ii*)

$$f'(z_{\max}) = \frac{3f(z_{\max}) - 4f(z_{\max} - \Delta_z) + f(z_{\max} - 2\Delta_z)}{2\Delta_z} + O(\Delta_z^2).$$

Proof. By Taylor's theorem,

$$\frac{f(-z_{\max} + \Delta_z) - f(-z_{\max})}{\Delta_z} = f'(-z_{\max}) + \frac{\Delta_z}{2} f''(-z_{\max}) + O(\Delta_z^2) \tag{58}$$

and

$$\begin{aligned} f''(-z_{\max}) &= f''(-z_{\max} + \Delta_z) + O(\Delta_z) \\ &= \frac{f(-z_{\max} + 2\Delta_z) - 2f(-z_{\max} + \Delta_z) + f(-z_{\max})}{\Delta_z^2} + O(\Delta_z). \end{aligned} \tag{59}$$

Substituting Equation (59) into Equation (58) gives

$$\begin{aligned} &\frac{f(-z_{\max} + \Delta_z) - f(-z_{\max})}{\Delta_z} \\ &\qquad = f'(-z_{\max}) + \frac{f(-z_{\max} + 2\Delta_z) - 2f(-z_{\max} + \Delta_z) + f(-z_{\max})}{2\Delta_z} + O(\Delta_z^2). \end{aligned}$$

Result (*i*) is obtained by isolating $f'(-z_{\max})$. The proof of (*ii*) is similar and therefore omitted. □

Proposition 0.14 motivates the approximations

$$\begin{aligned} &\gamma \frac{\Delta \hat{\varphi}_\alpha^{(0)}}{\Delta z}(t_{j+\sigma_\alpha}, z_i) \\ &:= \sigma_\alpha \gamma(\omega, z_i) \frac{-\hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i+2}) + 4\hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i+1}) - 3\hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_i)}{2\Delta_z} \\ &\quad + (1 - \sigma_\alpha)\gamma(\omega, z_i) \frac{-\hat{\varphi}_\alpha^{(0)}(t_j, z_{i+2}) + 4\hat{\varphi}_\alpha^{(0)}(t_j, z_{i+1}) - 3\hat{\varphi}_\alpha^{(0)}(t_j, z_i)}{2\Delta_z} \end{aligned} \tag{60}$$

when $i = -n_z$ and

$$\begin{aligned} \gamma \frac{\Delta \hat{\varphi}_\alpha^{(0)}}{\Delta z}(t_{j+\sigma_\alpha}, z_i) &:= \sigma_\alpha \gamma(\omega, z_i) \frac{3\hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_i) - 4\hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i-1}) + \hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i-2})}{2\Delta_z} \\ &+ (1 - \sigma_\alpha)\gamma(\omega, z_i) \frac{3\hat{\varphi}_\alpha^{(0)}(t_j, z_i) - 4\hat{\varphi}_\alpha^{(0)}(t_j, z_{i-1}) + \hat{\varphi}_\alpha^{(0)}(t_j, z_{i-2})}{2\Delta_z} \end{aligned} \tag{61}$$

when $i = n_z$. From Equation (42), we replace the approximations by their formulas of Equations (47), (50), (57), (60) and (61), and we express the values of time t_{j+1} in function of the values up to time t_j. This leads to

$$
\begin{aligned}
&\hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i+1}) \left[\frac{-\sigma_\alpha \Delta_t^{1-\alpha} \gamma(\omega, z_{i+\frac{1}{2}})}{2\Delta_z}\right] \\
&\quad + \hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_i) \left[\frac{c_0^{(\alpha,j)}}{\Gamma(1-\alpha)} - \Delta_t^{1-\alpha} \sigma_\alpha \beta(r, \omega, z_i) + \frac{\Delta_t^{1-\alpha} \sigma_\alpha \frac{\partial \gamma}{\partial z_i}(\omega, z_i)}{2\Delta_z}\right] \\
&\quad + \hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i-1}) \left[\frac{\sigma_\alpha \Delta_t^{1-\alpha} \gamma(\omega, z_{i-\frac{1}{2}})}{2\Delta_z}\right] \\
&= \hat{\varphi}_\alpha^{(0)}(t_j, z_i) \left[(1-\sigma_\alpha) \Delta_t^{1-\alpha} \beta(r, \omega, z_i) + \frac{c_0^{(\alpha,j)}}{\Gamma(1-\alpha)}\right] \\
&\quad - \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{j-1} c_{j-s}^{(\alpha,j)} \left(\hat{\varphi}_\alpha^{(0)}(t_{s+1}, z_i) - \hat{\varphi}_\alpha^{(0)}(t_s, z_i)\right) \\
&\quad + \frac{(1-\sigma_\alpha)\Delta_t^{1-\alpha}}{2\Delta_z} \left[\gamma(\omega, z_{i+\frac{1}{2}}) \hat{\varphi}_\alpha^{(0)}(t_j, z_{i+1}) - \Delta_z \frac{\partial \gamma}{\partial z_i}(\omega, z_i) \hat{\varphi}_\alpha^{(0)}(t_j, z_i) \right. \\
&\qquad\qquad \left. - \gamma(\omega, z_{i-\frac{1}{2}}) \hat{\varphi}_\alpha^{(0)}(t_j, z_{i-1})\right]
\end{aligned} \tag{62}
$$

in the case $i \in \{-n_z+1, \ldots, n_z-1\}$. Otherwise, it gives

$$
\begin{aligned}
&\hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i+2}) \left[\frac{\sigma_\alpha \Delta_t^{1-\alpha} \gamma(\omega, z_i)}{2\Delta_z}\right] + \hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i+1}) \left[\frac{-2\Delta_t^{1-\alpha} \sigma_\alpha \gamma(\omega, z_i)}{\Delta_z}\right] \\
&\quad + \hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i-1}) \left[\frac{c_0^{(\alpha,j)}}{\Gamma(1-\alpha)} - \Delta_t^{1-\alpha} \sigma_\alpha \beta(r, \omega, z_i) + 3\frac{\sigma_\alpha \Delta_t^{1-\alpha} \gamma(\omega, z_i)}{2\Delta_z}\right] \\
&= \hat{\varphi}_\alpha^{(0)}(t_j, z_i) \left[(1-\sigma_\alpha) \Delta_t^{1-\alpha} \beta(r, \omega, z_i) + \frac{c_0^{(\alpha,j)}}{\Gamma(1-\alpha)}\right] \\
&\quad - \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{j-1} c_{j-s}^{(\alpha,j)} \left(\hat{\varphi}_\alpha^{(0)}(t_{s+1}, z_i) - \hat{\varphi}_\alpha^{(0)}(t_s, z_i)\right) \\
&\quad + \frac{(1-\sigma_\alpha)\Delta_t^{1-\alpha} \gamma(\omega, z_i)}{2\Delta_z} \left[-\hat{\varphi}_\alpha^{(0)}(t_j, z_{i+2}) + 4\hat{\varphi}_\alpha^{(0)}(t_j, z_{i+1}) - 3\hat{\varphi}_\alpha^{(0)}(t_j, z_i)\right]
\end{aligned} \tag{63}
$$

when $i = -n_z$ and

$$
\begin{aligned}
&\hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_i) \left[\frac{c_0^{(\alpha,j)}}{\Gamma(1-\alpha)} - \Delta_t^{1-\alpha} \sigma_\alpha \beta(r, \omega, z_i) - 3\frac{\sigma_\alpha \Delta_t^{1-\alpha} \gamma(\omega, z_i)}{2\Delta_z}\right] \\
&\quad + \hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i-1}) \left[\frac{2\Delta_t^{1-\alpha} \sigma_\alpha \gamma(\omega, z_i)}{\Delta_z}\right] \\
&\quad + \hat{\varphi}_\alpha^{(0)}(t_{j+1}, z_{i-2}) \left[\frac{-\sigma_\alpha \Delta_t^{1-\alpha} \gamma(\omega, z_i)}{2\Delta_z}\right] \\
&= \hat{\varphi}_\alpha^{(0)}(t_j, z_i) \left[(1-\sigma_\alpha) \Delta_t^{1-\alpha} \beta(r, \omega, z_i) + \frac{c_0^{(\alpha,j)}}{\Gamma(1-\alpha)}\right] \\
&\quad - \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{j-1} c_{j-s}^{(\alpha,j)} \left(\hat{\varphi}_\alpha^{(0)}(t_{s+1}, z_i) - \hat{\varphi}_\alpha^{(0)}(t_s, z_i)\right) \\
&\quad + \frac{(1-\sigma_\alpha)\Delta_t^{1-\alpha} \gamma(\omega, z_i)}{2\Delta_z} \left[3\hat{\varphi}_\alpha^{(0)}(t_j, z_i) - 4\hat{\varphi}_\alpha^{(0)}(t_j, z_{i-1}) + \hat{\varphi}_\alpha^{(0)}(t_j, z_{i-2})\right]
\end{aligned} \tag{64}
$$

when $i = n_z$. Equations (62)-(64) actually describe a linear system of equations. Namely, if we define the matrix $\boldsymbol{\gamma} \in \mathbb{R}^{(2n_z+1)\times(2n_z+1)}$ as

$$\begin{pmatrix} 3\gamma_{n_z} & -4\gamma_{n_z} & \gamma_{n_z} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \gamma_{n_z-1+\frac{1}{2}} & -\Delta_z\partial\gamma_{n_z-1} & -\gamma_{n_z-1-\frac{1}{2}} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \gamma_{n_z-2+\frac{1}{2}} & -\Delta_z\partial\gamma_{n_z-2} & -\gamma_{n_z-2-\frac{1}{2}} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma_{-n_z+2+\frac{1}{2}} & -\partial\gamma_{-n_z+2} & -\gamma_{-n_z+2-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{-n_z+1+\frac{1}{2}} & -\Delta_z\partial\gamma_{-n_z+1} & -\gamma_{-n_z+1-\frac{1}{2}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\gamma_{-n_z} & 4\gamma_{-n_z} & -3\gamma_{-n_z} \end{pmatrix}$$

where γ_i and $\partial\gamma_i$ are respectively used as shorthands for $\gamma(\omega, z_i)$ and $\frac{\partial\gamma}{\partial z_i}(\omega, z_i)$, and the matrix $\boldsymbol{\beta} \in \mathbb{R}^{(2n_z+1)\times(2n_z+1)}$ as

$$\boldsymbol{\beta} := \operatorname{diag}\left(\beta(r,\omega,z_{n_z}), \beta(r,\omega,z_{n_z-1}), \dots, \beta(r,\omega,z_{-n_z})\right),$$

then we can rewrite Equations (62)-(64) in the matrix form

$$\begin{aligned} &\left[-\sigma_\alpha \Delta_t^{1-\alpha}\left(\boldsymbol{\beta} + \frac{\boldsymbol{\gamma}}{2\Delta_z}\right) + \mathbf{I}\frac{c_0^{(\alpha,j)}}{\Gamma(1-\alpha)}\right]\hat{\boldsymbol{\varphi}}^\alpha(t_{j+1}) \\ &= \left[(1-\sigma_\alpha)\Delta_t^{1-\alpha}\left(\boldsymbol{\beta} + \frac{\boldsymbol{\gamma}}{2\Delta_z}\right) + \mathbf{I}\frac{c_0^{(\alpha,j)}}{\Gamma(1-\alpha)}\right]\hat{\boldsymbol{\varphi}}^\alpha(t_j) + \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{j-1} c_{j-s}^{(\alpha,j)}\left(\hat{\boldsymbol{\varphi}}^\alpha(t_{s+1}) - \hat{\boldsymbol{\varphi}}^\alpha(t_s)\right) \end{aligned}$$

where $\mathbf{I} \in \mathbb{R}^{(2n_z+1)\times(2n_z+1)}$ is an identity matrix and

$$\hat{\boldsymbol{\varphi}}^\alpha(t_j) = (\hat{\varphi}_\alpha^{(0)}(t_j, z_{n_z}), \hat{\varphi}_\alpha^{(0)}(t_j, z_{n_z-1}), \dots, \hat{\varphi}_\alpha^{(0)}(t_j, z_{-n_z}))^\top \in \mathbb{R}^{(2n_z+1)\times 1}.$$

From the initial condition $\hat{\boldsymbol{\varphi}}_\alpha(t_0) = 1$, it is thus possible to compute recursively all the $\hat{\boldsymbol{\varphi}}_\alpha(t_j)$'s by using

$$\begin{aligned} &\hat{\boldsymbol{\varphi}}^\alpha(t_{j+1}) \\ &= \left[-\sigma_\alpha \Delta_t^{1-\alpha}\left(\boldsymbol{\beta} + \frac{\boldsymbol{\gamma}}{2\Delta_z}\right) + \mathbf{I}\frac{c_0^{(\alpha,j)}}{\Gamma(1-\alpha)}\right]^{-1}\left[(1-\sigma_\alpha)\Delta_t^{1-\alpha}\left(\boldsymbol{\beta} + \frac{\boldsymbol{\gamma}}{2\Delta_z}\right) + \mathbf{I}\frac{c_0^{(\alpha,j)}}{\Gamma(1-\alpha)}\right]\hat{\boldsymbol{\varphi}}^\alpha(t_j) \\ &+ \frac{1}{\Gamma(1-\alpha)}\left[-\sigma_\alpha \Delta_t^{1-\alpha}\left(\boldsymbol{\beta} + \frac{\boldsymbol{\gamma}}{2\Delta_z}\right) + \mathbf{I}\frac{c_0^{(\alpha,j)}}{\Gamma(1-\alpha)}\right]^{-1}\sum_{s=0}^{j-1} c_{j-s}^{(\alpha,j)}\left(\hat{\boldsymbol{\varphi}}^\alpha(t_{s+1}) - \hat{\boldsymbol{\varphi}}^\alpha(t_s)\right). \end{aligned}$$

0.5.2 Inversion of the Fourier transform via FFT

Now that we are armed to compute the values of the transform φ_α, this subsection describes in details how to obtain the call prices from this transform by inverting it. As in Carr and Madan (1999), this inversion is performed numerically with the help of a Fast Fourier Transform algorithm. For a fixed maturity $T > 0$, and an evaluation time $t \in [0, T)$, we wish to compute the call price

$$\mathbb{E}_\mathbb{Q}[e^{-rS_T}(e^{X_{S_T}} - e^k)_+ | \mathcal{G}_t],$$

where $(X_t)_{t\geqslant 0}$ is the logarithm of the stock price and $k = \ln K$ is the log-strike. By the Markov property, this call price can be considered as the function

$$C(T, k|t, y_1, y_2, v, u) = \mathbb{E}_\mathbb{Q}[e^{-rS_T}(e^{X_{S_T}} - e^k)_+ | (A_{S_t}, \lambda_{S_t}, S_t, U_{S_t}) = (y_1, y_2, v, u)].$$

Since this function is not square integrable (with respect to the log-strike k), Fourier theory does not apply. However, if we fix $\varepsilon > 1$, then the function

$$c(T, k|t, y_1, y_2, v, u) := e^{\varepsilon k}\mathbb{E}_\mathbb{Q}[e^{-rS_T}(e^{X_{S_T}} - e^k)_+ | (A_{S_t}, \lambda_{S_t}, S_t, U_{S_t}) = (y_1, y_2, v, u)]$$

is square integrable and its Fourier transform exists. Denote by $\hat{c}$ the Fourier transform with respect to the log-strike k,

$$\hat{c}(T,\omega|t,y_1,y_2,v,u) := \int_{\mathbb{R}} e^{\mathrm{i}\omega k} c(T,k|t,y_1,y_2,v,u)\mathrm{d}k.$$

This Fourier transform can be expressed in terms of the Fourier-Laplace transform of the couple (S_t, X_{S_t}) through the following computations

$$\begin{aligned}
\hat{c}(T,\omega|t,y_1,y_2,v,u) &= \int_{\mathbb{R}} e^{(\mathrm{i}\omega+\varepsilon)k} C(T,k|t,y_1,y_2,v,u)\mathrm{d}k \\
&= \int_{\mathbb{R}} e^{(\mathrm{i}\omega+\varepsilon)k} \int_{\mathbb{R}+} \int_{k}^{+\infty} e^{-rx_0} \left(e^{x_1} - e^{k}\right) p_\alpha(T,x_0,x_1|t,y_1,y_2,v,u)\mathrm{d}x_1\mathrm{d}x_0\mathrm{d}k \\
&= \int_{\mathbb{R}+} e^{-rx_0} \int_{\mathbb{R}} e^{x_1} p_\alpha(T,x_0,x_1|t,y_1,y_2,v,u) \int_{-\infty}^{x_1} e^{(\mathrm{i}\omega+\varepsilon)k}\mathrm{d}k\mathrm{d}x_1\mathrm{d}x_0 \\
&- \int_{\mathbb{R}+} e^{-rx_0} \int_{\mathbb{R}} p_\alpha(T,x_0,x_1|t,y_1,y_2,v,u) \int_{-\infty}^{x_1} e^{(\mathrm{i}\omega+\varepsilon+1)k}\mathrm{d}k\mathrm{d}x_1\mathrm{d}x_0 \\
&= \left(\frac{1}{\mathrm{i}\omega+\varepsilon} - \frac{1}{\mathrm{i}\omega+\varepsilon+1}\right) \int_{\mathbb{R}+} \int_{\mathbb{R}} e^{-rx_0+(\mathrm{i}\omega+\varepsilon+1)x_1} p_\alpha(T,x_0,x_1|t,y_1,y_2,v,u)\mathrm{d}x_1\mathrm{d}x_0 \\
&= \frac{1}{(\mathrm{i}\omega+\varepsilon)^2+(\mathrm{i}\omega+\varepsilon)} \varphi_\alpha(T,r,\mathrm{i}\omega+\varepsilon+1,0|t,y_1,y_2,v,u).
\end{aligned}$$

Then we can invert the Fourier transform to obtain the call price

$$C(T,k|t,y_1,y_2,v,u) = \frac{e^{-\varepsilon k}}{\pi} \int_{\mathbb{R}+} e^{-\mathrm{i}\omega k} \frac{\varphi_\alpha(T,r,\mathrm{i}\omega+\varepsilon+1,0|t,y_1,y_2,v,u)}{(\mathrm{i}\omega+\varepsilon)^2+(\mathrm{i}\omega+\varepsilon)}\mathrm{d}\omega. \tag{65}$$

We will approximate this integral with the help of Riemann sums and perform the computations with the help of a fast Fourier transform algorithm. Following Carr and Madan (1999), the approximation for the integral of Equation (65) is of the form

$$C(T,k_n|t,y_1,y_2,v,u) \approx \frac{e^{-\varepsilon k_n}}{\pi} \sum_{j=1}^{N} e^{\mathrm{i}\omega_j k_n} \frac{\varphi_\alpha(T,r,\mathrm{i}\omega_j+\varepsilon+1,0|t,y_1,y_2,v,u)}{(\mathrm{i}\omega_j+\varepsilon)^2+(\mathrm{i}\omega_j+\varepsilon)} \left[\frac{3+(-1)^j-\zeta_{j-1}}{3}\right] \Delta_\omega$$

for the range of log-strikes $k_n = -k_{\max} + \Delta_k(n-1)$, $k_{\max} > 0$, $n = 1,\ldots,N \in \mathbb{N}$ and $\Delta_k = 2k_{\max}/N$. The discretization of the variable ω is of the form $\omega_j = \Delta_\omega(j-1)$ for some $\Delta_\omega > 0$ and $\zeta_{j-1} = 0$ if $j = 1$ and $\zeta_{j-1} = 0$ when $j \neq 1$. Imposing $\Delta_k\Delta_\omega = 2\pi/N$ and performing some computations yields

$$\begin{aligned}
&C(T,k_n|t,y_1,y_2,v,u) \\
&\approx \frac{e^{-\varepsilon k_n}}{\pi} \sum_{j=1}^{N} \exp\left\{-\frac{2\mathrm{i}\pi(j-1)(n-1)}{N}\right\} \frac{\varphi_\alpha(T,r,\mathrm{i}\omega_j+\varepsilon+1,0|t,y_1,y_2,v,u)}{(\mathrm{i}\omega_j+\varepsilon)^2+(\mathrm{i}\omega_j+\varepsilon)} (-1)^{j-1} \left[\frac{3+(-1)^j-\zeta_{j-1}}{3}\right] \Delta_\omega,
\end{aligned}$$

which can be computed with a fast Fourier transform algorithm.

0.6 Hedging of a call option

In this section, we discuss the hedging strategy for contingent claims, and in particular call options. A first strategy could be to "Delta" hedge the call option, that is buying a quantity $\partial C/\partial A_{S_t}$ of the underlying asset $(A_{S_t})_{t\geqslant 0}$ to mimic the first order behavior of the call option price with respect to the asset. However, such an approach fails to properly take into account the jumps that occurs in the price. To better take into account the occurrences of jumps, we propose a hedging strategy that is determined through a variance minimization problem, as pioneered in Föllmer and Sondermann (1985). This approach is also referred to as quadratic hedging and is used in numerous studies. We can cite for example Föllmer and Schweizer (1991), Pham (2000) and Moraux and Hainaut (2018).

The variance minimization of the hedging error will be performed under a fixed risk neutral measure $\mathbb{Q}$, as described in Section 0.4. The choice of such a measure is questionable. As a matter of fact, this approach suffers from two drawbacks. The first is that the chosen risk neutral measure $\mathbb{Q}$ becomes an input of the optimization procedure that leads to the hedging strategy. Consequently, the hedging strategy provides neither a risk neutral measure nor a price for the contingent claim. The second drawback is that the profits and losses that will occur in practice are ruled by the real-world probability measure. From this point of view, the optimization should be performed under the real probability measure and not under a risk neutral one. However, hedging under a risk neutral measure has some benefits. On the one hand, quadratic hedging with discontinuous processes under probability measures that are not risk neutral does not admit a solution in general. On the other hand, the non-uniqueness of a risk neutral measure allows to adjust the parameters that appear in the optimization problem. In particular, the parameters could then be adjusted to reflect the uncertainty over the evolution of prices and the risk aversion of traders. As noted in L. P. Hansen and Sargent (1995) and Lars Peter Hansen and Sargent (2001), the risk neutral measure can be chosen so that the obtained hedging strategy is more conservative than any other strategy built under $\mathbb{P}$.

The quadratic hedging approach relies on predictable representation theorems for martingales. Such theorems allow to write contingent claims as stochastic integrals of predictable processes with respect to the risk factors that drive the underlying asset price. The first representation theorem is often called the Kunita-Watanabe decomposition theorem and was introduced in Kunita and Watanabe (1967). Similar results can be found in Kunita (2004) and in Chapter 3 of Jean Jacod and Shiryaev (2003). We start with the general case of the hedging of a square integrable contingent claims Y. More precisely, we establish the existence and uniqueness the quadratic hedging strategy under each fixed risk neutral measure $\mathbb{Q}$. Afterwards, we determine the particular hedging strategy when the contingent claim Y is the payoff of a call option.

In order to rely on the Kunita-Watanabe decomposition theorem for square integrable martingales, we work with the discounted asset prices. Recall that in the subdiffusive case, the discounted asset price is $(\tilde{A}_{S_t})_{t\geqslant 0}$, where $\tilde{A}_t = e^{-rt}A_t, t \geqslant 0$. Fix a risk neutral measure $\mathbb{Q}$ as in Section 0.4. Note that when talking about square integrable martingales, we refer to the terminology of Protter (2005) (P.180), i.e. an $(\mathcal{F}_t)_{t\geqslant 0}$-adapted stochastic process $(M_t)_{t\geqslant 0}$ that satisfies $\mathbb{E}[M_t^2] < +\infty$ and $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ for all $t \geqslant s \geqslant 0$. In particular, we do not necessarily mean that $\sup_{t\geqslant 0} \mathbb{E}[M_t^2] < +\infty$ (the processes that satisfy this additional condition are referred to as L^2 martingales in Protter (2005) and are of course square integrable martingales).

Let $T > 0$ be a deterministic maturity and Y be an $\mathcal{F}_T^{A\circ S}$-measurable positive and square integrable random variable that represents the payoff of a contingent claim to be paid at time T. We aim at finding a portfolio (or strategy) in order to hedge the contingent claim Y. A portfolio consists of a vector valued predictable stochastic process $(\pi_t^{(0)}, \pi_t^{(1)}, \ldots, \pi_t^{(d)})_{t\in[0,T]}$ where d is the number of risky assets in the market and $\pi_t^{(i)}$ is the quantity of assets i to be hold at time t. Asset $i = 0$ is the risk-free asset. In our case, $d = 1$ and the present value $(V_t(\boldsymbol{\pi}))_{t\in[0,T]}$ of the portfolio $\boldsymbol{\pi} = (\pi_t^{(0)}, \pi_t^{(1)})_{t\in[0,T]}$ is given by

$$V_t(\boldsymbol{\pi}) = \pi_t^{(0)} + \pi_t^{(1)}\tilde{A}_{S_t} \tag{66}$$

at time t. On top of being $(\mathcal{F}_t^{A\circ S})_{t\in[0,T]}$-predictable, we require the portfolio $\boldsymbol{\pi}$ to be self-financing. This concept refers to the fact that the instantaneous changes in the value $(V_t(\boldsymbol{\pi}))_{t\in[0,T]}$ are due to changes in the prices of the assets, and not to instantaneous rebalancing of the asset quantities $(\pi_t^{(0)}, \pi_t^{(1)})_{t\in[0,T]}$ we hold. Mathematically, this condition is expressed as

$$V_t(\boldsymbol{\pi}) = V_0(\boldsymbol{\pi}) + \int_0^t \pi_u^{(1)}\mathrm{d}\tilde{A}_{S_u}.$$

For more material about self-financing portfolios, we refer to the chapter 2 of Jeanblanc, Yor, and Chesney (2009). The goal is thus to find

$$\boldsymbol{\pi}^\star = \arg\min_{\boldsymbol{\pi}\in\boldsymbol{\Pi}} \mathbb{E}_{\mathbb{Q}}\left[(V_T(\boldsymbol{\pi}) - e^{-rS_T}Y)^2\right] \tag{67}$$

where $\boldsymbol{\Pi}$ is the set of all self-financing portfolios. The next result requires the notion of orthogonal martingales. This notion was given at Definition 0.1. Let us state the Kunita-Watanabe decomposition theorem.

Lemma 0.9. *Let $(B_t)_{t\geqslant 0}, (M_t)_{t\geqslant 0}$ be two square integrable $(\mathbb{Q},\mathbb{G})$-martingales, where $\mathbb{G}=(\mathcal{G}_t)_{t\geqslant 0}$ is a filtration that satisfies the usual conditions. There exists a unique choice of square integrable $(\mathbb{Q},\mathbb{G})$-martingales $(M'_t)_{t\geqslant 0}$ and $(M''_t)_{t\geqslant 0}$ such that*

(i) $(M'_t)_{t\geqslant 0}\in\mathbf{\Phi}(B)$, *where*

$$\mathbf{\Phi}(B):=\left\{\left(\int_0^t \phi_u \mathrm{d}B_u\right)_{t\geqslant 0} : (\phi_u)_{u\geqslant 0} \text{ is a } \mathbb{G}\text{-predictable process}\right\},$$

(ii) $(M''_t)_{t\geqslant 0}$ *is orthogonal to any process that belongs to* $\mathbf{\Phi}(B)$,

(iii) $(M_t)_{t\geqslant 0}$ *and* $(M'_t+M''_t)_{t\geqslant 0}$ *are indistinguishable.*

The statement and the proof can be found at Proposition 4.1 in Kunita and Watanabe (1967). In order to apply Lemma 0.9, we need the filtration $(\mathcal{G}_t)_{t\geqslant 0}$ to satisfy what are called the usual conditions. The first usual condition is that $\mathcal{G}_0$ must contain all the null sets of the probability space, which is here the case by assumption. The second is that the σ-algebras $\mathcal{G}_{t+}:=\bigcap_{\varepsilon>0}\mathcal{G}_{t+\varepsilon}$ and $\mathcal{G}_t$ should coincide for any $t\geqslant 0$. This property is called the right-continuity of the filtration. The two next lemmas show that the filtration $\mathbb{F}^{A\circ S}$ that we have chosen for the discounted asset price $(\tilde{A}_{S_t})_{t\geqslant 0}$ is right-continuous.

Lemma 0.10. *Let $(\mathcal{H}_t)_{t\geqslant 0}$ and $(\mathcal{F}_t)_{t\geqslant 0}$ be two right-continuous filtrations on the same probability space $(\Omega,\mathcal{F},\mathbb{P})$. Then the filtration $(\mathcal{H}_t\vee\mathcal{F}_t)_{t\geqslant 0}$ is right-continous.*

Proof. Fix $t\geqslant 0$. We begin by showing the inclusion

$$\bigcap_{\varepsilon>0}(\mathcal{H}_{t+\varepsilon}\cup\mathcal{F}_{t+\varepsilon})\subset\mathcal{H}_t\cup\mathcal{F}_t. \tag{68}$$

Note that this inclusion is trivial in the case $\mathcal{H}_t\cup\mathcal{F}_t=\mathcal{F}$, so that we assume $\mathcal{H}_t\cup\mathcal{F}_t\neq\mathcal{F}$, that is $\mathcal{H}_t\cup\mathcal{F}_t$ is strictly included in $\mathcal{F}$. In this proof, set complements are taken with respect to $\mathcal{F}$, i.e. for $\mathcal{A}\subset\mathcal{F}$, $\overline{\mathcal{A}}=\mathcal{F}\setminus\mathcal{A}$. Let $B\in\overline{\mathcal{H}_t\cup\mathcal{F}_t}$. By the right-continuity of the filtrations and two applications of De Morgan's laws, we have

$$\begin{aligned}B&\in\overline{\left(\bigcap_{\varepsilon>0}\mathcal{H}_{t+\varepsilon}\right)\cup\left(\bigcap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}\right)}\\&=\overline{\left(\bigcap_{\varepsilon>0}\mathcal{H}_{t+\varepsilon}\right)}\cap\overline{\left(\bigcap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}\right)}\\&=\left(\bigcup_{\varepsilon>0}\overline{\mathcal{H}_{t+\varepsilon}}\right)\cap\left(\bigcup_{\varepsilon>0}\overline{\mathcal{F}_{t+\varepsilon}}\right)\end{aligned}$$

so that there are $\varepsilon_{\mathcal{H}},\varepsilon_{\mathcal{F}}>0$ such that $B\in\overline{\mathcal{H}_{t+\varepsilon_{\mathcal{H}}}}\cap\overline{\mathcal{F}_{t+\varepsilon_{\mathcal{F}}}}$. Setting $\varepsilon=\varepsilon_{\mathcal{H}}\wedge\varepsilon_{\mathcal{F}}$, we have that $B\in\overline{\mathcal{H}_{t+\varepsilon}}\cap\overline{\mathcal{F}_{t+\varepsilon}}$, which implies

$$B\in\overline{\bigcap_{\varepsilon>0}(\mathcal{H}_{t+\varepsilon}\cup\mathcal{F}_{t+\varepsilon})}$$

and thereby proves the inclusion at Equation (68). As a consequence,

$$\sigma\left(\bigcap_{\varepsilon>0}(\mathcal{H}_{t+\varepsilon}\cup\mathcal{F}_{t+\varepsilon})\right)\subset\mathcal{H}_t\vee\mathcal{F}_t.$$

It remains to show that

$$\bigcap_{\varepsilon>0}(\mathcal{H}_{t+\varepsilon}\vee\mathcal{F}_{t+\varepsilon})=\sigma\left(\bigcap_{\varepsilon>0}(\mathcal{H}_{t+\varepsilon}\cup\mathcal{F}_{t+\varepsilon})\right).$$

To this end, note that

$B \in \bigcap_{\varepsilon>0}(\mathcal{H}_{t+\varepsilon} \vee \mathcal{F}_{t+\varepsilon})$

$\Leftrightarrow$ For any $\varepsilon > 0$, $B \in \mathcal{S}$ whenever $\mathcal{S}$ is a σ-algebra that satisfies $\mathcal{S} \supset \mathcal{H}_{t+\varepsilon} \cup \mathcal{F}_{t+\varepsilon}$

$\Leftrightarrow$ For any $\varepsilon > 0$, $B \in \mathcal{S}$ whenever $\mathcal{S}$ is a σ-algebra that satisfies $\mathcal{S} \supset \mathcal{H}_{t+\varepsilon} \vee \mathcal{F}_{t+\varepsilon}$

$\Leftrightarrow$ For any $\varepsilon > 0$, $B \in \mathcal{H}_{t+\varepsilon} \vee \mathcal{F}_{t+\varepsilon}$

$\Leftrightarrow$ $B \in \bigcap_{\varepsilon>0}(\mathcal{H}_{t+\varepsilon} \vee \mathcal{F}_{t+\varepsilon})$,

concluding the proof. □

Lemma 0.11. *The filtration $\mathbb{F}^{A\circ S}$ is right-continuous.*

Proof. Recall that $\mathcal{F}_t^{A\circ S} = \mathcal{F}_{S_t}^{A,U}$, where $\mathcal{F}_t^{A,U} = \mathcal{F}_t^{A} \vee \mathcal{F}_t^{U} = \mathcal{F}_t^{W} \vee \mathcal{F}_t^{J} \vee \mathcal{F}_t^{U}$.

First we show that $\mathbb{F}^{A,U}$ is right-continuous. According to Lemma 0.10, it is enough to show that $\mathbb{F}^{W}$, $\mathbb{F}^{J}$ and $\mathbb{F}^{U}$ are right-continuous. The right-continuity of both $\mathbb{F}^{U}$ and $\mathbb{F}^{W}$ is guaranteed by Theorem 31 in the chapter 1 of Protter (2005), which states that the natural filtrations of Lévy processes completed with the null sets are right-continuous. The proof of the right-continuity of natural filtrations of counting processes (Theorem 25 in the same chapter) is also valid for the right-continuity of $\mathbb{F}^{J}$. It follows that $\mathbb{F}^{A,U}$ is right-continuous. It remains to show that this conclusion extends to the time-changed filtration $\mathbb{F}^{A\circ S}$.

Fix $t \geqslant 0$ and assume that $A \in \bigcap_{n\in\mathbb{N}} \mathcal{F}^{A\circ S}_{t+\frac{1}{n}}$. Then for each $n \in \mathbb{N}$ and all $s \geqslant 0$, $A \cap \{S_{t+\frac{1}{n}} \leqslant s\} \in \mathcal{F}_s^{A,U}$. It implies that for any $k \in \mathbb{N}$,

$$\bigcap_{m\geqslant k} \bigcup_{n\in\mathbb{N}} A \cap \left\{S_{t+\frac{1}{n}} \leqslant s + \frac{1}{m}\right\} \in \mathcal{F}^{A,U}_{s+\frac{1}{k}}, \tag{69}$$

and thus the right-continuity of $\mathbb{F}^{A,U}$ entails that the set of Equation (69) is in $\mathcal{F}_s^{A,U}$. Finally, since the paths of $(S_t)_{t\geqslant 0}$ are nondecreasing and right-continuous, the set of Equation (69) equals $A \cap \{S_t \leqslant s\}$. This proves that $A \in \mathcal{F}_{S_t}^{A,U} = \mathcal{F}_t^{A\circ S}$. □

We can now state the existence and uniqueness of the quadratic hedging strategy.

Proposition 0.15. *Let Y be a $\mathcal{F}_T^{A\circ S}$- measurable square integrable random variable. Then there exists a unique choice of a $\mathbb{F}^{A\circ S}$-predictable process $(y_t)_{t\geqslant 0}$ and a martingale $(M_t)_{t\geqslant 0}$ such that*

$$e^{-rS_T} Y = \int_0^T y_s \mathrm{d}\tilde{A}_{S_s} + M_T \tag{70}$$

with $(M_t)_{t\geqslant 0}$ being orthogonal to any process in $\mathbf{\Phi}(\tilde{A} \circ S)$. Moreover, there exists a unique solution to the quadratic hedging problem (67) given by

$$\boldsymbol{\pi}^\star = (\mathbb{E}_{\mathbb{Q}}[e^{-rS_T} Y] - y_0 \tilde{A}_{S_0}, y_t)_{t\in[0,T]}.$$

Proof. Define the process $(Y_t)_{t\geqslant 0}$ as $Y_t := \mathbb{E}_{\mathbb{Q}}[e^{-rS_T} Y | \mathcal{F}_t^{A\circ S}]$. It is then clear that $(Y_t)_{t\geqslant 0}$ is a square integrable $(\mathbb{Q}, \mathbb{F}^{A\circ S})$-martingale. Lemma 0.9 entails the existence of a unique choice of a $\mathbb{F}^{A\circ S}$-predictable process $(y_t)_{t\geqslant 0}$ and a square integrable $(\mathbb{Q}, \mathbb{F}^{A\circ S})$-martingale $(M_t)_{t\geqslant 0}$ such that

$$Y_t = \int_0^t y_s \mathrm{d}\tilde{A}_{S_s} + M_t.$$

Moreover, the stochastic integral $(\int_0^t y_s \mathrm{d}\tilde{A}_{S_s})_{t\geqslant 0}$ is a square integrable $(\mathbb{Q}, \mathbb{F}^{A\circ S})$-martingale. Equation (70) follows. The orthogonality of $(M_t)_{t\geqslant 0}$ to any process in $\mathbf{\Phi}(\tilde{A} \circ S)$ corresponds to the second statement of

Lemma 0.9. Then, for any self-financing portfolio $\boldsymbol{\pi} = (\pi_t^{(0)}, \pi_t^{(1)})$, we can write

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[(V_T(\boldsymbol{\pi}) - e^{-rS_T}Y)^2] &= \mathbb{E}_{\mathbb{Q}}\left[\left(V_0(\boldsymbol{\pi}) + \int_0^T (\pi_u^{(1)} - y_u)\mathrm{d}\tilde{A}_{S_u} - M_T\right)^2\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[(V_0(\boldsymbol{\pi}) - M_T)^2\right] + \mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^T (\pi_u^{(1)} - y_u)\mathrm{d}\tilde{A}_{S_u}\right)^2\right] \\ &\quad - 2\mathbb{E}_{\mathbb{Q}}\left[M_T \int_0^T (\pi_u^{(1)} - y_u)\mathrm{d}\tilde{A}_{S_u}\right].\end{aligned} \tag{71}$$

Note that $\left(\int_0^t (\pi_u^{(1)} - y_u)\mathrm{d}\tilde{A}_{S_u}\right)_{t\geqslant 0} \in \boldsymbol{\Phi}(\tilde{A}\circ S)$ and thus this martingale is orthogonal to $(M_t)_{t\geqslant 0}$. It implies that

$$\mathbb{E}_{\mathbb{Q}}\left[M_T \int_0^T (\pi_u^{(1)} - y_u)\mathrm{d}\tilde{A}_{S_u}\right] = 0.$$

Therefore, with some more straightforward computations, Equation (71) becomes

$$\mathbb{E}_{\mathbb{Q}}[(V_T(\boldsymbol{\pi}) - e^{-rS_T}Y)^2] = (V_0(\boldsymbol{\pi}) - M_0)^2 + \mathbb{E}_{\mathbb{Q}}[(M_T - M_0)^2] + \mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^T (\pi_u^{(1)} - y_u)\mathrm{d}\tilde{A}_{S_u}\right)^2\right].$$

To this point, it is already clear that the optimal initial value for the hedging strategy is $V_0(\boldsymbol{\pi}) = M_0 = \mathbb{E}_{\mathbb{Q}}[e^{-rS_T}Y]$. It is also clear that the optimal $(\pi_t^{(1)})_{t\geqslant 0}$ is determined through the minimization of

$$\mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^T (\pi_u^{(1)} - y_u)\mathrm{d}\tilde{A}_{S_u}\right)^2\right].$$

Moreover,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^T (\pi_u^{(1)} - y_u)\mathrm{d}\tilde{A}_{S_u}\right)^2\right] &= \mathbb{E}_{\mathbb{Q}}\left[\left[(\pi^{(1)} - y)\cdot(\tilde{A}\circ S), (\pi^{(1)} - y)\cdot(\tilde{A}\circ S)\right]_T\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \left(\pi_t^{(1)} - y_t\right)^2 \mathrm{d}\left[\tilde{A}\circ S, \tilde{A}\circ S\right]_t\right]\end{aligned} \tag{72}$$

where $((\pi^{(1)} - y)\cdot(\tilde{A}\circ S))_t$ denotes the stochastic integral $\int_0^t (\pi_u^{(1)} - y_u)\mathrm{d}\tilde{A}_{S_u}$. The first and second equalities in Equation (72) respectively follow from Corollary 3 and Theorem 29 in Protter (2005), since $\left(\int_0^t (\pi_u^{(1)} - y_u)\mathrm{d}\tilde{A}_{S_u}\right)_{t\geqslant 0}$ is a martingale. Quadratic variations being increasing processes, the quantity at Equation (72) is minimized when $\pi_t^{(1)} = y_t$ for all $t \geqslant 0$. The result follows by combining our findings with Equation (66). ☐

Unfortunately, Proposition 0.15 does not give an explicit way to compute the optimal strategy $\boldsymbol{\pi}^\star$. The reason is that the Kunita-Watanabe decomposition theorem does not give an explicit formula for the predictable process $(y_t)_{t\geqslant 0}$ but only asserts its existence ans uniqueness. However, for any square-integrable contingent claim whose payoff depend only on the terminal value $\tilde{A}_{S_T}$ of the stock, one can use Ito's lemma to compute explicitly the solution of the quadratic hedging problem. The remainder of this section is thus dedicated the explicit quadratic hedging portfolio when $Y = f(A_{S_T})$. By the Markov property, there exists a function g such that the $(\mathcal{F}_t^{A\circ S})_{t\geqslant 0}$-martingale $(Y_t)_{t\geqslant 0} = (\mathbb{E}_{\mathbb{Q}}[e^{-rS_T}f(A_{S_T})|\mathcal{F}_t^{A\circ S}])_{t\geqslant 0}$ satisfies

$$\mathbb{E}_{\mathbb{Q}}[e^{-rS_T}f(A_{S_T})|\mathcal{F}_t^{A\circ S}] = g(t, \tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_t}), \tag{73}$$

or

$$g(t,y_1,y_2,v,u)=\mathbb{E}[e^{-rS_T}f(A_{S_T})|(\tilde{A}_{S_t},\lambda_{S_t},S_t,U_{S_t})=(y_1,y_2,v,u)]. \tag{74}$$

We now show that the argument t is actually irrelevant in the function g. The intuition behind this result is that in the time-changed setting, the relevant time is given by the last observed value of $(U_{S_t})_{t\geqslant 0}$, ruling out the original time scale $(t)_{t\geqslant 0}$. To prove this, we rely on two intermediary results.

Lemma 0.12. *For all $t\geqslant 0$, it holds that $S_{U_t}=t$ a.s.. Moreover $S\circ U=\mathrm{id}_{\mathbb{R}^+}$ a.s., that is the processes $((S\circ U)_t)_{t\geqslant 0}$ and $(t)_{t\geqslant 0}$ are indistinguishable.*

Proof. The first part of the statement follows readily from

$$S_{U_t(\omega)}(\omega)=\inf\{\tau>0:U_\tau\geqslant U_t(\omega)\}$$

and the fact that the paths of $(U_t)_{t\geqslant 0}$ are strictly increasing a.s.. The second part of the statement is a consequence of the càdlàg paths of $((S\circ U)_t)_{t\geqslant 0}$, combined with the first statement. ☐

Lemma 0.13. *For each $u\geqslant s\geqslant 0$, we have*

$$\{S_s=v\}\cap\{U_{S_s}=u\}=\{S_u=v\}\cap\{U_{S_s}=u\}. \tag{75}$$

As a consequence,

$$\{S_s=v\}\cap\{U_{S_s}=u\}=\{S_u=v\}\cap\{U_{S_s}=u\}\cap\{U_{S_u}=u\}. \tag{76}$$

Proof. Let ω be a member of the set at the left-hand side of Equation (75). Since $U_v(\omega)=u$, it is clear that

$$S_u(\omega):=\inf\{\tau>0:U_\tau(\omega)\geqslant u\}\leqslant v.$$

Assume by contradiction that $\inf\{\tau>0:U_\tau(\omega)\geqslant u\}<v$. Then there exists an $\varepsilon>0$ such that $U_{v-\varepsilon}(\omega)\geqslant u$. Hence, as $u\geqslant s$, this leads to $v-\varepsilon\geqslant S_s(\omega)=v$, which is a contradiction. We conclude that $S_u(\omega)=v$.

Conversely, let ω be a member of the set at the right-hand side of Equation (75). The equality $U_{S_s(\omega)}(\omega)=u$ implies that

$$S_u(\omega)=(S\circ U\circ S)_s(\omega)=v.$$

Moreover, Lemma 0.12 says that $(S\circ U\circ S)_s(\omega)=S_s(\omega)$ a.s., which proves that $S_s(\omega)=v$.

That the set at the left-hand side of Equation (76) contains the set at the right-hand side is an obvious consequence of Equation (75) that we have just established. Let $\omega\in\{S_s=v\}\cap\{U_{S_s}=u\}$. By Equation (75), $S_s(\omega)=S_u(\omega)=v$, so that $U_{S_u(\omega)}(\omega)=U_v(\omega)=u$. ☐

We are now able to prove the irrelevancy of the t argument in the function g of Equation (74).

Proposition 0.16. *Let g be the function of Equation (74). Then for any $t\in[0,T]$, $u\geqslant t$, $v\geqslant 0$, we have*

$$g(t,y_1,y_2,v,u)=g(u\wedge T,y_1,y_2,v,u)$$

so that g does not depend on t.

Proof. By Lemma 0.13,

$$\begin{aligned}g(t,y_1,y_2,v,u)&=\mathbb{E}_{\mathbb{Q}}[e^{-rS_T}f(A_{S_T})|(\tilde{A}_{S_t},\lambda_{S_t},S_t,U_{S_t})=(y_1,y_2,v,u)]\\&=\mathbb{E}_{\mathbb{Q}}\left[e^{-rS_T}f(A_{S_T})|(\tilde{A}_{S_u},\lambda_{S_u},S_u,U_{S_s},U_{S_u})=(y_1,y_2,v,u,u)\right].\end{aligned}$$

If $u\leqslant T$, the Markov property of $(S_t,U_{S_t})_{t\geqslant 0}$ implies that

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[e^{-rS_T}f(A_{S_T})|(\tilde{A}_{S_u},\lambda_{S_u},S_u,U_{S_s},U_{S_u})=(y_1,y_2,v,u,u)\right]\\&=\mathbb{E}_{\mathbb{Q}}\left[e^{-rS_T}f(A_{S_T})|(\tilde{A}_{S_u},\lambda_{S_u},S_u,U_{S_u})=(y_1,y_2,v,u)\right]\\&=g(u,y_1,y_2,v,u)\end{aligned}$$

whereas if $u > T$, one has

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[e^{-rS_T} f(A_{S_T}) | (A_{S_u}, \lambda_{S_u}, S_u, U_{S_s}, U_{S_u}) = (y_1, y_2, v, u, u)\right] \\
&= e^{-rS_T} f(A_{S_T}) \\
&= g(T, y_1, y_2, v, u).
\end{aligned}$$

Indeed, in that case, since $U_{S_s} = U_{S_u} = u$ and $s \leqslant T < u$, we have $v = S_s \leqslant S_T \leqslant S_u = v$. □

The previous result motivates that in the following, the function g will be defined as

$$g(y_1, y_2, v, u) = \mathbb{E}_{\mathbb{Q}}[e^{-rS_T} f(A_{S_T}) | (\tilde{A}_{S_u}, \lambda_{S_u}, S_u, U_{S_u}) = (y_1, y_2, v, u)].$$

Assuming that g is sufficiently regular, we apply Ito's lemma for semimartingales on the process $(g(\tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_t}))_{t\geqslant 0}$. To do so, note that all the processes that are involved are indeed semimartingales. As a matter of fact, since $(\tilde{A}_t)_{t\geqslant 0}$ and $(\lambda_t)_{t\geqslant 0}$ are semimartingales, Theorem 10.16 in J. Jacod (1979) implies that their time-changed counterparts are also semimartingales. Moreover, since $(S_t)_{t\geqslant 0}$ and $(U_{S_t})_{t\geqslant 0}$ have finite variation (because they are nondecreasing) and are càdlàg, Theorem 26 in the second chapter of Protter (2005) entails that they are quadratic pure jump semimartingales. In order to use Ito's lemma, we need to determine the quadratic (co)variations of (or between) the processes that are involved in the function g. We begin with the proof that the quadratic covariation between $(U_t)_{t\geqslant 0}$ and $(A_t)_{t\geqslant 0}$ or $(\lambda_t)_{t\geqslant 0}$ is zero. The proof consists in showing that these processes never jump together.

Proposition 0.17. *The quadratic covariation processes $([A,U]_t)_{t\geqslant 0}$ and $([\lambda,U]_t)_{t\geqslant 0}$ are indistinguishable from the null process.*

Proof. We give the proof for $([A,U]_t)_{t\geqslant 0}$ only, as the case of $([\lambda,U]_t)_{t\geqslant 0}$ is essentially the same.

The proof rely on the assumption regarding the structure of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ explained in the introduction of the model. Since $(U_t)_{t\geqslant 0}$ is a quadratic pure jump semimartingale[6], it holds that

$$[A,U]_t = \sum_{0<s\leqslant t} \Delta A_s \Delta U_s.$$

Let us define $\mathcal{J}_t^n(\omega) = \{s \in (0,t] : \Delta U_s(\omega) > n^{-1}\}$ and $\mathcal{J}_t(\omega) = \bigcup_{n\geqslant 1} \mathcal{J}_t^n(\omega) = \{s \in (0,t] : \Delta U_s(\omega) > 0\}$. Since $(U_t)_{t\geqslant 0}$ is almost surely strictly increasing and $U_t(\omega) < +\infty$ for almost all ω, the set $\mathcal{J}_t^n(\omega)$ is almost surely finite for all $n \geqslant 0$ and $t > 0$ (which also implies that the set of jumps $\mathcal{J}_t(\omega)$ is countable, as it is a countable union of finite sets). Similarly, we define $\tilde{\mathcal{J}}_t^n(\omega_2) = \{s \in (0,t] : \Delta U_s^{(2)}(\omega_2) > n^{-1}\}$ and $\tilde{\mathcal{J}}_t(\omega_2) = \bigcup_{n\geqslant 1} \tilde{\mathcal{J}}_t^n(\omega_2)$. Note that for all n, $\tilde{\mathcal{J}}_t^n(\omega_2) = \mathcal{J}_t^n(\omega_1, \omega_2)$. From the monotone convergence theorem

[6] Again, this process is càdlàg and since it is strictly increasing, it has finite variation. Theorem 26 in the second chapter of Protter (2005) therefore implies that it is a quadratic pure jump semimartingale.

and Tonelli's theorem, it follows that

$$\begin{aligned}
\mathbb{E}[|[A,U]_t|] &\leqslant \mathbb{E}\left[\sum_{0<s\leqslant t} |\Delta A_s|\Delta U_s\right] \\
&= \int_\Omega \sum_{s\in\mathcal{J}_t(\omega)} |\Delta A_s(\omega)|\Delta U_s(\omega)\mathbb{P}(\mathrm{d}\omega) \\
&= \lim_{n\to+\infty} \int_\Omega \sum_{s\in\mathcal{J}_t^n(\omega)} |\Delta A_s(\omega)|\Delta U_s(\omega)\mathbb{P}(\mathrm{d}\omega) \\
&= \lim_{n\to+\infty} \int_{\Omega_2}\int_{\Omega_1} \sum_{s\in\mathcal{J}_t^n(\omega_1,\omega_2)} |\Delta A_s(\omega_1,\omega_2)|\Delta U_s(\omega_1,\omega_2)\mathbb{P}^{(1)}(\mathrm{d}\omega_1)\mathbb{P}^{(2)}(\mathrm{d}\omega_2) \\
&= \lim_{n\to+\infty} \int_{\Omega_2}\int_{\Omega_1} \sum_{s\in\tilde{\mathcal{J}}_t^n(\omega_2)} |\Delta A_s^{(1)}(\omega_1)|\Delta U_s^{(2)}(\omega_2)\mathbb{P}^{(1)}(\mathrm{d}\omega_1)\mathbb{P}^{(2)}(\mathrm{d}\omega_2) \\
&= \lim_{n\to+\infty} \int_{\Omega_2} \sum_{s\in\tilde{\mathcal{J}}_t^n(\omega_2)} \Delta U_s^{(2)}(\omega_2)\left(\int_{\Omega_1} |\Delta A_s^{(1)}(\omega_1)|\mathbb{P}^{(1)}(\mathrm{d}\omega_1)\right)\mathbb{P}^{(2)}(\mathrm{d}\omega_2) \\
&= \lim_{n\to+\infty} \int_{\Omega_2} \sum_{s\in\tilde{\mathcal{J}}_t^n(\omega_2)} \Delta U_s^{(2)}(\omega_2)[\psi(1,0)-1]\mathbb{E}[A_{s-}]\mathbb{E}[\Delta N_s]\mathbb{P}^{(2)}(\mathrm{d}\omega_2) \\
&= 0,
\end{aligned}$$

where the last equality is a consequence of $\Delta N_s = 0$ $\mathbb{P}$-a.s., for all $s \geqslant 0$. We conclude that for all $t \geqslant 0$, $[A,U]_t = 0$ a.s., that is $([A,U]_t)_{t\geqslant 0}$ and the null process are modifications. The conclusion then follows from the fact that $([A,U]_t)_{t\geqslant 0}$ is a càdlàg process (which is a consequence of Theorem 22 in Protter (2005), combined with the polarization identity for quadratic variations). □

Corollary 0.5. *The quadratic covariation process* $([A\circ S, U\circ S]_t)_{t\geqslant 0}$ *and* $([\lambda\circ S, U\circ S]_t)_{t\geqslant 0}$ *are indistinguishable from the null process.*

Corollary 0.6. *The processes* $(|\Delta(A\circ S)_t\Delta(U\circ S)_t|)_{t\geqslant 0}$ *and* $(|\Delta(\lambda\circ S)_t\Delta(U\circ S)_t|)_{t\geqslant 0}$ *are indistinguishable from the null process.*

Proof. This follows from the proof of Proposition 0.17. It is shown that the process $(\sum_{0<s\leqslant t}|\Delta A_s|\Delta U_s)_{t\geqslant 0}$ is indistinguishable from the zero process. The result then follows from the inequalities

$$0 \leqslant |\Delta(A\circ S)_t\Delta(U\circ S)_t| \leqslant \sum_{0<s\leqslant S_t} |\Delta A_s|\Delta U_s$$

and

$$0 \leqslant |\Delta(\lambda\circ S)_t\Delta(U\circ S)_t| \leqslant \sum_{0<s\leqslant S_t} |\Delta \lambda_s|\Delta U_s.$$

□

Proposition 0.18. *For any semimartingale* $(X_t)_{t\geqslant 0}$*, the quadratic covariation process* $([X,S]_t)_{t\geqslant 0}$ *is indistinguishable from the null process.*

Proof. The quadratic covariation processes being càdlàg, it is enough to show that they are modifications of the null process. As observed above, the process $(S_t)_{t\geqslant 0}$ is a quadratic pure jump semimartingale. As a consequence, Theorem 28 of Protter (2005) (Chapter 2) yields

$$[X,S]_t = X_0S_0 + \sum_{0<s\leqslant t} \Delta X_s\Delta S_s,$$

which is 0 a.s., since $(S_t)_{t\geqslant 0}$ is continuous and $S_0 = 0$. □

Proposition 0.19. *Assume that the function g of Equation* (73) *is sufficiently regular. Then, we have that*

$$
\begin{aligned}
&g(\tilde{A}_{S_T}, \lambda_{S_T}, S_T, U_{S_T}) = g(\tilde{A}_0, \lambda_0, 0, 0) + \sigma \int_0^T \tilde{A}_{S_t} \frac{\partial g}{\partial y_1} \mathrm{d}W_{S_t} \\
&\quad + \int_0^T \left[\frac{\partial g}{\partial v} + \kappa(\theta - \lambda_{S_t}) \frac{\partial g}{\partial y_2} - \tilde{A}_{S_t} \mathbb{E}[e^{\xi} - 1] \lambda_{S_t} \frac{\partial g}{\partial y_1} + \frac{\sigma^2}{2} \tilde{A}_{S_t}^2 \frac{\partial^2 g}{\partial y_1^2} \right] \mathrm{d}S_t \\
&\quad + \int_0^T \int_{\mathbb{R}} (g(e^z \tilde{A}_{S_{t-}}, \lambda_{S_{t-}} + \eta|z|, S_t, U_{S_t}) - g(\tilde{A}_{S_{t-}}, \lambda_{S_{t-}}, S_t, U_{S_t})) \Xi(\mathrm{d}z) \mathrm{d}N_{S_t} \\
&\quad + \sum_{0<t\leqslant T} (g(\tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_t}) - g(\tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_{t-}})),
\end{aligned}
$$

where Ξ is the random measure associated with the jump process $(J_t)_{t\geqslant 0}$.

Proof. An application of Ito's lemma for semimartingales yields

$$
\begin{aligned}
&g(t, \tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_t}) - g(\tilde{A}_{S_0}, \lambda_{S_0}, S_0, U_{S_0}) \\
&= \int_0^t \frac{\partial g}{\partial y_1} (\tilde{A}_{S_{s-}}, \lambda_{S_{s-}}, S_{s-}, U_{S_{s-}}) \tilde{A}_{S_{s-}} \left[\sigma \mathrm{d}W_{S_s} + \lambda_{S_s} \mathbb{E}[e^{\xi} - 1] \mathrm{d}S_s \right] \\
&\quad + \int_0^t \frac{\partial g}{\partial y_2} (\tilde{A}_{S_{s-}}, \lambda_{S_{s-}}, S_{s-}, U_{S_{s-}}) \kappa(\theta - \lambda_{S_s}) \mathrm{d}S_s \\
&\quad + \int_0^t \frac{\partial g}{\partial v} (\tilde{A}_{S_{s-}}, \lambda_{S_{s-}}, S_{s-}, U_{S_{s-}}) \mathrm{d}S_s \\
&\quad + \frac{\sigma^2}{2} \int_0^t \frac{\partial^2 g}{\partial y_1^2} (\tilde{A}_{S_{s-}}, \lambda_{S_{s-}}, S_{s-}, U_{S_{s-}}) \tilde{A}_{S_{s-}}^2 \mathrm{d}S_s \\
&\quad + \sum_{0<s\leqslant t} \left[g(\tilde{A}_{S_s}, \lambda_{S_s}, S_s, U_{S_s}) - g(\tilde{A}_{S_{s-}}, \lambda_{S_{s-}}, S_{s-}, U_{S_-}) \right].
\end{aligned}
\tag{77}
$$

The jump part can be decomposed as follows

$$
\begin{aligned}
&\sum_{0<s\leqslant t} \left[g(\tilde{A}_{S_s}, \lambda_{S_s}, S_s, U_{S_s}) - g(\tilde{A}_{S_{s-}}, \lambda_{S_{s-}}, S_{s-}, U_{S_{s-}}) \right] \\
&= \sum_{0<s\leqslant t} \left[g(\tilde{A}_{S_s}, \lambda_{S_s}, S_s, U_{S_s}) - \lim_{u\uparrow s} g(\tilde{A}_{S_u}, \lambda_{S_u}, S_u, U_{S_u}) \right] \mathbf{1}_{\{\Delta(U\circ S)_s = 0\} \cap \{\Delta(\tilde{A}\circ S)_s > 0\}} \\
&\quad + \sum_{0<s\leqslant t} \left[g(\tilde{A}_{S_s}, \lambda_{S_s}, S_s, U_{S_s}) - \lim_{u\uparrow s} g(\tilde{A}_{S_u}, \lambda_{S_u}, S_u, U_{S_u}) \right] \mathbf{1}_{\{\Delta(U\circ S)_s > 0\} \cap \{\Delta(\tilde{A}\circ S)_s = 0\}} \\
&\quad + \sum_{0<s\leqslant t} \left[g(\tilde{A}_{S_s}, \lambda_{S_s}, S_s, U_{S_s}) - \lim_{u\uparrow s} g(\tilde{A}_{S_u}, \lambda_{S_u}, S_u, U_{S_u}) \right] \mathbf{1}_{\{\Delta(U\circ S)_s > 0\} \cap \{\Delta(\tilde{A}\circ S)_s > 0\}}
\end{aligned}
$$

and Corollary 0.6 implies that the last term is indistinguishable from the zero process. The continuity of g

is then used to obtain

$$
\begin{aligned}
&\sum_{0<s\leqslant t}\left[g(\tilde{A}_{S_s},\lambda_{S_s},S_s,U_{S_s})-g(\tilde{A}_{S_{s-}},\lambda_{S_{s-}},S_{s-},U_{S_{s-}})\right]\\
&=\sum_{0<s\leqslant t}\left[g(\tilde{A}_{S_s},\lambda_{S_s},S_s,U_{S_s})-\lim_{u\uparrow s}g(\tilde{A}_{S_u},\lambda_{S_u},S_u,U_{S_u})\right]\mathbf{1}_{\{\Delta(U\circ S)_s=0\}\cap\{\Delta(\tilde{A}\circ S)_s>0\}}\\
&\quad+\sum_{0<s\leqslant t}\left[g(\tilde{A}_{S_s},\lambda_{S_s},S_s,U_{S_s})-\lim_{u\uparrow s}g(\tilde{A}_{S_u},\lambda_{S_u},S_u,U_{S_u})\right]\mathbf{1}_{\{\Delta(U\circ S)_s>0\}\cap\{\Delta(\tilde{A}\circ S)_s=0\}}\\
&=\sum_{0<s\leqslant t}\left[g(\tilde{A}_{S_s},\lambda_{S_s},S_s,U_{S_s})-g(\lim_{u\uparrow s}(\tilde{A}_{S_u},\lambda_{S_u},S_u,U_{S_u}))\right]\mathbf{1}_{\{\Delta(U\circ S)_s=0\}\cap\{\Delta(\tilde{A}\circ S)_s>0\}}\\
&\quad+\sum_{0<s\leqslant t}\left[g(\tilde{A}_{S_s},\lambda_{S_s},S_s,U_{S_s})-g(\lim_{u\uparrow s}(\tilde{A}_{S_u},\lambda_{S_u},S_u,U_{S_u}))\right]\mathbf{1}_{\{\Delta(U\circ S)_s>0\}\cap\{\Delta(\tilde{A}\circ S)_s=0\}}\\
&=\sum_{0<s\leqslant t}\left[g(\tilde{A}_{S_s},\lambda_{S_s},S_s,U_{S_s})-g(\tilde{A}_{S_{s-}},\lambda_{S_{s-}},S_s,U_{S_s})\right]\mathbf{1}_{\{\Delta(U\circ S)_s=0\}\cap\{\Delta(\tilde{A}\circ S)_s>0\}}\\
&\quad+\sum_{0<s\leqslant t}\left[g(\tilde{A}_{S_s},\lambda_{S_s},S_s,U_{S_s})-g(\tilde{A}_{S_s},\lambda_{S_s},S_s,U_{S_{s-}})\right]\mathbf{1}_{\{\Delta(U\circ S)_s>0\}\cap\{\Delta(\tilde{A}\circ S)_s=0\}}.
\end{aligned}
$$

The result is finally obtained by rearranging the terms in Equation (77) and rewriting the first jumps term with the help of the random measure Ξ, which gives

$$
\begin{aligned}
&\sum_{0<s\leqslant t}\left[g(\tilde{A}_{S_s},\lambda_{S_s},S_s,U_{S_s})-g(\tilde{A}_{S_{s-}},\lambda_{S_{s-}},S_s,U_{S_s})\right]\mathbf{1}_{\{\Delta(U\circ S)_s=0\}\cap\{\Delta(\tilde{A}\circ S)_s>0\}}\\
&\qquad=\int_0^T\int_{\mathbb{R}}(g(e^z\tilde{A}_{S_{t-}},\lambda_{S_{t-}}+\eta|z|,S_t,U_{S_t})-g(\tilde{A}_{S_{t-}},\lambda_{S_{t-}},S_t,U_{S_t}))\Xi(\mathrm{d}z)\mathrm{d}N_{S_t}
\end{aligned}
$$

and concludes the proof. □

Using Proposition 0.19, we can derive the optimal hedging strategy $\boldsymbol{\pi}^\star$ of Equation (67) when $Y_T=f(A_{S_T})$. This strategy is given in the next proposition.

Proposition 0.20. *Assume the the function g of Equation (73) is sufficiently smooth. Then if $Y=f(A_{S_T})$ is square integrable, the hedging problem (67) admits the solution $\boldsymbol{\pi}^\star=(\pi_t^{(0\star)},\pi_t^{(1\star)})_{t\geqslant 0}$, where*

$$
\pi_u^{(1\star)}=\frac{\sigma^2\tilde{A}_{S_{u-}}\frac{\partial g}{\partial y_1}+\lambda_{S_{u-}}\int_{\mathbb{R}}(g(e^z\tilde{A}_{S_{u-}},\lambda_{S_{u-}}+\eta|z|,S_u,U_{S_u})-g(\tilde{A}_{S_{u-}},\lambda_{S_{u-}},S_u,U_{S_u})(e^z-1)\nu(\mathrm{d}z)}{\tilde{A}_{S_{u-}}\left[\sigma^2+\lambda_{S_{u-}}\int_{\mathbb{R}}(e^z-1)^2\nu(\mathrm{d}z)\right]}
$$

and

$$
\pi_u^{(0\star)}=\mathbb{E}_{\mathbb{Q}}[e^{-rS_T}f(A_{S_T})]-\pi_0^{(1\star)}\tilde{A}_{S_0}.
$$

Proof. Recall that the function g and the $\mathbb{F}^{A\circ S}$-martingale $(Y_t)_{t\geqslant 0}=(\mathbb{E}_{\mathbb{Q}}[e^{-rS_T}f(A_{S_T})|\mathcal{F}_t^{A\circ S}])_{t\geqslant 0}$ satisfy

$$
\mathbb{E}_{\mathbb{Q}}[e^{-rS_T}f(A_{S_T})|\mathcal{F}_t^{A\circ S}]=g(\tilde{A}_{S_t},\lambda_{S_t},S_t,U_{S_t}). \tag{78}
$$

For any self-financing portfolio $\boldsymbol{\pi}=(\pi_t^{(0)},\pi_t^{(1)})_{t\in[0,T]}$, we have the following decomposition of the quadratic hedging error

$$\mathbb{E}_{\mathbb{Q}}\left[\left(Y_T - V_0(\boldsymbol{\pi}) - \int_0^T \pi_t^{(1)}\mathrm{d}\tilde{A}_{S_t}\right)^2\right] = \mathbb{E}_{\mathbb{Q}}\left[\left(Y_T - Y_0 + (Y_0 - V_0(\boldsymbol{\pi})) - \int_0^T \pi_t^{(1)}\mathrm{d}\tilde{A}_{S_t}\right)^2\right]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[(Y_0 - V_0(\boldsymbol{\pi}))^2 + \left(Y_T - Y_0 - \int_0^T \pi_t^{(1)}\mathrm{d}\tilde{A}_{S_t}\right)^2 + 2(Y_0 - V_0(\boldsymbol{\pi}))\left(Y_T - Y_0 - \int_0^T \pi_t^{(1)}\mathrm{d}\tilde{A}_{S_t}\right)\right]$$
$$= (Y_0 - V_0(\boldsymbol{\pi}))^2 + \mathbb{E}_{\mathbb{Q}}\left[\left(Y_T - Y_0 - \int_0^T \pi_t^{(1)}\mathrm{d}\tilde{A}_{S_t}\right)^2\right],$$

where the last equality comes from

$$\mathbb{E}_{\mathbb{Q}}[Y_T - Y_0] = Y_0 - Y_0 = 0$$

and

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \pi_t^{(1)}\mathrm{d}\tilde{A}_{S_t}\right] = 0,$$

as $(Y_t)_{t\geqslant 0}$ and $(\tilde{A}_{S_t})_{t\geqslant 0}$ are $\mathbb{F}^{A\circ S}$-martingales. It is then clear that the optimal initial amount is $V_0(\boldsymbol{\pi}^\star) = Y_0 = \mathbb{E}_{\mathbb{Q}}[e^{-rS_T}f(A_{S_T})]$. Next, Proposition 0.19 implies that

$$Y_T - Y_0 - \int_0^T \pi_t^{(1)}\mathrm{d}\tilde{A}_{S_t} = \sigma\int_0^T \tilde{A}_{S_t}\left(\frac{\partial g}{\partial y_1} - \pi_t^{(1)}\right)\mathrm{d}W_{S_t}$$
$$+ \int_0^T \left[\frac{\partial g}{\partial v} + \kappa(\theta - \lambda_{S_t})\frac{\partial g}{\partial y_2} - \tilde{A}_{S_t}\mathbb{E}_{\mathbb{Q}}[e^{\xi} - 1]\lambda_{S_t}\left(\frac{\partial g}{\partial y_1} - \pi_t^{(1)}\right) + \frac{\sigma^2}{2}\tilde{A}_{S_t}^2\frac{\partial^2 g}{\partial y_1^2}\right]\mathrm{d}S_t$$
$$+ \int_0^T\int_{\mathbb{R}}\left(g(e^z\tilde{A}_{S_{t-}}, \lambda_{S_{t-}} + \eta|z|, S_t, U_{S_t}) - g(\tilde{A}_{S_{t-}}, \lambda_{S_{t-}}, S_t, U_{S_t}) - (e^z - 1)\tilde{A}_{S_{t-}}\pi_t^{(1)}\right)\Xi(\mathrm{d}z)\mathrm{d}N_{S_t}$$
$$+ \sum_{0<t\leqslant T}(g(\tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_t}) - g(\tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_{t-}}))$$

In order to shorten the notations, we define

$$H_t^{(1)} := \sigma\tilde{A}_{S_t}\left(\frac{\partial g}{\partial y_1}(\tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_t}) - \pi_t^{(1)}\right),$$

$$H_t^{(2)} := \frac{\partial g}{\partial v}(\tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_t}) + \kappa(\theta - \lambda_{S_t})\frac{\partial g}{\partial y_2}(\tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_t})$$
$$- \tilde{A}_{S_t}\mathbb{E}_{\mathbb{Q}}[e^{\xi} - 1]\lambda_{S_t}\left(\frac{\partial g}{\partial y_1}(\tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_t}) - \pi_t^{(1)}\right) + \frac{\sigma^2}{2}\tilde{A}_{S_t}^2\frac{\partial^2 g}{\partial y_1^2}(\tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_t}),$$

$$X_t^{(3)} := \int_0^t\int_{\mathbb{R}}\Big(g(e^z\tilde{A}_{S_{u-}}, \lambda_{S_{u-}} + \eta|z|, S_u, U_{S_u}) - g(\tilde{A}_{S_{u-}}, \lambda_{S_{u-}}, S_u, U_{S_u})$$
$$- (e^z - 1)\tilde{A}_{S_{u-}}\pi_u^{(1)}\Big)\Xi(\mathrm{d}z)\mathrm{d}N_{S_u}$$

and

$$X_t^{(4)} := \sum_{0<u\leqslant t}(g(\tilde{A}_{S_u}, \lambda_{S_u}, S_u, U_{S_u}) - g(\tilde{A}_{S_u}, \lambda_{S_u}, S_u, U_{S_{u-}})).$$

Moreover we denote

$$X_t^{(1)} := \int_0^t H_{u-}^{(1)}\mathrm{d}W_{S_u} \qquad X_t^{(2)} := \int_0^t H_{u-}^{(2)}\mathrm{d}S_u$$

and $(H_t)_{t\in[0,T]} := (Y_t - Y_0 - \int_0^t \pi_u^{(1)} \mathrm{d}\tilde{A}_{S_u})_{t\geqslant 0}$, which is a martingale. Using the notations we introduced, we find

$$\begin{aligned} H_T &= Y_T - Y_0 - \int_0^T \pi_t^{(1)} \mathrm{d}\tilde{A}_{S_t} \\ &= \int_0^T H_{t-}^{(1)} \mathrm{d}W_{S_t} + \int_0^T H_{t-}^{(2)} \mathrm{d}S_t + X_T^{(3)} + X_T^{(4)} \\ &= \sum_{k=1}^4 X_T^{(k)}. \end{aligned} \tag{79}$$

Since $(H_t)_{t\in[0,T]}$ is a martingale, we have

$$\mathbb{E}_{\mathbb{Q}}\left[\left(Y_T - Y_0 - \int_0^T \pi_t^{(1)} \mathrm{d}\tilde{A}_{S_t}\right)^2\right] = \mathbb{E}_{\mathbb{Q}}[H_T^2] = \mathbb{E}_{\mathbb{Q}}[[H,H]_T],$$

where the second equality follows from Corollary 3 in Protter (2005) (Chapter 2). Combining Equation (79) and the bilinearity of quadratic covariations, it follows that

$$[H,H]_t = \sum_{k=1}^4 \sum_{j=1}^4 [X^{(k)}, X^{(j)}]_t.$$

Note that the process $(X_t^{(k)})_{t\in[0,T]}$ is continuous when $k \in \{1,2\}$ and quadratic pure jump when $k \neq 1$. Indeed, Theorem 29 in Protter (2005) (Chapter 2) implies that

$$[X^{(2)}, X^{(2)}]_t = \int_0^t \left(H_{u-}^{(2)}\right)^2 \mathrm{d}[S,S]_u$$

which is indistinguishable from the null process, as implied by Proposition 0.18. It follows that if $k \neq 1$ and $j \in \{1,2,3,4\}$ then

$$[X^{(k)}, X^{(j)}]_t = [X^{(j)}, X^{(k)}]_t = \sum_{0<u\leqslant t} \Delta X_u^{(k)} \Delta X_u^{(j)}.$$

Since $(X_t^{(k)})_{t\in[0,T]}$ is continuous for $k \in \{1,2\}$, we infer that $([X^{(k)}, X^{(j)}]_t)_{t\in[0,T]}$ is indistinguishable from the null process whenever $k \in \{1,2\}$ and $j \neq 1$. In addition, Theorem 29 in the second chapter of Protter (2005) also entails that

$$[X^{(1)}, X^{(1)}]_t = \int_0^t \left(H_{u-}^{(1)}\right)^2 \mathrm{d}[W\circ S, W\circ S]_u = \int_0^t \left(H_{u-}^{(1)}\right)^2 \mathrm{d}S_u.$$

Next we have

$$\begin{aligned} [X^{(4)}, X^{(4)}]_t = \int_0^t \int_{\mathbb{R}} \Big(& g(e^z \tilde{A}_{S_{u-}}, \lambda_{S_{u-}} + \eta|z|, S_u, U_{S_u}) - g(\tilde{A}_{S_{u-}}, \lambda_{S_{u-}}, S_u, U_{S_u}) \\ & - (e^z - 1)\tilde{A}_{S_{u-}} \pi_u^{(1)} \Big)^2 \chi(\mathrm{d}z, \mathrm{d}S_u), \end{aligned}$$

$$[X^{(4)}, X^{(4)}]_t = \sum_{0<u\leqslant t} (g(\tilde{A}_{S_u}, \lambda_{S_u}, S_u, U_{S_u}) - g(\tilde{A}_{S_u}, \lambda_{S_u}, S_u, U_{S_{u-}}))^2$$

and

$$[X^{(3)}, X^{(4)}]_t = [X^{(4)}, X^{(3)}]_t = 0,$$

which follows from Corollary 0.6. As a consequence,

$$
\begin{aligned}
[H,H]_T &= \int_0^t \left(H_{u-}^{(2)}\right)^2 \mathrm{d}S_u + \sum_{0<u\leqslant t} (g(\tilde{A}_{S_u}, \lambda_{S_u}, S_u, U_{S_u}) - g(\tilde{A}_{S_u}, \lambda_{S_u}, S_u, U_{S_{u-}}))^2 \\
&+ \int_0^t \int_{\mathbb{R}} \bigg(g(e^z \tilde{A}_{S_{u-}}, \lambda_{S_{u-}} + \eta|z|, S_u, U_{S_u}) - g(\tilde{A}_{S_{u-}}, \lambda_{S_{u-}}, S_u, U_{S_u}) \\
&\qquad - (e^z-1)\tilde{A}_{S_{u-}} \pi_u^{(1)} \bigg)^2 \chi(\mathrm{d}z, \mathrm{d}S_u).
\end{aligned}
$$

and thus the strategy $(\pi_t^{(1)})_{t\in[0,T]}$ will be chosen so as to minimize the quantity

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[H_T^2] &= \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \left(\sigma \tilde{A}_{S_{u-}} \left(\frac{\partial g}{\partial y_1}(\tilde{A}_{S_{u-}}, \lambda_{S_{u-}}, S_{u-}, U_{S_{u-}}) - \phi_u\right)\right)^2 \mathrm{d}S_u\right] \\
&+ \mathbb{E}_{\mathbb{Q}}\bigg[\int_0^T \int_{\mathbb{R}} \bigg(g(e^z \tilde{A}_{S_{u-}}, \lambda_{S_{u-}} + \eta|z|, S_u, U_{S_u}) - g(\tilde{A}_{S_{u-}}, \lambda_{S_{u-}}, S_u, U_{S_u}) \\
&\qquad - (e^z-1)\tilde{A}_{S_{u-}} \pi_u^{(1)} \bigg)^2 \nu(\mathrm{d}z) \lambda_{S_u} \mathrm{d}S_u \bigg] \\
&+ \mathbb{E}_{\mathbb{Q}}\left[\sum_{0<u\leqslant T} (g(\tilde{A}_{S_u}, \lambda_{S_u}, S_u, U_{S_u}) - g(\tilde{A}_{S_u}, \lambda_{S_u}, S_u, U_{S_{u-}}))^2\right].
\end{aligned}
$$

Since the third term does not depend on $\pi_u^{(1)}$ and $(S_t)_{t\geqslant 0}$ is nondecreasing (thus $\mathrm{d}S_u \geqslant 0$), the problem amounts to finding

$$
\begin{aligned}
\pi_u^{(1\star)} = \arg\min_{\pi_u^{(1)}} \bigg\{ & \left(\sigma \tilde{A}_{S_{u-}} \left(\frac{\partial g}{\partial y_1}(\tilde{A}_{S_{u-}}, \lambda_{S_{u-}}, S_{u-}, U_{S_{u-}}) - \phi_u\right)\right)^2 \\
&+ \int_{\mathbb{R}} \bigg(g(e^z \tilde{A}_{S_{u-}}, \lambda_{S_{u-}} + \eta|z|, S_u, U_{S_u}) - g(\tilde{A}_{S_{u-}}, \lambda_{S_{u-}}, S_u, U_{S_u}) \\
&\qquad - (e^z-1)\tilde{A}_{S_{u-}} \pi_u^{(1)} \bigg)^2 \nu(\mathrm{d}z) \lambda_{S_u} \bigg\}.
\end{aligned}
$$

It is then easy to check the first and second derivatives with respect to $\pi_u^{(1)}$ to conclude that

$$
\pi_u^{(1\star)} = \frac{\sigma^2 \tilde{A}_{S_{u-}} \frac{\partial g}{\partial y_1} + \lambda_{S_{u-}} \int_{\mathbb{R}} (g(e^z \tilde{A}_{S_{u-}}, \lambda_{S_{u-}} + \eta|z|, S_u, U_{S_u}) - g(\tilde{A}_{S_{u-}}, \lambda_{S_{u-}}, S_u, U_{S_u})(e^z-1)\nu(\mathrm{d}z)}{\tilde{A}_{S_{u-}} \left[\sigma^2 + \lambda_{S_{u-}} \int_{\mathbb{R}} (e^z-1)^2 \nu(\mathrm{d}z)\right]},
$$

as announced. □

The optimal quadratic hedging strategy for a call option comes as a corollary.

Corollary 0.7. *The call price process $(Y_t)_{t\in[0,T]} = (\tilde{C}(t, \tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_t}))_{t\in[0,T]}$ is a martingale that satisfies the following SDE*

$$
\begin{aligned}
\tilde{C}(\tilde{A}_{S_T}, \lambda_{S_T}, S_T, U_{S_T}) &= \tilde{C}(\tilde{A}_0, \lambda_0, 0, 0) + \sigma \int_0^T \tilde{A}_{S_t} \frac{\partial \tilde{C}}{\partial y_1} \mathrm{d}W_{S_t} \\
&+ \int_0^T \left[\frac{\partial \tilde{C}}{\partial v} + \kappa(\theta - \lambda_{S_t}) \frac{\partial \tilde{C}}{\partial y_2} - \tilde{A}_{S_t} \mathbb{E}_{\mathbb{Q}}[e^{\xi} - 1] \lambda_{S_t} \frac{\partial \tilde{C}}{\partial y_1} + \frac{\sigma^2}{2} \tilde{A}_{S_t}^2 \frac{\partial^2 \tilde{C}}{\partial y_1^2}\right] \mathrm{d}S_t \\
&+ \int_0^T \int_{\mathbb{R}} (\tilde{C}(e^z \tilde{A}_{S_{t-}}, \lambda_{S_{t-}} + \eta|z|, S_t, U_{S_t}) - \tilde{C}(\tilde{A}_{S_{t-}}, \lambda_{S_{t-}}, S_t, U_{S_t})) \Xi(\mathrm{d}z) \mathrm{d}N_{S_t} \\
&+ \sum_{0<t\leqslant T} (\tilde{C}(\tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_t}) - \tilde{C}(\tilde{A}_{S_t}, \lambda_{S_t}, S_t, U_{S_{t-}})).
\end{aligned}
$$

Moreover, in the case of a call option, the hedging problem (67) *admits the solution* $\boldsymbol{\pi}^\star = (\pi_t^{(0\star)}, \pi_t^{(1\star)})_{t\geqslant 0}$, *where*

$$\pi_u^{(1\star)} = \frac{\sigma^2 \tilde{A}_{S_{u-}} \frac{\partial \tilde{C}}{\partial y_1} + \lambda_{S_{u-}} \int_{\mathbb{R}} (\tilde{C}(e^z \tilde{A}_{S_{u-}}, \lambda_{S_{u-}} + \eta|z|, S_u, U_{S_u}) - \tilde{C}(\tilde{A}_{S_{u-}}, \lambda_{S_{u-}}, S_u, U_{S_u})(e^z - 1)\nu(\mathrm{d}z)}{\tilde{A}_{S_{u-}} \left[\sigma^2 + \lambda_{S_{u-}} \int_{\mathbb{R}} (e^z - 1)^2 \nu(\mathrm{d}z)\right]}$$

and

$$\pi_u^{(0\star)} = \mathbb{E}_{\mathbb{Q}}[e^{-rS_T}(A_{S_T} - K)_+] - \pi_0^{(1\star)} \tilde{A}_{S_0}.$$

We close this section with a theoretical results about stopping-times.

Lemma 0.14. *For any a.s. finite* $(\mathcal{F}^U_{S_t})$*-stopping-time* τ*, the random variables* U_{S_τ} *and* $U_{S_{\tau-}}$ *are* $(\mathcal{F}^U_{S_t})_{t\geqslant 0}$*-stopping times.*

Proof. To prove that U_{S_τ} is a $(\mathcal{F}^U_{S_t})_{t\geqslant 0}$-stopping time, we need to show that for any $t \geqslant 0$, $\{U_{S_\tau} \leqslant t\} \in \mathcal{F}^U_{S_t}$. We have that

$$\{U_{S_\tau} \leqslant t\} = (\{U_{S_\tau} \leqslant t\} \cap \{\tau \leqslant t\}) \cup \underbrace{(\{U_{S_\tau} \leqslant t\} \cap \{\tau > t\})}_{=\emptyset} \tag{80}$$

where the emptiness of the second set follows from the inequality $U_{S_\tau} \geqslant \tau$. The right-continuous paths of $(U_{S_t})_{t\geqslant 0}$ imply that it is a progressively measurable process, from what it follows that U_{S_τ} is $\mathcal{F}^U_{S_\tau}$-measurable. Note that the two assertions of the previous sentence are respectively Propositions 4.8 and 4.9 in the chapter 1 of Revuz and Yor (2004). The $\mathcal{F}^U_{S_\tau}$-measurability of U_{S_τ} precisely means that $\{U_{S_\tau} \in B\} \cap \{\tau \leqslant t\} \in \mathcal{F}^U_{S_t}$ for all $t \geqslant 0$ and any Borel set $B \in \mathcal{B}_{\mathbb{R}}$. It is then obvious that Equation (80) implies that $\{U_{S_\tau} \leqslant t\} \in \mathcal{F}^U_{S_t}$. We conclude that U_{S_τ} is a $(\mathcal{F}^U_{S_t})_{t\geqslant 0}$-stopping time.

Let us prove now that $U_{S_{\tau-}}$ is a $(\mathcal{F}^U_{S_t})_{t\geqslant 0}$-stopping time. One has

$$\begin{aligned}\{U_{S_{\tau-}} \leqslant t\} &= \left(\{U_{S_{\tau-}} \leqslant t\} \cap \{\tau = 0\}\right) \cup \left(\{U_{S_{\tau-}} \leqslant t\} \cap \{\tau > 0\}\right) \\ &= \{\tau = 0\} \cup \left(\{U_{S_{\tau-}} \leqslant t\} \cap \{\tau > 0\}\right).\end{aligned}$$

As $\{\tau = 0\}$ is in $\mathcal{F}^U_{S_t}$ for any $t \geqslant 0$, it remains to show that it is also the case for $\{U_{S_{\tau-}} \leqslant t\} \cap \{\tau > 0\}$. To this end, we will establish that for any $k \in \mathbb{N}$,

$$\{U_{S_{\tau-}} \leqslant t\} \cap \{\tau > 0\} = \bigcap_{\substack{n\in\mathbb{N} \\ n\geqslant k}} \bigcup_{q\in\mathbb{Q}\cap(0,t]} \left\{q < \tau \leqslant q + \frac{1}{n}\right\} \cap \{U_{S_q} \leqslant t\}. \tag{81}$$

Let ω be a member of the set at the left-hand side of Equation (81) and fix $n \in \mathbb{N}$. Let $q \in ([\tau(\omega) - \frac{1}{2n}] \vee 0, \tau(\omega)) \cap \mathbb{Q}$. Then $\omega \in \left\{q < \tau \leqslant q + \frac{1}{n}\right\}$. Moreover, $q < \tau(\omega)$ implies that $U_{S_q}(\omega) \leqslant U_{S_{\tau-}}(\omega) \leqslant t$ and thus $\omega \in \{U_{S_q} \leqslant t\}$. Finally, note that $U_{S_q}(\omega) \geqslant q$ entails the inequality $q \leqslant t$. This proves the ω is also a member of the second set.

Let ω be a member of the set at the right-hand side of Equation (81). Then for each $n \geqslant k$, we have a $q_n \in (0,t] \cap \mathbb{Q}$ that satisfies $q_n < \tau(\omega) \leqslant q_n + \frac{1}{n}$ and $U_{S_{q_n}}(\omega) \leqslant t$. Up to a subsequence, we obtain that $q_n \uparrow \tau(\omega)$. Therefore,

$$\lim_{n\to+\infty} U_{S_{q_n}}(\omega) = U_{S_{\tau-}}(\omega) \leqslant t.$$

The second inclusion follows.

It remains to observe that Equation (81) establishes that $\{U_{S_{\tau-}} \leqslant t\} \cap \{\tau > 0\} \in \mathcal{F}^U_{S_{t+\frac{1}{k}}}$ for any $k \in \mathbb{N}$. The $\mathcal{F}^U_{S_t}$-measurability of this set is then a consequence of the right-continuity of $(\mathcal{F}^U_{S_t})_{t\geqslant 0}$ (see Lemma 0.11). □

Table 1: Parameters used for the computations.

θ	ρ_+	ρ_-	p	η	κ	σ	λ_0	r	S_0
6.44	30.47	-33.9	0.37	337.08	14.71	0.12	5.62	0.01	100

0.7 Numerical results

This section presents the results of our numerical experiments. These results consist of call prices and associated implied volatilities computed with the help of the method described in Section 0.5. The values of the parameters are from Moraux and Hainaut (2018) and are reported in Table 1.

Figure 2 displays the prices of call option for various strikes, fractional orders α and maturities from one to six months. Note that the case $\alpha = 1$ corresponds to the non-fractional case. All the call prices used in this section are reported in Tables 2 and 3. We observe that a lower fractional order leads to higher call prices for the considered maturities. This observation is reflected on Figure 3, which presents the implied volatilities computed on with those call prices. As a matter of fact, the implied volatilities are always higher for smaller fractional orders. Figure 4 presents the corresponding implied volatility surfaces. From these last two figures, we notice that the fractional model allows to introduce a special feature: the implied volatilities are especially high for the lowest maturities. This seems to indicate that such models correspond to a situation in which most of the movements of the price of the stock occur very shortly. A fractional model thus seems appropriate for use in a period of turbulent financial markets.

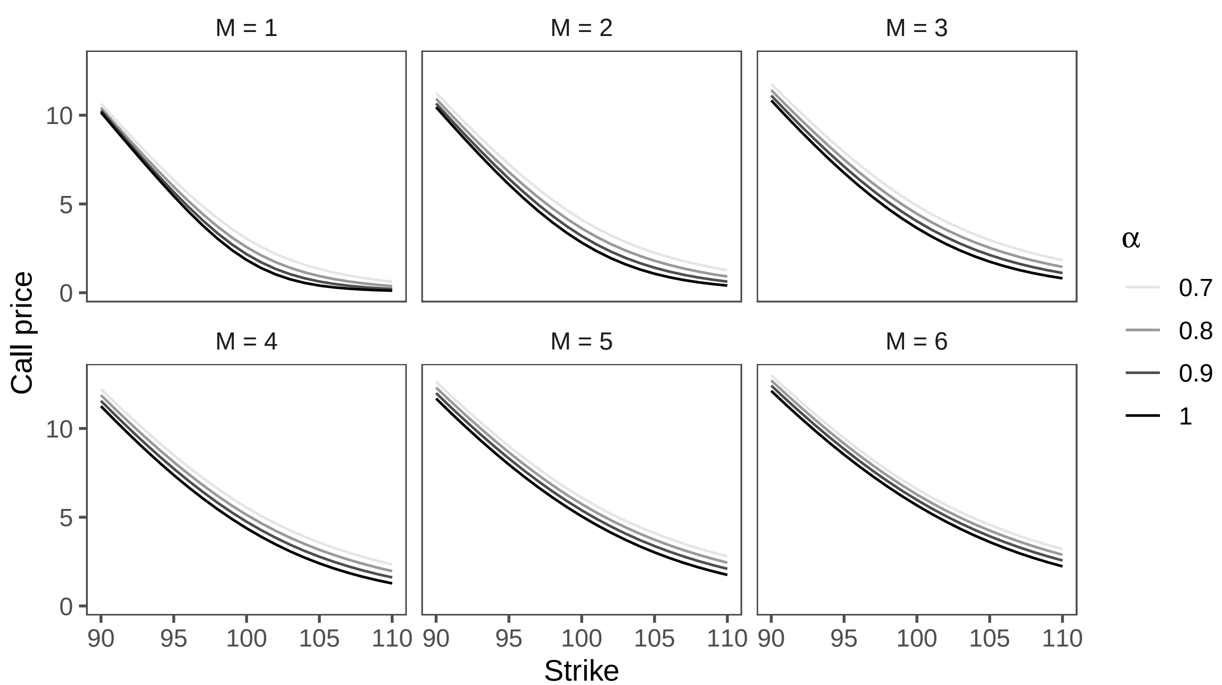

Figure 2: Comparison of the call option prices for various fractional orders α and for the non-fractional case ($\alpha = 1$). M corresponds to the maturity expressed in months.

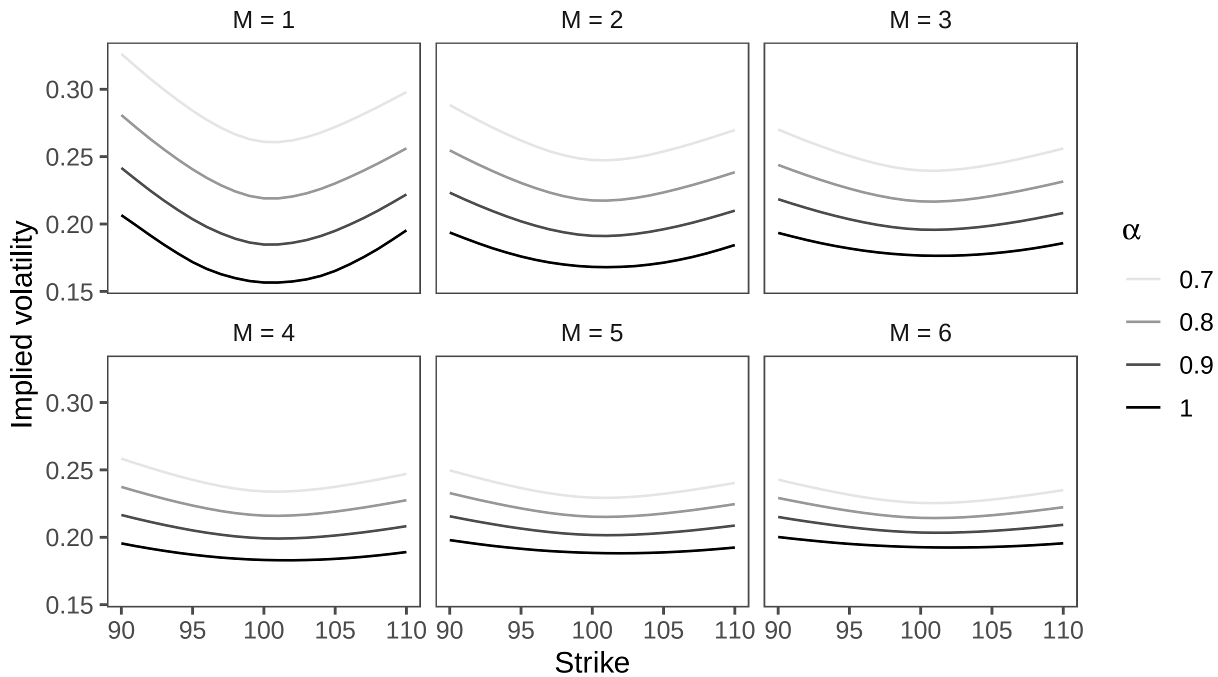

Figure 3: Comparison of the implied volatility smiles for various fractional orders α and for the non-fractional case ($\alpha = 1$). M corresponds to the maturity expressed in months.

Table 2: Call prices for strikes from 90 to 110.

	$\alpha = 0.7$						$\alpha = 0.8$					
	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$	$M = 6$	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$	$M = 6$
90	10.65	11.25	11.76	12.22	12.63	13.00	10.42	10.92	11.42	11.88	12.31	12.71
91	9.75	10.40	10.95	11.43	11.86	12.25	9.49	10.05	10.58	11.07	11.53	11.95
92	8.86	9.57	10.15	10.66	11.11	11.51	8.57	9.20	9.77	10.29	10.77	11.21
93	8.00	8.76	9.38	9.91	10.38	10.80	7.67	8.37	8.99	9.54	10.04	10.50
94	7.16	7.98	8.64	9.19	9.68	10.12	6.80	7.57	8.23	8.81	9.33	9.81
95	6.35	7.23	7.92	8.50	9.00	9.45	5.97	6.80	7.50	8.11	8.65	9.15
96	5.58	6.52	7.24	7.84	8.36	8.82	5.17	6.07	6.80	7.44	8.00	8.51
97	4.86	5.84	6.59	7.21	7.74	8.21	4.42	5.38	6.15	6.80	7.38	7.90
98	4.19	5.22	5.98	6.61	7.16	7.64	3.73	4.74	5.53	6.21	6.80	7.33
99	3.58	4.64	5.42	6.06	6.61	7.10	3.11	4.15	4.96	5.65	6.25	6.79
100	3.04	4.11	4.90	5.55	6.10	6.59	2.56	3.62	4.44	5.13	5.74	6.28
101	2.58	3.64	4.43	5.07	5.63	6.12	2.10	3.15	3.97	4.66	5.27	5.81
102	2.18	3.22	4.00	4.64	5.20	5.69	1.72	2.74	3.55	4.23	4.83	5.37
103	1.85	2.86	3.62	4.25	4.80	5.28	1.40	2.38	3.17	3.84	4.43	4.97
104	1.57	2.53	3.28	3.90	4.43	4.91	1.14	2.07	2.83	3.48	4.07	4.59
105	1.33	2.25	2.97	3.57	4.10	4.57	0.94	1.80	2.53	3.16	3.73	4.25
106	1.14	2.00	2.69	3.28	3.79	4.25	0.77	1.57	2.26	2.87	3.43	3.93
107	0.97	1.78	2.44	3.01	3.51	3.96	0.64	1.36	2.02	2.61	3.14	3.64
108	0.83	1.59	2.22	2.76	3.25	3.69	0.53	1.19	1.81	2.37	2.89	3.37
109	0.72	1.42	2.02	2.54	3.01	3.44	0.44	1.04	1.62	2.16	2.65	3.12
110	0.62	1.27	1.83	2.34	2.79	3.21	0.37	0.91	1.45	1.96	2.44	2.89

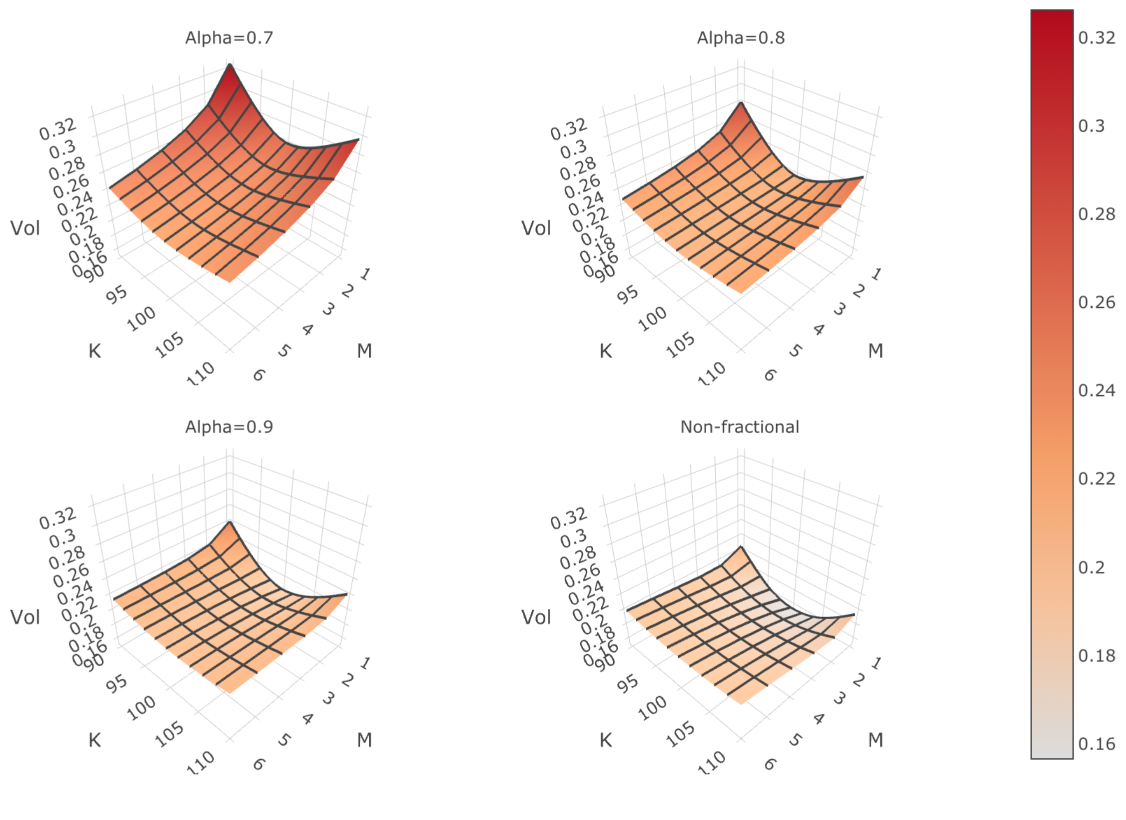

Figure 4: Comparison of the implied volatilities surfaces for various fractional orders α and for the non-fractional case. The M axis corresponds to the maturity expressed in months and the K axis corresponds to the strikes.

Table 3: Call prices for strikes from 90 to 110.

	$\alpha = 0.9$						$\alpha = 1$					
	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$	$M = 6$	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$	$M = 6$
90	10.26	10.66	11.11	11.56	11.99	12.41	10.16	10.45	10.83	11.25	11.69	12.12
91	9.30	9.76	10.25	10.74	11.21	11.65	9.19	9.53	9.96	10.42	10.89	11.35
92	8.36	8.89	9.43	9.95	10.44	10.91	8.22	8.63	9.11	9.62	10.11	10.60
93	7.44	8.03	8.62	9.18	9.70	10.19	7.28	7.75	8.29	8.84	9.37	9.88
94	6.54	7.21	7.85	8.44	8.99	9.50	6.35	6.91	7.50	8.09	8.65	9.19
95	5.67	6.42	7.11	7.73	8.31	8.84	5.44	6.10	6.75	7.37	7.97	8.52
96	4.84	5.67	6.41	7.06	7.66	8.20	4.58	5.33	6.04	6.70	7.31	7.89
97	4.06	4.97	5.74	6.42	7.04	7.60	3.78	4.61	5.37	6.06	6.69	7.29
98	3.35	4.32	5.12	5.82	6.45	7.02	3.05	3.95	4.74	5.46	6.11	6.72
99	2.72	3.73	4.55	5.26	5.90	6.48	2.40	3.36	4.17	4.90	5.56	6.18
100	2.17	3.20	4.03	4.75	5.39	5.97	1.84	2.82	3.64	4.38	5.05	5.67
101	1.71	2.73	3.55	4.27	4.92	5.50	1.39	2.35	3.17	3.90	4.57	5.19
102	1.34	2.32	3.13	3.84	4.48	5.06	1.03	1.94	2.74	3.47	4.13	4.75
103	1.04	1.96	2.75	3.45	4.08	4.65	0.75	1.60	2.37	3.07	3.73	4.34
104	0.81	1.66	2.42	3.09	3.71	4.28	0.55	1.31	2.04	2.72	3.36	3.96
105	0.63	1.41	2.12	2.77	3.37	3.93	0.40	1.07	1.75	2.40	3.02	3.60
106	0.50	1.19	1.86	2.49	3.07	3.61	0.30	0.87	1.50	2.11	2.71	3.28
107	0.39	1.01	1.63	2.23	2.79	3.31	0.23	0.71	1.28	1.86	2.43	2.98
108	0.31	0.86	1.44	2.00	2.53	3.04	0.18	0.59	1.10	1.64	2.18	2.71
109	0.25	0.73	1.26	1.79	2.30	2.79	0.14	0.49	0.94	1.44	1.95	2.46
110	0.21	0.63	1.11	1.61	2.09	2.56	0.11	0.40	0.81	1.27	1.75	2.23

Conclusions

In this article, we proposed a model for small capitalization stocks that are characterized by illiquidity periods. The motionless periods in their prices were modeled through the inverse of an α-stable subordinator. The occurrence of clustered sudden moves in the prices were captured by the use of self-exciting Hawkes processes. In a first section, we have showed that if we add some information about the path of the α-subordinator, its inverse satisfy the Markov property. In the next sections, we have presented a class of risk neutral measures for our model and described a numerical method to compute call option prices. Afterwards, we derived quadratic hedging strategy for contingent claims, and in particular for call options. We concluded this paper by giving the numerical results we obtained when computing call option prices with the method we proposed.

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