Existence of periodic solutions and bifurcation points for generalized ordinary differential equations

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3 authors:

Marcia Federson
University of São Paulo

Jean Mawhin
Université Catholique de Louvain - UCLouvain

Carolina Mesquita
University of São Paulo

Existence of periodic solutions and bifurcation points for generalized ordinary differential equations

M. Federson[*], J. Mawhin[†], C. Mesquita[‡]

Abstract

The generalized ordinary differential equations (shortly GODEs), introduced by J. Kurzweil in 1957, encompass other types of equations. The first main result of this paper extends to GODEs some classical conditions on the existence of a periodic solution of a nonautonomous ODE. By means of the correspondence between impulse differential equations (shortly IDEs) and GODEs, we translate the result to IDEs. Instead of the classical hypotheses that the functions on the righthand side of an IDE are piecewise continuous, it is enough to require that they are integrable in the sense of Lebesgue, allowing such functions to have many discontinuities. Our second main result provides conditions for the existence of a bifurcation point with respect to the trivial solution of a periodic boundary value problem for a GODE depending upon a parameter, and, again, we apply such result to IDEs. The machinery employed to obtain the main results are the topological degree theory, tools from the theory of compact operators and an Arzelà-Ascoli-type theorem for regulated functions.

Keywords: Periodic solutions; Bifurcation; Kurzweil-Henstock integral; Brouwer degree; Leray-Schauder degree.
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1 Introduction

The aim of this paper is to obtain theorems on the existence of a periodic solution and on the existence of a bifurcation point for periodic solutions, in the framework of J. Kurzweil's generalized ordinary differential equations (GODE, for short) (see [20, 21, 22, 30]). The main tool used here is the theory of topological degree [6, 11, 16, 34].

The application of topological methods to periodic solutions of ordinary differential systems is well-explored. In [25, 26], J. Mawhin has proved a continuation theorem for the existence of at least one periodic solution for nonautonomous systems involving Carathéodory functions using the coincidence degree (see [25], Theorem IV.13, [26], Theorem 4.1). In the present paper, extend this continuation theorem to the periodic boundary problem for GODEs. Because GODEs are presented as integral equations where the (non absolute) integral is of the type

[*]Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil. E-mail: federson@icmc.usp.br
[†]Research Institute in Mathematics and Physics, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium. E-mail: jean.mawhin@uclouvain.be
[‡]Federal University of São Carlos, Caixa Postal 676, 13565.905 São Carlos SP, Brazil. E-mail: mcmesquita@dm.ufscar.br

defined by J. Kurzweil in [20], our requirements of the righthand side of the equation only involve some integrability in the sense of Kurzweil (see Theorem 4.8 in the sequel). It is a known fact that finite dimensional GODEs encompass other types of equations, among others, functional differential equations, integral equations, dynamic equations on time scales, measure and impulsive differential equations. Here, we illustrate this fact with impulsive differential equations (IDE for short), and translate our result to these equations (see Theorem 6.5).

Concerning the existence of bifurcation points by means of degree theory, we mention among other ones the books [18], by M. A. Krasnosel'skiĭ et Zabreiko, and [19], by W. Krawcewicz and Jianhong Wu. In [18], Theorem 56.2, and in [19], Theorem 5.1.7, the proofs use the Leray-Schauder index for compact perturbations of identity in a Banach space. On the other hand, in the books [13] by R. E. Gaines and J. Mawhin, and [25] by J. Mawhin, the authors use the coincidence degree to extend the existence condition for a bifurcation point to relatively compact perturbations of linear Fredholm operators between Banach spaces (see [13], Theorem 10.6, [25], Theorem IX.3). The linearized case is also investigated in Theorem 10.8 of [13] and in Theorem IX.5 of [25]. It is worth mentioning that H. Amann, in his book [2], deals with finite dimensional operators specialized for ODEs and deduces his result on the existence of a bifurcation point from the Brouwer index (see [2], Theorem 26.5). Inspired by these ideas, we associate a fixed point operator to the periodic boundary value problem for a nonautonomous GODE and establish conditions under which the equation admits a bifurcation point with respect to the trivial solution (see Theorem 5.6 in the sequel). Then, we apply this result to periodic solutions of IDEs (Theorem 6.8) with Carathéodory right-hand members. It is worth mentioning that one could consider right-hand members involving Kurzweil-Henstock instead of Lebesgue integrability. Other approaches for bifurcation results for impulsive differential equations can be found in [1, 5, 23].

The theory of GODEs, introduced by J. Kurzweil in 1957 (see [20]), is based on the Kurzweil-Henstock non-absolute integration theory developed independently by J. Kurzweil and R. Henstock in the late nineteen fifties and early nineteen sixties (see, e.g., [14, 21]). This means that the righthand sides of the equations may be non-absolutely integrable with respect to t, hence coping with large oscillations and many jumps. For instance, the function $f : [0, 1] \longrightarrow \mathbb{R}$ given by

$$f(t) = \begin{cases} \dfrac{1}{t} \sin\left(\dfrac{1}{t^3}\right) \chi_{(0,1]}(t), & \text{if } t \in (0, 1], \\ 0, & \text{if } t = 0, \end{cases}$$

where χ_E denotes the characteristic function of a set $E \subset \mathbb{R}$, is neither Riemann nor Lebesgue integrable, but is integrable in the sense of Kurzweil-Henstock.

Because the solutions of a GODE are given in an integral form as

$$x(s) = x(0) + \int_0^s DF(x(\tau), t), \quad s \in [0, T],$$

where $T > 0$ and the integral is in the sense of Kurzweil (see Definition 2.1 in the sequel), they can easily be related to integral operators. We recall, in Section 2, a few basic properties of the Kurzweil integration theory, to make the paper is self-contained.

In Section 3, we consider the periodic solutions of GODEs. We introduce the operator $\mathcal{M}$, defined from the space G of the regulated functions $x : [0, T] \to \mathbb{R}^n$ into itself, by

$$\mathcal{M}(x)(s) = x(0) + \int_0^T DF(x(\tau), t) + \int_0^s DF(x(\tau), t), \quad s \in [0, T],$$

and prove that the fixed points of $\mathcal{M}$ are the periodic solutions of the nonautonomous GODE (Proposition 3.2). To apply Leray-Schauder degree, we need to prove that $\mathcal{M}$ is a compact operator (i.e. a continuous mapping which takes bounded sets into relatively compact sets). This is done in Propositions 3.3 and 3.4.

In Section 4, we state and prove an existence theorem for periodic solutions of GODEs, namely Theorem 4.8. A preliminary version was proposed in 1993 in the unpublished thesis of C. Gorez, written under the direction of J. Mawhin. To prove it, we employ results from degree theory (see e.g. [6, 11, 34]), and in particular the invariance under homotopy of the Leray-Schauder degree (see [11], page 179) applied to the family of operators $\mathcal{H} : \overline{\Delta} \times [0, 1] \to G$ given by

$$\mathcal{H}(x, \lambda)(s) = x(0) + \int_0^T DF(x(\tau), t) + \lambda \int_0^s DF(x(\tau), t), \quad s \in [0, T],$$

where Δ is an open bounded subset of G such that $\mathcal{H}(\cdot, \lambda)$ has no fixed point in $\partial\Delta$ for $\lambda \in [0, 1]$, and the link between the Leray-Schauder and the Brouwer degree for perturbations of identity with finite-dimensional range.

In Section 5, we introduce a concept of bifurcation point with respect to the trivial solution of some periodic boundary value problem of GODEs depending on a parameter $\lambda \in \Lambda_0 \subset \mathbb{R}$,

$$\frac{dx}{d\tau} = DF(\lambda, x, t).$$

We extend in Theorem 5.6 to this problem a classical sufficient condition for the existence of a bifurcation point introduced by Krasnosel'skiĭ (see e.g. [18]). More specifically, we consider the nonlinear operator $\mathcal{N} : \Lambda_0 \times G \to G$ given by

$$\mathcal{N}(\lambda, x)(s) = x(T) + \int_0^s DF(\lambda, x(\tau), t), \quad s \in [0, T],$$

whose fixed points are the periodic solutions of the GODE, and we prove that $\mathcal{N}$ is compact with respect to the second variable (see Proposition 5.2 in the sequel) and continuous with respect to the first variable (see Proposition 5.3). In addition, we show that $\mathcal{N}$ is a homotopy of compact transformations on the closure of a ball in G so that we can use, again, the invariance with respect to homotopy (see the proof of Theorem 5.6 in the sequel) in order to prove, by a contradiction argument, that a variation of the Leray-Schauder index between two values of λ leads to a bifurcation point between them.

Concerning applications, we translate Theorem 4.8 on the existence of periodic solutions of GODEs to the case of IDEs with Carathéodory righthand sides (see Theorem 6.5 in the sequel). Then we give an explicit example of an IDE for which we prove the existence of a periodic solution. We also translate our result on the existence of a bifurcation point to IDEs (see Theorem 6.8 in the sequel) and give an explicit example. Our results generalize results from the literature. See, for instance, [3, 23, 24, 29, 33].

2 Background on Kurzweil integration theory

Let $T > 0$ be a fixed number. Consider a function $\delta : [0, T] \to \mathbb{R}^+$ (called a *gauge* on $[0, T]$). A tagged division of the interval $[0, T]$ with division points $0 = t_0 \leq t_1 \leq \cdots \leq t_m = T$ and tags $\tau_j \in [t_{j-1}, t_j]$, $j = 1, \ldots, m$, is called *δ-fine*, whenever $[t_{j-1}, t_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j))$, $j = 1, \ldots, m$. We denote such tagged division by $d = (\tau_j, [t_{j-1}, t_j])_{j=1}^m$.

Throughout this paper, we denote respectively by $|\cdot|$ and $\|\cdot\|$ any norm in $\mathbb{R}$ and $\mathbb{R}^n$. The next definition is due to J. Kurzweil. See, for instance, [20, 21, 22, 30].

Definition 2.1. A function $U : [0, T] \times [0, T] \to \mathbb{R}^n$ is called *Kurzweil integrable* on $[0, T]$, if there exists an element $K \in \mathbb{R}^n$ such that, for every $\varepsilon > 0$, there exists a gauge δ on $[0, T]$ satisfying

$$\left\| \sum_{j=1}^{m} [U(\tau_j, t_j) - U(\tau_j, t_{j-1})] - K \right\| \leq \varepsilon.$$

for every δ-fine tagged division $d = (\tau_j, [t_{j-1}, t_j])_{j=1}^m$ of $[0, T]$. In this case, we write $K = \int_0^T DU(\tau, t)$.

Let us recall the original definition of a GODE introduced by Kurzweil in 1957, and presented in [30], Definition 2.11.

Definition 2.2. Let $\mathcal{O} \subset \mathbb{R}^n$ be open and $F : \mathcal{O} \times [0, T] \to \mathbb{R}^n$ be a function. We say that a function $x : [0, T] \to \mathbb{R}^n$ is a *solution* of the generalized ordinary differential equation (GODE)

$$\frac{dx}{d\tau} = DF(x, t) \tag{2.1}$$

whenever $x(t) \in \mathcal{O}$, for every $t \in [0, T]$, and

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t) \tag{2.2}$$

for every $s_1, s_2 \in [0, T]$.

Remark 2.3.

- The integral on the righthand side of (2.2) is in the sense of Definition 2.1, with $U : [0, T] \times [0, T] \to \mathbb{R}^n$ given by $U(\tau, t) = F(x(\tau), t)$;
- If $x(\tau) = a$ for all $\tau \in [0, T]$, where $a \in \mathbb{R}^n$, the Riemannian sum in Definition 2.1 becomes

$$\sum_{j=1}^{m} [F(a, t_j) - F(a, t_{j-1})] = F(a, T) - F(a, 0),$$

  for any tagged division $d = (\tau_j, [t_{j-1}, t_j])_{j=1}^m$ of $[0, T]$. Hence, the Kurzweil integral is

$$\int_0^T DF(a, t) = F(a, T) - F(a, 0). \tag{2.3}$$

In particular, when $x(\tau) = a$ for all $\tau \in [0, T]$ is a solution of equation (2.2), we obtain

$$0 = F(a, T) - F(a, 0).$$

We are interested in a suitable class of righthand sides F of (2.1), introduced in [30] and defined as follows. Thorough this paper, $B_R \subset \mathbb{R}^n$ denotes the open ball with center at zero and of radius $R > 0$. We assume that, for each $R > 0$, there exist a nondecreasing function $h_R : [0, T] \to \mathbb{R}$ and an increasing and continuous function $\omega_R : [0, +\infty) \to \mathbb{R}$, with $\omega_R(0) = 0$, such that

$$\|F(z, t_2) - F(z, t_1)\| \leq |h_R(t_2) - h_R(t_1)| \tag{2.4}$$

$$\|F(z, t_2) - F(z, t_1) - F(y, t_2) + F(y, t_1)\| \leq \omega_R(\|z - y\|)|h_R(t_2) - h_R(t_1)|, \tag{2.5}$$

for every z, $y \in B_R$, and every t_1, $t_2 \in [0, T]$. We denote by $\mathcal{F}(B_R \times [0, T], h_R, \omega_R)$, the set of all functions $F : B_R \times [0, T] \to \mathbb{R}^n$ satisfying (2.4) and (2.5).

Following N. Bourbaki and J. Dieudonné (see [4] and [7]), a function $x : [0, T] \to \mathbb{R}^n$ is called regulated, if the lateral limits

$$\lim_{s \to t^-} x(s), \; t \in (0, T] \;\text{ and }\; \lim_{s \to t^+} x(s), \;\; t \in [0, T)$$

exist. We denote by G the space of all regulated functions $x : [0, T] \to \mathbb{R}^n$ with the usual supremum norm $\|x\|_\infty = \sup_{t \in [0,T]} \|x(t)\|$. The fact that $(G, \| \cdot \|_\infty)$ is a Banach space is well-known. (see, e. g., [15], Theorem 3.6, p. 18).

It is worth recalling that all functions $x : [0, T] \to \mathbb{R}$ of bounded variation are also regulated functions (see, e.g. [15], Corollary 4, p. 18) which are, in turn, Darboux integrable ([15], Theorem 3.6, p. 18). As a matter of fact, the *raison d'être* of regulated functions lies on the fact that every regulated function $f : [0, T] \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ has a primitive, i.e., there exists a continuous function $F : [0, T] \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $\dfrac{dF}{dt}(t) = f(t)$ almost everywhere in $[0, T]$ (see [4], p.4, and [7], p.139).

The next proposition presents a characterization of relatively compact subsets (subsets with compact closure) of the space G of the regulated functions from $[0, T]$ to $\mathbb{R}^n$. Such result can be found in [32], Corollary 4.3.8 and is a consequence of a result by D. Franková in [12], Theorem 2.17.

Proposition 2.4. *Let $\mathcal{A} \subset G$ and assume that the set $\{x(0),\ x \in \mathcal{A}\}$ is bounded and that there exists a nondecreasing function $h : [0, T] \to \mathbb{R}^n$ such that*

$$\|x(t) - x(s)\| \leq |h(t) - h(s)|, \; \textit{for every} \;\; t, s \in [0, T] \; \textit{and} \; x \in \mathcal{A}.$$

Then $\mathcal{A}$ is relatively compact in G.

The following proposition, which compiles Lemma 3.9 and Corollaries 3.11 and 3.16 from [30], gives some information about the integral form (2.2) of the GODE (2.1).

Proposition 2.5. *If $F : B_R \times [0, T] \to \mathbb{R}^n$ satisfies* (2.4)*, then the following assertions hold.*

(i) *If* $x : [0,T] \to \mathbb{R}^n$*, with* $(x(s), s) \in B_R \times [0,T]$*, for all* $s \in [0,T]$*, and if the integral* $\int_{s_1}^{s_2} DF(x(\tau), t)$ *exists, then for any* $s_1, s_2 \in [0,T]$*, the inequality*

$$\left\| \int_{s_1}^{s_2} DF(x(\tau), t) \right\| \leq |h_R(s_2) - h_R(s_1)|.$$

holds.

(ii) *If* $x : [0,T] \to \mathbb{R}^n$ *is a solution of the GODE* (2.1)*, then* x *is of bounded variation in* $[0,T]$ *and* $var_0^T x \leq h_R(T) - h_R(0) < +\infty$*, where* $var_0^T x$ *denotes the variation of* x *in* $[0,T]$*.*

(iii) *If* $F \in \mathcal{F}(B_R \times [0,T], h_R, \omega_R)$*, for all* $R > 0$ *and* $x : [0,T] \to \mathbb{R}^n$ *is a regulated function such that* $(x(s), s) \in B_R \times [0,T]$ *for every* $s \in [0,T]$*, then the integral* $\int_0^T DF(x(\tau), t)$ *exists.*

The last result of this quick overview on the theory of GODEs is an important property of the class $F \in \mathcal{F}(B_R \times [0,T], h_R, \omega_R)$ which will be useful in the next section. A proof of it can be found in [28], Lemma 5.

Lemma 2.6. *Let* $F \in \mathcal{F}(B_R \times [0,T], h_R, w_R)$*, for all* $R > 0$*. If* $x, y : [0,T] \to B_R$ *are regulated functions, then*

$$\left\| \int_0^T D[F(x(\tau), t) - F(y(\tau), t)] \right\| \leq \int_0^T \omega_R(\|x(t) - y(t)\|) dh_R(t).$$

where the integral above is in the sense of Kurzweil.

The next estimate follows directly from the definition of the Kurzweil integral. See [31], Lemma 2.2.

Lemma 2.7. *Let* $U : [0,T] \times [0,T] \to \mathbb{R}^n$ *be Kurzweil integrable and let the functions* $z : [0,T] \to \mathbb{R}$ *be regulated and* $g : [0,T] \to \mathbb{R}$ *be nondecreasing such that*

$$\|U(\tau, t) - U(\tau, s)\| \leq z(\tau)\, |g(s) - g(t)| \quad \text{for all } t, s, \tau \in [0,T].$$

Then

$$\left\| \int_0^T DU(\tau, t) \right\| \leq \int_0^T z(\tau)\, dg(\tau).$$

3 Periodic solutions of GODEs

In this section, we introduce the concept of periodic solutions for GODEs and to establish an equivalence result.

Definition 3.1. Let $T > 0$ be a fixed number. We say that a function $x : [0,T] \to \mathbb{R}^n$ is a *T-periodic solution* of the GODE

$$\frac{dx}{d\tau} = DF(x, t) \tag{3.1}$$

if it is a solution of (3.1) such that $x(0) = x(T)$.

Let us introduce an operator

$$\mathcal{M} : G \to G, \quad x \ \mapsto \ \mathcal{M}(x),$$

given by

$$\mathcal{M}(x)(s) = x(0) + \int_0^T DF(x(\tau), t) + \int_0^s DF(x(\tau), t), \quad \text{for all } s \in [0, T], \tag{3.2}$$

where we assume that $F \in \mathcal{F}(B_R \times [0, T], h_R, \omega_R)$, for all $R > 0$.

By Proposition 2.5, item (iii), it is clear that the operator $\mathcal{M}$ is well-defined.

The next proposition describes a one-to-one correspondence between the T-periodic solutions of (3.1) and the fixed points of the operator $\mathcal{M}$ given by (3.2).

Proposition 3.2. *Suppose $F \in \mathcal{F}(B_R \times [0, T], h_R, \omega_R)$, for all $R > 0$. A function $x : [0, T] \to \mathbb{R}^n$ is a T-periodic solution of* (3.1) *if and only if x is a fixed point of the operator $\mathcal{M} : G \to G$ given by* (3.2)*.*

Proof. Suppose x is a T-periodic solution of (3.1). By (ii) of Proposition 2.5, $x \in G$ and

$$x(s) = x(0) + \int_0^s DF(x(\tau), t), \quad \text{for all } s \in [0, T],$$

and $x(T) = x(0)$. Hence

$$\int_0^T DF(x(\tau), t) = 0,$$

and,

$$x(s) = x(0) + \int_0^T DF(x(\tau), t) + \int_0^s DF(x(\tau), t), \quad \text{for all } s \in [0, T],$$

which implies $\mathcal{M}(x)(s) = x(s)$, for every $s \in [0, T]$, and x is a fixed point of t $\mathcal{M}$.

Conversely, let $x \in G$ be a fixed point of the operator $\mathcal{M}$. Then

$$x(s) = x(0) + \int_0^T DF(x(\tau), t) + \int_0^s DF(x(\tau), t), \quad \text{for all } s \in [0, T], \tag{3.3}$$

Taking $s = 0$ in (3.3), we obtain

$$\int_0^T DF(x(\tau), t) = 0. \tag{3.4}$$

Then, taking $s = T$ in (3.3) and using (3.4), we get $x(T) = x(0)$. Finally, from (3.4) and (3.3),

$$x(s) = x(0) + \int_0^s DF(x(\tau), t), \quad \text{for all } s \in [0, T],$$

which means that x is a T-periodic solution of (3.1) and the proof is complete. □

The next result ensures the continuity of the operator $\mathcal{M}$ on G.

Proposition 3.3. *Suppose $F \in \mathcal{F}(B_R \times [0, T], h_R, \omega_R)$, for all $R > 0$. Then, the operator $\mathcal{M} : G \to G$, defined in* (3.2)*, is continuous on G.*

Proof. Let x, $y \in G$ be such that $x(s)$, $y(s) \in B_R$, for some $R > 0$ and for all $s \in [0,T]$. Hence,

$$\begin{aligned}\|\mathcal{M}(y) - \mathcal{M}(x)\|_\infty &= \sup_{s\in[0,T]} \|\mathcal{M}(y)(s) - \mathcal{M}(x)(s)\| \\ &\leq \|y(0) - x(0)\| + \left\| \int_0^T D[F(y(\tau), t) - F(x(\tau), t)] \right\| \\ &+ \sup_{s\in[0,T]} \left\{ \left\| \int_0^s D[F(y(\tau), t) - F(x(\tau), t)] \right\| \right\}.\end{aligned}$$

By Lemma 2.6, we have

$$\left\| \int_0^T D[F(y(\tau), t) - F(x(\tau), t)] \right\| \leq \int_0^T \omega_R(\|y(t) - x(t)\|) dh_R(t). \tag{3.5}$$

Then, using (3.5), we obtain

$$\|\mathcal{M}(y) - \mathcal{M}(x)\|_\infty \leq \|y - x\|_\infty + 2\omega_R(\|y - x\|_\infty)[h_R(T) - h_R(0)].$$

and the proof is complete. □

The next result ensures that the operator $\mathcal{M} : G \to G$, defined in (3.2), takes bounded sets of G into relatively compact sets of G.

Proposition 3.4. *Suppose $F \in \mathcal{F}(B_R \times [0,T], h_R, \omega_R)$, for all $R > 0$. Then the operator $\mathcal{M}$ defined in* (3.2) *maps bounded sets of G into relatively compact sets of G (i.e. its closure in G is compact).*

Proof. It is enough to prove that the set $\mathcal{A} = \{\mathcal{M}(x),\ x \in M\}$ is relatively compact in G, for every bounded set $M \subset G$. It is clear that the set $\{\mathcal{M}(x)(0),\ x \in M\}$ is bounded.

By item (i) of Proposition 2.5, we have

$$\|\mathcal{M}(x)(s') - \mathcal{M}(x)(s)\| = \left\| \int_s^{s'} DF(x(\tau), t) \right\| \leq |h_R(s') - h_R(s)|,$$

for every $s, s' \in [0,T]$ and every $x \in M$. The statement follows by Proposition 2.4. □

4 Existence of periodic solutions of GODEs

Keeping the notations and terminology of the previous section, we now state and prove a result which ensures the existence of at least one T-periodic solution of the GODE

$$\frac{dx}{d\tau} = DF(x,t), \tag{4.1}$$

where $F \in \mathcal{F}(B_R \times [0,T], h_R, \omega_R)$ for all $R > 0$. To this end, we employ the Leray-Schauder degree theory. For the reader's convenience, we recall some basic concepts and results of the topological degree theory, both for finite and infinite dimensional spaces (see [2, 6, 11, 27]).

A proof of the following theorem can be found in [2], Theorem 21.5.

Theorem 4.1. *Let E be a finite dimensional Banach space. Then, for every open and bounded subset $\mathcal{D}$ of E and every $z \in E$, there exists a function*

$$\deg(\cdot\,, \mathcal{D}, z) : D_z(\mathcal{D}, E) \to \mathbb{Z}$$

called the Brouwer degree, where $D_z(\mathcal{D}, E) = \{f \in C(\overline{\mathcal{D}}, E);\ z \notin f(\partial\mathcal{D})\}$, which has the following properties:

(i) *(Normalization): If $z \in \mathcal{D}$, then $\deg(I, \mathcal{D}, z) = 1$, where $I : E \to E$ denotes the identity operator.*

(ii) *(Homotopy invariance): Let $J \subseteq \mathbb{R}$ be a nonempty compact interval. Moreover, assume that $H \in C(J \times \overline{\mathcal{D}}, E)$ and $y \in C(J, E)$ are such that*

$$y(\lambda) \notin H(\{\lambda\} \times \partial\mathcal{D}), \ \ \textit{for each} \ \ \lambda \in J.$$

Then,

$$\deg(H(\lambda, \cdot), \mathcal{D}, y(\lambda))$$

is well-defined and independent of $\lambda \in J$.

Corollary 4.2. *Let $\mathcal{D}$ be an open and bounded subset of a Banach space E of finite dimension n and, for $f \in C(\overline{\mathcal{D}}, E)$, assume that z does not belong to $f(\partial\mathcal{D})$. Then*

$$\deg(-f, \mathcal{D}, z) = (-1)^n \deg(f, \mathcal{D}, z).$$

In order to extend the concept of degree for functions whose domain is a subset of an arbitrary Banach space, we recall some elements of the Leray-Schauder's degree theory (see [6, 11, 16, 27]). As in [11], Definition 7.1, p. 174, a mapping $f : X \subset E \longrightarrow E$ (not necessarily linear) is compact on X, whenever f is continuous on X and takes bounded sets of X into relatively compact sets of E.

Definition 4.3. Let E be a Banach space and $\mathcal{D} \subset E$ be an open bounded set. Let $f : \overline{\mathcal{D}} \to E$ be a compact operator such that $z \notin (I - f)(\partial\mathcal{D})$. The Leray-Schauder degree is a function $\deg_{LS}$, which associates to each triple $(I - f, \mathcal{D}, z)$, an integer $\deg_{LS}(I - f, \mathcal{D}, z) \in \mathbb{Z}$ satisfying the following properties:

(i) $\deg_{LS}(I, \mathcal{D}, z) = 1$, for $z \in \mathcal{D}$, where $I : E \to E$ denotes the identity operator.

(ii) If $\deg_{LS}(I - f, \mathcal{D}, z) \neq 0$, then $z \in (I - f)(\mathcal{D})$.

We now recall the definition of a homotopy of compact transformations. For more details, see [11], page 178.

Definition 4.4. Let E be a Banach space and $\mathcal{D} \subset E$ be an open bounded set. Let $M \subset \mathcal{D}$ and $H : [0, 1] \times \overline{\mathcal{D}} \longrightarrow E$. We say that H is a homotopy of compact transformations on M, whenever

(a) For each $\lambda \in [0, 1]$ fixed, $H(\lambda, x)$ is compact on M.

(b) For every $\varepsilon > 0$ and for every bounded $L \subset M$, there exists $\delta > 0$ such that

$$\|H(\lambda_1, x) - H(\lambda_2, x)\| \leq \varepsilon$$

whenever $x \in L$ and $|\lambda_1 - \lambda_2| < \delta$.

The next result, borrowed from [11], p. 179, presents an important property of the Leray-Schauder degree, which will be useful in the following sections.

Theorem 4.5. *[Invariance under Homotopy] Let E be a Banach space and $\mathcal{D} \subset E$ be an open bounded set. Assume that $H : [0,1] \times \overline{\mathcal{D}} \longrightarrow E$ is a homotopy of compact transformations on $\overline{\mathcal{D}}$. Set*

$$\phi_\lambda = I - H(\lambda, \cdot\,)$$

for $\lambda \in [0,1]$ and assume that $z \notin \phi_\lambda(\partial\mathcal{D})$, for every $\lambda \in [0,1]$. Then, $\deg_{LS}[I - H(\lambda, \cdot\,), \mathcal{D}, z]$ is independent of λ.

Definition 4.6. Let E be a Banach space, $\mathcal{D} \subset E$ be an open bounded set and $f \in C(\overline{\mathcal{D}}, E)$ be a function whose range is contained in some finite dimensional subspace F of E. The *Leray-Schauder degree* of $I - f$ with respect to $\mathcal{D}$, at a point $z \notin (I - f)(\partial\mathcal{D})$, is defined by the integer

$$\deg_{LS}(I - f, \mathcal{D}, z) = \deg((I - f)\mid_{\overline{\mathcal{D}} \cap F}, \mathcal{D} \cap F, z),$$

where the righthand side is the Brouwer degree defined according to Theorem 4.1.

Next, we recall the concept of Leray-Schauder index as presented in [6, 11, 16].

Definition 4.7. Let $f : \overline{\mathcal{D}} \to E$ satisfy the conditions of Definition 4.3. If a is an isolated fixed point of f, for small $r > 0$, we define the *Leray-Schauder index* of $I - f$ at a by

$$\mathrm{ind}_{LS}[I - f, a] = \deg_{LS}[I - f, B(a, r), 0] \tag{4.2}$$

where $B(a, r)$ denotes the open ball of radius r and center $a \in D \subset E$.

The next result ensures that the GODE (4.1) has at least one T-periodic solution.

Theorem 4.8. *Assume that $F \in \mathcal{F}(B_R \times [0,T], h_R, \omega_R)$, for all $R > 0$. Suppose there exists an open and bounded subset Δ of G such that the following statements are valid:*

(i) *For every $\lambda \in (0, 1]$, the equation*

$$\frac{dx}{d\tau} = \lambda DF(x, t) \tag{4.3}$$

does not admit a T-periodic solution x on G with $x \in \partial\Delta$.

(ii) *The equation*

$$\Psi(a) := \int_0^T DF(a, t) = 0$$

does not admit a solution $a \in \partial\Delta \cap \mathbb{R}^n$ (where $\mathbb{R}^n$ is viewed as the set of constant functions in G).

(iii) $deg(\Psi, \Delta \cap \mathbb{R}^n, 0) \neq 0$.

Then the GODE (4.1) *has at least one T-periodic solution $x \in \Delta$.*

Proof. Let us define an operator $\mathcal{H} : [0,1] \times \overline{\Delta} \to G$ given by

$$\mathcal{H}(\lambda, x)(s) := x(0) + \int_0^T DF(x(\tau), t) + \lambda \int_0^s DF(x(\tau), t) \tag{4.4}$$

parametrized by $\lambda \in [0,1]$, for every $s \in [0,T]$.

Suppose $\lambda = 1$. Then, by Proposition 3.2, the fixed points of the operator $\mathcal{H}$ are the T-periodic solutions of (4.1). Thus, by hypothesis (i), $\mathcal{H}(1,x) \neq x$ for every $x \in \partial\Delta$.

Now, we consider $\lambda \in (0,1)$. If x is a fixed point of the operator $\mathcal{H}$, then taking $s = 0$ in (4.4), we obtain

$$\int_0^T DF(x(\tau), t) = 0 \tag{4.5}$$

and taking $s = T$ in (4.4) and using (4.5), we conclude that $x(T) = x(0)$ and, moreover,

$$x(s) = x(0) + \lambda \int_0^s DF(x(\tau), t), \quad \text{for all } s \in [0,T].$$

Therefore x is a T-periodic solution of (4.3). Hence, for every $\lambda \in (0,1)$, the fixed points of $\mathcal{H}(\lambda, \cdot\,)$ are T-periodic solutions of (4.3). Thus, by hypothesis (i), $\mathcal{H}(\lambda, x) \neq x$ for every $\lambda \in (0,1)$ and $x \in \partial\Delta$.

Now, we consider the case where $\lambda = 0$. If x is a fixed point of the operator $\mathcal{H}$, then

$$x(s) = x(0) + \int_0^T DF(x(\tau), t), \quad s \in [0,T], \tag{4.6}$$

which implies x is constant, that is, $x(s) = a$ in $[0,T]$. Thus, from (4.6), we obtain

$$\int_0^T DF(a, t) = 0.$$

By hypothesis (ii), it is clear that $\mathcal{H}(0,x) \neq x$ for every $x \in \partial\Delta$.

Then, combining all the previous cases, we conclude that

$$\mathcal{H}(\lambda, u) \neq u, \quad \text{for every pair } (\lambda, u) \in [0,1] \times \partial\Delta$$

and, hence, we have

$$0 \notin I - \mathcal{H}(\lambda, \cdot\,)(\partial\Delta), \ \lambda \in [0,1].$$

By Propositions 3.3 and 3.4, it is easy to conclude that the operator H is a homotopy of compact transformations on $\overline{\Delta}$. Therefore, by Theorem 4.5, we conclude that

$$\deg_{LS}(I - \mathcal{H}(1, \cdot\,), \Delta, 0) = \deg_{LS}(I - \mathcal{H}(0, \cdot\,), \Delta, 0).$$

Notice that $\mathcal{H}(\{0\}, x) \subset \mathbb{R}^n$, for every $x \in \overline{\Delta}$. Then, using Definition 4.6 and Corollary 4.2, we obtain

$$\begin{aligned}\deg_{LS}(I-\mathcal{H}(0,\cdot\,),\Delta,0) &= \deg((I-\mathcal{H}(0,\cdot\,)|_{\mathbb{R}^n},\Delta\cap\mathbb{R}^n,0)\\ &= \deg(-\Psi,\Delta\cap\mathbb{R}^n,0) = (-1)^n\deg(\Psi,\Delta\cap\mathbb{R}^n,0)\neq 0,\end{aligned}$$

where the last equation is different from zero, since hypothesis (iii) holds. Thus,

$$d_{LS}(I-\mathcal{H}(1,\cdot\,),\Delta,0)\neq 0.$$

By (ii) of Definition 4.3, there exists $x\in\Delta$ such that $\mathcal{H}(1,x)=x$ and, hence, $x\in\Delta$ is a fixed point of $\mathcal{H}(1,\cdot\,)$. Consequently, by Proposition 3.2, x is a T-periodic solution of (4.1). □

5 Bifurcation theory for GODEs

In this section, our first goal is to define the concept of a bifurcation point with respect to the trivial solution of the periodic boundary value problem for the nonautonomous GODE

$$\frac{dx}{d\tau} = DF(\lambda,x,t),\ \ x(0)=x(T), \tag{5.1}$$

where, for each $\lambda\in\Lambda_0\subset\mathbb{R}$ fixed, $F(\lambda,\cdot,\cdot)\in\mathcal{F}(B_R\times[0,T],h_R,\omega_R)$ for all $R>0$. We assume that the following equality holds:

(B1) $F(\lambda,0,t)-F(\lambda,0,s)=0,\quad$ for all $t,s\in[0,T]$ and $\lambda\in\Lambda_0$.

The assumption (B1) implies that equation (5.1) admits the trivial solution for each λ and we study the bifurcation with respect to the trivial solution.

In order to prove the existence of a bifurcation point with respect to the trivial solution of (5.1), let $\Omega\subset G$ be an open and bounded set containing 0. Inspired by [17], Appendix II, define the operator

$$\Phi:\Lambda_0\times\overline{\Omega}\to G,\quad (\lambda,x)\ \mapsto\ \Phi(\lambda,x)$$

by

$$\Phi(\lambda,x)(s) = x(s)-x(T)-\int_0^s DF(\lambda,x(\tau),t),\quad s\in[0,T]. \tag{5.2}$$

It follows from Proposition 2.5, item (iii), that the operator Φ is well-defined. Note that, from (B1), $\Phi(\lambda,0)=0$, for every $\lambda\in\Lambda_0$.

As $\Phi(\lambda,x)(0)=x(0)-x(T)$, it is clear that, for each $\lambda\in\Lambda_0$, there is a one-to-one correspondence between the zeros of the operator $\Phi(\lambda,\cdot)$, and the solutions of the GODE (5.1). By this fact, it is enough to define a bifurcation point for the equation $\Phi(\lambda,x)=0$, where Φ is given by (5.2).

This new concept for GODEs is inspired by [17], p. 143 and [2], p. 370.

Definition 5.1. A couple $(\lambda_0,0)\in\Lambda_0\times\overline{\Omega}$ is called a *bifurcation point* of the equation $\Phi(\lambda,x)=0$, if every neighborhood of $(\lambda_0,0)\in\Lambda_0\times\overline{\Omega}$ contains a solution (λ,x) of the equation $\Phi(\lambda,x)=0$ such that $x\neq 0$.

Now, let us consider an operator

$$\mathcal{N} : \Lambda_0 \times \overline{\Omega} \to G, \quad (\lambda, x) \ \mapsto \ \mathcal{N}(\lambda, x)$$

given by

$$\mathcal{N}(\lambda, x)(s) = x(T) + \int_0^s DF(\lambda, x(\tau), t), \quad s \in [0, T]. \tag{5.3}$$

Notice that the operator Φ, given by (5.2), can be written as

$$\Phi(\lambda, x)(s) = x(s) - \mathcal{N}(\lambda, x)(s), \quad \text{for every } \lambda \in \Lambda_0, \ s \in [0, T].$$

In order to prove the main result of this section, which ensures the existence of a bifurcation point for equation (5.1), we use the Leray-Schauder degree theory. At first, we need to prove that the operator $\mathcal{N} : \Lambda_0 \times \overline{\Omega} \to G$, given by (5.3), satisfies some conditions which are described by the next two propositions. The first proposition ensures that $\mathcal{N}$ is compact with respect to the second variable. A proof of it can be carried out by applying the proofs of Propositions 3.3 and 3.4.

Proposition 5.2. *If, for each $\lambda \in \Lambda_0$ fixed, $F(\lambda, \cdot, \cdot) \in \mathcal{F}(B_R \times [0, T], h_R, \omega_R)$ for all $R > 0$, then, the following properties are satisfied.*

(i) *For each fixed $\lambda \in \Lambda_0$, the operator $\mathcal{N}(\lambda, \cdot) : \overline{\Omega} \to G$ is continuous on $\overline{\Omega}$.*

(ii) *For each fixed $\lambda \in \Lambda_0$, the operator $\mathcal{N}(\lambda, \cdot) : \overline{\Omega} \to G$ takes bounded sets of G into relatively compact sets of $\overline{\Omega}$.*

Proposition 5.3. *Suppose for each fixed $\lambda \in \Lambda_0$, $F(\lambda, \cdot, \cdot) \in \mathcal{F}(B_R \times [0, T], h_R, \omega_R)$ for all $R > 0$, and assume that the following condition holds:*

(B2) *There is a nondecreasing function $g : [0, T] \to \mathbb{R}$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|F(\lambda_1, z, t) - F(\lambda_2, z, t) - F(\lambda_1, z, s) + F(\lambda_2, z, s)\| \leq \varepsilon |g(t) - g(s)|$$

for all $z \in B_R$, $t, s \in [0, T]$, and λ_1, $\lambda_2 \in \Lambda_0$ with $|\lambda_1 - \lambda_2| \leq \delta$.

Then,

$$\|\mathcal{N}(\lambda_1, x) - \mathcal{N}(\lambda_2, x)\|_\infty < \varepsilon \ [g(T) - g(0)]$$

wherever $|\lambda_1 - \lambda_2| \leq \delta$ and $x \in \overline{\Omega}$.

Proof. Let $\varepsilon > 0$ and consider $\delta > 0$ as in (B2). Therefore,

$$\begin{aligned} \|\mathcal{N}((\lambda_1, x) - \mathcal{N}(\lambda_2, x)\|_\infty &= \sup_{s \in [0,T]} \|\mathcal{N}(\lambda_1, x)(s) - \mathcal{N}(\lambda_2, x)(s)\| \\ &= \sup_{s \in [0,T]} \left\{ \left\| \int_0^s D[F(\lambda_1, x(\tau), t) - F(\lambda_2, x(\tau), t)] \right\| \right\} \\ &\leq \varepsilon \ [g(T) - g(0)] \end{aligned}$$

for every $x \in \overline{\Omega}$ and $\lambda_1, \lambda_2 \in \Lambda_0$ with $|\lambda_1 - \lambda_2| \leq \delta$, since Lemma 2.7 holds. □

Proposition 5.4. *Suppose for each $\lambda \in \Lambda_0$, $F(\lambda, \cdot, \cdot) \in \mathcal{F}(B_R \times [0,T], h_R, \omega_R)$ for all $R > 0$, and condition* (B2) *holds. Then, the operator $\mathcal{N} : \Lambda_0 \times \overline{\Omega} \to G$ is compact on $\Lambda_0 \times \overline{\Omega}$.*

Proof. A proof is carried out by applying the proofs of Propositions 3.3, 3.4 and 5.3. □

In the following lines, $B(0, r)$ denotes the open ball in G centered at the origin and of radius $r > 0$, that is,

$$B(0, r) = \{x \in G, \ \|x\|_\infty < r\}. \tag{5.4}$$

The next result is an adaptation of Lemma X.5 from [13] to the framework of GODEs.

Proposition 5.5. *Suppose for each $\lambda \in \Lambda_0$, $F(\lambda, \cdot, \cdot) \in \mathcal{F}(B_R \times [0,T], h_R, \omega_R)$ for all $R > 0$ and that conditions* (B1) *and* (B2) *hold. Assume that $[\lambda_1, \lambda_2] \subset \Lambda_0$ contains no bifurcation point for the equation $\Phi(\lambda, x) = 0$, where Φ is given by* (5.2)*. Then, there exists $\delta > 0$ such that, for each $\lambda \in [\lambda_1, \lambda_2]$ and each $x \in B(0, \delta) \cap \overline{\Omega}$, if we have*

$$x = \mathcal{N}(\lambda, x),$$

then $x = 0$, where the operator $\mathcal{N}$ is defined in (5.3)*.*

Proof. By Proposition 5.4, the operator $\mathcal{N}$ is compact on $\Lambda_0 \times \overline{\Omega}$. Then, the set

$$\mathcal{C} = \big\{(\lambda, x) \in [\lambda_1, \lambda_2] \times \overline{\Omega} \ ; \ x = \mathcal{N}(\lambda, x)\big\}$$

is also compact on $\Lambda_0 \times G$. Suppose the statement does not hold. Then, for each $n \in \mathbb{N}^*$, there exist sequences $\lambda_n \in [\lambda_1, \lambda_2]$ and $x_n \in B(0, \frac{1}{n}) \cap \overline{\Omega}$ such that

$$x_n = \mathcal{N}(\lambda_n, x_n), \quad x_n \neq 0. \tag{5.5}$$

Using the compactness of the set $\mathcal{C}$, there exists a subsequence (λ_{n_k}, x_{n_k}) of (λ_n, x_n), which we will denote by (λ_n, x_n), satisfying

$$(\lambda_n, x_n) \to (\lambda_0, x_0) \quad \text{as } n \to \infty.$$

Moreover, $(\lambda_0, x_0) \in [\lambda_1, \lambda_2] \times \overline{\Omega}$ and $x_0 = 0$. On the other hand, by (5.5), $(\lambda_0, 0)$ is a bifurcation point of the equation $\Phi(\lambda, x) = 0$, which contradicts the hypothesis. □

Now, we establish conditions for the existence of a bifurcation point for GODEs.

Theorem 5.6. *Suppose for each $\lambda \in \Lambda_0$ fixed, $F(\lambda, \cdot, \cdot) \in \mathcal{F}(B_R \times [0,T], h_R, \omega_R)$ for all $R > 0$ and* (B1) *and* (B2) *are valid. Let $\Phi : \Lambda_0 \times \overline{\Omega} \to G$ and $\mathcal{N} : \Lambda_0 \times \overline{\Omega} \to G$ be given by* (5.2) *and* (5.3) *respectively. If, we have $[\lambda_1, \lambda_2] \subset \Lambda_0$ and*

$$\text{ind}_{LS}[I - \mathcal{N}(\lambda_1, \cdot), 0] \neq \text{ind}_{LS}[I - \mathcal{N}(\lambda_2, \cdot), 0], \tag{5.6}$$

then there exists $\lambda_0 \in [\lambda_1, \lambda_2]$ such that $(\lambda_0, 0)$ is a bifurcation point of equation $\Phi(\lambda, x) = 0$.

Proof. By the definition of the Leray-Schauder index, there exist $\delta_1, \delta_2 > 0$ such that the equations

$$\mathcal{N}(\lambda_1, x) = x, \quad x \in B(0, \delta_1)$$

and

$$\mathcal{N}(\lambda_2, x) = x, \quad x \in B(0, \delta_2),$$

have the unique solution $x = 0$.

Suppose $[\lambda_1, \lambda_2] \subset \Lambda_0$ contains no bifurcation points with respect to the trivial solution of the equation $\Phi(\lambda, x) = 0$. By Proposition 5.5, there is $\delta > 0$ having the property that, for each $\lambda \in [\lambda_1, \lambda_2]$ the only $x \in B(0, \delta) \cap \overline{\Omega}$ such that

$$x = \mathcal{N}(\lambda, x)$$

is $x = 0$. Thus, taking $\alpha = \min\{\delta_1, \delta_2, \delta\}$, we obtain

$$\mathcal{N}(\lambda, x) \neq x, \quad \text{for every} \ \ (\lambda, x) \in [\lambda_1, \lambda_2] \times B(0, \alpha) \setminus \{0\}. \tag{5.7}$$

Define an operator

$$\overline{\mathcal{N}} : [0, 1] \times \overline{B(0, \alpha)} \to G, \quad (\mu, x) \ \mapsto \ \mathcal{N}(\mu, x)$$

given by

$$\overline{\mathcal{N}}(\mu, x)(s) = x(T) + \int_0^s DF(\mu\lambda_2 + (1 - \mu)\lambda_1), x(\tau), t), \quad \text{for all } s \in [0, T].$$

By Propositions 5.2 and 5.3, $\overline{\mathcal{N}}$ is a homotopy of compact transformations on $\overline{B(0, \alpha)}$. In order to prove that $\deg_{LS}(I - \overline{\mathcal{N}}(\mu, \cdot), B(0, \alpha), 0)$ is independent of $\mu \in [0, 1]$, it is enough to prove that

$$0 \ \notin \ (I - \overline{\mathcal{N}}(\mu, \cdot))(\partial B(0, \alpha)), \quad \text{for all } \mu \in [0, 1]. \tag{5.8}$$

By item (i) of Proposition 5.2 and by (5.7), we have

$$x - \overline{\mathcal{N}}(\mu, x) \neq 0, \quad \text{for every} \ \ (\mu, x) \in [0, 1] \times \overline{B(0, \alpha)} \setminus \{0\}. \tag{5.9}$$

Note that if $x \in \partial B(0, \alpha)$, then $\|x\|_\infty = \alpha \neq 0$. Thus,

$$x - \overline{\mathcal{N}}(\mu, x) \neq 0, \quad \text{for every} \ \ (\mu, x) \in [0, 1] \times \partial B(0, \alpha)$$

and (5.8) holds.

By Theorem 4.5, we obtain

$$\deg_{LS}[I - \overline{\mathcal{N}}(0, \cdot), B(0, \alpha), 0] = \deg_{LS}[I - \overline{\mathcal{N}}(1, \cdot), B(0, \alpha), 0].$$

By definition of the operator $\mathcal{N}$ in (5.3), we also have

$$\deg_{LS}[I - \overline{\mathcal{N}}(0, \cdot), B(0, \alpha), 0] = \deg_{LS}[I - \mathcal{N}(\lambda_1, \cdot), B(0, \alpha), 0] \quad \text{and}$$

$$\deg_{LS}[I - \overline{\mathcal{N}}(1,\cdot), B(0,\alpha), 0] = \deg_{LS}[I - \mathcal{N}(\lambda_2,\cdot), B(0,\alpha), 0].$$

Then, by Definition 4.7, we conclude that

$$\begin{aligned}
\operatorname{ind}_{LS}[I - \mathcal{N}(\lambda_1,\cdot), 0] &= \deg_{LS}[I - \mathcal{N}(\lambda_1,\cdot), B(0,\alpha), 0] \\
&= \deg_{LS}[I - \mathcal{N}(\lambda_2,\cdot), B(0,\alpha), 0] \\
&= \operatorname{ind}_{LS}[I - \mathcal{N}(\lambda_2,\cdot), 0],
\end{aligned}$$

which contradicts (5.6) and the proof is complete. □

6 Applications to IDEs

In this section, our goal is to apply the results from the previous sections to IDEs, using the correspondence between IDEs and GODEs, which can be found in [10], Theorems 3.4 and 3.5 (see also [30], Theorem 5.20).

Consider the following IDE

$$\begin{cases} x'(t) = f(x(t),t), & t \neq t_i \\ \Delta x(t_i) = I_i(x(t_i)), & i = 1,\ldots,m, \end{cases} \tag{6.1}$$

where $f : B_R \times [0,T] \to \mathbb{R}^n$, t_i, for $i = 1,\ldots,m$, are pre-assigned moments of impulse effects, with $0 < t_1 < \ldots < t_m < T$, the impulsive operators $I_i : B_R \subset \mathbb{R}^n \to \mathbb{R}^n$, for $i = 1,\ldots,m$ are continuous functions and

$$\Delta x(t_i) := x(t_i+) - x(t_i-) = x(t_i+) - x(t_i), \quad i = 1,\ldots,m, \tag{6.2}$$

that is, we assume that x is left continuous at $t = t_i$ and the lateral limits $x(t_i+)$ exist, for $i = 1,\ldots,m$.

The IDE (6.1) is equivalent to the integral equation

$$x(t) = x(0) + \int_0^t f(x(s),s)ds + \sum_{0<t_i<t} I_i(x(t_i)), \quad t \in [0,T] \tag{6.3}$$

where the integral exists in the some sense. We consider Lebesgue integrability here, but one could consider Kurzweil-Henstock integrability instead.

For $d \in [0,T)$, we define the left continuous Heaviside function H_d by

$$H_d(t) = \begin{cases} 0, & t \leq d \\ 1, & t > d \end{cases}$$

Then,

$$\sum_{0<t_i<t} I_i(x(t_i)) = \sum_{i=1}^{m} I_i(x(t_i)) H_{t_i}(t), \quad t \in [0,T].$$

Therefore equation (6.3) can be rewritten as

$$x(t) = x(0) + \int_0^t f(x(s), s)ds + \sum_{i=1}^{m} I_i(x(t_i))H_{t_i}(t), \quad t \in [0, T].$$

Definition 6.1. A function $x\colon [0, T] \to \mathbb{R}^n$ is called a Carathéodory solution or simply a solution of the IDE (6.1), if it satisfies:

(i) $x(t) \in B_R$, for all $t \in [0, T]$ and x is absolutely continuous on each interval $[0, t_1]$, $(t_i, t_{i+1}]$, for $i = 1, ..., m-1$, $(t_m, T]$,

(ii) $x'(t) = f(x(t), t)$, for almost all t, such that $t \neq t_i$,

(iii) $\Delta x(t_i) := x(t_i+) - x(t_i) = I_i(x(t_i))$, $i = 1, ..., m$.

We also denote by $L^1([0, T], \mathbb{R}^n)$ the space of Lebesgue integrable functions from $[0, T]$ to $\mathbb{R}^n$ with finite integral. Let the function $f : B_R \times [0, T] \to \mathbb{R}^n$ satisfies the following conditions :

(H1) for any $z \in B_R$, $f(z, \cdot) \in L^1([0, T], \mathbb{R}^n)$;

(H2) there exists a function $M_1 \in L^1([0, T], \mathbb{R})$ such that, for all $t_1, t_2 \in [0, T]$ and all $z \in B_R$,

$$\left\| \int_{t_1}^{t_2} f(z, s)\, ds \right\| \leq \int_{t_1}^{t_2} M_1(s)\, ds;$$

(H3) there exists a function $M_2 \in L^1([0, T], \mathbb{R})$ such that, for all $t_1, t_2 \in [0, T]$ and all $z, w \in B_R$,

$$\left\| \int_{t_1}^{t_2} [f(z, s) - f(w, s)]\, ds \right\| \leq \omega_R(\|z - w\|) \int_{t_1}^{t_2} M_2(s)\, ds.$$

Remark 6.2. It is important to mention here that hypotheses (H2) and (H3) above are Carathéorody- and Lipschitz-type conditions on the indefinite integral of f and not on the function f itself. Carathéorody- and Lipschitz-type conditions on f would read as

(H2′) there exists a function $M_1 \in L^1([0, T], \mathbb{R})$ such that, for all $s \in [0, T]$ and all $z \in B_R$,

$$\|f(z, s)\| \leq M_1(s);$$

(H3′) there exists a function $M_2 \in L^1([0, T], \mathbb{R})$ such that, for all $s \in [0, T]$ and all $z, w \in B_R$,

$$\|f(z, s) - f(w, s)\| \leq M_2(s)\, \omega_R(\|z - w\|).$$

Assumptions on the indefinite integral instead of the function f happen to allow that the integrand is nowhere continuous. Take the Dirichlet function on $[0, T]$, for instance. Although its indefinite integral is Carathéodory and Lipschitzian, the Dirichlet function is not continuous and, hence, not Lipschitzian.

Concerning the impulse operators, we assume that the following conditions hold:

(H4) There exists $K_1 > 0$ such that $\|I_i(z)\| \leq K_1$ for all $z \in B_R$ and $i = 1, \dots, m$.

(H5) There exists $K_2 > 0$ such that $\|I_i(z) - I_i(w)\| \leq K_2\|z - w\|$, for all $z, w \in B_R$ and $i = 1, \dots, m$.

For each pair (z, t) in $B_R \times [0, T]$, we define

$$F(z,t) := \int_0^t f(z,s)\,ds + \sum_{i=1}^{m} I_i(z) H_{t_i}(t), \tag{6.4}$$

Then, taking a function $h_R : [0, T] \to \mathbb{R}$ given by

$$h_R(t) := \int_0^t [M_1(s) + M_2(s)]\,ds + \max\{K_1, K_2\} \sum_{i=1}^{m} H_{t_i}(t), \quad t \in [0, T], \tag{6.5}$$

one can conclude that:

(a) h_R is nondecreasing and left continuous;

(b) $F \in \mathcal{F}(B_R \times [0, T], h_R, \omega_R)$, for all $R > 0$;

(c) $\displaystyle\int_0^t DF(x(\tau), s) = \int_0^t f(x(s), s)\,ds + \sum_{0<t_i<t} I_i(x(t_i))$, for every $t \in [0, T]$.

See the calculations in [10], and also in [30], Proposition 5.12. Under the above conditions, the following result holds true and establishes a one-to-one correspondence between the IDE (6.1) and its corresponding GODE. A proof of it follows as in [30], Theorem 5.20. See also [10], Theorems 3.4 and 3.5 for a more general situation.

Theorem 6.3. *Suppose conditions* (H1) *to* (H5) *hold. A function $x : [0, T] \to \mathbb{R}^n$ is a solution of the IDE* (6.1) *if and only if x is a solution of the GODE*

$$\frac{dx}{d\tau} = DF(x, t),$$

where F is given by (6.4).

Now, we present the definition of a T-periodic solution of the IDE (6.1). Let $G^-([0, T], \mathbb{R}^n)$ denote the set of all the regulated functions $x : [0, T] \to \mathbb{R}^n$ which are continuous from the left on $[0, T]$. It is clear that any solution of the IDE (6.1) is an element of $G^-([0, T], \mathbb{R}^n)$.

Definition 6.4. A function $x \in G^-([0, T], \mathbb{R}^n)$ is a T-periodic solution of the IDE (6.1), if it is a solution of (6.1) such that $x(0) = x(T)$.

The next result is the corresponding version of Theorem 4.8 for IDEs.

Theorem 6.5. *Suppose conditions* (H1) *to* (H5) *are satisfied. Assume that there exists an open bounded set $\Delta \subset G^-([0, T], \mathbb{R}^n)$ such that the following conditions hold:*

(i) *For any $\lambda \in (0, 1]$, the IDE*

$$x'(t) = \lambda f(x(t), t), \; t \neq t_i, \quad \Delta x(t_i) = \lambda I_i(x(t_i)) \quad i = 1, \dots, m \tag{6.6}$$

has no T-periodic solution $x \in G^-([0, T], \mathbb{R}^n) \cap \partial\Delta$.

(ii) *The equation*

$$\psi(a) := \int_0^T f(a,s)\,ds + \sum_{0<t_i<T} I_i(a) = 0 \tag{6.7}$$

has no solution on $\partial\Delta \cap \mathbb{R}^n$, where $\mathbb{R}^n$ is viewed as the set of constant functions in $G^-([0,T],\mathbb{R}^n)$.

(iii) $deg[\psi, \Delta \cap \mathbb{R}^n, 0] \neq 0$.

Then the IDE (6.1) *has at least one T-periodic solution in* Δ.

In the next lines, we present an example, which satisfies all the conditions of Theorem 6.5.

Example 6.6. Denote by $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$ the set of irrational numbers, and consider the periodic bounndary value problem with impulses

$$\begin{cases} \dot{x}(t) = \chi_{\mathbb{I}}(t)\frac{x(t)}{1+|x(t)|} - E'(t) := f(x(t),t), \quad t \in [0,1] \setminus \{\frac{1}{2}\}, \\ \Delta x(1/2) = \frac{1}{2}, \\ x(0) = x(1), \end{cases} \tag{6.8}$$

where $E : [0,1] \to \mathbb{R}$ is absolutely continuous and such that $E(0) = E(1)$. It is easy to check that the hypotheses (H2′) and (H3′) are fulfilled and, clearly, conditions (H2) and (H3) as well. Let $\lambda \in (0,1]$ and $x(t)$ be a possible solution of the problem

$$\begin{cases} \dot{x}(t) = \lambda\chi_{\mathbb{I}}(t)\frac{x(t)}{1+|x(t)|} - \lambda E'(t), \quad t \in [0,1] \setminus \{\frac{1}{2}\}, \\ \Delta x(1/2) = \frac{\lambda}{2}, \\ x(0) = x(1), \end{cases} \tag{6.9}$$

Equivalently, for $t \in [0,1]$, using (6.3),

$$x(t) = x(0) + \lambda \int_0^t \chi_{\mathbb{I}}(s)\frac{x(s)}{1+|x(s)|}\,ds + \lambda\frac{H_{1/2}(t)}{2} + \lambda[E(t) - E(0)],$$
$$x(0) = x(1),$$

or, as $\int_0^t \chi_{\mathbb{Q}}(s)\frac{x(s)}{1+|x(s)|}\,ds = 0$,

$$x(t) = x(0) + \lambda \int_0^t \frac{x(s)}{1+|x(s)|}\,ds + \lambda\frac{H_{1/2}(t)}{2} + \lambda[E(t) - E(0)],$$
$$x(0) = x(1).$$

For $t = 1$, this equation becomes

$$0 = \int_0^1 \frac{x(s)}{1+|x(s)|}\,ds + \frac{1}{2},$$

and there exists $\tau \in [0,1]$ such that

$$\frac{x(\tau)}{1+|x(\tau)|} = -\frac{1}{2},$$

that is such that $x(\tau) = -1$. Consequently,

$$x(t) = -1 + \lambda \int_\tau^t \frac{x(s)}{1+|x(s)|}\, ds + \lambda \frac{H_{1/2}(t) - H_{1/2}(\tau)}{2} + \lambda[E(t) - E(\tau)],$$
$$x(0) = x(1),$$

This implies that $|x(t)| < 3 + 2\|E\|_\infty$ for any possible solution of (6.9). So condition (i) of Theorem 6.5 holds for

$$\Delta = \{x \in G^-([0,1), \mathbb{R}) : \|x\|_\infty < 3 + 2\|E\|_\infty\}.$$

On the other hand, for each $a \in \mathbb{R}$,

$$\psi(a) = \int_0^1 \chi_{\mathbb{I}}(s) \frac{a}{1+|a|}\, ds + \frac{1}{2} = \frac{a}{1+|a|} + \frac{1}{2} \tag{6.10}$$

has the unique zero $a = -1 \in (-3 - 2\|E\|_\infty, 3 + 2\|E\|_\infty)$ with $\psi'(-1) = \frac{1}{4} > 0$, so that $\deg[\psi, (-3 - 2\|E\|_\infty, 3 + 2\|E\|_\infty, 0) = +1$. Therefore, conditions (ii) and (iii) of Theorem 6.5 also hold and the existence of a solution follows.

In what follows, we state the corresponding bifurcation result to Theorem 5.6 for IDEs.

Let $\Lambda_0 \subset \mathbb{R}$. Consider the following periodic boundary value problem for an IDE depending on $\lambda \in \Lambda_0$ given by

$$\begin{cases} x'(t) = f(\lambda, x(t), t), & t \neq t_i \\ \Delta x(t_i) = I_i(x(t_i)), & i = 1, \ldots, m \\ x(0) = x(T), \end{cases} \tag{6.11}$$

where $f \colon \Lambda_0 \times B_R \times [0,T] \to \mathbb{R}^n$, t_i, for $i = 1, \ldots, m$, with $0 < t_1 < \ldots < t_m < T$, are pre-assigned moments of impulse effects, $I_i \colon B_R \to \mathbb{R}^n$ are continuous impulse operators, for $i = 1, \ldots, m$, and

$$\Delta x(t_i) := x(t_i+) - x(t_i-) = x(t_i+) - x(t_i) = I_i(x(t_i)),$$

that is, we assume that x is left continuous at $t = t_i$ and the lateral limit $x(t_i+)$ exists, for $i = 1, \ldots, m$. Note that we are considering that the impulse operators do not depend on λ.

As before, the integral form of problem (6.11) is given, using Heaviside functions, by

$$x(t) = x(T) + \int_0^t f(\lambda, x(s), s) ds + \sum_{i=1}^m I_i(x(t_i)) H_{t_i}(t), \quad t \in [0,T].$$

Assume that $f : \Lambda_0 \times B_R \times [0,T] \to \mathbb{R}^n$ satisfies the following conditions:

($H_{1\lambda}$) for each $(\lambda, z) \in \Lambda_0 \times B_R$ fixed, the function $f(\lambda, z, \cdot) \in L^1([0,T], \mathbb{R}^n)$;

($H_{2\lambda}$) there exists a function $N_1 \in L^1([0,T], \mathbb{R})$ such that, for all $\lambda \in \Lambda_0$, all $t_1, t_2 \in [0,T]$, and all $z \in B_R$, we have

$$\left\| \int_{t_1}^{t_2} f(\lambda, z, s)\, ds \right\| \leq \int_{t_1}^{t_2} N_1(s)\, ds;$$

($H_{3\lambda}$) there exists a function $N_2 \in L^1([0,T], \mathbb{R})$ such that, for all $\lambda \in \Lambda_0$, all $t_1, t_2 \in [0,T]$, and all $z, w \in B_R$, we have

$$\left\| \int_{t_1}^{t_2} [f(\lambda, z, s) - f(\lambda, w, s)]\, ds \right\| \leq \omega_R(\|z - w\|) \int_{t_1}^{t_2} N_2(s)\, ds.$$

Moreover, assume that the impulse operators satisfy conditions (H4) and (H5).

For each triple (λ, z, t) in $\Lambda_0 \times B_R \times [0,T]$, we define

$$F(\lambda, z, t) := \int_0^t f(\lambda, z, s)\, ds + \sum_{i=1}^{m} I_i(z) H_{t_i}(t) \tag{6.12}$$

and a function $h_R : [0,T] \to \mathbb{R}$ by

$$h_R(t) := \int_0^t [N_1(s) + N_2(s)]\, ds + \max\{K_1, K_2\} \sum_{i=1}^{m} H_{t_i}(t), \quad t \in [0,T]. \tag{6.13}$$

As before, one can prove that

(a') h_R is nondecreasing and left continuous;

(b') for each $\lambda \in \Lambda_0$ fixed, $F(\lambda, \cdot, \cdot) \in \mathcal{F}(B_R \times [0,T], h_R, \omega_R)$, for all $R > 0$;

(c') $\displaystyle\int_0^t DF(\lambda, x(\tau), s) = \int_0^t f(\lambda, x(s), s)\, ds + \sum_{0<t_i<t} I_i(x(t_i)), \quad t \in [0,T].$

Analogously to Theorem 6.3, under the conditions above, one can prove that $x : [0,T] \to \mathbb{R}^n$ is a solution of periodic boundary value problem for the impulsive system (6.11), if and only if, it is a solution of the periodic boundary value problem for GODE

$$\frac{dx}{d\tau} = DF(\lambda, x, t), \quad x(0) = x(T),$$

where F is given by (6.12).

Let $\Omega_1 \subset G^-([0,T], \mathbb{R}^n)$ be an open and bounded set such that $0 \in \Omega_1$ and consider the operators

$$\phi : \Lambda_0 \times \overline{\Omega}_1 \to G^-([0,T], \mathbb{R}^n)$$

given by

$$\phi(\lambda, x)(t) = x(t) - x(T) - \int_0^t f(\lambda, x(s), s)\, ds - \sum_{0<t_i<t} I_i(x(t_i)), \quad t \in [0,T], \tag{6.14}$$

and $N : \Lambda_0 \times \overline{\Omega}_1 \to G^-([0,T],\mathbb{R}^n)$ given by

$$N(\lambda, x)(t) = x(T) + \int_0^t f(\lambda, x(s), s)\, ds + \sum_{0<t_i<t} I_i(x(t_i)), \quad t \in [0,T].$$

Notice that the operator ϕ given by (6.14) can be written as

$$\phi(\lambda, x)(t) = x(t) - N(\lambda, x)(t), \quad t \in [0,T],$$

As there is a one-to-one correspondence between the zeros of the operator $\phi(\lambda,\cdot)$, for each $\lambda \in \Lambda_0$ and the solutions of the periodic boundary value problem for the IDE (6.11), it is enough to define a bifurcation point (with respect to trivial solution) for the equation $\phi(\lambda, x) = 0$, where ϕ is given by (6.14).

Definition 6.7. A couple $(\lambda_0, 0) \in \Lambda_0 \times \overline{\Omega}_1$ is called a *bifurcation point* of the equation $\phi(\lambda, x) = 0$, if every neighborhood of $(\lambda_0, 0) \in \Lambda_0 \times \overline{\Omega}_1$ contains a solution (λ, x) of the equation $\phi(\lambda, x) = 0$ such that $x \neq 0$.

Our next result presents conditions on the existence of a bifurcation point of equation $\phi(\lambda, x) = 0$, with ϕ given by (6.14).

Theorem 6.8. *Let $\phi : \Lambda_0 \times \overline{\Omega}_1 \to G^-([0,T],\mathbb{R}^n)$ be given by (6.14). Suppose conditions $(\mathrm{H}_{1\lambda})$ to $(\mathrm{H}_{3\lambda})$, (H4), and (H5) are valid and $[\lambda_1, \lambda_2] \subset \Lambda_0$. Suppose the following properties hold:*

(i) $f(\lambda, 0, t) = 0$, *for all* $\lambda \in \Lambda_0$ *and* $t \in [0,T]$;

(ii) $I_i(0) = 0$, *for every* $i = \{1, ..., m\}$;

(iii) *Assume that there is a nondecreasing function $k : [0,T] \to \mathbb{R}$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ satisfying*

$$\left\| \int_s^t [f(\lambda_1, z, r) - f(\lambda_2, z, r)]\, dr \right\| \leq \varepsilon |k(t) - k(s)|$$

for every $t, s \in [0,T]$ and $\lambda_1, \lambda_2 \in \Lambda_0$ with $|\lambda_1 - \lambda_2| \leq \delta$.

If, moreover,

$$\mathrm{ind}_{LS}(I - N(\lambda_1, \cdot), 0) \neq \mathrm{ind}_{LS}(I - N(\lambda_2, \cdot), 0),$$

then there exists $\lambda_0 \in [\lambda_1, \lambda_2]$ such that $(\lambda_0, 0)$ is a bifurcation point of $\phi(\lambda, x) = 0$.

Proof. Let F be given by (6.12). Conditions (B1) and (5.6) of Theorem 5.6 are not difficult to prove. Then, we will only prove that condition (B2) from Proposition 5.3 holds.

By condition (iii) above, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\begin{aligned} &\|F(\lambda_1, z, t) - F(\lambda_2, z, t) - F(\lambda_1, z, s) + F(\lambda_2, z, s)\| \\ &= \left\| \int_s^t [f(\lambda_1, z, r) - f(\lambda_2, z, r)]\, dr \right\| \leq \varepsilon |k(t) - k(s)| \end{aligned}$$

wherever $|\lambda_1 - \lambda_2| \leq \delta$ and $s, t \in [0,T]$. Now, we can apply Theorems 5.6 and 6.3 and the statement follows. □

Next, we recall a few more useful concepts. Let X be a Banach space and consider $L : X \to X$ a compact linear operator. We say that $\lambda_0 \in \mathbb{R}$ is a characteristic number of L if there exists $x \in X$ satisfying

$$x = \lambda_0 L x,$$

such that $x \neq 0$. The algebraic multiplicity of λ_0 is equal to the number of linearly independent eigenvectors $x \in X$.

The following results whose proof can be found in [34] Section 13.7 and Corollary 14.6, will be employed in the example following them.

Proposition 6.9. *Let X be a Banach space, $L : X \to X$ a compact linear operator, and $R : B(0, R) \to X$ a compact operator such that*

$$\lim_{\|x\|\to 0} \frac{\|R(x)\|}{\|x\|} = 0.$$

If λ is not a characteristic value of L, then

$$\mathrm{ind}_{LS}(I - \lambda L - R, 0) = \mathrm{ind}_{LS}(I - \lambda L, 0) = \pm 1.$$

Proposition 6.10. *Let X be a Banach space, $L_1, L_2 : X \to X$ compact linear operators such that $I - L_1$ and $I - L_2$ are invertible. Then*

$$\mathrm{ind}_{LS}((I - L_1)(I - L_2), 0) = \mathrm{ind}_{LS}(I - L_1, 0) \cdot \mathrm{ind}_{LS}(I - L_2, 0).$$

Proposition 6.11. *Let X be a Banach space and $L : X \to X$ a compact linear operator. If λ_0 is a characteristic number of L of algebraic multiplicity $\alpha(\lambda_0)$, then*

$$\mathrm{ind}_{LS}(I - (\lambda_0 + \beta)L(\cdot), 0) = (-1)^{\alpha(\lambda_0)} \mathrm{ind}_{LS}(I - (\lambda_0 - \beta)L(\cdot), 0)$$

for all sufficiently small $\beta > 0$.

Example 6.12. Consider the periodic boundary value problem for the differential equation subject to a single impulse effect

$$\begin{cases} \dot{x}(t) = \lambda b(t)x(t) + c(t)x^2(t) = f(\lambda, x(t), t), & t \in [0, 1] \setminus \{\frac{1}{2}\}, \\ \Delta x(1/2)) = x^2(1/2), & \text{if } \ t = \frac{1}{2} \\ x(0) = x(1), \end{cases} \tag{6.15}$$

where $b, c \in L^1([0, 1], \mathbb{R})$ and $\int_0^1 b(t)\, dt \neq 0$. The linearized periodic boundary value problem

$$\dot{x}(t) = \lambda b(t)x(t), \;\; \Delta x(1/2) = 0, \;\; x(0) = x(1)$$

has a nontrivial solution if and only if $\lambda = 0$, and hence we can assume, say, that $\lambda \in [-1, 1]$ and $x(t) \in [-1, 1]$, as we are interested in solutions close to 0. It is clear that $|f(\lambda, x, \cdot)|$ is bounded by a Carathéodory function on $[-1, 1] \times [-1, 1] \times [0, 1]$ and conditions (H4) and (H5) on the impulse operators are trivially satisfied. It is not difficult to check that conditions ($\mathrm{H}_{2\lambda}$) and

($H_{3\lambda}$) are fulfilled and that hypotheses (i), (ii) and (iii) of Theorem 6.8 are trivially satisfied.

In this case, we take $\Omega_1 = B(0,1) \subset G^-([0,1],\mathbb{R})$ and study the corresponding fixed point operator

$$\phi : [-1,1] \times \overline{B(0,1)} \to G^-([0,1],\mathbb{R})$$

given by

$$\phi(\lambda,x)(t) = x(t) - x(1) - \lambda \int_0^t b(s)x(s)\,ds - \int_0^t c(s)x^2(s)\,ds + H_{1/2}(s)x^2(1/2), \quad t \in [0,1].$$

If we define the linear compact operators $L_1, P_1 : G^-([0,1],\mathbb{R}) \to G^-([0,1],\mathbb{R})$ by

$$(L_1x)(t) = \int_0^t b(s)x(s)\,ds, \quad (P_1x)(t) = x(1), \quad t \in [0,1],$$

the linear compact perturbation of identity $L_2 : G^-([0,1],\mathbb{R}) \to G^-([0,1],\mathbb{R})$ by

$$L_2 = I - P_1 - L_1,$$

and the nonlinear compact operator $R : \overline{B(0,1)} \to G^-([0,1],\mathbb{R})$ by

$$R(x)(t) = \int_0^t c(s)x^2(s)\,ds + H_{1/2}(s)x^2(1/2), \quad t \in [0,1],$$

then

$$L_2x = 0 \iff x(0) = x(1) \quad \text{and} \quad x'(t) = b(t)x(t) \iff x \equiv 0$$

because $\int_0^1 b(t)\,dt \neq 0$, and L_2 is invertible. Furthermore,

$$\lim_{x\to 0} \frac{\|R(x)\|_\infty}{\|x\|_\infty} = 0.$$

Hence,

$$\phi(\lambda,x) = L_2x - (\lambda-1)L_1x - R(x) = L_2[x - (\lambda-1)L_2^{-1}L_1x - L_2^{-1}R(x)],$$

for all $(\lambda,x) \in [-1,1] \times B(0,1)$. Consequently, if $-1 < \lambda_- < 0 < \lambda_+ < 1$, we have, from the Propositions 6.9, 6.10 and 6.11 on the Leray-Schauder index,

$$\begin{aligned}\mathrm{ind}_{LS}(\phi(\lambda_-,0)) &= \mathrm{ind}_{LS}(L_2[I - (\lambda_-)L_2^{-1}L_1],0) = \mathrm{ind}_{LS}(L_2,0)\cdot \mathrm{ind}_{LS}(I - (\lambda_-)L_2^{-1}L_1,0)\\ &= -\mathrm{ind}_{LS}(L_2,0)\cdot \mathrm{ind}_{LS}(I - (\lambda_+)L_2^{-1}L_1,0) = -\mathrm{ind}_{LS}(\phi(\lambda_+,0)),\end{aligned}$$

because -1 is a characteristic value of multiplicity one of the linear operator $L_2^{-1}L_1$, as it follows from the fact that

$$x + L_2^{-1}L_1x = 0 \iff L_2x + L_1x = 0 \iff x(t) - x(1) = 0 \ \ (t \in [0,1]).$$

Theorem 6.8 implies that $(0,0)$ is a bifurcation point for (6.15).

7 Final remark

One can easily check that, with appropriate adaptations, our results also apply to measure differential equations which can be regarded as GODEs (see [8, Theorems 3.8 and 3.9]). Moreover, since measure differential equations encompass not only differential equation with impulses (see [9, Theorem 3.1]), but also dynamic equations on time scales (see [8, Theorem 4.3]), our results apply to these types of equations as well. Furthermore, Kurzweil-Henstock-Stieltjes integration (= Perron-Stieltjes integration) could easily replace Lebesgue integration for the functions on the righthand sides of the equations and, hence, problems involving highly oscillating functions (i.e., functions of unbounded variation) would be also included.

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References

[1] M. Akhmet; Kashkynbayev, Ardak, *Bifurcation in Autonomous and Nonautonomous Differential Equations with Discontinuities.* Nonlinear Physical Science. Higher Education Press, Beijing; Springer, Singapore, 2017.

[2] H. Amann, *Ordinary Differential Equations. An Introduction to Nonlinear Analysis.* Gruyter Studies in Mathematics, 13. Walter de Gruyter Co., Berlin, 1990.

[3] M. Bohner, X. Lia, C. Wang, Impulsive differential equations: periodic solutions and applications. *J. Math. Anal. Appl.*, 52 (2015) 173–178.

[4] N. Bourbaki, *Éléments de mathématique.* (French) Fonctions d'une variable réelle. Théorie élémentaire. Springer-Verlag, Berlin-Heidelberg, 2007.

[5] K. M. Church; X. Liu, Bifurcation of bounded solutions of impulsive differential equations. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 26(14) (2016), 1650242, 20 pp.

[6] K. Deimling, *Nonlinear Functional Analysis,* Springer-Verlag, Berlin, 1985.

[7] J. Dieudonné, *Foundations of Modern Analysis.* Pure and Applied Mathematics, Vol. X Academic Press, New York-London, 1960.

[8] M. Federson; J.G. Mesquita; A. Slavík, Measure functional differential equations and functional dynamic equations on time scales. *J. Differential Equations* 252(6) (2012), 3816–3847.

[9] M. Federson; J.G. Mesquita; A. Slavík, Basic results for functional differential and dynamic equations involving impulses. *Math. Nachr.* 286(2-3) (2013), 181–204.

[10] M. Federson; Š. Schwabik, Generalized ODE approach to impulsive retarded functional differential equations. *Differential Integral Equations*, 19(11) (2006), 1201–1234.

[11] I. Fonseca, W. Gangbo, *Degree Theory in Analysis and Applications,* Clarendon Press, Oxford, 1995.

[12] D. Franková, Regulated functions, *Math Bohem.* 116 (1991) 20–59.

[13] R.E. Gaines; J. L. Mawhin,*Coincidence Degree, and Nonlinear Differential Equations.* Lecture Notes in Math., Vol. 568. Springer-Verlag, Berlin-New York, 1977.

[14] R. Henstock, Definitions of Riemann type of the variational integrals. *Proc. London Math. Soc.* 11(3) (1961), 402–418.

[15] C. S. Hönig, *Volterra Stieltjes-integral equations. Functional analytic methods; linear constraints.* Mathematics Studies, No. 16. Notas de Matematica, No. 56. [Notes on Mathematics, No. 56] North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975.

[16] J. Leray; J. Schauder, Topologie et équations fonctionnelles, *Ann. Scient. Éc. Norm. Sup.* 51 (1934), 45–78.

[17] M.A. Krasnosel'skiĭ, *The Operator of Translation along Trajectories of Differential Equations*, American Math. Soc., Providence, 1968.

[18] M. A. Krasnosel'skiĭ and P. P. Zabreĭko, *Geometrical Methods of Nonlinear Analysis.* Translated from the Russian by C. Fenske. Springer-Verlag, Berlin, 1984.

[19] W. Krawcewicz; Jianhong Wu, *Theory of Degrees with Applications to Bifurcations and Differential Equations.* Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1997.

[20] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter, *Czechoslovak Math. J.* 7(82) (1957), 418-449.

[21] J. Kurzweil, Generalized ordinary differential equations, *Czechoslovak Math. J.* 8(83) (1958), 360-388.

[22] J. Kurzweil, *Generalized Ordinary Differential Equations. Not Absolutely Continuous Solutions.* Series in Real Analysis, 11. World Scientific, Singapore, 2012.

[23] A. Lakmeche; O. Arino, Bifurcation of non trivial periodic solutions of impulsive differential equations arising chemotherapeutic treatment. *Dynam. Contin. Discrete Impuls. Systems* 7(2) (2000), 265-287.

[24] Ma, Ruyun; Yang, Bianxia; Wang, Zhenyan. Bifurcation of positive periodic solutions of first-order impulsive differential equations. *Bound. Value Probl.* (2012), 2012:83, 16 pp.

[25] J. Mawhin, Topological Degree Methods in Nonlinear Boundary Value Problems, CMBS Regional Conf. Series in Math. No. 40, Amer. Math. Soc., Providence, RI, 1978.

[26] J. Mawhin, Topological Degree and Boundary Value Problems for Nonlinear Differential Equations, *Topological methods for ordinary differential equations*, Lecture Notes in Math. Vol. 1537, Springer, Berlin, 1991, 74-142.

[27] J. Mawhin, Leray-Schauder degree - A half century of extensions and applications, *Topological Methods in Nonlinear Analysis*, 14 (1999), 195-228.

[28] J.G. Mesquita; A. Slavík, *Periodic averaging theorems for various types of equations, J. Math. Anal. Appl.* 387(2) (2012), 862-877.

[29] D. Qian, X. Li, Periodic solutions for ordinary differential equations with sublinear impulsive effects, *J. Math. Anal. Appl.* 303 (2005) 288-303.

[30] Š. Schwabik, *Generalized Ordinary Differential Equations.* Series in Real Analysis, 5. World Scientific Publishing Co., Inc., River Edge, NJ, 1992.

[31] A. Slavík, Generalized differential equations: differentiability of solutions with respect to initial conditions and parameters, *J. Math. Anal. Appl.* 402(1) (2013), 261-274.

[32] M. Tvrdý; A. Slavík; G. Monteiro, *Kurzweil-Stieltjes Integral Theory and Applications,* Series in Real Analysis, Vol 14, World Scientific Publishing Co., Inc., River Edge, NJ, 2018.

[33] D. Yujun, Z. Erxin, An Application of Coincidence Degree Continuation Theorem in Existence of Solutions of Impulsive Differential Equations. *Journal of Mathematical Analysis and Applications,* (1996), 875-889.

[34] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems,* Springer-Verlag, Berlin, 1986.