MARKOV MOVES, L^2-BURAU MAPS AND LEHMER'S CONSTANTS

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ABSTRACT. We study the effect of Markov moves on L^2-Burau maps of braids, in order to construct link invariants from these maps with a process mirroring the well-known Alexander-Burau formula.

We prove such a Markov invariance for the L^2-Burau maps which descend to the groups of the braid closures or lower, and for these maps we establish that the associated link invariants are twisted L^2-Alexander torsions. This last point generalizes a previous result of A. Conway and the author.

Furthermore, we find two counter-examples to Markov invariance, meaning two families of L^2-Burau maps that cannot yield link invariants with the process described in our paper. The proofs use relations between Fuglede-Kadison determinants, Mahler measures, and random walks on Cayley graphs, as well as works of Boyd, Bartholdi and Dasbach-Lalin.

Along the way, we compute new values for Fuglede-Kadison determinants over non-cyclic free groups. As a consequence, we partially answer a question of Lück, as we provide new upper bounds for Lehmer's constants for all torsionfree groups which have non-cyclic free subgroups.

Our results suggest that twisted L^2-Alexander torsions are the only link invariants we can hope to build from L^2-Burau maps with the present approach.

CONTENTS

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1. Introduction

The L^2-Burau maps and reduced L^2-Burau maps of braids were introduced in 2016 in [5] by A. Conway and the author. These maps generalise the Burau representation of braid groups (introduced in [7]) and are indexed by a positive real number t and a group epimorphism γ starting at a free group $\mathbb{F}_n$ of finite rank. In a sense, all the L^2-Burau maps are contained between the Burau representation and the Artin injection of the braid group B_n in the automorpishm group $\mathrm{Aut}(\mathbb{F}_n)$ of the free group $\mathbb{F}_n$. Moreover, A. Conway and the author proved in [5] that for a braid $\beta \in B_n$, its image by the reduced L^2-Burau map $\overline{\mathcal{B}}^{(2)}_{t,\gamma_\beta}$ associated to the epimorphism $\gamma_\beta \colon \mathbb{F}_n \to G_\beta \cong \pi_1\left(S^3 \setminus \hat{\beta}\right)$ yields the *L^2-Alexander torsion* of the braid closure $\hat{\beta}$. The L^2-Alexander torsion is an invariant of links introduced by Li-Zhang and Dubois-Friedl-Lück [12, 9], whose construction can be compared with those of the twisted Alexander polynomials, and which detects various topological information. As a consequence of the main result of [5], the reduced L^2-Burau map $\overline{\mathcal{B}}^{(2)}_{t,\gamma_\beta}$ thus contains deep topological information about β such as the hyperbolic volume or the genus of the link $\hat{\beta}$.

It is then natural to wonder whether L^2-Burau maps associated to other epimorphisms can similarly provide link invariants and detect topological information of the braid, and this article provides a partial positive answer to this question.

We first introduce the notion of *Markov-admissibility* of a family $\mathcal{Q}$ of group epimorphisms $Q_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow G_{Q_\beta}$ indexed by braids $\beta \in \sqcup_{n\geqslant 1} B_n$, in Section 4. Roughly speaking, we say that such a family is Markov-admissible when for any two braids α, β that have isotopic closures (and thus are related by Markov moves), the associated epimorphisms Q_α and Q_β descend "to the same depth" and are related by a sequence of commutative diagrams. Markov admissibility appears to be a necessary condition in order to construct Markov functions and link invariants from general families of epimorphisms indexed by braids.

In this paper we focus on a specific candidate for being a Markov function, namely the function

$$F_{\mathcal{Q}} := \left(\begin{array}{ccc} \sqcup_{n\geqslant 1} B_n & \to & \mathcal{F}(\mathbb{R}_{>0}, \mathbb{R}_{>0})/\{t \mapsto t^m, m \in \mathbb{Z}\} \\ \beta & \mapsto & \left[t \mapsto \dfrac{\det^r_{G_{Q_\beta}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right)}{\max(1,t)^n} \right] \end{array} \right),$$

where $\mathcal{Q}$ is a Markov-admissible family of epimorphisms and $\det^r_G$ is the *regular Fuglede-Kadison determinant* for the group G, a version of the determinant for infinite-dimensional G-equivariant operators on $\ell^2(G)$ such as the L^2-Burau maps (see Section 2 for a definition).

The first main result of this article is the following theorem (stated here without technical details for readability):

Theorem 1.1 (Theorem 5.1)**.** *Let $\mathcal{Q}$ be a Markov-admissible family of epimorphisms that descends to the groups of the braid closures or deeper. Then $F_{\mathcal{Q}}$ is a Markov function, and thus defines an invariant of links. Moreover, this link invariant is a twisted L^2-Alexander torsion.*

As detailed in Section 5, to prove Theorem 1.1 we study how Markov moves on braids modify reduced L^2-Burau maps and we use properties of the Fuglede-Kadison determinant.

Part of Theorem 1.1 was already proven in [5], without discussing Markov invariance. Indeed, when $\mathcal{Q}$ is the family that descends to the groups of the braid closures G_β, [5, Theorem 4.9] directly linked $F_{\mathcal{Q}}$ to the L^2-Alexander torsions of the braid closures, as a variant of the well-known Alexander-Burau formula [7]; and since the L^2-Alexander torsions were already known link invariants, Markov invariance was thus reciprocally guaranteed, and carefully studying Markov moves was unnecessary.

However, as stated in Theorem 1.1, the methods of the current article provide several new link invariants, which (unsurprisingly) happen to be *twisted* L^2-Alexander torsions of links.

Theorem 1.1 is not an equivalence in the sense that $F_\mathcal{Q}$ could theoretically be a Markov function for other families $\mathcal{Q}$, but the specific convenient cancellations that occur in matrix coefficients in the proof of Theorem 1.1 make this seem unlikely. As further evidence, we establish two families $\mathcal{Q}$ such that $F_\mathcal{Q}$ is not a Markov function, in the second main result of this article:

Theorem 1.2 (Theorems 6.1 and 6.2)**.** *Let $\mathcal{Q}$ be either the family of identities of the free groups or the family of abelianizations of the free groups. Then $F_\mathcal{Q}$ is not a Markov function.*

Our main tools to prove Theorem 1.2 are relations between Fuglede-Kadison determinants (which are technical to define and difficult to compute), Mahler measures of polynomials (notably studied by Boyd [6]) and combinatorics on Cayley graphs of free groups (specifically works of Bartholdi and Dasbach-Lalin [2, 8]). Connections between Fuglede-Kadison determinants and random walks on Cayley graphs were surveyed and studied in [11]. On the way to the proof of Theorem 1.2, we compute new values for Fuglede-Kadison determinants of operators over the free groups, which are interesting in their own right:

Theorem 1.3 (Theorem 7.3)**.** *Let $d \geqslant 3$. Let $x_1, \ldots, x_{d-1}$ be $d-1$ generators of the free group $\mathbb{F}_{d-1}$. Let $\zeta_1, \ldots, \zeta_{d-1} \in \mathbb{C}$ such that $|\zeta_1| = \ldots = |\zeta_{d-1}| = 1$. Then:*

$$\det{}_{\mathbb{F}_{d-1}}(\mathrm{Id} + \zeta_1 R_{x_1} + \ldots + \zeta_{d-1} R_{x_{d-1}}) = \frac{(d-1)^{\frac{d-1}{2}}}{d^{\frac{d-2}{2}}}.$$

Theorem 1.3 has a consequence which is independent of the rest of the themes of this paper, in that it yields new upper bounds for Lehmer's constants $\Lambda_1^w(\mathbb{F}_d)$ of free groups (see [14] for a survey of Lehmer's constants and Corollary 8.5 for more details). Hence the non-cyclic free groups $\mathbb{F}_d$ (as well as any torsionfree group having non-cyclic free subgroups, such as groups of hyperbolic 3-manifolds) are now the first torsionfree groups G for which Lehmer's constant $\Lambda_1^w(G)$ is known to be lesser than 1.176. This was known to be satisfied for the smaller Lehmer's constant $\Lambda^w(G)$, for G the fundamental group of the Weeks manifold (see Example 8.4).

As the reader will probably agree, the initial question (how to build link invariants from L^2-Burau maps) is still far from answered. We restricted ourselves to studying the most intuitive form of a potential Markov function, namely $F_\mathcal{Q}$, and we found that twisted L^2-Alexander torsions appeared to be the best link invariants we could obtain with it. This last point is unsurprising considering the form of $F_\mathcal{Q}$ and its natural connection with the Alexander polynomial and its variations.

However, there may well be new link invariants to discover via other functions of the L^2-Burau maps, and we hope that our computations of how these maps change under Markov moves can be of use for future research in this vein.

The present article arose as a part of a wider project in collaboration with C. Anghel aiming to construct new knot invariants from L^2-versions of the Burau and Lawrence representations of braid groups. To attain this end, studying the influence of Markov moves on L^2-Burau maps appears to be a natural step.

The article is organised as follows: in Section 2 we recall preliminaries on braid groups and L^2-invariants; in Section 3, we cover some improvements of classical properties of L^2-torsions; in Section 4 we introduce the notion of a Markov-admissible family of group epimorphisms and we study several examples; in Section 5 we state and prove Theorem 5.1 on Markov invariance; in Section 6 we present counter-examples to Markov invariance; in Section 7 we compute new values of Fuglede-Kadison determinants over free groups; finally, in Section 8, we present new upper bounds on Lehmer's constants for a large class of torsionfree groups.

Sections 7 and 8 can be read independently from Sections 3 to 6 (the only relation lies in Theorem 7.3 being used in the proof of Theorem 6.2).

2. Preliminaries

In this section, we will set some notation and recall some fundamental properties. We will mostly follow the conventions of [5] and [13].

2.1. **Braid groups.** The braid group B_n can be seen as the set of isotopy classes of orientation-preserving homeomorphisms of the punctured disk $D_n := D^2 \setminus \{p_1, ..., p_n\}$ which fix the boundary pointwise. Recall that B_n admits a presentation with $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ following the relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for each i, and $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 2$. Topologically, the generator σ_i is the braid whose i-th component passes over the $(i+1)$-th component.

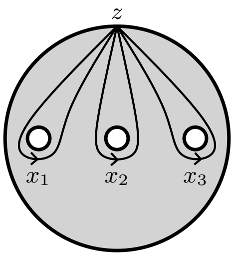


Figure 1. The punctured disk D_3.

Fix a base point z of D_n and denote by x_i the simple loop based at z turning once around p_i counterclockwise for $i = 1, 2, \ldots, n$ (see Figure 1). The group $\pi_1(D_n, z)$ can then be identified with the free group $\mathbb{F}_n$ on the x_i. If H_β is a homeomorphism of D_n representing a braid β, then the induced automorphism h_β of the free group $\mathbb{F}_n$ depends only on β. It follows from the way we compose braids that $h_{\alpha\beta} = h_\beta \circ h_\alpha$, and the resulting *right* action of B_n on $\mathbb{F}_n$ (named the *Artin action*) can be explicitly described by

$$h_{\sigma_i}(x_j) = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } j = i, \\ x_i & \text{if } j = i+1, \\ x_j & \text{otherwise,} \end{cases} \qquad h_{\sigma_i^{-1}}(x_j) = \begin{cases} x_{i+1} & \text{if } j = i, \\ x_{i+1}^{-1} x_i x_{i+1} & \text{if } j = i+1, \\ x_j & \text{otherwise.} \end{cases}$$

In this paper we will also use a second set of generators of $\mathbb{F}_n$, namely

$$g_1 := x_1, \ g_2 := x_1 x_2, \ \ldots, \ g_n := x_1 \ldots x_n.$$

Looking at Figure 1, g_i represents the class of the loop that circles the first i punctures. On these generators, B_n acts in the following way:

$$h_{\sigma_i}(g_j) = \begin{cases} g_{i+1} g_i^{-1} g_{i-1} & \text{if } j = i, \\ g_j & \text{otherwise,} \end{cases} \qquad h_{\sigma_i^{-1}}(g_j) = \begin{cases} g_{i-1} g_i^{-1} g_{i+1} & \text{if } j = i, \\ g_j & \text{otherwise,} \end{cases}$$

where we use the convention $g_0 := 1$.

2.2. **Fox calculus.** Denoting by $\mathbb{F}_n$ the free group on $x_1, x_2, \ldots, x_n$, and for $i \in \{1, \ldots, n\}$, the i-th *Fox derivative* $\dfrac{\partial}{\partial x_i} : \mathbb{Z}[\mathbb{F}_n] \to \mathbb{Z}[\mathbb{F}_n]$ (first introduced in [10]) is the linear extension of the map defined on $\mathbb{F}_n$ by: $\forall i, j \in \{1, \ldots, n\}, \forall u, v \in \mathbb{F}_n$,

$$\frac{\partial x_j}{\partial x_i} = \delta_{i,j}, \qquad \frac{\partial x_j^{-1}}{\partial x_i} = -\delta_{i,j} x_j^{-1}, \qquad \frac{\partial (uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} + u \frac{\partial v}{\partial x_i}.$$

The following formula is often called the *fundamental formula of Fox calculus*:

Proposition 2.1. *Let $u \in \mathbb{Z}\mathbb{F}_n$ and $\epsilon \colon \mathbb{Z}\mathbb{F}_n \to \mathbb{Z}$ the ring epimorphism defined by $\epsilon \colon x_i \mapsto 1$ for all $i \in \{1, \ldots, n\}$. Then:*

$$u - \epsilon(u) \cdot 1 = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \cdot (x_i - 1).$$

2.3. **Mahler measure of a polynomial.** Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ denote a d-variable polynomial. Then its *Mahler measure* $\mathcal{M}(P)$ is the nonnegative real number

$$\mathcal{M}(P) := \exp\left(\frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \ln\left(\left|P(e^{i\theta_1}, \ldots, e^{i\theta_d})\right|\right) d\theta_1 \ldots d\theta_d \right) \in \mathbb{R}_{\geqslant 0}.$$

Remark that this definition immediately extends to d-variable *Laurent* polynomials $P \in \mathbb{C}\left[X_1^{\pm 1}, \ldots, X_d^{\pm 1}\right]$. This will be useful in the next section, where the group algebra $\mathbb{C}[\mathbb{Z}^d]$ will be naturally identified with the algebra of Laurent polynomials $\mathbb{C}\left[X_1^{\pm 1}, \ldots, X_d^{\pm 1}\right]$. We refer to [6] (among others) for a survey on Mahler measures.

Example 2.2. For $d = 1$, and $P(X) = C \cdot X^{-l} \cdot \prod_{j=1}^{r} (X - \alpha_j) \in \mathbb{C}[X^{\pm 1}]$ (where $C, \alpha_1, \ldots, \alpha_r \in \mathbb{C}$ and $l \in \mathbb{N}$), there is a closed formula

$$\mathcal{M}(P) = |C| \cdot \prod_{j=1}^{r} \max\{1, |\alpha_j|\}.$$

When P has two or more variables, the Mahler measure is known for several classes of examples, such as the following one:

Example 2.3 ([6], Section 4). The two-variable polynomial $1 + X + Y \in \mathbb{C}[X, Y]$ has Mahler measure $\mathcal{M}(1 + X + Y) = e^{\frac{1}{\pi} \Im \mathrm{Li}_2\left(e^{i\pi/3}\right)} = 1.38135...$

Example 2.3 will be used later in the proof of Theorem 6.1.

2.4. **Fuglede-Kadison determinant.** In this section we will give short definitions of the von Neumann trace and the Fuglede-Kadison determinant. More details can be found in [13] and [5].

Let G be a finitely generated group. The Hilbert space $\ell^2(G)$ is the completion of the group algebra $\mathbb{C}G$, and the space of bounded operators on it is denoted $B(\ell^2(G))$. We will focus on *right-multiplication operators* $R_w \in B(\ell^2(G))$, where $R_.$ denotes the right regular action of G on $\ell^2(G)$ extended to the group ring $\mathbb{C}G$ (and further extended to the rings of matrices $M_{p,q}(\mathbb{C}G)$).

For any element $w = a_0 \cdot 1_G + a_1 g_1 \ldots + a_r g_r \in \mathbb{C}G$, the *von Neumann trace* tr_G of the associated right multiplication operator is defined as

$$\mathrm{tr}_G(R_w) = \mathrm{tr}_G\left(a_0 \mathrm{Id}_{\ell^2(G)} + a_1 R_{g_1} \ldots + a_r R_{g_r}\right) := a_0,$$

and the von Neumann trace for a finite square matrix over $\mathbb{C}G$ is given as the sum of the traces of the diagonal coefficients.

Now the most concise definition of the *Fuglede-Kadison determinant* $\det_G(A)$ of a right-multiplication operator A is probably

$$\det{}_G(A) := \lim_{\varepsilon \to 0^+} \left(\exp \circ \left(\frac{1}{2}\mathrm{tr}_G\right) \circ \ln\right) \left((A_\perp)^*(A_\perp) + \varepsilon \mathrm{Id}\right) \geqslant 0,$$

where $A_\perp$ is the restriction of A to a supplementary of its kernel, $*$ is the adjunction and ln the logarithm of an operator in the sense of the holomorphic functional calculus. Compare with [13, Theorem 3.14] and Proposition 2.4 below. We call the operator A *of determinant class* if $\det_G(A) \neq 0$.

The following properties concern the classical Fuglede-Kadison determinant $\det_G$ described in [14, Chapter 3], which is not always multiplicative if one deals with non injective operators. Moreover, this determinant forgets about the influence of the spectral value 0, which surprisingly makes it take the value 1 for the zero operator. More recent articles have used the *regular Fuglede-Kadison determinant* $\det^r_G$ instead, which is defined for square injective operators, is zero for non injective operators, and is always multiplicative. In this paper we will work with both types of determinants, but the reader should be reassured that most of the statements we will make remain unchanged while replacing one determinant with the other (up to assumptions on injectivity usually). Similarly, the statements of the following Proposition 2.4 admit immediate variants with $\det^r_G$. All statements of Proposition 2.4 follow from [14][Section 3], except for (6), which directly follows from the others.

Proposition 2.4 ([14]). *Let G be a countable discrete group and let*

$$A, B, C, D \in \sqcup_{p,q\in\mathbb{N}} R_{M_{p,q}(\mathbb{C}G)}$$

be general right multiplication operators. The Fuglede-Kadison determinant satisfies the following properties:

(1) *(multiplicativity) If A, B are injective, square and of the same size, then*

$$\det{}_G(A \circ B) = \det{}_G(A)\det{}_G(B).$$

(2) *(block triangular case) If A, B are injective and square, then*

$$\det{}_G \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det{}_G(A)\det{}_G(B),$$

where C has the appropriate dimensions.

(3) *(induction) If $\iota\colon G \hookrightarrow H$ is a group monomorphism, then*

$$\det{}_H(\iota(A)) = \det{}_G(A).$$

(4) *(relation with the von Neumann trace) If A is a positive operator, then*

$$\det{}_G(A) = (\exp \circ \mathrm{tr}_G \circ \ln)(A).$$

(5) *(simple case) If $g \in G$ is of infinite order, then for all $t \in \mathbb{C}$ the operator $\mathrm{Id} - tR_g$ is injective and*

$$\det_G(\mathrm{Id} - tR_g) = \max(1, |t|).$$

(6) *(2×2 trick) For all $A, B, C, D \in N(G)$ such that B is invertible, $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is injective if and only if $DB^{-1}A - C$ is injective, and in this case one has:*

$$\det{}_G \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det{}_G(B)\det{}_G(DB^{-1}A - C).$$

(7) *(relation with Mahler measure) Let $G = \mathbb{Z}^d$, and $P \in \mathbb{C}[X_1^{\pm1}, \ldots, X_d^{\pm1}]$ denote the Laurent polynomial associated to the operator $A \in R_{\mathbb{C}\mathbb{Z}^d}$. Then*

$$\det_{\mathbb{Z}^d}(A) = \mathcal{M}(P) = \exp\left(\frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int_0^{2\pi} \ln\left(\left|P(e^{i\theta_1}, \ldots, e^{i\theta_d})\right|\right) d\theta_1 \ldots d\theta_d\right),$$

where $\mathcal{M}$ is the Mahler measure.

(8) *(limit of positive operators) If A is injective, then*

$$\det{}_G(A) = \lim_{\varepsilon\to0^+}\sqrt{\det{}_G(A^*A + \varepsilon\mathrm{Id})}.$$

(9) *(dilations) Let $\lambda \in \mathbb{C}^*$. Then:*

$$\det{}_G\left(\lambda\ \mathrm{Id}^{\oplus n}\right) = |\lambda|^n.$$

Remark 2.5. If the group G satisfies the *strong Atiyah conjecture* (see [13, Chapter 10]), then the right multiplication operator by any non-zero element of $\mathbb{C}G$ is injective, which makes it convenient to apply some parts of Proposition 2.4. Note that free groups and free abelian groups satisfy the strong Atiyah conjecture.

2.5. **L^2-Burau maps on braids.** Let $n \in \mathbb{N}^*$, $t > 0$, let $\Phi_n \colon \mathbb{F}_n \twoheadrightarrow \mathbb{Z}$ denote the projection that sends all free generators to 1, and let $\gamma \colon \mathbb{F}_n \twoheadrightarrow G$ denote an epimorphism such that Φ_n factors through γ. Let $\kappa(t, \Phi_n, \gamma) : \mathbb{Z}\mathbb{F}_n \to \mathbb{R}G$ denote the ring homomorphism that sends $g \in \mathbb{F}_n$ to $t^{\Phi_n(g)}\gamma(g) \in \mathbb{R}G$.

Then, following [5], the associated L^2*-Burau map* on B_n is

$$\mathcal{B}^{(2)}_{t,\gamma} \colon B_n \ni \beta \ \mapsto \ R_{\kappa(t,\Phi_n,\gamma)(J)} \in B\left(\ell^2(G)^{\oplus n}\right),$$

where $J = \left(\dfrac{\partial h_\beta(x_j)}{\partial x_i}\right)_{1\leqslant i,j\leqslant n}$ is the Fox jacobian of h_β for the base of the x_i.

The *reduced* L^2*-Burau map* on B_n (associated to the same parameters t, γ) is

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma} \colon B_n \ni \beta \ \mapsto \ R_{\kappa(t,\Phi_n,\gamma)(J')} \in B\left(\ell^2(G)^{\oplus (n-1)}\right),$$

where $J' = \left(\dfrac{\partial h_\beta(g_j)}{\partial g_i}\right)_{1\leqslant i,j\leqslant n-1}$ is the Fox jacobian of h_β for the base of the g_i.

In the remainder of this article we will focus on reduced L^2-Burau maps.

Observe that L^2-Burau maps (reduced or not) can also be defined as maps over a certain homology of a cover of the punctured disk (see [5] for details). Although these homological definitions may be more natural and useful for further generalizations, the current article will only use the previous definitions via Fox jacobians.

Let us now state an (anti-)multiplication formula for the reduced L^2-Burau maps.

Proposition 2.6 ([5])**.** *For any n, t, γ as above and any two braids $\alpha, \beta \in B_n$, we have:*

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha\beta) = \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\beta) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_\beta}(\alpha).$$

Note that the unreduced L^2-Burau maps satisfy an identical formula.

It follows from Proposition 2.6 that a reduced L^2-Burau map can be computed for any braid via knowing the values on the generators σ_i of the braid group. For the reader's convenience and since they will be used in the remainder of this article, we now provide the image of the generators σ_i:

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_1) = \begin{pmatrix} -tR_{\gamma(g_2 g_1^{-1})} & 0 \\ \mathrm{Id} & \mathrm{Id} \end{pmatrix} \oplus \mathrm{Id}^{\oplus(n-3)},$$

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_i) = \mathrm{Id}^{\oplus(i-2)} \oplus \begin{pmatrix} \mathrm{Id} & tR_{\gamma(g_{i+1} g_i^{-1})} & 0 \\ 0 & -tR_{\gamma(g_{i+1} g_i^{-1})} & 0 \\ 0 & \mathrm{Id} & \mathrm{Id} \end{pmatrix} \oplus \mathrm{Id}^{\oplus(n-i-2)} \ \textit{for } 1 < i < n-1,$$

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_{n-1}) = \mathrm{Id}^{\oplus(n-3)} \oplus \begin{pmatrix} \mathrm{Id} & tR_{\gamma(g_n g_{n-1}^{-1})} \\ 0 & -tR_{\gamma(g_n g_{n-1}^{-1})} \end{pmatrix},$$

and the images of the inverses σ_i^{-1} of the generators:

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_1^{-1}) = \begin{pmatrix} -\frac{1}{t}R_{\gamma(g_1^{-1})} & 0 \\ \frac{1}{t}R_{\gamma(g_1^{-1})} & \mathrm{Id} \end{pmatrix} \oplus \mathrm{Id}^{\oplus(n-3)},$$

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_i^{-1}) = \mathrm{Id}^{\oplus(i-2)} \oplus \begin{pmatrix} \mathrm{Id} & \mathrm{Id} & 0 \\ 0 & -\frac{1}{t}R_{\gamma(g_{i-1}g_i^{-1})} & 0 \\ 0 & \frac{1}{t}R_{\gamma(g_{i-1}g_i^{-1})} & \mathrm{Id} \end{pmatrix} \oplus \mathrm{Id}^{\oplus(n-i-2)} \ for\ 1 < i < n-1,$$

$$\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_{n-1}^{-1}) = \mathrm{Id}^{\oplus(n-3)} \oplus \begin{pmatrix} \mathrm{Id} & \mathrm{Id} \\ 0 & -\frac{1}{t}R_{\gamma(g_{n-2}g_{n-1}^{-1})} \end{pmatrix}.$$

Remark 2.7. It follows from what precedes and from Proposition 2.4 that for all $t > 0$, we have $\det_G\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\sigma_i^{\pm 1})\right) = t^{\pm 1}$.

2.6. **L^2-torsions.** This section covers some necessary definitions to state the results in Section 3, which in turn will be used to prove Theorem 5.1 (2) in Section 5. We refer to [13] and [4] for more details.

A *finitely generated Hilbert* $\mathcal{N}(G)$*-module* is an Hilbert space V on which there is a left G-action by isometries, and such that there exists a positive integer m and an embedding ϕ of V into $\bigoplus_{i=1}^m \ell^2(G)$ (in this paper, such spaces V will always be of the form $\ell^2(G)^{\oplus n}$ for $n \in \mathbb{N}$).

For U and V two finitely generated Hilbert $\mathcal{N}(G)$-modules, we will call $f\colon U \to V$ a *morphism of finitely generated Hilbert* $\mathcal{N}(G)$*-modules* if f is a linear G-equivariant map, bounded for the respective scalar products of U and V (in this paper, these morphisms will simply be right multiplication operators).

A *finite Hilbert* $\mathcal{N}(G)$*-chain complex* C_* is a sequence of morphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules

$$C_* = 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

such that $\partial_p \circ \partial_{p+1} = 0$ for all p (in this paper, n will be at most 3).

The *p-th L^2-homology of* C_* $H_p^{(2)}(C_*) := Ker(\partial_p)/\overline{Im(\partial_{p+1})}$ is a finitely generated Hilbert $\mathcal{N}(G)$-module. We say that C_* is *weakly acyclic* if its L^2-homology is trivial. We say that C_* is of *determinant class* if all the operators ∂_p are of determinant class.

Let C_* be a finite Hilbert $\mathcal{N}(G)$-chain complex as above. Its *L^2-torsion* is

$$T^{(2)}(C_*) := \prod_{i=1}^{n} \det{}_{\mathcal{N}(G)}(\partial_i)^{(-1)^i} \in \mathbb{R}_{>0}$$

if C_* is weakly acyclic and of determinant class, and is $T^{(2)}(C_*) := 0$ otherwise.

Let π be a group and $\phi\colon \pi \to \mathbb{Z}$, $\gamma\colon \pi \to G$ two group homomorphisms. We say that (π, ϕ, γ) *forms an admissible triple* if $\phi\colon \pi \to \mathbb{Z}$ factors through γ. For X a CW-complex, we say that $(X, \phi\colon \pi_1(X) \to \mathbb{Z}, \gamma\colon \pi_1(X) \to G)$ *forms an admissible triple* if $(\pi_1(X), \phi, \gamma)$ forms one. Let (X, ϕ, γ) be such an admissible triple, $\pi = \pi_1(X)$ and $t > 0$. We define a ring homomorphism

$$\kappa(\pi, \phi, \gamma, t)\colon \begin{pmatrix} \mathbb{Z}[\pi] & \longrightarrow & \mathbb{R}[G] \\ \sum_{j=1}^r m_j g_j & \longmapsto & \sum_{j=1}^r m_j t^{\phi(g_j)} \gamma(g_j) \end{pmatrix}$$

and we also denote $\kappa(\pi, \phi, \gamma, t)$ its induction over the $M_{p,q}(\mathbb{Z}[\pi])$.

Assume X is compact. The cellular chain complex of $\widetilde{X}$ denoted $C_*(\widetilde{X}, \mathbb{Z}) = \left(\ldots \to \bigoplus_i \mathbb{Z}[\pi]\widetilde{e}_i^k \to \ldots\right)$ is a chain complex of left $\mathbb{Z}[\pi]$-modules. Here the $\widetilde{e}_i^k$ are lifts of the cells e_i^k of X. The group π acts on the right on $\ell^2(G)$ by $g \mapsto R_{\kappa(\pi,\phi,\gamma,t)(g)}$, an action which induces a structure of right $\mathbb{Z}[\pi]$-module on $\ell^2(G)$. Let

$$C_*^{(2)}(X,\phi,\gamma,t) = \ell^2(G) \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X},\mathbb{Z})$$

denote the finite Hilbert $\mathcal{N}(G)$-chain complex obtained by tensor product via these left- and right-actions; we call $C_*^{(2)}(X,\phi,\gamma,t)$ *a* $\mathcal{N}(G)$*-cellular chain complex of* X.

If $C_*^{(2)}(X,\phi,\gamma,t)$ is a $\mathcal{N}(G)$-cellular chain complex of X, then denote

$$T^{(2)}(X,\phi,\gamma)(t) = T^{(2)}\left(C_*^{(2)}(X,\phi,\gamma,t)\right)$$

the *L^2-Alexander torsion of* (X,ϕ,γ) *at* $t > 0$. It is non-zero if and only if $C_*^{(2)}(X,\phi,\gamma,t)$ is weakly acyclic and of determinant class.

Let $L = L_1 \cup \ldots \cup L_c$ be a link in S^3, M_L its exterior and $\alpha_L \colon G_L = \pi_1(M_L) \to \mathbb{Z}^c$ the abelianization of its group. Any homomorphism $\phi\colon G_L \to \mathbb{Z}$ factors through α_L and thus is written $\phi = (n_1,\ldots,n_c)\circ\alpha_L$ where $n_1,\ldots,n_c \in \mathbb{Z}$. Any admissible triple (M_L,ϕ,γ) can thus be written $(M_L,(n_1,\ldots,n_c)\circ\alpha_L,\gamma)$, and we will denote

$$T^{(2)}_{L,(n_1,\ldots,n_c)}(\gamma)(t) := T^{(2)}(M_L,(n_1,\ldots,n_c)\circ\alpha_L,\gamma)(t)$$

the *twisted L^2-Alexander torsion* associated to L, the coefficients $(n_1,\ldots,n_c)$, the morphism γ (the *twist*), at the value t. We sometimes omit the *twisted* when $\gamma = \mathrm{id}$.

3. Some useful properties of L^2-torsions

In this section we recall and generalize several natural properties of L^2-torsions, that are used in the proof of Theorem 5.1 (2).

The following Proposition 3.1 is a rephrasing of several results in [13], which concern short exact sequences of finite Hilbert $\mathcal{N}(G)$-chain complexes. Recall that $0 \to C_* \overset{\eta_*}{\to} D_* \overset{\rho_*}{\to} E_* \to 0$ is a *short exact sequence of finite Hilbert* $\mathcal{N}(G)$*-chain complexes* if $0 \to C_p \overset{\eta_p}{\to} D_p \overset{\rho_p}{\to} E_p \to 0$ is exact for every p and if η_*,ρ_* commute with the boundary operators of C_*, D_*, E_*.

Proposition 3.1 ([13]). *Let $0 \to C_* \overset{\eta_*}{\to} D_* \overset{\rho_*}{\to} E_* \to 0$ be a short exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes, such that for every $p \in \mathbb{Z}$, η_p and ρ_p are of determinant class. Then the following hold:*

1. *If two among C_*, D_*, E_* are weakly acyclic, then the third one is as well.*
2. *If C_*, D_*, E_* are all weakly acyclic, and if two of them are of determinant class, then the third one is as well.*
3. *If either C_*, D_*, E_* are all weakly acyclic and of determinant class, or if D_* is not, then the L^2-torsions satisfy*

$$T^{(2)}(D_*)\cdot\left(\prod_{p\in\mathbb{Z}}\left(\frac{\det_G(\rho_p)}{\det_G(\eta_p)}\right)^{(-1)^p}\right) = T^{(2)}(C_*)\cdot T^{(2)}(E_*).$$

Proof. Let us prove (1). If two among C_*, D_*, E_* are weakly acyclic, then the long weakly exact homology sequence of finite Hilbert $\mathcal{N}(G)$-modules

$$LHS_* = \ \ldots \to H^{(2)}_{n+1}(E_*) \to H^{(2)}_n(C_*) \to H^{(2)}_n(D_*) \to H^{(2)}_n(E_*) \to \ldots$$

of [13, Theorem 1.21] is trivial, and thus C_*, D_*, E_* are all weakly acyclic.

Now, (2) and (3) follow from [13, Theorem 3.35 (1)], the assumptions on ι_*, ρ_*, and the fact that LHS_* is trivial (which implies that LHS_* is of determinant class and that its L^2-torsion is equal to 1). $\square$

The following proposition is a slight generalization of [3, Theorem 2.12], and concerns the invariance under simple homotopy equivalence.

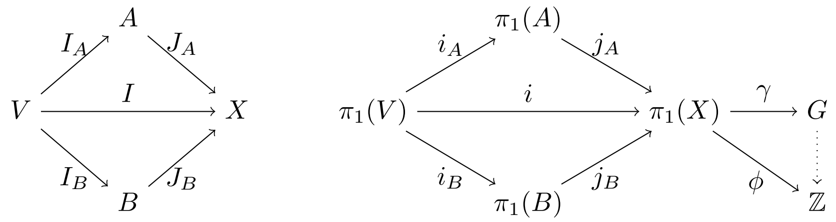
Proposition 3.2. *Let $f : X \to Y$ be a simple homotopy equivalence between two finite CW-complexes inducing the group isomorphism $f_\sharp \colon \pi_1(X) \to \pi_1(Y)$. The triple (Y, ϕ, γ) is an admissible triple if and only if $(X, \phi \circ f_\sharp, \gamma \circ f_\sharp)$ is one. Moreover, for all $t > 0$:*

1. *$C_*^{(2)}(X, \phi \circ f_\sharp, \gamma \circ f_\sharp, t)$ is weakly acyclic if and only if $C_*^{(2)}(Y, \phi, \gamma, t)$ is,*
2. *$C_*^{(2)}(X, \phi \circ f_\sharp, \gamma \circ f_\sharp, t)$ is weakly acyclic and of determinant class if and only if $C_*^{(2)}(Y, \phi, \gamma, t)$ is,*
3. *$T^{(2)}(X, \phi \circ f_\sharp, \gamma \circ f_\sharp)(t) \doteq T^{(2)}(Y, \phi, \gamma)(t)$.*

Proof. The proof is almost exactly as the one of [3, Theorem 2.12]: we study the case where f is an elementary expansion, and we relate $C_*^{(2)}(X, \phi \circ f_\sharp, \gamma \circ f_\sharp, t)$ and $C_*^{(2)}(Y, \phi, \gamma, t)$ through an exact sequence. Then we apply Proposition 3.1. □

The following Proposition 3.3 is a slight generalization of the gluing formula of [3, Theorem 3.1] and [4, Proposition 3.5] for L^2-Alexander torsions (which only stated that (1) and (2) together imply (3)).

Proposition 3.3. *Let X, A, B, V be finite CW-complexes such that $X = A \cup B$ and $V = A \cap B$. We denote the various inclusions (which are assumed to be cellular) and their inductions on fundamental groups as in the following diagrams:*



Let $(\pi_1(X), \phi\colon \pi_1(X) \to \mathbb{Z}, \gamma\colon \pi_1(X) \to G)$ be an admissible triple, and $t > 0$. If any two of the following properties are satisfied,

1. *$C_*^{(2)}(V, \phi \circ i, \gamma \circ i, t)$ is weakly acyclic (resp. weakly acyclic and of determinant class),*
2. *$C_*^{(2)}(A, \phi \circ j_A, \gamma \circ j_A, t)$ and $C_*^{(2)}(B, \phi \circ j_B, \gamma \circ j_B, t)$ are weakly acyclic (resp. weakly acyclic and of determinant class),*
3. *$C_*^{(2)}(X, \phi, \gamma, t)$ is weakly acyclic (resp. weakly acyclic and of determinant class),*

then the third property is satisfied as well, and we have

$$T^{(2)}(X, \phi, \gamma)(t) \doteq \frac{T^{(2)}(A, \phi \circ j_A, \gamma \circ j_A)(t) \cdot T^{(2)}(B, \phi \circ j_B, \gamma \circ j_B)(t)}{T^{(2)}(V, \phi \circ i, \gamma \circ i)(t)}.$$

Proof. The proof works similarly as the one of [3, Theorem 3.1]. Let us now sketch the modified arguments. Let V_*, X_* denote the finite Hilbert $\mathcal{N}(G)$-chain complexes of properties (1) and (3), and C_* the direct sum of the two finite Hilbert $\mathcal{N}(G)$-chain complexes in property (2). Observe that property (2) can be rephrased as C_* being weakly acyclic (resp. weakly acyclic and of determinant class). As explained in [3, Theorem 3.1] (via classical arguments), we have an exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes $0 \to V_* \to C_* \to X_* \to 0$, where the horizontal operators are of determinant class. The result then follows from Proposition 3.1 and the computation of Fuglede-Kadison determinants of the horizontal operators. □

The following Proposition 3.4 is a slight generalization of the L^2-Torres formula of [4, Theorem 4.4] (which only stated that (1) implies (2) instead of their equivalence).

Proposition 3.4. *Let $L = L_1 \cup \ldots \cup L_c$ be a c-component link, and $L' = L \cup L_{c+1}$ a $(c+1)$-component link link admitting L as a sublink. Let $M_L, M_{L'}$ denote the exteriors of L and L'. Let $Q\colon \pi_1(M_{L'}) \twoheadrightarrow \pi_1(M_L)$ denote the group epimorphism induced by removing the component L_{c+1}. Let $\lambda \in \pi_1(M_{L'})$ denote the class of a preferred longitude of L_{c+1}.*

Let $\phi : \pi_1(M_L) \to \mathbb{Z}$ and $\gamma : \pi_1(M_L) \to G$ be group homomorphisms such that $(\pi_1(M_L), \phi, \gamma)$ forms an admissible triple. We can write $\phi = (n_1, \ldots, n_c) \circ \alpha_L$ and thus $\phi \circ Q = (n_1, \ldots, n_c, 0) \circ \alpha_{L'}$ for some non zero vector $(n_1, \ldots, n_c) \in \mathbb{Z}^c$.

Assume that $(\gamma \circ Q)(\lambda)$ is of infinite order in G. Then for all $t > 0$, the following are equivalent:

1. *$C^{(2)}_*(M_{L'}, (n_1, \ldots, n_c, 0) \circ \alpha_{L'}, \gamma \circ Q)(t)$ is weakly acyclic (resp. weakly acyclic and of determinant class),*
2. *$C^{(2)}_*(M_L, (n_1, \ldots, n_c) \circ \alpha_L, \gamma)(t)$ is weakly acyclic (resp. weakly acyclic and of determinant class).*

Moreover, we have

$$T^{(2)}_{L,(n_1,\ldots,n_c)}(\gamma)(t) \doteq \frac{T^{(2)}_{L',(n_1,\ldots,n_c,0)}(\gamma \circ Q)(t)}{\max(1,t)^{|\mathrm{lk}(L_1,L_{c+1})n_1 + \ldots + \mathrm{lk}(L_c,L_{c+1})n_c|}}.$$

Proof. The proof is similar to [4, Section 4]. Here we use the generalized gluing formula of Proposition 3.3 instead of the weaker version of [4, Proposition 3.5]. □

4. Markov admissibility of a family of epimorphisms

In this section we introduce the notion of Markov-admissibility for a family of epimorphisms indexed by braids, and we discuss several examples of such families.

For each $n \geqslant 1$ and each braid $\beta \in B_n$, let us denote

- $n(\beta) := n$ the number of strands,
- h_β the (Artin) group automorphism on $\mathbb{F}_{n(\beta)}$ (recall that $\beta \mapsto h_\beta$ is anti-multiplicative),
- $\gamma_\beta\colon \mathbb{F}_{n(\beta)} \twoheadrightarrow G_\beta$ the quotient by all relations of the form $\star = h_\beta(\star)$,
- $\hat\beta$ the closure of β, a link in S^3,
- $G_{\hat\beta} = \pi_1\left(S^3 \setminus \hat\beta\right)$ the group of the link $\hat\beta$,
- $\Phi_n\colon \mathbb{F}_n \twoheadrightarrow \mathbb{Z}$ the epimorphism which sends the n generators to 1,
- $\iota_n\colon \mathbb{F}_n \hookrightarrow \mathbb{F}_{n+1}$ the group inclusion sending the n generators of $\mathbb{F}_n$ to the first n generators of $\mathbb{F}_{n+1}$.

Definition 4.1. A family $\mathcal{Q}$ of group epimorphisms of the form

$$\mathcal{Q} = \left\{ Q_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow G_{Q_\beta} \mid \beta \in \sqcup_{n \geqslant 1} B_n \right\}$$

will be called *Markov-admissible* if it satisfies the following conditions:

1. For all $n \geqslant 1$ and all $\alpha, \beta \in B_n$, there exists a group homomorphism $\chi^{\mathcal{Q}}_{\beta,\alpha}\colon G_{Q_\beta} \to G_{Q_{\alpha^{-1}\beta\alpha}}$ such that $Q_{\alpha^{-1}\beta\alpha} \circ h_\alpha = \chi^{\mathcal{Q}}_{\beta,\alpha} \circ Q_\beta$. Observe that this implies that each $\chi^{\mathcal{Q}}_{\beta,\alpha}$ is uniquely defined and is an isomorphism.
2. For all $n \geqslant 1$, $\beta \in B_n$ and $\varepsilon \in \{\pm 1\}$, there exists a group monomorphism $\sigma^{\mathcal{Q}}_{\beta,\varepsilon}\colon G_{Q_\beta} \hookrightarrow G_{Q_{\sigma_n^\varepsilon \beta}}$ such that $Q_{\sigma_n^\varepsilon \beta} \circ \iota_n = \sigma^{\mathcal{Q}}_{\beta,\varepsilon} \circ Q_\beta$. Observe that each $Q_{\sigma_n^\varepsilon \beta}$ is uniquely defined.

These two conditions are illustrated in Figure 2.

Roughly speaking, the epimorphisms in a Markov-admissible family will be compatible in a way that lets us hope to compute (L^2-)knot invariants by using them. Note that less restrictive definitions may be preferred in the future if we find better ways of computing L^2-objects such as Fuglede-Kadison determinants.

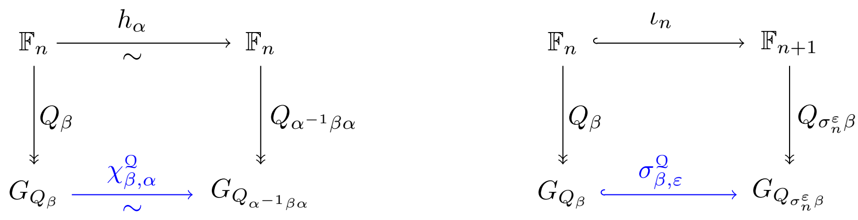


FIGURE 2. Conditions for $\mathcal{Q}$ to be Markov-admissible

We will now present several examples of Markov-admissible families, which are all displayed in Figure 4 for clarity. Moreover, Figure 4 summarizes the results of Sections 5 and 6 concerning Markov invariance (✓ standing for yes and ✗ for no).

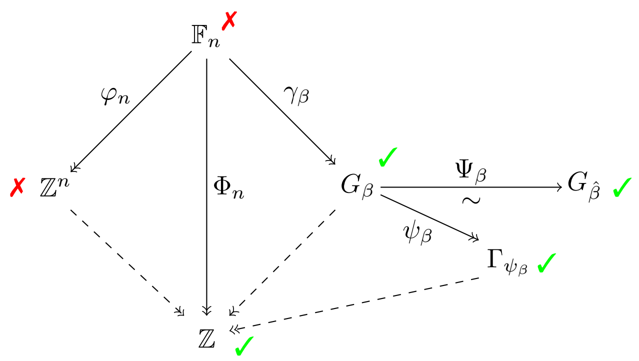


FIGURE 3. When does $\overline{\mathcal{B}}^{(2)}_{t,\gamma}$ yield a map on braids which is invariant under Markov moves, for $\gamma\colon \mathbb{F}_n \twoheadrightarrow G$?

Example 4.2. The family of identity morphisms $\mathcal{Q} = \left\{\mathrm{id}_{\mathbb{F}_{n(\beta)}}\right\}$ is Markov-admissible, with $\chi^{\mathcal{Q}}_{\beta,\alpha} = \mathrm{id}_{\mathbb{F}_{n(\beta)}}$ and $\sigma^{\mathcal{Q}}_{\beta,\varepsilon} = \iota_{n(\beta)}$.

Example 4.3. The family $\mathcal{Q} = \left\{\Phi_{n(\beta)}\right\}$ is Markov-admissible, with $\chi^{\mathcal{Q}}_{\beta,\alpha} = \sigma^{\mathcal{Q}}_{\beta,\varepsilon} = \mathrm{id}_{\mathbb{Z}}$.

Example 4.4. The family of abelianizations $\mathcal{Q} = \left\{\varphi_{n(\beta)}\colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \mathbb{Z}^{n(\beta)}\right\}$ is Markov-admissible, with $\sigma^{\mathcal{Q}}_{\beta,\varepsilon}$ the inclusion $\mathbb{Z}^{n(\beta)} \hookrightarrow \mathbb{Z}^{n(\beta)+1}$ induced by $\iota_{n(\beta)}$, and $\chi^{\mathcal{Q}}_{\beta,\alpha}$ the permutation on the canonical generators of $\mathbb{Z}^n$ corresponding to the permutation of $n(\alpha)$ strands induced by $\alpha \in B_n$.

The following proposition is an elementary result in group theory, but is stated for the reader's convenience.

Proposition 4.5. *Let $f\colon G \to H$ be a group homomorphism, and N a normal subgroup of G. If f is surjective and $Ker(f) \subset N$ (in particular if f is an isomorphism), then f induces an isomorphism between G/N and $H/f(N)$.*

Let us now consider epimorphisms that descend to the fundamental groups of the braid closure complements.

Proposition 4.6. *The family $\mathcal{Q} = \{\gamma_\beta\colon \mathbb{F}_{n(\beta)} \twoheadrightarrow G_\beta\}$ is Markov-admissible. Moreover the monomorphisms $\sigma^{\mathcal{Q}}_{\beta,\varepsilon}$ are isomorphisms.*

Proof. First step: Markov 1:
Take $n \geqslant 1$ and $\alpha, \beta \in B_n$. Note that

$$h_\alpha(Ker(\gamma_\beta)) = h_\alpha\left(\ \langle\langle h_\beta(x)x^{-1}; x \in \mathbb{F}_n\rangle\rangle\ \right) = \langle\langle h_{\beta\alpha}(x)h_\alpha(x)^{-1}; x \in \mathbb{F}_n\rangle\rangle$$
$$= \langle\langle h_{\alpha^{-1}\beta\alpha}(y)y^{-1}; y \in \mathbb{F}_n\rangle\rangle = Ker(\gamma_{\alpha^{-1}\beta\alpha}),$$

thus it follows from Proposition 4.5 that the isomorphism $h_\alpha \colon \mathbb{F}_n \xrightarrow{\sim} \mathbb{F}_n$ induces the required isomorphism $\chi^{\mathbb{Q}}_{\beta,\alpha} \colon G_\beta \xrightarrow{\sim} G_{\alpha^{-1}\beta\alpha}$.

Second step: Markov 2:
Take $n \geqslant 1$ and $\beta \in B_n$. Let $\beta_+ = \sigma_n^{-1}\iota(\beta) \in B_{n+1}$. First notice that

$$h_{\beta_+}(x_j)x_j^{-1} = h_{\iota(\beta)}(h_{\sigma_n^{-1}}(x_j))x_j^{-1} = h_{\iota(\beta)}(x_j)x_j^{-1} \quad \text{for } 1 \leqslant j \leqslant n-1,$$
$$h_{\beta_+}(x_n)x_n^{-1} = h_{\iota(\beta)}(h_{\sigma_n^{-1}}(x_n))x_n^{-1} = h_{\iota(\beta)}(x_{n+1})x_n^{-1} = x_{n+1}x_n^{-1},$$
$$h_{\beta_+}(x_{n+1})x_{n+1}^{-1} = h_{\iota(\beta)}(h_{\sigma_n^{-1}}(x_{n+1}))x_{n+1}^{-1} = h_{\iota(\beta)}(x_{n+1}^{-1}x_nx_{n+1})x_{n+1}^{-1} = x_{n+1}^{-1}h_{\iota(\beta)}(x_n).$$

Hence $Ker(\gamma_{\beta_+}) = \langle\langle\ x_{n+1}x_n^{-1}\ ;\ \iota_n(Ker(\gamma_\beta))\ \rangle\rangle$.

We can now define $\sigma^{\mathbb{Q}}_{\beta,-1} := \Big(G_\beta \ni [x]_{G_\beta} \mapsto [\iota_n(x)]_{G_{\beta_+}} \in G_{\beta_+}\Big)$, where $x \in \mathbb{F}_n$ and $[\cdot]_G$ is the quotient class in G. Since $\iota_n(Ker(\gamma_\beta)) \subset Ker(\gamma_{\beta_+})$, then $\sigma^{\mathbb{Q}}_{\beta,-1}$ is a well-defined group homomorphism.

Let us prove that $\sigma^{\mathbb{Q}}_{\beta,-1}$ is surjective. Let $[y]_{G_{\beta_+}} \in G_{\beta_+}$, with $y \in \mathbb{F}_{n+1}$. Let $y' \in \iota_n(\mathbb{F}_n)$ be the word constructed from y by replacing all letters x_{n+1} with x_n. Hence $[y]_{G_{\beta_+}} = [y']_{G_{\beta_+}} \in Im(\sigma^{\mathbb{Q}}_{\beta,-1})$ and $\sigma^{\mathbb{Q}}_{\beta,-1}$ is surjective.

Let us prove that $\sigma^{\mathbb{Q}}_{\beta,-1}$ is injective. Let $[\iota_n(x)]_{G_{\beta_+}} \in G_{\beta_+}$ (with $x \in \mathbb{F}_n$) be trivial. Then $\iota_n(x) \in Ker(\gamma_{\beta_+}) = \langle\langle\ x_{n+1}x_n^{-1}\ ;\ \iota_n(Ker(\gamma_\beta))\ \rangle\rangle$. Thus $\iota_n(x)$ is a product of conjugates (in $\mathbb{F}_{n+1}$) of terms $(x_{n+1}x_n^{-1})^{\pm 1}$ and/or terms in $\iota_n(Ker(\gamma_\beta))$. But since $\iota_n(x)$ is a free word without the letter x_{n+1}, we conclude that the conjugates of terms $(x_{n+1}x_n^{-1})^{\pm 1}$ in $\iota_n(x)$ cancel each other. Thus $\iota_n(x) \in \iota_n(Ker(\gamma_\beta))$ and $[x]_{G_\beta} = 1$. Hence $\sigma^{\mathbb{Q}}_{\beta,-1}$ is injective.

Third step: Markov 2 again:
Take $n \geqslant 1$, $\beta \in B_n$, and let $\beta_+ = \sigma_n\iota(\beta) \in B_{n+1}$. The proof is similar as in the Second step, with the following differences:

$$h_{\beta_+}(x_n)x_n^{-1} = h_{\iota(\beta)}(x_n)x_{n+1}\left(h_{\iota(\beta)}(x_n)\right)^{-1}x_n^{-1},$$
$$h_{\beta_+}(x_{n+1})x_{n+1}^{-1} = h_{\iota(\beta)}(x_n)x_{n+1}^{-1},$$
$$Ker(\gamma_{\beta_+}) = \langle\langle\ h_{\iota(\beta)}(x_n)x_{n+1}^{-1}\ ;\ \iota_n(Ker(\gamma_\beta))\ \rangle\rangle,$$

and we replace the x_{n+1} with $h_{\iota(\beta)}(x_n)$ in the proof of the surjectivity of $\sigma^{\mathbb{Q}}_{\beta,1}$. □

Remark 4.7. In the second step of the previous proof, one can alternatively prove that $\sigma^{\mathbb{Q}}_{\beta,-1}$ is an isomorphism by observing that it corresponds to the sequence of Tietze transformations going from the presentation

$$\langle x_1, \ldots, x_n | h_\beta(x_1)x_1^{-1}, \ldots, h_\beta(x_{n-1})x_{n-1}^{-1}, h_\beta(x_n)x_n^{-1}\rangle$$

of G_β to the presentation

$$\langle x_1, \ldots, x_n, x_{n+1} | h_{\iota(\beta)}(x_1)x_1^{-1}, \ldots, h_{\iota(\beta)}(x_{n-1})x_{n-1}^{-1}, x_{n+1}x_n^{-1}, x_{n+1}^{-1}h_{\iota(\beta)}(x_n)\rangle =$$
$$\langle x_1, \ldots, x_n, x_{n+1} | h_{\beta_+}(x_1)x_1^{-1}, \ldots, h_{\beta_+}(x_{n-1})x_{n-1}^{-1}, h_{\beta_+}(x_n)x_n^{-1}, h_{\beta_+}(x_{n+1})x_{n+1}^{-1}\rangle$$

of G_{β_+}.

Finally, let us fix notations for epimorphisms that go lower than the groups of the braid closures.

Example 4.8. Let $\left\{\psi_\beta \colon G_\beta \twoheadrightarrow \Gamma_{\psi_\beta}\right\}_{\beta\in\sqcup_{n\geqslant 1}B_n}$ be a family of epimorphisms such that

- $\Phi_{n(\beta)}$ factors through $\psi_\beta\circ\gamma_\beta$,
- $\chi^{\mathcal{Q}}_{\beta,\alpha}\left(Ker(\psi_\beta)\right) = Ker(\psi_{\alpha^{-1}\beta\alpha})$,
- $\sigma^{\mathcal{Q}}_{\beta,\varepsilon}\left(Ker(\psi_\beta)\right) = Ker(\psi_{\sigma_n^\varepsilon\iota(\beta)})$.

Then it follows from Propositions 4.5 and 4.6 that the family

$$\mathcal{Q} = \left\{\psi_\beta\circ\gamma_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \Gamma_{\psi_\beta}\right\}$$

is Markov-admissible.

Note that the first assumption (that $\Phi_{n(\beta)}$ factors through $\psi_\beta\circ\gamma_\beta$) is not necessary for $\mathcal{Q}$ to be Markov-admissible, but is relevant in order to compare the function $F_{\mathcal{Q}}$ of the next section with L^2-Alexander torsions (for which such a factoring property is assumed, see [5]).

5. A sufficient condition for Markov invariance

In this section, we state and prove Theorem 5.1. For $\mathcal{Q} = \{Q_\beta\}$ a Markov-admissible family, we define the function

$$F_{\mathcal{Q}} := \left(\begin{array}{ccc} \sqcup_{n\geqslant 1}B_n & \to & \mathcal{F}(\mathbb{R}_{>0},\mathbb{R}_{\geqslant 0})/\{t\mapsto t^m, m\in\mathbb{Z}\} \\ \beta & \mapsto & \left[t\mapsto \dfrac{\det{}^r_{G_{Q_\beta}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right)}{\max(1,t)^n}\right] \end{array}\right),$$

taking values in equivalence classes $[t\mapsto f(t)]$ of selfmaps of $\mathbb{R}_{>0}$, up to multiplication by monomials with integer exponents.

The following theorem gives a sufficient condition on $\mathcal{Q}$ for $F_{\mathcal{Q}}$ to be a Markov function and thus yield a invariant of links.

Theorem 5.1. *Let $\mathcal{Q} = \{\psi_\beta\circ\gamma_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \Gamma_{\psi_\beta}\}$ be as in Example 4.8. Then:*

1. *$F_{\mathcal{Q}}$ is a Markov function, and thus defines an invariant of links, that we will denote $\mathcal{T}_{\mathcal{Q}}$.*
2. *For all $n\geqslant 1$, $t>0$, and $\beta\in B_n$, we have*

$$F_{\mathcal{Q}}(\beta)(t) \doteq T^{(2)}_{\hat{\beta},(1,\ldots,1)}(\psi_\beta\circ(\Psi_\beta)^{-1})(t).$$

In particular, given a link L, then for any braid β such that $\hat{\beta} = L$, the invariant $\mathcal{T}_{\mathcal{Q}}(L)$ of L is equal to the equivalence class of selfmaps of $\mathbb{R}_{>0}$

$$\left[t\mapsto T^{(2)}_{L,(1,\ldots,1)}(\psi_\beta\circ(\Psi_\beta)^{-1})(t)\right]$$

of twisted L^2-Alexander torsions of the exterior of L, where the twist is the epimorphism $\psi_\beta\circ(\Psi_\beta)^{-1} : G_L \twoheadrightarrow \Gamma_{\psi_\beta}$.

Remark that for $\mathcal{Q} = \{\Psi_\beta\circ\gamma_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow G_{\hat{\beta}}\}$, where $\Psi_\beta \colon G_\beta \xrightarrow{\sim} G_{\hat{\beta}}$ is an isomorphism, it follows from [5] that the link invariant $\mathcal{T}_{\mathcal{Q}}$ is the L^2-Alexander torsion. Hence, Theorem 5.1 (2) is a generalization of the main result of [5].

To prove Theorem 5.1 (1), we will use two lemmas, one for each Markov move.

Lemma 5.2. *Let $\mathcal{Q} = \{\psi_\beta\circ\gamma_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \Gamma_{\psi_\beta}\}$ be as in Example 4.8. Then $F_{\mathcal{Q}}$ is invariant under the first Markov move.*

Proof. Let $\mathcal{Q} = \{Q_\beta := \psi_\beta\circ\gamma_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \Gamma_{\psi_\beta}\}$. Let $n\geqslant 1$ be an integer, $t>0$, and $\alpha,\beta\in B_n$. We will prove that

$$\det{}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha^{-1}\beta\alpha) - \mathrm{Id}^{\oplus(n-1)}\right) = \det{}^r_{\Gamma_{\psi_\beta}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right).$$

Observe that for any epimorphism $\gamma\colon \mathbb{F}_n \to G$, Proposition 2.6 implies that:

$$\mathrm{Id}^{\oplus(n-1)} = \overline{\mathcal{B}}^{(2)}_{t,\gamma}(1) = \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha\alpha^{-1}) = \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha^{-1}) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\alpha^{-1}}}(\alpha),$$

thus $\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha^{-1}) = \left(\overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\alpha^{-1}}}(\alpha)\right)^{-1}$.

Consequently, we have for any epimorphism $\gamma\colon \mathbb{F}_n \to G$:

$$\begin{aligned}\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha^{-1}\beta\alpha) &= \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_\alpha}(\beta) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\beta\alpha}}(\alpha^{-1}) \\ &= \overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_\alpha}(\beta) \circ \left(\overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1}.\end{aligned}$$

Hence, for any epimorphism $\gamma\colon \mathbb{F}_n \to G$:

$$\begin{aligned}&{\det}^r_G\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha^{-1}\beta\alpha) - \mathrm{Id}^{\oplus(n-1)}\right) \\ &= {\det}^r_G\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha) \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_\alpha}(\beta) \circ \left(\overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1} - \mathrm{Id}^{\oplus(n-1)}\right) \\ &= {\det}^r_G\left(\overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_\alpha}(\beta) - \left(\overline{\mathcal{B}}^{(2)}_{t,\gamma}(\alpha)\right)^{-1} \circ \overline{\mathcal{B}}^{(2)}_{t,\gamma\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right),\end{aligned}$$

where the second equality follows from Remark 2.7 and Proposition 2.4 (1).

For $\gamma = Q_{\alpha^{-1}\beta\alpha}$, we thus have:

$$\begin{aligned}&{\det}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha^{-1}\beta\alpha) - \mathrm{Id}^{\oplus(n-1)}\right) \\ &= {\det}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}\circ h_\alpha}(\beta) - \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1} \circ \overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right) \\ &= {\det}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,\chi^{\mathcal{Q}}_{\beta,\alpha}\circ Q_\beta}(\beta) - \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1} \circ \overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right).\end{aligned}$$

Now, since for every braid $\sigma \in B_n$, $\gamma_\sigma \circ h_\sigma = \gamma_\sigma$ and $Q_\sigma = \psi_\sigma \circ \gamma_\sigma$, we obtain:

$$\begin{aligned}&{\det}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha^{-1}\beta\alpha) - \mathrm{Id}^{\oplus(n-1)}\right) \\ &= {\det}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,\chi^{\mathcal{Q}}_{\beta,\alpha}\circ Q_\beta}(\beta) - \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1} \circ \overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}\circ h_{\alpha^{-1}\beta\alpha}}(\alpha)\right) \\ &= {\det}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,\chi^{\mathcal{Q}}_{\beta,\alpha}\circ Q_\beta}(\beta) - \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha)\right)^{-1} \circ \overline{\mathcal{B}}^{(2)}_{t,Q_{\alpha^{-1}\beta\alpha}}(\alpha)\right) \\ &= {\det}^r_{\Gamma_{\psi_{\alpha^{-1}\beta\alpha}}}\left(\overline{\mathcal{B}}^{(2)}_{t,\chi^{\mathcal{Q}}_{\beta,\alpha}\circ Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right) \\ &= {\det}^r_{\Gamma_{\psi_\beta}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right),\end{aligned}$$

where the last equality follows from Proposition 2.4 (3). □

Lemma 5.3. *Let $\mathcal{Q} = \{\psi_\beta \circ \gamma_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \Gamma_{\psi_\beta}\}$ be as in Example 4.8. Then $F_{\mathcal{Q}}$ is invariant under the second Markov move.*

Proof. For any braid β let us denote $Q_\beta := \psi_\beta \circ \gamma_\beta \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \Gamma_{\psi_\beta}$.

<u>First step: negative crossing:</u>
Let $n \in \mathbb{N}_{\geqslant 1}$, $\beta \in B_n$ and $\beta_- = \sigma_n^{-1}\iota(\beta) \in B_{n+1}$. Let $t > 0$. We will prove that

$$\frac{{\det}^r_{\Gamma_{\psi_{\beta_-}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-) - \mathrm{Id}^{\oplus n}\right)}{\max(1,t)^{n+1}} = \frac{1}{t} \cdot \frac{{\det}^r_{\Gamma_{\psi_\beta}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right)}{\max(1,t)^n}.$$

In order to do this, we will compose $\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-) - \mathrm{Id}^{\oplus n}$ with two operators (one named $\mathfrak{G}$ on the left, one named $\mathfrak{D}$ on the right) so that we obtain a block triangular operator with the upper left block "mostly" equal to $\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}$.

First, it follows from Proposition 2.6 that

$$\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-) = \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}\left(\sigma_n^{-1}\iota(\beta)\right) = \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\iota(\beta))\, \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ h_{\iota(\beta)}}\left(\sigma_n^{-1}\right).$$

Recall that $\overline{\mathcal{B}}^{(2)}_{t,id}\left(\sigma_n^{-1}\right) = \begin{pmatrix} \mathrm{Id} & & & & 0 \\ & \ddots & & & \vdots \\ & & \mathrm{Id} & & 0 \\ & & & \mathrm{Id} & \mathrm{Id} \\ 0 & \dots & 0 & 0 & -\frac{1}{t}R_{g_{n-1}g_n^{-1}} \end{pmatrix}$, thus the operator defined as $\mathfrak{D} := \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ h_{\iota(\beta)}}\left(\sigma_n^{-1}\right)\right)^{-1}$ is equal to:

$$\mathfrak{D} := \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ h_{\iota(\beta)}}\left(\sigma_n^{-1}\right)\right)^{-1} = \begin{pmatrix} \mathrm{Id} & & & & 0 \\ & \ddots & & & \vdots \\ & & \mathrm{Id} & & 0 \\ & & & \mathrm{Id} & tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_n g_{n-1}^{-1})} \\ 0 & \dots & 0 & 0 & -tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_n g_{n-1}^{-1})} \end{pmatrix}.$$

We therefore compute:

$$\begin{aligned} \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-)\circ\mathfrak{D} &= \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\iota(\beta))\circ \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ h_{\iota(\beta)}}\left(\sigma_n^{-1}\right)\circ\mathfrak{D} \\ &= \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\iota(\beta)) \\ &= \begin{pmatrix} & 0 \\ \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ\iota_{\mathbb{F}_n}}(\beta) & \vdots \\ & 0 \\ \mathcal{R} & \mathrm{Id} \end{pmatrix}, \end{aligned}$$

where the row $\mathcal{R}$ is the right multiplication operator by the row

$$\left(\kappa(t,\Phi_{n+1},Q_{\beta_-})\left(\frac{\partial h_{\iota(\beta)}(g_j)}{\partial g_n}\right)\right) = \left(\kappa(t,\Phi_{n+1}\circ\iota_{\mathbb{F}_n},Q_{\beta_-}\circ\iota_{\mathbb{F}_n})\left(\frac{\partial h_\beta(g_j)}{\partial g_n}\right)\right),$$

that has $n-1$ coefficients in $\mathbb{R}\Gamma_{\psi_{\beta_-}}$.

Now, the fundamental formula of Fox calculus (Proposition 2.1) implies that:

$$\begin{pmatrix} R_{g_1-1} & \dots & R_{g_n-1}\end{pmatrix}\cdot\left(R\frac{\partial h_{\iota(\beta)}(g_j)}{\partial g_i}\right)_{1\leqslant i,j\leqslant n} = \begin{pmatrix} R_{h_{\iota(\beta)}(g_1)-1} & \dots & R_{h_{\iota(\beta)}(g_n)-1}\end{pmatrix},$$

thus, by applying $\kappa(t,\Phi_{n+1},Q_{\beta_-})$ and by definition of $\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\iota(\beta))$, we obtain:

$$\begin{aligned} &\begin{pmatrix} tR_{Q_{\beta_-}(g_1)}-\mathrm{Id} & \dots & t^nR_{Q_{\beta_-}(g_n)}-\mathrm{Id}\end{pmatrix}\cdot\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\iota(\beta)) \\ &= \begin{pmatrix} tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_1)}-\mathrm{Id} & \dots & t^nR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_n)}-\mathrm{Id}\end{pmatrix} \\ &= \begin{pmatrix} tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_1)}-\mathrm{Id} & \dots & t^{n-1}R_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_{n-1})}-\mathrm{Id} & t^nR_{Q_{\beta_-}(g_n)}-\mathrm{Id}\end{pmatrix}, \end{aligned}$$

where the second equality follows from the fact that $\beta \in B_n$ leaves g_n unchanged in the Artin action. Let us thus define the following block triangular operator $\mathfrak{G}$:

$$\mathfrak{G} := \begin{pmatrix} \mathrm{Id} & & & 0 \\ & \ddots & & \vdots \\ & & \mathrm{Id} & 0 \\ tR_{Q_{\beta_-}(g_1)} - \mathrm{Id} & \dots & t^{n-1}R_{Q_{\beta_-}(g_{n-1})} - \mathrm{Id} & t^n R_{Q_{\beta_-}(g_n)} - \mathrm{Id} \end{pmatrix}.$$

It immediately follows from what precedes that

$$\mathfrak{G} \circ \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-)\right) \circ \mathfrak{D} = \mathfrak{G} \circ \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\iota(\beta))\right) =$$

$$\begin{pmatrix} & \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ\iota_{\mathbb{F}_n}}(\beta) & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_1)} - \mathrm{Id} & \dots & t^{n-1}R_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_{n-1})} - \mathrm{Id} & t^n R_{Q_{\beta_-}(g_n)} - \mathrm{Id} \end{pmatrix}.$$

On the other hand, we compute $\mathfrak{G} \circ \left(\mathrm{Id}^{\oplus n}\right) \circ \mathfrak{D} =$

$$\begin{pmatrix} \mathrm{Id} & & & 0 \\ & \ddots & & \vdots \\ & & & 0 \\ & & \mathrm{Id} & tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_n g_{n-1}^{-1})} \\ tR_{Q_{\beta_-}(g_1)} - \mathrm{Id} & \dots & t^{n-1}R_{Q_{\beta_-}(g_{n-1})} - \mathrm{Id} & \star \end{pmatrix},$$

with $\star = t^n R_{Q_{\beta_-}\left(g_n h_{\iota(\beta)}(g_{n-1}^{-1}) g_{n-1}\right)} - t^{n+1} R_{Q_{\beta_-}\left(g_n h_{\iota(\beta)}(g_{n-1}^{-1}) g_n\right)}$.

Hence

$$\mathfrak{G} \circ \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-) - \mathrm{Id}^{\oplus n}\right) \circ \mathfrak{D} =$$

$$\begin{pmatrix} & \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ\iota_{\mathbb{F}_n}}(\beta) - \mathrm{Id}^{\oplus(n-1)} & & \begin{matrix} 0 \\ \vdots \\ 0 \\ -tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_n g_{n-1}^{-1})} \end{matrix} \\ \dots & t^j R_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_j)} - t^j R_{Q_{\beta_-}(g_j)} & \dots & \square \end{pmatrix},$$

where $j \in \{1, \dots, n-1\}$ and

$$\square = t^n R_{Q_{\beta_-}(g_n)} - \mathrm{Id} - t^n R_{Q_{\beta_-}\left(g_n h_{\iota(\beta)}(g_{n-1}^{-1}) g_{n-1}\right)} + t^{n+1} R_{Q_{\beta_-}\left(g_n h_{\iota(\beta)}(g_{n-1}^{-1}) g_n\right)}.$$

Now we use the fact that our epimorphism Q_{β_-} descends deeper than the braid closure group: indeed, for every $j \in \{1, \dots, n-1\}$, we have:

$$Q_{\beta_-}(g_j) = \left(Q_{\beta_-} \circ h_{\beta_-}\right)(g_j) = \left(Q_{\beta_-} \circ h_{\iota(\beta)} \circ h_{\sigma_n^{-1}}\right)(g_j) = \left(Q_{\beta_-} \circ h_{\iota(\beta)}\right)(g_j).$$

Thus $\mathfrak{G} \circ \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-) - \mathrm{Id}^{\oplus n}\right) \circ \mathfrak{D}$ is actually upper block triangular and equal to:

$$\begin{pmatrix} \overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ\iota_{\mathbb{F}_n}}(\beta) - \mathrm{Id}^{\oplus(n-1)} & \begin{matrix} 0 \\ \vdots \\ 0 \\ -tR_{\left(Q_{\beta_-}\circ h_{\iota(\beta)}\right)(g_n g_{n-1}^{-1})} \end{matrix} \\ 0 & -\mathrm{Id} + t^{n+1} R_{Q_{\beta_-}\left(g_n h_{\iota(\beta)}(g_{n-1}^{-1}) g_n\right)} \end{pmatrix}.$$

By applying $\det^r_{\Gamma_{\psi_{\beta_-}}}$ to the previous equality, it therefore follows from Proposition 2.4 and Remark 2.7 that

$$\begin{aligned}&\max(1,t)^n \cdot \det{}^r_{\Gamma_{\psi_{\beta_-}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}}(\beta_-) - \mathrm{Id}^{\oplus n}\right)\cdot t\\ &= \det{}^r_{\Gamma_{\psi_{\beta_-}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_-}\circ \iota_{\mathbb{F}_n}}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right)\cdot \max(1,t)^{n+1}.\end{aligned}$$

We conclude by using the fact that $Q_{\beta_-}\circ\iota_{\mathbb{F}_n} = \sigma^{\mathcal{Q}}_{\beta,-1}\circ Q_\beta$ (by Markov-admissibility of $\mathcal{Q}$) and Proposition 2.4 (3).

Second step: positive crossing:
Let $\beta_+ = \sigma_n\iota(\beta) \in B_{n+1}$. We will proceed almost exactly as in the first step, except for the following differences:

- We now aim to prove that

$$\frac{\det{}^r_{\Gamma_{\psi_{\beta_+}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_+}}(\beta_+) - \mathrm{Id}^{\oplus n}\right)}{\max(1,t)^{n+1}} = \frac{\det{}^r_{\Gamma_{\psi_\beta}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_\beta}(\beta) - \mathrm{Id}^{\oplus(n-1)}\right)}{\max(1,t)^{n}}.$$

- The operator $\mathfrak{D}$ becomes:

$$\mathfrak{D} := \left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_+}\circ h_{\iota(\beta)}}(\sigma_n)\right)^{-1} = \begin{pmatrix}\mathrm{Id} & & & & 0\\ & \ddots & & & \vdots\\ & & \mathrm{Id} & & 0\\ & & & \mathrm{Id} & \mathrm{Id}\\ 0 & \dots & 0 & 0 & -\frac{1}{t}R_{Q_{\beta_+}(g_n g_{n+1}^{-1})}\end{pmatrix}.$$

- The final lower right coefficient $\star$ of $\mathfrak{G}\circ\mathfrak{D}$ becomes

$$\star = t^{n-1}R_{Q_{\beta_+}(g_{n-1})} - \mathrm{Id} - t^{n-1}R_{Q_{\beta_+}\left(g_n g_{n+1}^{-1} g_n\right)} + \frac{1}{t}R_{Q_{\beta_+}\left(g_n g_{n+1}^{-1}\right)}.$$

- The final lower right coefficient $\square$ of $\mathfrak{G}\circ\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_+}}(\beta_+) - \mathrm{Id}^{\oplus n}\right)\circ\mathfrak{D}$ becomes

$$\square = -t^{n-1}R_{Q_{\beta_+}(g_{n-1})} + t^n R_{Q_{\beta_+}(g_n)} + t^{n-1}R_{Q_{\beta_+}\left(g_n g_{n+1}^{-1} g_n\right)} - \frac{1}{t}R_{Q_{\beta_+}\left(g_n g_{n+1}^{-1}\right)}.$$

- The simplification

$$\square = t^n R_{Q_{\beta_+}(g_n)} - \frac{1}{t}R_{Q_{\beta_+}\left(g_n g_{n+1}^{-1}\right)}$$

  comes from the fact that in the ring $\mathbb{Z}\mathbb{F}_{n+1}$ we have the equalities:

$$\begin{aligned} g_{n-1} - g_n g_{n+1}^{-1} g_n &= g_n\left(g_n^{-1} - g_{n+1}^{-1} g_n g_{n-1}^{-1}\right) g_{n-1}\\ &= g_n\left(g_n^{-1} - h_{\sigma_n}^{-1}\left(g_n^{-1}\right)\right) g_{n-1}\\ &= g_n\left(g_n^{-1} - h_{\sigma_n}^{-1}\left(h_{\iota(\beta)}^{-1}(g_n^{-1})\right)\right) g_{n-1}\\ &= g_n\left(g_n^{-1} - h_{\beta_+}^{-1}\left(g_n^{-1}\right)\right) g_{n-1}\\ &= g_n\left(h_{\beta_+}\left(h_{\beta_+}^{-1}\left(g_n^{-1}\right)\right) - h_{\beta_+}^{-1}\left(g_n^{-1}\right)\right) g_{n-1}.\end{aligned}$$

  Hence, by composing with $\gamma_{\beta_+}: \mathbb{Z}[\mathbb{F}^n] \to \mathbb{Z}[G_{\beta_+}]$, the quotient by all relations $h_{\beta_+}(*) = *$, we get $\gamma_{\beta_+}\left(g_{n-1} - g_n g_{n+1}^{-1} g_n\right) = 0$. The previous equality to 0 remains true through any deeper epimorphism $Q_{\beta_+} = \psi_{\beta_+}\circ\gamma_{\beta_+}$.

- Applying $\det^r_{\Gamma_{\psi_{\beta_+}}}$ to the final equality now yields (as expected):

$$\begin{aligned}
&\max(1,t)^n \cdot \det{}^r_{\Gamma_{\psi_{\beta_+}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_+}}(\beta_+) - \mathrm{Id}^{\oplus n}\right)\cdot \frac{1}{t}\\
&= \det{}^r_{\Gamma_{\psi_{\beta_+}}}\left(\overline{\mathcal{B}}^{(2)}_{t,Q_{\beta_+}\circ \iota_{\mathbb{F}_n}}(\beta) - \mathrm{Id}^{\oplus (n-1)}\right)\cdot \frac{1}{t}\max(1,t)^{n+1}.
\end{aligned}$$

□

We now have all the tools to prove Theorem 5.1.

Proof of Theorem 5.1. Part (1) follows immediately from Lemmas 5.2 and 5.3. The last part follows immediately from parts (1) and (2). Let us prove part (2).

Let L be a link in S^3, of exterior $M_L = S^3 \setminus \nu L$. Let $n \geqslant 1$ and $\beta \in B_n$ such that $L = \hat{\beta}$. We will mostly follow the way of the proof of [5, Theorem 4.9], except for the fact that ψ_β is now a general epimorphism and not the identity. This is why we need the generalized properties of Section 3. We will skip over some details already covered in [5].

Recall that

$$P = \langle g_1, \ldots, g_n | r_1 := h_\beta(g_1)g_1^{-1}, \ldots, r_{n-1} := h_\beta(g_{n-1})g_{n-1}^{-1}\rangle$$

is a presentation of G_β, and also of G_L, through the isomorphism $\Psi_\beta : G_\beta \xrightarrow{\sim} G_L$. Let W_P be the 2-dimensional CW-complex constructed from P. Recall that W_P has a single 0-cell, one 1-cell for each generator of P, and one 2-cell for each relator of P, each 2-cell being glued on the wedge of circles that is the 1-skeleton following the word in the generators formed by the relator in question.

Now denote $L' = L \cup C_\beta$, where C_β is the boundary circle of D_n not coming from one of the punctures, when drawing L as the closure of β. Then

$$P' = \langle g_1, \ldots, g_n, y | r_1' := h_\beta(g_1)yg_1^{-1}y^{-1}, \ldots, r_n' := h_\beta(g_n)yg_n^{-1}y^{-1}\rangle$$

is a presentation of $G_{L'}$, with y a meridian of C_β. Let $W_{P'}$ be the 2-dimensional CW-complex constructed from P'. Since L' is not split, $W_{P'}$ and $M_{L'}$ are therefore $K(G_{L'}, 1)$ spaces, and since the Whitehead group of $G_{L'}$ is trivial, we have that $W_{P'}$ is simple homotopy equivalent to $M_{L'}$.

To simplify notations, let us denote $G := \Gamma_{\psi_\beta}$, $\psi := \psi_\beta \circ (\Psi_\beta)^{-1} : G_L \twoheadrightarrow G$ and $\phi_L := (1, \ldots, 1) \circ \alpha_L : G_L \twoheadrightarrow \mathbb{Z}$. Let $t > 0$. Let $Q\colon \pi_1(M_{L'}) \twoheadrightarrow \pi_1(M_L)$ denote the group epimorphism induced by removing the component C_β. Let us also denote as follows the five finite Hilbert $\mathcal{N}(G)$-chain complexes we will use in the proof:

- $E_* := C_*^{(2)}(M_L, \phi_L, \psi, t)$,
- $E_*' := C_*^{(2)}(M_{L'}, \phi_L \circ Q, \psi \circ Q, t)$,
- $W_* := C_*^{(2)}(W_P, \phi_L, \psi, t) =$

$$\bigoplus_{j=1}^{n-1} \ell^2(G)\widetilde{r_j} \xrightarrow[\begin{pmatrix}\overline{\mathcal{B}}^{(2)}_{t,\psi}(\beta) - \mathrm{Id}^{n-1}\\ * \end{pmatrix}]{\partial_2} \bigoplus_{i=1}^{n} \ell^2(G)\widetilde{g_i} \xrightarrow[R_{(t\psi(g_1)-1,\ldots,t^n\psi(g_n)-1)}]{\partial_1} \ell^2(G),$$

- $W_*' := C_*^{(2)}(W_{P'}, \phi_L \circ Q, \psi \circ Q, t) =$

$$\bigoplus_{j=1}^{n} \ell^2(G)\widetilde{r_j'} \xrightarrow[\begin{pmatrix}\overline{\mathcal{B}}^{(2)}_{t,\psi}(\beta) - \mathrm{Id}^{n-1} & 0\\ * & 0\\ tR_{\psi(h_\beta(g_1))} - \mathrm{Id}\ \ldots & t^nR_{\psi(g_n)} - \mathrm{Id}\end{pmatrix}]{\partial_2} \bigoplus_{i=1}^{n} \ell^2(G)\widetilde{g_i} \oplus \ell^2(G)\widetilde{y}$$

$$\xrightarrow[R_{(t\psi(g_1)-1,\ldots,t^n\psi(g_n)-1,0)}]{\partial_1} \ell^2(G),$$

- $D_* := 0 \to \ell^2(G)\widetilde{r'_n} \xrightarrow[\mathrm{Id}-t^n R_{\psi(g_n)}]{\partial_2} \ell^2(G)\widetilde{y} \xrightarrow[0]{\partial_1} 0 \to 0.$

The forms of W_* and W'_* follow from the fact that we can use Fox calculus to describe the boundary operators in celullar chain complexes associated to the universal covers $\widetilde{W_P}$ and $\widetilde{W'_P}$.

Observe that we have $T^{(2)}(W_*) = F_{\mathcal{Q}}(\beta)(t)$. Moreover, we have $T^{(2)}(E_*) = T^{(2)}_{L,(1,\ldots,1)}(\psi)(t)$ by definition. Thus we only need to prove that $T^{(2)}(W_*) = T^{(2)}(E_*)$. Let us state the two following facts.

<u>Fact 1:</u> For λ the class of a preferred longitude of C_β in $G_{L'}$, $\psi(Q(\lambda))$ has infinite order in G. This is a consequence of ϕ_L factoring through ψ and $\phi(Q(\lambda)) = n \neq 0$.

<u>Fact 2:</u> There exists

$$0 \to W_* \xrightarrow{\eta_*} W'_* \xrightarrow{\rho_*} D_* \to 0,$$

a short exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes, with η_2, η_1 the obvious inclusions, $\eta_0 = 0$, ρ_2, ρ_1 the obvious projections, and $\rho_0 = \mathrm{Id}$. This Fact follows from comparing the cells of W_P and $W_{P'}$, and from the definitions of L^2-Burau maps with Fox calculus.

We can now establish the following equalities of L^2-torsions:

$$T^{(2)}(E_*) \doteq \frac{T^{(2)}(E'_*)}{\max(1,t)^n} \doteq \frac{T^{(2)}(W'_*)}{\max(1,t)^n} \doteq T^{(2)}(W_*),$$

where the first equality follows from Proposition 3.4 and Fact 1, the second one follows from Proposition 3.2, and the third one follows from Proposition 3.1, Fact 2 and Proposition 2.4 (5) and (8).

Observe that if any one of E_*, E'_*, W'_*, W_* is not weakly acyclic (resp. is weakly acyclic but not of determinant class), then no one is (resp. they all are), and in this case their L^2-torsions are all 0 (and the previous equalities still stand). ☐

Remark 5.4. In the previous proof, the exact sequence in Fact 2 is reversed from the one in the proof of [5, Theorem 4.9], and the latter is incorrect. Our proof of Theorem 5.1 (2), which generalizes the one of [5, Theorem 4.9], can thus be considered as an erratum of this mistake.

Remark 5.5. The proof of Theorem 5.1 (2) can be shortened if L is non split, directly by using the simple homotopy equivalence between M_L and W_P, like in the proof of [5, Theorem 4.9].

6. Counter-examples to Markov invariance

Theorem 5.1 (1) established that $F_{\mathcal{Q}}$ is a Markov function when the Markov-admissible family of epimorphisms $\mathcal{Q}$ descends to the link groups or lower. It is now natural to ask if those families $\mathcal{Q}$ are the only ones for which $F_{\mathcal{Q}}$ is a Markov function. We do not have an answer to this question at the time of writing.

However, by looking at the details of the proof of Theorem 5.1 (more precisely those of Lemmas 5.2 and 5.3), it appears that applying the epimorphism γ_β (or a strictly deeper epimorphism) is necessary in order to obtain the cancellations in the matrices that yield Markov invariance for $F_{\mathcal{Q}}$. As additional evidence for this hypothesis, we discovered two families $\mathcal{Q}$ such that $F_{\mathcal{Q}}$ is not a Markov function: one family (the identities of the free group) lives strictly higher than the γ_β, and the other one (the abelianizations of the free groups) is not comparable to the γ_β (Figure 4 can help visualizing this). See the following Theorems 6.1 and 6.2.

These two counter-examples illustrate some of the difficulties in computing general Fuglede-Kadison determinants, and some transversal techniques one might need to use in order to do so (techniques such as computation of Mahler measures of polynomials, or combinatorics of closed paths on Cayley graphs). Notably, the proof of Theorem 6.2 uses the new values for Fuglede-Kadison determinants over free groups computed in Theorem 7.3 via combinatorial and analytical techniques. These new values have separate interesting consequences which are listed in Section 8.

Of course, appeareances might still deceive, and it could happen that unexpected identities of Fuglede-Kadison determinants occur even with families of epimorphisms not encompassed by Theorem 5.1, especially since computing such determinants remains a daunting task today.

This section is split into two parts, which cover Theorems 6.1 and 6.2 respectively.

6.1. **Descending to the free abelian group.** In the following theorem, we prove that when the family $\mathcal{Q}$ descends to the free abelian groups, $F_{\mathcal{Q}}$ is not a Markov function and thus cannot yield link invariants. The proof uses properties of the Fuglede-Kadison determinant and a value of the Mahler measure due to Boyd [6].

Theorem 6.1. *For the family of abelianizations $\mathcal{Q} = \left\{\varphi_{n(\beta)} \colon \mathbb{F}_{n(\beta)} \twoheadrightarrow \mathbb{Z}^{n(\beta)}\right\}$, the value at $t = 1$ of the function $F_{\mathcal{Q}}$ is not invariant under Markov moves. In particular $F_{\mathcal{Q}}$ is not a Markov function.*

Proof. Let $t > 0,\ n = 2,\ \beta = \sigma_1^{-1} \in B_2$, and $\beta_+ = \sigma_1^{-1}\sigma_2 \in B_3$. Following Section 2.1, we compute that β_+ acts on $\mathbb{F}_3$ by

$$\begin{cases} g_1 \\ g_2 \end{cases} \overset{h_{\beta_+}}{\mapsto} \begin{cases} g_1^{-1} g_2 \\ g_3 g_2^{-1} g_1^{-1} g_2 \end{cases},$$

where g_1, g_2, g_3 denote the generators as in Section 2.1. Thus Fox calculus gives:

$$\overline{\mathcal{B}}^{(2)}_{t,id}(\beta_+) - \mathrm{Id}^{\oplus 2} = \begin{pmatrix} -\frac{1}{t} R_{g_1^{-1}} - \mathrm{Id} & -R_{g_3 g_2^{-1} g_1^{-1}} \\ \frac{1}{t} R_{g_1^{-1}} & -\mathrm{Id} - t R_{g_3 g_2^{-1}} + R_{g_3 g_2^{-1} g_1^{-1}} \end{pmatrix}.$$

Hence, from Remark 2.5 and Proposition 2.4 (6), we have:

$$\begin{aligned}
&\det{}_{\mathbb{F}_3}\left(\overline{\mathcal{B}}^{(2)}_{t,id}(\beta_+) - \mathrm{Id}^{\oplus 2}\right) \\
&= \det{}_{\mathbb{F}_3}\left(\left(-\mathrm{Id} - tR_{g_3 g_2^{-1}} + R_{g_3 g_2^{-1} g_1^{-1}}\right)\left(-R_{g_1 g_2 g_3^{-1}}\right)\left(-\frac{1}{t} R_{g_1^{-1}} - \mathrm{Id}\right) - \frac{1}{t} R_{g_1^{-1}}\right) \\
&= \det{}_{\mathbb{F}_3}\left(-\left(tR_{g_1} + R_{g_1 g_2 g_3^{-1}} + \frac{1}{t} R_{g_2 g_3^{-1}}\right)\right) \\
&= \frac{1}{t}\det{}_{\mathbb{F}_3}\left(\mathrm{Id} + tR_{g_1} + t^2 R_{g_1 g_3 g_2^{-1}}\right).
\end{aligned}$$

Let z_1, z_2, z_3 denote the canonical generators of $\mathbb{Z}^3$. Then it follows from the same arguments as in the previous computation that:

$$\det{}_{\mathbb{F}_3}\left(\overline{\mathcal{B}}^{(2)}_{t,\varphi_3}(\beta_+) - \mathrm{Id}^{\oplus 2}\right) = \frac{1}{t}\det{}_{\mathbb{Z}^3}\left(\mathrm{Id} + tR_{z_1} + t^2 R_{z_1 z_3}\right).$$

Let H be the subgroup of $\mathbb{Z}^3$ isomorphic to $\mathbb{Z}^2$ and generated by $\lambda = z_1$ and $\mu = z_1 z_3$. Recall that $F_{\mathcal{Q}}(\beta_+)$ is an equivalence class of functions of $t > 0$ up to multiplication by a monomial, thus the value at $t = 1$ is always the same regardless of the representant of the equivalence class. Let us denote this value $F_{\mathcal{Q}}(\beta_+)(1)$. We then have:

$$\begin{aligned}
F_{\mathcal{Q}}(\beta_+)(1) &= \det{}_{\mathbb{Z}^3}\left(\mathrm{Id} + R_{z_1} + R_{z_1 z_3}\right) \\
&= \det{}_H\left(\mathrm{Id} + R_\lambda + R_\mu\right) \\
&= \mathcal{M}(1 + X + Y) = 1.38135... \neq 1.
\end{aligned}$$

where $\mathcal{M}$ is the Mahler measure, the second equality follows from Proposition 2.4 (3), the third one from Proposition 2.4 (7) and the fourth one from Example 2.3.

Now, since by Proposition 2.4 (5) we have:

$$F_{\mathcal{Q}}(\beta)(1) = \det{}_{\mathbb{Z}}\left(-R_{\varphi_2(g_1^{-1}g_2)} - \mathrm{Id}\right) = 1,$$

we conclude that $\beta \mapsto F_{\mathcal{Q}}(\beta)(1)$ is not invariant under Markov moves. □

6.2. **Remaining at the free group.** We cannot expect Markov invariance by remaining at the level of the free group either, as the following theorem shows:

Theorem 6.2. *For the family of identities $\mathcal{Q} = \{\mathrm{id}_{\mathbb{F}_{n(\beta)}}\}$, the function $F_{\mathcal{Q}}$ is not invariant under Markov moves.*

The proof will use part of Theorem 7.3, which will be proved in the next section.

Proof. Let us take $t = 1,\ n = 2,\ \beta = \sigma_1^{-1} \in B_2,\ \beta_+ = \sigma_1^{-1}\sigma_2 \in B_3$. Then, as in the proof of Theorem 6.1, we obtain

$$\begin{aligned} F_{\mathcal{Q}}(\beta_+)(1) &= \det{}_{\mathbb{F}_3}\left(\mathcal{B}^{(2)}_{1,id}(\beta_+) - \mathrm{Id}^{\oplus 2}\right) \\ &= \det{}_{\mathbb{F}_3}\left(\mathrm{Id} + R_{g_1} + R_{g_1 g_3 g_2^{-1}}\right) \\ &= \det{}_F\left(\mathrm{Id} + R_x + R_y\right), \end{aligned}$$

where F is the free group on two generators x, y that embeds in $\mathbb{F}_3$ via $x \mapsto g_1, y \mapsto g_1 g_3 g_2^{-1}$, and the last equality follows from Proposition 2.4 (3). Now, since

$$F_{\mathcal{Q}}(\beta)(1) = \det{}_{\mathbb{F}_2}\left(-R_{g_1^{-1}g_2} - \mathrm{Id}\right) = 1,$$

it remains to prove that $\det_F(\mathrm{Id} + R_x + R_y) \neq 1$. This follows from Theorem 7.3, which establishes that

$$\det{}_F\left(\mathrm{Id} + R_x + R_y\right) = \frac{2}{\sqrt{3}} \neq 1.$$

□

7. Fuglede-Kadison determinants over free groups

In this section, we present a general method to compute Fuglede-Kadison determinants via counting paths on Cayley graphs, and we apply this method to symmetric operators over free groups (studied by Bartholdi and Dasbach-Lalin).

7.1. **Computing Fuglede-Kadison determinants from Cayley graphs.** The following lemma gives a method to compute the Fuglede-Kadison determinant $\det_G(A)$ using the generating series $u_{A^*A}(t)$ associated to the number of closed paths on a Cayley graph associated to G and A^*A. This method was studied by Dasbach-Lalin in [8] and by Lück in [13, Section 3.7] with slight differences in notation, and we provide a complete proof here for the reader's convenience.

Lemma 7.1. *Let G be a finitely presented group and let $A \in R_{\mathbb{C}G}$ denote an injective right multiplication operator on $\ell^2(G)$ by a non-zero element of the group algebra. Then for any $\lambda \in \left(0, \|A\|^{-2}\right)$, we have:*

$$\begin{aligned} \det{}_G(A) &= \lim_{\varepsilon \to 0^+} \frac{1}{\sqrt{\lambda}} \exp\left(-\frac{1}{2}\int_0^1 \frac{w_{\lambda,\varepsilon}(t) - 1}{t} dt\right) \\ &= \lim_{\varepsilon \to 0^+} \frac{1}{\sqrt{\lambda}} \exp\left(-\frac{1}{2}\int_0^1 \frac{\frac{1}{1-(1-\lambda\varepsilon)t}\, u_{A^*A}\left(\frac{-\lambda t}{1-(1-\lambda\varepsilon)t}\right) - 1}{t} dt\right), \end{aligned}$$

where

$$u_{A^*A}(t) := \sum_{k=0}^{\infty} \mathrm{tr}_G\left((A^*A)^k\right) t^k$$

is a well-defined power series for t small enough, and

$$w_{\lambda,\varepsilon}(t) = \sum_{n=0}^{\infty} \mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id} - \lambda A^*A)^n\right) t^n = \frac{1}{1-(1-\lambda\varepsilon)t}\, u_{A^*A}\left(\frac{-\lambda t}{1-(1-\lambda\varepsilon)t}\right)$$

is a well-defined power series for ε small enough and $|t| < 1$.

Proof. First equality: Let $\lambda \in \left(0, \|A\|^{-2}\right)$. Since A is injective, Proposition 2.4 (8) implies that

$$\det{}_G(A) = \lim_{\varepsilon\to 0^+} \sqrt{\det{}_G(A^*A + \varepsilon\mathrm{Id})}.$$

Let $\varepsilon > 0$ such that $\lambda < \dfrac{1}{\|A\|^2+\varepsilon} < \dfrac{1}{\|A\|^2}$. Since $A^*A + \varepsilon\mathrm{Id}$ is positive, we obtain:

$$\begin{aligned}
\det{}_G(A^*A + \varepsilon\mathrm{Id}) &= (\exp\circ\mathrm{tr}_G\circ\ln)\left(A^*A + \varepsilon\mathrm{Id}\right)\\
&= \frac{1}{\lambda}\left(\exp\circ\mathrm{tr}_G\circ\ln\right)\left(\lambda A^*A + \lambda\varepsilon\mathrm{Id}\right)\\
&= \frac{1}{\lambda}\left(\exp\circ\mathrm{tr}_G\circ\ln\right)\left(\mathrm{Id} - (\mathrm{Id} - (\lambda A^*A + \lambda\varepsilon\mathrm{Id}))\right)\\
&= \frac{1}{\lambda}\left(\exp\circ\mathrm{tr}_G\right)\left(-\sum_{n=1}^{\infty}\frac{1}{n}(\mathrm{Id} - (\lambda A^*A + \lambda\varepsilon\mathrm{Id}))^n\right)\\
&= \frac{1}{\lambda}\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id} - \lambda A^*A)^n\right)\right),
\end{aligned}$$

where the first equality follows from Proposition 2.4 (4), the fourth one from holomorphic functional calculus and the fact that the spectrum of the positive operator $\lambda A^*A + \lambda\varepsilon\mathrm{Id}$ is inside $(0,1)$ (since $\lambda < \dfrac{1}{\|A\|^2+\varepsilon}$).

Now, since the series $\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id} - \lambda A^*A)^n\right)$ converges, then

$$w_{\lambda,\varepsilon}(t) := \sum_{n=0}^{\infty} \mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id} - \lambda A^*A)^n\right) t^n$$

is a well-defined power series for $|t| < 1$, and moreover that for all $T \in (0,1)$,

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id} - \lambda A^*A)^n\right) T^n = \int_0^T \frac{w_{\lambda,\varepsilon}(t)-1}{t}dt.$$

Finally, once again since $\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id} - \lambda A^*A)^n\right)$ converges, we can apply Abel's theorem and make $T \to 1^-$ in the previous equality. Hence:

$$\begin{aligned}
\det{}_G(A) &= \lim_{\varepsilon\to 0^+}\sqrt{\det{}_G(A^*A + \varepsilon\mathrm{Id})}\\
&= \lim_{\varepsilon\to 0^+}\sqrt{\frac{1}{\lambda}\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id} - \lambda A^*A)^n\right)\right)}\\
&= \lim_{\varepsilon\to 0^+}\sqrt{\frac{1}{\lambda}\exp\left(-\int_0^1\frac{w_{\lambda,\varepsilon}(t)-1}{t}dt\right)},
\end{aligned}$$

and the first equality follows.

Second equality: Now we compute, for $\varepsilon > 0$ small enough and $t \in [0,1)$:

$$\begin{aligned}
w_{\lambda,\varepsilon}(t) &= \sum_{n=0}^{\infty} \mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id} - \lambda A^*A)^n\right) t^n \\
&= \sum_{n=0}^{\infty} \mathrm{tr}_G\left(\sum_{k=0}^{n}\binom{n}{k}(1-\lambda\varepsilon)^{n-k}(-\lambda)^k (A^*A)^k\right) t^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(1-\lambda\varepsilon)^{n-k}(-\lambda)^k \mathrm{tr}_G\left((A^*A)^k\right) t^n \\
&= \sum_{k=0}^{\infty}\left(\frac{-\lambda}{1-\lambda\varepsilon}\right)^k \mathrm{tr}_G\left((A^*A)^k\right)\sum_{n=k}^{\infty}\binom{n}{k}(1-\lambda\varepsilon)^n t^n \\
&= \sum_{k=0}^{\infty}\left(\frac{-\lambda}{1-\lambda\varepsilon}\right)^k \mathrm{tr}_G\left((A^*A)^k\right)\frac{((1-\lambda\varepsilon)t)^k}{(1-(1-\lambda\varepsilon)t)^{k+1}} \\
&= \frac{1}{1-(1-\lambda\varepsilon)t}\sum_{k=0}^{\infty}\left(\frac{-\lambda t}{1-(1-\lambda\varepsilon)t}\right)^k \mathrm{tr}_G\left((A^*A)^k\right),
\end{aligned}$$

where the fifth equality follows from the binomial formula. Thus, by denoting $u(t) := \sum_{k=0}^{\infty}\mathrm{tr}_G\left((A^*A)^k\right)t^k$, we have

$$w_{\lambda,\varepsilon}(t) = \frac{1}{1-(1-\lambda\varepsilon)t}\, u\left(\frac{-\lambda t}{1-(1-\lambda\varepsilon)t}\right)$$

and the second equality follows. ☐

7.2. **The non-cyclic free groups.** The following proposition follows from works of Bartholdi and Dasbach-Lalin [2, 8] on counting paths on regular trees.

Proposition 7.2. *Let $d \geqslant 3$, let $x_1, \ldots, x_{d-1}$ be $d-1$ generators of the free group $\mathbb{F}_{d-1}$, and let $\zeta_1, \ldots, \zeta_{d-1} \in \mathbb{C}$ such that $|\zeta_1| = \ldots = |\zeta_{d-1}| = 1$.*

For $G = \mathbb{F}_{d-1}$, $A = \mathrm{Id} + \zeta_1 R_{x_1} + \ldots + \zeta_{d-1}R_{x_{d-1}}$, and t small enough, the following generating series is equal to:

$$u_{A^*A}(t) = \sum_{k=0}^{\infty}\mathrm{tr}_G\left((A^*A)^k\right)t^k = \frac{2d-2}{d-2+d\sqrt{1-4(d-1)t}}.$$

Proof. The case where $\zeta_1 = \ldots = \zeta_{d-1} = 1$ is proven in [2, 8]. The main idea is the fact that $\mathrm{tr}_G\left((A^*A)^k\right)$ counts the number of closed paths on the Cayley graph associated to G and A^*A, which is a d-regular tree in the present case. The generating series $u_{A^*A}(t)$ is then determined as the solution of a functional equation, which is found by listing every possible form of a closed path on the graph.

Let us consider the general case. For $k \in \mathbb{N}$, the coefficient $\mathrm{tr}_G\left((A^*A)^k\right)$ is equal to the sum of all terms $\zeta_{j_1}\zeta_{j_2}^* \ldots \zeta_{j_{2k-1}}\zeta_{j_{2k}}^*$ (where $j_1, \ldots, j_{2k} \in \{0, 1, \ldots, d-1\}$ and $\zeta_0 := 1$) such that $x_{j_1}x_{j_2}^{-1}\ldots x_{j_{2k-1}}x_{j_{2k}}^{-1} \in \mathbb{F}_{d-1}$ is trivial (with the convention $x_0 := 1$). Now, if $x_{j_1}x_{j_2}^{-1}\ldots x_{j_{2k-1}}x_{j_{2k}}^{-1}$ is trivial, then each non-zero index j_i is paired with another index $j_{i'}$ with $i' \neq i, j_{i'} = j_i$. Hence the associated term $\zeta_{j_1}\zeta_{j_2}^* \ldots \zeta_{j_{2k-1}}\zeta_{j_{2k}}^*$ is equal to 1, like in the first case, and the result follows similarly. ☐

In the following theorem, we compute Fuglede-Kadison determinants of basic operators on the free groups, using the explicit generating series of Proposition 7.2.

Theorem 7.3. *Let $d \geqslant 3$, and let $x_1, \ldots, x_{d-1}$ be $d-1$ generators of the free group $\mathbb{F}_{d-1}$. Let $\zeta_1, \ldots, \zeta_{d-1} \in \mathbb{C}$ such that $|\zeta_1| = \ldots = |\zeta_{d-1}| = 1$. Then we have:*

$$\det{}_{\mathbb{F}_{d-1}}(\mathrm{Id} + \zeta_1 R_{x_1} + \ldots + \zeta_{d-1}R_{x_{d-1}}) = \frac{(d-1)^{\frac{d-1}{2}}}{d^{\frac{d-2}{2}}}.$$

In particular, for any two generators x, y of the free group $\mathbb{F}_2$, we have:

$$\det{}_{\mathbb{F}_2}(\mathrm{Id} + R_x + R_y) = \frac{2}{\sqrt{3}} = 1.15...$$

Proof. Let $d \geqslant 3$ and $\zeta_1, \ldots, \zeta_{d-1}$ in the unit circle. First we use Lemma 7.1, by taking $G = \mathbb{F}_{d-1}$, $A = \mathrm{Id} + \zeta_1 R_{x_1} + \ldots + \zeta_{d-1} R_{x_{d-1}}$, $\lambda \in (0, \frac{1}{d^2})$, and we obtain:

$$\begin{aligned}\det{}_{\mathbb{F}_{d-1}}(\mathrm{Id} + \zeta_1 R_{x_1} + \ldots + \zeta_{d-1} R_{x_{d-1}}) &= \det{}_G(A) \\ &= \lim_{\varepsilon \to 0^+} \frac{1}{\sqrt{\lambda}} \exp\left(-\frac{1}{2}\int_0^1 \frac{w_{\lambda,\varepsilon}(t) - 1}{t} dt\right),\end{aligned}$$

where

$$w_{\lambda,\varepsilon}(t) = \sum_{n=0}^{\infty} \mathrm{tr}_G\left(((1-\lambda\varepsilon)\mathrm{Id} - \lambda A^* A)^n\right) t^n$$

for $\varepsilon > 0$ small enough and $t \in [0, 1)$.

From Lemma 7.1, by denoting $u_{A^*A}(t) := \sum_{k=0}^{\infty} \mathrm{tr}_G\left((A^*A)^k\right) t^k$, we have

$$w_{\lambda,\varepsilon}(t) = \frac{1}{1 - (1-\lambda\varepsilon)t}\; u_{A^*A}\left(\frac{-\lambda t}{1 - (1-\lambda\varepsilon)t}\right).$$

Since it suffices to prove that

$$\lim_{\varepsilon \to 0^+} \int_0^1 \frac{w_{\lambda,\varepsilon}(t) - 1}{t} dt = \ln\left(\frac{d^{d-2}}{(d-1)^{d-1}\lambda}\right),$$

let us denote $I_{\lambda,\varepsilon} := \int_0^1 \frac{w_{\lambda,\varepsilon}(t)-1}{t} dt$ and prove that $I_{\lambda,\varepsilon} \underset{\varepsilon \to 0^+}{\to} \ln\left(\frac{d^{d-2}}{(d-1)^{d-1}\lambda}\right)$.

It follows from Proposition 7.2 that

$$u_{A^*A}(t) = \frac{2d-2}{d - 2 + d\sqrt{1 - 4(d-1)t}},$$

thus

$$I_{\lambda,\varepsilon} = \int_0^1 \frac{\frac{1}{1-(1-\lambda\varepsilon)t}\; \frac{2d-2}{d-2+d\sqrt{1+\frac{4(d-1)\lambda t}{1-(1-\lambda\varepsilon)t}}} - 1}{t} dt.$$

With the change of variables $x = (1-\lambda\varepsilon)t$ and by denoting $a = \frac{4(d-1)\lambda}{1-\lambda\varepsilon}$, we find $I_{\lambda,\varepsilon} = \int_0^{1-\lambda\varepsilon}\left(2(d-1)R(x) - \frac{1}{x}\right) dx$, where

$$R(x) := \frac{1}{x(1-x)\left(d - 2 + d\sqrt{1 + \frac{ax}{1-x}}\right)} = \frac{d\sqrt{1 + \frac{ax}{1-x}} - (d-2)}{x\left((d^2 a - 4d + 4)x + 4d - 4\right)}.$$

To find an antiderivative of $R(x)$, we first split it into a sum of partial fractions:

$$R(x) = \frac{1}{4(d-1)}\left(-\frac{d-2}{x} + \frac{d-2}{x + \frac{4d-4}{d^2a-4d+4}} + d\frac{\sqrt{1+\frac{ax}{1-x}}}{x} - d\frac{\sqrt{1+\frac{ax}{1-x}}}{x + \frac{4d-4}{d^2a-4d+4}}\right).$$

For any generic constant C, an antiderivative of $x \mapsto \dfrac{\sqrt{1+\frac{ax}{1-x}}}{x + \frac{4C-4}{C^2a-4C+4}}$ is given by

$$x \mapsto 2\sqrt{1-a}\cdot\mathrm{arsinh}\left(\sqrt{\frac{(1-x)(1-a)}{a}}\right) - 2\frac{C-2}{C}\,\mathrm{artanh}\left(\frac{C-2}{C}\sqrt{\frac{1-x}{ax-x+1}}\right),$$

hence the cases $C=1$ and $C=d$ provide the antiderivatives of the third and fourth terms in the previous expression of $R(x)$. The arsinh terms cancel, and we find the following antiderivative $F(x)$ for the function $\left(2(d-1)R(x)-\frac{1}{x}\right)$:

$$\begin{aligned}F(x)=&-\frac{d}{2}\ln(x)+\frac{d-2}{2}\ln\left|x+\frac{4d-4}{d^2a-4d+4}\right|\\&-d\,\mathrm{artanh}\left(\sqrt{\frac{1-x}{ax-x+1}}\right)+(d-2)\,\mathrm{artanh}\left(\frac{d-2}{d}\sqrt{\frac{1-x}{ax-x+1}}\right).\end{aligned}$$

From what precedes, we therefore have $I_{\lambda,\varepsilon}=F(1-\lambda\varepsilon)-F(0)$.

Recall that a,R,F all depend on ε, although it is not apparent in the notation. Since $\lim_{\varepsilon\to 0^+}(a)=4(d-1)\lambda$, we can compute

$$\lim_{\varepsilon\to 0^+}F(1-\lambda\varepsilon)=\frac{d-2}{2}\ln\left|\frac{d^2 4(d-1)\lambda}{d^2 4(d-1)\lambda-4d+4}\right|.$$

To compute $F(0)$, we first need to remark that

$$\begin{aligned}\mathrm{artanh}\left(\sqrt{\frac{1-x}{ax-x+1}}\right)&=\frac{1}{2}\ln\left(\frac{1+\sqrt{\frac{1-x}{ax-x+1}}}{1-\sqrt{\frac{1-x}{ax-x+1}}}\right)\\&=\ln\left(1+\sqrt{\frac{1-x}{ax-x+1}}\right)-\frac{1}{2}\ln\left(1-\frac{1-x}{ax-x+1}\right)\\&=\ln\left(1+\sqrt{\frac{1-x}{ax-x+1}}\right)+\frac{1}{2}\ln\left(ax-x+1\right)-\frac{1}{2}\ln\left(ax\right).\end{aligned}$$

The only terms in $F(x)$ that diverge in $x=0$ are $-\frac{d}{2}\ln(x)$ and $(-d)\left(-\frac{1}{2}\ln(ax)\right)$, which cancel (leaving the term $\frac{d}{2}\ln(a)$). Hence we have:

$$F(0)=\frac{d-2}{2}\ln\left|\frac{4d-4}{d^2a-4d+4}\right|-d\ln(2)+\frac{d}{2}\ln(a)+(d-2)\,\mathrm{artanh}\left(\frac{d-2}{d}\right).$$

Since $\mathrm{artanh}\left(\frac{d-2}{d}\right)=\frac{1}{2}\ln(d-1)$, we find in the limit $\varepsilon\to 0^+$:

$$F(0)\underset{\varepsilon\to 0^+}{\to}\frac{d-2}{2}\ln\left|\frac{4d-4}{d^2 4(d-1)\lambda-4d+4}\right|-d\ln(2)+\frac{d}{2}\ln(4(d-1)\lambda)+\frac{d-2}{2}\ln(d-1),$$

and thus $\lim_{\varepsilon\to 0^+}I_{\lambda,\varepsilon}=\lim_{\varepsilon\to 0^+}F(1-\lambda\varepsilon)-F(0)$ is equal to

$$\lim_{\varepsilon\to 0^+}=\ln\left(\frac{\left(d^2 4(d-1)\lambda\right)^{\frac{d-2}{2}}}{(4d-4)^{\frac{d-2}{2}}\,2^{-d}\left(4(d-1)\lambda\right)^{\frac{d}{2}}(d-1)^{\frac{d-2}{2}}}\right)=\ln\left(\frac{d^{d-2}}{(d-1)^{d-1}\lambda}\right),$$

which concludes the proof. □

The methods used in Proposition 7.2 and Theorem 7.3 can be generalised to other operators and Cayley graphs, as in the following corollary.

Corollary 7.4. *Let $d\geqslant 2$, and let $x_1,\ldots,x_d$ be d generators of the free group $\mathbb{F}_d$. Let $\zeta_1,\xi_1,\ldots,\zeta_d,\xi_d\in\mathbb{C}$ such that $|\zeta_1|=|\xi_1|=\ldots=|\zeta_d|=|\xi_d|=1$. Let $A=\zeta_1R_{x_1}+\xi_1R_{x_1^{-1}}+\ldots+\zeta_dR_{x_d}+\xi_dR_{x_d^{-1}}$. Then we have:*

(1) *For t small enough, the following generating series is equal to:*

$$u_{A^*A}(t)=\sum_{k=0}^{\infty}\mathrm{tr}_{\mathbb{F}_d}\left((A^*A)^k\right)t^k=\frac{4d-2}{2d-2+2d\sqrt{1-4(2d-1)t}}.$$

(2) *The Fugelede-Kadison determinant of A is equal to*

$$\det{}_{\mathbb{F}_d}\left(\zeta_1 R_{x_1} + \xi_1 R_{x_1^{-1}} + \ldots + \zeta_d R_{x_d} + \xi_d R_{x_d^{-1}}\right) = \frac{(2d-1)^{\frac{2d-1}{2}}}{(2d)^{d-1}}.$$

Proof. (1) The case where $\zeta_1 = \xi_1 = \ldots = \zeta_d = \xi_d = 1$ follows from [2, 8] (this time the circuits are considered in a $(2d)$-regular tree).

The general case follows from a similar argument as in the proof of Proposition 7.2, with a slight difference: this time no letter is trivial, thus any trivial word must be of even length, and therefore any coefficient ζ_i (resp. ξ_i) is necessarily multiplied with a coefficient equal to ζ_i^* (resp. ξ_i^*), and vice-versa.

(2) The result follows from (1) (i.e. the value of $u_{A^*A}(t)$) as in the proof of Theorem 7.3, except that each d is replaced with $2d$. □

8. New upper bounds for Lehmer's constants

In this section, we establish new upper bounds for Lehmer's constants for a large class of torsionfree groups, as a consequence of Theorem 7.3.

Given a group G, its *Lehmer's constants*, as defined by Lück in [14], are:

- $\Lambda(G) := \inf\left\{\det_G(A) \mid A \in \sqcup_{p,q\in\mathbb{N}} R_{M_{p,q}(\mathbb{Z}G)},\ \det_G(A) > 1\right\}$,
- $\Lambda^w(G) := \inf\left\{\det_G(A) \mid A \in \sqcup_{n\in\mathbb{N}} R_{M_n(\mathbb{Z}G)},\ A \text{ injective},\ \det_G(A) > 1\right\}$,
- $\Lambda_1(G) := \inf\left\{\det_G(A) \mid A \in R_{\mathbb{Z}G},\ \det_G(A) > 1\right\}$,
- $\Lambda_1^w(G) := \inf\left\{\det_G(A) \mid A \in R_{\mathbb{Z}G},\ A \text{ injective},\ \det_G(A) > 1\right\}$.

Observe that $1 \leqslant \Lambda(G) \leqslant \Lambda^w(G) \leqslant \Lambda_1^w(G)$ and $1 \leqslant \Lambda(G) \leqslant \Lambda_1(G) \leqslant \Lambda_1^w(G)$.

Remark 8.1. If H is a subgroup of G, it follows from Proposition 2.4 (3) that $\lambda(G) \leqslant \lambda(H)$ for any $\lambda \in \{\Lambda, \Lambda_1, \Lambda^w, \Lambda_1^w\}$.

Lehmer's constants are inspired by the well-known Lehmer problem, that we re-state as a conjecture as follows:

Conjecture 8.2 (Lehmer's problem, [14] Problem 1.3)**.** $\Lambda_1^w(\mathbb{Z})$ *is equal to the Mahler measure of Lehmer's polynomial* $L(z) = z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1$*:*

$$\Lambda_1^w(\mathbb{Z}) = \mathcal{M}(L) = \mathcal{M}(z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1) = 1.176...$$

As explained in [14], Lehmer's constants are especially interesting to study for torsionfree groups. For this class of groups, Lück proposed the following question:

Question 8.3 ([14], Introduction)*.* For which torsionfree groups G do we have

$$\Lambda(G) = \Lambda^w(G) = \Lambda_1(G) = \Lambda_1^w(G) = \mathcal{M}(L) = 1.176...\ ?$$

Example 8.4 ([14], Example 13.2)*.* Lück provided a partial negative answer to Question 8.3 by proving that $\Lambda(G) = \Lambda^w(G) \leqslant 1.06 < \mathcal{M}(L)$ for G the fundamental group of the hyperbolic Weeks manifold. It follows from Remark 8.1 that the same is true for any group G' containing G as a subgroup.

Now, as a consequence of Theorem 7.3, we obtain new upper bounds on Lehmer's constants and a negative answer to Question 8.3 for a large class of groups:

Corollary 8.5. *For every $d \geqslant 2$, Lehmer's constants $\Lambda(\mathbb{F}_d), \Lambda_1(\mathbb{F}_d), \Lambda^w(\mathbb{F}_d)$ and $\Lambda_1^w(\mathbb{F}_d)$ do not depend on d. Moreover, for every $d \geqslant 2$, we have*

$$\Lambda(\mathbb{F}_d) \leqslant \Lambda^w(\mathbb{F}_d) \leqslant \Lambda_1^w(\mathbb{F}_d) = \Lambda_1(\mathbb{F}_d) \leqslant \frac{2}{\sqrt{3}} = 1.15... < \mathcal{M}(L) = 1.176...$$

In particular, any torsionfree group G containing a subgroup $\mathbb{F}_d$ for $d \geqslant 2$ (such as the fundamental group of a hyperbolic 3-manifold, see [1, C.3, C.26]*) also satisfies*

$$\Lambda(G), \Lambda_1(G), \Lambda^w(G), \Lambda_1^w(G) \in \left[1, \frac{2}{\sqrt{3}}\right].$$

Proof. The first statement follows from Remark 8.1 and the fact every free group $\mathbb{F}_d$ (for $d \geqslant 2$) injects in $\mathbb{F}_2$ and vice-versa.

In the second statement, the first two inequalities follow immediately from the definitions, and the first equality from the fact that free groups satisfy the Strong Atiyah Conjecture (see Remark 2.5 and [13, Theorem 10.19]). The third inequality follows from Theorem 7.3 and the first statement (observe that $d \mapsto \frac{(d-1)^{\frac{d-1}{2}}}{d^{\frac{d-2}{2}}}$ is increasing for $d \geqslant 3$, thus $\frac{2}{\sqrt{3}}$ is the best upper bound available).

The third statement follows from the second one and Remark 8.1: for any torsion-free group G with $\mathbb{F}_d < G$, we have $\lambda(G) \leqslant \lambda(\mathbb{F}_d)$ for any $\lambda \in \{\Lambda, \Lambda_1, \Lambda^w, \Lambda_1^w\}$. □

Remark 8.6. The fundamental group of the Weeks manifold contains free subgroups (see [1, C.3, C.26]). In this sense, Corollary 8.5 can be seen as a generalization of the class of counterexamples to Question 8.3 mentioned in Example 8.4. However, for this same class, the upper bound 1.06... in Example 8.4 is better than the bound $\frac{2}{\sqrt{3}}$ of Corollary 8.5, for the constants $\Lambda(G), \Lambda^w(G)$.

From what precedes, we find that the combinatorial and analytical techniques used in Theorem 7.3 show promise for solving problems such as Question 8.3.

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