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Torsion theories in simplicial groups and homology

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Contents

Introduction

The main objective of this thesis is to apply some known techniques of torsion theories and preradicals to different non-abelian categories, and in particular to the category $Simp(Grp)$ of simplicial groups. Two families of torsion theories in simplicial groups are introduced and they constitute a lattice $\mu(Grp)$ of torsion theories in $Simp(Grp)$. Connections between the lattice $\mu(Grp)$ and the homotopical aspects of $Simp(Grp)$ are studied. For instance, the torsion theories of $\mu(Grp)$ are defined by truncations of the Moore chain complex and, moreover, we show how the homotopy groups of a simplicial group can be computed with the preradicals of the torsion theories in $\mu(Grp)$.

Furthermore, we will have an easier description of the torsion theories of $\mu(Grp)$ when restricted to certain subcategories of $Simp(Grp)$. Restricted to the category of internal groupoids, the torsion theories of $\mu(Grp)$ yield the already known examples of (Abelian groups, equivalence relations) and (Connected groupoids, discrete groupoids). As new examples, we will study the subcategory of simplicial groups $\mathcal{M}_{2\geq}$ of simplicial groups whose Moore complex vanishes for $n > 2$ and the subcategory of Dakin's group T-complexes, as these two examples generalize the torsion theories already studied in internal groupoids in [BG06], [EG10] and [Man15]. In these cases torsion theories can be studied as torsion theories in Conduché's 2-crossed modules and in Ashley's reduced crossed complexes, respectively.

Categorical algebra

Already in the early years of category theory, when abelian categories were introduced to capture the categorical aspects of abelian groups to provide a proper setting to develop homological algebra, the following natural question arose in Mac Lane's article [Lan50]: what is a suitable categorical setting that captures many fundamental properties of the category of groups (and of other categories such as Lie algebras or categories of universal algebras) in the same way as abelian categories do for the category of abelian groups?

A solution to this problem, and our context for this thesis, is given by the notion of a semi-abelian category in the sense of [JMT02]. In order to understand the notion of a semi-abelian category it is useful to recall that abelian categories are characterized by the so-called 'Tierney equation'([Bar71]):

$$\text{(abelian)} = \text{(Barr-exact)} + \text{(additive)}.$$

Here, a regular category is a finitely complete category with coequalizers where each morphism $f : A \to B$ admits a pullback stable 'image' factorization:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\ & f(A) \arrow[ur, "m"'] & \end{tikzcd}$$

where e is a regular epimorphism and m is a monomorphism, and a regular category is called (Barr-)exact if, moreover, internal equivalence relations are effective. The next ingredient of a semi-abelian category is protomodularity in the sense of Bourn ([Bou91]). In a pointed category $\mathbb{X}$, protomodularity is equivalent to the validity of the Split Short Five Lemma in $\mathbb{X}$. In addition, if $\mathbb{X}$ is a pointed, regular and protomodular category then the classical basic results of homological algebra hold in $\mathbb{X}$, such as the snake lemma, the 3×3 lemma, etc. Finally, putting all these properties together we get the definition of a semi-abelian category:

$$\text{(semi-abelian)} = \text{(pointed)} + \text{(exact)} + \text{(protomodular)} + \text{(binary coproducts)}.$$

More generally, we will be interested in normal categories [Jan10], that is regular categories where regular epimorphisms are normal epimorphisms. In the general context of normal categories, torsion theories and preradicals have nice properties.

Torsion theories

Since their introduction by Dickson [Dic66], torsion theories serve to study abelian categories. A torsion theory in a pointed category $\mathbb{X}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ that satisfy the axioms:

TT1 Any morphism $f : T \to F$ with T in $\mathcal{T}$ and F in $\mathcal{F}$ is zero.

TT2 Any object X in $\mathbb{X}$ has a short exact sequence:

$$0 \longrightarrow T(X) \longrightarrow X \longrightarrow F(X) \longrightarrow 0$$

with $T(X)$ in $\mathcal{T}$ and $F(X)$ in $\mathcal{F}$.

In a torsion theory the subcategories $\mathcal{T}$ and $\mathcal{F}$ determine each other uniquely. The subcategory $\mathcal{T}$ is a mono-coreflective subcategory of $\mathbb{X}$ and $\mathcal{F}$ is an epi-reflective subcategory of $\mathbb{X}$:

$$\begin{tikzcd} \mathcal{T} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathbb{X} \arrow[l, bend left] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, bend left] \end{tikzcd}.$$

The first example of a torsion theory is given by torsion and torsions-free abelian groups. In the category Ab of abelian groups we define $\mathcal{T}$ as the subcategory of torsion abelian groups X, the abelian groups where all elements have finite order:

$$\text{for} \quad x \in X, \quad \text{then there is} \quad n \in \mathbb{N} \quad \text{such} \quad nx = 0.$$

And $\mathcal{F}$ is the category of torsion-free abelian groups, abelian groups X where only the trivial element has finite order:

$$\text{if} \quad nx = 0, \quad \text{then} \quad x = 0.$$

From the axiom TT2, each torsion theory defines a 'torsion subobject' $T(X)$ for each object X. This construction is functorial and is a first example of a preradical of $\mathbb{X}$. By a preradical on $\mathbb{X}$ we mean a subfunctor $r : \mathbb{X} \to \mathbb{X}$ of the identity functor on $\mathbb{X}$; this means we have a subobject $\sigma_X : r(X) \to X$ and for a morphism $f : X \to Y$ we have a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ r(X) \arrow[u, "\sigma_X"] \arrow[r, "f'"'] & r(Y) \arrow[u, "\sigma_Y"'] \end{tikzcd}.$$

A preradical r in $\mathbb{X}$ defines two subcategories in $\mathbb{X}$, the r-torsion and r-torsion-free subcategory:

$$\mathcal{T}_r = \{X | r(X) = X\}, \quad \mathcal{F}_r = \{X | r(X) = 0\}.$$

In a normal category $\mathbb{X}$ torsion theories are in bijection with *idempotent radicals*, preradicals such that $rr(X) = r(X)$ and $r(X/r(X)) = 0$. The proofs presented here closely follow those of [BG06] and [CDT06], which study torsion theories in different non-abelian settings, namely homological categories and the so-called $(\mathcal{E}, \mathcal{M})$-normal categories, respectively.

If the category $\mathcal{T}$ is closed under subobjects in $\mathbb{X}$ then the torsion theory is called hereditary. For example, the torsion theory of torsion abelian groups is hereditary. For the category $RMod$ of modules over an associative ring R the torsion theories are characterized by a theorem proved by P. Gabriel that states bijections between:

1. Hereditary torsion theories;
2. Left exact radicals $r : RMod \to RMod$;
3. Gabriel topologies on the ring R;
4. Localizations of $RMod$.

As a generalization, in a locally finitely presentable abelian category $\mathbb{A}$ we have a bijection between:

1. Hereditary torsion theories in $\mathbb{A}$;
2. Left exact radicals in $\mathbb{A}$;
3. Universal closure operators in $\mathbb{A}$;
4. Localizations of $\mathbb{A}$.

By a localization of $\mathbb{A}$ we mean a subcategory $\mathbb{B}$ of $\mathbb{A}$ where the inclusion I has a finite-limit preserving left adjoint L:

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbb{B} \arrow[l, bend left, "I"] \end{tikzcd} .$$

For the case of non-abelian categories, in [BG06] the bijections between hereditary torsion theories, hereditary closure operators and hereditary radicals is established for homological categories. And in [CDT06] torsion theories and radicals are studied in other non-abelian settings. However, the connection with localizations has not been studied yet. Here, we present how to construct a hereditary torsion theory $(\mathcal{T}_L, \mathcal{F}_L)$ from a localization $L \dashv I$ in the context of normal categories.

It is easy to see that if we have torsion theories $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{S}, \mathcal{G})$ in a category $\mathbb{X}$,

$$\mathcal{T} \subset \mathcal{S} \quad \text{if and only if} \quad \mathcal{G} \subset \mathcal{F}.$$

This allows one to introduce an order in the class of torsion theories for a category $\mathbb{X}$, $(\mathcal{T}, \mathcal{F}) \leq (\mathcal{S}, \mathcal{G})$ if $\mathcal{T} \subset \mathcal{S}$ and hence, we have a lattice $\mathbb{X}tors$ of torsion theories of a category $\mathbb{X}$.

It has been a major tool in ring theory to study the lattice $Rtors$ of torsion theories of the category $RMod$ of modules over a R, as well as the sublattice $R_h tors$ of hereditary torsion theories. These lattices have interesting properties, for instance, for any ring R the lattice $R_h tors$ is small (a set) and it has the structure of a frame and, moreover, some rings can be characterized by the lattice $R_h tors$. We will study a particular sublattice $\mu(Grp)$ of torsion theories in simplicial groups.

Simplicial objects and homotopy

Introduced by S. Eilenberg and J. A. Zilber in the 50's [EZ50], a simplicial set X is a functor

$$X : \ \Delta^{op} \longrightarrow Sets$$

where Δ^{op} is the simplicial category. A simplicial set encodes the homotopical properties of 'well-behaved' topological spaces in a combinatorial way. However, it must also be recalled that in order to define a homotopy equivalence relation on a simplicial set X, and hence define the homotopy groups $\pi_n(X)$, X should satisfy the Kan condition. It was proved by D. Kan that the singular simplicial set $S(X)$ of a topological space X satisfies this property.

A simplicial object X in a category $\mathbb{X}$ is a functor $X : \Delta^{op} \to \mathbb{X}$. Simplicial objects in the category of groups, called simplicial groups, appear naturally in topology, since the singular simplicial set of a loop space $\Omega(X)$ (of a pointed topological space X) is a simplicial group. It was proved by Moore that simplicial groups satisfy the Kan condition and that the homotopy groups $\pi_n(X)$ of a simplicial group are isomorphic to the homology groups of the normalized chain complex $M(X)$ of X:

$$\pi_n(X) \cong H_n(M(X)).$$

Moreover, these groups are abelian for $n \geq 1$.

For the case of abelian groups, the Dold-Kan theorem states an equivalence via the normalization functor between simplicial abelian groups and chain complexes in abelian groups,

$$Simp(Ab) \cong chn(Ab)$$

in such a way that simplicial homotopy corresponds to homology.

A breakthrough in order to study the homotopical aspects of simplicial objects in a categorical way was presented by A. Carboni and G.M. Kelly and M.C. Pedicchio in [CKP93]. There, the Kan condition is introduced for simplicial objects in a regular category $\mathbb{X}$ and it was proved that simplicial objects in $\mathbb{X}$ satisfy the Kan condition if and only if $\mathbb{X}$ is a Mal'tsev category. Mal'tsev categories are categories where internal reflexive relation are in fact equivalence relations [CPP92]. These categories extend Mal'tsev varieties of universal algebras, which have been characterized by A. Mal'tsev ([Mal54]) as varieties with a ternary operation $p(x, y, z)$ such that:

$$p(x, y, y) = x \quad \text{and} \quad p(x, x, y) = y.$$

The category of groups is a Mal'tsev category, where such an operation $p(x, y, z)$ is given by $p(x, y, z) = xy^{-1}z$. It is the existence of this operation that makes the proof of Moore possible (see [Bar71]), so simplicial objects in a Mal'tsev category satisfy the Kan condition and this includes simplicial objects in groups, rings, abelian groups, modules over a ring, and others. Semi-abelian categories in particular are Mal'tsev categories.

In [EdL04] the results of Moore for simplicial groups are generalized for simplicial objects in semi-abelian categories. The Moore normalization functor

$$M : \ Simp(\mathbb{X}) \longrightarrow ch(\mathbb{X})$$

is defined as

$$M(X)_n = \bigcap_{i=0}^{n-1} ker(\delta_i)$$

where δ_i are the face operators of the simplicial object X. Then, the nth-homology object of X is defined as the homology object of the normalization chain complex $H_n(M(X))$ and for $n \geq 1$ the $H_n(M(X))$ are abelian objects in $\mathbb{X}$.

Our interest in torsion theories in simplicial groups lies in the following. First, the category of simplicial groups has embedded the category $Grpd(Grp)$ of internal groupoids in groups (via the nerve functor). Examples of torsion theories in internal groupoids have been studied in detail, for example in [BG06], [EG15] and [Man15], so it is natural to look for generalizations of these torsion theories in simplicial groups.

Secondly, as mentioned before there are the homotopical aspects of simplicial groups, so in our work we will give applications of torsion theories to this subject.

Finally, simplicial groups as well as other algebraic categories from algebraic topology, like Dakin's T-complexes or Conduché's 2 -crossed modules (chapter 5), are semi-abelian but they have not been studied in depth from the point of view of categorical algebra.

Structure of the text

Chapter 1 serves as an introduction to the categorical concepts we will be using throughout this thesis. Moreover, it briefly introduces the reader to different aspects of categorical algebra, as we will encounter different kind of categories. The relation between the various contexts that will be mentioned in this thesis can summarised in the following diagram:

$$\begin{tikzcd}
\text{semi-abelian} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{homological} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{Mal'tsev} & \\
& \text{ideal determined} \arrow[r, Rightarrow] & \text{normal} \arrow[r, Rightarrow] & \text{regular}
\end{tikzcd}$$

Chapter 2 studies torsion theories in normal categories. Sections 2.1 and 2.2 introduces preradicals and their connections with torsion theories. In particular we recall the bijection:

$$\{\text{torsion theories in}\,\mathbb{X}\} \cong \{\text{idempotent radicals in }\ \mathbb{X}\}$$

and its restriction

$$\{\text{hereditary torsion theories in}\,\mathbb{X}\} \cong \{\text{hereditary radicals in }\ \mathbb{X}\}.$$

In section 2.3 some basic examples are recalled, some classical abelian examples as well as some more recent developments in non-abelian contexts. Following the observations in 2.3, section 2.4 introduces two theorems for torsion theories in normal categories that are well known for abelian categories but have been somehow overlooked in recent works:

1. Let $\mathbb{X}$ be a normal exact category, if $(\mathcal{T}, \mathcal{F})$ is an hereditary torsion theory then $\mathcal{T}$ is a normal exact category. Moreover, if $\mathbb{X}$ is semi-abelian then $\mathcal{T}$ is semi-abelian.

2. Let $\mathbb{X}$ be a normal category and $L \dashv I$ a localization of $\mathbb{X}$, then the subcategory
$$\mathcal{T}_L = \{X | L(X) \cong 0\}$$
is a hereditary torsion subcategory of $\mathbb{X}$.

As an application of the second theorem above, following [Pop73] an example of a hereditary torsion theory is given in the semi-abelian category of sheaves of groups over a topological space given by the sheafification/localization of presheaves.

In chapter 3 we study the category $Grpd(\mathbb{X})$ of internal groupoids in a normal Mal'tsev category $\mathbb{X}$ and we recall two torsion theories in $Grpd(\mathbb{X})$ already studied in [BG06] and [EG10]:

1. $(Ab(\mathbb{X}), Eq(\mathbb{X}))$ given by internal abelian objects in $\mathbb{X}$ and internal equivalence relations in $\mathbb{X}$;

2. and $(Conn(Grpd(\mathbb{X})), Dis(\mathbb{X}))$ given by connected groupoids in $\mathbb{X}$ and discrete groupoids.

If $\mathbb{X}$ is the category Grp of groups, it is known that internal groupoids are equivalent to Whitehead's crossed modules ([Whi41]). A crossed module is a morphism $\partial : A \to B$ with an action of B on A, written ${}^{b}a$ for $a \in A$ and $b \in B$, satisfying the identities:

$$\partial({}^{b}a) = b\partial(a)b^{-1} \quad \text{and} \quad {}^{\partial(a)}a' = aa'a^{-1}.$$

so the previous torsion theories in internal groupoids correspond to torsion theories in crossed modules:

1. $(Ab(\mathbb{X}), Eq(\mathbb{X})) = (Ab, NMono)$ where the objects of Ab are crossed modules of the form $A \to 0$ with A an abelian group, and the objects of $NMono$ are inclusions of normal subgroups $i : N \to G$.

2. $(Conn(Grpd(\mathbb{X})), Dis(\mathbb{X})) = (CExt, Dis)$ where $CExt$ is the category of central extensions in groups and the objects Dis are discrete crossed modules, i.e. crossed modules of the form $0 \to G$ for a group G.

Chapter 4 is the core of this work. Since we can consider $Grpd(\mathbb{X})$ as a full subcategory of $Simp(\mathbb{X})$ under the nerve functor, our first objective is to generalise the torsion theories in internal groupoids to a family of torsion theories in $Simp(\mathbb{X})$. As a first step we consider the category $pch(\mathbb{X})$ of proper chain complexes in an ideal determined $\mathbb{X}$ so that we obtain a lattice $\mathcal{COT}$ of torsion theories given by cotruncations in $pch(\mathbb{X})$

$$\mathcal{COK}_n \quad \text{and} \quad \mathcal{KER}_n$$

for each $n \geq 0$. We then define the torsion theories $\mu_{n\geq}$ and $\mu_{\geq n}$ in simplicial groups. More precisely, $\mu_{n\geq}$ has as torsion-free category the category of simplicial groups with trivial Moore complex for $i > n$, and similarly, $\mu_{\geq n}$ has as torsion category the category of simplicial groups with trivial Moore complex for $i < n$. These constitute a lattice $\mu(Grp)$ of torsion theories. The lattice $\mu(Grp)$ and $\mathcal{COT}$ are related via the Moore normalization:

$$\begin{array}{ccccccccc}
\mu(Grp) = & \ldots \leq & \mu_{1\geq} & \leq & \mu_{\geq 1} & \leq & \mu_{0\geq} & \leq & Simp(Grp) \\
\big\downarrow{\scriptstyle M} & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{COT}(pch(Grp)_0) = & \ldots \leq & \mathcal{COK}_2 & \leq & \mathcal{KER}_1 & \leq & \mathcal{COK}_1 & \leq & pch(Grp)_{\geq 0}
\end{array}$$

so that each torsion/torsion-free subcategory of $\mu_{n\geq}$ and $\mu_{\geq n}$ is mapped into the torsion/free subcategory of $\mathcal{COK}_n$ and $\mathcal{KER}_n$, accordingly.

The lattice $\mu(Grp)$ defines a lattice of idempotent radicals of $Simp(X)$ (denoted by $\mu_{n\geq}$ and $\mu_{\geq n}$) and, hence for a simplicial group X we have a sublattice of torsion subobjects $\mu_{n\geq}(X)$ and $\mu_{\geq n}(X)$. We can study homotopical properties of X with these subobjects, for example the quotient

$$\Pi^{\geq n+1}_{n+1\geq}(X) = \mu_{\geq n+1}(X)/\mu_{n+1\geq}(X)$$

is isomorphic to $K(\pi_{n+1}(X), n+1)$ the $(n+1)$-st Eilenberg-Mac Lane simplicial group of the $(n+1)$-st homotopy group of X.

Chapter 5 brings more applications of the lattice $\mu(Grp)$ by restricting it to certain subcategories of $Simp(Grp)$. In the first place we consider the subcategory $\mathcal{M}_{n\geq}$ of simplicial groups with trivial Moore complex for $i > n$. For the case $n = 2$, $\mathcal{M}_{2\geq}$ is equivalent to the category of Conduché's 2-crossed modules. Following the examples in chapter 3, we observe that the subcategories of discrete simplicial groups Dis, internal equivalence relations $Eq(Grp)$ and internal groupoids $Grpd(Grp)$ are still torsion-free subcategories of $\mathcal{M}_{n\geq}$. And similarly, the category Ab of abelian groups and categories of special kinds of central extensions are torsion subcategories of $\mathcal{M}_{n\geq}$.

A second example, we study the subcategory of Dakin's group T-complexes, simplicial groups where there is a canonical filler of horns. This category is equivalent to the category $Crs(Grp)$ of Ashley's reduced crossed complexes. A reduced crossed complex is a proper chain complex in the categoy of groups:

$$M = \ \ldots \longrightarrow M_n \xrightarrow{\delta_n} M_{n-1} \longrightarrow \ldots \qquad M_2 \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} M_0$$

with actions $M_0 \to Aut(M_i)$ where M_n is an abelian group for $n > 2$, the morphism $\delta_1 : M_1 \to M_0$ is a crossed modules and the actions $\delta_1(M_1) \to Aut(M_i)$ are trivial. Both the subcategories $\mathcal{M}_{n\geq}$ and $Crs(Grp)$ naturally extend the examples for internal groupoids.

As a second kind of application, we study torsion torsion-free categories or TTF-categories for short. In an abelian category $\mathbb{X}$, a subcategory $\mathcal{T}$ of $\mathbb{X}$ is a TTF-subcategory if there are subcategories $\mathcal{C}$ and $\mathcal{F}$ such that $(\mathcal{C}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{F})$ are torsion theories, the triplet $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is called a TTF-theory. Semi-abelian examples of TTF-theories in $chn(Grp)$ are given in chapter 4:

$$(Ker(\mathbf{cot}_{n-1}), ch(\mathbb{X})_{n-1\geq}, ch(\mathbb{X})_{\geq n}), \quad (ch(\mathbb{X})_{n-1\geq}, ch(\mathbb{X})_{\geq n}, \mathcal{F}_{\mathbf{tr}_{n-1}}).$$

However, these examples cannot be generalized to simplicial groups, but the triplets

$$(CExt, Dis, Ab)$$

in $\mathbb{X}Mod$ and

$$(Ker(\mathbf{cot}_{n-1}), Crs(Grp)_{n-1\geq}, Crs(Grp)_{\geq n})\,.$$

in $Crs(Grp)$ behave like TTF-theories in a weak sense. This means that they are triplets $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ of subcategories such that:

1. the pair $(\mathcal{C}, \mathcal{T})$ is a torsion theory in the usual sense.

2. the torsion-free category $\mathcal{T}$ is mono-coreflective (but not normal mono-coreflective).

3. the pair $(\mathcal{T}, \mathcal{F})$ satisfies axiom TT1 of a torsion theory.

4. the pair $(\mathcal{T}, \mathcal{F})$ satifies axiom TT2 of a torsion theory only for a class $\mathcal{E}$ of objects i.e. for X there is a short exact sequence:

$$0 \longrightarrow T(X) \longrightarrow X \longrightarrow F(X) \longrightarrow 0$$

with $T(X)$ in $\mathcal{T}$ and $F(X)$ in $\mathcal{F}$ if and only if X belongs to $\mathcal{E}$.

Chapter 1

Categories

In this first chapter, we will introduce the categorical foundations and the basic terminology we will be using.

1.1 Morphisms

Definition 1.1.1. An object 0 in a category $\mathbb{X}$ is called a zero object if it is both initial and terminal, i.e. for every object A there is exactly one morphism $\alpha_A : 0 \to A$ and $\tau_A : A \to 0$. We will write $\alpha_A \tau_A = 0_A$ (or only 0 if there is no confusion), and we say that a morphism $f : A \to B$ is a zero morphism if f factors through 0, i.e. the diagram commutes

$$\begin{tikzcd} & 0 \arrow[dr, "\alpha_B"] & \\ A \arrow[ur, "\tau_B"] \arrow[rr, "f"'] & & B \end{tikzcd}$$

A category is called *pointed* if it has a zero object (necessarily unique up to isomorphism).

1.1.2. In a pointed category $\mathbb{X}$, provided that they exist, the kernel of a morphism $f : A \to B$ (resp. cokernel) is defined as the morphism $k(f) : ker(f) \to A$, (resp. $c(f) : B \to cok(f)$) where:

$$\begin{tikzcd} ker(f) \arrow[r, "k(f)"] \arrow[d] & A \arrow[d, "f"] \\ 0 \arrow[r, "\alpha_B"'] & B \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\tau_A"'] & B \arrow[d, "c(f)"] \\ 0 \arrow[r] & cok(f) \end{tikzcd}$$

is a pullback, respectively a pushout.

Definition 1.1.3. A morphism $f : A \to B$ in $\mathbb{X}$ is called:

- *monomorphism*: if for every morphisms $g, h : C \to A$ such that $fg = fh$ then $g = h$;
- *regular monomorphism* if it is the equalizer of a pair of morphisms

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C \ ;$$

- *normal monomorphism* if it is the kernel of a morphism in $\mathbb{X}$;
- *split monomorphism* or a section if there is a morphism $p : B \to A$ such that $pf = 1_A$.

Dually, a morphism $f : A \to B$ is called (regular/normal/split) epimorphism if it satisfies the dual condition.

Monomorphisms capture the categorical aspect of an "injective morphism" in a category. In the category of sets and functions *Sets* they all coincide (monomophisms, regular and split) as injective functions but in other algebraic categories they may all be very different and in fact may not even correspond to injective functions. For instance, this is the case of the category of abelian divisible groups *Div*, where an abelian group X is divisible if it has the property:

$$\text{for all } x \in X,\ n \in \mathbb{N}_+ \Rightarrow \text{ there is } y \in X \text{ such that } x = ny.$$

The morphism $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ in *Div* is a monomorphism but is not injective.

Another basic discrepancy among monomorphisms is that in abelian categories such as *Ab* the category of abelian groups or $RMod$ of modules over a ring R all monomorphisms are normal, i.e. kernel of some morphism. However, in *Grp* the monomorphisms are injective group morphisms $f : G \to H$ but they are only normal if the image $f(G)$ is a normal subgroup of H, i.e. closed under conjugation in H.

In general, in a category $\mathbb{X}$ we have:

$$\text{split monomorphism} \Rightarrow \text{regular monomorphism} \Rightarrow \text{monomorphism}$$

and, in a pointed category $\mathbb{X}$,

$$\text{normal monomorphism} \Rightarrow \text{regular monomorphism} \Rightarrow \text{monomorphism.}$$

To see this, notice that given a section $s : A \to B$ of $p : B \to A$, s is the equalizer of the arrows $B \overset{1_B}{\underset{sp}{\rightrightarrows}} B$, and that a kernel of a morphism $f : A \to B$ is actually the equalizer of $A \overset{f}{\underset{0}{\rightrightarrows}} B$. However, a split monomorphism may fail to be a normal one: for instance in groups, the diagonal morphism $\Delta : G \to G \times G$, $\Delta(g) = (g, g)$ may not have a normal image:

$$(a, b)(g, g)(a^{-1}, b^{-1}) = (aga^{-1}, bgb^{-1})$$

for a, b, g in G. This is the case if and only if G is abelian.

The distinction between the different types of epimorphisms plays a central part for the study of exactness properties of a category. For the category *Sets* of sets, the epimorphisms are exactly surjective functions and they also coincide with regular and, as a consequence of the Axiom of Choice, with split epimorphisms. In contrast, in the category of rings with unit, *Rings*, the inclusion $\mathbb{Z} \to \mathbb{Q}$ is a epimorphism but is not surjective. However, in varieties in universal algebra, like *Rings* and monoids, *Mon*, regular epimorphisms are exactly surjective morphisms. Regular epimorphisms are more adequate to study quotients.

1.1.4. For an object A of $\mathbb{X}$ and two morphisms $m : M \to A$ and $n : N \to A$ such that m factors through n, i.e. there is $t : M \to N$ with $m = nt$, we write $m \leq n$:

$$\begin{tikzcd} N \arrow[dr, "n"] & \\ & A \\ M \arrow[uu, "t"] \arrow[ur, "m"'] & \end{tikzcd}$$

A *subobject* of A is an equivalence class of momorphisms, where $n \equiv m$ if and only if $n \leq m$ and $m \leq n$. We write $sub(A)$ for the order category of subobjects over A. Similarly, a *quotient* of A is an equivalence class of regular epimorphisms with domain A, where for two regular epimorphisms $p : A \to Q$ and $q : A \to Q$, $p \equiv q$ if each one factors through the other one.

1.2 Relations and Regular categories

Regular and (Barr) exact categories allows us to have factorization of morphisms with subobjects and quotients. They have played a central part in category theory as abelian categories, elementary toposes and varieties of universal algebras are exact.

In order to study regular categories we must introduce internal relations and their relation with regular epimorphisms. Their connection is better understood if we recall that in varieties in universal algebra, quotients $p : X \to Y$ are in bijection with congruences on X, equivalence relations compatible with the operations of the algebra. For example, in the case of groups, if $H \leq G$ is a subgroup we can define a relation in G,

$$x \equiv y \quad \text{if and only if} \quad xy^{-1} \in H,$$

this gives an equivalence relation in *Sets* but not in *Grp*; i.e.

$$R = \{(x, y) \in G \times G \mid x \equiv y\}$$

is a subgroup of $G \times G$ if and only if H is a normal subgroup in G. And this allows the quotient set G/H to have a well defined group structure. So in this case there is a bijection of quotients, congruences and normal monomorphisms. In more general categories, we will work with internal relations instead of congruences.

Regular categories are everywhere in category theory, we recommend [Joh02] and [Bor94] for a more in depth treatment.

1.2.1. In a finitely complete category $\mathbb{X}$, a *relation* R on a object X is a subobject $r = (r_0, r_1) : R \to X \times X$:

$$\begin{tikzcd} & X \times X \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\ R \arrow[ur, "r"] \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_0"'] & X . \end{tikzcd}$$

A relation is called *reflexive* if there is a morphism σ such that the diagram commutes:

$$\begin{tikzcd} & R \arrow[d, "r"] \\ X \arrow[ur, "\sigma"] \arrow[r, "\Delta_X = (1_X, 1_X)"'] & X \times X . \end{tikzcd}$$

A relation is called *symmetric* if there is a morphism i such that the diagram commutes:

$$\begin{tikzcd} X \times X \arrow[r, "t = (p_2, p_1)"] & X \times X \\ R \arrow[u, "r"] \arrow[r, "i"'] & R . \arrow[u, "r"'] \end{tikzcd}$$

A relation is called *transitive* if there is a morphism $m : R_2 \to R$ such that

$$\begin{tikzcd} R_2 \arrow[r, shift left=2, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_0"'] & R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_0"'] & X \end{tikzcd}$$

satisfies $r_0 m = r_0 \pi_0$ and $r_1 m = r_1 \pi_2$ where R_2 is the pullback:

$$\begin{tikzcd} R_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_0"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_0"'] & X\,. \end{tikzcd}$$

A relation is an *equivalence relation* if it is reflexive, symmetric and transitive.

1.2.2. For a morphism $f : A \to B$ we can construct an equivalence relation on A in $\mathbb{X}$, the kernel pair of f:

$$\begin{tikzcd} Eq_2[f] \arrow[r, shift left=2, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_0"'] & Eq[f] \arrow[loop above, "i"] \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] & A \arrow[l, "\sigma" description] \arrow[r, "f"] & B \end{tikzcd} \tag{1.1}$$

where $Eq[f]$ is the pullback of f along itself. In *Sets* this is the equivalence relation on A saying that $x \equiv y$ if and only if $f(x) = f(y)$ for a function $f : A \to B$.

A relation R is called *effective* if it is the kernel pair of a morphism f in $\mathbb{X}$, $Eq[f] = R$.

1.2.3. We write $Eq(\mathbb{X})$ for the category of equivalence relations and for an object X, $Eq_X(X)$ for the category of equivalence relations over X. Since $r : R \to X \times X$ in $Eq_X(\mathbb{X})$ is reflexive we have $\Delta_X \leq r$, so $Rel(X)$ have a initial (or bottom) object $\Delta_X = (1_X, 1_X) : X \to X \times X$. Moreover, there is terminal object, the largest equivalence on X, $\nabla_X = 1 : X \times X \to X \times X$:

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "\Delta_X"'] & R \arrow[r, "r"] \arrow[d, "r"] & X \times X \arrow[dl, "\nabla_X = 1_{X \times X}"] \\ & X \times X & \end{tikzcd}$$

and composing with projections $p_0, p_1 : X \times X \to X$, this yields a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "1_X"] & R \arrow[r, "(r_0, r_1)"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & X \times X \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\ X \arrow[r, "1"'] & X \arrow[r, "1"'] & X \end{tikzcd}$$

Both the equivalence relations Δ_X and ∇_X are effective. The equivalence relation $\Delta_X = Eq[1_X]$ is called the discrete relation on X, and $\nabla_X = Eq[\tau_X]$ is called the indiscrete relation on X.

Monomorphisms can be characterized by their kernel pair:

Lemma 1.2.4. Consider a morphism f and its kernel pair $Eq[f]$, as in 1.1. The following are equivalent for f:

1. f is a monomorphism.

2. the morphisms δ_0, δ_1 are equal.

3. δ_0 or δ_1 is an isomorphism.

4. σ is an isomorphism.

It is also useful to remember that for a morphism f and the pullback of f along any morphism p in $\mathbb{X}$:

$$\begin{tikzcd}
ker(p^*(f)) \arrow[r, "\cong"] \arrow[d] & ker(f) \arrow[d] \\
P \arrow[r] \arrow[d, "p^*(f)"'] & A \arrow[d, "f"] \\
Y \arrow[r, "p"'] & B
\end{tikzcd}$$

when have $ker(p^*(f)) \cong ker(f)$.

Definition 1.2.5. A finitely complete category $\mathbb{X}$ is *regular* if:

- coequalizers of kernel pairs exist in $\mathbb{X}$;

- regular epimorphisms are pullback stable, that is:

for any pullback

$$\begin{tikzcd}
C \times_B A \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "f"] \\
C \arrow[r, "g"'] & B
\end{tikzcd}$$

of an regular epimorphim f along a morphism g, p_1 is a regular epimorphism.

A regular category is called *exact* (in the sense of Barr [Bar71]) if all equivalence relations are effective.

1.2.6. The category *Sets* of sets is exact. It is well-know that abelian categories like *Ab*, *Rmod* and Grothendieck categories are exact ([Bor94]). Also algebras,

Alg_T, for an algebraic theory T, in the sense of Lawvere, are exact. These include varieties of universal algebras (see [ARV10]).

The category $\mathbb{F}$ of torsion-free abelian groups, is given by the abelian groups where only the trivial element has a finite order. Then $\mathbb{F}$ is regular since it is a normal epireflective subcategory of Ab (section 1.4). But it is not exact; in $\mathbb{Z}$ we can define the equivalence relation R as

$$x \simeq_R y \quad \text{if and only if} \quad x - y = 2k \text{ for some } k \in \mathbb{Z}.$$

This equivalence relation is not effective. If R were effective, in particular, $R \rightrightarrows \mathbb{Z}$ it should be the equivalence of its coequalizer. However, in the quotient $\mathbb{Z} \to \mathbb{Z}/\simeq_R$ we have that $[x] + [x] = [0]$ for all $x \in \mathbb{Z}$. Since $\mathbb{Z}/\simeq_R$ is torsion-free, it must be trivial. But the kernel pair of $\mathbb{Z} \to 0$ is the indiscrete relation $\nabla_{\mathbb{Z}} = \mathbb{Z} \times \mathbb{Z} \rightrightarrows \mathbb{Z}$.

The category Top of topological spaces, is not regular, as regular epimorphisms are not pullback stable. However, the category $Grp(Top)$ of topological groups and Haussdorf groups, $Grp(Haus)$, are regular, but the category $Grp(Cmp)$ of compact Haussdorf groups is exact, this has been studied in [BC05].

Theorem 1.2.7. ([Bar71]) Let $\mathbb{X}$ be a regular category. Then $\mathbb{X}$ admits (regular epi, mono)-factorization, i.e. any arrow $f : A \to B$ has a factorization $f = mp$ with p a regular epimorphism and m a monomorphism. Moreover, these factorizations are pullback stable and for a morphism f the factorization is necessarily unique up isomorphism.

Proof. Let $f : A \to B$ an arrow in $\mathbb{X}$ and consider the kernel pair $Eq[f]$:

$$\begin{tikzcd} Eq[f] \arrow[r, shift left, "\delta_1"] \arrow[r, shift right, "\delta_0"'] & A \arrow[r, "f"] \arrow[dr, "p"'] & B \\ & & I \arrow[u, dotted, "m"'] \end{tikzcd}$$

Then p is the coequalizer of the kernel pair and m is given by the universal property of p. We will show that m is a monomorphism using 1.2.4. Consider the diagram

$$\begin{tikzcd}
Eq[f] \arrow[r, "b"] \arrow[d, "a"] & Eq[m] \times_I A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & A \arrow[d, "p"] \\
A \times_I Eq[m] \arrow[r, "\phi_2"] \arrow[d, "\phi_1"] & Eq[m] \arrow[r, "p_2"] \arrow[d, "p_1"] & I \arrow[d, "m"] \\
A \arrow[r, "p"] & I \arrow[r, "m"] & B
\end{tikzcd}$$

where each square is a pullback. The whole square is a pullback and we can assume $\delta_0 = \phi_1 a$ and $\delta_1 = \pi_2 b$. The arrow $\phi_2 a = \pi_1 b$ is an epimorphism as a consequence of pullback stability of regular epimorphisms.

Now, we have

$$p_1 \phi_2 a = p \phi_1 a = p \delta_0 = p \delta_1 = q \pi_2 b = p_2 \pi_1 b = p_2 \phi_2 a$$

and since $\phi_2 a$ is epic then $p_1 = p_2$ and finally m is a monomorphism.

The uniqueness of the factorization follows from the fact that regular epimorphisms have the property that for a square

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "a"'] & B \arrow[d, "b"] \arrow[dl, dotted, "t"'] \\
C \arrow[r, "m"'] & D
\end{tikzcd}$$

with p a regular epimorphism and m a monomorphism there exists a unique t such that $tp = a$ and $mt = b$. $\square$

Using this property we can give an alternative definition of a regular category.

Corollary 1.2.8. (See [BG04]) Let $\mathbb{X}$ be a finitely complete category. Then $\mathbb{X}$ is a regular category if and only if any arrow admits a pullback stable (regular epi, mono)-factorization.

1.2.9. Consider a morphism $f : A \to B$ and its (regular epi, mono)-factorization as in 1.2.7

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads, "e_f"'] & & B \\
& f(A) \arrow[ur, "m_f"'] &
\end{tikzcd}$$

The *image* of f is the subobject $m_f : f(A) \to B$.

We introduce some useful properties of regular epimorphisms, the proofs can be found in [BG04].

Proposition 1.2.10. Let $\mathbb{X}$ be a regular category. Then:

- if f is a regular epimorphism and a monomorphism then it is an isomorphism.
- if gf is a regular epimorphism then g is a regular epimorphism.
- if f and g are regular composable epimorphisms then gf is a regular epimorphism.
- if $f : A \to B$ and $g : X \to Y$ are regular epimorphisms then $f \times g : A \times X \to B \times Y$ is also a regular epimorphism.

1.3 Normal categories and exact sequences

As mentioned before, in a pointed category $\mathbb{X}$, normal and regular epimorphisms might be different. Since we will often use short exact sequences it will be useful to work with normal categories instead of only regular ones.

Normal categories were introduced in [Jan10], see also [CDT06] and [BJ09].

Definition 1.3.1. A pointed regular category $\mathbb{X}$ is *normal* if every regular epimorphism is normal. Equivalently, every morphism admits a pullback stable (normal epi, mono)-factorization.

A sequence:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is *exact* at Y if both

1. m_f is kernel of g
2. e_g is a cokernel of f.

where (e_f, m_f) and (e_g, m_g) are the regular epi-mono factorizations of f and g. Consider

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e_f"'] & & Y \arrow[rr, "g"] \arrow[dr, "e_g"'] & & Z \\ & f(X) \arrow[ur, "m_f"'] & & g(Y) \arrow[ur, "m_g"'] & \end{tikzcd}$$

notice that $m_f = ker(g)$ if and only if $m_f = ker(e_g)$. When all regular epimorphisms are normal, then condition 2) follows from 1); if e_g is a normal epimorphism, it is a cokernel of some morphism but, in particular, it is also a cokernel of its kernel m_f. So, for a normal category, we can define:

Definition 1.3.2. In a normal category $\mathbb{X}$ a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact at Y if m_f is a kernel of g.

With this definition we recall some natural properties that will hold in any normal category, like the category Grp of groups.

Lemma 1.3.3. ([Jan10]) For a morphism $f : X \to Y$ in a normal category the following conditions are equivalent:

1. f is regular epimorphism.
2. The sequence
$$X \xrightarrow{f} Y \longrightarrow 0$$
is exact a Y.

Proof. The kernel of $\tau_Y : Y \to 0$ is given by 1_Y and it is equal to m_f if and only if f is a regular epimophism. □

Proposition 1.3.4. ([Jan10]) For a normal category $\mathbb{X}$ with pullbacks the following conditions are equivalent:

1. Any morphism having a trivial kernel is a monomorphism;
2. Any regular epimorphism having a trivial kernel is an isomorphism;
3. Any split epimorphism having a trivial kernel is an isomorphism.

Proof. The implications 1) ⇒ 2) ⇒ 3) are trivial. To prove 3) ⇒ 1) consider $f : X \to Y$ a morphism with trivial kernel and the pullback of f along itself, i.e. the kernel pair of f:

$$\begin{tikzcd} Eq[f] \arrow[r, "\delta_0"] \arrow[d, "\delta_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y . \end{tikzcd}$$

The morphism δ_0 also has trivial kernel and it is a split epimorphism (since $Eq[f]$ is a reflexive relation), so δ_0 is an isomorphism. Similarly δ_1 is isomorphism. This implies that f is a monomorphism by lemma 1.2.4. □

Remark that since a cokernel is always the cokernel of its kernel, if we take a cokernel of the kernel of $f : X \to Y$ with trivial kernel then the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & Y \end{array}$$

is both a pullback and a pushout, so f is an isomorphism.

Corollary 1.3.5. ([Jan10]) For a normal category $\mathbb{X}$ the following conditions are satisfied:

1. Any morphism in $\mathbb{X}$ having a trivial kernel is a monomorphism.
2. A sequence
$$0 \longrightarrow X \xrightarrow{f} Y$$
is exact at X if and only if f is a monomorphism.
3. A sequence
$$0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0$$
is exact at X and Y if and only if f is an isomorphism.

1.3.6. For a morphism $f : X \to Y$ in a normal category $\mathbb{X}$ we have two distinguished short sequences given by the image factorization. Firstly, the sequence

$$0 \longrightarrow ker(f) \xrightarrow{k(f)} X \xrightarrow{e_f} f(X) \longrightarrow 0$$

is always exact. For a normal suboject $k : K \to X$ we will write the object representing the cokernel of k as X/K. Similarly, we have a sequence

$$f(X) \xrightarrow{m_f} Y \xrightarrow{q(f)} cok(f) \longrightarrow 0 \tag{1.2}$$

but this may not be exact at $f(X)$. This is expected even in the category of groups, *Grp*, just notice that for a (not normal) subgroup, $H \leq G$ the cokernel exists and is given by making quotient with $N_G(H)$, the smallest normal subgroup of G containing H (we are implicity using that the cokernel of a subobject exists in any normal category).

A morphism $f : X \to Y$ is called *proper* if the monomorphism m_f is normal, and this happens if and only if the short exact sequence (1.2) is exact.

We introduce two important properties of normal categories:

Lemma 1.3.7. ([BJ09]). In a normal category $\mathbb{X}$, pullbacks reflect monomorphisms, i.e. in a pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "p^*(f)"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

f is a monomorphism whenever $p^*(f)$ is a monomorphism.

Proof. Since a monomorphism is characterized by the property of having a trivial kernel, just consider the diagram:

$$\begin{tikzcd} ker(p^*(f)) \arrow[r] \arrow[d, "\cong"] & P \arrow[r, "p^*(f)"] \arrow[d] & E \arrow[d, "p"] \\ ker(f) \arrow[r] & A \arrow[r, "f"] & B \end{tikzcd}$$

Since $p^*(f)$ is monic, $ker(p^*(f))$ is zero and so is $ker(f)$. □

In normal categories the Noether's third isomorphism theorem holds:

Lemma 1.3.8. ([EG13]). Let $\mathbb{X}$ be a normal category. Then, given two normal subobjects $k : K \to A$ and $l : L \to A$ such that $k \leq l$, i.e. k factors through l, then there is an isomorphism

$$A/L \cong \frac{A/K}{L/K}$$

Proof. Consider the diagram given by the exact sequences of k, l and $i : K \to L$:

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "id"] & L \arrow[r, "q_i"] \arrow[d, "l"] & L/K \arrow[r] \arrow[d, dotted, "l'"] & 0 \\ 0 \arrow[r] & K \arrow[r, "k"] & A \arrow[r, "q_k"] \arrow[d, "q_l"] & A/K \arrow[r] \arrow[d, dotted, "q'"] & 0 \\ & & A/L \arrow[r, "1"] \arrow[d] & A/L & \\ & & 0 & & \end{tikzcd}$$

Taking $k = li$ the arrow l' is induced by the universal property of q_i and, respectively, q' is induced by q_k. First, l' is a monomorphism since if we take the pullback:

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_0"'] & ker(l') \arrow[d, "k(l')"] \\ L \arrow[r, "q_i"'] & L/K \end{tikzcd}$$

p_0 must factors through K, this gives that $k(l')p_1 = q_i p_0 = 0$. By stability of pullbacks p_1 is a normal epimorphism and $ker(l') = 0$. Now q' is a normal epimorphism since when considering:

$$\begin{tikzcd} L \arrow[r, "l"] \arrow[d, "\phi"] & A \arrow[r, "q_l"] \arrow[d, "q_k"] & A/L \arrow[d, "id"] \\ ker(q') \arrow[r, "k(q')"] & A/K \arrow[r, "q'"] & A/L \end{tikzcd}$$

the left hand square is a pullback and ϕ, induced by $ker(q')$, turns out to be a normal epimorphism since q_k is a normal epimorphism. It follows that the sequence

$$0 \longrightarrow L/K \longrightarrow A/K \longrightarrow A/L \longrightarrow 0$$

is exact. $\square$

1.4 Reflective subcategories

For a category $\mathbb{X}$, when considering a subcategory $\mathbb{A}$ of $\mathbb{X}$, we actually mean a full and replete subcategory of $\mathbb{X}$, unless we explicitly say otherwise. By a replete subcategory we mean "closed under isomorphisms": if $A \in \mathbb{A}$ and $A \cong B$ then $B \in \mathbb{A}$.

1.4.1. Consider a subcategory $\mathbb{A}$ of $\mathbb{X}$ given by the inclusion, a fully faithful embedding, $i : \mathbb{A} \to \mathbb{X}$. Then $\mathbb{A}$ is called a *reflective* subcategory of $\mathbb{X}$ if there is a left adjoint $R : \mathbb{X} \to \mathbb{A}$, called the reflector, of the inclusion $i : \mathbb{A} \to \mathbb{X}$:

$$\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "R"] \arrow[r, phantom, "\perp"] & \mathbb{A} \arrow[l, bend left, "i"] \end{tikzcd}. \tag{1.3}$$

The counit $\epsilon_A : Ri(A) \to A$ is always an isomorphism and, for an object X in $\mathbb{X}$, the componenet of the unit $\eta_X : X \to iR(X)$ is called the *reflection* of X.

The morphism η_X is universal: for any morphism $g : X \to A$ with A in $\mathbb{A}$ there is a unique morphism γ such

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "\eta_X"'] & A \\ & iR(X). \arrow[u, "\gamma"'] \end{tikzcd}$$

The subcategory $\mathbb{A}$ is said to be *closed under subobjects* if every subobject in $\mathbb{X}$ of an object in $\mathbb{A}$ lies in $\mathbb{X}$. Similarly, $\mathbb{A}$ is *closed under quotients* if for any regular epimorphism $p : A \to B$ with A in $\mathbb{A}$, then B is also in $\mathbb{A}$.

Definition 1.4.2. Consider a reflective subcategory $\mathbb{A}$ of $\mathbb{X}$, 1.3. We say that:

- $\mathbb{A}$ is *regular-epireflective* (or *normal-epireflective*) in $\mathbb{X}$ if the reflection η_X is a regular (normal) epimorphisms for all X.
- ([CDT06]) $\mathbb{A}$ is a *Birkhoff* subcategory of $\mathbb{X}$ if it is normal-epireflective and closed under quotients in $\mathbb{X}$.
- $\mathbb{A}$ is a *localization* of $\mathbb{X}$ if the left adjoint r preserves finite limits.

The dual notions are also used. In particular, a subcategory $\mathbb{B}$ of $\mathbb{X}$ is coreflective if the inclusion j has a right adjoint t. And

- $\mathbb{B}$ is *normal-monocoreflective* in $\mathbb{X}$ if the coreflection of $j \dashv t$, each component of the counit, $\epsilon_X : jt(X) \to X$ is a normal monomorphism.
- ([CDT06]) $\mathbb{B}$ is a *coBirkhoff* subcategory of $\mathbb{X}$ if it is normal monocoreflective closed under subobjects in $\mathbb{X}$.
- $\mathbb{B}$ is a *colocalization* of $\mathbb{X}$ if the right adjoint t preserves finite colimits.

Its easy to see that $\mathbb{A}$ is complete (cocomplete) whenever $\mathbb{X}$ is so. More precisely:

Lemma 1.4.3. Let $\mathbb{A}$ be a reflective subcategory of category $\mathbb{X}$. For a small category $\mathcal{J}$ and $j : \mathcal{J} \to \mathbb{A}$ then

$$lim\ j \cong lim\ ij \quad colim\ j \cong R(colim\ ij),$$

provided that $lim\ ij$ and, respectively, $colim\ ij$ exists in $\mathbb{X}$.

The dual of the lemma also holds for coreflective subcategories.

1.4.4. For a reflective subcategory $\mathbb{A}$ of regular category $\mathbb{X}$, if the reflections η_X are regular epimorphisms then $\mathbb{A}$ is closed under subobjects. Indeed, for a subobject $s : S \to A$, from the diagram

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[d, "s"'] & iR(S) \arrow[d, "iR(s)"] \\ A \arrow[r, "\eta_A"'] & iR(A) \end{tikzcd}$$

if A is in $\mathbb{A}$, η_A is an isomorphism and then η_S is a monomorphism since s is so. Then η_S is both a regular epimorphism and a monomorphism, hence an isomorphism.

Another property of regular-epireflective categories is that $ir(f)$ is a regular epimorphism in $\mathbb{X}$ when f is a regular epimorphism in $\mathbb{X}$,

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & iR(A) \arrow[d, "iR(f)"] \\ B \arrow[r, "\eta_B"'] & iR(B) \end{tikzcd}$$

since both f and η_B are regular epimorphisms then so is $iR(f)$. Moreover, since $\mathbb{A}$ is closed in $\mathbb{X}$ under subobjects, image of morphisms in $\mathbb{A}$ are in $\mathbb{A}$ (1.2.9), so a morphism f in $\mathbb{A}$ is a regular epimorphism if and only if $i(f)$ is a regular epimorphism in $\mathbb{X}$.

1.5 Homological and semi-abelian categories

Even if in a normal category we can work with short exact sequences this context is too general to properly reflect some more typical aspects of *Grp*. In particular, we will be interested in a suitable non-abelian categorical setting to adapt results from homological algebra, for example the five lemma, the nine lemma, the snake lemma, etc. that still hold true in *Grp*.

Protomodularity and homological categories will be of most importance to this end.

Definition 1.5.1. Let $\mathbb{X}$ be a category. The category of points $Pt(\mathbb{X})$ is defined as follows:

- Objects: Split epimorphisms (p, s) of $\mathbb{X}$ with a given splitting i.e. we have morphisms
$$p : \; X \longrightarrow Y \;, \quad s : \; Y \longrightarrow X$$
and $ps = 1_Y$.

- Morphisms: Pairs of morphisms $(f_1, f_0) : (p, s) \to (p', s')$:

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, shift right, "p"'] & X' \arrow[d, shift right, "p'"'] \\ Y \arrow[r, "f_0"] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"'] \end{tikzcd}$$

such that $p'f_1 = f_0 p$ and $f_1 s = s' f_0$.

Let $\mathbb{X}$ have pullbacks, the codomain functor:

$$\pi : \; Pt(\mathbb{X}) \longrightarrow \mathbb{X} \; ,$$

sending a split epimorphism (p, s) as above to the codomain Y of p. This functor is a fibration called *fibration of points* ([Bou91]).

The fiber of π of an object Y refers to the subcategory $Pt_Y(\mathbb{X})$ of $Pt(\mathbb{X})$ of split epimorphisms with codomain Y. And a morphism $\alpha : B \to Y$ induces a *change-of-base* functor among fibers:

$$\alpha^* : \; Pt_Y(\mathbb{X}) \longrightarrow Pt_B(\mathbb{X})$$

given by "pulling back" along α, so for (p, s) in $Pt_Y(\mathbb{X})$ consider the pullback of p along α

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\alpha^*(p)"'] & X \arrow[d, shift right, "p"'] \\ B \arrow[r, "\alpha"] & Y. \arrow[u, shift right, "s"'] \end{tikzcd}$$

So, $\alpha^*(p)$ is a split epimorphism with the section induced by $s' = (1, s\alpha)$ and so $\alpha^*(p, s) = (\alpha^*(p), s')$.

Definition 1.5.2. ([Bou91]) A category $\mathbb{X}$ with pullbacks is called *protomodular* if the functor α^* is conservative for all morphisms α in $\mathbb{X}$.

For a pointed category $\mathbb{X}$, protomodularity is actually equivalent to the Split Short Five Lemma to be valid in $\mathbb{X}$. To recall, the Split Short Five Lemma states that for a diagram

$$\begin{tikzcd} ker(p) \arrow[r] \arrow[d, "\alpha"] & X \arrow[r, shift right, "p"'] \arrow[d, "\beta"] & Y \arrow[l, shift right, "s"'] \arrow[d, "\gamma"] \\ ker(f) \arrow[r] & A \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "t"'] \end{tikzcd}$$

where (p, s) and (f, t) are split epimorphisms, so if α and γ are isomorphisms, then so is β.

Proposition 1.5.3. For a pointed category $\mathbb{X}$ with pullbacks, the following conditions are equivalent:

1. $\mathbb{X}$ is protomodular;
2. In $\mathbb{X}$ the Split Short Five Lemma holds.

Proof. Consider $\alpha_Y : 0 \to Y$ and $p : X \to Y$ with a section s, applying $\alpha_Y^*(p, s)$ is actually given by $ker(p)$,

$$\begin{tikzcd} ker(p) \arrow[r, "k(p)"] \arrow[d, shift right, "\alpha_Y^*(p,s)"'] & X \arrow[d, shift right, "p"'] \\ 0 \arrow[r, "\alpha_Y"'] \arrow[u, shift right] & Y \arrow[u, shift right, "s"'] \end{tikzcd} .$$

So given a morphism $\beta : X \to A$ in $Pt_Y(\mathbb{X})$

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d, shift right, "p"'] & A \arrow[d, shift right, "f"'] \\ Y \arrow[r, "1_Y"'] \arrow[u, shift right, "s"'] & Y \arrow[u, shift right, "t"'] \end{tikzcd} ,$$

we apply $\alpha_Y^*(\beta)$, to obtain a diagram:

$$\begin{tikzcd} ker(p) \arrow[r] \arrow[d, "\alpha^*(\beta)"] & X \arrow[r, shift right, "p"'] \arrow[d, "\beta"] & Y \arrow[l, shift right, "s"'] \arrow[d, "1_Y"] \\ ker(f) \arrow[r] & A \arrow[r, shift right, "f"'] & Y \arrow[l, shift right, "t"'] \end{tikzcd}$$

then if, α_Y^* is conservative then if $\alpha^*(\beta)$ is an isomorphism then β is so. This shows that the Split Short Five Lemma is equivalent to the functors α^* being conservative for the morphism $\alpha : 0 \to Y$. It remains to prove that for a morphism $f : X \to Y$, the functor f^* is conservative. Since $\alpha_Y = f \circ \alpha_X$,

$$\begin{tikzcd} 0 \arrow[r, "\alpha_Y"] \arrow[d, "\alpha_X"'] & Y \\ X \arrow[ur, "f"'] & \end{tikzcd}$$

we have, $\alpha_Y^* = \alpha_X^* f^*$. So f^* is conservative when both α_X^* and α_Y^* are conservative. $\square$

We present the main definitions:

Definition 1.5.4. A category $\mathbb{X}$ is called

- *homological* if it is pointed, regular and protomodular.
- *semi-abelian* if it is homological, exact and has binary coproducts.

Semi-abelian categories were introduced in [JMT02], they have been study widely. See [BG06], [BB04] and [BC05] for instance.

Examples 1.5.5. As expected, Grp and abelian categories are semi-abelian. Examples are vast and range over many areas, firstly many varieties of universal algebras like Lie Algebras, ${}_K Lie$, and the category of rings without unit, Rng, not necessarily unital rings are semi-abelian ([JMT02]).

It was proved in [Bou96] that the category of Heyting algebras is protomodular. It is also exact being an algebraic variety but not pointed so $Pt_Y(Heyt)$ is semi-abelian, the same for $Bool$, boolean algebras. Following this, for a topos $\mathcal{E}$ like $Sets$ or $LoCo$ locally connected spaces, the dual categories $\mathcal{E}^{op}$ are protomodular, exact and cocomplete ([Bou96], [LM92]).

The category of topological spaces is not regular but the category $Grp(Top)$ of topological groups is homological and compact Hausdorff groups $Grp(Hcmp)$ is semi-abelian. Moreover, taking a semi-abelian algebraic theory $\mathbb{T}$, the category of models of $\mathbb{T}$ in compact Haussdorf spaces is semi-abelian ([BC05]).

Other examples, include ${}_K Hopf_{coc.}$ cocommutative Hopf algebras and $\mathbb{C}^*$-algebras that are semi-abelian ([GR04] ,[GSV19]).

Remark 1.5.6. It is useful to recall from [Bou91], that for a regular category $\mathbb{X}$, the condition of protomodularity is equivalent to the next property: given a commutative diagram in $\mathbb{X}$:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & B \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "(2)"] & C \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

with p a regular epimorphism, if the external rectangle, $(1+2)$, is a pullback then, the left hand square (1) is a pullback if and only if the right hand square (2) is a pullback.

Definition 1.5.7. Let be $\mathbb{X}$ a normal category, and consider a diagram

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "p"'] & B \arrow[d, "q"] \\ C \arrow[r, "n"'] & D \end{tikzcd}$$

then $\mathbb{X}$:

- is *ideal determined* if each time we have p and q normal epimorphisms, m a normal monomophism and n a monomorphism, then n is a normal monomorphism.

- satisfies the *Corresponce Theorem* (CT) if each time we have p and q normal epimorphisms and m, n monomorphisms such that $ker(p) \cong ker(q)$ then the diagram is a pullback.

Remark 1.5.8. It is known that a homological category is normal and it can be proved that it also satisfies (CT), applying 1.5.6 to

$$\begin{tikzcd} ker(q) \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & A \arrow[r, "m"] \arrow[d, "p"] \arrow[dr, phantom, "(2)"] & B \arrow[d, "q"] \\ 0 \arrow[r] & C \arrow[r, "n"] & D, \end{tikzcd}$$

since n is mono, $ker(n) = 0$ then $ker(p) = ker(q)$. Finally, (1) and $(1+2)$ are pullbacks and the results follows. Moreover, a homological category with binary coproducts that is ideal determined is semi-abelian [JMT02].

A reflective subcategory $\mathbb{A}$ of $\mathbb{X}$ has limits and colimits, provided that they exist in $\mathbb{X}$. In particular, if $\mathbb{X}$ is pointed and/or protomodular then so is $\mathbb{A}$. Being regular or exact is a different situation.

Lemma 1.5.9. If $\mathbb{A}$ is a reflective subcategory of $\mathbb{X}$ as in 1.3, then:

- if $\mathbb{A}$ is regular epireflective and $\mathbb{X}$ is regular then $\mathbb{A}$ is regular.

- if $\mathbb{A}$ is Birkhoff and $\mathbb{X}$ is exact then $\mathbb{A}$ is exact.

- if $\mathbb{A}$ is a localization and $\mathbb{X}$ is regular/exact then $\mathbb{A}$ is regular/exact.

Proof. If $\mathbb{A}$ is regular epireflective, it suffices to prove pullback stability of regular epimorphisms. But since the inclusion $i : \mathbb{A} \to \mathbb{X}$ preserves limits/pullbacks, if p is a regular epimorphism in $\mathbb{A}$ for a pullback $f^*(p)$ then $i(p)$ (recall 1.4.4) and $i(f^*(p))$ are regular epimorphisms in $\mathbb{X}$ and then so is $i(f^*(p))$ in $\mathbb{A}$.

If $\mathbb{A}$ is Birkhoff, consider an equivalence relation E over X in $\mathbb{A}$. Since $\mathbb{X}$ is exact E is the kernel pair of its coequalizer in $\mathbb{X}$ but since $\mathbb{A}$ is closed under quotients, the coequalizer is in $\mathbb{A}$. And E is effective in $\mathbb{A}$.

The case of localizations is different. Clearly, there is pullback stability of regular epimorphisms since R is exact. Now, an equivalence relation E in $\mathbb{A}$ is effective in $\mathbb{X}$, $E = Eq[f]$ for some f in $\mathbb{X}$, since R preserves limits then $E = Eq[R(f)]$. □

Corollary 1.5.10. If $\mathbb{A}$ is a regular epireflective subcategory of a homological category $\mathbb{X}$ then $\mathbb{A}$ is homological. If $\mathbb{A}$ is a Birkhoff subcategory of a semi-abelian category $\mathbb{X}$ then $\mathbb{A}$ is semi-abelian. If $\mathbb{A}$ is a localization of a semi-abelian category $\mathbb{X}$ then $\mathbb{A}$ is semi-abelian.

To illustrate these lasts results, consider $\mathbb{F}$ the category of torsion-free abelian groups, where an abelian group X is torsion-free if satisfies

$$x^n = 0 \Rightarrow x = 0 \text{ for all } x \in X.$$

$\mathbb{F}$ is clearly not closed under quotients in *Ab*, but is regular-epireflective with the reflector given by

$$\begin{tikzcd} Ab \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{F} \arrow[l, bend left] \end{tikzcd}, \quad F(X) = X/T(X)$$

where $T(X)$ is the torsion subgroup of X, the subgroup of elements of finite order of X.

The category *Ab* of abelian groups is a Birkhoff subcategory of *Grp*. The reflector is called the *abelianization* functor

$$\begin{tikzcd} Grp \arrow[r, bend left, "ab"] \arrow[r, phantom, "\perp"] & Ab \arrow[l, bend left] \end{tikzcd}, \quad ab(X) = X/X'$$

where $X' = [X, X]$ is the commutator subgroup.

Finally, *Grp*, our canonical semi-abelian example, does not have any non-trivial localisations, i.e. apart from 0 or *Grp* itself (originally proved in [Bor80]). In contrast to abelian categories, Grothendieck categories or categories of modules over rings that have plenty of localizations, actually characterized by hereditary torsion theories ([Ste75], [Bor94]). For example, consider the category $\mathbb{Q}Vect$ of vector spaces over $\mathbb{Q}$ is a localization of *Ab*,

$$\begin{tikzcd} Ab \arrow[r, bend left, "\mathbb{Q}\otimes\_"] \arrow[r, phantom, "\perp"] & \mathbb{Q}Vect \arrow[l, bend left] \end{tikzcd}.$$

Since $\mathbb{Q}$ is a field, it is a flat abelian group and $\mathbb{Q} \otimes \_$ is an exact functor. From this, we can notice that localizations and epireflective categories are very different; $\mathbb{Q}vect$ is not epireflective in *Ab*, the unit is given by the mappings $x \to 1 \otimes x$. And $\mathbb{F}$, the torsion-free abelian groups in *Ab* has not a exact reflector, it suffices to apply F to the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \ .$$

Chapter 2

Torsion theories

Torsion theories were introduced by Dickson in [Dic66] for abelian categories and quickly provided diverse applications to categories of modules over rings, [Ste75] and [Pop73]. Recently, they have been (re)-introduced in different non-abelian contexts, for instance see [BG06], [CDT06] or [JT07]. Here we develop an introductory study of torsion theories in our particular context following [BG06], [CDT06].

Probably the easiest way is to start with the associated torsion subobject of a torsion theory, which will be given by preradicals.

2.1 Preradicals

Through out this section we will fix a normal category $\mathbb{X}$ with cokernels. Unless otherwise stated all the results in this section are proved in [CDT06] in a more general setting, we include the proofs we consider relevant in our context of normal categories.

It will be useful to remember that for a normal category $\mathbb{X}$ and $Arr(\mathbb{X})$ the category of morphisms of $\mathbb{X}$ and where morphisms are commutative squares, taking kernel and cokernel is functorial and even gives an adjunction

$$\begin{tikzcd} Arr(\mathbb{X}) \arrow[r, bend left, "Cok"] \arrow[r, phantom, "\perp"] & Arr(\mathbb{X}) \arrow[l, bend left, "Ker"] \end{tikzcd}$$

and for a fixed object X in $\mathbb{X}$ this adjunction restricts to an adjunction:

$$\begin{tikzcd} \mathbb{X}/X \arrow[r, bend left, "Cok"] \arrow[r, phantom, "\perp"] & X/\mathbb{X} \arrow[l, bend left, "Ker"] \end{tikzcd} .$$

The next definition is a natural generalization of this adjunction when we change $\mathbb{X}$ for the category $End(\mathbb{X})$ of endofunctors of $\mathbb{X}$ with the object 1_X.

2.1.1. For a *pointed* endofunctor (S, ρ) we mean an endofunctor $S : \mathbb{X} \to \mathbb{X}$ provided with a natural transformation $\rho : id_{\mathbb{X}} \to S$. And a copointed endofunctor (r, σ) is $r : \mathbb{X} \to \mathbb{X}$ with $\sigma : r \to id_{\mathbb{X}}$:

$$\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "id"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\Downarrow\rho"] & \mathbb{X} \end{tikzcd} , \qquad \begin{tikzcd} \mathbb{X} \arrow[r, bend left, "r"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbb{X} \end{tikzcd} .$$

For a pointed endofunctor (S, ρ) we define a copointed endofunctor by taking the kernel of ρ:

$$Ker(S, \rho) = (ker(\rho), k(\rho)), \qquad ker(\rho_X) \xrightarrow{k(\rho_X)} X \xrightarrow{\rho} S(X) \ .$$

Dually, the cokernel of a copointed functor gives a pointed functor:

$$Coker(r, \sigma) = (cok(\sigma), q(\sigma)), \qquad r(X) \xrightarrow{\sigma_X} X \xrightarrow{q(\sigma_X)} cok(\sigma_X) \ .$$

Moreover, Ker and $Coker$ are functorial and give an adjunction:

$$\begin{tikzcd} \{\text{copointed endofunctors}\} \arrow[r, bend left, "Coker"] \arrow[r, phantom, "\perp"] & \{\text{pointed endofunctors}\} \arrow[l, bend left, "Ker"] \end{tikzcd} . \tag{2.1}$$

Pointed and copointed functors define subcategories of $\mathbb{X}$ by fixed objects; $Fix(S, \rho) = \{X \mid \rho_x \text{ iso}\}$ and $Fix(r, \sigma) = \{X \mid \sigma_x \text{ iso}\}$, ([CDT06]).

We shall focus on preradicals, a special kind of copointed functors.

Definition 2.1.2. A (normal) preradical in $\mathbb{X}$ is a normal subfunctor r of the identity of $\mathbb{X}$, equivalently, a copointed functor (r, σ) with a normal σ; so $\sigma_X : r(X) \to X$ is a normal monomorphism for all objects X.

In other words, for $f : X \to Y$ there is a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ r(X) \arrow[u, "\sigma_X"] \arrow[r, "r(f)"'] & r(Y) \arrow[u, "\sigma_Y"'] \end{tikzcd} \tag{2.2}$$

2.1.3. Since each kernel is the kernel of its cokernel and dually each cokernel necessarily is the cokernel of its kernel; the adjunction of 2.1 restricts to a equivalence:

$$\begin{tikzcd} \{\text{preradicals}\} \arrow[rr, bend left, "Coker"] & \cong & \{\text{normal pointed endofunctors}\} \arrow[ll, bend left, "Ker"] \end{tikzcd} .$$

where by a normal pointed endofunctor (s, ρ) we mean that each object ρ_X is a normal epimorphism.

With this in mind, with a preradical r and its counterpart $R = Coker(r)$ $R(X) = X/r(X)$, we define the r-*Torsion* subcategory of r-torsion objects:

$$\mathcal{T}_r = Fix(r, \sigma) = \{X \mid r(X) = X\}.$$

And the r-*Torsion-free* subcategory of torsion-free objects

$$\mathcal{F}_r = Fix(R, \rho) = \{X \mid R(X) = X\} = \{X \mid r(X) = 0\}.$$

For a fixed preradical we will always have a short exact sequence for each object

$$0 \longrightarrow r(X) \xrightarrow{\sigma_X} X \xrightarrow{\rho_X} R(X) \longrightarrow 0 .$$

Note that $r(X)$ is not necessarily a torsion object or $R(X)$ torsion-free, further assumptions on r are needed for this.

Definition 2.1.4. ([CDT06]) A preradical r of $\mathbb{X}$ is called

- *idempotent* if $r(r(X)) = r(X)$ for all objects X.
- *radical* if $r(X/r(X)) = 0$ for all objects X.
- *hereditary* if $f^{-1}(r((Y))) = r(X)$ for every monomorphism $f : X \to Y$.
- *cohereditary* if $f(r(X)) = r(Y)$ for every normal epimorphism $f : X \to$

Y.[1]

Note that a hereditary preradical r is necessarily idempotent, simply take $f = r(X) \to X$; similarly, a cohereditary preradical is a radical by taking the quotient $f = X \to X/r(X)$.

2.1.5. For a preradical r, r is a radical if and only if $R(\rho_X)$ is an isomorphism: just consider the diagram

$$\begin{tikzcd} & & & r(X/r(X)) \arrow[d] & \\ 0 \arrow[r] & r(X) \arrow[r] & X \arrow[r, "\rho_X"] & R(X) \arrow[r] \arrow[d, "R(\rho_X)"] & 0 \\ & & & RR(X)\,. & \end{tikzcd}$$

So, since ρ_x is a normal epimorphism and $R(\rho_X)$ is an isomorphism, the subcategory $\mathcal{F}_r = Fix(R, \rho)$ is actually normal epireflective with the functor R as the reflection.

On the other hand an idempotent preradical r exhibitis $\mathcal{T}_r$ as a normal monocoreflective subcategory. In summary we have

Lemma 2.1.6. In $\mathbb{X}$ there are bijective correspondences:

$$\{\text{radicals}\} \simeq \{\text{normal-epireflective subcategories}\}^{op}$$

and

$$\{\text{idempotent preradicals}\} \simeq \{\text{normal-monocoreflective subcategories}\}.$$

A subcategory $\mathbb{A}$ of $\mathbb{X}$ is closed under *extensions* if whenever we have a short exact sequence in $\mathbb{X}$

$$0 \longrightarrow K \longrightarrow X \longrightarrow Q \longrightarrow 0$$

with K and Q in $\mathbb{A}$ then so is X in $\mathbb{A}$.

Proposition 2.1.7. Let r be a preradical of $\mathbb{X}$, then

1. $\mathcal{T}_r \cap \mathcal{F}_r = 0$

[1] This definition is adapted from [CDT] where $\mathbb{X}$ is chosen with a factorization system $(\mathcal{E}, \mathcal{M})$, so the terminology $\mathcal{E}$-cohereditary/$\mathcal{M}$-hereditary is used and hence the lack of duality or more precisely there is duality up to a factorization system. However, in normal categories we have a canonical factorization system $(NormEpi, Mono)$.

2. $\mathcal{T}_r$ is closed under quotients.

3. $\mathcal{F}_r$ is closed under subobjects.

4. if r is a radical then $\mathcal{F}_r$ is closed under extensions.

5. if r is idempotent then $\mathcal{T}_r$ is closed under extensions.

Proof. 1) is trivial since $X = r(X)$ being torsion and $0 = r(X) = X$ being torsion-free. 2) and 3) are immediate from 2.2. If f is a quotient and $r(X) = X$ then $\sigma_Y r(f)$ is a regular epimorphism and so is σ_Y a monomorphism and a regular epimorphism, $r(Y) = Y$. And if f is a monomorphism and $r(Y) = 0$ then $f\sigma_X = 0$ but σ_X is monic. So $r(X) = X$.

4) Consider the commutative diagram for radical r

$$\begin{tikzcd}
 & r(K) \arrow[r] \arrow[d] & r(X) \arrow[r] \arrow[d] & r(Q) \arrow[d] & \\
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "\rho_K"] & X \arrow[r, "p"] \arrow[d, "\rho_X"] & Q \arrow[r] \arrow[d, "\rho_Q"] & 0 \\
 & R(K) \arrow[r] & R(X) \arrow[r] & R(Q) &
\end{tikzcd}$$

If K and Q are in $\mathcal{T}_r$ then $r(K) = K$ and $R(K) = 0$, So ρ_X factors through p by $p' : Q \to R(X)$ that is a normal epimorphism. Now since Q is in $\mathcal{T}_r$ then so is $R(X)$ being a quotient of Q, separately r is radical, $r(X/r(X)) = 0$ so $R(X) = X/r(X)$ is also torsion-free, then $R(X) = 0$. 5) is similar to 4). $\square$

Proposition 2.1.8. Let r be a preradical of $\mathbb{X}$, then

1. r is hereditary if and only if r is idempotent and $\mathcal{T}_r$ is closed under subobjects.

2. provided that $\mathbb{X}$ is ideal determined, r is cohereditary if and only if r is a radical and $\mathcal{F}_r$ is closed under quotients.

Proof. 1) We have noticed that hereditary preradicals are idempotent. Next, for a monomorphism $m : X \to Y$ with Y in $\mathcal{T}_R$, since

$$\begin{tikzcd}
r(X) \arrow[r, "r(m)"] \arrow[d, "\sigma_X"] & r(Y) \arrow[d, "\sigma_Y"] \\
X \arrow[r, "m"'] & Y
\end{tikzcd}$$

is a pullback and σ_Y is an iso then $r(X) = X$. Conversely, for a monomorphism $m : X \to Y$ since $r(Y) = rr(Y)$ is torsion and $\mathcal{T}_r$ is closed under subobjects then $m^{-1}(r(Y))$ is torsion. Then notice that $r(X) \leq m^{-1}(r(Y))$, and if we apply r to $m^{-1}(r(Y)) \leq X$ we have $m^{-1}(r(Y)) = r(m^{-1}(r(Y))) \leq r(X)$.

2) For a normal epimorphism $f : X \to Y$ if $r(X) = 0$, since r is cohereditary $fr(X) = r(Y) = 0$ and $\mathcal{F}_r$ is closed under images. Conversely, consider the diagram,

$$\begin{tikzcd}
r(X) \arrow[r] \arrow[d] & X \arrow[r, "\rho_x"] \arrow[d, "f"] & R(X) \arrow[d, "f'"] \\
f(r(X)) \arrow[r] & Y \arrow[r, "q"] & Y/f(r(X)) := Z
\end{tikzcd}$$

now since $\mathbb{X}$ is ideal determined $r(X)$ is normal in X and so is $f(r(X))$ in Y. Now for the induced arrow f' since $f'\rho_X = qf$ is a normal epimorphism, also f' is a normal epimorphism. Since r is a radical and $R(X)$ is torsion-free so is Z. As expected we always have $f(r(X))) \leq r(Y)$ and since Z is torsion free, $r(Z) = 0$ and the composite $r(Y) \to Y \to Z$ is zero. Finally, $r(Y) \leq f(r(X))$. $\square$

Applying the 2.1.6 with 2.1.8 we have:

Corollary 2.1.9. Following the definitions in 1.4.2. In a normal category $\mathbb{X}$ there are biyective correspondences:

$$\{\text{hereditary preradicals}\} \cong \{\text{co-Birkhoff subcategories}\}$$

and if $\mathbb{X}$ is ideal determined

$$\{\text{cohereditary preradicals }\} \cong \{\text{Birkhoff subcategories}\}.$$

We conclude this section with a characterization of hereditary radicals proved in [BG06] for homological categories.

Corollary 2.1.10. A radical r is hereditary if and only if the reflector R of $\mathcal{F}_r$ preserves monomorphisms.

2.2 Torsion Theories

Definition 2.2.1. A *torsion theory* in a pointed category $\mathbb{X}$ is a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of $\mathbb{X}$ such that:

TT1 for all $X \in \mathcal{T}$ and $Y \in \mathcal{F}$, every morphism $f : X \to Y$ is zero,

TT2 for every object $X \in \mathbb{X}$ exists a short exact sequence

$$0 \longrightarrow T_X \xrightarrow{t_X} X \xrightarrow{\eta_X} F_X \longrightarrow 0$$

with $T_X \in \mathcal{T}$ and $F_X \in \mathcal{F}$.

$\mathcal{T}$ is the torsion part and $\mathcal{F}$ is the torsion-free part of the torsion theory. A subcategory of $\mathbb{X}$ is called torsion/torsion-free if it is a torsion/torsion-free part of a torsion theory.

Proposition 2.2.2. Given a torsion theory $(\mathcal{T}, \mathcal{F})$ for the object X the sequence

$$0 \longrightarrow T_X \xrightarrow{t_X} X \xrightarrow{\eta_X} F_X \longrightarrow 0$$

is necessarily unique (up to isomorphism).

Proof. Let us consider

$$\begin{tikzcd}
0 \arrow[r] & T_X \arrow[r, "t_X"] & X \arrow[r, "\eta_X"] \arrow[d, "id"] & F_X \arrow[r] & 0 \\
0 \arrow[r] & T'_X \arrow[r, "t'_X"] & X \arrow[r, "\eta'_X"] & F'_X \arrow[r] & 0
\end{tikzcd}$$

exact sequences with T_X, T'_X in $\mathcal{T}$ and F_X, F'_X in $\mathcal{F}$. The composites $\eta'_X t_X$ and $\eta_X t'_X$ are zero morphism by 1) of 2.2.1, so they induce morphisms $\alpha : F_X \to F'_X$ and $\beta : F'_X \to F_X$ with $\alpha\eta_X = \eta'_X$ and $\eta_X = \beta\eta'_X$. By the universal properties of cokernels p, p'; α and β are inverses. So $F_X \cong F'_X$. Similarly, $T'_X \cong T_X$. $\square$

Theorem 2.2.3. ([CDT06]) A pair of subcategories $(\mathcal{T}, \mathcal{F})$ of a normal category $\mathbb{X}$ is a torsion theory if and only if there is a unique idempotent radical r such that $\mathcal{T}_r = \mathcal{T}$ and $\mathcal{F}_r = \mathcal{F}$

Proof. Given a torsion theory $(\mathcal{T}, \mathcal{F})$, first notice that, for a morphism $f : X \to Y$, taking the torsion or torsion-free part is functorial:

$$\begin{tikzcd}
0 \arrow[r] & T_X \arrow[r, "t_X"] \arrow[d, "T_f"] & X \arrow[r, "\eta_X"] \arrow[d, "f"] & F_X \arrow[r] \arrow[d, "F_f"] & 0 \\
0 \arrow[r] & T_Y \arrow[r, "t'_X"'] & Y \arrow[r, "\eta'_X"'] & F_Y \arrow[r] & 0\,.
\end{tikzcd}$$

In particular, $\eta'_X f t_X = 0$ then T_Y induces an arrow T_f. Analogously, we have F_f making the right hand square commute. This allows one to define a preradical $r(X) = T_X$ and $R(X) = F_X = X/T_X$.

Given a preradical r and a morphism $f : X \to Y$, is clear from

$$\begin{tikzcd} r(X) \arrow[r, "r(f)"] \arrow[d] & r(Y) \arrow[d] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

that if $r(X) = X$ and $r(Y) = 0$ then f factors through zero. So there are no non-trivial morphism from $\mathcal{T}_r$ and $\mathcal{F}_r$. Now given the exact sequence

$$0 \longrightarrow r(X) \xrightarrow{\sigma_X} X \xrightarrow{\rho_X} R(X) \longrightarrow 0$$

if r is idempotent then $r(X)$ is torsion and if r is a radical then $X/r(X)$ is torsion-free. Now we have $\mathcal{T}_r = \mathcal{T}$ from the exact sequence. If t_X is an isomorphism then X is torsion and if X is torsion then $X \to F_X$ is zero and its kernel is an isomorphism. From $\mathcal{T}_r = \mathcal{T}$ then r is idempotent. Analogously, $\mathcal{F}_r = \mathcal{F}$ and r is a radical. □

2.2.4. In a torsion theory $(\mathcal{T}, \mathcal{F})$ the torsion part and the torsion-free part determine each other. $\mathcal{T}$ is a normal monocoreflectvie subcategory as in 1.4.2, and a normal monocoreflective subcategory with a radical coreflector (i.e. the idempotent preradical) is a torsion subcategory. And a full subcategory $\mathcal{F}$ is torsion-free if and only if it is a normal epireflective with reflector F and the kernel of F as in 2.1.1 is an idempotent radical, $t = JT$.

$$\begin{tikzcd} \mathcal{T} \arrow[r, bend left, "J"] \arrow[r, phantom, "\perp"] & \mathbb{X} \arrow[l, bend left, "T"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, bend left, "I"] \end{tikzcd} \tag{2.3}$$

Now, for a normal category $\mathbb{X}$ and a torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathbb{X}$, applying 2.1.8, we have that $\mathcal{T}$ is closed under quotients, $\mathcal{F}$ is closed under subobjects and both are closed under extensions. Moreover, the torsion theory $(\mathcal{T}, \mathcal{F})$ will be called *hereditary* if $\mathcal{T}$ is closed under subobjects and it will be called *cohereditary* if $\mathcal{F}$ is closed under quotients. In other words, $(\mathcal{T}, \mathcal{F})$ is hereditary when $\mathcal{T}$ is a coBirkhoff subcategory of $\mathbb{X}$ and, similarly, is cohereditary if $\mathcal{F}$ is a Birkhoff subcategory of $\mathbb{X}$, (1.4.2).

Theorem 2.2.5. ([BG06]) Let $\mathbb{X}$ be a normal category. A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathbb{X}$ is a torsion theory in $\mathbb{X}$ if and only if

1. $\mathcal{T} \cap \mathcal{F} = 0$.

2. $\mathcal{T}$ is closed under quotients and $\mathcal{F}$ is closed under subobjects.

3. For every object X there is a short exact sequence

$$0 \longrightarrow T_X \xrightarrow{t_X} X \xrightarrow{\eta_X} F_X \longrightarrow 0$$

with T_X in $\mathcal{T}$ and F_X in $\mathcal{F}$.

Proof. Clearly a torsion theory $(\mathcal{T}, \mathcal{F})$ satisfies all conditions. Conversely, consider an arrow $f : X \to Y$ with X in $\mathcal{T}$ and Y in $\mathcal{F}$. Taking the image factorization, $f(X)$ is both torsion since X is in $\mathcal{T}$ and torsion-free since Y is in $\mathcal{F}$. So $f(X) = 0$ and $f = 0$. $\square$

2.2.6. In order to give a useful characterization of torsion-free subcategories among reflective ones we need to introduce some definitions. Consider $\mathcal{F}$ a normal epireflective subcategory of $\mathbb{X}$ with reflector F. Since η_X is a normal epimorphism we have a short exact sequence

$$0 \longrightarrow ker(\eta_X) \xrightarrow{k_X} X \xrightarrow{\eta_X} F(X) \longrightarrow 0 \ . \tag{2.4}$$

So we can define the full subcategories of $\mathbb{X}$:

1. $\mathcal{T}_{\mathcal{F}} = \{T \mid T \cong ker(\eta_X) \text{ for some } X\}$

2. $\mathcal{F}^{\leftarrow} = \{T \mid Hom(T, F) = 0 \text{ for all } F \text{ in } \mathcal{F}\}$

3. $Ker(F) = \{T \mid F(T) \cong 0\}$

It is easy to see that indeed we always have $\mathcal{F}^{\leftarrow} = Ker(F)$ and $\mathcal{F}^{\leftarrow} \subseteq \mathcal{T}_{\mathcal{F}}$. Also, notice that $\mathcal{T}_{\mathcal{F}} = \mathcal{F}^{\leftarrow}$ if and only if the reflector F is a *normal* functor, i.e. for all objects X, $F(ker(\eta_X)) \cong 0$.

Theorem 2.2.7. ([BG06]) Let $\mathcal{F}$ be a full subcategory of a normal category $\mathbb{X}$. The following conditions are equivalent:

1. $\mathcal{F}$ is a torsion-free subcategory of $\mathbb{X}$.

2. $\mathcal{F}$ is normal-epireflective (as in 1.4.2) of $\mathbb{X}$ and $\mathcal{T}_{\mathcal{F}} = \mathcal{F}^{\leftarrow}$.

3. $\mathcal{F}$ is normal-epireflective of $\mathbb{X}$ and $Hom(T, F) = 0$ for all T in $\mathcal{T}_{\mathcal{F}}$ and F in $\mathcal{F}$.

4. $\mathcal{F}$ is normal-epireflective of $\mathbb{X}$, $\mathcal{F}$ is closed under extensions and $\mathcal{T}_{\mathcal{F}}$ is closed under quotients.

Proof. 1) $\Rightarrow$ 2). Given a torsion theory $(\mathcal{T}, \mathcal{F})$, we recall that $\mathcal{F}$ is always normal epireflective. Let see that $\mathcal{T}_{\mathcal{F}} = \mathcal{F}^{\leftarrow} = \mathcal{T}$. We always have $\mathcal{F}^{\leftarrow} \subset \mathcal{T}_{\mathcal{F}}$.

Now let us see that $\mathcal{T}_{\mathcal{F}} \subset \mathcal{T}$. For T in $\mathcal{T}_{\mathcal{F}}$, T is isomorphic to kernel of η_X in TT2 for some X. Since $\mathcal{T}$ is closed under isomorphisms then T is in $\mathcal{T}$. And finally $\mathcal{T} \subset \mathcal{F}^{\leftarrow}$ just by the definition of of torsion theory, TT1. 2) $\Rightarrow$ 3), is trivial from $\mathcal{T}_{\mathcal{F}} = \mathcal{F}^{\leftarrow}$. For 3) $\Rightarrow$ 4) consider a quotient $p : T \to Y$ with T in $\mathcal{T}_{\mathcal{F}}$. By hypothesis $\eta_Y p = 0$ so there is i making the diagram commute:

$$\begin{tikzcd} & & T \arrow[dl, "i"'] \arrow[d, "p"] & & \\ 0 \arrow[r] & T_Y \arrow[r, "t_Y"] & Y \arrow[r, "\eta_Y"] & F_Y \arrow[r] & 0 \,. \end{tikzcd}$$

Since p is a normal epimorphism then so is t_Y, hence it is an iso and Y is in $\mathcal{T}_{\mathcal{F}}$. On the other hand, given a short exact sequence

$$0 \longrightarrow F_1 \xrightarrow{f} X \xrightarrow{g} F_2 \longrightarrow 0$$

with F_1, F_2 in $\mathcal{F}$, there is $l : ker(\eta_X) \to F_1$ such that $fl = k_X$ but $l = 0$ and hence $k_X = 0$. Finally η_X is an isomorphism.

4) $\Rightarrow$ 1). By 2.2.5 it is only required to prove $\mathcal{T}_{\mathcal{F}} \cap \mathcal{F} = 0$. Given X in $\mathcal{T}_{\mathcal{F}} \cap \mathcal{F}$ consider

$$0 \longrightarrow X = ker(\eta_Y) \xrightarrow{k_Y} Y \xrightarrow{\eta_Y} F(Y) \longrightarrow 0 \ .$$

Since $\mathcal{F}$ is closed under extensions Y is in $\mathcal{F}$ so $k_Y = 0$ and $X = 0$. $\square$

Under these hypothesis of the theorem 2.2.7 the torsion theory is given by $(\mathcal{T}_{\mathcal{F}}, \mathcal{F})$ or, equivalently $(\mathcal{F}^{\leftarrow}, \mathcal{T})$. Moreover, we shall be interested in the conditions 1) and 2) of 2.2.7, so we have:

Corollary 2.2.8. Let $\mathcal{F}$ be a full subcategory of a normal category $\mathbb{X}$. The following conditions are equivalent:

1. $\mathcal{F}$ is a torsion-free subcategory.

2. $\mathcal{F}$ is normal-epireflective and the reflection F is normal, $F(ker(\eta_X)) = 0$.

As hinted from 2.2.7 closure under extensions is characteristic for both torsion and torsion-free subcategories, and even in different settings as seen in [CDT06] and [JT07]. For normal categories we will only mention:

Theorem 2.2.9. (see [CDT06]) Let $\mathbb{X}$ be a normal category. Then a normal monocoreflective $\mathcal{T}$ of $\mathbb{X}$ is torsion if and only if it is closed under extensions. [2]

[2] The correponding characterization for torsion-free subcategories also holds but additional hypotheses are needed, for example some stability of normal monomorphisms along pushouts of normal epimorphisms and also being ideal determined.

Proof. Using the notation of section 2.1 since with a normal monocoreflective subcategory is associated an idempotent preradical r, we shall prove that if $\mathcal{T} = \mathcal{T}_r$ is closed under extensions then r is also radical. Consider the pullback:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "p"'] & X \arrow[d, "\rho_X"] \\ r(X/r(X)) \arrow[r] & X/r(X). \end{tikzcd}$$

Since it is a pullback, $ker(p) \cong ker(\rho_X) \cong r(X)$ and since normal epimorphisms are pullback stable we have a short exact sequence

$$0 \longrightarrow r(X) \longrightarrow P \xrightarrow{p} r(X/r(X)) \longrightarrow 0$$

so P is in $\mathcal{T}$. Since $\mathcal{T}$ is coreflective then $P \leq r(X)$ and by the pullback diagram also $r(X)$ factors through P; so $P \cong r(X)$. Finally $p = 0$ and $r(X/r(X)) = 0$. □

2.2.1 The lattice of torsion theories

We will introduce an order in the class of torsion theories in a category $\mathbb{X}$.

Lemma 2.2.10. Let $\tau = (\mathcal{T}, \mathcal{F})$ and $\sigma = (\mathcal{S}, \mathcal{G})$ torsion theories in a normal category $\mathbb{X}$. Then

$$\mathcal{S} \subseteq \mathcal{T} \text{ if and only if } \mathcal{F} \subseteq \mathcal{G}.$$

Proof. Let t, s the idempotent radicals associated with τ and σ. Since torsion subcategories are coreflective we have that $\mathcal{S} \subseteq \mathcal{T}$ if and only if $s(X) \leq t(X)$ for all objects X. So for Y in $\mathcal{G}$, $t(Y) = 0$ and then $s(Y) = 0$ so finally $Y \cong \mathcal{G}(Y)$. The converse is similar. □

Definition 2.2.11. Given torsion theories $\tau = (\mathcal{T}, \mathcal{F})$ and $\sigma = (\mathcal{S}, \mathcal{G})$ in a normal category $\mathbb{X}$, we define the partial order:

$$\sigma \leq \tau \text{ if and only if } \mathcal{S} \subseteq \mathcal{T}.$$

Equivalently, if and only if $s(X) \leq t(X)$ for all objects X. This order has a top element $(\mathbb{X}, 0)$ and a bottom element $(0, \mathbb{X})$, we will denote them as $\mathbb{X}$, 0 respectively if there is no risk of confusion.

We will denote $\mathbb{X}tors$, for the (possibly big) lattice of all torsion theories of $\mathbb{X}$ and $\mathbb{X}_h tors$ for the sublattice of hereditary torsion theories.

In the particular case of $\mathbb{X} = RMod$, the categories of modules over a ring R, the notation $(RMod)_h tors = R_h tors$ is used and it is always a small frame. This lattice has been used extensively in the characterization of rings R (see [Gol86]).

Lemma 2.2.12. Given torsion theories $\tau = (\mathcal{T}, \mathcal{F})$ and $\sigma = (\mathcal{S}, \mathcal{G})$ in a normal category $\mathbb{X}$ with idempotent radicals t, s. If $\sigma \leq \tau$ then for all objects X there is a short exact sequence in $\mathbb{X}$:

$$0 \longrightarrow t(X)/s(X) \longrightarrow G(X) \longrightarrow F(X) \longrightarrow 0$$

where F, G are the reflectors of $\mathcal{F}$ and $\mathcal{G}$.

Proof. It follows form Noether's third isomorphisms theorem 1.3.8 for the normal subobjects $t(X)$, $s(X)$ of X. More precisely, since $F(X) = X/t(X)$ and $G(X) = X/s(X)$ we have

$$X/t(X) \cong \frac{X/s(X)}{t(X)/s(X)}.$$

□

2.2.13. For torsion theories $\sigma \leq \tau$ as in 2.2.12, we will be interested in the functor given by the quotient of the preradicals, s, t:

$$t/s : \ \mathbb{X} \longrightarrow \mathbb{X} \ , \quad t/s(X) = t(X)/s(X).$$

By the previous lemma we have two short exact sequences in endofunctors of $\mathbb{X}$:

$$0 \longrightarrow s \longrightarrow t \longrightarrow t/s \longrightarrow 0$$

and

$$0 \longrightarrow t/s \longrightarrow G \longrightarrow F \longrightarrow 0 \ .$$

where G and F are the endofunctors given by the reflectors to the torsion-free subcategories.

2.3 Torsion theories in algebra

Our most important example, actually, will be studied in the next chapter. However, we provide some basic examples in order to clarify some of the theory of this chapter and recall some historical results in order to give an overview of some of the applications provided by torsion theories.

2.3.1 Abelian categories

Torsion theories were introduce early in the 60's by Dickson ([Dic66]) for abelian categories. Quickly, they were used to study and even characterize noncommutative rings via the torsion theories of their category of modules, *RMod* (see, for example, the extensive monograph [Gol86] or [Ste75]). Historically, a first major contribution is:

Theorem 2.3.1. [3](Maranda, Gabriel, see [Ste75]) For a ring R, there is a bijective correspondence between:

1. Hereditary torsion theories.
2. Left exact radicals in *Rmod*.
3. Left Gabriel topologies.
4. Localizations of *Rmod*. [4]

Here a Gabriel topology can be seem as the additive counter part of a Grothendieck topology ([LM92]) for a ring R when it is considered as a category of one element and *Rmod* as a category of additive presheaves over R. In this case, a Gabriel topology reduces to a family of left ideals of R satisfying some axioms.

Moreover, this is the basis for the Gabriel-Popescu theorem, [GP64] that presents any Grothendieck category as a localization of a category of modules with respect to a particular torsion theory (see [Ste75], [Pop73]).

2.3.2. As a first example: in *Ab*, the categories $\mathbb{T}$ of torsion abelian groups and $\mathbb{F}$ of torsion-free abelian groups gives a hereditary torsion theory $(\mathbb{T}, \mathbb{F})$ in *Ab*. So for a group X taking the torsion subgroup $T(X)$, the subgroup of X of elements of finite order, defines a hereditary radical.

$$\begin{tikzcd} \mathbb{T} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & Ab \arrow[l, bend left, "T"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{F} \arrow[l, bend left] \end{tikzcd}$$

In a similar way, for a prime number p we have a hereditary torsion theory given by the torsion subcategory $\mathbb{T}_p$ of p-torsion groups, where a group is said to be of p-torsion if its elements have order a power of p.

[3]In [Bor94], the case of locally finitely presentable abelian categories is exposed. Using Gabriel topologies is not possible, however the theorem includes the bijection with universal closure operators.

[4]Originally, the term Giraud subcategory was used, a reflective subcategory with a reflection that preserves kernels. For abelian categories this is the same as preserving all finite limits, hence a localization.

Also in Ab consider the category Div of divisible abelian groups and the category Red of reduced abelian groups, where a group is called reduced if the only divisible subgroup is 0; then (Div, Red) is a non-hereditary torsion theory of Ab. Clearly, $\mathbb{Q}$ is divisible but $\mathbb{Z}$ is reduced. Now, for a group X taking the largest divisible subgroup $d(X)$ of X, is only an idempotent radical.

$$\begin{tikzcd} Div \arrow[r, bend left] \arrow[r, phantom, "\perp"] & Ab \arrow[l, bend left, "d"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & Red \arrow[l, bend left] \end{tikzcd}$$

2.3.3. For a prime p and $n \in \mathbb{N}$ we define the full replete subcategory $\mathbb{T}_{p,n}$ of Ab, whose objects are abelian groups whose elements have order p^m with $m \leq n$. Clearly, $\mathbb{T}_{p,n}$ is closed under subobjects and quotients but it is not closed under extensions in Ab. To see this, take $p = 2$ and $n = 1$, $\mathbb{T}_{2,1} \cong \mathbb{Z}_2 Vect$ the category of vector spaces over $\mathbb{Z}_2$, is suffices to consider

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4/\mathbb{Z}_2 \longrightarrow 0\ .$$

So $\mathbb{T}_{p,n}$ determines a hereditary preradical. Now, it is clear that in the lattice $Ab_h tors$ of hereditary of torsion theories of Ab the category $\mathbb{T}$ of torsion abelian groups is maximal, this means that there in no non-trivial hereditary torsion categories that contain $\mathbb{T}$. This follows, that since a torsion subcategory $\mathcal{T}$ of Ab containing $\mathbb{T}$ would have an abelian group with elements without finite order and thus containing a copy of $\mathbb{Z}$ since $\mathcal{T}$ is closed under subobjects. Now, since torsion categories are closed under coproducts and quotients in Ab, having a generator implies that the torsion category $\mathcal{T} \cong Ab$.

On the other hand, the lattice $Ab_h tors$ the subcategories $\mathbb{T}_p$ are minimal, i.e. there no non-trivial hereditary torsion category contained in $\mathbb{T}_p$. This is easy to see, since a non-trivial torsion subcategory $\mathcal{T}$ of Ab contained in $\mathbb{T}_p$ must have a p-abelian group and thus a copy of $\mathbb{Z}_p$ since $\mathcal{T}$ is closed under extensions. Hence, $\mathcal{T}$ contains all cyclics groups $\mathbb{Z}_{p^n}$, a family of generators of $\mathbb{T}_p$. This also proves that $\mathbb{T}_p$ is the smallest hereditary torsion category containing the categories $\mathbb{T}_{p,n}$.

2.3.4. In abelian categories, it is very natural to ask when the torsion theory splits, i.e. when the torsion subobject $t(X)$ is a direct summand of X, so

$$\begin{tikzcd} 0 \arrow[r] & t(X) \arrow[r, shift left, "\sigma_x"] & X \arrow[l, shift left] \arrow[r, shift left, "\eta_X"] & F(X) \arrow[l, shift left] \arrow[r] & 0 \end{tikzcd}$$

is a split short exact sequence and $X \cong t(X) \oplus F(X)$. Clearly, our most basic example, $(\mathbb{T}, \mathbb{F})$ in Ab, does not satisfy this. But since $\mathbb{Z}$ is a noetherian ring, the category $Ab_{f.g.}$ of finitely generated abelian groups is an abelian subcategory of Ab and, indeed, the restriction of $(\mathbb{T}, \mathbb{F})$ to $Ab_{f.g.}$ splits.

A second elementary example, torsion abelian groups $\mathbb{T}$ being a hereditary torsion subcategory of an abelian category is itself abelian. Now, recall that a torsion group is a direct sum of its p-components or p-torsion,

$$T(X) = \bigoplus_p p(X).$$

So each torsion theory of $\mathbb{T}_p$ for each prime p splits in $\mathbb{T}$. In fact, every hereditary torsion theory in $\mathbb{T}$ splits since the torsion subgroup is determined by choosing a family of primes $L = \{p_i\}_i$ and thus defining the torsion subgroup as

$$T_L(X) = \bigoplus_{i \in L} p_i(X).$$

Moreover, $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory in $\mathbb{T}$ if and only if $(\mathcal{F}, \mathcal{T})$ is a torsion theory in $\mathcal{T}$.

([Ste75]) For *Rmod* and a torsion theory $(\mathcal{T}, \mathcal{F})$, if there is a central idempotent e of the ring R such that $t(X) = eX$ then $(\mathcal{T}, \mathcal{F})$ splits. Closely related to this problem, is to study when a torsion category $\mathcal{T}$ is also torsion-free, called a torsion torsion-free subcategory or TTF for short. So $\mathcal{T}$ is a TTF subcategory of $\mathbb{X}$ when there is $\mathcal{F}$ and $\mathcal{C}$ subcategories such that $(\mathcal{C}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{F})$ are torsion theories for $\mathbb{X}$. TTF subcategories can be found outside abelian categories, for example in triangulated categories ([BR07]) and in chapter 4 we will introduce an example of a TTF subcategory in a normal ideal detemermined category.

2.3.2 Non-abelian categories

2.3.5. In *Grp*, taking the commutator subgroup G' of a group G, the subgroup generated by all the commutator elements $[a, b] = aba^{-1}b^{-1}$, is a radical. The category Ab is a normal epireflective subcategory of *Grp* where the reflector is the abelianization functor $ab(G) = G/G'$. Ab is reflective in *Grp* but not torsion-free, clearly the commutator is not idempontent. If we consider the categories introduced in 2.2.6 we have

$$ker(ab()) = Ab^{\leftarrow} = \{G \mid G' = G\}$$

i.e. its the category of perfect groups (groups such that $G = G'$). And any non-abelian group with order p^3 with p prime, like the dihedral D_4 or the quaternion group $\mathbb{H}$, is in $\mathcal{T}_{\mathcal{F}}$ but not in $ker(ab())$.[5]

Definition 2.3.6. Let $\mathbb{A}$ be a reflective subcategory of a normal category $\mathbb{X}$ as in 1.3. Then the reflector R is called:

- [BCGS08] a *protolocalization* if R preserves short exact sequences;
- [EG10] a *protoadditive reflector* if R preserves split short exact sequences.

For abelian categories a localization and protolocalization are the same, and being protoadditive means to preserve finite products.

It was proved in [EG15] (Theorem 2.6), that hereditary torsion theories in homological categories have a protoadditive reflection and in [CGJ18] is shown that Grp does not have any non-trivial protoadditive reflector or protolocalizations. So in Grp any hereditary torsion theory is trivial. But a non hereditary torsion theory example in Grp is given by adapting $(\mathbb{T}', \mathbb{F}')$ where torsion-free groups are taken in the usual sense and torsion groups are groups generated by elements of finite order. In order to clarify why this torsion theory is not hereditary take $GL_2(\mathbb{Q})$, the invertible matrix group of 2×2 with coefficients in $\mathbb{Q}$. Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 1/2 & 0 \end{pmatrix}, \quad AB = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

thus $H =< A, B >$ is torsion since both A, B have order 2 but but AB has no finite order, so $< AB > \cong \mathbb{Z}$ is torsion-free.

2.3.7. In [GKV16] it is proved that the category $Hopf_{K,coc}$ of cocommutative Hopf algebras over a field K with characteristic zero is a semi-abelian category and moreover there is a hereditary torsion theory given by

$$\begin{tikzcd} Lie_K \arrow[r, bend left] \arrow[r, phantom, "\perp"] & Hopf_{K,coc} \arrow[l, bend left] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & Grp \arrow[l, bend left] \end{tikzcd}$$

the torsion subcategory as Lie_K the category of Lie alegbras over K and the torsion-free subcategory is Grp. Moreover, the reflector of this torsion theory is also a localization ([GKV18]).

[5]It follows from the fact that for a non-abelian group G with order p^3 with p prime it has $G' = Z(G) \simeq \mathbb{Z}_p$

2.4 Torsion theories and localizations

We shall study some properties that are well-known in abelian categories but have been overlooked in the non-abelian context and need special attention.

2.4.1 Localizations and hereditary torsion subcategories

[6]

The next result is based on 2.3.1. First, notice that given a reflection $R \dashv i$ of a pointed category $\mathbb{X}$ with unit η, for an object X then $R(X) \cong 0$ if and only if η_X is a zero morphism.

Theorem 2.4.1. Let $\mathbb{A}$ be a protolocalization as in 2.3.6 of a normal category $\mathbb{X}$ with reflector L and unit η. The subcategories of $\mathbb{X}$

$$\mathcal{T}_L = ker(L) = \{X \mid L(X) \cong 0\} = \{X \mid \eta_X = 0\}$$

and

$$\mathcal{F}_L = \{X \mid \eta_X : X \to iL(X) \text{ is monic}\}$$

define a torsion theory $(\mathcal{T}_L, \mathcal{F}_L)$ in $\mathbb{X}$.

Proof. TT1). For a morphism $f : X \to Y$ consider the diagram given by the naturality of η:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ iL(X) \arrow[r, "iL(f)"'] & iL(Y)\,. \end{tikzcd}$$

Now, if $\eta_X = 0$ and η_Y is monic is clear that f is zero.

TT2). Consider for an object X the regular epi-mono factorization (p, m) of η_X and the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & ker(\eta_X) \arrow[r, "k"] & X \arrow[r, "p"] \arrow[dr, "\eta_X"'] & \eta_X(X) \arrow[r] \arrow[d, "m"] & 0 \\ & & & iL(X)\,. & \end{tikzcd}$$

[6]This section is expanded from [Lop22b].

To see that $ker(\eta_X)$ is torsion, consider the commutative diagram:

$$\begin{tikzcd}
ker(\eta_X) \arrow[r, "k"] \arrow[d, "\eta_{ker(\eta_X)}"'] & X \arrow[d, "\eta_X"] \\
iL(ker(\eta_X)) \arrow[r, "iL(k)"'] & iL(X)\,.
\end{tikzcd}$$

Since L is preserves short exact sequences then $iL(k)$ is monic and since $\eta_x k = 0$ this implies that $\eta_{ker(\eta_X)} = 0$. To see that $\eta_X(X)$ is torsion-free consider the diagram:

$$\begin{tikzcd}
X \arrow[rr, "\eta_X"] \arrow[dr, "p"] \arrow[dd, "\eta_X"] & & iL(X) \arrow[dd, "\eta_{iL(X)}"] \\
& \eta_X(X) \arrow[ur, "m"] \arrow[dd, "\eta_{\eta_X(X)}"] & \\
iL(X) \arrow[rr, "iL(\eta_X)"] \arrow[dr, "iL(p)"'] & & iLiL(X) \\
& iL(\eta_X(X))\,. \arrow[ur, "iL(m)"'] &
\end{tikzcd}$$

Notice that since L is a reflector then $iL(\eta_X)$ and $\eta_{iL(X)}$ are isomorphisms. Since $iL(M)\eta_{\eta_X(X)}$ is a monomorphism then $\eta_{\eta_X(X)}$ is also a monomorphism. $\square$

Corollary 2.4.2. Let $\mathbb{A}$ be a localization of a normal category $\mathbb{X}$ then there is a torsion theory $(\mathcal{T}_L, \mathcal{F}_L)$ as in 2.4.1.

Corollary 2.4.3. The torsion theory $(\mathcal{T}_L, \mathcal{F}_L)$ induced by a protolocalization $L \dashv i$ of a normal category $\mathbb{X}$ is normally hereditary (see 2.4.5 below), i.e. the category $\mathcal{T}$ is closed under normal subobjects.

Proof. Since L preserves short exact sequences it preseves kernels, so if $m : S \to X$ is a normal monomorphism with X torsion then $L(S) = 0$. $\square$

Corollary 2.4.4. The torsion theory $(\mathcal{T}_L, \mathcal{F}_L)$ induced by a localization $L \dashv i$ of a normal category $\mathbb{X}$ is hereditary.

Proof. Since L is exact it preserves monomorphism so if $m : S \to X$ with $L(X) = 0$ then $L(S) = 0$. $\square$

It is also worth mentioning that, under the hypothesis from above, a localization $L : \mathbb{X} \to \mathbb{A}$ induces a preradical on $\mathbb{X}$ as $r = ker(\eta)$, so it can be proved that $\mathcal{T}_L = \mathcal{T}_r$.

Since we are working with different kind of monomorphisms we can introduce weaker notions of hereditary torsion theories.

Definition 2.4.5. A preradical r in a normal category $\mathbb{X}$ is called *normally hereditary* if for all normal subobjects $m : X \to Y$ the diagram

$$\begin{tikzcd} r(X) \arrow[r, "r(m)"] \arrow[d, "\sigma_X"'] & r(Y) \arrow[d, "\sigma_Y"] \\ X \arrow[r, "m"] & Y \end{tikzcd}$$

is a pullback. Equivalently, form 2.1.8 (the same proof can be easily adapted), a preradical r is normally hereditary if and only if r is idempotent and the subcategory $\mathcal{T}_r$ is closed under normal subobjects.

Moreover, we say:

- A torsion theory $(\mathcal{T}, \mathcal{F})$ in a normal category $\mathbb{X}$ is called *normally hereditary* if $\mathcal{T}$ is closed under normal subobjects;
- ([GR07]) A torsion theory $(\mathcal{T}, \mathcal{F})$ in a normal category $\mathbb{X}$ is called *quasi-hereditary* if $\mathcal{T}$ is closed under regular monomorphisms, i.e. for every equalizer $e : E \to T$ with T in $\mathcal{T}$ then so is E in $\mathcal{T}$.

Lemma 2.4.6. For r a preradical in a normal category $\mathbb{X}$ the following are equivalent:

1. r is normally hereditary;
2. For all normal subobject $m : X \to Y$, $r(X) = X \cap r(Y)$;
3. The preradical $r : \mathbb{X} \to \mathbb{X}$ is left exact, i.e. it preserves left exact sequences.

Proof. 1) $\Leftrightarrow$ 2) is trivial from the definition. 2) $\Leftrightarrow$ 3) is also easy to see, just consider the commutative diagram in $\mathbb{X}$

$$\begin{tikzcd} 0 \arrow[r] & r(X) \cap K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & r(X) \arrow[r, "r(f)"] \arrow[d] & r(Y) \arrow[d] & \\ 0 \arrow[r] & K \arrow[r, "k"] & X \arrow[r, "f"] & Y \arrow[r] & 0\,. \end{tikzcd}$$

Since $r(X) \cap K$ is the kernel of $r(f)$ then r is left exact if and only if $r(X) \cap K = r(K)$. $\square$

We recall that for a coreflective subcategory and in particular for a torsion subcategory $\mathcal{T}$ of $\mathbb{X}$, colimits in $\mathcal{T}$ are computed as in $\mathbb{X}$ but limits are not necessarily computed as in $\mathbb{X}$, recall 1.4.3.

Lemma 2.4.7. Let $(\mathcal{T}, \mathcal{F})$ be a normally hereditary torsion theory of a normal category $\mathbb{X}$. Then $\mathcal{T}$ is closed under kernel pairs of arrows $f : A \to B$ with A in $\mathcal{T}$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} ker(p_1) \arrow[r] \arrow[d, "\cong"] & Eq(f) \arrow[r, "p_1"] \arrow[d, "p_0"] & A \arrow[d, "f"] \\ ker(f) \arrow[r] & A \arrow[r, "f"] & B\,. \end{tikzcd}$$

Since $\mathcal{T}$ is closed under normal subobjects and A is torsion then $ker(f)$ is torsion. Now, since $\mathcal{T}$ is closed under extensions in $\mathbb{X}$ and we have the isomorphism $ker(f) \cong ker(p_1)$ then $Eq(f)$ is torsion. □

Lemma 2.4.8. Let $(\mathcal{T}, \mathcal{F})$ be a normally hereditary torsion theory of a normal category $\mathbb{X}$ and the inclusion $i : \mathcal{T} \to \mathbb{X}$ preserves pullbacks. Then $\mathcal{T}$ is also a normal category. Moreover, if $\mathbb{X}$ is a normal exact category then $\mathcal{T}$ is also a exact normal category.

Proof. We will first prove that an arrow q in $\mathcal{T}$ is a regular epimorphism in $\mathbb{X}$ if and only if it is a regular epimorphism in $\mathcal{T}$. Clearly, if q is a regular epimorphism in $\mathcal{T}$ and the inclusion $i : \mathcal{T} \to \mathbb{X}$ preserves colimits then q is a regular epimorphism in $\mathbb{X}$. Now, if q is a regular epimorphism in $\mathbb{X}$ it is a coequalizer of its kernel pair $Eq(q)$ in $\mathbb{X}$ and, since $\mathcal{T}$ is closed under kernel pairs in $\mathbb{X}$ we have the isomorphism $Eq(q) \cong t(Eq(q))$, so q is a regular epimorphism in $\mathcal{T}$. Moreover, since $\mathcal{T}$ is normally hereditary normal epimorphisms coincide with regular epimorphisms in $\mathcal{T}$.

To prove pullback stability of regular epimorphism consider the pullback diagram in $\mathcal{T}$:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ A \arrow[r] & B \end{tikzcd}$$

with p a regular epimorphism in $\mathcal{T}$. Since the inclusion $i : \mathcal{T} \to \mathbb{X}$ preserves pullbacks and quotients, we have that p is a regular epimorphism in $\mathbb{X}$, and so p' is a regular epimorphism in $\mathcal{T}$.

Finally, since $\mathcal{T}$ is closed under quotients and $\mathbb{X}$ is exact any equivalence relation in $\mathcal{T}$ must be effective and $\mathcal{T}$ is exact. □

Notice that if $i : \mathcal{T} \to \mathbb{X}$ preserves pullbacks, then it preserves finite limits since it obviously preserve the zero object. In the context of a homological category $\mathbb{X}$ we have:

Proposition 2.4.9. ([GR07], Theorem 3.3) Let $(\mathcal{T}, \mathcal{F})$ a torsion theory in a homological category $\mathbb{X}$. The following conditions are equivalent:

1. $(\mathcal{T}, \mathcal{F})$ is quasi-hereditary
2. the associated idempotent radical $t : \mathbb{X} \to \mathbb{X}$ preserves finite limits.
3. the associated idempotent radical $t : \mathbb{X} \to \mathbb{X}$ preserves equalizers.
4. for every regular subobject $e : E \to A$ in $\mathbb{X}$, then $F(e)$ is a monomorphism in $\mathcal{F}$.

In the abelian context, hereditary, normally hereditary and quasi-hereditary are the same and the idempotent radical t is left exact if and only if it preserves finite limits as it is shown in 2.3.1.

Theorem 2.4.10. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in a homological category $\mathbb{X}$. If $(\mathcal{T}, \mathcal{F})$ is quasi-hereditary then $\mathcal{T}$ is a homological category. Accordingly, if $(\mathcal{T}, \mathcal{F})$ is hereditary then $\mathcal{T}$ is a homological category.

Proof. It follows form 2.4.8 and 2.4.9 that $\mathcal{T}$ is regular. It remains to see that $\mathcal{T}$ is protomodular, but this follows form the fact that t preserves finite limits and so the inclusion $i : \mathcal{T} \to \mathbb{X}$ preserves exact sequences. Notice that i is also conservative. $\square$

Corollary 2.4.11. If $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory in a semi-abelian category $\mathbb{X}$ then $\mathcal{T}$ is a semi-abelian category.

Proof. Since $\mathcal{T}$ is coreflective in $\mathbb{X}$ it is finitely complete and finitely cocomplete. It follows from 2.4.8 and 2.4.9 that $\mathcal{T}$ is exact. Since the inclusion $i : \mathcal{T} \to \mathbb{X}$ preserves exact sequences $\mathcal{T}$ is protomodular and finally $\mathcal{T}$ is semi-abelian. $\square$

2.4.2 Sheaves over a topological space

Beside 2.3.7 not many examples of (non-abelian) localizations of semi-abelian categories have been studied. We will study a large family of examples using the category of sheaves over a topological space. We recall the fundamentals using [LM92] and [CV04] as main references.

2.4.12. We fix a topological space X. Consider the category given by the topology $\mathcal{O}(X)$ of X, i.e. the category given by the preordered set of open sets of X. The objects of $\mathcal{O}(X)$ are the open sets of X and there is exactly one morphism $U \to V$ if $U \subseteq V$ for U, V open sets of X.

The category of *presheaves over* X in $Sets$ is the functor category

$$psh(X, sets) = [\mathcal{O}(X)^{op}, Sets].$$

So a presheaf P is a contravariant functor:

$$P: \begin{array}{ccc} \mathcal{O}(X)^{op} & \longrightarrow & Sets \\ U \subseteq V & & P(V) \to P(U)\,. \end{array}$$

Usually, the morphism $P(V) \to P(U)$ is written as restriction of elements, $f \mapsto f|_U$ for $f \in P(V)$.

Definition 2.4.13. A presheaf P is called a *sheaf* if is satisfies: for each open covering $U = \cup_i U_i$, $i \in I$ for an open U of X there is an equalizer diagram:

$$P(U) \xrightarrow{e} \prod_i P(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} P(U_i \cap U_j)$$

where for $f \in P(U)$, $e(f) = \{f|_{U_i}\}_i$ and for a family $f_i \in P(U_i)$

$$p(f_i) = \{f_i|_{U_i \cap U_j}\}, \quad q(f_i) = \{f_j|_{U_i \cap U_j}\}.$$

We write $sh(X, Sets)$ for the full subcategory of sheaves of $psh(X, Sets)$.

Sheaves capture the local properties of X. So, for example, for a topological space Y we have the sheaf over X of continuous functions

$$Cont(\_, Y) : \mathcal{O}(X) \to Sets, \quad Cont(U, Y) = \{f : U \to Y \mid \text{is continuous}\}.$$

But in general, the presheaf of constant functions $Cte(\_, Y) : \mathcal{O}(X) \to Sets$ is not a sheaf.

The category $sh(X, Sets)$ is a localization of $psh(X, Sets)$, we will describe the reflector usually called the "sheafification" functor.

2.4.14. ([LM92]) For a continuous function $p : Y \to X$, we have the sheaf of *local sections*:

$$\Gamma(p) : \mathcal{O}(X)^{op} \to Sets, \quad \Gamma(p)(U) = \{s : U \to E \mid ps = 1_U\}.$$

This defines a functor:

$$\Gamma : \ Top/X \longrightarrow sh(X, Sets)\,.$$

Now, consider a presheaf P over X and for $x \in X$, the set of open sets of X that contain x is a filtered family, so we define the *Stalk* P_x of P at x as the "downwards" colimit:

$$P_x = colim_{x \in U} P(U)\,.$$

For p we can define a continuous function $p : \Lambda_P \to X$ as follows. Consider the disjoint union of all stalks of p and the morphism p as:

$$\Lambda_P = \coprod_x P_x, \qquad p : \begin{array}{ccc} \Lambda_P & \longrightarrow & X \\ ([f] \in P_x) & & x \end{array}$$

Then for each $s \in P(U)$ for an open U we define a function $\dot{s}$:

$$\dot{s} : \begin{array}{ccc} U & \longrightarrow & \Lambda_P \\ x & & ([s] \in P_x)\,, \end{array}$$

then we give Λ_P the topology by taking as base of opens all the sets $\dot{s}(U) \subset \Lambda_P$. This topology makes p and each $\dot{s}$ continuous. Moreover, the morphism $p : \Lambda_P \to X$ is a *local homeomorphism* (also called étale); each element of Λ_P has a open neighborhood which is mapped by p homeomorphically onto an open subset of X.

The construction of Λ_P is functorial and we have a diagram:

$$\begin{tikzcd}
psh(X, Sets) \arrow[r, shift left, "\Gamma"] \arrow[drr, "\Gamma"] & & Top/X \arrow[ll, shift left, "\Lambda"] \arrow[dll, "\Lambda"'] \\
sh(X, Sets) \arrow[u] & & Et/X\,. \arrow[u]
\end{tikzcd}$$

Theorem 2.4.15. (see [LM92]) For any topological space X there is an adjunction:

$$\begin{tikzcd}
psh(X, Sets) \arrow[r, bend left, "\Lambda"] \arrow[r, phantom, "\perp"] & Top/X \arrow[l, bend left, "\Gamma"]
\end{tikzcd}\,.$$

This adjunction restricted to $sh(X, Sets)$ and Et/X yields an equivalence:

$$\begin{tikzcd}
sh(X, Sets) \arrow[r, bend left, "\Lambda"] \arrow[r, phantom, "\cong"] & Et/X \arrow[l, bend left, "\Gamma"]
\end{tikzcd}\,.$$

Moreover, $Sh(X, Sets)$ is a reflective subcategory of $psh(X, Sets)$ and Et/X is a coreflective subcategory Top/X.

Thus the sheafification functor $\mathcal{S}$ is given by

$$\mathcal{S} = \Gamma\Lambda : \ psh(X, Sets) \longrightarrow sh(X, Sets) \ .$$

The reflector $\mathcal{S}$ is a localization since both Γ and Λ preserve finite limits. It follows from the fact that in $psh(X, Sets)$ limits and colimits are computed component-wise and that in $Sets$ finite limits and filtered colimits commute, in particular taking stalk at a point $x \in X$ commutes with finite limits.

2.4.16. The category $psh(X, Sets)$ is exact since it is a functor category over $Sets$, and thus $sh(X, Sets)$ being a localization of an exact category is also exact. Now, we will consider the category $psh(X, Grp)$ of presheaves over X on groups, a presheaf P over X on groups is nothing than a contravariant functor $P : \mathcal{O}(X)^{op} \to Grp$. So now each sets $P(U)$ is a group and the restrictions $P(V) \to P(U)$ are group morphisms.

We will introduce a definition that will also be used in the next chapters.

2.4.17. A *group object* or internal group in a category $\mathbb{X}$ with products and terminal object 0 is a object G with morphisms: $e : 0 \to G$, $m : G^2 \to G$ and $i : G \to G$ such that the diagrams commute: the identities

$$\begin{tikzcd} G \arrow[r, "{(1,e)}"] \arrow[dr, "1"'] & G^2 \arrow[d, "m"] & G \arrow[l, "{(e,1)}"'] \arrow[dl, "1"] \\ & G & \end{tikzcd}$$

the associativity

$$\begin{tikzcd} G^3 \arrow[r, "p_0 \times m"] \arrow[d, "m \times p_2"'] & G^2 \arrow[d, "m"] \\ G^2 \arrow[r, "m"'] & G \end{tikzcd}$$

and the inverses

$$\begin{tikzcd} G \arrow[r, "{(1,i)}"] \arrow[dr, "1"'] & G^2 \arrow[d, "m"] & G \arrow[l, "{(i,1)}"'] \arrow[dl, "1"] \\ & G & \end{tikzcd}$$

We write $Grp(\mathbb{X})$ for the category of internal group objects in $\mathbb{X}$.

2.4.18. Since in $psh(X, Sets)$ limits are computed component wise we have $Grp(psh(X, Sets)) \cong psh(X, Grp)$. Now, recall that limits in Grp are computed as in $Sets$, so for a P in $psh(X, Grp)$ the diagram in 2.4.13:

$$P(U) \xrightarrow{e} \prod_i P(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} P(U_i \cap U_j)$$

is an equalizer in Grp if and only if it is an equalizer in $Sets$. In other words P is a sheaf if $UP : \mathcal{O}(X)^{op} \to Grp \to Sets$ is a sheaf in $Sets$, where U is the forgerful functor. This also proves that $Grp(sh(X, Sets)) \cong sh(X, Grp)$. Since both $psh(X, Sets)$ and $sh(X, Sets)$ are exact categories we have that $psh(X, Grp)$ and $sh(X, Grp)$ are semi-abelian (see [BB04]), since $Grp(\mathbb{X})$ is exact protomodular whenever $\mathbb{X}$ is exact.

It is also worth studying $Grp(Top/X)$. In Top/X the terminal object is 1_X and the product of two objects $f : A \to X$, $g : B \to X$ is given by the pullback in Top:

$$\begin{tikzcd} A \times_X B \arrow[r, "p_1"] \arrow[d, "p_0"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & X . \end{tikzcd} \tag{2.5}$$

If we have a group object $(p : G \to X, e, i, m)$ in Top/X, notice that in $G \times_X G = \{(a, b) \in G \times G \mid p(a) = p(b)\}$ the elements of the product are pairs (a, b) of elements in G that are in the same fibre $p^{-1}(x)$ for some $x \in X$. From the commutative diagram:

$$\begin{tikzcd} G \times_X G \arrow[rr, "m"] \arrow[dr, "p'"'] & & G \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

we have that $pm(a, b) = p'(a, b) = p(a) = p(b) = x$. So the operation m is restricted to each fibre:

$$\begin{tikzcd} G \times_X G \arrow[r, "m"] & G \\ p^{-1}(x) \times_X p^{-1}(x) \arrow[u] \arrow[r, "m"] & p^{-1}(X) . \arrow[u] \end{tikzcd}$$

Similarly, also the morphism i is restricted to each fibre. So a group object in Top/X can be described as a continuous function $p : G \to X$ where each fibre $p^{-1}(x)$ is a group and the induced operations $m : G \times_X G \to G$, $i : G \to G$ and $e : X \to G$ are continuous.

It is worth mentioning that the pullback of two local homeomorphisms is a local homeomorphism.

2.4.19. The definition of a group object only requires finite products, so a functor $F : \mathbb{A} \to \mathbb{B}$ that preserves finite products induces a functor $F : Grp(\mathbb{A}) \to Grp(\mathbb{B})$. The functor $\Gamma : Top/X \to psh(X, Sets)$ preserves binary products. Just consider 2.5 and for an open sets U and sections $s_f : U \to A$ and $S_g : U \to B$ such that $fs_f = 1_U$, $gs_g = 1_U$ define a section $(s_f, s_g) : U \to A \times_X B$ and, conversely, a local section of $s : U \to A \times_X B$ with projections induces $p_0 s$ and $p_1 s$ local sections of f and g; so $\Gamma(fp_0 = gp_1) \cong \Gamma(f) \times \Gamma(g)$.

The functor $\Lambda : psh(X, Sets) \to Top/X$ also preserves finite products. Indeed, in $Sets$, filtered colimits commute with finite products and coproducts commute with pullbacks, we have that a product of presheaves $P \times Q$ is mapped to

$$\begin{tikzcd}
\coprod_x P_x \times Q_x \arrow[r] \arrow[d] & \coprod_x Q_x \arrow[d] \\
\coprod_x P_x \arrow[r] & X.
\end{tikzcd}$$

in Top/X.

Finally the adjunction from 2.4.15

$$\begin{tikzcd}
psh(X, Grp) \arrow[r, bend left, "\Lambda"] \arrow[r, phantom, "\perp"] & Grp(Top/X) \arrow[l, bend left, "\Gamma"]
\end{tikzcd}.$$

gets restricted to

$$\begin{tikzcd}
psh(X, Grp) \arrow[r, bend left, "\Gamma\Lambda"] \arrow[r, phantom, "\perp"] & sh(X, Grp) \arrow[l, bend left, "i"]
\end{tikzcd} ;$$

which is a localization of semi-abelian categories.

Let $\mathcal{N}$ be the full subcategory of $psh(X, Sets)$ given by the presheaves P such that the stalks are trivial i.e. $P_x = 0$ for all $x \in X$, such presheaves are called *negligible* ([Pop73]). The previous localization induces a hereditary torsion theory as in 2.4.1. We can characterize the torsion subcategory as follows.

Proposition 2.4.20. For a topological space X. Let

$$\mathcal{S} = \Gamma\Lambda : psh(X, Grp) \to sh(X, Grp)$$

the localization as in 2.4.19 and the torsion theory $(\mathcal{T}, \mathcal{F})$ in $psh(X, Grp)$ as in 2.4.1. Then $\mathcal{T} = Ker(\mathcal{S}) \cong \mathcal{N}$ where $\mathcal{N}$ is the subcategory of negligible presheaves over X.

Proof. Since $psh(X, Grp) \cong Grp(Top/X)$, a presheaf P in $psh(X, Grp)$ is in $\mathcal{T}$ if and only if $\Gamma(P) \cong 1_X$:

$$\begin{tikzcd} \Gamma(P) = \coprod_x P_x \arrow[rr, "\cong"] \arrow[dr, "p"'] & & X \arrow[dl, "1_X"] \\ & X & \end{tikzcd}$$

Moreover, the fibres of p and 1_X are isomorphic for each $x \in X$, so P is in $\mathcal{T}$ if and only if each $P_x = 0$ for each x. $\square$

In 2.4.18 we can replace Grp by any other algebraic theory $\mathbb{T}$. The category $\mathbb{T}Alg$ of models in $Sets$ also will satisfy $psh(X, \mathbb{T}Alg) \cong \mathbb{T}Alggpsh(X, Sets)$. The reason is due mainly to the fact that all limits and filtered colimits in $\mathbb{T}Alg$ are computed as in $Sets$ ([ARV10]) and for a $\mathbb{T}$-algebra in Top/X this means that each operation is restricted to fibres. In [BJ02] algebraic theories $\mathbb{T}$ such that $\mathbb{T}Alg$ is semi-abelian are characterized. Any semi-abelian algebraic theory will then satisfy 2.4.20.

Chapter 3

Torsion theories and internal groupoids

As a first step towards simplicial objects, we shall focus on the category of internal groupoids in $\mathbb{X}$. We will introduce two examples of torsion theories for internal groupoids. For the particular case of *Grp*, internal groupoids correspond to Whitehead's crossed modules so the torsion theories in internal groupoids are also studied via normalization as torsion theories in crossed modules.

The concept of Mal'tsev category is introduced, first, as a more general context for categorical algebra, since homological/semi-abelian categories are Mal'tsev categories. And equally important, the Mal'tsev axiom has important consequences in the study of internal groupoids. Firstly, reflexive relations are necessarily equivalence relations and, second internal categories are actually internal groupoids. Moreover, simplicial objects in Mal'tsev categories always satisfy the Kan condition which is needed to define an homotopy relation in simplicial objects (this shall be discussed in the next chapter). We recommend [BB04] as a complete reference for Mal'tsev categories.

Crossed modules were introduced by Whitehead ([Whi41]), mainly as a structure useful in algebraic topology. Now they provide an example of a semi-abelian category that is studied in its own right. We will review the fundamentals of crossed modules of groups and expose the relations between crossed modules, precrossed modules and groups. As the main reference in crossed modules we will use [BHS10] and for a more in-depth treatment [LLR04] ,[ACL07], [CCG02], [LG94]

Torsion theories in internal grupoids have been studied since their first appearances in the non-abelian context and have provided different applications since, for example [BG06], [EG10], [EG13], [EG15] and [Man15].

3.1 Internal groupoids in Mal'tsev categories

Definition 3.1.1. An *internal category* X in a category with pullbacks $\mathbb{X}$ is given by a diagram in $\mathbb{X}$:

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "\pi_2"] \arrow[r, "m"] \arrow[r, shift right=2, "\pi_0"'] & X_1 \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] & X_0 \arrow[l, "e"'] \end{tikzcd}$$

where

1. X_0 is the "object of objects";
2. X_1 is the "object of morphisms";
3. X_2 is the "object of composable morphisms".

and the morphisms δ_0, δ_1, e, m that are called the domain, codomain, identity and composition morphisms respectively. These morphisms are required to satisfy the "equations of a category" such as domain/codomain of the identities, associativity of the composition, etc. More precisely, the square

$$\begin{tikzcd} X_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_0"'] & X_1 \arrow[d, "\delta_0"] \\ X_1 \arrow[r, "\delta_1"'] & X_0 \end{tikzcd}$$

is required to be a pullback and the morphisms satisfy the equations

$$\delta_0 e = \delta_1 e = 1_{X_0},\ \ \delta_1\pi_2 = \delta_1 m,\ \ \delta_0\pi_0 = \delta_0 m,\ \ m(e\delta_0, 1_{X_1}) = 1_{X_1} = m(1_{X_1}, e\delta_1)$$

where $(e\delta_0, 1_{X_1}), (1_{X_1}, e\delta_1)$ are induced by the pullback X_2 and taking the pullback

$$\begin{tikzcd} X_3 \arrow[r, "p_1"] \arrow[d, "p_0"'] & X_2 \arrow[d, "\pi_0"] \\ X_2 \arrow[r, "\pi_1"'] & X_1\,. \end{tikzcd}$$

the equation $m(mp_o, \delta_1 p_1) = m(\delta_0 p_0, mp_1)$ is satisfied.

A morphism $f : X \to Y$ of internal categories is given by a pair of morphisms (f_0, f_1) in $\mathbb{X}$ such that

$$\begin{tikzcd}
X_2 \arrow[r, shift left=2, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_0"'] \arrow[d, "(f_1,f_1)"'] & X_1 \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] \arrow[d, "f_1"] & X_0 \arrow[l, "e" description] \arrow[d, "f_0"] \\
Y_2 \arrow[r, shift left=2, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_0"'] & Y_1 \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] & Y_0 \arrow[l, "e" description]
\end{tikzcd}$$

commutes serially, i.e. $f_0\delta_0 = \delta_0 f_1,\ f_1 e = e f_0,\ f_1 m = m(f_1, f_1)$ and so on. A morphism f of internal categories is a normal monomorphism if and only if f_0 and f_1 are kernels in $\mathbb{X}$ and, if $\mathbb{X}$ is a semi-abelian or homological category, f is a normal epimorphism if and only if f_0 and f_1 are normal epimorphisms (see [BG00]).

Definition 3.1.2. An internal category X is a *groupoid*, "if every arrow is invertible", i.e. if it is provided with $i : X_1 \to X_1$,

$$\begin{tikzcd}
X_2 \arrow[r, shift left=2, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_0"'] & X_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] & X_0 \arrow[l, "\sigma_0" description]
\end{tikzcd}$$

such that it "inverts morphisms", i.e.:

$$\delta_0 i = \delta_1,\ \delta_1 i = \delta_0,\ m(1_{X_1}, i) = e\delta_0,\ m(i, 1_{X_1}) = e\delta_1.$$

When $\mathbb{X}$ has finite products and a terminal object 0:

- an *internal group* or group object in $\mathbb{X}$ is a groupoid X in $\mathbb{X}$ such that $X_0 = 0$ (so the pullback X_2 is the product X_1^2), i.e. it looks like

  $$\begin{tikzcd}
  G^2 \arrow[r, shift left=2, "p_2"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_0"'] & G \arrow[loop above, "i"] \arrow[r, shift left=2] \arrow[r, shift right=2] & 0 \arrow[l]
  \end{tikzcd}$$

  for an object G, where p_0, p_2 are the product projections.

- An *abelian group object* is a group object such that m is abelian, i.e. $mt = m$ where $t : G^2 \to G^2$ is the twist morphism, $t = (p_2, p_0)$.

An internal group G in $\mathbb{X}$, is usually defined as an object G with morphisms $m : G^2 \to G$, $e : 0 \to G$ and $i : G \to G$ so we have the commutative diagrams for the “equations of groups”: the identities

$$\begin{tikzcd} G \arrow[r, "{(1,e)}"] \arrow[dr, "1"'] & G^2 \arrow[d, "m"] & G \arrow[l, "{(e,1)}"'] \arrow[dl, "1"] \\ & G & \end{tikzcd}$$

the associativity

$$\begin{tikzcd} G^3 \arrow[r, "p_0 \times m"] \arrow[d, "m \times p_2"'] & G^2 \arrow[d, "m"] \\ G^2 \arrow[r, "m"'] & G \end{tikzcd}$$

and the inverses

$$\begin{tikzcd} G \arrow[r, "{(1,i)}"] \arrow[dr, "1"'] & G^2 \arrow[d, "m"] & G \arrow[l, "{(i,1)}"'] \arrow[dl, "1"] \\ & G & \end{tikzcd}$$

which correspond exactly to the diagrams given by the definition of an internal groupoid.

For a category $\mathbb{X}$ with finite limits, we write $Cat(\mathbb{X})$, $Grpd(\mathbb{X})$, $Grp(\mathbb{X})$, $Ab(\mathbb{X})$ and $Eq(\mathbb{X})$, $Rrel(\mathbb{X})$ for the categories of internal categories, internal groupoids, internal groups, internal abelian groups, internal equivalences relations and internal reflexive relations in $\mathbb{X}$, respectively. Clearly, we have

$$Eq(\mathbb{X}) \subseteq Grpd(\mathbb{X}) \subseteq Cat(\mathbb{X}), \quad Eq(\mathbb{X}) \subseteq Rrel(\mathbb{X})$$

and

$$Ab(\mathbb{X}) \subseteq Grp(\mathbb{X}) \subseteq Grpd(\mathbb{X})\,.$$

Definition 3.1.3. ([CLP91]) A category $\mathbb{X}$ is called a *Mal'tsev* category or is said to satisfy the Mal'tsev property if any internal reflexive relation in $\mathbb{X}$ is an equivalence relation, i.e., $Rrel(\mathbb{X}) = Eq(\mathbb{X})$.

3.1.4. A varieties of universal algebras a category is Mal'tsev if and only if there is a ternary operation p called the Mal'tsev operation that satisfies $p(x, y, y) = x$ and $p(x, x, y) = y$ for all x and y. So Grp is a Mal'tsev category since we have $p(x, y, z) = xy^{-1}z$ and using the same operation any variety having an under-lying group structure is still a Mal'tsev category. If a category $\mathbb{X}$ is protomodular then $\mathbb{X}$ also satisfies the Mal'tsev property, this includes homological and semi-abelian categories. But being normal/regular and the Mal'tsev property are completely independent conditions, we can even have a normal ideal determined category that fails to be Mal'tsev (see [JMTU10] and [BB04]).

Clearly, a group G is exactly an internal group in *Sets*. However, it is not an internal group in *Grp* unless it is abelian, m is required to be a group morphism, so $Grp(Grp) = Ab(Grp) = Ab$. More generally, in a Mal'tsev category $\mathbb{X}$ internal groups are always abelian, so $Ab(\mathbb{X}) = Grp(\mathbb{X})$[1].

A crucial result for Mal'tsev categories is:

Theorem 3.1.5. ([CPP92]) If $\mathbb{X}$ is a Mal'tsev category then any every internal category is a groupoid, $Cat(\mathbb{X}) = Grpd(\mathbb{X})$[2].

Similar to the fundamental groups of topological spaces/simplicial sets we introduce:

Definition 3.1.6. Let $\mathbb{X}$ be a normal Mal'tsev category. The connected components functors defined as

$$\pi_0 : \; Grpd(\mathbb{X}) \longrightarrow \mathbb{X} \; , \quad \pi_0(X) = coeq(\delta_0, \delta_1)$$

and a groupoid X is called connected if the morphism $(\delta_0, \delta_1) : X_1 \to X_0^2$ is a normal epimorphism. So a groupoid is *connected* if the morphism (δ_0, δ_1) is a normal epimorphism if and only if $coeq(\delta_0, \delta_1) = 0$. The subcategory of connected internal groupoids will de denoted as $Conn(Grpd(\mathbb{X}))$

And the "object of automorphisms of 0" functor:

$$\pi_1 : \; Grpd(\mathbb{X}) \longrightarrow \mathbb{X} \; , \quad \pi_1(X) = ker(\; (\delta_0, \delta_1) : X_1 \to X_0^2 \;).$$

3.1.7. A *reflexive graph* in $\mathbb{X}$ is a diagram

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] & X_0 \arrow[l, "\sigma_0" description] \end{tikzcd}$$

where $\delta_0 \sigma_0 = \delta_1 \sigma_0 = 1_{X_0}$. Given a finitely cocomplete regular Mal'tsev category $\mathbb{X}$, if a reflexive graph admits a structure of a groupoid this is necessarily unique. Moreover, given a morphism of reflexive graphs between groupoids is necessarily a groupoid morphism. If $\mathbb{X}$ is a regular Mal'tsev category the forgetful functor $U : Grpd(\mathbb{X}) \to Rgph(\mathbb{X})$ admits a left adjoint functor. With

[1] In a Mal'tsev category, internal abelian groups are exactly the objects who admits an internal Mal'tsev operation p, a morphism satisfying "$p(x, y, y) = x$" and "$p(x, x, y) = y$", so we can define the group structure with $m = p(x, 1, z)$ and $i = p(1, y, 1)$.

[2] For $\mathbb{X} = Grp$, this is verified by noticing that the composition m factors through the Mal'tsev operation $p(x, y, z) = xy^{-1}z$, as $m(f, g) = p(f, \sigma_0\delta_1(f), g)$. And even the inverse of arrows i can be defined with p, $i(f) = p(\sigma_0\delta_0(f), f, \sigma_0\delta_1(f))$.

this in mind we will usually write a groupoid only with its components X_1, X_0 (see [BB04] and [Ped95]).

3.1.8. Among the subcategory $Eq(\mathbb{X})$ of $Grpd(\mathbb{X})$ it is important to distinguish two important subcategories:

- $Dis(\mathbb{X})$ of discrete equivalence relations: $1_G = G \to G$

$$\begin{tikzcd} G \arrow[r, shift left=2, "1"] \arrow[r, "1" description] \arrow[r, shift right=2, "1"'] & G \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "1"'] & G \arrow[l] \end{tikzcd} .$$

- $Ind(\mathbb{X})$ of indiscrete equivalence relations: $0 = G \to 0$

$$\begin{tikzcd} G^3 \arrow[r, shift left=2, "\phi_2"] \arrow[r, "\phi_1" description] \arrow[r, shift right=2, "\phi_0"'] & G^2 \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] & G \arrow[l, "\Delta" description] \end{tikzcd} .$$

Where $p_0, p_1 : G^2 \to G$ are the product projections and for G^3 we write $\phi_0 = (p_0, p_1)$, $\phi_1 = (p_0, p_2)$ and $\phi_2 = (p_1, p_2)$ where $p_0, p_1, p_2 : G^3 \to G$ are the product projections.

We can define the full embeddings given by the discrete and indiscrete relations of an object G in $\mathbb{X}$:

$$D : \mathbb{X} \longrightarrow Grpd(\mathbb{X}) \ , \quad Ind : \mathbb{X} \longrightarrow Grpd(\mathbb{X}) \ .$$

Actually, we have a string of adjuctions[3]:

$$\pi_0 \dashv D \dashv ()_0 \dashv Ind : \begin{tikzcd} Grpd(\mathbb{X}) \arrow[d, bend right=60, "\pi_0"'] \arrow[d, phantom, shift right=5, "\dashv"] \arrow[d, bend right=20] \arrow[d, phantom, "\dashv"] \arrow[d, phantom, shift left=5, "\dashv"] \\ \mathbb{X} \arrow[u, bend right=20] \arrow[u, bend right=60] \end{tikzcd}$$

We write $(X)_0 = X_0$ and $(X)_1 = X_1$. Since D and Ind are full embeddings, we recall quickly the non-trivial units/counits of these adjunctions. Consider a groupoid X. The unit of $\pi_0 \dashv D$:

$$\begin{tikzcd} X_1 \arrow[r, "q\delta_0 = q\delta_1"] \arrow[d, shift right, "\delta_0"'] \arrow[d, shift left, "\delta_1"] & \pi_0(X) \arrow[d, shift right, "1"'] \arrow[d, shift left, "1"] \\ X_0 \arrow[r, "q"'] & \pi_0(X) = coeq(\delta_0, \delta_1) \end{tikzcd} .$$

[3]The adjunction $\pi_0 \dashv D$ is studied in [Bou87] for the case of internal groupoids in an exact category $\mathbb{X}$. The adjunctions $D \dashv ()_0$ and $()_0 \dashv Ind$ are well-known, for example the case of $\mathbb{X} = Sets$ is mentioned in [GZ67]

The counit of $D \dashv ()_0$:

$$\begin{tikzcd} X_0 \arrow[r, "\sigma_0"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "1"] & X_1 \arrow[d, shift right, "\delta_0"'] \arrow[d, shift left, "\delta_1"] \\ X_0 \arrow[r, "1"'] & X_0 \,. \end{tikzcd}$$

The unit of $()_0 \dashv Ind$:

$$\begin{tikzcd} X_1 \arrow[r, "(\delta_0,\delta_1)"] \arrow[d, shift right, "\delta_0"'] \arrow[d, shift left, "\delta_1"] & X_0^2 \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\ X_0 \arrow[r, "1"'] & X_0 \,. \end{tikzcd}$$

All these subcategories of $Grpd(\mathbb{X})$ give rise to non-abelian torsion theories.

Proposition 3.1.9. ([BG06]) For a normal Mal'tsev category $\mathbb{X}$, the pair $(Ab(\mathbb{X}), Eq(\mathbb{X}))$ is a hereditary torsion theory in $Grpd(\mathbb{X})$.

$$\begin{tikzcd} Ab(\mathbb{X}) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & Grpd(\mathbb{X}) \arrow[l, bend left, "\pi_1"] \arrow[r, bend left, "supp"] \arrow[r, phantom, "\perp"] & Eq(\mathbb{X}) \arrow[l, bend left] \end{tikzcd}$$

Proof. For a groupoid X, the reflector is the support functor *supp* given by the image factorization of the induced morphism $\eta = (\delta_0, \delta_1)$:

$$\begin{tikzcd} X_1 \arrow[rr, "\eta=(\delta_0,\delta_1)"] \arrow[dr, "e_\eta"'] & & X_0 \times X_0 \\ & \eta_X(X_1) \arrow[ur, "m_\eta"'] & \end{tikzcd}$$

Then this gives a factorization of the groupoid X through e_η:

$$\begin{tikzcd} X_1 \arrow[rr, "e_\eta"] \arrow[dr, shift left, "\delta_1"] \arrow[dr, shift right, "\delta_0"'] & & \eta(X_1) = supp(X) \arrow[dl, shift right, "d_1"'] \arrow[dl, shift left, "d_0"] \\ & X_0 & \end{tikzcd}$$

Clearly, $supp(X)$ is an equivalence relation since m_η is a monomorphism. So the short exact sequence of the torsion theory is

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & X_1 \arrow[r, "e_\eta"] \arrow[d, shift right, "\delta_0"'] \arrow[d, shift left, "\delta_1"] & \eta(X_1) \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ 0 \arrow[r] & X_0 \arrow[r, "id"'] & X_0 \,. \end{tikzcd}$$

The $\pi_1(X)$ inherits an internal abelian group structure from X. To see that there are no morphisms between $Ab(X)$ and $Eq(X)$ different from zero, just consider a morphism $f = (f_0, f_1)$:

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, shift right] \arrow[d, shift left] & E_1 \arrow[d, shift right, "\delta_0"'] \arrow[d, shift left, "\delta_1"] \\ 0 \arrow[r, "f_0"'] & E_0 , \end{tikzcd}$$

the morphism f induces a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f_1"'] & 0 \times 0 = 0 \arrow[d] \\ E_1 \arrow[r, "(\delta_0,\delta_1)"'] & E_0 \times E_0 \end{tikzcd}$$

so if (δ_0, δ_1) is monic, then f_1 must be zero and so is f. Since $Ab(\mathbb{X})$ is always closed under subobjects in $Grpd(\mathbb{X})$ if $\mathbb{X}$ is a Mal'tsev category, then the torsion theory is hereditary. $\square$

Proposition 3.1.10. ([EG10], [EG15]) For a normal category $\mathbb{X}$, the pair $(Conn(Grpd(\mathbb{X})), Dis(\mathbb{X}))$, where $Conn(Grpd(\mathbb{X}))$ is the full subcategory of connected groupoids, is a cohereditary torsion theory in $Grpd(\mathbb{X})$.

$$\begin{tikzcd} Conn((Grpd(\mathbb{X}))) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & Grpd(\mathbb{X}) \arrow[l, bend left, "\Gamma"] \arrow[r, bend left, "\pi_0"] \arrow[r, phantom, "\perp"] & Dis(\mathbb{X}) \simeq \mathbb{X} \arrow[l, bend left, "D"] \end{tikzcd}$$

Proof. Writting $q = coeq(\delta_0, \delta_1) : X_0 \to coeq(\delta_0, \delta_1)$, the short exact sequence is given by

$$\begin{tikzcd} \Gamma_1(X) \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & X_1 \arrow[r, "q\delta_0 = q\delta_1"] \arrow[d, shift right, "\delta_0"'] \arrow[d, shift left, "\delta_1"] & coeq(\delta_0, \delta_1) \arrow[d, shift right, "1"'] \arrow[d, shift left, "1"] \\ \Gamma_0(X) \arrow[r] & X_0 \arrow[r, "q"'] & coeq(\delta_0, \delta_1) \end{tikzcd}$$

The kernel groupoid Γ of $(q\delta_0, q)$ correspond to the full subgroupoid of the connected component of 0, it is easy to see that indeed it is connected as in the definition 3.1.6. $\square$

Notice that $(Ab(\mathbb{X}), Eq(\mathbb{X})) \leq (Conn(Grpd(\mathbb{X})), Dis(\mathbb{X}))$.

3.2 Crossed modules

In this section we study the particular case when $\mathbb{X}$ is the category *Grp* of groups. Then the category of internal groupoids in groups is equivalent to the category of Whitehead's crossed modules; therefore the previous torsion theories in internal groupoids can be studied as torsion theories in crossed modules.

A classical problem in topology is to study the second homotopy group. In this direction the work of S. Mac Lane and J. H. C. Whitehead shows that crossed modules are weak pointed homotopy 2-types, for instance the standard geometric example of a crossed module is the boundary morphism of the second relative homotopy group

$$\partial : \ \pi_2(X, X_1, x) \longrightarrow \pi_1(X_1, x) \ .$$

Later, the investigation on double groupoids and, in particular, the version of the 2-dimensional Seifert-van Kampen Theorem by Brown and Spencer (1971-1973) led back to crossed modules. The categories of crossed modules and internal groupoids are equivalent, this result was known to J. L. Verdier was later expanded by J. L. Loday to the categories of 1-cat groups and simplicial groups with Moore complex of lenght 1. However, it seems that it was first R. Lavendhomme, that discovered the equivalence of internal categories in groups and crossed modules. Now, for the 'category-theorists' crossed modules and internal categories are studied in their own right, for instance internal crossed modules in a semi-abelian category (instead of groups) were introduced by G. Janelidze in [Jan03], and they form themselves a semi-abelian category. [4]

All group actions will be written on the left as ${}^{g}()$ and each group G will act on itself by conjugation as ${}^{g}(a) = g^{-1}ag$.

Definition 3.2.1. A *crossed module* in *Grp* is a group morphism: $\partial : A \to B$ with an action of B on A, written ${}^{b}(a)$, such that:

XM1 $\partial({}^{b}(a)) = {}^{b}(\partial(a)) = b^{-1}\partial(a)b$. ($\partial$ is equivariant)

XM2 ${}^{\partial(a)}(a') = a^{-1}a'a$. (Peiffer identity)

for all a and a' in A and b in B. If ∂ only satisfies XM1 is called a *precrossed module*. A morphism $f : \partial \to \partial'$ of precrossed/crossed modules is a pair of

[4] These historical remarks are taken from [BHS10], [Jan03] and [BS76].

morphisms (f_0, f_1) such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\partial"] \arrow[d, "f_1"'] & B \arrow[d, "f_0"] \\ X \arrow[r, "\partial'"'] & Y \end{tikzcd}$$

and is compatible with the actions: $f_1({}^b(a)) = {}^{f_0(b)}(f_1(a))$. The categories of crossed modules and precrossed modules will be written as $\mathbb{X}mod$ and $P\mathbb{X}Mod$ respectively.

3.2.2. Notice that for a precrossed module $\partial : A \to B$, the image $\partial(A)$ is a normal subgroup of B. Moreover, if ∂ is a crossed module then $ker(\partial) \leq Z(A)$, where $Z(A)$ is the center subgroup of A, so it is an abelian subgroup. So, a morphism $G \to 0$ is a precrossed module but it is only a crossed module if and only if G is abelian.

In the category of $\mathbb{X}Mod$, the monomorphisms are exactly where (f_0, f_1) are both injective. A morphism (f_0, f_1) is a regular epimorphism if and only if f_0, f_1 are surjective but epimorphisms are not necessarily surjective. A normal crossed submodule (A, B, ∂) of (X, Y, ∂') is a submodule where A and B are normal in X and Y, ${}^y(a) \in A$ and ${}^b(x)x^{-1} \in X$ for all a, y, b, x. The quotient is then given by $(X/A, Y/B, \partial'')$ (see [LLR04]).

The full embedding of $\mathbb{X}Mod$ into $P\mathbb{X}Mod$ admits a left adjoint, thus $\mathbb{X}Mod$ is a reflective subcategory. For a precrossed module $\partial : A \to B$ an element of the form ${}^{\partial(a)}a'(aa'a^{-1})^{-1}$ for a, a' in A is called a *Peiffer element* and the subgroup generated by them, $< A, A >$ is the *Peiffer commutator*. It can be shown that the reflector $R : P\mathbb{X}Mod \to \mathbb{X}Mod$ is defined by

$$R : \begin{array}{ccc} P\mathbb{X}Mod & \longrightarrow & \mathbb{X}Mod \\ \partial : A \to B & & R(\partial) : \frac{A}{<A,A>} \to B \end{array}$$

Note that ∂ is a crossed module if and only if $< A, A >= 0$.

3.2.3. In $\mathbb{X}mod$ we have the full replete subcategories:

- *Ab* of abelian groups, i.e. $\partial : A \to 0$ for A an abelian group.
- *NMono* of inclusion of normal subgroups, i.e. $\partial = i : N \to G$, the action is given by conjugation.
- *CExt* of central extensions, given by the surjective crossed modules ∂ :

$A \to B$. So

$$0 \longrightarrow ker(\partial) \longrightarrow A \xrightarrow{\partial} B \longrightarrow 0$$

is indeed a central extension in groups (the kernel is central in A).

- Dis of discrete crossed modules, i.e. $\partial : 0 \to G$ for a group G.
- Ind of indiscrete crossed modules, i.e. $\partial = 1_G : G \to G$ for a group G.

In conjunction we have the functors, for a precrossed module $\partial : A \to B$ and a group G :

- $H_0 : P\mathbb{X}Mod \to Grp$, $H_0(\partial) = cok(\partial) = B/\partial(A)$;
- $D : Grp \to P\mathbb{X}Mod$, $D(G) = 0 \to G$;
- $()_0 : P\mathbb{X}Mod \to Grp$, $(\partial)_0 = B$;
- $Ind = Grp \to P\mathbb{X}Mod$, $Ind(G) = 1_G : G \to G$;
- $()_1 = P\mathbb{X}Mod \to Grp$, $(\partial)_1 = A$;
- $D' = Grp \to P\mathbb{X}Mod$, $D'(G) = G \to 0$;
- $H_1 = P\mathbb{X}Mod \to Grp$, $H_1(\partial) = ker(\partial)$.

Moreover, these functors form a string of adjunctions (see [CCG02]):

$$H_0 \dashv D \dashv ()_0 \dashv Ind \dashv ()_1 : \begin{tikzcd}[row sep=large] \mathbb{X}Mod \arrow[d, bend right=40] \arrow[d, bend left=40] \\ Grp \arrow[u, bend left=80] \arrow[u] \arrow[u, bend right=80] \end{tikzcd}$$

Similar to the groupoid case, since D and Ind are full embeddings we recall the non-trivial unit/counit components (written vertically) for a crossed module $\partial : A \to B$:

The unit of $H_0 \dashv D$:

$$\begin{tikzcd} A \arrow[r, "\partial"'] \arrow[d] & B \arrow[d, "cok(\partial)"] \\ 0 \arrow[r] & cok(\partial) \end{tikzcd}$$

The counit of $D \dashv ()_0$:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B \arrow[d, "1_B"] \\ A \arrow[r, "\partial"'] & B \end{tikzcd}$$

The unit of $()_0 \dashv Ind$:

$$\begin{tikzcd} A \arrow[r, "\partial"] \arrow[d, "\partial"'] & B \arrow[d, "1_B"] \\ B \arrow[r, "1_B"'] & B \end{tikzcd}$$

The counit of $Ind \dashv ()_1$:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "\partial"] \\ A \arrow[r, "\partial"'] & B \end{tikzcd}$$

It is worth noticing, that the diagram of the unit of $()_0 \dashv Ind$ commutes if and only if ∂ satisfies axiom XM1 and the diagram of the counit of $Ind \dashv ()_1$ commutes if and only if ∂ satisfies XM2. So for precrossed modules, $P\mathbb{X}Mod$, only $H_0 \dashv D \dashv ()_0 \dashv Ind$ are adjoints. Also for $P\mathbb{X}Mod$ there are the adjunctions $()_1 \dashv D' \dashv H_1$ that do not hold when we restrict to $\mathbb{X}Mod$. However, there will be a torsion theory when restricted Grp to Ab as follows.[5]

Proposition 3.2.4. In $\mathbb{X}Mod$, the pair $(Ab, NMono)$ is an hereditary torsion theory.

$$\begin{tikzcd} Ab \arrow[r, bend left, "D'"] \arrow[r, phantom, "\perp"] & \mathbb{X}Mod \arrow[l, bend left, "H_1"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & NMono \arrow[l, bend left] \end{tikzcd}$$

Proof. The short exact sequence for a crossed module $\partial : A \to B$ is

$$\begin{tikzcd} ker(\partial) \arrow[r] \arrow[d] & A \arrow[r, "e_\partial"] \arrow[d, "\partial"] & \partial(A) \arrow[d, "m_\partial"] \\ 0 \arrow[r] & B \arrow[r, "id"] & B \end{tikzcd}$$

where (e_∂, m_∂) is the normal epi-mono factorization of ∂. Clearly, there are no non-trivial morphisms between Ab and $NMono$. $\square$

Proposition 3.2.5. In $\mathbb{X}Mod$, the pair $(CExt, Dis)$ is a cohereditary torsion theory.

$$\begin{tikzcd} CExt \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathbb{X}Mod \arrow[l, bend left] \arrow[r, bend left, "H_0"] \arrow[r, phantom, "\perp"] & Dis \simeq Grp \arrow[l, bend left, "D"] \end{tikzcd}$$

[5]In an even more limiting case, for a pointed category with kernels and cokernels $\mathbb{C}$, we can define these functors between $\mathbb{C}$ and the category of arrows $Arr(\mathbb{C})$ and yield the adjunctions $H_0 \dashv D \dashv ()_0 \dashv Ind \dashv ()_1 \dashv D' \dashv H_1$.

Proof. Similarly, for a crossed module $\partial : A \to B$ the short exact sequence is

$$\begin{array}{ccccc} A & \xrightarrow{id} & A & \longrightarrow & 0 \\ {\scriptstyle e_\partial}\downarrow & & \downarrow{\scriptstyle \partial} & & \downarrow \\ \partial(A) & \xrightarrow[m_\partial]{} & B & \longrightarrow & cok(\partial)\,, \end{array}$$

where e_∂, m_∂ is the normal epi-mono factorization of ∂. It is also easy to see that there are no non-trivial morphisms between $CExt$ and Dis. □

The next two results are due to Loday [Lod82], however here the proofs follow the calculations in [BHS10]. It is worth mentioning that in these works there is no mention of the Mal'tsev operation in groups, and yet it is used implicitly in the calculations.

Proposition 3.2.6. (see [Lod82]) A reflexive graph

$$\delta_0, \sigma_0, \delta_1 : \; X_1 \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} X_0$$

in Grp, admits a groupoid structure if and only if $[ker(\delta_0), ker(\delta_1)] = 0$.

Proof. We begin with a useful observation. If X is a groupoid the composition morphism m factorises through the Mal'tsev operation by $(\pi_0, \sigma_0\delta_0\pi_2, \pi_2)$

$$m : \begin{array}{ccccc} X_2 & \longrightarrow & X_1^3 & \xrightarrow{\quad p \quad} & X_1 \\ (f,g) & \longmapsto & & & f - \sigma_0\delta_0(g) + g\,. \end{array}$$

This follows from the calculations:

$$\begin{aligned} m(f,g) = m(f+0, 0+g) &= m(f+0, \sigma_0\delta_0(g) - \sigma_0\delta_0(g) + g) \\ &= m(f, \sigma_0\delta_0(g)) + m(0, -\sigma_0\delta_0(g) + g) = f - \sigma_0\delta_0(g) + g \end{aligned}$$

Similarly, we can write $m(f,g) = g - \sigma_0\delta_0(g) + f$. From this, if $f \in ker(\delta_1)$ and $g \in ker(\delta_0))$ we have $[ker(\delta_0), ker(\delta_1)] = 0$.

For the converse, for a reflexive graph X, we can take X_2 as the pullback and even define m as before. However, m is not a group morphism unless $[ker(\delta_0), ker(\delta_1)] = 0$ as seen from

$$\begin{aligned} m((f,g) + (a,b)) = m(f+a, g+b) &= f + a - \sigma_0\delta_0(g+b) + g + b \\ & f + a - \sigma_0\delta_0(b) - \sigma_0\delta_0(g) + g + b \end{aligned}$$

and

$$
\begin{aligned}
m(f,g)+m(a,b) &= f-\sigma_0\delta_0(g)+g+a-\sigma_0\delta_0(b)+b \\
&= f-\sigma_0\delta_0(g)+g+a-\sigma_0\delta_1(a)+b
\end{aligned}
$$

since $+a-\sigma_0\delta_1(a)$ is in $ker(\delta_1)$ and $-\sigma_0\delta_0(g)+g$ in $ker(\delta_0)$. $\square$

Since the work of Whitehead on crossed modules, it has been know that the category $\mathbb{X}Mod$ is equivalent to the category of $Cat(Grp) = Grpd(Grp)$.

Theorem 3.2.7. (see [Lod82]) There is an equivalence between $P\mathbb{X}Mod$ and $Rgph(Grp)$ given by the normalization functor M.

$$
M:\ Rgph(Grp) \longrightarrow P\mathbb{X}Mod\ ,\quad M(X)=\delta_1|_{ker(\delta_0)} : ker(\delta_0)\to X_1\to X_0
$$

for a reflexive graph X, $\delta_0,\sigma_0,\delta_1:\ X_1 \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} X_0$.

The inverse functor is

$$
L:\ P\mathbb{X}Mod \longrightarrow Rgph(Grp)
$$

where

$$
L(\partial) =\ A\rtimes B \begin{smallmatrix}\xrightarrow{\partial'}\\ \xleftarrow[i]{}\\ \xrightarrow[p]{}\end{smallmatrix} B
$$

for a precrossed module $\partial : A\to B$ and where p and i are the projection and inclusion of B of the semidirect product and $\partial'(a,b)=\partial(a)b$. Moreover, this equivalence is restricted to an equivalence $\mathbb{X}Mod \cong Grpd(Grp)$

Proof. Since $ker(\delta_0)$ is normal we can define the action of X_0 over $ker(\delta_0)$ by conjugation, ${}^{x}(k)=\sigma_0(x)k\sigma_0(x)^{-1}$, clearly it is a precrossed module. Then for a reflexive graph X there is an isomorphism:

$$
\begin{array}{ccc}
X_1 & \begin{smallmatrix}\xrightarrow{\delta_1}\\ \xleftarrow[\sigma_0]{}\\ \xrightarrow[\delta_0]{}\end{smallmatrix} & X_0 \\
\cong\Big\downarrow\mu & & \Big\downarrow id \\
ker(\delta_0)\rtimes X_0 & \begin{smallmatrix}\xrightarrow{\partial}\\ \xleftarrow[i]{}\\ \xrightarrow[p]{}\end{smallmatrix} & X_0
\end{array}
$$

where $\mu(x_1)=(x_1\sigma_0\delta_0(x_1)^{-1},\delta_0(x_1))$ and $\mu^{-1}(k,x_0)=k\sigma_0(x)$. Notice that $M(X)$ satisfies the Peiffer condition if $[\mathrm{ker}(\delta_0), ker(\delta_1)]=0$:

$${}^{\delta_1(a)}a' = \sigma_0\delta_1(a)a'\sigma_0\delta_1(a)^{-1} = aa^{-1}\sigma_0\delta_1(a)a'\sigma_0\delta_1(a)^{-1} = aa'a^{-1}$$

since $a^{-1}\sigma_0\delta_1(a)$ is in $ker(\delta_1)$.

And finally if ∂ is a crossed module notice that $[ker(p), ker(\partial')] = 0$ since

$$(k, 0)(j^{-1}, \delta_1(j)) = (kj^{-1}, \delta_1(j))$$

and

$$(j^{-1}, \delta_1(j))(k, 0) = ((j^{-1})^{\delta(j)}k, \delta_1(j)) = (j^{-1}jkj^{-1}, \delta_1(j))$$

for j, k in $ker(\delta_0)$. $\square$

Proposition 3.2.8. ([BG06], [EG10]) The normalization functor

$$M : \ Grpd(Grp) \longrightarrow \mathbb{X}Mod$$

restricts to an equivalence of torsion theories:

$$M : \begin{array}{c} (Ab(Grp), Eq(Grp)) \xrightarrow{\cong} (Ab, NMono) \\ \\ (Conn(Grp), Dis(Grp)) \xrightarrow{\cong} (CExt, Dis) \end{array}$$

Proof. We will show that each subcategory is mapped equivalently to its counterpart. Clearly, $Ab(Grp) \simeq Ab$

$$M : (\begin{tikzcd} A \arrow[r, shift left=2, "0"] \arrow[r, shift right=2] & 0 \arrow[l] \end{tikzcd}) = \ A \longrightarrow 0$$

and $Dis(Grp) \simeq Dis \simeq Grp$

$$M : (\begin{tikzcd} G \arrow[r, shift left=2, "id"] \arrow[r, shift right=2, "id"'] & G \arrow[l] \end{tikzcd}) = \ 0 \longrightarrow G\,.$$

Let X be a groupoid in Grp and its normalization

$$M : (\begin{tikzcd} X_1 \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] & X_0 \arrow[l] \end{tikzcd}) = \ ker(\delta_0) \longrightarrow X_0$$

If X is an equivalence relation then $(\delta_0, \delta_1) : X_1 \to X_0^2$ is injective, so for

$k \in ker(\delta_0)$ with $\delta_1(k) = 0$ the pair $(\delta_0(k), \delta_1(k)) = (0, 0)$, so $k = 0$. Also notice that $\delta_0(\sigma_0(x)k\sigma_0(x)^{-1}) = xx^{-1} = 0$, then $M(X)$ is a normal monomorphism.

Conversely, for a crossed module $i : N \to G$ the induced morphism

$$\begin{aligned} N \rtimes G &\longrightarrow G^2 \\ (n, g) &\longmapsto (g, ng) \end{aligned}$$

is clearly injective, so $L(i)$ is an equivalence relation.

If X is connected, $(\delta_0, \delta_1) : X_1 \to X_0^2$ is surjective. So for the pair $(0, x) \in X_0^2$ there is $k \in X_1$ such that $\delta_0(k) = 0$ and $\delta_1(k) = x$. Then the crossed module $ker(\delta_0) \to X_0$ is a central extension.

For a surjective crossed module $\partial : A \to B$ the induced morphism

$$\begin{aligned} A \rtimes B &\longrightarrow B^2 \\ (a, b) &\longmapsto (\partial(a)b, b) \end{aligned}$$

is surjective, since for a $(b_0, b_1) \in B^2$ there is a such that $\partial(a) = b_0 b_1^{-1}$ so the pair (a, b) is mapped onto (b_0, b_1). So $L(\partial)$ is a connected groupoid. $\square$

Remark 3.2.9. From the adjunctions in 3.2.3 some observations can be made. The subcategory of discrete crossed modules, Dis is a torsion-free subcategory of $\mathbb{X}Mod$ that is also coreflective in $\mathbb{X}Mod$ with $D \dashv ()_0$. The counit ϵ of the coreflection, for a crossed module $\partial : A \to B$, is

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "0"'] & B \arrow[d, "1_B"] \\ A \arrow[r, "\partial"'] & B \end{tikzcd}$$

which is monic since the arrows $\epsilon_B = (0, 1_B)$ are injective, but it is not a normal monomorphism, since this would imply ${}^b(a) = a$ for all $a \in A$ and $b \in B$. With 2.2.9 in mind, this shows that Dis is closed under extensions and monocoreflective in $\mathbb{X}Mod$ that is not a torsion subcategory of $\mathbb{X}Mod$. In chapter 4 we will study TTF-subcategories, subcategories that are both torsion and torsion-free subcategories, so Dis is an example of a weaker version of a TTF-subcategory.

Also from 3.2.3, the subcategory of indiscrete crossed modules Ind is reflective and coreflective in $\mathbb{X}Mod$, $()_0 \dashv Ind \dashv ()_1$ so the inclusion $Ind \to \mathbb{X}Mod$ is exact and we have a localization $()_0 \dashv Ind$.

Let be η the unit of $()_0 \dashv Ind$:

$$\begin{array}{ccc} A & \xrightarrow{\partial} & B \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle 1} \\ B & \xrightarrow[1]{} & B \end{array}$$

so $\eta_\partial = (\partial, 1_B)$. The torsion theory in $\mathbb{X}mod$ induced from the localization as in 2.4.1 actually corresponds to $(Ab, NMono)$:

$$\mathcal{T} = Ker((\_)_0 Ind) = \{X \mid (Ind(X))_0 \cong 0\} \cong Ab$$

and

$$\mathcal{F} = \{X \mid \eta_\partial = (\partial, 1_B) \text{ is monic}\} \cong NMono\,.$$

Remark 3.2.10. Given a semi-abelian category $\mathbb{X}$ the notion of an internal action is introduced in [BJK05]. For objects B and X an B-action over X is given by a morphism $\alpha_{B,X} : B\flat X \to X$ where the object $B\flat X$ is the kernel of $(0, 1) : B + X \to X$ that satisfy some conditions. Later in [Jan03] the notion of an internal crossed module in a semi-abelian category $\mathbb{X}$ is introduced. It is also proved the equivalences between internal precrossed modules/internal reflexives graphs and crossed modules/internal categories.

The previous torsion theories in crossed modules in groups can be studied in the internal case of a semi-abelian category as mention in [BG06]. For the purpose of this work we will restrict to the case of groups.

Chapter 4

Torsion theories in simplicial objects

[1]

Following the examples in groupoids and crossed modules, we will try to generalize the torsion theories to simplicial objects. The principal technique is by using truncations in the Moore normalization. A particular lattice of torsion theories given by 'good' truncations is studied and connections with the homology/homotopy objects are given.

4.1 Preliminaries

4.1.1 Simplicial homotopy

We recall the basics of simplicial homotopy. In particular, we introduce the Kan condition and recall its connection with Mal'tsev categories.

Definition 4.1.1. The *simplicial category* Δ is given by

- Objects: the ordered sets $[n] = \{0 < 1 < 2 < \cdots < n\}$ for $n \in \mathbb{N}$.
- Morphisms: the non-decreasing monotone functions.

In particular, we have the morphisms:

- $\delta_i^n : [n-1] \to [n]$ the injection which does not take the value $i \in [n]$.
- $\sigma_i^n : [n+1] \to [n]$ the surjection which takes twice the value $i \in [n]$.

[1] This chapter is adapted from [Lop22b].

If there is no risk of confusion we only write δ_i and σ_i. These morphisms satisfy the relations, called the *simplicial identities*:

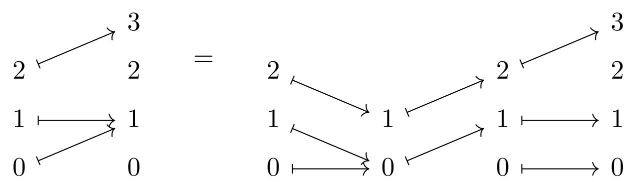
$$
\begin{aligned}
\delta_j\delta_i &= \delta_i\delta_{j-1} \quad \text{if} \quad i < j \\
\sigma_j\sigma_i &= \sigma_i\sigma_{j+1} \quad \text{if} \quad i \leq j \\
\sigma_j\delta_i &= \begin{cases} \delta_i\sigma_{j-1} & \text{if} \quad i < j \\ 1 & \text{if} \quad i = j \text{ or } i = j+1 \\ \delta_{i-1}\sigma_j & \text{if} \quad i > j+1\,. \end{cases}
\end{aligned}
$$

The morphisms δ_i^n, σ_i^n are the generators of Δ, i.e. any morphism $\mu : [m] \to [n]$ can be written in a unique way as

$$\mu = \delta_{i_s}^n \delta_{i_{s-1}}^{n-1} \dots \delta_{i_1}^{n-t+1} \sigma_{j_t}^{m-t} \dots \sigma_{j_2}^{m-2} \sigma_{j_1}^{m-1}\,.$$

such that $n \geq i_s > \cdots > i_1 \geq 0$, $0 \leq j_t < \cdots < j_1 < m$ and $n = m - t + s$. For example:

$$[2] \xrightarrow{\mu} [3] \qquad\qquad [2] \xrightarrow{\sigma_0} [1] \xrightarrow{\delta_0} [2] \xrightarrow{\delta_2} [3]$$



Definition 4.1.2. A *simplicial object* X in a category $\mathbb{X}$ is a functor

$$X : \ \Delta^{op} \longrightarrow \mathbb{X}\ .$$

As a consequence of the simplicial identities X is determined by a family of objects X_i in $\mathbb{X}$ for $0 \leq i$ and morphisms $d_i^n : X_n \to X_{n-1}$ called the *face* morphisms and $s_i^n : X_n \to X_{n+1}$ called the *degeneracy* morphisms. If there is no risk of confusion we will simply write d_i and s_i for the face and degeneracy morphisms.

$$X = \quad \dots X_n \begin{array}{c} \xrightarrow{d_n} \\ \xleftarrow{s_{n-1}} \\ \vdots \\ \xleftarrow{s_0} \\ \xrightarrow[d_0]{} \end{array} \quad \dots \quad \begin{array}{c} \xrightarrow{d_4} \\ \xleftarrow{s_3} \\ \vdots \\ \xleftarrow{s_0} \\ \xrightarrow[d_0]{} \end{array} X_3 \begin{array}{c} \xrightarrow{d_3} \\ \xleftarrow{s_2} \\ \xrightarrow{d_2} \\ \xleftarrow{s_1} \\ \xrightarrow{d_1} \\ \xleftarrow{s_0} \\ \xrightarrow[d_0]{} \end{array} X_2 \begin{array}{c} \xrightarrow{d_2} \\ \xleftarrow{s_1} \\ \xrightarrow{d_1} \\ \xleftarrow{s_0} \\ \xrightarrow[d_0]{} \end{array} X_1 \begin{array}{c} \xrightarrow{d_1} \\ \xleftarrow{s_0} \\ \xrightarrow[d_0]{} \end{array} X_0$$

such that they satisfy:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{if} \quad i < j \\ s_i s_j &= s_{j+1} s_i \quad \text{if} \quad i \leq j \\ d_i s_j &= \begin{cases} s_{j-1} d_i & \text{if} \quad i < j \\ 1 & \text{if} \quad i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if} \quad i > j+1 \,. \end{cases} \end{aligned}$$

A morphism of simplicial objects $f : X \to Y$ is a sequence of morphisms $f_n : X_n \to Y_n$ such that

$$d_i^n f_n = f_{n-1} d_i^n \quad \text{and} \quad s_i^n f_n = f_{n+1} s_i^n \,.$$

We denote the category of simplicial objects in $\mathbb{X}$ as $Simp(\mathbb{X}) = [\Delta^{op}, \mathbb{X}]$.

We introduce the Kan property for simplicial objects in a regular category, this generalizes the classical well-known cases for the categories of *Sets*, *Grp*, *Rmod* and other ones.

Definition 4.1.3. (see [CKP93]) Consider a simplicial object X in a regular category $\mathbb{X}$. For $n \geq 1$ and $k \in [n]$ the object of (n, k)-horns in $\mathbb{X}$ is given by

$$x_i : \; K(n,k) \longrightarrow X_{n-1} \qquad i \in \{0, \dots, n\} - \{k\}$$

such that $d_i x_j = d_{j-1} x_i$ for all $i < j$ with $i, j \neq k$ which is universal with respect to this property. For $n = 1$ we define $K(1,0) = K(1,1) = X_0$. The universal property yields comparison morphisms

$$l(n,k) : \; X_n \longrightarrow K(n,k) \;.$$

We say that X satisfies the *Kan condition* or that is *Kan* when the $l(n,k)$ are regular epimorphisms. In particular, the comparison morphisms to the $(1,k)$-horns are $d_0, d_1 : X_1 \to X_0$.

4.1.4. The importance of the Kan condition lies mainly in that it implies that the homotopy relation is an equivalence relation ([GZ67]). Given a basepoint 0 in X_0 of a Kan simplicial set, we consider $Z_n = \{x \in X_n : d_i(x) = 0 \text{ for all } i = 0, \ldots, n\}$ where $0 = \sigma_i(0)$ for all i. For elements x, x' of Z_n we say that they are homotopic if there is y in X_{n+1}, called an homotopy, such that

$$d_i(y) = \begin{cases} 0 & i < n \\ x & i = n \\ x' & i = n+1\,. \end{cases}$$

The Kan condition implies this is indeed an equivalence relation and then the homotopy groups are defined as $\pi_n(X) = Z_n/\sim$. It was observed by Kan that the singular simplicial set of a topological space is Kan, and Moore ([Moo55]) proved that a simplicial group considered as a simplicial set is also Kan. The same holds for abelian groups and R-modules (see [Wei94], [May67]). In fact, what is needed in these cases to make Moore's proof work is a Malt'sev operation $p(x, y, z)$ such that $p(x, y, y) = x$ and $p(x, x, y) = y$.

We now introduce 'the nerve functor'. Introduced by Grothendieck, the nerve functor can be applied to internal categories in a category $\mathbb{X}$ with finite limits, see for example [Dus75].

Definition 4.1.5. Let $\mathbb{X}$ be a category with finite limits and X an internal category in $\mathbb{X}$:

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, shift left=3, "\delta_2"] \arrow[r] \arrow[r, shift right=3, "\delta_1" description, "\delta_0"'] & X_1 \arrow[r, shift left=2, "\delta_1"] \arrow[r, shift right=2, "\delta_0"'] & X_0 \arrow[l, "s" description] \end{tikzcd}$$

The nerve of X, denoted by $Ner(X)$, is the simplicial object in $\mathbb{X}$ defined as follows:

1. $Ner(X)_0 = X_0$, $Ner(X)_1 = X_1$ and for $n \geq 2$, $Ner(X)_n = X_1^{\times_{X_0}^n} = X_1 \times_{X_0} X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is the object of n-composable arrows, i.e. the limit of

$$\begin{tikzcd}[column sep=small] & X_1 \arrow[dl, "\delta_0"'] \arrow[dr, "\delta_1"] & & X_1 \arrow[dl, "\delta_0"'] \arrow[dr, "\delta_1"] & & & X_1 \arrow[dl, "\delta_0"'] \arrow[dr, "\delta_1"] & \\ X_0 & & X_0 & & X_0 \ldots X_0 & & & X_0\,. \end{tikzcd}$$

2. For $n \geq 3$, the face morphisms $d_0^n = (p_1, p_2, \ldots, p_n)$, $d_n^n = (p_0, p_1 \ldots, p_{n-1})$ forget the first and the last arrow respectively.

   And for $0 < i < n$, $d_i^n = (p_0, \ldots, \delta_1(p_i, p_{i+1}), \ldots, p_n)$ composes the ith-arrow with the $(i+1)$th-arrow.

3. The degeneracy morphism $s_i^n = (p_0, \dots, p_{i-1}, s\delta_0 p_i, p_i, \dots, p_n)$ inserts an identity for the ith-arrow.

This defines the fully faithful nerve functor

$$Ner: \; Cat(\mathbb{X}) \longrightarrow Simp(\mathbb{X}) \; .$$

An internal groupoid in $\mathbb{X}$ always satisfy the Kan condition and moreover an internal category is Kan if and only if it is an internal groupoid ([Dus75]). Recall that if $\mathbb{X}$ is a regular Mal'tsev category internal categories are in fact internal groupoids.

Theorem 4.1.6. ([CKP93] THEOREM 4.2) A regular category $\mathbb{X}$ is a Mal'tsev category if and only if every simplicial object in $\mathbb{X}$ is Kan.

For $\mathbb{X} = Grp$, it was proved by Moore ([Moo55]) that the homotopy groups can be calculated by homology of a chain complex given by normalization. We introduce the Moore normalization functor in a more general context following [EdL04].

Definition 4.1.7. Let X be a simplicial object in a pointed category with pullbacks $\mathbb{X}$. The normalized, or Moore, chain complex $M(X)$ is the chain complex

$$M(X) = \; \dots \xrightarrow{\delta_{n+2}} M(X)_{n+1} \xrightarrow{\delta_{n+1}} M(X)_n \xrightarrow{\delta_n} M(X)_{n-1} \xrightarrow{\delta_{n-1}} \dots$$

with $M(X)_0 = X_0$ and for $n > 0$

$$M(X)_n = \bigcap_{i=0}^{n-1} ker(d_i : \; X_n \longrightarrow X_{n-1} \;)$$

with the differentials

$$\delta_n = d_n \cap_i ker(d_i) : \; M(X)_n \longrightarrow M(X)_{n-1}$$

and $M(X)_i = 0$ for $i < 0$. This defines a functor:

$$M: \; Simp(\mathbb{X}) \longrightarrow ch(\mathbb{X}) \; .$$

Given a simplicial object X and its associated Moore chain complex $M(X)$

they can be represented as:

$$\begin{tikzcd}
X = & \dots \arrow[r, "\vdots", "d_i"'] & X_3 \arrow[r, shift left=4.5, "d_3"] \arrow[r, shift left=1.5, "d_2"] \arrow[r, shift right=1.5, "d_1"] \arrow[r, shift right=4.5, "d_0"'] & X_2 \arrow[r, shift left=3, "d_2"] \arrow[r, "d_1"] \arrow[r, shift right=3, "d_0"'] & X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X_0 \\
M(X) = & \dots \arrow[r] & M(X)_3 \arrow[u, hook] \arrow[r, "\delta_3"] & M(X)_2 \arrow[u, hook] \arrow[r, "\delta_2"] & M(X)_1 \arrow[u, hook] \arrow[r, "\delta_1"] & M(X)_0 \arrow[u, "1"']
\end{tikzcd}$$

4.1.8. In a pointed category $\mathbb{X}$, a chain complex X is a collection of morphisms:

$$X = \ \dots \xrightarrow{\delta_{n+2}} X_{n+1} \xrightarrow{\delta_{n+1}} X_n \xrightarrow{\delta_n} X_{n-1} \xrightarrow{\delta_{n-1}} \dots$$

such that $\delta_n \delta_{n+1} = 0$ for all $n \in \mathbb{Z}$. If $\mathbb{X}$ has kernels and cokernels we can define the n-homology objects of X in two different ways:

$$H_n(X) = cok(X_{n+1} \to ker(\delta_n)), \quad K_n(X) = ker(cok(\delta_{n+1}) \to X_{n-1})\,.$$

In abelian categories the objects $H_n(X)$ and $K_n(X)$ are isomorphic and it was proved in [EdL04] that this is also the case for a pointed regular protomodular category, provided that the chain complex X is proper, i.e. the differential morphisms δ_n have normal images. The category of chain complexes will be studied in-depth in the next section.

For the case of groups, $\mathbb{X} = Grp$, the homotopy groups of a simplicial group X are calculated by the homology of the Moore chain complex:

$$\pi_n(X) \cong H_n(M(X))$$

for all $n \in \mathbb{N}$. The Moore chain complex is proper and even if it is not necessarily a chain complex of abelian groups the homology groups $H_n(M(X))$ are always abelian for $n \geq 1$ ([Moo55]).

In more general categories the following facts are known:

- ([EdL04]) If $\mathbb{X}$ is semi-abelian and X is a simplicial object in $\mathbb{X}$, then the homology objects $H_n(M(X))$ are abelian for $n \geq 1$.
- ([EdL04]) If $\mathbb{X}$ is pointed, regular and protomodular, then the Moore normalization functor M preserves regular epimorphisms. As a consequence it preserves short exact sequences.
- ([Bou07]) If $\mathbb{X}$ is pointed and protomodular, then the Moore normalization functor M is conservative, i.e. it reflects isomorphisms.

- ([Bou07]) If $\mathbb{X}$ is semi-abelian, then the Moore normalization functor M is monadic.
- In [dL09], if $\mathbb{X}$ is semi-abelian with enough projectives the homotopy groups are defined are indeed they can be calculated via the homology objects of the Moore complex.

4.2 Torsion theories in chain complexes

As a first step we will study torsion theories in chain complexes.

4.2.1 Chain complexes in normal categories

Definition 4.2.1. Let $\mathbb{X}$ be a normal category. A chain complex X is a family of objects X_i and morphisms δ_i with $i \in \mathbb{Z}$:

$$X = \quad \dots \longrightarrow X_{i+1} \xrightarrow{\delta_{i+1}} X_i \xrightarrow{\delta_i} X_{i-1} \longrightarrow \dots$$

with the condition $\delta_i\delta_{i+1} = 0$ for all i. A morphism $f : X \to Y$ is a family of morphisms f_i in $\mathbb{X}$:

$$\begin{tikzcd}
\dots \arrow[r, "\delta_{i+2}"] & X_{i+1} \arrow[r, "\delta_{i+1}"] \arrow[d, "f_{i+1}"] & X_i \arrow[r, "\delta_i"] \arrow[d, "f_i"] & X_{i-1} \arrow[r, "\delta_{i-1}"] \arrow[d, "f_{i-1}"] & \dots \\
\dots \arrow[r] & Y_{i+1} \arrow[r, "\delta_{i+1}"'] & Y_i \arrow[r, "\delta_i"'] & Y_{i-1} \arrow[r] & \dots
\end{tikzcd}$$

such that $f_{i-1}\delta_i = \delta_i f_i$ for all i.

Throughout this section for a chain complex X, we will write e_i, m_i for the normal epi/mono factorization of the morphisms δ_i for each i.

We will denote the category of chain complexes in $\mathbb{X}$ as $ch(\mathbb{X})$ and $pch(\mathbb{X})$ for the full subcategory of proper chain complexes, those complexes where each δ_i is a proper morphism i.e. $m_i : \delta_i(X_i) \to X_{i-1}$ is a normal monomorphism:

$$\begin{tikzcd}
X_i \arrow[rr, "\delta_i"] \arrow[dr, "e_i"'] & & X_{i-1} \\
& \delta_i(X_i) \arrow[ur, "m_i"'] &
\end{tikzcd}$$

For a fixed $n \in \mathbb{N}$ we write $ch(\mathbb{X})_{\geq n}$, $ch(\mathbb{X})_{n\geq}$, $pch(\mathbb{X})_{n\geq}$, $pch(\mathbb{X})_{\geq n}$ for the categories of truncated complexes at order n. The categories $ch(\mathbb{X})_{\geq n}$ and $pch(\mathbb{X})_{\geq n}$ are the complexes bounded below at n:

$$X = \ \dots \xrightarrow{\delta_{n+3}} X_{n+2} \xrightarrow{\delta_{n+2}} X_{n+1} \xrightarrow{\delta_{n+1}} X_n$$

and $ch(\mathbb{X})_{n\geq}$ and $pch(\mathbb{X})_{n\geq}$ are bounded above at n:

$$X = \ X_n \xrightarrow{\delta_n} X_{n-1} \xrightarrow{\delta_{n-1}} X_{n-2} \xrightarrow{\delta_{n-2}} \dots$$

and as a special case we write $ch(\mathbb{X})_{n\geq m}$ for the category of bounded complexes for fixed n and m. So, for the categories of arrows/proper arrows, we have $Arr(\mathbb{X}) = ch(\mathbb{X})_{1\geq 0}$ and $PArr(\mathbb{X}) = pch(\mathbb{X})_{1\geq 0}$. Finally, we write $\mathbb{X}_n$ for the subcategory of $pch(\mathbb{X})$ that have trivial objects for all $i \neq n$.

The category $ch(\mathbb{X})$ has all limits and colimits of $\mathbb{X}$, and these are computed component-wise. The category $pch(\mathbb{X})$ is not so well-behaved, for example the kernel in the category of chain complexes may not be a proper one. For example, the proper arrow morphism $f = (f_1, f_0)$:

$$\begin{tikzcd}
< a^2, r > \arrow[r, hook] \arrow[d, "f_1"'] & D_4 \arrow[d, "f_0"] \\
< a^2, r > / < r > \arrow[r] & 0
\end{tikzcd}$$

where D_4 is the dihedral group with generators $a^4 = r^2 = 1$ and $ar = ra^{-1}$. The kernel is given by the inclusion $< r > \to D_4$ which in not a normal subgroup.

By a short exact sequence in $pch(\mathbb{X})$ we mean a short exact sequence in $ch(\mathbb{X})$ where the objects are proper chain complexes.

Lemma 4.2.2. If $\mathbb{X}$ is an ideal determined category then the category $pch(\mathbb{X})$ has cokernels and they are computed as in $ch(\mathbb{X})$.

Proof. We will prove the case for the category $PArr(\mathbb{X})$. For a morphism $f : X \to Y$ of proper arrows (X, d), (Y, δ) consider the commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[dd, "d"] & Y_1 \arrow[rr, "p"] \arrow[dd, "\delta"] \arrow[dr, "e_\delta"] & & cok(f_1) \arrow[dd, "\delta'"] \arrow[dr, "e_{\delta'}"] & \\
 & & \delta(Y_1) \arrow[rr, "q'"] \arrow[dl, "m_\delta"] & & \delta'(cok(f_1)) \arrow[dl, "m_{\delta'}"] \\
X_0 \arrow[r, "f_0"] & Y_0 \arrow[rr, "q"] & & cok(f_0) &
\end{tikzcd}$$

where δ' is induced by universal property of the cokernel p and m_δ, e_δ and

$m_{\delta'}$, $e_{\delta'}$ are the normal epi-mono image factorizations of δ and δ' respectively. Now, since taking images is functorial we have q' such that $q'e_\delta = e_{\delta'}p$ and $m_{\delta'}q' = qm_\delta$, then q' is a normal epimorphism since p and $e_{\delta'}$ are also normal epimorphisms. Finally, since $\mathbb{X}$ is ideal determined and m_δ is a normal monomorphism and $m_{\delta'}$ is a monomorphism, then $m_{\delta'}$ is a normal monomorphism. So $\delta' : cok(f_1) \to cok(f_0)$ is the cokernel of f in $pch(\mathbb{X})$. $\square$

For a fixed n and X an n-truncated chain complex in $pch(\mathbb{X})_{n\geq}$,

$$X = \; X_n \xrightarrow{\delta_n} X_{n-1} \xrightarrow{\delta_{n-1}} X_{n-2} \xrightarrow{\delta_{n-2}} \dots ,$$

there are two convenient ways to extend X to an infinite chain complex in $pch(\mathbb{X})$, so $pch(\mathbb{X})_{n\geq}$ has two different embeddings into $pch(\mathbb{X})$. Also, there are two different ways to n-truncate a chain complex into $pch(\mathbb{X})_{n\geq}$. Different authors use different notations and differ in which are the 'good' truncations. We choose to follow the terminology of [BHS10], although it is used in a very different setting.

4.2.3. For a fixed $n \in \mathbb{Z}$ we define the functors:

- $\mathbf{tr}_n : \; ch(\mathbb{X}) \longrightarrow ch(\mathbb{X})_{n\geq}$ is the canonical (upward) truncation:

$$\mathbf{tr}_n(X) = \; X_n \longrightarrow X_{n-1} \longrightarrow X_{n-2} \longrightarrow \dots$$

- $\mathbf{sk}_n : \; ch(\mathbb{X})_{n\geq} \longrightarrow ch(\mathbb{X})$ is the canonical inclusion or skeleton functor:

$$\mathbf{sk}_n(Y) = \; \dots \longrightarrow 0 \longrightarrow 0 \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \dots$$

- $\mathbf{cosk}_n : \; ch(\mathbb{X})_{n\geq} \longrightarrow ch(\mathbb{X})$ the coskeleton functor is given by:

$$\mathbf{cosk}_n(Y) = \; \dots \longrightarrow 0 \longrightarrow ker(\delta_n) \xrightarrow{k(\delta_n)} Y_n \xrightarrow{\delta_n} Y_{n-1} \dots$$

- $\mathbf{cot}_n : \; ch(\mathbb{X}) \longrightarrow ch(\mathbb{X})_{n\geq}$ the (upward) cotruncation functor:

$$\mathbf{cot}_n(X) = \; cok(\delta_{n+1}) \xrightarrow{\delta'_n} X_{n-1} \longrightarrow X_{n-2} \longrightarrow \dots .$$

Dually, for the bounded below chain complexes at n:

- $\mathbf{tr}'_n : \ ch(\mathbb{X}) \longrightarrow ch(\mathbb{X})_{\geq n}$ is the (downward) truncation:

$$\mathbf{tr}'_n(X) = \ \ldots \longrightarrow X_{n+2} \longrightarrow X_{n+1} \longrightarrow X_n$$

- $\mathbf{sk}'_n : \ ch(\mathbb{X})_{\geq n} \longrightarrow ch(\mathbb{X})$ is the canonical inclusion or skeleton functor:

$$\mathbf{sk}'_n(Y) = \ \ldots \longrightarrow Y_{n+1} \longrightarrow Y_n \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

- $\mathbf{cosk}'_n : \ ch(\mathbb{X})_{\geq n} \longrightarrow ch(\mathbb{X})$ is the coskeleton functor:

$$\mathbf{cosk}'_n(Y) = \ \ldots Y_{n+1} \longrightarrow Y_n \xrightarrow{cok(\delta_{n+1})} cok(\delta_{n+1}) \longrightarrow 0 \ldots$$

- $\mathbf{cot}'_n : \ ch(\mathbb{X}) \longrightarrow ch(\mathbb{X})_{n\geq}$ the (downward) cotruncation functor:

$$\mathbf{cot}'_n(X) = \ \ldots \longrightarrow X_{n+2} \longrightarrow X_{n+1} \xrightarrow{\delta'_{n+1}} ker(\delta_n) \ .$$

We write $\mathbf{Sk}_n = \mathbf{sk}_n\mathbf{tr}_n$ and $\mathbf{Cosk}_n = \mathbf{cosk}_n\mathbf{tr}_n$ and $\mathbf{Cot}_n = \mathbf{sk}_n\mathbf{cot}_n$, $\mathbf{Cot}'_n = \mathbf{sk}'_n\mathbf{cot}'_n$.

We will consider $ch(\mathbb{X})_{n\geq}$ and $ch(\mathbb{X})_{\geq n}$ as full subcategories of $ch(\mathbb{X})$ given by the skeleton functors $\mathbf{sk}$ and $\mathbf{sk}'$, respectively.

Lemma 4.2.4. Let $\mathbb{X}$ be a normal category. The functors $\mathbf{cot}_n$ and $\mathbf{cot}'_n$ can be restricted to proper chain complexes:

$$\mathbf{cot}_n = \ pch(\mathbb{X}) \longrightarrow pch(\mathbb{X})_{n\geq} \ , \quad \mathbf{cot}'_n = \ pch(\mathbb{X}) \longrightarrow pch(\mathbb{X})_{\geq n} \ .$$

Proof. Since $\mathbb{X}$ is normal $\mathbf{cot}_n(X)$ and $\mathbf{cot}'_n(X)$ are in fact proper complexes if X is proper. Indeed, for $\mathbf{cot}_n$ consider:

$$\begin{tikzcd}
X_n \arrow[rr, "\delta_n"] \arrow[dr, two heads, "e_n"'] \arrow[dd, two heads, "q_n"'] & & X_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[dd, "1"] & \ldots \\
& \delta_n(X_n) \arrow[ur, hook, "m_n"'] \arrow[dd, "q'_n"] & & \\
cok(\delta_{n+1}) \arrow[rr, "\delta'_n"] \arrow[dr, two heads, "e'_n"'] & & X_{n-1} \arrow[r, "\delta_{n-1}"] & \ldots \\
& \delta'_n(cok(\delta_{n+1})) \arrow[ur, "m'_n"'] & &
\end{tikzcd}$$

where δ'_n is induced by the cokernel $q_n = cok(\delta_{n+1})$ and consider the image factorizations of δ_n and δ'_n. q'_n is a normal epimorphism since q_n and e'_n are also. Now, since $m_n = m'_n q'_n$ is a monomorphism then so is q'_n a monomorphism and hence, an isomorphism. So, $\mathbf{cot}_n$ gets restricted to proper chains.

For $\mathbf{cot}'$, consider a proper chain complex X, so if $\delta_{n+1}(X_{n+1})$ is normal in X_n then it is normal in $ker(\delta_n)$:

$$\begin{tikzcd}
X_{n+2} \arrow[r, "\delta_{n+2}"] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "e_{n+1}"'] & X_n \\
 & \delta(X_{n+1}) \arrow[r] \arrow[ur, "m_{n+1}"] & ker(\delta_n) \arrow[u]
\end{tikzcd}$$

So, the corestriction of δ_{n+1}, $\delta'_{n+1} = X_{n+1} \to ker(\delta_n)$, is proper. $\square$

Lemma 4.2.5. For a normal category $\mathbb{X}$ we have the adjunctions:

$$\mathbf{cot}_n \dashv \mathbf{sk}_n \dashv \mathbf{tr}_n \dashv \mathbf{cosk}_n : \begin{tikzcd}
ch(\mathbb{X}) \arrow[d, shift right=9] \arrow[d, phantom, shift right=6, "\dashv"] \arrow[d, leftarrow, shift right=3] \arrow[d, phantom, "\dashv"] \arrow[d, shift left=3] \arrow[d, phantom, shift left=6, "\dashv"] \arrow[d, leftarrow, shift left=9] \\
ch(\mathbb{X})_{n\geq}
\end{tikzcd} .$$

Proof. Since **sk** and **cosk** are full and faithful embeddings, we shall only describe the non-trivial units and counits. Consider X in $ch(\mathbb{X})$, the counit α of $\mathbf{sk}_n \dashv \mathbf{tr}_n$ is given by:

$$\begin{tikzcd}
\mathbf{Sk}_n(X) = \arrow[d, "\alpha"] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & X_n \arrow[r, "\delta_n"] \arrow[d, "1"] & X_{n-1} \arrow[r] \arrow[d, "1"] & \dots \\
X = & \dots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] & X_n \arrow[r, "\delta_n"] & X_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

and the unit β of $\mathbf{tr}_n \dashv \mathbf{cosk}_n$:

$$\begin{tikzcd}
X = \arrow[d, "\beta"] & \dots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d] & X_n \arrow[r, "\delta_n"] \arrow[d, "1"] & X_{n-1} \arrow[r] \arrow[d, "1"] & \dots \\
\mathbf{Cosk}_n(X) = & \dots \arrow[r] & ker(\delta_n) \arrow[r, "k(\delta_n)"] & X_n \arrow[r, "\delta_n"] & X_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

and the unit γ of $\mathbf{cot}_n \dashv \mathbf{sk}_n$:

$$\begin{tikzcd}
X = \arrow[d, "\gamma"] & \dots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d] & X_n \arrow[r, "\delta_n"] \arrow[d, "cok(\delta_{n+1})"] & X_{n-1} \arrow[r] \arrow[d, "1"] & \dots \\
\mathbf{Cot}_n(X) = & \dots \arrow[r] & 0 \arrow[r] & cok(\delta_{n+1}) \arrow[r, "\delta'_n"] & X_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

It is straightforward to see that α, β and γ are universal arrows. For example in the case for γ, consider $f : X \to \mathbf{Sk}_n(A)$ where A is a n-truncated chain complex:

$$\begin{tikzcd}
X = \arrow[d, "f"] & \dots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}"] & X_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\mathbf{Sk}_n(A) = & \dots \arrow[r] & 0 \arrow[r] & A_n \arrow[r, "\delta'_n"] & A_{n-1} \arrow[r] & \dots ,
\end{tikzcd}$$

since $f_n\delta_{n+1} = 0$ then by the universal property of $cok(\delta_{n+1})$ there is f'_n such that $f_n = f'_n cok(\delta_{n+1})$ and thus we define $f' : \mathbf{cot}_n(X) \to A$ as $(f')_n = f'_n$ and $(f')_i = f_i$ for $i < n$. Finally, we have that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\gamma"] \arrow[dr, "f"'] & \mathbf{Cot}_n(X) \arrow[d, "f'"] \\
& \mathbf{Sk}_n(A) .
\end{tikzcd}$$

□

Similarly, we have adjunctions for the bounded below chain complexes. The proof is completely analogue to the previous one.

Lemma 4.2.6. For a normal category $\mathbb{X}$ we have the adjunctions:

$$\mathbf{cosk}'_n \dashv \mathbf{tr}'_n \dashv \mathbf{sk}'_n \dashv \mathbf{cot}'_n : \begin{tikzcd} ch(\mathbb{X})_{\geq n} \arrow[d, bend right=60] \arrow[d, phantom, "\dashv"] \arrow[d, bend left=60] \\ ch(\mathbb{X}) . \arrow[u, bend right=30] \arrow[u, bend left=60] \end{tikzcd}$$

Proof. Consider X in $ch(\mathbb{X})$. The unit α' of $\mathbf{tr}'_n \dashv \mathbf{sk}'_n$ is:

$$\begin{tikzcd}
X = \arrow[d, "\alpha'"] & \dots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "1"] & X_n \arrow[r, "\delta_n"] \arrow[d, "1"] & X_{n-1} \arrow[r] \arrow[d] & \dots \\
\mathbf{Sk}'_n(X) = & \dots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] & X_n \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

and the counit β' of $\mathbf{cosk}'_n \dashv \mathbf{tr}'_n$:

$$\begin{tikzcd}[row sep=small]
\mathbf{Cosk}'_n(X) = \arrow[d, "\beta'"] & \dots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "1"] & X_n \arrow[r, "cok(\delta_{n+1})"] \arrow[d, "1"] & cok(\delta_{n+1}) \arrow[r] \arrow[d] & \dots \\
X = & \dots \arrow[r] & X_{n+1} \arrow[r] & X_n \arrow[r, "\delta_n"] & X_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

and the counit γ' of $\mathbf{sk}' \dashv \mathbf{cot}'$:

$$\begin{tikzcd}[row sep=small]
\mathbf{Cot}'_n(X) = \arrow[d, "\gamma'"] & \dots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "1"] & ker(\delta_n) \arrow[r] \arrow[d, "k(\delta_n)"] & 0 \arrow[r] \arrow[d] & \dots \\
X = & \dots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] & X_n \arrow[r, "\delta_n"] & X_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

$\square$

By 4.2.4 we immediately have:

Corollary 4.2.7. Let $\mathbb{X}$ be a normal category then the adjunctions $\mathbf{cot}_n \dashv \mathbf{sk}_n \dashv \mathbf{tr}_n \dashv \mathbf{cosk}_n$ and $\mathbf{cosk}'_n \dashv \mathbf{tr}'_n \dashv \mathbf{sk}'_n \dashv \mathbf{cot}'_n$ for chain complexes $ch(\mathbb{X})$ can be restricted to proper chain complexes:

$$\mathbf{cot}_n \dashv \mathbf{sk}_n \dashv \mathbf{tr}_n \dashv \mathbf{cosk}_n : \begin{tikzcd}[row sep=large]
pch(\mathbb{X}) \arrow[d, bend right=70, ""{name=A}] \arrow[d, leftarrow, bend right=25, ""{name=B}] \arrow[d, bend left=25, ""{name=C}] \arrow[d, leftarrow, bend left=70, ""{name=D}] \arrow[from=A, to=B, phantom, "\dashv"] \arrow[from=B, to=C, phantom, "\dashv"] \arrow[from=C, to=D, phantom, "\dashv"] \\
pch(\mathbb{X})_{n\geq}
\end{tikzcd}$$

and

$$\mathbf{cosk}'_n \dashv \mathbf{tr}'_n \dashv \mathbf{sk}'_n \dashv \mathbf{cot}'_n : \begin{tikzcd}[row sep=large]
pch(\mathbb{X})_{\geq n} \arrow[d, bend right=70, ""{name=A}] \arrow[d, leftarrow, bend right=25, ""{name=B}] \arrow[d, bend left=25, ""{name=C}] \arrow[d, leftarrow, bend left=70, ""{name=D}] \arrow[from=A, to=B, phantom, "\dashv"] \arrow[from=B, to=C, phantom, "\dashv"] \arrow[from=C, to=D, phantom, "\dashv"] \\
pch(\mathbb{X})\,.
\end{tikzcd}$$

Remark 4.2.8. Note that the embeddings $\mathbf{sk}_n$ and $\mathbf{cosk}_n$ ($\mathbf{sk}'_n$ and $\mathbf{cosk}'_n$) only differ in one degree.

In [BHS10] the adjunctions $\mathbf{cot}_n \dashv \mathbf{sk}_n \dashv \mathbf{tr}_n \dashv \mathbf{cosk}_n$ are used in a very different context, mainly applied to crossed complexes over groupoids and higher groupoids and other homotopy models. However, $\mathbf{cosk}'_n \dashv \mathbf{tr}'_n \dashv \mathbf{sk}'_n \dashv \mathbf{cot}'_n$ are not mentioned and do seem to have any application in those contexts.

Remark 4.2.9. [2] The category of chain complexes/proper chain complexes present a dual behaviour as above. However, for a category $\mathbb{X}$ being normal

[2]The author would like to thank prof. T. Van der Linden for suggesting this remark.

is not a self-dual property and neither for a morphism to be proper. On the other hand, we could work in a more general setting of a pointed category $\mathbb{X}$ with kernels and cokernels; and we call a morphism $f : A \to B$ proper if f is admits a factorization $f = me$ with e a normal epimomphism and m a normal epimorphism and this factorization is unique in the following sense: given another factorization $f = m'e'$ with e' a normal epirmophism and m' a normal monorphism then there is a morphism t such that $te = e'$ and $m't = m$.

In this general setting in the category $\mathbb{X}$ lemma 4.2.4 holds and lemmas 4.2.5 and 4.2.6 are formally dual. It is worth mentioning that (formally) dual properties of torsion theories in pointed categories with kernels and cokernels have been studied in [JT07].

4.2.2 Torsion theories given by cotruncations

Through this section $\mathbb{X}$ is a normal category.

4.2.10. We will study torsion theories in $pch(\mathbb{X})$ given by the 'good' truncations, in our case the cotruncations. Recall that a proper morphism $f : A \to B$ defines two truncated morphisms, $ker(f) \to 0$ and $0 \to cok(f)$ and defines two canonical short exact sequence given by the normal epi-mono factorization of f:

$$0 \longrightarrow ker(f) \xhookrightarrow{k(f)} A \xrightarrow{e_f} f(A) \longrightarrow 0$$

$$0 \longrightarrow f(A) \xhookrightarrow{m_f} B \xrightarrow{cok(f)} cok(f) \longrightarrow 0 \ .$$

We will be particularly interested in the case of proper chain complexes, but we will first define the torsion theories for $ch(\mathbb{X})$ (which is a normal category when $\mathbb{X}$ is so, since limits and colimits are computed component-wise) and then consider the restriction of each torsion theory to $pch(\mathbb{X})$.

Definition 4.2.11. Let $\mathbb{X}$ be a normal category, we define the full subcategories in $pch(\mathbb{X})$ for each $n \in \mathbb{Z}$:

$$\mathcal{EP}_n = \{X \mid \delta_n \text{ is a normal epi and } X_i = 0 \text{ for } n-1 > i\}.$$

For example, a proper chain complex X in $\mathcal{EP}_n$ looks like this:

$$\ldots \longrightarrow X_{n+1} \xrightarrow{\delta_{n+1}} X_n \xrightarrow{\delta_n} X_{n-1} \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

with δ_n a normal epimorphism. And, similarly,

$$\mathcal{MN}_n = \{X \mid \delta_n \text{ is a normal mono and } X_i = 0 \text{ for } i > n\}.$$

For example, a proper chain complex X in $\mathcal{NM}_n$ looks like this:

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow X_n \xhookrightarrow{\delta_n} X_{n-1} \xrightarrow{\delta_{n-1}} X_{n-2} \longrightarrow \ldots$$

where δ_n is a normal monomorphism.

Note that $\mathcal{EP}_n$ is a full subcategory of $pch(\mathbb{X})_{\geq n-1}$ and $\mathcal{MN}_n$ is a full subcategory of $pch(\mathbb{X})_{n\geq}$.

4.2.12. Let $\mathbb{X}$ be a normal category. The unit of $\mathbf{cot}_{n-1} \dashv \mathbf{sk}_{n-1}$ for a chain complex X is given by

$$\begin{tikzcd}
X = \arrow[d, "\eta_X"] & \ldots \arrow[r] & X_n \arrow[r, "\delta_n"] \arrow[d] & X_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "cok(\delta_n)"] & X_{n-2} \arrow[r] \arrow[d, "1"] & \ldots \\
\mathbf{Cot}_{n-1}(X) & \ldots \arrow[r] & 0 \arrow[r] & cok(\delta_n) \arrow[r, "\delta'_{n-1}"] & X_{n-2} \arrow[r, "\delta_{n-2}"] & \ldots
\end{tikzcd}$$

which is a normal epimorphism component-wise, so $ch(\mathbb{X})_{n-1\geq}$ is a normal epireflective subcategory of $ch(\mathbb{X})$. It is easy to see that $ch(\mathbb{X})_{n-1\geq}$ is closed under subobjects, quotients and extensions in $ch(\mathbb{X})$, however it is not a torsion-free subcategory, the functor $\mathbf{cot}_{n-1}$ is not normal as in 2.2.8. Indeed, it may happen that $\mathbf{cot}_{n-1}(ker(\eta_X))$ may not be trivial for a chain complex X.

As a counter-example we can consider the truncated case $\mathbf{cot}_0 = coker : Arr(\mathbb{X}) \to \mathbb{X}$. Let D_4 be the dihedral group with generators $a^2 = b^4 = 1$ and $aba = b^{-1}$ and consider $X =< a >\to D_4$ the inclusion morphism and the unit η_X:

$$\begin{tikzcd}
< a > \arrow[r] \arrow[d, "\eta_{X,1}"] & D_4 \arrow[d, "\eta_{X,0}"] \\
0 \arrow[r] & D_4/ < a, b^2 >
\end{tikzcd}$$

so $ker(\eta_X)$ is the inclusion $< a >\to< a, b^2 >$ which does not have a trivial cokernel. Accordingly, the functor $coker$ is not a normal functor.

Remember that in a normal category $\mathbb{X}$ a morphism is a monomorphism if and only if it has trivial kernel. The dual does not hold for epimorphisms, there are morphisms with trivial cokernel and that are not epimorphisms. So, a chain complex X is in $Ker(\mathbf{cot}_{n-1})$ if and only if it is a truncated (below)

$(n-1)$ chain complex

$$X = \ \dots \longrightarrow X_{n+1} \xrightarrow{\delta_{n+1}} X_n \xrightarrow{\delta_n} X_{n-1} \longrightarrow 0 \longrightarrow \dots$$

and where the morphism δ_n has a trivial cokernel[3].

Following 2.2.6 and the fact that **cot** is not a normal functor we have that $Ker(\mathbf{cot}_{n-1}) \subset \mathcal{T}_{ch(\mathbb{X})_{n-1\geq}}$. Also notice that $\mathcal{EP}_n \subset Ker(\mathbf{cot}_{n-1})$. This is an example of a normal epireflective subcategory $ch(\mathbb{X})_{n\geq}$ closed under extensions in a semi-abelian category, $ch(\mathbb{X})$, which is not a torsion-free subcategory.

We will prove that when restricted to proper chain complexes the subcategories $Ker(\mathbf{cot}_{n-1})$ and $\mathcal{T}_{pch(\mathbb{X})_{n-1\geq}}$ are equivalent, $\mathbf{cot}_{n-1}$ is normal and we have a torsion theory $(\mathcal{EP}_n, pch(\mathbb{X})_{n-1\geq})$ in $pch(\mathbb{X})$.

Corollary 4.2.13. Let $\mathbb{X}$ be a normal category. The pair of subcategories $(Ker(\mathbf{cot}_{n-1}), ch(\mathbb{X})_{n-1\geq})$ in $ch(\mathbb{X})$ gets restricted to a cohereditary torsion theory in $pch(\mathbb{X})$ given by the pair $(\mathcal{EP}_n, pch(\mathbb{X})_{n-1\geq})$.

Proof. We will prove that $Ker(\mathbf{cot}_{n-1}) \cap pch(\mathbb{X}) = \mathcal{EP}_n$. It suffices to prove that a proper morphism $f : A \to B$ with trivial cokernel is a normal epimorphism. Indeed, consider e, m the normal epi/mono factorization and the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "e"'] & B \arrow[r, "q"] & cok(f) \\
f(A) \arrow[ur, "m"] \arrow[r, "m'"'] & ker(q) \arrow[u, "k"'] &
\end{tikzcd}$$

where m' is induced by the kernel $ker(q)$. Then if f is a proper morphism then m' is an isomorphism since the normal monomorphism m is the kernel of its cokernel q. Also, if $cok(f) = 0$ then k is also an isomorphism. Finally, m is an isomorphism and f is a normal epimorphism.

Now we prove that $(\mathcal{EP}_n, pch(\mathbb{X})_{n-1\geq})$ is a torsion theory. The axiom TT1 is trivial since a morphism f:

$$\begin{tikzcd}
X_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & X_{n-1} \arrow[d, "f_{n-1}"] \\
0 \arrow[r] & Y_{n-1}
\end{tikzcd}$$

with δ_n a regular epi must be trivial. And, for a proper chain complex X the

[3]Recall that for a functor $F : \mathbb{A} \to \mathbb{B}$ between pointed categories the kernel of F, $Ker(F)$ is the replete full subcategory of A of objects A such that $F(A) \cong 0$.

short exact sequence of the torsion theory $(\mathcal{EP}_n, pch(\mathbb{X})_{n-1\geq})$ is given by

$$\begin{tikzcd}
\dots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "id"] & X_n \arrow[r, two heads, "e_n"] \arrow[d, "id"] & \delta_n(X_n) \arrow[r] \arrow[d, hook, "m_n"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d] & X_n \arrow[r, "\delta_n"] \arrow[d] & X_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, two heads, "c(\delta_n)"] & X_{n-2} \arrow[r] \arrow[d, "id"] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & cok(\delta_n) \arrow[r] & X_{n-2} \arrow[r] & \dots
\end{tikzcd}$$

and we have $cok(\delta_n) \cong X_{n-1}/\delta_n(X_n)$. $\square$

For $n \in \mathbb{Z}$ the adjunction $\mathbf{tr}_n \dashv \mathbf{cosk}_n$ is a localization: indeed, the fact that $\mathbf{tr}_n$ admits a left adjoint $\mathbf{sk}_n$ implies that $\mathbf{tr}_n$ it preserves finite limits. From 2.4.1 and 2.4.4, $\mathbf{tr}_n$ induces an hereditary torsion theory $(\mathcal{T}_{\mathbf{tr}_n}, \mathcal{F}_{\mathbf{tr}_n})$.

Theorem 4.2.14. Let $\mathbb{X}$ be a normal category. The localization $\mathbf{tr}_{n-1} \dashv \mathbf{cosk}_{n-1}$ induces a hereditary torsion theory $(\mathcal{T}_{\mathbf{tr}_{n-1}}, \mathcal{F}_{\mathbf{tr}_{n-1}})$ in $ch(\mathbb{X})$ as in 2.4.1. This torsion theory can be restricted to a hereditary torsion theory $(pch(\mathbb{X})_{\geq n}, \mathcal{MN}_n)$ in $pch(\mathbb{X})$.

Proof. From 2.4.1, the localization $\mathbf{tr}_{n-1} \dashv \mathbf{cosk}_{n-1}$ induces a torsion theory as

$$\mathcal{T}_{\mathbf{tr}_{n-1}} = \{X \mid \mathbf{tr}_{n-1}(X) = 0\} = \{X \mid X_i = 0 \,\text{for}\, n-1 \leq i\} = ch(\mathbb{X})_{\geq n}$$

and

$$\mathcal{F}_{\mathbf{tr}_{n-1}} = \{X \mid \eta_X \text{ monic}\}.$$

where η_X is the unit of $\mathbf{tr}_{n-1} \dashv \mathbf{cosk}_{n-1}$. The short exact sequence of the torsion theory is given by

$$0 \longrightarrow ker(\eta_X) \xrightarrow{ker(\eta_X)} X \xrightarrow{e} \eta_X(X) \longrightarrow 0$$

where e is the normal epimorphism of the image factorization of $\eta_X = me$. Consider the commutative diagram

$$\begin{tikzcd}
X_n \arrow[r, "\delta_n"] \arrow[d, "e_n"'] & X_{n-1} \arrow[r, "\delta_{n-1}"] & X_{n-2} \\
\delta_n(X_n) \arrow[r, "m'_n"'] \arrow[ur, "m_n"] & ker(\delta_{n-1}) \arrow[u, "k"'] &
\end{tikzcd}$$

where m'_n is induced by the kernel $ker(\delta_{n-1})$, so the normal epi/mono factorization of η_X at the level n is given by $\eta_X = m'_n e_n$. So, the short exact sequence is

$$\begin{tikzcd}
\dots \arrow[r] & X_{n+1} \arrow[r] \arrow[d, "id"] & ker(\delta_n) \arrow[r] \arrow[d, hook, "k(\delta_n)"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d] & X_n \arrow[r, "\delta_n"] \arrow[d, two heads, "e_n"] & X_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "id"] & X_{n-2} \arrow[r] \arrow[d, "id"] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \delta(X_n) \arrow[r, hook, "m_n"] & X_{n-1} \arrow[r] & X_{n-2} \arrow[r] & \dots
\end{tikzcd}$$

Notice that the torsion subobject of X is given the downward cotruncation $\mathbf{Cot}'_n$, and it restricts to proper chain complexes. It follows that $(\mathcal{T}_{\mathbf{tr}_{n-1}}, \mathcal{F}_{\mathbf{tr}_{n-1}})$ restricts to proper chain complexes as $(pch(\mathbb{X})_{\geq n}, \mathcal{MN}_n)$. $\square$

4.2.15. We will denote the previous torsion theories in $pch(\mathbb{X})$ as

$$\mathcal{COK}_n = (\mathcal{EP}_n, pch(\mathbb{X})_{n-1\geq}), \quad \mathcal{KER}_n = (pch(\mathbb{X})_{\geq n}, \mathcal{MN}_n),$$

and we write ker_n and cok_n for the associated idempotent radicals of each torsion theory $\mathcal{KER}_n$ and $\mathcal{COK}_n$ respectively.

In summary, we have that $pch(\mathbb{X})_{\geq n}$ is a torsion subcategory in $pch(\mathbb{X})$ and the coreflector is given by $\mathbf{cot}'_n : pch(\mathbb{X}) \to pch(\mathbb{X})_{\geq n}$. Similarly, $pch(\mathbb{X})_{n-1\geq}$ is a torsion-free subcategory of $pch(\mathbb{X})$ with the reflector $\mathbf{cot}_n$:

$$\mathcal{COK}_n = \quad \begin{tikzcd}
Ker(\mathbf{cot}) \arrow[r] & ch(\mathbb{X}) \arrow[r, bend left, "\mathbf{cot}_{n-1}"] \arrow[r, phantom, "\perp"] & ch(\mathbb{X})_{n-1\geq} \arrow[l, bend left, "\mathbf{sk}_{n-1}"] \\
\mathcal{EP}_n \arrow[u] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & pch(\mathbb{X}) \arrow[u] \arrow[l, bend left] \arrow[r, bend left, "\mathbf{cot}_{n-1}"] \arrow[r, phantom, "\perp"] & pch(\mathbb{X})_{n-1\geq} \arrow[u] \arrow[l, bend left, "\mathbf{sk}_{n-1}"]
\end{tikzcd}$$

and

$$
\begin{tikzcd}
 & ch(\mathbb{X})_{\geq n} \arrow[r, bend left, "\mathbf{sk}'_n"] \arrow[r, phantom, "\perp"] & ch(\mathbb{X}) \arrow[l, bend left, "\mathbf{cot}'_n"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{F}_{\mathbf{tr_{n-1}}} \arrow[l, bend left] \\
\mathcal{KER}_n = & pch(\mathbb{X})_{\geq n} \arrow[u] \arrow[r, bend left, "\mathbf{sk}'_n"] \arrow[r, phantom, "\perp"] & pch(\mathbb{X}) \arrow[u] \arrow[l, bend left, "\mathbf{cot}'_n"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{MN}_n\,. \arrow[u] \arrow[l, bend left]
\end{tikzcd}
$$

4.2.3 The lattice $\mathcal{COT}$

Proposition 4.2.16. Let $\mathbb{X}$ be a normal category. In $pch(\mathbb{X})$ for each $n \in \mathbb{Z}$ we have the inclusions as full subcategories

$$\ldots \leq pch(\mathbb{X})_{\geq n+1} \leq \mathcal{EP}_{n+1} \leq pch(\mathbb{X})_{\geq n} \leq \mathcal{EP}_n \leq pch(\mathbb{X})_{\geq n-1} \leq \ldots$$

Equivalently,

$$\ldots \geq \mathcal{MN}_{n+2} \geq pch(\mathbb{X})_{n+1\geq} \geq \mathcal{MN}_{n+1} \geq pch(\mathbb{X})_{n\geq} \geq \mathcal{MN}_n \geq \ldots .$$

Moreover, this gives a linearly ordered lattice of torsion theories in $pch(\mathbb{X})$:

$$O \leq \ldots \leq \mathcal{KER}_{n+1} \leq \mathcal{COK}_{n+1} \leq \mathcal{KER}_n \leq \mathcal{COK}_n \leq \ldots \leq pch(\mathbb{X})$$

Proof. By definition we have $\mathcal{EP}_n \leq pch(\mathbb{X})_{\geq n-1}$ and since a morphism $X_{n+1} \to 0$ is a normal epimorphism we have $pch(\mathbb{X})_{\geq n} \leq \mathcal{EP}_n$. Recall that the order is reverse for the torsion-free subcategories. □

This construction works with truncated or bounded chains complexes, in particularly we will be interested in the case for $pch(\mathbb{X})_{\geq 0}$, $pch(\mathbb{X})_{n\geq 0}$ the category of proper chain complexes bounded below 0 and above n for a fixed n.

Corollary 4.2.17. Let $\mathbb{X}$ be a normal category. In $pch(\mathbb{X})_{\geq 0}$ there is a linearly ordered lattice of torsion theories given by:

$$0 \leq \ldots \leq \mathcal{KER}_n \leq \mathcal{COK}_n \leq \ldots \leq \mathcal{KER}_1 \leq \mathcal{COK}_1 \leq pch(\mathbb{X})_{\geq 0}$$

Corollary 4.2.18. Let $\mathbb{X}$ be a normal category. In $pch(\mathbb{X})_{n\geq 0}$ there is a linearly ordered lattice of torsion theories given by:

$$0 \leq \mathcal{KER}_n \leq \mathcal{COK}_n \leq \ldots \leq \mathcal{COK}_2 \leq \mathcal{KER}_1 \leq \mathcal{COK}_1 \leq pch(\mathbb{X})_{n\geq 0}$$

Definition 4.2.19. For a normal category $\mathbb{X}$, we write $\mathcal{COT}(pch(\mathbb{X}))$ for the lattice of torsion theories given by the cotruncations functors **cot**, **cot$'$** in $pch(\mathbb{X})$ as in 4.2.15, 4.2.16, 4.2.17, 4.2.18.

Similarly, for $\mathcal{COT}(pch(\mathbb{X})_{\geq 0})$ and $\mathcal{COT}(pch(\mathbb{X})_{n\geq 0})$.

4.2.20. Each torsion theory has its associated idempotent radical, and thus $\mathcal{COT}$ also defines a lattice of idempotent radicals. Moreover, for each proper chain complex X there is the lattice $cot(X)$ of subobjects of X given by the torsion subobjects $ker_n(X)$, $cok_n(X)$ of X.

As an example, consider X in $pch(\mathbb{X})_{2\geq 0}$:

$$X = \quad X_2 \xrightarrow{\delta_2} X_1 \xrightarrow{\delta_1} X_0\,,$$

and the lattice $\mathcal{COT}(pch(\mathbb{X})_{2\geq 0})$:

$$O \leq \mathcal{KER}_2 \leq \mathcal{COK}_2 \leq \mathcal{KER}_1 \leq \mathcal{COK}_1 \leq pch(\mathbb{X})_{2\geq 0}\,,$$

so for X the lattice of subobjects $cot(X)$ is

$$\begin{array}{rccccc}
X = & X_2 & \xrightarrow{\delta_2} & X_1 & \xrightarrow{\delta_1} & X_0 \\
cok_1(X) = & X_2 & \xrightarrow{\delta_2} & X_1 & \xrightarrow{\delta_1} \!\!\!\!\rightarrow & \delta_1(X_1) \\
ker_1(X) = & X_2 & \xrightarrow{\delta_2} & ker(\delta_1) & \longrightarrow & 0 \\
cok_2(X) = & X_2 & \xrightarrow{\delta_2} \!\!\!\!\rightarrow & \delta_2(X_2) & \longrightarrow & 0 \\
ker_2(X) = & ker(\delta_2) & \longrightarrow & 0 & \longrightarrow & 0 \\
0 = & 0 & \longrightarrow & 0 & \longrightarrow & 0\,.
\end{array}$$

Remark 4.2.21. Working with bounded above complexes will give a maximal element in the lattice $\mathcal{COT}$, namely $\mathcal{COK}_1$. And bounded below complexes have $\mathcal{KER}_n$ as a minimal element in $\mathcal{COT}$. Moreover, in this case $\mathcal{COK}_1 = (\mathcal{EP}_1, \mathbb{X}_0)$ and $\mathcal{KER}_n = (\mathbb{X}_n, \mathcal{NM}_n)$ where $\mathbb{X}_n$ is the subcategory of chain complexes that are zero for all $i \neq n$.

It is worth mentioning that $\mathcal{COT}(pch(\mathbb{X}))$ is a sublattice of $pch(\mathbb{X})tors$ (2.2.11) the lattice of all torsion theories in $pch(\mathbb{X})$, but they may not need to coincide. For example, if $\mathbb{X} = Ab$ and $(\mathbb{T}, \mathbb{F})$ the torsion theory given by torsion/torsion-free abelian groups then $(Arr(\mathbb{T}), Arr(\mathbb{F}))$ is a torsion theory in $Arr(Ab)$ not considered in $\mathcal{COT}(Arr(Ab))$.

4.2.4 Homology

In abelian categories, the homology objects of a chain complex X are defined as

$$H_n(X) = ker(\delta_n)/\delta_{n+1}(X_{n+1})$$

or in other words $cok(X_{n+1} \to ker(\delta_n))$. We can also consider its dual $K_n(M) = ker(cok(\delta_{n+1}) \to X_{n-1})$, these definitions can be defined using only limits and colimits. In the abelian case the objects $H_n(X)$ and $K_n(X)$ are isomorphic and then in [EdL04] this is proved for homological categories provided that the chain complex X is proper. We will prove this fact in a different way for normal categories.

Theorem 4.2.22. For a normal category $\mathbb{X}$ and X a proper chain complex then the objects $H_n(X)$, $K_n(X)$ are isomorphic and are given by

$$H_n(X) \cong K_n(X) \cong ker_n(X)/cok_{n+1}(X)$$

where $cok_{n+1}(X)$, $ker_n(X)$ are the torsion subobjects of X given by the torsion theories of $\mathcal{COK}$ and $\mathcal{KER}$ (as defined in 4.2.15) and where $H_n(X)$, $K_n(X)$ are considered as trivial chain complexes except at the order n that have the object $H_n(X)$, $K_n(X)$ respectively.

Proof. The objects $H_n(X)$ and $K_n(X)$ are defined as follows: consider the diagram

$$\begin{tikzcd}
X_{n+1} \arrow[rr, "\delta_{n+1}"] \arrow[d, "e_{n+1}"'] & & X_n \arrow[rr, "\delta_n"] \arrow[dr, "q_{n+1}"'] \arrow[drr, "e_n"] & & X_{n-1} \\
\delta_{n+1}(X_{n+1}) \arrow[r, "m'_{n+1}"'] \arrow[urr, "m_{n+1}"] & ker(\delta_n) \arrow[ur, "k_n"'] & & cok(\delta_{n+1}) \arrow[r, "e'_n"'] & \delta_n(X_n) \arrow[u, "m_n"']
\end{tikzcd}$$

where $e_{n+1},\ m_{n+1}$ and $e_n,\ m_n$ are the epi/mono factorization of δ_{n+1}, δ_n; and m'_{n+1} and e'_n are induced by the universal properties of $ker(\delta_n)$ and $cok(\delta_{n+1})$, respectively. So $H_n(X) = cok(m'_{n+1})$ and $K_n(X) = ker(e'_n)$.

Since $ch(\mathbb{X})$ is normal this follows from the third isomorphism theorem (1.3.8) for the normal subobjects $cok_{n+1}(X) \leq ker_n(X)$ of X as we are going to explain: We have a short exact sequence in $ch(\mathbb{X})$:

$$0 \longrightarrow ker_n(X)/cok_{n+1}(X) \longrightarrow X/cok_{n+1}(X) \longrightarrow X/ker_n(X) \longrightarrow 0 \ .$$

To be more precise observe the short exact sequences that define $H_n(X)$ and $K_n(X)$. Then $H_n(X)$ is the cokernel of the inclusion $cok_{n+1}(X) \leq ker_n(X)$:

$$\begin{tikzcd}
0 \arrow[d] & & & & & \\
cok_{n+1}(X) = \arrow[d] & \ldots \arrow[r] & X_{n+1} \arrow[r] \arrow[d] & \delta_{n+1}(X_{n+1}) \arrow[r] \arrow[d, "m'_{n+1}"] & 0 \arrow[r] \arrow[d] & \ldots \\
ker_n(X) = \arrow[d] & \ldots \arrow[r] & X_{n+1} \arrow[r] \arrow[d] & ker(\delta_n) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
H_n(X) = \arrow[d] & \ldots \arrow[r] & 0 \arrow[r] & H_n(X) \arrow[r] & 0 \arrow[r] & \ldots \\
0 & & & & &
\end{tikzcd}$$

so $H_n(X) \cong ker_n(X)/cok_{n+1}(X)$; and on the other hand

$$\begin{tikzcd}
0 \arrow[d] & & & & & \\
K_n(X) = \arrow[d] & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & K_n(X) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
X/cok_{n+1}(X) = \arrow[d] & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & cok(\delta_{n+1}) \arrow[r] \arrow[d, "e'_n"] & X_{n-1} \arrow[r] \arrow[d] & \ldots \\
X/ker_n(X) = \arrow[d] & \ldots \arrow[r] & 0 \arrow[r] & \delta_n(X) \arrow[r] & X_{n-1} \arrow[r] & \ldots \\
0 & & & & &
\end{tikzcd}$$

so $K_n(X) \cong ker(X/cok_{n+1}(X) \to X/ker_n(X))$. The third isomorphism theorem then yields the isomorphism $H_n(X) = K_n(X)$. $\square$

The coskeleton functors $\mathbf{cosk}_n$, $\mathbf{cosk}'_n$ can be used to characterize the torsion subobjects/torsion-free quotients of the torsion theories in $\mathcal{COT}$ as follows:

Proposition 4.2.23. Let $\mathbb{X}$ be a normal category. For X in $pch(\mathbb{X})$ the following are equivalent:

1. $H_n(X) = 0$;

2. $X/ker_{n+1}(X) \cong \mathbf{Cosk}_n(X)$.

Similarly, the following are equivalent:

1. $H_n(X) = 0$;

2. $cok_n(X) \cong \mathbf{Cosk}'_n(X)$.

where $ker_{n+1}(X)$ and $cok_n(X)$ are the torsion subobjects of X given by the torsion theories in 4.2.15.

Proof. First, recall from 4.2.14 that the unit $X \to \mathbf{Cosk}_n(X)$ factors through the reflection of $\mathcal{MN}_{n+1}$:

$$\begin{tikzcd}
X = \arrow[d, two heads] & \dots \arrow[r] & X_{n+2} \arrow[r, "\delta_{n+2}"] \arrow[d] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, two heads, "e_{n+1}"] & X_n \arrow[r, "\delta_n"] \arrow[d] & \dots \\
X/ker_{n+1}(X) = \arrow[d] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & \delta_{n+1}(X_{n+1}) \arrow[r, hook, "m_{n+1}"] \arrow[d] & X_n \arrow[r, "\delta_n"] \arrow[d] & \dots \\
\mathbf{Cosk}_n(X) = & \dots \arrow[r] & 0 \arrow[r] & ker(\delta_n) \arrow[r, hook, "k(\delta_n)"] & X_n \arrow[r, "\delta_n"] & \dots
\end{tikzcd}$$

And, by definition, $\delta_{n+1}(X_{n+1}) \cong ker(\delta_n)$ if and only if $H_n(X) = 0$. The second part is similar, since $cok(\delta_{n+1}) \cong \delta_n(X)$ if and only if $H_n(X) = 0$. $\square$

Consider the category $pch(\mathbb{X})_{\geq 0}$, the lattice $\mathcal{COT}(pch(\mathbb{X})_{\geq 0})$ induces a lattice of idempotent radicals:

$$0 \leq \dots \leq cok_{n+1} \leq ker_n \leq cok_n \leq \dots \leq cok_2 \leq ker_1 \leq cok_1 \leq Id\,,$$

and hence, for each chain complex M there is a lattice $cot(M)$ of the torsion subobjects of M:

$$0 \leq \dots \leq ker_n(M) \leq cok_n(M) \leq \dots \leq cok_2(M) \leq ker_1(M) \leq cok_1(M) \leq M.$$

We will be interested in all possible quotients of torsion subobjects of M (including 0 and M itself), or equivalently the quotients of preradicals of the lattice $\mathcal{COT}(pch(\mathbb{X})_{\geq 0})$ as in 2.2.13. The homology of these quotients is calculated as follows.

Lemma 4.2.24. Let $\mathbb{X}$ be a normal category and M a chain complex in $pch(\mathbb{X})_{\geq 0}$. Consider the lattice $cot(M)$ (as in 4.2.19) then we have:

1. For all $n > 0$

$$H_i(cok_n(M)) = H_i(ker_n(M)) = \begin{cases} H_i(M) & i \geq n \\ 0 & n > i\,. \end{cases}$$

2. For all $n > 0$

$$H_i\left(\frac{M}{cok_n(M)}\right) = H_i\left(\frac{M}{ker_n(M)}\right) = \begin{cases} 0 & i \geq n \\ H_i(M) & n > i\,. \end{cases}$$

3. For all $n > 0$

$$H_i\left(\frac{cok_n(M)}{ker_n(M)}\right) = 0 \text{ for all } i\,.$$

4. For $m > n$

$$H_i\left(\frac{cok_n(M)}{ker_m(M)}\right) = H_i\left(\frac{cok_n(M)}{cok_m(M)}\right) = \begin{cases} H_i(M) & m > i \geq n \\ 0 & \text{otherwise}\,. \end{cases}$$

5. Moreover, for $m > n$

$$H_i\left(\frac{cok_n(M)}{cok_m(M)}\right) = H_i\left(\frac{ker_n(M)}{cok_m(M)}\right) = H_i\left(\frac{ker_n(M)}{ker_m(M)}\right)\,.$$

6. In particular, for $m = n + 1$

$$H_i\left(\frac{cok_n(M)}{ker_{n+1}(M)}\right) = \begin{cases} H_i(M) & i = n \\ 0 & i \neq n\,. \end{cases}$$

Proof. It is straightforward to calculate the homology of each chain complex. For 1) consider:

$$\begin{array}{ll} cok_n(M) = & \ldots \longrightarrow M_{n+1} \xrightarrow{\delta_{n+1}} M_n \xrightarrow{e_n} \delta_n(M_n) \longrightarrow 0 \longrightarrow \ldots \\ ker_n(M) = & \ldots \longrightarrow M_{n+1} \xrightarrow{\delta'_{n+1}} ker(\delta_n) \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots \end{array}$$

For 2) consider:

$$\begin{array}{ll} \frac{M}{cok_n(M)} = & \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow \frac{M_{n-1}}{\delta_n(M_n)} \xrightarrow{\delta'_{n-1}} M_{n-2} \longrightarrow \ldots \\ \frac{M}{ker_n(M)} = & \ldots \longrightarrow 0 \longrightarrow \delta_n(M_n) \xrightarrow{m_n} M_{n-1} \longrightarrow M_{n-2} \longrightarrow \ldots \end{array}$$

For 3) consider:

$$\frac{cok_n(M)}{ker_n(M)} = \qquad \ldots \longrightarrow 0 \longrightarrow \frac{M_n}{ker(\delta_n)} \xrightarrow{\cong} \delta_n(M_n) \longrightarrow 0 \longrightarrow \ldots$$

For 4) and 5) the following chain complexes have the same homology:

$$\frac{ker_n(M)}{ker_m(M)} = \quad \delta_m(M_m) \to M_{m-1} \to \ldots \to M_{n+1} \to ker(\delta_n) \longrightarrow 0$$

$$\frac{ker_n(M)}{cok_m(M)} = \quad 0 \longrightarrow \frac{M_{m-1}}{\delta_m(M_m)} \to \ldots \to M_{n+1} \to ker(\delta_n) \longrightarrow 0$$

$$\frac{cok_n(M)}{ker_m(M)} = \quad \delta_m(M_m) \to M_{m-1} \to \ldots \to M_{n+1} \longrightarrow M_n \longrightarrow \delta_n(M_n)$$

$$\frac{cok_n(M)}{cok_m(M)} = \quad 0 \longrightarrow \frac{M_{m-1}}{\delta_m(M_m)} \to \ldots \to M_{n+1} \longrightarrow M_n \longrightarrow \delta_n(M_n)$$

□

4.2.25. Notice that for a chain complex M the quotients

$$\frac{cok_n(M)}{cok_{n+1}(M)}, \quad \frac{cok_n(M)}{ker_{n+1}(M)}, \quad \frac{ker_n(M)}{ker_{n+1}(M)}, \quad \frac{ker_n(M)}{cok_{n+1}(M)}$$

have trivial homology objects except for $H_n(M)$ at order n. In addition to this property, $\frac{ker_n(M)}{cok_{n+1}(M)}$ is a trivial chain complex except at order n that have the object $H_n(M)$ (see 4.2.22):

$$\frac{ker_n(M)}{cok_{n+1}(M)} = \qquad \ldots \longrightarrow 0 \longrightarrow H_n(M) \longrightarrow 0 \longrightarrow \ldots$$

4.2.5 $ch(\mathbb{X})_{n\geq}$ and $ch(\mathbb{X})_{\geq n}$ as TTF subcategories

In addition to the torsion theories $\mathcal{KER}_n$, $\mathcal{COK}_n$ there is another family of torsion theories in $ch(\mathbb{X})$ and, by restriction, in $pch(\mathbb{X})$. In fact, the subcategories of truncated chain complexes, $ch(\mathbb{X})_{\geq n}$, are simultaneously torsion and torsion-free subcategories of $ch(\mathbb{X})$. Furthermore, in $pch(\mathbb{X})$ both the subcategories $pch(\mathbb{X})_{n\geq}$ and $pch(\mathbb{X})_{\geq n}$ will be torsion and torsion-free. We recall the definition of a TTF subcategory.

TTF-theories were introduced in [Jan65] for abelian categories and provided useful applications, for example some TTF-theories allow to decompose an object into a direct sum of torsion subobjects of two different torsion theories (see also [Ste75]).

Definition 4.2.26. A full subcategory $\mathcal{T}$ of a normal category $\mathbb{X}$ is a *torsion torsion-free* or a *TTF* (for short) subcategory if there are full subcategories $\mathcal{C}$ and $\mathcal{F}$ of $\mathbb{X}$ such that $(\mathcal{C},\mathcal{T})$ and $(\mathcal{T},\mathcal{F})$ are torsion theories in $\mathbb{X}$. It is convenient to call such a triplet $(\mathcal{C},\mathcal{T},\mathcal{F})$ a *TTF theory* of $\mathbb{X}$.

Notice that in a TTF theory $(\mathcal{C},\mathcal{T},\mathcal{F})$ the torsion theory $(\mathcal{T},\mathcal{F})$ is hereditary and $(\mathcal{C},\mathcal{T})$ is cohereditary.

Remember from the previous torsion theories $(Ker(\mathbf{cot}_{n-1\geq}), pch(\mathbb{X})_{n-1\geq})$ and $(ch(\mathbb{X})_{\geq n}, \mathcal{F}_{\mathbf{tr}_{n-1}})$, that $pch(\mathbb{X})_{n\geq}$ is already a torsion-free subcategory of $pch(\mathbb{X})$ and respectively $ch(\mathbb{X})_{\geq n}$ is a torsion subcategory of $ch(\mathbb{X})$.

Theorem 4.2.27. Let $\mathbb{X}$ be a normal category. For each $n \in \mathbb{Z}$ the pair $(ch(\mathbb{X})_{n-1\geq}, ch(\mathbb{X})_{\geq n})$ is a hereditary cohereditary torsion theory in $ch(\mathbb{X})$. The reflector and coreflector are given by $\mathbf{sk}_{n-1} \dashv \mathbf{tr}_{n-1}$ and $\mathbf{tr}'_n \dashv \mathbf{sk}'_n$ defined in 4.2.3:

$$\begin{tikzcd} ch(\mathbb{X})_{n-1\geq} \arrow[r, phantom, "\perp"] \arrow[r, bend left, "\mathbf{sk}_{n-1}"] & ch(\mathbb{X}) \arrow[l, bend left, "\mathbf{tr}_{n-1}"] \arrow[r, phantom, "\perp"] \arrow[r, bend left, "\mathbf{tr}'_n"] & ch(\mathbb{X})_{\geq n} \arrow[l, bend left, "\mathbf{sk}'_n"] \end{tikzcd}$$

Proof. For X in $ch(\mathbb{X})_{n-1\geq}$ and Y in $ch(\mathbb{X})_{\geq n}$ it is clear that there is only the trivial morphism from $\mathbf{sk}_{n-1}(X) \to \mathbf{sk}'_n(Y)$:

$$\begin{tikzcd} X = & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & X_{n-2} \arrow[d] \\ Y = & \dots \arrow[r] & Y_{n+1} \arrow[r] & Y_n \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

Since limits and colimits are computed component-wise in $ch(\mathbb{X})$, the short exact sequence of the torsion theory for a chain complex X in $ch(\mathbb{X})$ is given by:

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & X_{n-2} \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & X_{n+1} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & X_{n-2} \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & X_{n+1} \arrow[r] & X_n \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

□

Corollary 4.2.28. Let $\mathbb{X}$ be a normal category. For each $n \in \mathbb{Z}$ the triplets of full subcategories

$$(ch(\mathbb{X})_{n-1\geq}, ch(\mathbb{X})_{\geq n}, \mathcal{F}_{\mathbf{tr}_{n-1}})$$

are TTF theories in $ch(\mathbb{X})$. Moreover, by restriction these determine the TTF theories in $pch(\mathbb{X})$

$$(pch(\mathbb{X})_{n-1\geq}, pch(\mathbb{X})_{\geq n}, \mathcal{MN}_n).$$

Similarly, the triplet of subcategories in $ch(\mathbb{X})$

$$(Ker(\mathbf{cot}_{n-1\geq}), ch(\mathbb{X})_{n-1\geq}, ch(\mathbb{X})_{\geq n})$$

determines the TTF theories in $pch(\mathbb{X})$

$$(\mathcal{EP}_n, pch(\mathbb{X})_{n-1\geq}, pch(\mathbb{X})_{\geq n}).$$

4.3 Torsion theories in simplicial groups

We introduce torsion theories in $Simp(\mathbb{X})$ defined in a similar way as the torsion theories $\mathcal{COK}_n$ and $\mathcal{KER}_n$ of $pch(\mathbb{X})$. Moreover, for the particular case of the category of simplicial groups $Simp(Grp)$, these torsion theories are defined by the simplicial objects who have truncated above and below Moore chain complexes.

4.3.1 Torsion theories induced by $tr_n \dashv cosk_n$

We recall some useful properties and constructions that will be useful for simplicial objects in a category $\mathbb{X}$. References for more elementary facts of simplicial objects include: [GZ67], [Dus75] and [May67].

4.3.1. We denote Δ_n the full subcategory of Δ whose objects are the ordered sets $[m]$ for $m \leq n$. So an n-truncated simplicial object X (or a simplicial object truncated at level n) is a functor

$$X = \ \Delta_n^{op} \longrightarrow \mathbb{X} \ .$$

Such truncated simplicial object X is equivalent to a family of objects X_i for $0 \leq i \leq n$ and morphisms d_i, s_i:

$$X = \ X_n \ \begin{array}{c} \xrightarrow{d_n} \\ \xleftarrow{s_{n-1}} \\ \vdots \\ \xleftarrow{s_0} \\ \xrightarrow[d_0]{} \end{array} \cdots \begin{array}{c} \xrightarrow{d_4} \\ \xleftarrow{s_3} \\ \vdots \\ \xleftarrow{s_0} \\ \xrightarrow[d_0]{} \end{array} X_3 \ \begin{array}{c} \xrightarrow{d_3} \\ \xleftarrow{s_2} \\ \longrightarrow \\ \xleftarrow{d_2} \\ \xrightarrow{s_1} \\ \xleftarrow{d_1} \\ \xrightarrow[d_0]{s_0} \end{array} X_2 \ \begin{array}{c} \xrightarrow{d_2} \\ \xleftarrow{s_1} \\ \longrightarrow \\ \xleftarrow{d_1} \\ \xrightarrow[d_0]{s_0} \end{array} X_1 \ \begin{array}{c} \xrightarrow{d_1} \\ \xleftarrow{s_0} \\ \xrightarrow[d_0]{} \end{array} X_0$$

such that they satisfy the simplicial identities where they are defined. The category of n-truncated simplicial objects will be denoted as $Simp_n(\mathbb{X})$.

4.3.2. (see [Dus75]) For each $n \in \mathbb{N}$ we have the truncation functor

$$tr_n : \; Simp(\mathbb{X}) \longrightarrow Simp_n(\mathbb{X})$$

which simply forgets the objects X_i of a simplicial object X for dimensions higher than n. It is a standard application of Kan extensions that if $\mathbb{X}$ has finite limits then tr_n admits a right adjoint $cosk_n$ called the *n-coskeleton* functor, while if $\mathbb{X}$ has finite colimits then tr_n admits a left adjoint sk_n called the *n-skeleton*:

$$sk_n \dashv tr_n \dashv cosk_n : \quad \begin{tikzcd} Simp(\mathbb{X}) \arrow[d, bend right, ""'] \arrow[d, bend left, ""] \\ Simp_n(\mathbb{X}) \arrow[u] \end{tikzcd}$$

The endofunctors of $Simp(\mathbb{X})$, $Cosk_n = cosk_n tr_n$ and $Sk_n = sk_n tr_n$ are called the n-coskeleton and n-skeleton functors and they give an adjunction $Sk_n \dashv Cosk_n$.

We can describe the n-coskeleton functor as follows. For an n-truncated simplicial object X the *simplicial kernel* of the face morphisms $d_0, \ldots, d_n : X_n \to X_{n-1}$ is an object Δ_{n+1} in $\mathbb{X}$ with morphisms $\phi_0, \ldots, \phi_{n+1} : \Delta_{n+1} \to X_n$ such that $d_i\phi_j = d_{j-1}\phi_i$ for all $i < j$ and it is universal with this property: given a family of morphisms $p_0, \ldots, p_{n+1} : Y \to X_n$ such that $d_i p_j = d_{j-1} p_i$ for all $i < j$ then there is a unique morphism $\alpha : Y \to \Delta_{n+1}$ such that $\phi_i \alpha = p_i$:

$$\begin{tikzcd} Y \arrow[r, shift left=2, "p_{n+1}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "p_0"'] \arrow[d, "\alpha"] & X_n \arrow[d, "1"] & \\ \Delta_{n+1} \arrow[r, shift left=2, "\phi_{n+1}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\phi_0"'] & X_n \arrow[r, shift left=2, "d_n"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "d_0"'] & X_{n-1} \ldots \end{tikzcd}$$

Moreover, the universal property of the simplicial kernel Δ_{n+1} allows one to define degeneracies morphisms $s_i : X_n \to \Delta_{n+1}$. So the simplicial kernel of X defines an $(n+1)$-truncated simplicial object.

We may define the n-coskeleton of the n-truncated simplicial object as iteration of successive simplicial kernels:

$$cosk_n(X) = \quad \dots \begin{array}{c} \xrightarrow{\phi_{n+3}} \\ \xleftarrow{s_{n+2}} \\ \vdots \\ \xleftarrow[s_0]{} \\ \xrightarrow[\phi_0]{} \end{array} \Delta_{n+2} \begin{array}{c} \xrightarrow{\phi_{n+2}} \\ \xleftarrow{s_{n+1}} \\ \vdots \\ \xleftarrow[s_0]{} \\ \xrightarrow[\phi_0]{} \end{array} \Delta_{n+1} \begin{array}{c} \xrightarrow{d_{n+1}} \\ \xleftarrow{s_n} \\ \vdots \\ \xleftarrow[s_0]{} \\ \xrightarrow[d_0]{} \end{array} X_n \begin{array}{c} \xrightarrow{d_n} \\ \xleftarrow{s_{n-1}} \\ \vdots \\ \xleftarrow[s_0]{} \\ \xrightarrow[d_0]{} \end{array} X_{n-1} \dots$$

For $\mathbb{X} = Grp$ the simplicial kernel Δ_{n+1} of an n-truncated simplicial group can be described as the subgroup of X_n^{n+2} of $(n+2)$-tuples $(x_0, \dots, x_{n+1})$ such that $\phi_i(x_j) = \phi_{j-1}(x_i)$ for $i < j$ and where ϕ_i are the product projections.

The next result is proved in [Con84] for simplicial groups.

Theorem 4.3.3. Let $\mathbb{X}$ be a pointed category with finite limits. For a simplicial object X, the Moore normalization of the n-coskeleton $Cosk_n(X)$ satifies:

- $M(Cosk_n(X))_i = M(X)_i$ for $i \leq n$;
- $M(Cosk_n(X))_{n+1} = ker(\delta_n : M(X)_n \to M(X)_{n-1})$;
- $M(Cosk_n(X))_i = 0$ for $i > n+1$.

i.e. we have:

$$Cosk_n = \qquad \dots \longrightarrow \Delta_{n+2} \begin{array}{c} \xrightarrow{\phi_{n+2}} \\ \vdots \\ \xrightarrow[\phi_0]{} \end{array} \Delta_{n+1} \begin{array}{c} \xrightarrow{\phi_{n+1}} \\ \vdots \\ \xrightarrow[\phi_0]{} \end{array} X_n \begin{array}{c} \xrightarrow{d_n} \\ \vdots \\ \xrightarrow[d_0]{} \end{array} X_{n-1}$$

$$M(Cosk_n) = \qquad \dots \longrightarrow 0 \longrightarrow ker(\delta_n) \xrightarrow{k(\delta_n)} M(X)_n \xrightarrow{\delta_n} M(X)_{n-1}$$

Proof. The case for $i \leq n$ is trivial. For $i = n+1$ since Δ_{n+1} and the morphisms ϕ_i satisfy the simplicial identities then $\phi'_{n+1} : M(Cosk_n(X))_{n+1} \to M(X)_n$ factors through $ker(\delta_n)$.

Notice that by definition $M(X)_n = \cap_0^{n-1} ker(d_i)$ then $ker(\delta_n) = \cap_0^n ker(d_i)$, so we can consider the family of morphisms $\{p_i : ker(\delta_n) \to X_n\}$ with $p_i = 0$ for $i \leq n$ and p_{n+1} the inclusion of $ker(\delta_n)$ to X_n. We have $d_i p_j = d_{j-1} p_i$ if $i < j$ so the universal property of Δ_{n+1} gives α such that $p_i = \phi_i \alpha$. By construction $\phi_i \alpha = p_i = 0$ for $i \leq n$ so α factors through $M(Cosk_n(X))_{n+1}$. It follows that $M(Cosk_n(X))_{n+1} \cong ker(\delta_n)$.

For $i > n+1$, it follows by induction and since $k(\delta_n)$ is monic and that $Cosk_m(Cosk_n(X)) \cong Cosk_n(X)$ if $m > n$. □

Also recall that semi-abelian categories are finitely complete and cocomplete, so the functors sk_n and $cosk_n$ are defined and they are left and right adjoint of the truncation functor tr_n for each $n \geq 0$; $sk_n \dashv tr_n \dashv cosk_n$.

When $\mathbb{X}$ is a normal category we can introduce the simplicial analogues of the torsion theories $\mathcal{KER}_n$.

4.3.4. If $\mathbb{X}$ is a normal category with finite limits the adjunction $tr_n \dashv cosk_n$ is a localization: indeed, it is clear that tr_n preserves limits since they are computed component-wise in $Simp(X)$ and $Simp_n(X)$.

From 2.4.1 the adjunction $tr_n \dashv cosk_n$ with unit η, induces a hereditary torsion theory $(\mathcal{T}_{tr_n}, \mathcal{F}_{tr_n})$, where

$$\mathcal{T}_{tr_n} \cong ker(tr_n) = \{X \mid X_i = 0 \quad \text{for} \quad i \leq n\}$$

and

$$\mathcal{F}_{tr_n} = \{X \mid \eta_X : X \to Cosk_n(X) \quad \text{is monic}\}.$$

This torsion theory extends the torsion theory $(Ab(\mathbb{X}), Eq(\mathbb{X}))$ of $Grpd(\mathbb{X})$ (3.1.9) if $\mathbb{X}$ is a Mal'tsev category. Notice that the adjunction $()_0 \dashv Ind$ for internal groupoids (3.1.8) still hold in $Simp(\mathbb{X})$, in particular for an object X the indiscrete simplicial object is

$$Ind(X) = \qquad \cdots \; \substack{\xrightarrow{\phi_4} \\ \vdots \\ \xrightarrow[\phi_0]{}} \; X^4 \; \substack{\xrightarrow{\phi_4} \\ \vdots \\ \xrightarrow[\phi_0]{}} \; X^3 \; \substack{\xrightarrow{\phi_4} \\ \vdots \\ \xrightarrow[\phi_0]{}} \; X^2 \; \substack{\xrightarrow{\phi_1} \\ \xleftarrow[s_0]{} \\ \xrightarrow[\phi_0]{}} \; X \, .$$

where X^n is the n-fold product of X and degeneracies are defined by the product projections.

Proposition 4.3.5. Let $\mathbb{X}$ be a normal Mal'tsev category. For $n = 0$, consider the torsion theory $(\mathcal{T}_{tr_0}, \mathcal{F}_{tr_0})$ in $Simp(\mathbb{X})$. The torsion-free subcategory $\mathcal{F}_{tr_0}$ is equivalent to $Eq(\mathbb{X})$.

Proof. Clearly, we have $tr_0 = ()_0$ and the right adjoint, the 0-coskeleton functor, coincides with the indiscrete functor, so $tr_0 \dashv cosk_0 \cong ()_0 \dashv Ind$.

The unit of $()_0 \dashv Ind$ for a simplicial object X is given by

$$\begin{array}{ccccccccc}
X = & \cdots \; \substack{\xrightarrow{d_4} \\ \vdots \\ \xrightarrow[d_0]{}} & X_3 & \substack{\xrightarrow{d_3} \\ \vdots \\ \xrightarrow[d_0]{}} & X_2 & \substack{\xrightarrow{d_2} \\ \vdots \\ \xrightarrow[d_0]{}} & X_1 & \substack{\xrightarrow{d_1} \\ \xleftarrow[s_0]{} \\ \xrightarrow[d_0]{}} & X_0 \\
\Big\downarrow{\scriptstyle \eta_X} & & \Big\downarrow{\scriptstyle (d_0,d_1,d_2,d_3)} & & \Big\downarrow{\scriptstyle (d_0,d_1,d_2)} & & \Big\downarrow{\scriptstyle (d_0,d_1)} & & \Big\downarrow{\scriptstyle 1} \\
Ind(X) = & \cdots \; \substack{\xrightarrow{\phi_4} \\ \vdots \\ \xrightarrow[\phi_0]{}} & X_0^4 & \substack{\xrightarrow{\phi_4} \\ \vdots \\ \xrightarrow[\phi_0]{}} & X_0^3 & \substack{\xrightarrow{\phi_4} \\ \vdots \\ \xrightarrow[\phi_0]{}} & X_0^2 & \substack{\xrightarrow{\phi_1} \\ \xleftarrow[s_0]{} \\ \xrightarrow[\phi_0]{}} & X_0 \, .
\end{array}$$

Since $\mathbb{X}$ is a Mal'tsev category, it is proved in [Duv21] that internal groupoids are closed under subobjects in $Simp(\mathbb{X})$. So, if X has a monic unit η_X, then X

is a groupoid since $Ind(X)$ is an equivalence relation. Finally, since $(d_0, d_1) : X_1 \to X_0^2$ is monic then X is an equivalence relation and $\mathcal{F}_{tr_0} \subseteq Eq(\mathbb{X})$. On the other hand, clearly an equivalence relation always has a monic unit η_X. $\square$

Notice also that $Ab(\mathbb{X}) \subset \mathcal{T}_{tr_0}$, since an internal abelian group X always has $X_0 = 0$.

4.3.2 Torsion theories and truncated Moore normalizations

We recall that in [dL09], the homotopy groups in simplicial objects with enough projectives are defined in such way that the category of simplicial objects has a Quillen model structure. So, the next definition makes sense in this context.

Definition 4.3.6. Let $\mathbb{X}$ be a semi-abelian category with enough projectives and $n \in \mathbb{N}$, a simplicial object X in $Simp(\mathbb{X})$ is called an n-type if $\pi_i(X) = 0$ for all $i > n$.

A simplicial object X with a Moore complex such that $M(X)_i = 0$ for $i > n$ is an n-type.

We will write $\mathcal{M}_{n\geq}$ for the full subcategory of $Simp(X)$ of simplicial objects with trivial Moore complex for $i > n$, $M(X)_i = 0$. Similarly, we write $\mathcal{M}_{\geq n}$ for the category of simplicial objects with trivial Moore complex for $i < n$.

A simplicial object is a $K(A, n)$-simplicial group (or has type $K(A, n)$) if $\pi_n(X) = A$ and $\pi_i(X) = 0$ for $i \neq n$.

In order to introduce the simplicial analogues of the torsion theories $\mathcal{COK}_n$ we will restrict to the case of $\mathbb{X} = Grp$. We will review some properties of the category $\mathcal{M}_{n\geq}$ in Grp.

Theorem 4.3.7. ([Con84]) The category $\mathcal{M}_{n\geq}$ is equivalent to the full subcategory of $Simp_n(Grp)$ given by the n-truncated simplicial groups such that

$$[\bigcap_{i\in I} ker(d_i), \bigcap_{j\in J} ker(d_j)] = 0$$

for all subsets I, J of $n = \{0, 1, \ldots, n\}$ such that $I \cap J = \emptyset$ and $I \cup J = n$.

We have seen, from 3.2.7, that the category of internal groupoids in groups is equivalent to the category of reflexive graphs such that $[ker(d_0), ker(d_1)] = 0$. Moreover, also from Loday's [Lod82] it had been noticed that $Grpd(Grp) \cong \mathcal{M}_{1\geq}$. So, the previous theorem by Conduché is a generalization of these results.

We will prove that the subcategories $\mathcal{M}_{n\geq}$ are torsion-free subcategories of $Simp(\mathbb{X})$, respectively, that the subcategories $\mathcal{M}_{\geq n}$ are the torsion subcategories of $(\mathcal{T}_{tr_{n-1}}, \mathcal{F}_{tr_{n-1}})$. To this end we need to introduce the Conduché decomposition of a simplicial group as semi-direct product of degenerate images of the lower order Moore complex objects. This decomposition is given as follows:

4.3.8. ([Con84]) In order to avoid multiple subscripts we will write $\sigma_i = \bar{i}$ for the degeneracy maps of Δ.

For any object $[n] = \{0 < 1 < \cdots < n\}$ of the simplicial category Δ we will introduce an order in $S(n)$ the set of surjective maps of Δ with domain $[n]$. Any surjective map $\sigma : [n] \to [m]$ is written uniquely as $\sigma = \bar{i_1}\bar{i_2}\dots\bar{i}_{n-m}$ with $i_1 < i_2 < \cdots < i_{n-m}$. We introduce the inverse lexicographic order in $S(n,m)$ the set of surjective maps form $[n]$ to $[m]$:

$$\bar{i_1}\bar{i_2}\dots\bar{i}_{n-m} < \bar{j_1}\bar{j_2}\dots\bar{j}_{n-m} \quad \text{if} \quad i_{n-m} = j_{n-m}, \dots, i_{s+1} = j_{s+1}, \text{ and } i_s > j_s.$$

This order extends to $S(n)$ by setting $S(n,m) < S(n,l)$ if $m > l$.

As an example, for $S(4)$ we have:

$$\begin{aligned} id_{[4]} < \bar{3} < \bar{2} < \bar{2}\bar{3} < \bar{1} < \bar{1}\bar{3} < \bar{1}\bar{2} < \bar{1}\bar{2}\bar{3} \\ < \bar{0} < \bar{0}\bar{3} < \bar{0}\bar{2} < \bar{0}\bar{2}\bar{3} < \bar{0}\bar{1} < \bar{0}\bar{1}\bar{3} < \bar{0}\bar{1}\bar{2} < \bar{0}\bar{1}\bar{2}\bar{3} \end{aligned}$$

For a simplicial group X and a surjective map $\mathbf{i} = \bar{i_1}\bar{i_2}\dots\bar{i_r}$ we have $s_{\mathbf{i}} = s_{i_r}\dots s_{i_1}$ and $d_{\mathbf{i}} = d_{i_1}\dots d_{i_r}$. Using the order of $S(n)$ we have a filtration of X_n by the subgroups

$$G_{n,\mathbf{i}} = \bigcap_{\mathbf{j}\geq\mathbf{i}} ker(d_{\mathbf{j}}).$$

Notice that $G_{n,id} = 0$ and $G_{n,\bar{n-1}} = M(X)_n$.

4.3.9. ([Con84]) The order $S(n)$ satisfies that for a surjective map $\mathbf{i} : [n] \to [r]$ and its successor $\mathbf{j}$ we have the semidirect product

$$G_{n,\mathbf{j}} \cong G_{n,\mathbf{i}} \rtimes_{s_{\mathbf{i}}} M(X)_r.$$

Finally, this implies that X_n decomposes as a succession of semi-direct products:

$$X_n = (\dots(M(X)_n \rtimes_{s_{n-1}} M(X)_{n-1}) \rtimes_{s_{n-2}} \dots) \rtimes_{s_{p-1}\dots s_0} M(X)_0$$

As an example, for $n = 2$ consider a simplicial group X with Moore complex M:

$$X_2 \;\begin{smallmatrix} \xrightarrow{d_2} \\ \xleftarrow{} \\ \xrightarrow[d_1]{s_1} \\ \xleftarrow{s_0} \\ \xrightarrow[d_0]{} \end{smallmatrix}\; X_1 \;\begin{smallmatrix} \xrightarrow{d_1} \\ \xleftarrow{} \\ \xrightarrow[d_0]{s_0} \end{smallmatrix}\; X_0\,,$$

and the order $S(2) = \{id < \bar{1} < \bar{0} < \bar{0}\bar{1}\}$. Then we have the subgroups of X_2:

$$G_{2,\bar{0}\bar{1}} = ker(d_0 d_1)$$

$$G_{2,\bar{0}} = ker(d_0) \cap ker(d_0 d_1)$$

$$G_{2,\bar{1}} = ker(d_1) \cap ker(d_0) \cap ker(d_0 d_1) = M_2$$

$$G_{2,id} = ker(id) \cap ker(d_1) \cap ker(d_0) \cap ker(d_0 d_1) = 0.$$

and thus we have the split short exact sequences:

$$\begin{tikzcd}
M_2 = G_{2,\bar{1}} \arrow[r] & G_{2,\bar{0}} \arrow[r, shift right, "d_1"'] \arrow[d] & M_1 \arrow[l, shift right, "s_1"'] & \\
& G_{2,\bar{0}\bar{1}} \arrow[r] \arrow[d, shift right, "d_0"'] & X_2 \arrow[r, shift right, "d_0 d_1"'] & X_0 = M_0 \arrow[l, shift right, "s_1 s_0"'] \\
& M_1 \arrow[u, shift right, "s_0"'] & &
\end{tikzcd}$$

And it is clear that

$$X_2 = G_{2,\bar{0}\bar{1}} \rtimes_{s_1 s_0} M_0 = (G_{2,\bar{0}} \rtimes_{s_0} M_1) \rtimes_{s_1 s_0} M_0 = (((M_2 \rtimes_{s_1} M_1) \rtimes_{s_0} M_1) \rtimes_{s_1 s_0} M_0.$$

Similarly, for $n = 3$ we have:

$$X_3 = M_3 \rtimes M_2 \rtimes M_2 \rtimes M_1 \rtimes M_2 \rtimes M_1 \rtimes M_1 \rtimes M_0$$

with the parenthesis nested to the left as before and each M_i is included into X_3 with respective degeneracy map $s_{\mathbf{i}}$ following the order of $S(3)$.

Corollary 4.3.10. For each $n \in \mathbb{N}$, the category $\mathcal{M}_{\geq n+1}$ and $\mathcal{T}_{tr_n}$ are equivalent.

Moreover, the category $\mathcal{M}_{\geq n+1}$ is a torsion subcategory in $Simp(\mathbb{X})$;

$$(\mathcal{T}_{tr_n}, \mathcal{F}_{tr-n}) \cong (\mathcal{M}_{\geq n+1}, \mathcal{F}_{tr_n}).$$

Proof. It follows inmmediately from the semidirect decomposition that a simplicial group X has $X_i = 0$ for $n > i$ if and only if $M(X)_i = 0$ for $n > i$. $\square$

Proposition 4.3.11. ([Por93]) There is a 'truncation' functor

$$Cot_n : Simp(Grp) \to Simp(Grp)$$

such that there is a natural isomorphism:

$$Cot_n M \cong M\mathbf{Cot}_n$$

where M is the Moore normalization functor.

4.3.12. The functor Cot_n in 4.3.11 will be called the *n-cotruncation*[4]. The functor $Cot_n(X)$ is defined as follows:

$$Cot_n(X)_i = X_i \quad \text{for} \quad n > i\,,$$

$$Cot_n(X)_n = X_n/\delta_{n+1}(M_{n+1})\,,$$

and for $i > n$ the object $Cot_n(X)$ is obtained by deleting all $M(X)_k$ for $k > n$ and replacing $M(X)_n$ by $M(X)_n/\delta_{n+1}(M(X)_{n+1})$ in the semidirect decomposition.

For example, for a simplicial group X with Moore complex M, the first objects of $Cot_1(X)$ are:

$$Cot_1(X)_0 = X_0 = M_0$$

$$Cot_1(X)_1 = M_1/\delta_2(M_2) = M_1/\delta_2(M_2) \rtimes M_0$$

$$Cot_1(X)_2 = 0 \rtimes M_1/\delta_2(M_2) \rtimes M_1/\delta_2(M_2) \rtimes M_0$$

$$Cot_1(X)_3 = 0 \rtimes 0 \rtimes 0 \rtimes M_1/\delta_2(M_2) \rtimes 0 \rtimes M_1/\delta_2(M_2) \rtimes M_1/\delta_2(M_2) \rtimes M_0\,.$$

Proposition 4.3.13. ([Por93]) Let $\mathcal{M}_{n\geq}$ be the full subcategory of *Simp*(*Grp*) defined by those groups whose Moore complex is trivial for dimensions greater than n. Let $i_n : \mathcal{M}_{n\geq} \to Simp(Grp)$ the inclusion functor then

1. Cot_n is left adjoint of i_n;

[4]This is not the original terminology from [Por93]. There the simplicial groups of $\mathcal{M}_{n\geq}$ are called n-truncated and the functor $Cot_n = t\mid_n$ is called the n-truncation. The name cotruncation is suitable here, firstly, because it behaves similarly to the cotruncation functor of chain complexes introduced in section 4.2 and, secondly, because we will be working with both the truncation functors Cot_n and tr_n of simplicial groups.

2. the unit $\eta_X : X \to Cot_n(X)$ of the adjunction is a natural epimorphism which induces an isomorphism in $\pi_i(X)$ for $i \leq n$;

$$\begin{tikzcd}
\dots \arrow[r, shift left, "d_{n+2}"] \arrow[r, shift right, "d_0"'] & X_{n+1} \arrow[r, shift left, "d_{n+1}"] \arrow[r, shift right, "d_0"'] \arrow[d, "\eta_{n+1}"] & X_n \arrow[r, shift left, "d_n"] \arrow[r, shift right, "d_0"'] \arrow[d, "\eta_n"] & X_{n-1} \arrow[r, shift left, "d_{n-1}"] \arrow[r, shift right, "d_0"'] \arrow[d, "1"] & \dots \\
\dots \arrow[r, shift left, "d_{n+2}"] \arrow[r, shift right, "d_0"'] & Cot(X)_{n+1} \arrow[r, shift left, "d_{n+1}"] \arrow[r, shift right, "d_0"'] & X_n/\delta_{n+1}(M_{n+1}) \arrow[r, shift left, "d_n"] \arrow[r, shift right, "d0"'] & X_{n-1} \arrow[r, shift left, "d_n"] \arrow[r, shift right, "d_0"'] & \dots .
\end{tikzcd} \tag{4.1}$$

3. for any simplicial group X, $\pi(Cot_n(X)) = 0$ for $i > n$;

4. the inclusion $\mathcal{M}_{n\geq} \to \mathcal{M}_{n+1\geq}$ correspond to a natural epimorphism

$$\eta_n : Cot_{n+1} \to Cot_n$$

and for a simplicial group X then $ker(\eta_n(X))$ is a $K(\pi_{n+1}(X), n+1)$-simplicial group.

Unlike the case for $ch(\mathbb{X})$ this cotruncation functor for simplicial groups is normal and thus defines a torsion theory in $Simp(Grp)$.

Corollary 4.3.14. The subcategory $\mathcal{M}_{n\geq}$ of $Simp(Grp)$ given by the simplicial groups with trivial Moore complex for dimension greater than n is a torsion-free subcategory of $Simp(Grp)$; the torsion theory is given by the pair

$$(Ker(Cot_n), \mathcal{M}_{n\geq}).$$

Proof. By 4.3.13 and by 2.2.8 it suffices to prove that the functor Cot_n is normal. For a simplicial group X with a Moore complex M, since taking normalization preseves exact sequences we have that the Moore complex of $ker(\eta_X)$ is:

$$\dots \longrightarrow M_{n+1} \xrightarrow{e_{n+1}} d_{n+1}(M_{n+1}) \longrightarrow 0 \longrightarrow \dots$$

which is trivial under the chain cotruncation $\mathbf{cot}_n$. Since the functor $\mathbf{cot}_n$ and Cot_n commute with the Moore normalization we have that $Cot_n(\eta_X) = 0$ for any simplicial group X. $\square$

Corollary 4.3.15. A simplicial group X with Moore complex M belongs to

$Ker(Cot_n)$ if and only if $M_i = 0$ for $n-1 \geq i$ and δ_{n+1} is a regular epimorphism:

$$\dots \longrightarrow M_{n+1} \xrightarrow{\delta_{n+1}} M_n \longrightarrow 0 \longrightarrow \dots$$

Corollary 4.3.16. The category $\mathcal{M}_{0\geq}$ is equivalent to the category $Dis(Grp)$ of discrete simplicial groups.

The category $\mathcal{M}_{1\geq}$ is equivalent to the category of internal grupoids.

Moreover, the categories $Dis(Grp) \cong Grp$ and $Grpd(Grp)$ are torsion-free subcategories of $Simp(\mathbb{X})$.

Proof. For $n = 0$, it is clear from the definition of the cotruncation functor Cot_0 that $Cot_0(X) = Dis(\pi_0(X))$.

For $n = 1$, it has been mentioned that $\mathcal{M}_{1\geq}$ is equivalent to the category of internal grupoids. $\square$

4.3.3 The lattice $\mu(Grp)$ and the fundamental simplicial groups

Theorem 4.3.17. Let $\mathbb{X}$ be a semi-abelian category and X a simplicial object with Moore chain complex N. Then the normalization functor M maps the short exact sequence of X given by the torsion theory $(\mathcal{T}_{tr_n}, \mathcal{F}_{tr_n})$ in $Simp(\mathbb{X})$ into the short exact sequence of N given by the torsion theory $\mathcal{KER}_{n+1}$ in $pch(\mathbb{X})$.

Moreover, M maps the torsion theory $(\mathcal{T}_{tr_n}, \mathcal{F}_{tr_n})$ into $\mathcal{KER}_{n+1}$:

$$M : \begin{array}{ccc} Simp(\mathbb{X}) & \longrightarrow & pch(\mathbb{X}) \\ (\mathcal{T}_{tr_n}, \mathcal{F}_{tr_n}) & & (pch(\mathbb{X})_{\geq n+1}, \mathcal{MN}_n) \end{array}$$

i.e. the subcategory $\mathcal{T}_{tr_n}$ is mapped into $pch(\mathbb{X})_{\geq n+1}$ and $\mathcal{F}_{tr_n}$ into $\mathcal{MN}_n$.

Proof. Since $\mathbb{X}$ is semi-abelian the normalization functor M is exact, it preserves short exact sequences ([EdL04]) and also preserves the normal epi/mono factorization of morphisms in $Simp(X)$. From 4.3.3 notice that M commutes (up to isomorphism) with the truncation and coskeleton functors:

$$\begin{tikzcd} Simp(\mathbb{X}) \arrow[r, "M"] \arrow[d, bend right, "tr_n"'] \arrow[d, phantom, "\dashv"] & ch(\mathbb{X}) \arrow[d, bend right, "\mathbf{tr}_n"'] \arrow[d, phantom, "\dashv"] \\ Simp_n(\mathbb{X}) \arrow[r, "M"] \arrow[u, bend right, "cosk_n"'] & ch(\mathbb{X})_{n\geq} \arrow[u, bend right, "\mathbf{cosk}_n"'] \end{tikzcd}.$$

So, for a simplicial object X and its Moore complex N, the functor M maps the short exact sequence in $Simp(\mathbb{X})$:

$$\begin{tikzcd} 0 \arrow[r] & ker(\eta_X) \arrow[r, "k"] & X \arrow[r, "e"] \arrow[dr, "\eta_X"'] & \eta_X(X) \arrow[r] \arrow[d, "m"] & 0 \\ & & & Cosk_n(X) & \end{tikzcd}$$

into

$$\begin{tikzcd} M(ker(\eta_X)) = \arrow[d, hook, "m(k)"] & \dots \arrow[r] & N_{n+2} \arrow[r, "\delta'_{n+2}"] \arrow[d] & ker(\delta_{n+1}) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \dots \\ M(X) = \arrow[d, two heads, "M(e)"] \arrow[dd, bend right, "M(\eta_X)"'] & \dots \arrow[r] & N_{n+2} \arrow[r, "\delta_{n+2}"] \arrow[d] & N_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, two heads, "e_{n+1}"] & N_n \arrow[r, "\delta_n"] \arrow[d] & \dots \\ M(\eta_X(X)) = \arrow[d, "M(m)"] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & \delta_{n+1}(N_{n+1}) \arrow[r, hook, "m_{n+1}"] \arrow[d] & N_n \arrow[r, "\delta_n"] \arrow[d] & \dots \\ M(Cosk_n(X)) = & \dots \arrow[r] & 0 \arrow[r] & ker(\delta_n) \arrow[r, hook, "k(\delta_n)"] & N_n \arrow[r, "\delta_n"] & \dots \end{tikzcd}$$

by M.

Since the short exact sequence of the torsion theories are preserved, it follows that $M(\mathcal{T}_{tr_n}) \subseteq pch(\mathbb{X})_{\geq n+1}$ and $M(\mathcal{F}_{tr_n}) \subseteq \mathcal{MN}_n$. $\square$

Theorem 4.3.18. Let X be a simplicial group with Moore complex N. The normalization functor M maps the short exact sequence of X given by the torsion theory $(Ker(Cot_n), \mathcal{M}_{n\geq})$ in $Simp(Grp)$ into the short exact sequence of N given by $\mathcal{COK}_{n+1}$.

Moreover, M maps the torsion category $Ker(Cot_n)$ into the torsion category $\mathcal{EP}_{n+1}$ and, respectively, the torsion-free category $\mathcal{M}_{n\geq}$ into $pch(Grp)_{n\geq}$;

$$M : \begin{array}{ccc} Simp(Grp) & \longrightarrow & pch(Grp) \\ (Ker(Cot_n), \mathcal{M}_{n\geq}) & \longmapsto & (\mathcal{EP}_{n+1}, pch(Grp)_{n\geq}) \end{array}$$

Proof. Since the cotruncation functors commute (up to isomorphism) with the Moore normalization

$$\begin{tikzcd} Simp(Grp) \arrow[r, "M"] \arrow[d, bend right, "Cot_n"'] \arrow[d, phantom, "\dashv"] & pch(Grp) \arrow[d, bend right, "\mathbf{Cot}_n"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{M}_{n\geq} \arrow[r, "M"] \arrow[u, bend right] & pch(Grp)_{n\geq} \arrow[u, bend right, "\mathbf{Sk}_n"'] \end{tikzcd}$$

and the normalization M preserves kernels (it is an exact functor), for a simplicial group X and its Moore complex N the short exact sequence

$$0 \longrightarrow ker(\eta_X) \longrightarrow X \xrightarrow{\eta_X} Cot_n(X) \longrightarrow 0$$

is mapped by M into the short exact sequence (written vertically)

$$\begin{array}{ccccccccccc}
M(ker(\eta_x)) = & \dots \to & N_{n+1} & \xrightarrow{e_{n+1}} & \delta_{n+1}(N_{n+1}) & \longrightarrow & 0 & \longrightarrow \dots \\
\downarrow & & \downarrow & & \downarrow m_{n+1} & & \downarrow & \\
M(X) = & \dots \to & N_{n+1} & \xrightarrow{\delta_{n+1}} & N_n & \xrightarrow{\delta_n} & N_{n-1} & \longrightarrow \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
M(Cot_n(X)) = & \dots \longrightarrow & 0 & \longrightarrow & N_n/\delta_{n+1}(N_{n+1}) & \xrightarrow{\delta'_n} & N_{n-1} & \longrightarrow \dots .
\end{array}$$

Since the associated short exact sequence of the torsion theory is preserved by M it follows that $M(Ker(Cot_n)) \subset \mathcal{EP}_{n+1}$ and $M(\mathcal{M}_{n\geq}) \subset pch(Grp)_{n\geq}$. $\square$

4.3.19. For each $n \geq 0$ we will denote the previous torsion theories as

$$\mu_{n\geq} = (Ker(Cot_n), \mathcal{M}_{n\geq}), \quad \mu_{\geq n+1} = (\mathcal{M}_{\geq n+1}, \mathcal{F}_{tr_n}).$$

For a simplicial group X we will write $\mu_{n\geq}(X)$ and $\mu_{\geq n+1}(X)$ for the torsion subobject of X correspondig to each torsion theory. Since the Moore normalization functor is exact the subcategories $\mathcal{M}_{n\geq}$ and $\mathcal{M}_{\geq n+1}$ are always closed under subobjects and quotients in $Simp(Grp)$, so the torsion theories $\mu_{n\geq}$ are cohereditary and $\mu_{\geq n+1}$ are hereditary.

We will write $\mu(Grp)$ for the set of these torsion theories. The next result shows that $\mu(Grp)$ is indeed a lattice similar to the lattice $\mathcal{COT}(pch(Grp)_{\geq 0})$.

Theorem 4.3.20. The torsion subcategories of the torsion theories $\mu_{n\geq}$ and $\mu_{\geq n+1}$ in $Simp(Grp)$ are linearly ordered as:

$$0 \subseteq \cdots \subseteq Ker(Cot_{n+1}) \subseteq \mathcal{M}_{\geq n+1} \subseteq Ker(Cot_n) \subseteq \mathcal{M}_{\geq n} \subseteq \cdots \subseteq Simp(Grp).$$

Moreover, the torsion theories $\mu_{n\geq}$ and $\mu_{\geq n+1}$ form a linearly ordered lattice $\mu(Grp)$:

$$0 \leq \cdots \leq \mu_{n+1\geq} \leq \mu_{\geq n+1} \leq \mu_{n\geq} \leq \mu_{\geq n} \leq \dots$$
$$\cdots \leq \mu_{\geq 2} \leq \mu_{1\geq} \leq \mu_{\geq 1} \leq \mu_{0\geq} \leq Simp(Grp)\,.$$

Proof. First we will prove $\mathcal{M}_{\geq n+1} \subseteq Ker(Cot_n)$. For a simplicial group X and M its Moore normalization and η the counit as in 4.1, if $M_i = 0$ for $n \geq i$ then $X_i = 0$ for $n \geq i$. Since η_X is a normal epimorphism, we conclude that $Cot_n(X)_i = 0$ for $n \geq i$. It follows from the semidirect decomposition that $Cot_n(X) = 0$.

Now we will prove $Ker(Cot_n) \subseteq \mathcal{M}_{\geq n}$. From 4.1 it is clear that if $Cot_n(X) = 0$ we have $X_i = 0$ for $n-1 \geq i$, then $M_i = 0$ for $n-1 \geq i$ and, accordingly, X is in $\mathcal{M}_{\geq n}$. $\square$

Theorem 4.3.21. For all $n \geq 0$ we have

1. the subcategory $\mathcal{M}_{n\geq}$ of simplicial groups with trivial Moore complexes for $i > n$ is a Birkhoff subcategory of $Simp(Grp)$. Moreover, $\mathcal{M}_{n\geq}$ is semi-abelian.

2. the subcategories $\mathcal{M}_{\geq n}$ of simplicial groups with trivial Moore Complexes for $n > i$ are semi-abelian.

Proof. This follows from the fact that $\mathcal{M}_{n\geq}$ is a torsion-free subcategories of the cohereditary torsion theory $\mu_{n\geq}$, so $\mathcal{M}_{n\geq}$ is closed under quotients in $Simp(Grp)$. $\mathcal{M}_{n\geq}$ is semi-abelian from Corollary 1.5.10.

On the other hand, $\mathcal{M}_{\geq n}$ is the torsion category of the hereditary torsion theory $\mu_{\geq n}$ in a semi-abelian category, so it follows the second statement from 2.4.11. $\square$

4.3.22. In summary, the normalization functor M maps each torsion theory of $\mu(Grp)$ into a torsion theory of $\mathcal{COT}(pch(Grp)_{\geq 0})$:

$$M : \begin{array}{ccc} \mu_{n\geq} & \longrightarrow & \mathcal{COK}_{n+1} \\ \mu_{\geq n} & \longrightarrow & \mathcal{KER}_n \end{array},$$

i.e. each torsion/torsion-free subcategory is mapped into the corresponding torsion/torsion-free subcategory, accordingly, and the associated short exact sequence is preserved:

$$\begin{array}{cc} \mu(Grp) = & \ldots \leq \mu_{1\geq} \leq \mu_{\geq 1} \leq \mu_{0\geq} \leq Simp(Grp) \\ \big\downarrow{\scriptstyle M} & \\ \mathcal{COT}(pch(Grp)_{\geq 0}) = & \ldots \leq \mathcal{COK}_2 \leq \mathcal{KER}_1 \leq \mathcal{COK}_1 \leq pch(Grp)_{\geq 0} \end{array}$$

Definition 4.3.23. Let $\mu(Grp)$ be the lattice defined as in 4.3.19. For $m \geq n$ and the idempotent radicals of the torsion theories of $\mu(Grp)$:

$$\mu_{m\geq} \quad \leq \quad \mu_{\geq m} \quad \leq \cdots \leq \quad \mu_{n\geq} \quad \leq \quad \mu_{\geq n}\,,$$

consider the quotients of preradicals of $Simp(Grp)$:

$$\Pi^{\geq n}_{m\geq} = \frac{\mu_{\geq n}}{\mu_{m\geq}}\,, \quad \Pi^{n\geq}_{\geq m} = \frac{\mu_{n\geq}}{\mu_{\geq m}}\,, \quad \Pi^{\geq n}_{\geq m} = \frac{\mu_{\geq n}}{\mu_{\geq m}}\,, \quad \Pi^{n\geq}_{m\geq} = \frac{\mu_{n\geq}}{\mu_{m\geq}}\,;$$

and for all n the trivial quotients:

$$\Pi^{\geq n} = \frac{\mu_{\geq n}}{0} \cong \mu_{\geq n}\,, \quad \Pi_{\geq n} = \frac{Id}{\mu_{\geq n}}\,, \quad \Pi^{n\geq} = \frac{\mu_{n\geq}}{0} \cong \mu_{n\geq}\,, \quad \Pi_{n\geq} = \frac{Id}{\mu_{n\geq}}\,.$$

For a simplicial group X the objects $\Pi^{\geq n}_{m\geq}(X), \Pi^{n\geq}_{\geq m}(X), \Pi^{\geq n}_{\geq m}(X)$ and $\Pi^{n\geq}_{m\geq}(X)$ as well as $\Pi^{\geq n}(X), \Pi_{\geq n}(X), \Pi^{n\geq}(X)$ and $\Pi_{n\geq}(X)$ will be called the *fundamental simplicial groups* of X.

Accordingly, the family of functors:

$$\Pi^{\geq n}_{m\geq}, \Pi^{n\geq}_{\geq m}, \Pi^{\geq n}_{\geq m}, \Pi^{n\geq}_{m\geq}, \Pi^{\geq n}, \Pi_{\geq n}, \Pi^{n\geq}, \Pi_{n\geq} : \; Simp(Grp) \longrightarrow Simp(Grp)$$

will be called *fundamental simplicial functors*.

The rest of the section will be devoted to justify the choice of the name 'fundamental'. The homotopy groups of the fundamental simplicial groups of a simplicial group X are the same as X but only in certain dimension, otherwise they are trivial. The following calculations include 3) 4.3.13.

Theorem 4.3.24. Let $\mu(Grp)$ be the lattice in 4.3.19. Let X be a simplicial group with Moore normalization M. The homotopy groups of the fundamental simplicial group of X are calculated as follows:

1. For all $n \geq 0$

$$\pi_i(\Pi^{n\geq}(X)) = \pi_i(\Pi^{\geq n+1}) = \begin{cases} \pi_i(M) & i \geq n+1 \\ 0 & n+1 > i\,. \end{cases}$$

2. For all $n \geq 0$

$$\pi_i(\Pi_{n\geq}(X)) = \pi_i(\Pi^{\geq n+1}(X)) = \begin{cases} 0 & i \geq n+1 \\ \pi_i(X) & n+1 > i\,. \end{cases}$$

3. For all $n \geq 0$
$$\pi_i(\Pi^{n\geq}_{\geq n+1}(X)) = 0 \text{ for all } i\,.$$

4. For $m > n \geq 0$
$$\pi_i(\Pi^{n\geq}_{\geq m+1}(X)) = \begin{cases} \pi_i(X) & m+1 > i \geq n+1 \\ 0 & \text{otherwise}\,. \end{cases}$$

5. Moreover, for $m > n \geq 0$ and for all i
$$\pi_i(\Pi^{n\geq}_{\geq m+1}(X)) = \pi_i(\Pi^{n\geq}_{m\geq}(X)) = \pi_i(\Pi^{\geq n+1}_{\geq m+1}(X)) = \pi_i(\Pi^{\geq n+1}_{m\geq}(X))\,.$$

6. In particular, for $m = n+1$
$$\pi_i(\Pi^{n\geq}_{\geq n+2}(X)) = \begin{cases} \pi_i(X) & i = n+1 \\ 0 & i \neq n\,. \end{cases}$$

Proof. From 4.3.11 and 4.3.3 it is easy to see that the preradicals commute with normalization:

$$\begin{array}{ccc} Simp(Grp) & \xrightarrow{\mu_{\geq n+1}} & Simp(Grp) \\ \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} \\ pch(Grp)_{\geq 0} & \xrightarrow{ker_{n+1}} & pch(Grp)_{\geq 0} \end{array} \quad , \quad \begin{array}{ccc} Simp(Grp) & \xrightarrow{\mu_{n\geq}} & Simp(Grp) \\ \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M} \\ pch(Grp)_{\geq 0} & \xrightarrow{cok_{n+1}} & pch(Grp)_{\geq 0}\,. \end{array}$$

Since the homotopy groups of a simplicial group are calculated with the homology of the Moore normalization the result follows from 4.2.24. $\square$

4.3.25. For an abelian group A, a simplicial group X is an Eilenberg-Mac Lane simplicial group of type $K(A,n)$, or a $K(A,n)$-simplicial group, if it has $\pi_n(X) = A$ and all other homotopy groups trivial.

In particular, the n-th Eilenberg-Mac Lane simplicial group $K(A,n)$ for an abelian group A (in symmetric form) is defined as follows. Consider the $(n+1)$-truncated simplicial group $k(A,n)$:

$$k(A,n) = \;\; A^{n+1} \; \underset{d_0}{\overset{d_{n+1}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}}}} \; A \; \underset{0}{\overset{0}{\underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}}}} \; 0 \; \underset{0}{\overset{0}{\underset{\longrightarrow}{\overset{\longrightarrow}{}}}} \; \cdots \; \underset{0}{\overset{0}{\underset{\longrightarrow}{\overset{\longrightarrow}{}}}} \; 0$$

where the non-trivial face morphisms are

$$(d_0, d_1, \ldots, d_{n+1}) = (p_0, p_0 + p_1, p_1 + p_2, \ldots, p_{n-1} + p_n, p_n)\,,$$

where p_i are the product projections, and the degeneracies are given by $s_i = (0, \ldots, 1_A, \ldots, 0)$ with 1_A in the ith-place for $0 \leq i \leq n$. Then we define

$$K(A, n) = cosk_{n+1}(k(A, n)).$$

It can be observed that $m \geq n+1$ $K(A, n)_m = A^{\binom{p}{n}}$, where $\binom{p}{n}$ is the binomial coefficient.

Indeed, it is easy to see that the the Moore complex of $K(A, n)$ is:

$$M(K(A, n)) = \ \ldots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \ . \tag{4.2}$$

The Dold-Kan Theorem (see [Wei94] for instance) gives an equivalence between the categories of simplicial abelian groups $Simp(Ab)$ and chain complexes in abelian groups $chn(Ab)_{\geq 0}$, where the equivalence is given by the Moore normalization. In [CC91] this equivalence was further extended to an equivalence between $Simp(Grp)$ and the category of hypercrossed modules in Grp. A hypercrossed module is a group chain complex M with group actions for all n:

$$\Phi^n_\alpha : \ M_{r(\alpha)} \longrightarrow Aut(M_n) \ \text{ for } \alpha \in S(n)$$

and binary operations

$$\Gamma^n_{\alpha,\sigma} : \ M_{r(\alpha)} \times M_{r(\sigma)} \longrightarrow M_n \ \text{ for } \alpha, \sigma \in S(n),\ 1 < \sigma < \alpha,\ \alpha \cap \sigma = \emptyset$$

satisfying some equations. $S(n)$ is the set of surjective maps of Δ with domain $[n]$ and has the order introduced in 4.3.8 and $\alpha \cap \sigma = \emptyset$ means that the maps α, σ do not share a common index in their factorization by the degeneracies $\bar{i}$.

Lemma 4.3.26. Let X be a simplicial group with Moore complex

$$M = \ \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

then X isomorphic to the Eilenberg- Mac Lane simplicial group $K(A, n)$.

Proof. Consider the chain complex M as above. Since all degrees of M are trivial except one, all morphisms $M_i \to Aut(M_j)$ with $j > i$ are trivial as well all binary mappings $M_i \times M_j \to M_k$ with $k > i, j$. This means that the structure of Hypercrossed module of M is necessarily unique. Thus, it follows from the equivalence of Hypercrossed modules and simplicial groups ([CC91]) that $X \cong K(A, n)$. $\square$

The next corollary generalizes part 4) of 4.3.13.

Corollary 4.3.27. For $n \geq 0$ and a simplicial group X the simplicial groups:

$$\Pi^{n\geq}_{\geq n+2}(X)\,, \quad \Pi^{n\geq}_{n+1\geq}(X)\,, \quad \Pi^{\geq n+1}_{\geq n+2}(X)\,, \quad \Pi^{\geq n+1}_{n+1\geq}(X)$$

are $K(\pi_{n+1}(X), n+1)$-simplicial groups.

Moreover, $\Pi^{\geq n+1}_{n+1\geq}(X)$ is isomorphic to $(n+1)$-th Eilenberg-Mac Lane simplicial group $K(\pi_{n+1}(X), n+1)$.

Proof. It follows by definition from 6) of 4.3.24. Moreover, the Moore complex of $\Pi^{\geq n+1}_{n+1\geq}(X)$ isomorphic to the chain complex $\frac{ker_{n+1}(M)}{cok_{n+2}(M)}$, that is a trivial chain complex except at order n that has $\pi_{n+1}(X)$. It follows from the observation in 4.3.25 that $\Pi^{\geq n+1}_{n+1\geq}(X) \cong K(\pi_{n+1}(X), n+1)$. □

We conclude with some observations.

4.3.28. 1. For a simplicial group X the connected components group is calculated as $\pi_0(X) \cong coeq(d_0, d_1) \cong X_0/\delta_1(M_1)$, so $Cot_0 \cong \pi_0$. Hence, the adjunction $\pi_0 \dashv D$ is the torsion-free reflection of the torsion theory $\mu_{0\geq}$ and $\mathcal{M}_{0\geq} \cong Dis(Grp)$ (4.3.16). Moreover, the following diagram commutes (up to isomorphism):

$$\begin{tikzcd} Simp(Grp) \arrow[rr, "\Pi_{0\geq}"] \arrow[dr, "\pi_0"'] & & Dis(Grp) \\ & Grp \arrow[ur, "D"'] & \end{tikzcd}$$

2. For $n \geq 1$, we can consider the category Ab_n the full subcategory of $Simp(Grp)$ given by the n-Eilenberg Mac Lane objects $K(A, n)$ for all abelian groups A. Clearly, $Ab \cong Ab_n$ and by 4.3.27 and the observations in 4.3.25 the following diagram commutes (up to isomorphism):

$$\begin{tikzcd} Simp(Grp) \arrow[rr, "\Pi^{\geq n+1}_{n+1\geq}"] \arrow[dr, "\pi_n"'] & & Ab_n \\ & Ab \arrow[ur, "K(\_{,}n)"'] & \end{tikzcd}$$

3. The *fundamental groupoid* or Poincaré groupoid can be defined categorically as the groupoid given by the left adjoint Π_1 of the nerve functor $\mathcal{N} : Grpd(\mathbb{X}) \to Simp(\mathbb{X})$ (see [GZ67]). For the case of $\mathbb{X} = Grp$, we have noticed that $\mathcal{M}_{1\geq} \cong Grpd(Grp) \cong \mathbb{X}Mod$ is the torsion-free part of $\mu_{1\geq}$, it follows from the definition of the cotruncation functor Cot_1 and the semi-direct decomposition that the inclusion $\mathcal{M}_{1\geq} \to Simp(Grp)$ is

naturally isomorphic to the nerve functor $\mathcal{N} : Grpd(Grp) \to Simp(Grp)$, so the diagram

$$\begin{tikzcd} Simp(Grp) \arrow[rr, "\Pi_{1\geq}"] \arrow[dr, "\Pi_1"'] & & \mathcal{M}_{1\geq} \\ & Grpd(Grp) \arrow[ur, "\mathcal{N}"'] & \end{tikzcd}$$

commutes up to isomorphism.[5]

[5]In [Duv21], the fundamental groupoid is studied in the case of Mal'tsev exact categories. Similar to the group case, the left adjoint of the nerve functor is given by the unique groupoid structure of the reflexive graph $X_1/H(X) \rightrightarrows X_0$ where $H(X) = d_2(D_0 \cap D_1)$ and D_i is the kernel pair of the face morphisms $d_1, d_0 : X_2 \to X_1$.

Chapter 5

Applications

5.1 Chain complexes with operations

[1]

As seen in chapter 3, the torsion theories in internal groupoids can be studied in a simpler way as torsion theories in crossed modules, we can also study torsion theories in simplicial groups as torsion theories in categories given by the Moore chain complexes with operations.

In particular, the torsion/torsion-free objects of these chain complexes with operators are constructed as the torsion/torsion-free objects of the underlying proper chain complex of the torsion theories in $\mathcal{COT}$ in $pch(Grp)$.

We will restrict the lattice $\mu(Grp)$ to the semi-abelian subcategories of $Simp(Grp)$. Firstly, we consider the subcategories $\mathcal{M}_{n\geq}$ where the case for $n = 1$ reduces to the torsion theories in crossed modules and for $n = 2$ they correspond to Conduché's 2-crossed modules. As a second kind of example we consider the category of T-complexes and Ashley's crossed complexes in groups. We will show that different kinds of central extensions appear as torsion categories.

5.1.1. In [CC91], the notion of a hypercrossed module is introduced as a chain complex M with group actions for all n:

$$\Phi_\alpha^n : \ M_{r(\alpha)} \longrightarrow Aut(M_n) \ \text{ for}\, \alpha \in S(n)$$

[1]This chapter is adapted from [Lop22a].

and binary operations

$$\Gamma^n_{\alpha,\sigma} : \; M_{r(\alpha)} \times M_{r(\sigma)} \longrightarrow M_n \;\; \text{for}\, \alpha, \sigma \in S(n),\, 1 < \sigma < \alpha,\, \alpha \cap \sigma = \emptyset$$

satisfying some equations. The Moore complex of a simplicial group is an hypercrossed module and, thus we have a generalized version of the Dold-Kan theorem. The categories of hypercrossed modules and simplicial groups are equivalent under the Moore normalization functor M', so the diagram commutes up to isomorphism:

$$\begin{tikzcd} Simp(Grp) \arrow[rr, "M' \cong"] \arrow[dr, "M"'] & & {}_H\mathbb{X}mod \arrow[dl, "U"] \\ & chn(Grp) & \end{tikzcd}$$

where U forgets the group actions Φ^n_α and binary operations $\Gamma^n_{\alpha,\sigma}$. This equivalence M' generalises the following equivalences:

- The classical Dold-Kan Theorem between $chn(Ab)_{\geq 0}$ the category of chain complexes in abelian groups and the category $Simp(Ab)$ of simplicial abelian groups;
- the equivalence between the categories $Grpd(Grp)$ of internal groupoids in groups and $\mathbb{X}Mod$ of crossed modules;
- the equivalence between the categories $\mathcal{M}_{2\geq}$ of simplicial groups with trivial Moore complexes for $i > 2$ and ${}_2\mathbb{X}Mod$ of Conduché's 2-crossed modules;
- the equivalence between the categories of Dakin's group T-complexes and Ashley's crossed complexes in groups, and as well as the hypercrossed complexes with all the binary operations trivial $\Gamma^n_{\alpha,\sigma} = 0$.

5.1.1 Simplicial groups with truncated Moore complex

We will study the restriction of the torsion theories of $\mu(Grp)$ to the subcategories $\mathcal{M}_{n\geq}$ of $Simp(Grp)$. The category of abelian groups and special kinds of categories of central extensions will appear as torsion categories of $\mathcal{M}_{n\geq}$.

Recall that group actions are written in the left as ${}^b(a)$. We write $[a, b] = aba^{-1}b^{-1}$ for the commutator elements for a, b in a group A, and for a precrossed module $\delta : A \to B$ the Peiffer elements are $< a, a' >= aa'a^{-1\delta(a)}a'^{-1}$ and so, a precrossed module is a crossed module if and only if all Peiffer elements are trivial.

We will write $\mu(\mathcal{M}_{n\geq})$ for the lattice of the restricted torsion theories of $\mu(Grp)$ to the subcategory $\mathcal{M}_{n\geq}$. Will first study the case $n = 2$.

We recall the category of 2-crossed modules introduced by D. Conduché.

Definition 5.1.2. ([Con84]) A group chain complex

$$L \xrightarrow{\delta_2} M \xrightarrow{\delta_1} N$$

is called a 2-*crossed module* if N acts on L and M and the differentials δ_2, δ_1 are equivariant (N acts over itself with conjugation), and there is a mapping

$$\{\ ,\ \}:\ M \times M \longrightarrow L$$

satisfying:

2XM1 $\delta_2\{m_0, m_1\} = m_0 m_1 m_0^{-1}\ {}^{\delta_1(m_0)}(m_1^{-1})$;

2XM2 $\{\delta_2(l_0), \delta_2(l_1)\} = [l_0, l_1]$;

2XM3 $\{\delta_2(l), m\}\{m, \delta_2(l)\} = l\ {}^{\delta_1(m)}l^{-1}$;

2XM4 $\{m_0, m_1m_2\} = \{m_0, m_1\}\{m_0, m_2\}\{\delta_2\{m_0, m_2\}^{-1},\ {}^{\delta_1(m_0)}m_1\}$;

2XM5 $\{m_0m_1, m_2\} = \{m_0, m_1m_2m_1^{-1}\}\ {}^{\delta_1(m_0)}\{m_1, m_2\}$;

2XM6 ${}^n\{m_0, m_1\} = \{{}^nm_0, {}^nm_1\}$.

The map $\{\ ,\ \}$ is called the Peiffer Lifting.

A morphism of 2-crossed modules is a morphism of chain complexes that preserves the group action and the Peiffer lifting. The category of 2-crossed complexes will be denoted as ${}_2\mathbb{X}Mod$.

5.1.3. ([Con84]) From the definition we observe that given a 2-crossed module

$$L \xrightarrow{\delta_2} M \xrightarrow{\delta_1} N$$

we have:

1. The Peiffer lifting defines a group action of M over L as

$$^{m}(l) = l\{\delta_2(l)^{-1}, m\}$$

   and δ_2 is an equivariant morphism.

2. It follows from 2XM2 that $\delta_2 : L \to M$ is a crossed module but in general $\delta_1 : M \to N$ is only a precrossed module.

3. 2XM1 and 2XM2 basically mean that the Peiffer lifting $\{\ ,\ \}$ factors through the commutator of L and the Peiffer commutator of $\delta_1 : M \to N$:

$$\begin{tikzcd} L \times L \arrow[r, "\delta_2\times\delta_2"] \arrow[d, "{[\ ,\ ]}"'] & M \times M \arrow[dl, "{\{\ ,\ \}}"'] \arrow[d, "{<\ ,\ >}"] & \\ L \arrow[r, "\delta_2"] & M \arrow[r, "\delta_1"] & N\,. \end{tikzcd}$$

4. If $L = 0$, from 2XM1 $\delta_1 : M \to N$ is a crossed module.

5. For a precrossed module $\delta : A \to B$ since the Peiffer elements $< a, a' >= aa'a^{-1\delta(a)}a'^{-1}$ are contained in $ker(\delta)$, so the coskeleton:

$$\mathbf{cosk}_1(\delta) : \ ker(\delta) \longrightarrow A \xrightarrow{\delta} B$$

is a 2-crossed module with $\{\ ,\ \} =<\ ,\ >$.

6. In addition, if $\delta : A \to B$ is a crossed module, besides $\mathbf{cosk}_1(\delta)$, the skeleton of δ is a 2-crossed module with trivial Peiffer Lifting:

$$\mathbf{sk}_1(\delta) : \ 0 \longrightarrow A \xrightarrow{\delta} B\,.$$

7. The 1-cotruncation of a 2-crossed module is a crossed module:

$$\mathbf{cot}_1(\delta_2, \delta_1) : \ 0 \longrightarrow M/\delta_2(L) \longrightarrow N\,.$$

8. Form the previous observations we have an adjunction:

$$\begin{tikzcd} {}_2\mathbb{X}mod \arrow[r, bend left, "\mathbf{cot}_1"] \arrow[r, phantom, "\perp"] & \mathbb{X}mod\,. \arrow[l, bend left, "\mathbf{sk}_1"] \end{tikzcd}$$

9. Also, we have the adjunction:

$$\begin{tikzcd} {}_2\mathbb{X}Mod \arrow[r, bend left, "\mathbf{cot}_0"] \arrow[r, phantom, "\perp"] & Grp \arrow[l, bend left, "\mathbf{sk}_0"] \end{tikzcd} = \begin{tikzcd} {}_2\mathbb{X}Mod \arrow[r, bend left, "\mathbf{cot}_1"] \arrow[r, phantom, "\perp"] & \mathbb{X}Mod \arrow[l, bend left, "\mathbf{sk}_1"] \arrow[r, bend left, "H_0"] \arrow[r, phantom, "\perp"] & Grp\,. \arrow[l, bend left, "D"] \end{tikzcd}$$

10. The coskeletal adjunctions hold as well:

$$
\begin{tikzcd}
{}_2\mathbb{X}Mod \arrow[r, bend left, "\mathbf{tr}_1"] \arrow[r, phantom, "\perp"] & P\mathbb{X}Mod \arrow[l, bend left, "\mathbf{cosk}_1"]
\end{tikzcd}, \qquad
\begin{tikzcd}
{}_2\mathbb{X}Mod \arrow[r, bend left, "\mathbf{tr}_0"] \arrow[r, phantom, "\perp"] & Grp \arrow[l, bend left, "\mathbf{cosk}_0"]
\end{tikzcd}.
$$

For completeness sake and to further exhibit the similarities between crossed modules and 2-crossed modules we include the following results.

Proposition 5.1.4. (see [Lod82], [Con84] and [Con03]) For a simplicial group X with Moore complex M:

1. If $M_i = 0$ for $i \geq 2$, then the morphism $\delta_1 : M_1 \to M_2$ is a crossed module where M_0 acts on M_1 by conjugation with the degeneracy s_0.

   Conversely, there is a functor (the nerve functor) associating to a crossed module $\delta : A \to B$ a simplicial group X having δ as its Moore complex.

2. If $M_i = 0$ for $i \geq 3$, the complex $M_2 \to M_1 \to M_0$ is a 2-crossed module where M_0 acts on M_1 and M_2 by conjugation via the degeneracies and the Peiffer lifting

$$\{x, y\} = s_1[x, y][s_1(y), s_0(x)]\,.$$

   Conversely, there is a functor associating to a 2-crossed module (δ_2, δ_1) a simplicial group whose Moore complex is naturally isomorphic to (δ_2, δ_1).

Corollary 5.1.5. The category ${}_2\mathbb{X}Mod$ of 2-crossed modules is semi-abelian.

Proof. ${}_2\mathbb{X}Mod$ is equivalent to $\mathcal{M}_{2\geq}$ a torsion-free category of a cohereditary torsion theory of a semi-abelian category $Simp(\mathbb{X})$. In particular, ${}_2\mathbb{X}Mod$ is a Birkhoff subcategory of $Simp(Grp)$ so the result follows from 1.5.10. $\square$

The following categories were also introduced by Conduché.

Definition 5.1.6. ([Con84])

1. A *reduced 2-crossed module* is a group morphism $\delta : L \to M$ with a map $\{\ ,\ \} : M \times M \to L$ satisfying:

   (a) $\delta\{m_0, m_1\} = [m_0, m_1]$,

   (b) $\{\delta(l_0), \delta(l_1)\} = [l_0, l_1]$,

   (c) $\{\delta(l), m\}\{m, \delta(l)\} = 1$,

(d) $\{m_0, m_1m_2\} = \{m_0, m_1\}\{m_0, m_2\}\{[m_2, m_0], m_1\}$,

(e) $\{m_0m_1, m_2\} = \{m_0, m_1m_2m_1^{-1}\}\{m_1, m_2\}$.

The category of reduced 2-crossed modules will be denoted as $R_2\mathbb{X}Mod$.

2. A *stable crossed module* is a group morphism $\delta : L \to M$ with a map $\{\ ,\ \} : M \times M \to L$ satisfying:

   (a) $\delta\{m_0, m_1\} = [m_0, m_1]$,

   (b) $\{\delta(l_0), \delta(l_1)\} = [l_0, l_1]$,

   (c) $\{m_1, m_0\} = \{m_0, m_1\}^{-1}$,

   (d) $\{m_0m_1, m_2\} = \{m_0m_1m_0^{-1}, m_0m_2m_0^{-1}\}\{m_0, m_2\}$.

   We will denote $St\mathbb{X}Mod$ for the category of stable crossed modules.

The underlying morphism $\delta : L \to M$ of reduced 2-crossed module/stable crossed module is in fact a crossed module.

The terminology reduced 2-crossed module comes from the fact that a simplicial group X is called reduced if $X_0 = 0$ and from the next theorem[2].

Theorem 5.1.7. ([Con84]) The category of 2-crossed modules $L \to M \to N$ with $N = 0$ is equivalent to the category of simplicial groups with trivial Moore complex except for degrees $1, 2$ and it is also equivalent to the category $R_2\mathbb{X}Mod$ of reduced 2-crossed modules.

Theorem 5.1.8. ([Con84]) For $n \geq 2$ the category of simplicial groups with trivial Moore complex except for degrees n and $n + 1$ is equivalent to the category $St\mathbb{X}Mod$ of stable crossed modules.

[2] In [Con84], no name is given to these 2-crossed modules.

5.1.9. Recall the lattice $\mu(Grp)$:

$$
\begin{tikzcd}
Simp(Grp) = & Simp(Grp) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & Simp(Grp) \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & 0 \arrow[l, shift left] \arrow[d, hook] \\
\mu_{0\geq} = & Ker(Cot_0) \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & Simp(Grp) \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{M}_{0\geq} \cong Grp \arrow[l, shift left] \arrow[d, hook] \\
\mu_{\geq 1} = & \mathcal{M}_{\geq 1} \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & Simp(Grp) \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{F}_{tr_0} \cong Eq(Grp) \arrow[l, shift left] \arrow[d, hook] \\
\mu_{1\geq} = & Ker(Cot_1) \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & Simp(Grp) \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{M}_{1\geq} \cong Grpd(Grp) \arrow[l, shift left] \arrow[d, hook] \\
\mu_{\geq 2} = & \mathcal{M}_{\geq 2} \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & Simp(Grp) \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{F}_{tr_1} \arrow[l, shift left] \arrow[d, hook] \\
\mu_{2\geq} = & Ker(Cot_2) \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & Simp(Grp) \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{M}_{2\geq} \arrow[l, shift left] \\
\dots & \dots & \dots & \dots
\end{tikzcd}
$$

Lemma 5.1.10. For $n = 1$, the restriction of the lattice $\mu(Grp)$ to the subcategory $\mathcal{M}_{1\geq}$ is the lattice $\mu(\mathcal{M}_{1\geq})$:

$$0 \;\leq\; \mu'_{\geq 1} \;\leq\; \mu'_{0\geq} \;\leq\; \mathcal{M}_{1\geq}$$

where $\mu'_{\geq i}$ and $\mu'_{i\geq}$ are the restrictions of $\mu(Grp)$ to $\mathcal{M}_{1\geq}$. These torsion theories are equivalent to

$$0 \;\leq\; (Ab, Eq(Grp)) \;\leq\; (Conn(Grpd(Grp)), Dis(Grp)) \;\leq\; \mathcal{M}_{1\geq}\,.$$

And thus, equivalent to the lattice of torsion theories in crossed modules:

$$0 \;\leq\; (Ab, NMono) \;\leq\; (Cext, Dis) \;\leq\; \mathbb{X}Mod\,.$$

Proof. First, notice that besides $\mu_{\geq 1}$ and $\mu_{0\geq}$ the intersection of any torsion theory of $\mu(Grp)$ to $\mathcal{M}_{1\geq}$ is the bottom element $(0, \mathcal{M}_{1\geq})$. Since the torsion and the torsion-free category of a torsion theory determine each other and we already have $Dis(Grp) = Grp$ and $Eq(Grp)$ as torsion-free categories then the torsion categories much be equivalent, i.e. $Ker(Cot_0) \cap \mathcal{M}_{1\geq} = Conn(Grpd(Grp))$ and $\mathcal{M}_{\geq 1} \cap \mathcal{M}_{1\geq} = Ab$. By 3.2.8, this lattice corresponds to the lattice above in crossed modules. $\square$

Following the case for $n = 1$, for $n = 2$, if we write $\mu'_{i\geq}$, $\mu'_{\geq i}$ for the restrictions of the torsion theories, the only non-trivial torsion theories of $\mu(\mathcal{M}_{2\geq})$ are:

$$\begin{tikzcd}
\mu'_{0\geq} = & Ker(Cot_0) \cap \mathcal{M}_{2\geq} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{M}_{2\geq} \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & Grp \arrow[l, shift left] \arrow[d, hook] \\
\mu'_{\geq 1} = & \mathcal{M}_{\geq 1} \cap \mathcal{M}_{2\geq} \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{M}_{2\geq} \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & Eq(Grp) \arrow[l, shift left] \arrow[d, hook] \\
\mu'_{1\geq} = & Ker(Cot_1) \cap \mathcal{M}_{2\geq} \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{M}_{2\geq} \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & Grpd(Grp) \arrow[l, shift left] \arrow[d, hook] \\
\mu'_{\geq 2} = & \mathcal{M}_{\geq 2} \cap \mathcal{M}_{2\geq} \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{M}_{2\geq} \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{F}_{tr_1} \arrow[l, shift left]
\end{tikzcd}$$

Similar to the case internal groupoids/crossed modules, for $n = 2$ we can study these torsion theories with 2-crossed modules.

Lemma 5.1.11. For a 2-crossed module $X = L \to M \to N$ the chain complexes torsion objects:

$$\begin{aligned}
\mu_{\geq 2}(X) &= & ker(\delta_2) &\longrightarrow 0 \longrightarrow 0 \\
\mu_{1\geq}(X) &= & L &\longrightarrow \delta_2(L) \longrightarrow 0 \\
\mu_{\geq 1}(X) &= & L &\longrightarrow ker(\delta_1) \longrightarrow 0 \\
\mu_{0\geq}(X) &= & L &\longrightarrow M \longrightarrow \delta_1(X)
\end{aligned}$$

and the torsion-free objects:

$$\begin{aligned}
X/\mu_{\geq 2}(X) &= & \delta_2(L) &\longrightarrow M \longrightarrow N \\
X/\mu_{1\geq}(X) &= & 0 &\longrightarrow M/\delta_2(L) \longrightarrow N \\
X/\mu_{\geq 1}(X) &= & 0 &\longrightarrow \delta_1(M) \longrightarrow N \\
X/\mu_{0\geq}(X) &= & 0 &\longrightarrow 0 \longrightarrow N/\delta_1(M)\,.
\end{aligned}$$

are 2-crossed modules with the structure induced from X.

Proof. It is straightforward to verify of the axioms of 2-crossed modules. $\square$

Theorem 5.1.12. The torsion theories in $\mathcal{M}_{2\geq}$ are equivalent under the Moore normalization functor to the torsion theories in ${}_2\mathbb{X}Mod$:

1. $\mu'_{0\geq} \cong (Ker(\mathbf{cot}_0), Dis)$

2. $\mu'_{\geq 1} \cong (R_2\mathbb{X}Mod, NMono)$

3. $\mu'_{1\geq} \cong (CExt \cap R_2\mathbb{X}Mod, \mathbb{X}Mod)$

4. $\mu'_{\geq 2} \cong (Ab_2, \mathcal{F}_{\mathbf{tr}_1})$

where

- Ab_2 is the subcategory given by 2-crossed modules of the form $A \to 0 \to 0$ with A an abelian group.

- Dis, $NMono$, $\mathbb{X}Mod$ are respectively the categories of discrete crossed modules, normal subgroup crossed modules and crossed modules considered as 2-crossed modules under $\mathbf{sk}_1$.

- $R_2\mathbb{X}Mod$ is the category of 2-reduced crossed modules as in 5.1.7.

- $CExt \cap R_2\mathbb{X}Mod$ is the category of 2-reduced crossed modules who are also central extensions, i.e. the morphism δ is surjective;

- the functors $\mathbf{cot}_i, \mathbf{sk}_i, \mathbf{tr}_i, \mathbf{cosk}_i$ are defined for 2-crossed complexes as in 5.1.3.

Proof. Firstly, let M' the Conduché equivalence between the category $\mathcal{M}_{2\geq}$ and and the category 2-crossed modules, so the diagram

$$\begin{tikzcd} \mathcal{M}_{2\geq} \arrow[rr, "M'\cong"] \arrow[dr, "M"'] & & {}_2\mathbb{X}mod \arrow[dl, "U"] \\ & pch(Grp)_{2\geq} & \end{tikzcd}$$

commutes up to isomorphism where U is the forgetful functor. Now for the equivalences:

1. Is trivial.

2. Since $\mathcal{M}_{2\geq} \cap \mathcal{M}_{\geq 1}$ is the category of simplicial groups with non-trivial Moore complex at degrees 1,2 it must be equivalent to the category of reduced 2-crossed modules. As for what concerns to the torsion-free category it has been noticed that $Eq(Grp) \cong NMono$.

3. Since the morphism $\delta_2 : L \to M$ is a crossed module, then $L \to \delta_2(L)$ is a central extension. The Moore normalization functor is conservative, so a simplicial group X with a Moore complex $M_2 \to \delta_2(M_2) \to 0$ necessarily belongs to $Ker(Cot_1)$, then $Ker(Cot_1) \cap \mathcal{M}_{\geq 2} \cong CExt \cap R_2\mathbb{X}Mod$. For the torsion-free category it has been noticed that $Gpds(Grp) \cong \mathbb{X}Mod$.

4. Since $\mathcal{M}_{2\geq} \cap \mathcal{M}_{\geq 2}$ is the category of simplicial groups with the only non-trivial Moore complex at degree 2, it must be equivalent to Ab_2.

□

We will now consider the case for $n > 2$ and $\mu(\mathcal{M}_{n\geq})$. First, we can observe that the top non-trivial elements

$$\mu'_{1\geq} \ \leq \ \mu'_{\geq 1} \ \leq \ \mu'_{0\geq}$$

have as torsion-free categories the categories $Grpd(Grp)$, $Eq(Grp)$ and Dis, respectively.

$\mu(\mathcal{M}_{n\geq})$, unlike $\mu(Grp)$, has a minimal element $\mu'_{\geq n}$. The torsion categories of the first three bottom non-trivial elements of $\mu(\mathcal{M}_{n\geq})$ are characterized as follows.

Theorem 5.1.13. Consider $n > 2$, In $\mu(\mathcal{M}_{n\geq})$ consider the torsion theories:

1. $\mu_{\geq n-1} = (\mathcal{M}_{\geq n-1} \cap \mathcal{M}_{n\geq} \ , \ \mathcal{F}_{tr_{n-2}} \cap \mathcal{M}_{n\geq})$;
2. $\mu'_{n-1\geq} = (Ker(Cot_{n-1}) \cap \mathcal{M}_{n\geq} \ , \ \mathcal{M}_{n-1\geq} \cap \mathcal{M}_{n\geq})$;
3. $\mu'_{\geq n} = (\mathcal{M}_{\geq n} \cap \mathcal{M}_{n\geq} \ , \ \mathcal{F}_{tr_{n-1}} \cap \mathcal{M}_{n\geq})$.

Then for the torsion categories we have the equivalences:

1. $\mathcal{M}_{\geq n-1} \cap \mathcal{M}_{n\geq} \cong St\mathbb{X}Mod$.
2. $Ker(Cot_{n-1}) \cap \mathcal{M}_{n\geq} \cong St\mathbb{X}Mod \cap CExt$, where $St\mathbb{X}mod \cap Cext$ is the category of stable crossed modules that have a surjective map δ, i.e. they are central extensions.
3. $\mathcal{M}_{\geq n} \cap \mathcal{M}_{n\geq} \cong Ab_n$, where Ab_n is the subcategory of simplicial groups given by the Eilenberg Mac-Lane objects at degree n.

Proof. 1) Notice $\mathcal{M}_{\geq n-1} \cap \mathcal{M}_{n\geq}$ is the category of simplicial groups whose Moore complex is trivial except at degrees n, $n-1$, so by 5.1.8 it is equivalent to the category of crossed modules.

2) If X is a simplicial group in $Ker(Cot_{n-1}) \cap \mathcal{M}_{n\geq}$ its Moore complex M is trivial except at degrees n and $n-1$. Also, since X belongs to $Ker(Cot_{n-1})$ under the normalization $\delta_n : M_n \to M_{n-1}$ is surjective so X equivalent to a stable crossed module central extension.

3) Is trivial since a simplicial group in $\mathcal{M}_{\geq n} \cap \mathcal{M}_{n\geq}$ has a trivial Moore complex except at degree n, which must be an abelian group. □

Corollary 5.1.14. The category $St\mathbb{X}Mod$ of stable crossed modules and the category $R_2\mathbb{X}Mod$ of reduced 2-crossed modules are semi-abelian.

Proof. $St\mathbb{X}Mod$ is the torsion category of a hereditary torsion theory of $\mathcal{M}_{n\geq}$ which is semi-abelian. So, the result follows from 2.4.11. Similarly for $R_2\mathbb{X}Mod$. □

5.1.2 Reduced crossed complexes

Introduced by Dakin [Dak77], a T-complex is a Kan simplicial object that admits a canonical filler for horns, for example groupoids have this property. Following the work of Ashley's [Ash78] and the observations in [CC91] a group T-complex (a T-complex in simplicial groups) can be defined as simplicial group X with $M_n \cap D_n = 0$ where M is the Moore complex of X and D is the graded subgroup of X generated by the degenerated elements of X.

Definition 5.1.15. ([Ash78]) A *reduced crossed complex* or a group crossed complex M is a proper chain complex[3]

$$M = \ \ldots \longrightarrow M_n \xrightarrow{\delta_n} M_{n-1} \longrightarrow \ldots \longrightarrow M_2 \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} M_0$$

where

1. M_n is abelian for $n \geq 2$;
2. M_0 acts on M_n for $n \geq 1$ and the restriction to $\delta_1(M_1)$ acts trivially on M_n for $n \geq 2$;
3. δ_n preserves the action of M_0 and $\delta_1 : M_1 \to M_0$ is a crossed module.

A morphism of reduced crossed complexes is a chain complex morphism that preserves all actions. We will write $Crs(Grp)$ for the category of reduced crossed complexes.

It is proved in [Ash78] that the category of group T-complexes is equivalent to the category of reduced crossed complexes. And in [EP97] is shown that the category of reduced crossed complexes is an epi-reflective subcategory of simplicial groups[4].

[3]Usually, a (reduced) crossed complex is written with the indexes $i \geq 1$ since in the general case of crossed complex the object M_0 is used for the object of objects of the base groupoid M_1, M_0. We will not be working with crossed complexes over groupoids. Following [CC91], we start with $i = 0$ since reduced crossed complexes are the Moore complexes of group T-complexes.

[4]This result is consequence of more general result. Crossed complexes over groupoids (over sets) is a variety of simplicial groupoids, reduced crossed complexes are groupoid crossed complexes as a group is a groupoid with only one object.

Corollary 5.1.16. The category of Dakin's group T-complexes, and hence the category $Crs(Grp)$ of reduced crossed complexes, are semi-abelian.

Proof. From [EP97], we have that group T-complexes is a normal epi-reflective subcategory of $Simp(Grp)$. In fact it is a Birkhoff subcategory, we just need to prove that it is closed under regular epimorphism in $Simp(Grp)$. So let be X a group T-complex, a simplicial group with $M_n^X \cap D_n^X = 0$ and $f : X \to Y$ a regular epimorphism where M_n^X, M_n^Y are the Moore subobjects and D_n^X, D_n^Y are the graded subgroups generated by degenerated elements of X and Y respectively.

Since the Moore normalization preserves regular epimorphisms then the restriction $f_M : M_n^X \to M_n^Y$ is also a regular epimorphism. Now the restriction $f_D : D_n^X \to D_n^Y$ is surjective, since we have a commutative diagram:

$$\begin{tikzcd} X_n \arrow[d, "f"] & X_m \arrow[l, "s_\mathbf{i}"'] \arrow[d, "f"] \\ Y_n & Y_m \arrow[l, "s_\mathbf{i}"'] \end{tikzcd}$$

so for a degenerate element $s_\mathbf{i}(y_m)$ in Y_n ($s_\mathbf{i}$ a composition of degenerate morphisms), there is x_m in X_M such that $s_\mathbf{i} f(x_m) = f s_\mathbf{i}(x_m)$. Finally, the morphism:

$$f_M \times f_D : \ M_n^X \times_{X_n} D_n^X \longrightarrow M_n^Y \times_{Y_n} D_n^Y$$

is a regular epimorphism (see 1.2.10), so if $M_n^X \times_{X_n} D_n^X = M_n^X \cap D_n^X = 0$ then so $M_n^Y \cap D_n^Y = 0$. $\square$

5.1.17. As first examples, we have that for a crossed module $\delta : A \to B$ the 1-skeleton and 1-coskeleton are crossed complexes:

$$\mathbf{sk}_1(\delta) = \quad \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow A \longrightarrow B$$

$$\mathbf{cosk}_1(\delta) = \quad \ldots \longrightarrow 0 \longrightarrow ker(\delta) \longrightarrow A \longrightarrow B\,.$$

We can define an n-reduced crossed complex as a n-truncated chain complex M:

$$M = \ M_n \xrightarrow{\delta_n} M_{n-1} \longrightarrow \ldots \qquad M_2 \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} M_0$$

satisfying all the axioms of a reduced crossed complex that make sense, thus we have a category $Crs(Grp)_{n\geq}$ of n-truncated crossed complexes.

It is straightforward to see that for a n-truncated reduced crossed complex M the n-skeleton $\mathbf{sk}_n(M)$ and the n-coskeleton $\mathbf{cosk}_n(M)$ are reduced crossed complexes. We consider $Crs(Grp)_{n\geq}$ as a full subcategory of $Crs(Grp)$ under $\mathbf{sk}_n$. Clearly, $Crs(Grp)_{1\geq} \cong \mathbb{X}Mod$.

Also, the n-truncation and n-cotruncation of a reduced crossed complex are n-truncated reduced crossed complexes. So, we have the adjuctions [5]:

$$\mathbf{cot}_n \dashv \mathbf{sk}_n \dashv \mathbf{tr}_n \dashv \mathbf{cosk}_n : \quad \begin{tikzcd} Crs(Grp) \arrow[d, bend right=60] \arrow[d, phantom, shift right=4, "\dashv"] \arrow[d, phantom, "\dashv"] \arrow[d, bend left=20] \arrow[d, phantom, shift left=4, "\dashv"] \\ Crs(Grp)_{n\geq} \arrow[u, bend right=20] \arrow[u, bend right=60] \end{tikzcd} .$$

5.1.18. Consider the lattice $\mu(Crs(Grp))$ given by the restriction of $\mu(Grp)$ to $Crs(Grp)$

$$\mu(Crs(Grp)) = \quad \ldots \leq \mu'_{\geq 2} \leq \mu'_{1\geq} \leq \mu'_{\geq 1} \leq \mu'_{0\geq} \leq Crs(Grp)\,.$$

Then the torsion theories $\mu'_{n\geq}$ and $\mu'_{\geq n}$ can be expressed with the functors $\mathbf{cot}_n \dashv \mathbf{sk}_n \dashv \mathbf{tr}_n \dashv \mathbf{cosk}_n$:

$$\mu'_{n\geq} = (Ker(\mathbf{cot}_n), Crs(Grp)_{n\geq}), \quad \mu'_{\geq n} = (Crs(Grp)_{\geq n}, \mathcal{F}_{\mathbf{tr}_n})\,.$$

Here, $Crs(Grp)_{\geq n}$ is the subcategory of reduced crossed complexes trivial for degrees below n, so basically, $Crs(Grp)_{\geq n} \cong chn(Ab)_{\geq n}$ for $n \geq 1$. It is easy to see that $Grp \cong Dis(Grp)$, $Eq(Grp)$, $Grpd(Grp)$ are still torsion-free categories of $Crs(Grp)$.

Since $Crs(Grp)$ is a semi-abelian category then so are $Crs(Grp)_{n\geq}$ and $Crs(Grp)_{\geq n}$.

Theorem 5.1.19. For $n > 1$, let $\mu(Crs(Grp)_{n\geq})$ be the lattice given by restriction of $\mu(Grp)$ to $Crs(Grp)_{n\geq}$:

$$\mu(Crs(Grp)_{n\geq}) = 0 \leq \mu'_{\geq n} \leq \mu'_{n-1\geq} \leq \ldots \leq \mu'_{\geq 1} \leq \mu'_{0\geq} \leq Crs(Grp)_{n\geq}\,.$$

Where the bottom torsion theories are given by

1. $\mu'_{n-1\geq} = (Ker(\mathbf{cot}_{n-1}) \cap Crs(Grp)_{n\geq}\ ,\ \mathcal{M}_{n-1\geq} \cap Crs(Grp)_{n\geq})$;
2. $\mu'_{\geq n} = (\mathcal{M}_{\geq n} \cap Crs(Grp)_{n\geq}\ ,\ \mathcal{F}_{tr_{n-1}} \cap Crs(Grp)_{n\geq})$.

Then for the torsion categories we have the equivalences:

[5]([BHS10]) Once again, this functors are also defined in the more general setting of crossed complexes over groupoids, also the adjunctions hold.

1. $Ker(\mathbf{cot}_{n-1}) \cap Crs(Grp)_{n\geq} \cong CExt(Ab)$, where $CExt(Ab)$ is the category of central extension in abelian groups, i.e. surjective morphisms.

2. $\mathcal{M}_{\geq n} \cap Crs(Grp)_{n\geq} \cong Ab_n$; where Ab_n is the subcategory of simplicial groups given by the n-th Eilenberg-Mac Lane objects or, equivalently, the category of abelian groups.

Proof. The proof is similar to 5.1.13. □

5.2 A weak semi-abelian TTF-theory in simplicial groups

In 4.2.28, if we write $\mathbb{X} = Grp$ (or any semi-abelian category) we have introduced two different examples of TTF-theories in $ch(\mathbb{X})$ and in $pch(\mathbb{X})$:

$$(Ker(\mathbf{cot}_{n-1}), pch(\mathbb{X})_{n-1\geq}, pch(\mathbb{X})_{\geq n}), \quad (ch(\mathbb{X})_{n-1\geq}, ch(\mathbb{X})_{\geq n}, \mathcal{F}_{\mathbf{tr}_{n-1}}).$$

These examples rely on the fact that in $ch(\mathbb{X})$ we have the string of adjunctions $\mathbf{cot} \dashv \mathbf{sk} \dashv \mathbf{tr} \dashv \mathbf{cosk}$ which is not the case for simplicial objects.

We have seen that in internal groupoids $Grpd(Grp)$, and in $Simp(Grp)$, the subcategory of discrete groupoids $Dis(Grp) \cong \mathcal{M}_0$ is a torsion-free subcategory and also mono-coreflective but not normal mono-coreflective, so it is not a torsion subcategory. In general, the subcategories $\mathcal{M}_{n\geq}$ are only torsion-free.

However, restricted to $Crs(Grp)$ since we have the functors and adjunctions $\mathbf{cot} \dashv \mathbf{sk} \dashv \mathbf{tr} \dashv \mathbf{cosk}$ similarly defined as in $ch(\mathbb{X})$, so we will have weaker versions of a TTF-theories. More precisely, the categories $Crs(Grp)_{n-1\geq}$ are torsion-free mono-reflective subcategories of $Crs(Grp)$ and the pairs

$$(Crs(Grp)_{n-1\geq}, Crs(Grp)_{\geq n})$$

satisfy the axiom TT1 of a torsion theory and they only satisfy TT2 (the short exact sequence) for a special class of objects of $Crs(Grp)$.

Two different kinds of torsion theories are introduced.

5.2.1. Let $\mathbb{X}$ be a normal category.

1. We will call a torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathbb{X}$ a CTF-theory if $\mathcal{F}$ is a mono-coreflective category of $\mathbb{X}$. Trivially, in a TTF-theory $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ the pair $(\mathcal{C}, \mathcal{T})$ is a CTF-theory.

2. For a class of objects $\mathcal{E}$ of $\mathbb{X}$, a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathbb{X}$ will be called a $\mathcal{E}$-torsion theory if:

TT1 for all $X \in \mathcal{T}$ and $Y \in \mathcal{F}$, every morphism $f : X \to Y$ is zero,

TT2' for every object $X \in \mathcal{E}$ exists a short exact sequence

$$0 \longrightarrow T_X \xrightarrow{t_X} X \xrightarrow{f_X} F_X \longrightarrow 0$$

with $T_X \in \mathcal{T}$ and $F_X \in \mathcal{F}$.

As a first example of a $\mathcal{E}$-torsion theory we can consider given by the torsion theories $\mathcal{COK}_n$.

Lemma 5.2.2. In $ch(\mathbb{X})$ the category of chain complexes the pair

$$(Ker(\mathbf{cot}_n), ch(\mathbb{X})_{n\geq})$$

and $\mathcal{E}$ the class of proper chain complexes $pch(\mathbb{X})$ is a $\mathcal{E}$-torsion theory in $ch(\mathbb{X})$.

Proof. To verify TT1 it suffices to notice that given a commutative diagram:

$$\begin{tikzcd} X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}"] & X_n \arrow[d, "f_n"] \\ 0 \arrow[r] & Y_n \end{tikzcd}$$

with δ_n a morphism with trivial cokernel then the morphism f must be trivial. TT2' holds since it has been stablished that the restriction of the pair $(Ker(\mathbf{cot}_n), ch(\mathbb{X})_{n\geq})$ to proper chain gives a torsion theory $(\mathcal{EP}_n, pch(\mathbb{X})_{n\geq})$ in $pch(\mathbb{X})$. □

As a second example we have the subcategory of discrete crossed modules Dis in $\mathbb{X}Mod$ which behaves almost as a torsion torsion-free subcategory.

Proposition 5.2.3. In $\mathbb{X}Mod$ consider the triplet of subcategories:

$$(CExt, Dis, Ab)$$

then:

1. The pair $(CExt, Dis)$ is a CTF theory in $\mathbb{X}Mod$.
2. The pair (Dis, Ab) is an $\mathcal{E}$-torsion theory where $\mathcal{E}$ is the class of crossed modules $\delta : A \to B$ where the action $B \to Aut(A)$ is trivial.

Proof. 1) It has been noticed that the discrete functor D has right adjoint $(\,)_0$ with the counit for a crossed module $\delta : A \to B$:

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d] & A \arrow[d, "\delta"] \\ B \arrow[r, "1"] & B \end{tikzcd} \tag{5.1}$$

which is a monomorphism since the pair $(0,1)$ are injective morphisms.

2) It is clear that the pair (Dis, Ab) satisfies TT1 since in a commutative diagram

$$\begin{tikzcd} 0 \arrow[r, "f_1"] \arrow[d] & A \arrow[d] \\ G \arrow[r, "f_0"] & 0 \end{tikzcd}$$

the morphism $f = (f_1, f_0)$ is zero. For TT2', recall that the unit (5.1) is a normal monomorphism in $\mathbb{X}Mod$ if and only if ${}^{b}(a)a^{-1} = 0$, i.e. the action of B over A is trivial. From the Peiffer identity ${}^{\delta(a)}(a') = aa'a^{-1}$, a crossed module with trivial action also has A as an abelian group then we have the short exact sequence in $\mathbb{X}Mod$:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\delta"] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & B \arrow[r] & 0 \arrow[r] & 0\,. \end{tikzcd}$$

□

5.2.4. In $Crs(Grp)$ for each $n \geq 0$ consider the full subcategory $Crs(Grp)_{\geq n}$ of reduced crossed complexes M who are trivial in degrees below n:

$$M = \ \ldots \longrightarrow M_{n+1} \longrightarrow M_n \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots\,;$$

for all $n > 0$ the category $Crs(Grp)_{\geq n}$ is equivalent to the category $ch(Ab)_{\geq n} \cong ch(Ab)$ of chain complexes in abelian groups.

Thus for $n \geq 2$ we have a functor $\mathbf{tr'}_n : Crs(Grp) \to Crs(Grp)_{\geq n}$, defined for a crossed complex M by

$$\mathbf{tr'}_n(M) = \ \ldots \longrightarrow M_{n+1} \longrightarrow M_n \longrightarrow 0 \longrightarrow \ldots\,.$$

The natural chain complex morphism $f : M \to \mathbf{tr'}_n(M)$:

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & M_{n+1} & \longrightarrow & M_n & \longrightarrow & M_{n-1} & \longrightarrow & \dots & \longrightarrow & M_1 & \longrightarrow & M_0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
\dots & \longrightarrow & M_{n+1} & \longrightarrow & M_n & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

is a morphism in $Crs(Grp)$ if and only if all the actions $M_0 \to Aut(M_i)$ are trivial for $i \geq n$ since it must satisfy ${}^{m_0}m_n = f_n({}^{m_0}m_n) = {}^{f_0(m_0)} f_n(m_n) = m_n$ for all $m_0 \in M_0$ and $m_n \in M_n$. In particular, this condition holds if $\delta : M_1 \to M_0$ is a central extension, since in a crossed complex the restrictions of the actions $\delta_1(M_1) \to Aut(M_0)$ are trivial.

Proposition 5.2.5. For $n \geq 2$, consider the triplet of subcategories:

$$(Ker(\mathbf{cot}_{n-1}), Crs(Grp)_{n-1\geq}, Crs(Grp)_{\geq n}) .$$

in $Crs(Grp)$. Then

1. The pair $(Ker(\mathbf{cot}_{n-1}), Crs(Grp)_{n-1\geq})$ is CTF theory, i.e. the subcategory $Crs(Grp)_{n-1\geq}$ is mono-coreflective.
2. The pair $(Crs(Grp)_{n-1\geq}, Crs(Grp)_{\geq n})$ is an $\mathcal{E}$-torsion theory where $\mathcal{E}$ is the class of crossed complex M with all action $M_0 \to Aut(M_i)$ trivial for $i \geq n$.
3. If M if a crossed complex with $\delta_1 : M_1 \to M_0$ a crossed module central extension then M belongs to $\mathcal{E}$.

In particular, for $n = 2$ this holds for the triplet:

$$(Ker(\mathbf{cot}_1), \mathbb{X}Mod, chn(Ab)_{\geq 2}) .$$

Proof. 1) From 5.1.18, $\mu'_{n-1\geq} = (Ker(\mathbf{cot}_{n-1}), Crs(Grp)_{n-1\geq})$ is a torsion theory. It suffices to notice that the counit of $\mathbf{sk}_{n-1} \dashv \mathbf{tr}_{n-1}$ given by

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & M_{n-1} & \longrightarrow & M_{n-2} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & M_n & \longrightarrow & M_{n-1} & \longrightarrow & M_{n-2} & \longrightarrow & \dots
\end{array}$$

is monic since each component is an injective morphism.

2) It is clear that the pair $(Crs(Grp)_{n-1\geq}, Crs(Grp)_{\geq n})$ satisfies TT1 of the definition of a $\mathcal{E}$-torsion theory. Now, let M be a crossed complex with

trivial actions $M_0 \to Aut(M_i)$ and consider the morphisms $\mathbf{tr}_{n-1}(M) \to M \to \mathbf{tr'}_n(M)$ in $Crs(Grp)$:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M_{n-1} & \longrightarrow & M_{n-2} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & M_{n+1} & \longrightarrow & M_n & \longrightarrow & M_{n-1} & \longrightarrow & M_{n-2} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & M_{n+1} & \longrightarrow & M_n & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

it is a short exact sequence in $Crs(Grp)$ since it is a short exact sequence as chain complexes and the forgetful functor is conservative.

3) It follows from the definition of crossed complex that if δ_1 is surjective the actions $\delta_1(M_1) = M_0 \to Aut(M_i)$ are trivial. $\square$

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