Algebras with representable representations

Tim Van der Linden[1]

Fonds de la Recherche Scientifique–FNRS
Université catholique de Louvain

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[1] Joint work with Xabier García-Martínez, Matsvei Tsishyn and Corentin Vienne.

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- $\mathscr{V}$-algebras: morphism $\xi\colon B \to$ **???**

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Can the concept of a derivation be generalised?

Action representability [Borceux–Janelidze–Kelly, 2005]

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Equivalence classes of split extensions in $\mathscr{V}$ by an object X are representable, by an object $[X]$.

$$SplitExt(-, X) \cong Hom(-, [X]) \colon \mathscr{V}^{\mathrm{op}} \to Set$$

For each X, an object $[X]$ exists such that equivalence classes of split extensions of B by X correspond to morphisms $B \to [X]$, naturally in B.

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Then the object $[X]$ plays the role of $Der(X)$,
so it contains "generalised derivations".

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$Der(X)$ **cannot be generalised in a way which is globally valid.**

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B acts on an **abelian** algebra X (where $xy = 0$: X is just a vector space).

(We make representability fail for a simpler subclass of the actions.)

In the present context, such an action happens to be the same thing as a **representation** or **Beck module**: an abelian group (f, m, s) in $(\mathscr{V} \downarrow B)$.

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For instance:

- $xy + yx + x(yz) + (xy)z = 0$ entails
  $xy = -yx$ and $x(yz) = -(xy)z$;
- $xx = x$ entails $x = 0$, so that Boolean rings are excluded.

Structure of the proof

We know that any $\mathscr{V} \leqslant \mathit{Alg}_{\mathbb{K}}$ with $\mathbb{K}$ infinite and (RR)
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When this is done, proving that there are no subvarieties
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the λ/μ-rules in $X \rtimes_{\xi} (B^1 + B^2 + B^3)$ give rise to
a system $(f_i = 0)_{1 \leqslant i \leqslant 224}$ of 224 polynomial equations.

Finding non-trivial identities of degree 2

Proposition

Let $\mathbb{K}$ be a field of characteristic 0 and $\mathscr{V} \leqslant Alg_{\mathbb{K}}$.
If (RR) holds in $\mathscr{V}$, then $\mathscr{V}$ satisfies a non-trivial identity of degree 2.

Sketch of proof.

In $\mathscr{V}$ with (RR), from the λ/μ-rules we deduce a system of polynomial equations, which is inconsistent when there are no degree 2 identities.

We let X be a certain 79-dimensional vector space and consider three copies B^1, B^2 and B^3 of the free $\mathscr{V}$-algebra on a single generator, each with a chosen (abelian) action on X. The isomorphism

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Assuming there are no identities of degree 2, we may show
that there exist $\alpha_i \in \mathbb{Z}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$ such that
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31566240653468532985722705486804720454721162503860068674
68944668940571897397172623649920632890267296075654343504
20878447841877721000590295768558884307124148778177377876
83006491666659252329159304174905496937087738581344349487
06688018655694558517577569557620995746293278480812431412
26057440402445559832044185972204201826090047396984682860
86456278718305987356616291033342422821290606589434367054
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Checking that the equality does indeed hold is relatively simple.

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Checking that the equality does indeed hold is relatively simple.
The proof may be extended to infinite fields of prime characteristic.

Conclusion

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- Our proof depends on computer algebra. Currently, we know of no alternative argument.
- Likely, the condition that we work over an infinite field may be avoided by considering *operadic* varieties of $\mathbb{K}$-algebras (which have *multilinear* identities).

Thank you!