2021 60th IEEE Conference on Decision and Control (CDC)
December 13-15, 2021. Austin, Texas

Energy savings alone cannot explain the emergence of birds echelon formations

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Abstract— **We consider a two-dimensional multi-agent echelon formation, where each agent receives a benefit that depends on its position relative to others, and adjusts its position to increase this benefit. We analyze the selfish case where each agent maximizes its own benefit, leading to a Nash-equilibrium problem, and the collaborative case in which agents maximize the global benefit of the group. We provide conditions on the benefit function under which the echelon formations cannot be Nash equilbriums or group optimums.**

We then show that these conditions are satisfied by the conventionally used fixed-wing wake benefit model. This implies that energy saving alone is not sufficient to explain the emergence of the observed migratory formations, based on the fixed-wing model. Hence, either non-aerodynamic aspects or a more accurate model of bird dynamics should be considered to construct such formations.

I. INTRODUCTION

Formation control where multiple agents collaborate to move in certain shapes received extensive interest in the literature, see e.g., the survey [1]. Different methods based on available sensor measurements, e.g., position [2], distance [3] and bearing [4], have been proposed to achieve formations for various agent dynamics. There is also a research line focusing on imitating the collective behavior of animals, e.g., birds flock or fish school, by designing simple local interaction rules [5]-[7]. Despite these fruitful results, the existing research mostly focus on the actions agents take in order to form and maintain specific shapes, or on how phenomenological behavior models may result in formation-like behaviors. But, the benefits of these formations and their influence on formations emergence are rarely addressed.

In nature, it is accepted that migrating birds adopt the eye-catching line formation to save energy: each follower bird reduces energy expenditure by exploiting the extra supportive lift from the wake of the front neighboring bird [9]-[11]. By regarding birds as fixed wings, early researches [11]-[13] have tested the energy saving mechanism. Though the predicted relative position of neighboring birds is consistent with the observations of migrating birds, the position of each bird was always pre-fixed, neglecting birds' incentive to pick the preferred position. Some papers in the last decade [14], [15] also seek to construct line formations based on modified fixed wing models. However, their modification violates the wake evolution in aircraft experiments [16], and non-aerodynamic factors are also considered. Hence the conclusion could be questioned and the actual emergence of the specific formation shapes (echelon or V) remains unexplained on multiple aspects, such as birds interests in energy saving, sensing ability, and action strategies. A first fundamental question is whether migrating formations emerge purely based on energy saving? To answer this, we have tried employing the fixed-wing model to numerically constructing the echelon formation for birds by assuming all of them are either selfish or cooperative in energy optimization (see Section II-B for detailed explanation about these behaviors). Surprisingly, no observation-similar formation has been found in any situation.

Our contribution in this paper is to theoretically confirm this result. We study the two-dimensional (2D) multi-agent echelon formations based on benefit optimization. In our setting, each agent can receive from any other agent a benefit that depends on its relative position to that agent. A leader is fixed at the front of the group, while other followers can adjust their positions and their behaviors are purely guided by benefit optimization. Same as in our trial in constructing migratory formations, we consider that all agents are either selfish or cooperative, resulting in a self-benefit maximization non-cooperative game or a cooperative total benefit optimization problem, respectively. Related to the emergence of echelon formations, our focus is to derive conditions on the inter-agent benefit, under which there cannot exist a Nash equilibrium of the self-benefit game and/or the (local) optimum of the total benefit optimization, at which the relative position of each neighboring-agents lies within some proper set.

This question is close to constrained non-cooperative games and maximization, where the existence of equilibriums or optimums could be guaranteed by requiring objective functions to be continuous or concave [18], [19]. But, unlike these problems, we focus on whether the unconstrained game or maximization happen to have some equilibriums or optimums that are within the desired set. We derive several results by analyzing the necessary condition for the existence of the Nash equilibrium/the maximum. Based on these results, we confirm the numerical observations, that, in the context of fixed-wing models, bird purely trying to maximize their energy savings would not create V-formations.

The rest of the paper is organized as follows: In the next section, we formulated the problem of interest. Section III and IV present conditions on the inter-agent benefit such that the considered Nash equilibrium and optimum, respectively, cannot exist. In Section V, we apply the proposed theoretical conditions to analyzing the fixed-wing wake model and justify our numerical results. At last, Section VI concludes the paper and discuss the implication of the results. Due to

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space reason, the proof of all the formal results are omitted and will be made available in a journal publication.

II. PRELIMINARIES

A. Notations

Let $\mathrm{Id}(\cdot)$ be the identity map. For an interval $\mathcal{P} \subset \mathbb{R}_-$, we denote by $-\mathcal{P}$ and $2\mathcal{P}$ the image of $\mathcal{P}$ by a multiplication by -1 and 2, respectively (i.e. $-\mathcal{P} = -a : a \in \mathcal{P}$). For a differentiable function $f(x) : \mathbb{R}^m \to \mathbb{R}$, we denote by $\frac{\partial f(x^*)}{\partial x_i} = \frac{\partial f(x)}{\partial x_i}|_{x=x^*}$ the partial derivative of $f(x)$ with respect to x_i at $x^* \in \mathbb{R}^m$. Moreover, if $m = 2$, we denote $f_x(a, b) = \frac{\mathrm{d}f(x,b)}{\mathrm{d}x}|_{x=a}$, with $a, b \in \mathbb{R}$.

B. Echelon formation, agents benefits and interests

We consider $n+1$ agents with one leader labeled 0, and $n \geq 2$ followers labeled from 1 to n. Let $V = \{1, ..., n\}$. For each agent $i \in V \cup \{0\}$, we call agent j, with $j = i \mp k \geq 0$ with $1 \leq k \leq n$, the k-hop front (back) neighbor of i. Each agent $i \in V \cup \{0\}$ has a position $p_i = [x_i \ y_i]^\top \in \mathbb{R}^2$ and specifically $p_0 = 0_2$ in this paper. Let $X = [x_1 \ ... \ x_n]^\top$, $Y = [y_1 \ ... \ y_n]^\top$ and $p = [X^\top \ Y^\top]^\top$. In the paper, the x and y directions are also called the longitudinal and lateral directions, respectively. Backward motion means moving at the negative x direction. We denote by $p_{ij} = p_i - p_j$ and $x_{ij} = x_i - x_j$ for $i, j \in V \cup \{0\}$.

Each agent $i \in V \cup \{0\}$ gains a benefit f^i that depends on birds position p. In bird formations [9], the net energy saving of the bird can be approximated by the sum of the saved energy induced from the two front and back birds, and each additive is homogeneous and only depends on the relative position of birds. Thus, we assume that f^i can be decomposed as the sum of the inter-agent benefits agent i gets from each $j \in \mathcal{N}_i$, which is characterized by the same function $f(\cdot) : \mathbb{R}^2 \to \mathbb{R}$. Namely,

$$f^i(p) = \sum_{j \in \mathcal{N}_i} f(p_{ij}) = \sum_{j \in V \cup \{0\}}^{|i-j| \leq 2, \ i \neq j} f(p_{ij}) \tag{1}$$

where $\mathcal{N}_i$ is the set of 1- and 2-hop neighbors of i.

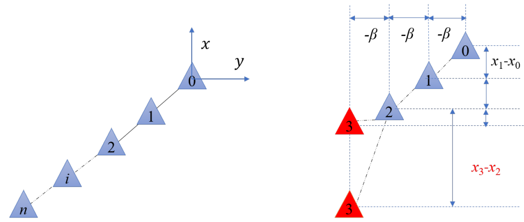
The total benefit $J(p)$ of the group is the sum of all agents' benefits:

$$J(p) = \sum_{i \in V \cup \{0\}} f^i(p) = \sum_{i \in V \cup \{0\}} \sum_{j \in \mathcal{N}_i} f(p_{ij}) \tag{2}$$

We focus on echelon formations as shown in Fig. 1(a), where agents are aligned diagonally behind the leader in one side with equal neighboring-agents distance. Motivated by the line formation of migrating birds where neighboring birds' lateral distances are almost the same but longitudinal distances are varied within proper range [9], [11], we allow the formation to be deviated from the strict echelon shape. Specifically, we focus on the left echelon formation where the position p_i of each follower $i \in V$ satisfies

$$y_i = -i\beta, \quad x_{i(i-1)} \in \mathcal{P} \tag{3}$$

where $\beta > 0$ and $\mathcal{P} = [-\alpha_l, -\alpha_s]$ with $\alpha_s, \alpha_l > 0$ and $\alpha_s \leq \alpha_l$. Let $Y(\beta) = [-\beta \ ... \ -n\beta]^\top$. Then $f^i(p)$ and $J(p)$ are also denoted as $f^i(X, Y(\beta))$ and $J(X, Y(\beta))$, respectively.



(a) (b)

Fig. 1. 2D echelon formation and formations that are weird. The flight is along the x direction. (a) $n+1$ agents create an echelon formation in a plane. (b) Weird formation that should be excluded. Agent 3 in red color is either too close or too far from the front neighbor longitudinally.

Motivated by the almost same lateral distance of neighboring birds in migrating formation [9], [11], we fix β and consider to construct the echelon formation by assuming that followers can adjust their longitudinal position x_i based on benefit maximization. The behavior of agents are assumed to affected by two different attitudes: selfishness and cooperativeness. Accordingly, we considered two scenarios.

In the first, all followers are selfish and would like to maximize their own benefits $f^i(p)$. This leads to a non-cooperative game and we are interested in the Nash equilibrium (NE) of the longitudinal positions, which is defined as the vector $X^* = [x_1^* \ \cdots \ x_n^*]^\top \in \mathbb{R}^n$ satisfying the condition below for each $i \in V$:

$$f^i(x_i^*, x_{-i}^*, Y(\beta)) \geq f^i(x_i, x_{-i}^*, Y(\beta)), \quad \forall x_i \in \mathbb{R} \tag{4}$$

where $x_{-i}^* = [x_1^* \ \cdots \ x_{i-1}^* \ x_{i+1}^* \ \cdots \ x_n^*]^\top$. The NE, if exists, corresponds to agents longitudinal positions with the property that no agent can increases its own benefit by choosing a different position unilaterally.

In the second, all agents cooperate to maximize the group total benefit J and we are interested in the cooperative equilibrium (CE), agents' longitudinal positions $\bar{X}^* = [\bar{x}_1^* \ \cdots \ \bar{x}_n^*]^\top \in \mathbb{R}^n$ that reaches a local maximum of J.

$$\bar{X}^* := \arg \max_{X \in \mathcal{B}_X} J(p) = \arg \max_{X \in \mathcal{B}_X} J(X, Y(\beta)) \tag{5}$$

where $\mathcal{B}_X$ is a neighborhood of X. It is proper to consider the local maximum since without prior knowledge of the global maximum, the cooperative followers have no incentive to shift away from a local maximum. Moreover, the global maximum is also a local maximum, thereby satisfies (5).

We denote by $p_i^* = [x_i^* \ \ -i\beta]^\top$, $\bar{p}_i^* = [\bar{x}_i^* \ \ -i\beta]^\top$, $p^* = [(X^*)^\top \ \ Y(\beta)^\top]^\top$, and $\bar{p}^* = [(\bar{X}^*)^\top \ \ Y(\beta)^\top]^\top$. In birds echelon formations, the relative longitudinal positions of neighboring birds are within a range. Thus, a stable echelon formation should correspond to an NE X^* and/or a CE $\bar{X}^*$, with $x_{i(i-1)}^* \in \mathcal{P}$ and $\bar{x}_{i(i-1)}^* \in \mathcal{P}, i \in V$, respectively, for a proper negative interval $\mathcal{P}$ that is consistent with observations. Hence, we only focus on such equilibriums. Whether a considered echelon formation can be constructed based on benefit maximization should relate to if there exist the equilibriums of interest.

C. Problem formulation

In view of $f^i(p)$ and $J(p)$, the existence of X^* and $\bar{X}^*$ should depend on the properties of the inter-agent benefit f. In our efforts to reconstruct the migratory formation of birds based purely on benefit maximization, no equilibrium that corresponds to observation-similar echelon formations has been found. To explain this, we focus on the following problem in the paper.

Problem 1. Given $n \geq 2$ agents, an interval $\mathcal{P} = [-\alpha_l, -\alpha_s]$ with $0 < \alpha_s \leq \alpha_l$ and a $\beta > 0$, under what conditions on f, the NE $X^* \in \mathbb{R}^n$ with $x^*_{i(i-1)} \in \mathcal{P}, i \in \mathcal{V}$ and/or the CE $\bar{X}^* \in \mathbb{R}^n$ with $\bar{x}^*_{i(i-1)} \in \mathcal{P}, i \in V$ are impossible.

At this stage, we make the following assumption on the inter-agent benefit f, allowing to consider its derivative.

Assumption 1: (a) $f(x, y)$ with $x, y \in \mathbb{R}$ is continuous in $\mathbb{R}^2$ and is continuously differentiable when $x \neq 0$ (b) $f(x, y) = f(x, -y)$ for $x, y \in \mathbb{R}$.

The derivative of f is not assumed to be continuous at $x = 0$ to allow for the potential sharp change of the inter-agent benefit when an agent shifts longitudinally from the back of another agent to the front.

Note that $f(p_{ij})$ takes $p_{ij} = [x_{ij}\ y_{ij}]^\top$, then by the chain rule, if a NE X^* with $x^*_{i(i-1)} \in \mathcal{P}, i \in V$ exists, it should satisfy the equation below for each $i \in V$

$$0 = \frac{\partial f^i(p^*)}{\partial x_i} = \sum_{j \in \mathcal{N}_i} \frac{\partial f(p_{ij})}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial x_i}\bigg|_{p^*} = \sum_{j \in \mathcal{N}_i} \frac{\partial f(p^*_{ij})}{\partial x_{ij}} \quad (6)$$

By contrast, if there exists a CE $\bar{X}^*$ with $\bar{x}^*_{i(i-1)} \in \mathcal{P}, i \in V$, it should satisfy the following equation for each $i \in V$

$$\begin{aligned} 0 = \frac{\partial J(\bar{p}^*)}{\partial x_i} &= \sum_{k \in V \cup \{0\}} \sum_{j \in \mathcal{N}_k} \frac{\partial f(p_{kj})}{\partial x_{kj}} \frac{\partial x_{kj}}{\partial x_i}\bigg|_{\bar{p}^*} \\ &= \sum_{j \in \mathcal{N}_i} \frac{\partial f(\bar{p}^*_{ij})}{\partial x_{ij}} - \frac{\partial f(\bar{p}^*_{ji})}{\partial x_{ji}} \end{aligned} \quad (7)$$

where the last equality comes from (1) and (2).

Hence, checking if the equilibriums of interest exist is equivalent to testing if there exist a solution X^* with $x^*_{i(i-1)} \in \mathcal{P}, i \in V$ to equation (6) and/or a solution $\bar{X}^*$ with $\bar{x}^*_{i(i-1)} \in \mathcal{P}, i \in V$ to (7), respectively.

III. Nonexistence of the NE of Interest

In this section, we focus on the selfish situation and discuss the conditions on f such that there exists no NE of interest.

A. Three agents formation

We first consider the simple case of three agents: 1 leader followed by the 1st follower and the 2nd follower. Intuitively, if the increment of agent 1's benefit from its back neighbor (agent 2) is more than its benefit loss from the front neighbor (agent 0) when agent 1 moves backward, then agent 1 would like to move backward to get more benefit. If this always holds when $x_{21}, x_{10} \in \mathcal{P}$, then agent 1 cannot be static. In other words, the NE of interest cannot exist. The theorem below formulates this intuition.

Theorem 1: For $n = 2$, $\beta > 0$, a closed interval $\mathcal{P} \subset \mathbb{R}_-$, if $f(\cdot)$ satisfies Assumption 1 and

$$\max_{x \in -\mathcal{P}} f_x(x, -\beta) < -\max_{x \in \mathcal{P}} f_x(x, -\beta) \quad (8)$$

then there exists no NE $X^* \in \mathbb{R}^2$ with $x^*_{10}, x^*_{21} \in \mathcal{P}$.

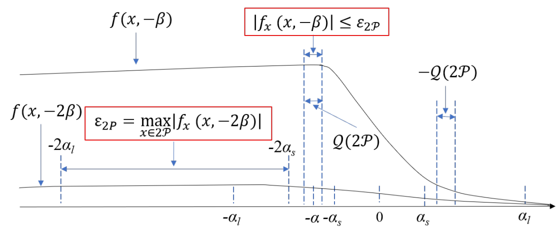


Fig. 2. Illustration of f, $\varepsilon_{2\mathcal{P}}$ and $\mathcal{Q}(2\mathcal{P})$, where $f(x, -2\beta)$ is flat in $2\mathcal{P}$.

Theorem 1 is based on the analysis of agent 1. If we consider the benefit change of both agents 1 and 2, a different result can be obtained. Consider just agent 1 and 2, namely, $f^2(p) = f(p_{21})$. If $f(x, -\beta)$ peaks at $x = -\alpha$, then agent 2 should be α behind agent 1. Now take the leader 0 into account, $f^2(p) = f(p_{21}) + f(p_{20})$. If $f(p_{20})$ changes very little when agent 2 moves along the longitudinal direction, then the best x_{21} that maximizes $f^2(p)$ would deviate very little from $-\alpha$. Hence when agent 1 moves longitudinally, if agent 2 wants to maximize the benefit, it should also move such that x_{21} is within a very narrow interval $\mathcal{Q} \ni -\alpha$. Suppose this is true and $x_{12} \in -\mathcal{Q}$ always holds. Now assume that the increment of agent 1's benefit from agent 2 is more than its benefit loss from agent 0 when agent 1 moves backward but $x_{10} \in \mathcal{P}$ still holds, then agent 1 will want to move backward, until eventually $x_{10} \notin -\mathcal{P}$. This implies that there exists no NE X^* with $x_{i(i-1)} \in \mathcal{P}, i \in \mathcal{V}$. This analysis can be formulated as another result, whose rigorous presentation relies on an assumption and several notations in the following.

Assumption 2: (a) $f(x, -\beta)$ has a global maximum $-\alpha$, and is strictly increasing when $x < -\alpha$ and strictly decreasing when $x > -\alpha$. (b) $-\alpha \in \mathcal{P}$.

Assumption 2(a) can be regarded as the attribute of the benefit f. It is very mild and satisfied by at least the benefit considered in Section V. Assumption 2(b) relates to the choice of the interval $\mathcal{P}$. It is reasonable since otherwise a trivial conclusion could be obtained for the case of two agents that the follower 1 would never stay statically behind the leader 0, with $x_{10} \in \mathcal{P}$.

We then characterize the narrow interval around $-\alpha$ mentioned in the intuitive analysis before Assumption 2. For any non-empty closed interval $\mathcal{I}$, we denote

$$\varepsilon_{\mathcal{I}} := \max_{x \in \mathcal{I}} |f_x(x, -2\beta)| \quad (9)$$

$$\mathcal{Q}(\mathcal{I}) = \{x \in \mathcal{P} |\, |f_x(x, -\beta)| \leq \varepsilon_{\mathcal{I}}\} \quad (10)$$

Then, if $\mathcal{I} = 2\mathcal{P}$, $\mathcal{Q}(\mathcal{I})$ relates to the narrow interval around $-\alpha$, though it may cover that. See Fig. 2 to get some vision of $\mathcal{Q}(2\mathcal{P})$. Formalizing the intuition, we have,

Theorem 2: For $n = 2$, $\beta > 0$ and $\mathcal{P} = [-\alpha_l, -\alpha_s]$ with $0 < \alpha_s \leq \alpha_l$, assume f satisfies Assumption 1 and 2. If

$$\max_{x \in -\mathcal{Q}(\mathcal{I})} f_x(x, -\beta) < -\max_{x \in \mathcal{P}} f_x(x, -\beta) \quad (11)$$

with $\mathcal{I} = 2\mathcal{P}$, then there exists no NE $X^* \in \mathbb{R}^2$ with $x^*_{10}, x^*_{21} \in \mathcal{P}$.

In Theorem 2, condition (11) should be satisfied for $\mathcal{I} = 2\mathcal{P}$. Based on the analysis before Assumption 2, it may implicitly require the variation of $f(x, -2\beta)$ for x in entire $2\mathcal{P}$ to be small. For those inter-agent benefits f that do not satisfy this condition, we can have another result if f additionally satisfies Assumption 3 as follows.

Assumption 3: $f(x, -2\beta)$ is strictly decreasing for $x \geq -2\alpha$.

Theorem 3: For $n = 2$, $\beta > 0$ and $\mathcal{P} = [-\alpha_l, -\alpha_s]$ with $0 < \alpha_s \leq \alpha_l$, assume that $f(\cdot)$ satisfies Assumption 1, 2 and 3, If (11) holds with $\mathcal{I} = [-2\alpha_l, -2\alpha]$, then there exists no NE $X^* \in \mathbb{R}^2$ with $x^*_{10}, x^*_{21} \in \mathcal{P}$.

Remark 1: Assumptions 1, 2 and 3 could be weakened. First, since $\mathcal{P}$ is finite, the conditions on f in these assumptions could be imposed just for sets that cover all the intervals concerned. For instance, in Assumption 2 (a), requiring $f_x(x, -\beta)$ to monotonically increase in $[-\alpha_l, -\alpha)$ and decrease in $(-\alpha, -\alpha_s] \cup -\mathcal{P}$ is sufficient to draw the conclusion of Theorem 2 and Theorem 3. Second, the conclusion of Theorem 2 would still hold if Assumption 2(b) is discarded. In that situation, $\mathcal{Q}(2\mathcal{P})$ may be empty. But this is not a problem since it implies a trivial case.

There is no strict advantage of using one theorem over others. On the one hand, by (9) and (10), $\mathcal{Q}(\mathcal{I}) \subseteq \mathcal{Q}(2\mathcal{P}) \subseteq \mathcal{P}$ with $\mathcal{I} = [-2\alpha_l, -2\alpha]$. Hence, condition (11) in Theorem 3 is easier to satisfy than that in Theorem 2, and than condition (8) in Theorem 1. On the other hand, Theorems 2 and 3 require more knowledge and assumptions on f_x than Theorem 1, which may not be satisfied by the benefit f. In addition, unless $\varepsilon_{\mathcal{I}}$ with $\mathcal{I} = 2\mathcal{P}$ or $\mathcal{I} = [-2\alpha_l, -2\alpha]$ is much small, $\mathcal{Q}(\mathcal{I})$ may not be a small subset of $\mathcal{P}$ such that $\max_{x \in -\mathcal{Q}(\mathcal{I})} f_x(x, -\beta)$ is much less than $\max_{x \in -\mathcal{P}} f_x(x, -\beta)$. In that case, Theorems 2 and Theorem 3 may not be more useful than Theorem 1.

B. General case

The following results extend Theorems 1, 2 and 3 to the multiple agents case.

Proposition 1: For $n \geq 3$, a closed interval $\mathcal{P} \subset \mathbb{R}_-$ and a $\beta > 0$, assume f satisfies Assumption 1. If $\max_{x \in -\mathcal{P}} f_x(x, -\beta) < -\max_{x \in \mathcal{P}} f_x(x, -\beta) - \varepsilon_{2\mathcal{P}}$, then there exists no NE $X^* \in \mathbb{R}^n$ with $x^*_{i(i-1)} \in \mathcal{P}$ for each $i \in V$.

Proposition 2: For $n \geq 3$, $\mathcal{P} = [-\alpha_l, -\alpha_s]$ with $0 < \alpha_s \leq \alpha_l$ and $\beta > 0$, assume that f satisfies Assumption 1 and 2. If $\max_{x \in -\mathcal{Q}(2\mathcal{P})} f_x(x, -\beta) < -\max_{x \in \mathcal{P}} f_x(x, -\beta) - \varepsilon_{2\mathcal{P}}$, then there exists no NE $X^* \in \mathbb{R}^n$ with $x^*_{i(i-1)} \in \mathcal{P}$ for each $i \in V$.

Proposition 3: For $n \geq 3$, $\mathcal{P} = [-\alpha_l, -\alpha_s]$ with $0 < \alpha_s \leq \alpha_l$ and $\beta > 0$, assume that f satisfies Assumption 1, 2 and 3. If $\max_{x \in -\mathcal{Q}(\mathcal{I})} f_x(x, -\beta) < -\max_{x \in \mathcal{P}} f_x(x, -\beta)$ with $\mathcal{I} = [-2\alpha_l, -2\alpha]$, then there exists no NE $X^* \in \mathbb{R}^n$ with $x^*_{i(i-1)} \in \mathcal{P}$ for each $i \in V$.

IV. Nonexistence of the CE of Interest

This section shows a simple condition on the benefit function f under which the CE of interest cannot exist. It is based on the intuition that if the sum of the benefit of any two agents $f(p_{ij}) + f(p_{ji})$, decreases as the longitudinal distance $|x_{ij}|$ between them increases, then agents being cohesive longitudinally will increase the total benefit $J(p)$. In particular, if $x_0 = \cdots = x_n$, then the total benefit J attains the maximum. However, in this situation $x_{i(i-1)} \notin \mathcal{P}$ for any negative interval $\mathcal{P}$. Hence, even all agents stop with the same x_i, it is not a CE of interest. Analogously, if $f(p_{ij}) + f(p_{ji})$ always increases as $|x_{ij}|$ increases, then there would not exist the CE of interest. Based on these intuitions, the following result can be obtained.

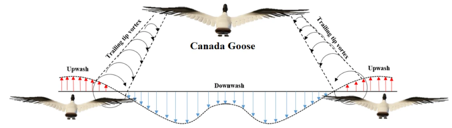


Fig. 3. The movement of airflow around a bird.

Theorem 4: For $n \geq 2$, $\mathcal{P} = [-\alpha_l, -\alpha_s]$ with $0 < \alpha_s \leq \alpha_l$ and $\beta > 0$, assume that f satisfies Assumption 1(a). If there exists a positive $\underline{\beta} \leq \beta$ such that

$$|f_x(x, y) + f_x(-x, -y))| > 0 \qquad (12)$$
$$\forall x \in (0, \alpha_l], \ \forall |y| \in [\underline{\beta}, +\infty)$$

then there exists no CE $\bar{X}^* \in \mathbb{R}^n$ with $\bar{x}^*_{i(i-1)} \in \mathcal{P}$ for each $i \in \mathcal{V}$.

Remark 2: Since condition (12) is proposed for every y with $|y| \in [\underline{\beta}, +\infty)$, it could also be used to show the non-existence of the cooperative equilibrium of interest for the situation where agents adjust relative position in both directions within proper intervals.

Remark 3: A simple class of benefit functions that satisfy condition (12) is $f(p) = g(x)h(y)$, where $h(y)$ is positive and differentiable with continuous derivative for all $y \in \mathbb{R}$, and $g(x)$ with $x \in \mathbb{R}$ is differentiable with continuous derivative, symmetric about the origin and strictly increasing or decreasing as $|x|$ increases, e.g., $|x|$, x^2, $\frac{1}{x^2}$ and the standard Gaussian function.

V. An Application to Line Migratory Formation

In this section, we apply the results presented above to analyzing the emergence of the line formation of migrating birds. In most of existing researches, each bird is approximated by a fixed wing, whose forward motion stirs the air around upward and downward. If a bird positions properly relative to another bird, it can get extra lift from the upward airflow generated by that bird (see Fig. 3) and reduce the drag. This can be regarded as the wake benefit from one bird to another. The movement of the stirred air is usually depicted by the horseshoe vortex model [11]. We only present the derivation of the wake benefit here. Readers can refer to [11] for more details on the model.

Assume that two birds $i = 0, 1$, with the same weight W and wingspan $2b$ fly together along the x direction with constant speed U in the plane. If bird 0 is at the origin $[0 \ 0]^\top$, the upward airflow velocity $v(x, y)$ at $[x \ y]^\top \in \mathbb{R}^2$ generated by bird 0 can be given as

$$v(x, y) = v_b(x, y) + v_t(x, y) \qquad (13)$$

$$v_b = \frac{\Gamma}{4\pi} \frac{x}{x^2 + r_0^2} \left[\frac{y + a}{\sqrt{(y + a)^2 + x^2 + r_0^2}} - \frac{y - a}{\sqrt{(y - a)^2 + x^2 + r_0^2}} \right]$$

$$v_t = \frac{\Gamma}{4\pi}\frac{y-a}{(y-a)^2+R(x)}\left[1-\frac{x}{\sqrt{(y-a)^2+x^2+R(x)}}\right]$$
$$-\frac{\Gamma}{4\pi}\frac{y+a}{(y+a)^2+R(x)}\left[1-\frac{x}{\sqrt{(y+a)^2+x^2+R(x)}}\right]$$

where $a=\frac{\pi}{4}b$, ρ is the air density, $\Gamma = W/(2\rho a U)$, $R(x) = r_0^2 + D_f|x|/U$ with $r_0 = 0.04b$ and D_f is a diffusion term to model wake dissipation when $|x| \to \infty$ [16], [14]. We select $D_f = 1.05\times 10^{-4}Ub$ such that $\sqrt{R(x)}$ increases from $0.04b$ to $0.1b$ when $|x|$ grows from 0 to $80b$. This is fairly realistic for fixed-wing wake [20]. The model is valid for a sufficiently long longitudinal distance that covers the range of distances of neighboring birds in migratory formation. Beyond that range, it is not accurate due to wake instability.

Consider bird 1 locating at $[x\ y]^\top$. After neglecting the momentum induced by the vertical airflow as in [11], the wake benefit of bird 1 received from bird 0 can be given as

$$f(x,y) = \frac{1}{2b}\int_{y-b}^{y+b} v(x,\eta)\mathrm{d}\eta \tag{14}$$

This function satisfies Assumption 1 except at the y axis. Computation shows that $f(x,y)$ has a maximum $(-\alpha,-\beta)$ in the negative orthant, with $\alpha \approx 3.468b$ and $\beta \approx (1+\frac{\pi}{4})b = a+b$, Moreover, it peaks around the line $y=-\beta$ in the negative orthant, which is argued to be the best relative lateral position of a follower to its front neighbor [9]. Hence, we fix β as this value. Fig. 4(a) shows $f(x,y)$ for $y=-\beta,-2\beta$ when $2b=1.5$. Normally, neighboring birds' longitudinal distance in migratory formation ranges within $[0.5,4]$ wingspans, implying $x_{i(i-1)} \in [-8b,-1b]$.

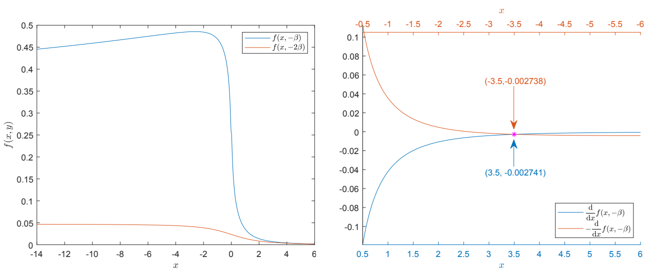
As mentioned before, we have been working on reconstructing the line formation of migrating birds based on the assumption that birds behavior are purely guided by wake benefit maximization, taking into account birds attitudes. We have not numerically found the NE and CE with $x^*_{i(i-1)} \in \mathcal{P}$ and $\bar{x}^*_{i(i-1)} \in \mathcal{P}$, respectively, for a much wide interval $\mathcal{P}$, e.g., $[-20b,-b]$ [17]. In the following, we confirm this numerical result.

A. Absence of the NE of interest

We first look at the selfish agents case. An example is presented for Canadian geese, for which averagely $W = 36.75$ N, $2b = 1.5$m, $U = 18$ m/s, and $\rho \approx 1.112$ kg/m^3. Recall condition (8) and (11), in the following, we always denote $\delta_1 = \max_{x\in\mathcal{P}} f_x(x,-\beta)$, $\delta_2 = \max_{x\in-\mathcal{P}} f_x(x,-\beta)$ and $\delta_3 = \max_{x\in-\mathcal{Q}(\mathcal{I})} f_x(x,-\beta)$ for the corresponding interval $\mathcal{P}$ and $\mathcal{Q}(\mathcal{I})$ accordingly.

By Fig. 4(b), we can find that if we select $\mathcal{P} = [-\alpha_l,-\alpha_s] = [-3.5,-0.5]$, $\delta_2 \le -\delta_1$. Hence condition (8) in Theorem 1 holds, and there should exist no NE X^* with $x^*_{i(i-1)} \in \mathcal{P}, i \in V$. Note that α_l cannot be increased more as the condition $\delta_2 < -\delta_1$ would not be satisfied.

On the other hand, $f(x,-\beta)$ peaks at $-\alpha = -3.468b = -2.601$ and we can see from Fig. 5(a) and 6(a) that Assumption 2 is satisfied. Let $\mathcal{P} = [-\alpha_l,-\alpha_s] = [-7,-2.5] \ni -\alpha$, then $\mathcal{I} = 2\mathcal{P} = [-14,-5]$. The value of $\varepsilon_{\mathcal{I}}$ can be found in Fig. 6(a). Moreover, the set $\mathcal{Q}(\mathcal{I}) = [-\alpha'_l,-\alpha'_s]$ satisfying



(a) (b)

Fig. 4. (a) $f(p)$ for $y=-\beta,-2\beta$ when $2b=1.5$ m. (b) The derivative of $f(x,-\beta)$. The point mark and x-axis with the same color corresponds to the derivative curve in that color. $\delta_2 = -0.002741$ and $-\delta_1 = -0.002738$.

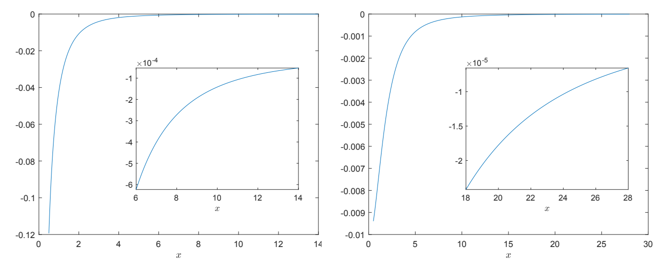
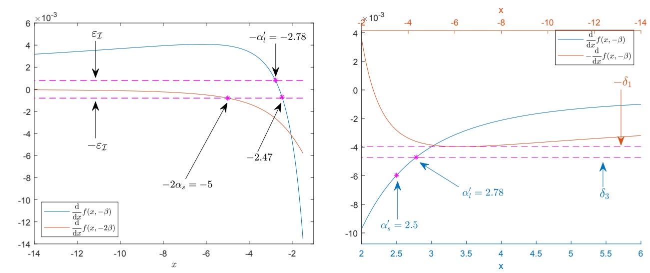
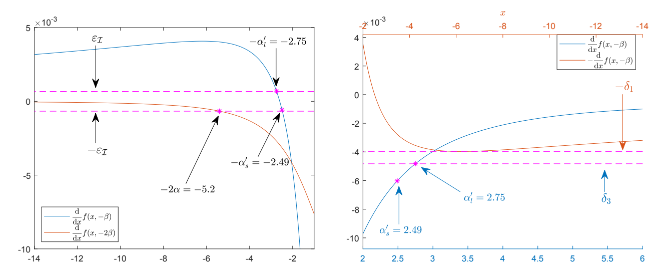

(a) $f_x(x,-\beta)$. (b) $f_x(x,-2\beta)$.

Fig. 5. $f_x(x,y)$ for positive x.



(a) (b)

Fig. 6. $f_x(x,y)$ for $y=-\beta$ and -2β. The value $\varepsilon_{\mathcal{I}} = \max_{x\in\mathcal{I}} f_x(x,-2\beta)$ with $\mathcal{I} = 2\mathcal{P} = [-14,-5]$ is represented as the dashed magenta lines. The two ends of $\mathcal{Q}(\mathcal{I})$ are $-\alpha'_l$ and $-\alpha'_s$.



(a) (b)

Fig. 7. $f_x(x,y)$ for $y=-\beta$ and -2β.

(10) is a neighborhood of $-\alpha$ and given as $[-2.78,-2.5]$[1]. Then by Fig. 6(b), $\delta_3 \le -\delta_1$ or condition (11) holds. Hence, by Theorem 2, there exists no NE X^* with $x^*_{i(i-1)} \in \mathcal{P}, i \in V$ for $\mathcal{P} = [-7,-2.5]$. The value of α_s cannot be much smaller than 2.5, since from Fig. 6(a), it would imply a wider $\mathcal{I}$, larger $\varepsilon_{\mathcal{I}}$, wider interval $\mathcal{Q}(\mathcal{I})$, and δ_3 that would be larger than $-\delta_1$, making the condition in Theorem 2 unsatisfied.

Note as mentioned before this subsection, no NE X^* of

[1]Though the lower magenta line in Fig. 6(a) intersects $f_x(x,-\beta)$ at -2.47, $\mathcal{Q} = [-\alpha'_l,-\alpha'_s]$ should be the sub-set of $\mathcal{P}$.

interest has been found for a wide $\mathcal{P}$, e.g., $[-20b, -b] = [-15, -0.75]$. Hence, this cannot be explained by Theorem 1 and 2. We then turn to Theorem 3. By Fig. 5(b) and 7(a), Assumption 3 is indeed satisfied for $x \geq -2\alpha$ (The curve of $f_x(x, -2\beta) < 0$ for $x \leq -14$ is not shown for a clear vision of $\varepsilon_{\mathcal{I}}$). Let $\mathcal{P} = [-14, -0.5] \ni -\alpha$, then $\mathcal{I} = [-28, -1]$. In Fig. 7(a), we can find $\mathcal{Q}(\mathcal{I}) = [-\alpha_l', -\alpha_s']$ that satisfies (10). Then by Fig. 7(b), $\delta_3 < -\delta_1$. Hence conditions (11) in Theorem 3 holds for $\mathcal{P} = [-14, -0.5]$, from which, we should have that there exists no NE X^* of interest for this interval. This indeed explains our numerical search of the NE of interest.

B. Absence the CE of interest

We then show that there exists no CE $\bar{X}^*$ of interest for a negative closed interval $\mathcal{P}$ with the wake benefit function (14). We can show that this function satisfies condition (12) with $\alpha_l \leq \frac{U}{D_f}(2ab - r_0^2), \underline{\beta} \in (\sqrt{a^2 + b^2}, \beta)$. Accordingly, we have,

Proposition 4: Consider f in (14) and $n \geq 2$, then there exists no CE $\bar{X}^* \in \mathbb{R}^n$ such that $\bar{x}^*_{i(i-1)} \in \mathcal{P}$ for any $\mathcal{P} = [-\alpha_l, -\alpha_s]$ with $0 < \alpha_s \leq \alpha_l \leq \frac{U}{D_f}(2ab - r_0^2)$.

Putting $D_f = 1.05 \times 10^{-4} Ub$, $a = \frac{\pi}{4} b$ and $r_0 = 0.04b$ into $\frac{U}{D_f}(2ab - r_0^2)$, we have that α_l is larger than seven thousands wingspans. Based on the wake model (14), the proposition predicts that no $\bar{X}^*$ of interest for a interval $\mathcal{P} \subset [-7000, 0)$ wingspans exists. Hence the echelon formation where neighboring birds have lateral distances of $(\frac{1}{2} + \frac{\pi}{8})$ wingspan and longitudinal distances less than 7000 wingspans cannot emerge, when birds cooperate to maximize the total wake benefit. It should be noticed that echelon formation with the longitudinal distance of neighboring birds larger than seven thousands wingspans is impractical, as birds never fly so far from each other. Furthermore, the fixed wing wake model might be invalid for such large longitudinal distance.

VI. Conclusion

In this paper, we focus the 2D echelon formation of multi-agents that behave to maximize relative-position dependent benefits. All agents can be either selfish to maximize its own benefit from others or cooperative to optimize the total benefit of the group. We discuss the conditions on the inter-agent benefit such that echelon formation cannot appear, regardless of agents attitudes. The theoretical conditions are employed to analyze the fixed-wing model that is usually used to study line formations of migrating birds, and justify our failure in numerically reconstructing migratory formations. This shows that this kind formation may not emerge if birds behavior is purely guided by energy savings.

Our results imply multiple possibilities for the emergence reason of the migratory formations. First, remember that we employ the fixed-wings to model birds and ignore the slow undulatory motion of birds wings, conventionally as in [9], [10], a natural hypothesis is that the wing-flapping of birds plays more important roles than expected. Nevertheless, fixed-wings are proper to represent the glide of birds in formation flight. Hence, a second hypothesis from our result is that non-aerodynamic factors, such as collision avoidance and vision enhancement [9] could also take parts in developing the migratory formation. Moreover, from the perspective of multi-agent control systems, more complex dynamics, the actual sensing and information processing ability of the bird, and the communication capacity among birds (for cooperative birds) may need to be considered to see if the current result would still hold. Finally, it is also interesting to theoretically characterize the condition on the benefit function f such that equilibriums corresponding to echelon formation exist.

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