Technical, Allocative and Overall Efficiency: Estimation and Inference

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Abstract

Nonparametric data envelopment analysis and free-disposal hull estimators are frequently used to estimate cost, revenue and profit efficiency as well as the corresponding allocative efficiencies. Papers in the literature often report sample means of such estimates along with sample standard deviations, inviting readers to make inference about means of these efficiencies using classical methods based on the standard Lindeberg-Feller central limit theorem (CLT). A number of papers explicitly make inference using the classical methods. However, the statistical properties of these estimators are (until now) unknown. This paper establishes rates of convergence and existence of limiting distributions for the various estimators. These properties are needed in order to make inference about individual producers using subsampling methods. In addition, properties of the first two moments of the estimators are derived, and these results are subsequently used to establish new CLTs for the estimators, providing formal justification for inference-making. The results reveal that the classical CLTs and methods do not provide valid inference when FDH estimators are used, and provide valid inference when DEA estimators only in a few restrictive, special cases.

Keywords: data envelopment analysis, allocative efficiency, overall efficiency, FDH, inference.

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1 Introduction

Nonparametric envelopment estimators such as the data envelopment analysis (DEA) estimators due to Farrell (1957) and Charnes et al. (1978) as well as the free-disposal hull (FDH) estimator introduced by Deprins et al. (1984) are widely used to estimate technical efficiency of firms and other organizations. The statistical properties of these estimators of technical efficiency are by now well-developed (see Simar and Wilson, 2013 and 2015 for recent surveys), and methods exist for making inference about the technical efficiency of a single firm as well as mean technical efficiency for a group (or population) of firms (e.g., see Kneip et al., 2015). In addition, results enabling tests of convexity versus non-convexity of the production set or constant versus variable returns to scale have been developed (see Kneip et al., 2016 for details).

When data on prices of inputs are available, one can estimate cost efficiency (also called input overall efficiency) or input allocative efficiency as proposed by Färe et al. (1985). Alternatively, when data on prices of outputs are available, one can estimate revenue efficiency (also called output overall efficiency) or output allocative efficiency as also proposed by Färe et al. (1985). When both input and output prices are available, one can estimate profit efficiency or profit allocative efficiency as discussed by Chambers et al. (1998), Färe and Grosskopf (2006) and Färe et al. (2008). Unfortunately, no statistical results exist for these estimators; to date, neither convergence rates nor existence of limiting distributions have been derived, nor has consistency been proved for any of these estimators. Consequently, inference—either for individual firms or for mean, expected values—has until now been impossible. Many empirical papers have estimated cost, revenue, or profit efficiency or the corresponding allocative efficiencies using either FDH or DEA estimators.[1] All of these papers have either ignored statistical inference, or have used classical methods that fail to provide valid inference except in a few vary specific, restrictive settings due to the results presented below.

The results obtained in this paper address this deficiency. DEA and FDH estimators of cost efficiency are examined and shown to have a non-degenerate limiting distribution, as well as a convergence rate that is faster than the rate of the corresponding technical efficiency estimator

[1] A Google Scholar search on 22 February 2015 finds approximately 17,500 papers using the keywords "DEA" and "cost efficiency". Replacing "cost" by "allocative," "revenue" or "profit" results in approximately 28,200, 1,440, or 5,130 papers, respectively. Repeating these four searches substituting "FDH" for "DEA" results in 2,160, 1,290, 246 and 702 papers, respectively.

when there is more than one input.[2] Similar to Kneip et al. (2015), it is shown that standard central limit theorem (CLT) results (e.g., the Lindeberg-Feller CLT) do not provide valid inference if there is more than one output when the DEA estimator is used, and never hold when the FDH is used. New CLT results are provided, enabling inference about mean cost efficiency. Similar results are developed for an estimator of input allocative efficiency. This estimator does not achieve the faster convergence rate of the cost efficiency estimator, and instead has the same convergence rate as the corresponding technical efficiency estimator. Standard CLT results are shown to provide invalid inference about input allocative efficiency whenever there is more than one input and one output, and new CLT results are provided to enable inference in settings with arbitrary numbers of inputs and outputs.

These results are next extended to estimators of revenue efficiency and output allocative efficiency. Similar to the estimator of cost efficiency, the estimator of revenue efficiency is shown to converge at a faster rate than the corresponding technical efficiency estimator. Standard CLT results are shown not to hold for mean revenue efficiency whenever there is more than one input, nor for mean output allocative efficiency whenever there is more than one input or one output. New CLT results are provided to enable inference in general settings.

Similar results are developed for an estimator of profit efficiency and of profit allocative efficiency. For sample size n, it is shown that profit efficiency can be estimated with rate n as $n \to \infty$ regardless of the number of inputs and outputs. Consequently, existing methods can be used for inference about mean profit efficiency. However, the estimator of profit allocative efficiency converges at the same rate as the corresponding estimator of technical efficiency. Consequently, standard CLT results cannot be used if there is more than one input or one output. Again, a new CLT is provided to enable inference in general settings.

The next section establishes notation and provides a statistical model. Precise definitions of the various measures discussed above are also given. Section 3.1 briefly reviews estimation of technical efficiency and mentions the results available for inference about technical efficiency. Sections 3.3–3.4 develop results for cost efficiency and input allocative efficiency, and these results are extended to revenue efficiency and output overall efficiency in Appendices B–C after a brief mention in Section

[2] We consider only the variable returns to scale (VRS) version of the DEA estimator, as the constant return to scale (CRS) version is seldom used. In addition, the VRS version of the DEA estimator remains consistent and attains the faster rate of the (CRS) DEA estimator under CRS (see Kneip et al., 2016 for details and a proof). Moreover, with globally constant returns to scale, profit is maximized at infinity.

3.5.[3] Sections 3.6–3.7 deal with profit efficiency and profit allocative efficiency. An empirical illustration using data from Aly et al. (1990) is presented in Section 4. Section 5 concludes. Proofs are given in Appendix A.

2 The Statistical Model

Let $X \in \mathbb{R}_+^p$ and $Y \in \mathbb{R}_+^q$ denote (random) vectors of input and output quantities, respectively. Similarly, let $x \in \mathbb{R}_+^p$ and $y \in \mathbb{R}_+^q$ denote fixed, nonstochastic vectors of input and output quantities. The production set

$$\Psi := \{(x, y) \mid x \text{ can produce } y\} \tag{2.1}$$

gives the set of feasible combinations of inputs and outputs. Several assumptions on Ψ are common in the literature. The assumptions of Shephard (1970) and Färe (1988) are typical and are used here.

Assumption 2.1. Ψ *is closed.*

Assumption 2.2. $(x, y) \notin \Psi$ *if* $x = 0,\ y \geq 0,\ y \neq 0$*; i.e., all production requires use of some inputs.*

Assumption 2.3. *Both inputs and outputs are strongly disposable, i.e.,* $\forall\,(x, y) \in \Psi$*, (i)* $\widetilde{x} \geq x \Rightarrow (\widetilde{x}, y) \in \Psi$ *and (ii)* $\widetilde{y} \leq y \Rightarrow (x, \widetilde{y}) \in \Psi$.

Here and throughout, inequalities involving vectors are defined on an element-by-element basis, as is standard. Assumption 2.1 permits definition of the the *technology* or *efficient frontier* Ψ^∂ of Ψ as the set of extreme points of Ψ, i.e.,

$$\Psi^\partial := \left\{(x, y) \mid (x, y) \in \Psi,\ (\gamma^{-1}x, \gamma y) \notin \Psi \text{ for any } \gamma \in (1, \infty)\right\}. \tag{2.2}$$

Assumption 2.2 rules out free lunches; i.e., production of any output quantities greater than 0 requires use of some inputs. Assumption 2.3 imposes weak monotonicity on the frontier, and is standard in microeconomic theory of the firm.

The Farrell (1957) input efficiency measure

$$\theta(x, y \mid \Psi) := \inf\{\theta \mid (\theta x, y) \in \Psi\} \tag{2.3}$$

[3] Appendices B–C are available separately as supplementary online material. Alternatively, Appendices B–C are available from the authors on request.

indicates the amount by which input levels can be proportionately scaled downward by the same factor without reducing output levels. The Farrell (1957) output efficiency measure gives the feasible, proportionate expansion of output quantities and is defined by

$$\lambda(x, y \mid \Psi) := \sup\left\{\lambda \mid (x, \lambda y) \in \Psi\right\}. \tag{2.4}$$

This gives a *radial* measure of efficiency since all output quantities are scaled by the same factor λ. Clearly, $\lambda(x, y \mid \Psi) \geq 1$ and $\theta(x, y \mid \Psi) \leq 1$ for all $(x, y) \in \Psi$.

Chambers et al. (1998) proposed the directional measure

$$\delta(x, y \mid d_x, d_y, \Psi) = \sup\left\{\delta \mid (x - \delta d_x, y + \delta d_y) \in \Psi\right\}, \tag{2.5}$$

which measures the distance from a point (x, y) to the frontier in the given direction $d = (-d_x, d_y)$, where $d_x \in \mathbb{R}_+^p$ and $d_y \in \mathbb{R}_+^q$. This measure is flexible in the sense that some values of the direction vector can be set to zero. A value $\delta(x, y \mid d_x, d_y, \Psi) = 0$ indicates an efficient point lying on the boundary of Ψ. Note that as a special case, the Farrell-Debreu radial distances can be recovered; e.g. if $d = (-x, 0)$ then $\delta(x, y \mid d_x, d_y, \Psi) = 1 - \theta(x, y \mid \Psi)^{-1}$ or if $d = (0, y)$ then $\delta(x, y \mid d_x, d_y, \Psi) = \lambda(x, y \mid \Psi) - 1$. Another interesting feature is that directional distances are additive measures, hence they permit negative values of x and y (e.g., in finance, an output y may be the return of a fund, which can be, and often is, negative).[4] Many choices of the direction vector are possible (e.g., a common one for all firms, or a specific direction for each firm; see Färe et al., 2008 for discussion), although care should be taken to ensure that the chosen direction vector maintains invariance with respect to units of measurement for input and output quantities.

Given a vector $w_x \in \mathbb{R}_+^p$ of input prices, the minimum cost of producing a specific vector y_0 of output quantities from a given vector x_0 of input quantities is

$$\mathcal{C}_{\min}(x_0, y_0 \mid \Psi, w_x) = \min_x \{w_x' x \mid (x, y_0) \in \Psi,\ x \in \mathbb{R}_+^p,\ w_x \in \mathbb{R}_{++}^p\}. \tag{2.6}$$

Cost efficiency (sometimes called input overall efficiency) for the firm operating at $(x_0, y_0) \in \Psi$ and facing input prices w_x is then defined by

$$\mathcal{C}(x_0, y_0 \mid \Psi, w_x) := \frac{\mathcal{C}_{\min}(x_0, y_0 \mid \Psi, w_x)}{w_x' x_0} = \frac{w_x' x_*}{w_x' x_0} \tag{2.7}$$

[4] The measure in (2.5) differs from the "additive" measure $\eta(x, y \mid \Psi) = \sup\{\eta \mid \eta = i_p' s_x + i_q' s_y,\ (x - s_x, y + s_y) \in \Psi\}$ estimated by Charnes et al. (1985), where $i_p,\ i_q$ denote $(p \times 1)$ and $(q \times 1)$ vectors of ones and $s_x,\ s_y$ denote $(p \times 1)$ and $(q \times 1)$ vectors of weights to be optimized. Charnes et al. (1985) present only an estimator, and do not define the object that is estimated. Moreover, the additive measure is not in general invariant to units of measurement.

where x_* is the argmin of the expression on the right-hand side (RHS) of (2.6). The cost efficiency measure in (2.7) gives the fraction by which cost of producing output quantities y_0 could be reduced when facing input prices w_x; achieving this reduction might require altering the mix of inputs used to produce y_0.

Färe et al. (1985) define *input allocative efficiency* as

$$\mathcal{A}_x(x_0, y_0 \mid \Psi, w_x) := \frac{\mathcal{C}(x_0, y_0 \mid \Psi, w_x)}{\theta(x_0, y_0 \mid \Psi)}. \tag{2.8}$$

Clearly, for any $(x_0, y_0) \in \Psi$ we have $\mathcal{A}_x(x_0, y_0 \mid \Psi, w_x) \leq 1$. The input allocative efficiency measure gives the part of cost inefficiency that would remain if input quantities x_0 were reduced to the technically-efficient level $\theta(x_0, y_0 \mid \Psi)x_0$.

Alternatively, given a vector $w_y \in \mathbb{R}^q_+$ of output prices, the maximum revenue from producing a specific vector y_0 of output quantities using a given vector x_0 of input quantities is

$$\mathcal{R}_{\max}(x_0, y_0 \mid \Psi, w_y) = \max_y \{w_y' y \mid (x_0, y) \in \Psi,\ y \in \mathbb{R}^q_+,\ w_y \in \mathbb{R}^q_{++}\}. \tag{2.9}$$

Revenue efficiency (sometimes called overall output efficiency) for the firm operating at $(x_0, y_0) \in \Psi$ and facing output prices w_y then

$$\mathcal{R}(x_0, y_0 \mid \Psi, w_y) := \frac{\mathcal{R}_{\max}(x_0, y_0 \mid \Psi, w_y)}{w_y' y_0} = \frac{w_y' y_*}{w_y' y_0} \tag{2.10}$$

where y_* is the argmin of the expression on the RHS of (2.9).

Analogous to the input allocative efficiency measure, Färe et al. (1985) define *output allocative efficiency* as

$$\mathcal{A}_y(x_0, y_0 \mid \Psi, w_y) := \frac{\mathcal{R}(x_0, y_0 \mid \Psi, w_y)}{\lambda(x_0, y_0 \mid \Psi)}. \tag{2.11}$$

By construction, $\mathcal{A}_y(x_0, y_0 \mid \Psi, w_y) \geq 1$ for $(x_0, y_0) \in \Psi$. Output allocative efficiency corresponds to the amount of revenue inefficiency that would remain after increasing output levels y_0 to the technically efficient levels $\lambda(x_0, y_0 \mid \Psi)y_0$.

Maximum profit for a firm operating at $(x_0, y_0) \in \Psi$ and facing prices $w_x,\ w_y$ is given by

$$\begin{aligned}\mathcal{P}_{\max}(x_0, y_0 \mid \Psi, w_x, w_y) = \max_{x,y} \Big\{ w_y' y - w_x' x \mid &(x, y) \in \Psi,\ x \in \mathbb{R}^p_+,\ y \in \mathbb{R}^q_+, \\ &w_x \in \mathbb{R}^p_{++},\ w_y \in \mathbb{R}^q_{++} \Big\}.\end{aligned} \tag{2.12}$$

However, defining profit efficiency as the ratio of maximum to observed profit, analogous to cost or revenue efficiency, is problematic because profit can be negative, particularly during periods of economic distress. Chambers et al. (1998) propose a Nerlovian profit efficiency measure for the firm operating at $(x_0, y_0) \in \Psi$ given by

$$\mathcal{P}(x_0, y_0 \mid \Psi, d_x, d_y, w_x, w_y) := \frac{\mathcal{P}_{\max}(x_0, y_0 \mid \Psi, w_x, w_y) - (w_y y_0 - w_x x_0)}{w_y d_y + w_x d_x} \tag{2.13}$$

where $d_x,\ d_y$ are the direction vectors used in (2.5) to measure technical efficiency. Profit efficiency amounts to the difference between maximum and observed profit (thereby accommodating negative observed profits), normalized by the "value" of the direction (d_x, d_y). Because the directional measure is additive, the corresponding measure of *profit allocative efficiency* is given by the difference

$$\mathcal{A}_\pi(x_0, y_0 \mid \Psi, d_x, d_y, w_x, w_y) := \mathcal{P}(x_0, y_0 \mid \Psi, d_x, d_y, w_x, w_y) - \delta(x, y \mid d_x, d_y, \Psi). \tag{2.14}$$

All of the quantities and model features defined so far are unobservable, and hence must be estimated. In addition, inference is needed in order to know what might be learned from data. Some additional assumptions are needed to complete the statistical model. The following assumptions are analogous to Assumptions 3.1–3.4 of Kneip et al. (2015). In order to draw upon previous results, we state the assumptions below in terms of the input-oriented measure of efficiency. The assumptions can also be stated in terms of the output and directional measures of efficiency, and the results of Kneip et al. (2015) extend to those measures after trivial (but tedious) changes in notation in Kneip et al. (2015). The first two assumptions that follow are needed for both DEA and FDH estimators.

Assumption 2.4. *(i) The random variables (X, Y) possess a joint density f with support $\mathcal{D} \subset \Psi$; and (ii) f is continuously differentiable on $\mathcal{D}$.*

Assumption 2.5. *(i) $\mathcal{D}^* := \{\theta(x, y \mid \Psi)x,\ y) \mid (x, y) \in \mathcal{D}\} \subset \mathcal{D}$; (ii) $\mathcal{D}^*$ is compact; and (iii) $f(\theta(x, y)x, y) > 0$ for all $(x, y) \in \mathcal{D}$.*

The next two assumptions are needed when DEA estimators are used. Assumption 2.6 imposes some smoothness on the frontier. Kneip et al. (2008) required only two-times differentiability to establish the existence of a limiting distribution for DEA estimators, by the stronger assumption that follows is needed to establish results on moments of the DEA estimators.

Assumption 2.6. *$\theta(x, y \mid \Psi)$ is three times continuously differentiable on $\mathcal{D}$.*

Recalling that the strong (i.e., free) disposability assumed in Assumption 2.3 implies that the frontier is weakly monotone, the next assumption strengthens this by requiring the frontier to be strictly monotone with no constant segments. This is also needed to establish properties of moments of the DEA estimators.

Assumption 2.7. *$\mathcal{D}$ is* almost strictly convex*; i.e., for any $(x,y),(\widetilde{x},\widetilde{y}) \in \mathcal{D}$ with $(\frac{x}{\|x\|},y) \neq (\frac{\widetilde{x}}{\|\widetilde{x}\|},\widetilde{y})$, the set $\{(x^*,y^*) \mid (x^*,y^*) = (x,y) + \alpha((\widetilde{x},\widetilde{y}) - (x,y))$ for some $0 < \alpha < 1\}$ is a subset of the* interior *of $\mathcal{D}$.*

Assumptions 2.1–2.7 comprise a statistical model similar to the one defined in Kneip et al. (2015) and where DEA estimators have desirable properties. Alternatively, when FDH estimators are used, Assumptions 2.6 and 2.7 can be replaced by the following assumption.

Assumption 2.8. *(i) $\theta(x,y)$ is twice continuously differentiable on $\mathcal{D}$; and (ii) all the first-order partial derivatives of $\theta(x,y)$ with respect to x and y are nonzero at any point $(x,y) \in \mathcal{D}$.*

Assumption 2.8 strengthens the assumption of strong disposability in 2.3 by requiring that the frontier is strictly monotone and does not possess constant segments (which would be the case, for example, if outputs are discrete as opposed to continuous, as in the case of ships produced by shipyards). Finally, part (i) of Assumption 2.8 is weaker than Assumption 2.6; here the frontier is required to be smooth, but not as smooth as required by Assumption 2.6.[5] Assumptions 2.1–2.5 and Assumption 2.8 comprise a statistical model appropriate for use of FDH estimators of technical efficiency, while Assumption 2.1–2.7 comprise a statistical model appropriate for use of DEA estimators of technical efficiency.

In applications where cost, revenue or profit efficiency are estimated, firms are often observed to face different prices. In order to consider properties of moments of estimators of cost, revenue or profit efficiency, an additional assumption is needed.

Assumption 2.9. *(i) The random variables (W_x, W_y) possess a joint density f_{W_x,W_y} with compact support $\mathcal{D}_W \subset \mathbb{R}^p_{++} \times \mathbb{R}^q_{++}$, and (ii) The random variables (X, Y, W_x, W_y) are defined on an appropriate probability space such that the joint density $f_{X,Y,W_x,W_y}(x, y, w_x, w_y)$ exists and is well-defined with support $\mathcal{D} \times \mathcal{D}_W$.*

Of course, prices of inputs and outputs are determined in markets. One might expect that the price of financial capital, which is mobile, might be constant, but this requires that markets

[5] Assumption 2.8 is slightly stronger, but much simpler than assumptions AII–AIII in Park et al. (2000).

reach a spatial equilibrium. Moreover, the price of physical capital, which is immobile, should be expected to vary across space. In addition, the prices of labor as well as banks' outputs may vary due to differences in local market conditions. Treating prices of both inputs and outputs as random variables in Assumption 2.9 provides some mathematical structure needed to define a statistical model. As will be seen below, estimates of cost, revenue and profit efficiency are in each case conditioned on observed prices. When considering mean efficiencies, expectations are over inputs and outputs as well as prices. Assumption 2.9 provides the mathematical structure needed to make inference about mean cost, revenue and profit efficiencies as well as the corresponding measures of allocative efficiencies.

Assumption 2.9 ensures that all prices are strictly positive and have finite upper bounds. Of course, in some situations firms may face the same prices, in which case f_{W_x,W_y} is degenerate with mass at a single point. In other situations, it may be the case that only input prices or output prices are observed. In such cases, the input or output prices can be viewed as being drawn from marginal distributions f_{W_x} or f_{W_y} corresponding to f_{W_x,W_y}. The joint density $f_{X,Y,W_x,W_y}(x,y,w_x,w_y)$ implies existence of the corresponding marginal distributions f_{X,Y,W_x} of inputs, outputs and input prices and f_{X,Y,W_y} of inputs, outputs and output prices.

3 Estimation and Inference

3.1 Technical Efficiency

Given a random sample $\mathcal{S}_n = \{(X_i, Y_i)\}$, the production set Ψ can by estimated by the free disposal hull of the sample observations in $\mathcal{S}$,

$$\widehat{\Psi}_{\text{FDH},n} := \bigcup_{(X_i,Y_i)\in\mathcal{S}_n} \Big\{(x,y)\in\mathbb{R}_+^{p+q} \mid x\geq X_i,\ y\leq Y_i\Big\}, \tag{3.1}$$

proposed by Deprins et al. (1984). Alternatively, Ψ can be estimated by the convex hull of $\widehat{\Psi}_{\text{FDH},n}$ of the free-disposal hull of the sample observations in $\mathcal{S}$, i.e., by

$$\widehat{\Psi}_{\text{DEA},n} := \big\{(x,y)\in\mathbb{R}^{p+q} \mid y\leq \boldsymbol{Y}\boldsymbol{\upsilon},\ x\geq \boldsymbol{X}\boldsymbol{\upsilon},\ \boldsymbol{i}_n'\boldsymbol{\upsilon}=1,\ \boldsymbol{\upsilon}\in\mathbb{R}_+^n\big\}, \tag{3.2}$$

where $\boldsymbol{X} = \big(X_1,\ \ldots,\ X_n\big)$ and $\boldsymbol{Y} = \big(Y_1,\ \ldots,\ Y_n\big)$ are $(p\times n)$ and $(q\times n)$ matrices of input and output vectors, respectively; $\boldsymbol{i}_n$ is an $(n\times 1)$ vector of ones, and $\boldsymbol{\upsilon}$ is a $(n\times 1)$ vector of weights. This is the (VRS) DEA estimator of Ψ, proposed by Farrell (1957) and Banker et al. (1984). FDH or DEA estimators of $\theta(x,y\mid\Psi)$, $\lambda(x,y\mid\Psi)$ and $\delta(x,y\mid d_x,d_y,\Psi)$ defined in Section 2 are obtained by substituting $\widehat{\Psi}_{\text{FDH},n}$ or $\widehat{\Psi}_{\text{DEA},n}$ for Ψ in (2.3)–(2.5) (respectively). In the case of DEA estimators, this results in

$$\theta(x, y \mid \widehat{\Psi}_{\text{DEA},n}) = \min_{\theta,\boldsymbol{v}} \left\{\theta \mid y \leq \boldsymbol{Y}\boldsymbol{v},\ \theta x \geq \boldsymbol{X}\boldsymbol{v},\ \boldsymbol{i}_n'\boldsymbol{v} = 1,\ \boldsymbol{v} \in \mathbb{R}_+^n\right\}, \tag{3.3}$$

$$\lambda(x, y \mid \widehat{\Psi}_{\text{DEA},n}) = \max_{\lambda,\boldsymbol{v}} \left\{\lambda \mid \lambda y \leq \boldsymbol{Y}\boldsymbol{v},\ x \geq \boldsymbol{X}\boldsymbol{v},\ \boldsymbol{i}_n'\boldsymbol{v} = 1,\ \boldsymbol{v} \in \mathbb{R}_+^n\right\} \tag{3.4}$$

and

$$\delta(x, y \mid d_x, d_y, \widehat{\Psi}_{\text{DEA},n}) = \max_{\delta,\boldsymbol{v}} \left\{\delta \mid (y + \delta d_y) \leq \boldsymbol{Y}\boldsymbol{v},\ (x - \delta d_x) \geq \boldsymbol{X}\boldsymbol{v},\ \boldsymbol{i}_n'\boldsymbol{v} = 1,\ \boldsymbol{v} \in \mathbb{R}_+^n\right\}. \tag{3.5}$$

Substituting $\widehat{\Psi}_{\text{FDH},n}$ leads to integer programming problems, but the estimators can be computed using simple numerical methods (e.g., see Simar and Wilson, 2013 and 2015 for details).

The statistical properties of these estimators are well-developed. Kneip et al. (1998) derive the rate of convergence of the input-oriented DEA estimator, while Kneip et al. (2008) derive its limiting distribution. Park et al. (2000) and Daouia et al. (2017) derive both the rate of convergence and limiting distribution of the input-oriented FDH estimator. Kneip et al. (2015) derive moment properties of both the input-oriented FDH and DEA estimators and establish central limit theorem (CLT) results for mean input-oriented efficiency after showing that the usual CLT results (e.g., the Lindeberg-Feller CLT) do not hold unless $(p + q) < 3$. All of these results extend trivially to the output-oriented estimator $\lambda(x, y \mid \widehat{\Psi}_n)$ after straightforward (but tedious) changes in notation. Simar et al. (2012) extend the results of Kneip et al. (1998) and Kneip et al. (2008) to the DEA directional efficiency estimator using the results of Wilson (2011), while Simar and Vanhems (2012) extend the results of Park et al. (2000) to the FDH directional efficiency estimator. Using similar ideas it can be shown that the moment results of Kneip et al. (2015) also extend to the directional case.

In all cases, the estimators are consistent, converge at rate n^{κ} (where $\kappa = 2/(p + q + 1)$ for the DEA estimators and $\kappa = 1/(p + q)$ for the FDH estimators) and possess non-degenerate limiting distributions under the appropriate set of assumptions. In addition, the bias of each of the three estimators is of order $O(n^{-\kappa})$. Bootstrap methods proposed by Kneip et al. (2008, 2011) and Simar and Wilson (2011a) provide consistent inference about $\theta(x, y \mid \Psi)$, $\lambda(x, y \mid \Psi)$ and $\delta(x, y \mid d_x, d_y, \Psi)$ for a fixed point $(x, y) \in \Psi$, and Kneip et al. (2015) provide CLT results enabling inference about the expected values of these measures over the random variables (X, Y).

3.2 Some Preliminary Results

The preliminary results developed here are used below in the discussion of estimation of cost, revenue and profit efficiency as well as input, output and profit allocative efficiency. First, consider the function $h_{w_x} : \mathbb{R}_+^{p+q} \mapsto \mathbb{R}_+^{1+q}$ such that $h_{w_x}(x, y) = A_{w_x} \begin{bmatrix} x' & y' \end{bmatrix}'$ where

$$A_{w_x} = \begin{bmatrix} w_x' & 0_q' \\ 0_{p\times q}' & I_q \end{bmatrix}, \tag{3.6}$$

0_q is a $(q\times 1)$ vector of zeros, $0_{p\times q}$ is a $(p\times q)$ matrix of zeros, and I_q is a $(q\times q)$ identity matrix. Then h_{w_x} is an affine function in the sense of Williamson and Trotter (1974) and Boyd and Vandenberghe (2004), and the image of Ψ under h_{w_x} is

$$\Psi_{w_x} := h_{w_x}(\Psi) = \{(c, y) \mid (c, y) = h_{w_x}(x, y) \;\forall\; (x, y) \in \Psi\}. \tag{3.7}$$

Clearly, $\Psi_{w_x} \subset \mathbb{R}_+^{1+q}$. It is well-known (e.g., see Boyd and Vandenberghe, 2004, pp. 36–38) that since h_{w_x} is affine, Ψ_{w_x} is convex if and only if Ψ is convex.

Next, consider the function $h_{w_y} : \mathbb{R}_+^{p+q} \mapsto \mathbb{R}_+^{p+1}$ such that $h_{w_y}(x, y) = A_{w_y} \begin{bmatrix} x' & y' \end{bmatrix}'$ where

$$A_{w_y} = \begin{bmatrix} I_p & 0_{p\times q} \\ 0_p' & w_y' \end{bmatrix} \tag{3.8}$$

is a $(p+1)\times(p+q)$ matrix. Similar to h_{w_x} defined above, h_{w_y} is an affine function, and the image of Ψ under h_{w_y} is

$$\Psi_{w_y} := h_{w_y}(\Psi) = \big\{(x, r) \mid (x, r) = h_{w_y}(x, y) \;\forall\; (x, y) \in \Psi\big\}. \tag{3.9}$$

Clearly $\Psi_{w_y} \subset \mathbb{R}_+^{p+1}$. Again, due to the properties of affine functions, Ψ_{w_y} is convex if and only if Ψ is convex.

Finally, define the function $h_{w_x,w_y} : \mathbb{R}_+^{p+q} \mapsto \mathbb{R}$ such that $h_{w_x,w_y}(x, y) = A_{w_x,w_y} \begin{bmatrix} x' & y' \end{bmatrix}'$ where

$$A_{w_x,w_y} = \begin{bmatrix} -w_x' & w_y' \end{bmatrix} \tag{3.10}$$

is a $1\times(p+q)$ matrix. Similar to h_{w_x} and h_{w_y} defined above, h_{w_x,w_y} is an affine function, and the image of Ψ under h_{w_x,w_y} is

$$\Psi_{w_x,w_y} := h_{w_x,w_y}(\Psi) = \big\{\pi \mid \pi := h_{w_x,w_y}(x, y) \;\forall\; (x, y) \in \Psi\big\}. \tag{3.11}$$

Clearly, $\Psi_{w_x,w_y} \subset \mathbb{R}$, and Ψ_{w_x,w_y} is trivially convex. In addition, the affine transformations of Ψ described above also preserve strong disposability as confirmed by the following result.

Lemma 3.1. *Assume Ψ is closed but not necessarily convex, and Assumption 2.3 holds. Then (i) Ψ_{w_x} satisfies strong disposability of cost and outputs, i.e., $\forall\ (c,y) \in \Psi_{w_x}$, $\widetilde{c} \geq c \Rightarrow (\widetilde{c}, y) \in \Psi_{w_x}$ and $\widetilde{y} \leq y \Rightarrow (c, \widetilde{y}) \in \Psi_{w_x}$; (ii) Ψ_{w_y} satisfies strong disposability of inputs and revenue, i.e., $\forall\ (x,r) \in \Psi_{w_y}$, $\widetilde{x} \geq x \Rightarrow (\widetilde{x}, r) \in \Psi_{w_y}$ and $\widetilde{r} \leq r \Rightarrow (x, \widetilde{r}) \in \Psi_{w_y}$; and (iii) Ψ_{w_x,w_y} satisfies strong disposability of profit, i.e., $\forall\ (\pi) \in \Psi_{w_x,w_y}$, $\widetilde{\pi} \leq \pi \Rightarrow (\widetilde{\pi}) \in \Psi_{w_x,w_y}$.*

It is obvious that Assumption 2.1 ensures that Ψ_{w_x} is closed. Hence, since strong disposability is preserved, under Assumptions 2.2–2.5 and Assumption 2.8, Ψ_{w_x} can be estimated by

$$\widehat{\Psi}_{\text{FDH},w_x,n} := \bigcup_{(C_i,Y_i)\in\mathcal{S}_{w_x,n}} \{(c,y) \mid c \geq C_i,\ y \leq Y_i\} \tag{3.12}$$

where $\mathcal{S}_{w_x,n} = \{(C_i, Y_i)\}_{i=1}^n$ results from applying the transformation h_{w_x} to each $(X_i, Y_i) \in \mathcal{S}_n$. Similar reasoning leads to the conclusion that Ψ_{w_y} and Ψ_{w_x,w_y} can be estimated by the FDH estimators $\widehat{\Psi}_{\text{FDH},w_y,n}$ and $\widehat{\Psi}_{\text{FDH},w_x,w_y,n}$, respectively, after applying the transformations h_{w_y} and h_{w_x,w_y} to the observations in $\mathcal{S}_n$.

The fact that the affine transformations of Ψ described above preserve convexity (when it exists) as well as strong disposability means that under Assumptions 2.1–2.7, DEA estimators can be used to estimate the transformed sets Ψ_{w_x}, Ψ_{w_y} and Ψ_{w_x,w_y}. In particular, let $\widehat{\Psi}_{\text{DEA},w_x,n} = h_{w_x}(\widehat{\Psi}_{\text{DEA},n})$; i.e, $\widehat{\Psi}_{\text{DEA},w_x,n}$ is the image of $\widehat{\Psi}_{\text{DEA},n}$ under h_{w_x}. This leads to the DEA estimator of Ψ_{w_x} in cost-output space, i.e., the convex hull of the free disposal hull of the observations in $\mathcal{S}_{w_x,n}$ given by

$$\widehat{\Psi}_{\text{DEA},w_x,n} = \left\{(c,y) \in \mathbb{R}^{1+q} \mid y \leq \boldsymbol{Y}\boldsymbol{\upsilon},\ c \geq \boldsymbol{C}\boldsymbol{\upsilon},\ \boldsymbol{i}_n'\boldsymbol{\upsilon} = 1,\ \boldsymbol{\upsilon} \in \mathbb{R}_+^n\right\}, \tag{3.13}$$

where $\boldsymbol{C}$ is the $(1 \times n)$ vector of costs with $(1,i)$-th element $C_i = w_x' X_i$. Similar reasoning leads to DEA estimators $\widehat{\Psi}_{\text{DEA},w_y,n}$ and $\widehat{\Psi}_{\text{DEA},w_x,w_y,n}$ of Ψ_{w_y} and Ψ_{w_x,w_y}.

3.3 Cost Efficiency

The usual approach to estimating cost efficiency given by (2.7) is to first estimate the vector of input levels that minimize cost by employing an empirical analog of (2.6). DEA estimators are typically used. In practice, given the vector w_x of input prices, this amounts to replacing Ψ in (2.6) with $\widehat{\Psi}_{\text{DEA},n}$ to obtain

$$\begin{aligned}\mathcal{C}_{\min}(x_0, y_0 \mid \widehat{\Psi}_{\text{DEA},n}, w_x) &= \min_{x,\boldsymbol{\upsilon}} \left\{w_x' x \mid \boldsymbol{Y}\boldsymbol{\upsilon} \geq y_0,\ \boldsymbol{X}\boldsymbol{\upsilon} \leq x,\ \boldsymbol{1}_n'\boldsymbol{\upsilon} = 1,\ \boldsymbol{\upsilon} \in \mathbb{R}_+^n\right\} \\ &= w_x' \widehat{x}_{\min}\end{aligned} \tag{3.14}$$

where $\widehat{x}_{\min}$ is the solution to the optimization problem in the first line of (3.14). Then cost efficiency is estimated by

$$\mathcal{C}(x_0, y_0 \mid \widehat{\Psi}_{\mathrm{DEA},n}, w_x) := \frac{\mathcal{C}_{\min}(x_0, y_0 \mid \widehat{\Psi}_n, w_x)}{w_x' x_0} = \frac{w_x' \widehat{x}_{\min}}{w_x' x_0}. \tag{3.15}$$

This is the suggested approach of Farrell (1957), Färe et al. (1985), Färe and Grosskopf (1995), Coelli et al. (1997) and Ray (2004), and has been used in empirical settings by Aly et al. (1990), Byrnes and Valdmanis (1994), Cummins et al. (1999), Sharmaa et al. (1999), Kohersa et al. (2000), Worthington (2000), Björkgren et al. (2001), Hartman et al. (2001), Coelli et al. (2002) Isik and Hassan (2002), Wadud (2003), Barros and Sampaio (2004), Barros and Mascarenhas (2005), Camanho and Dyson (2005, 2008), Chen et al. (2005), Cinemre et al. (2006) Havrylchyk (2006), Asmild et al. (2007), Ariff and Can (2008), Hansson and Öhlmér (2008) Hu et al. (2009), Cummins et al. (2010), Hsu and Petchsakulwong (2010), Kader et al. (2010), Kaur and Kaur (2010), Kwadjo Ansah-Adu (2011) Lozano (2011), Al-Khasawneh et al. (2012), Haelermans and Ruggiero (2013), Nedelea and Fannin (2013) Kočišová (2014), Nguyen et al. (2016), Ghiyasi (2017) and many others. Unfortunately, the statistical properties of both the estimator of minimum cost given in (3.14) as well as the estimator of cost efficiency given in (3.15) are unknown. Consequently, researchers often either (i) report only point estimates and perhaps sample means of the estimates in applications without making inference, or (ii) report sample means of estimate as well as sample standard deviations of cost efficiency estimates, implicitly inviting readers to use standard CLT results to assess statistical significance. Some (e.g., Kohersa et al., 2000) explicitly use standard CLT results to test whether means are different across groups of producers. Hartman et al. (2001) use the Kruskal-Wallis test to test for whether cost efficiency has the same distribution across groups, but the true cost efficiencies, as well as their ranks, are unobserved, casting doubt on the properties of their test. Several papers, including Cummins et al. (2010), Coelli et al. (2002) and Nedelea and Fannin (2013), make conventional inference in second-stage Tobit regressions where cost efficiency estimates are regressed on some explanatory variables. As will be seen shortly, this results in invalid inference. Moreover, these papers do not specify a coherent model where Tobit regression in a second-stage regression would be sensible.[6]

The next result establishes a distance-function characterization of the cost efficiency measure introduced in (2.7).

[6]See Simar and Wilson (2007) for further discussion on this point.

Lemma 3.2. *Let $c_0 = w_x' x_0$. Then for $(x_0, y_0) \in \Psi$,*

$$\mathcal{C}(x_0, y_0 \mid \Psi, w_x) = \theta(c_0, y_0 \mid \Psi_{w_x}). \tag{3.16}$$

Since it is apparent from (3.14) that Ψ is estimated by the convex hull of the sample observations in $\mathcal{S}_n$, one might imagine that the convergence rate of the estimator of cost efficiency is $n^{2/(p+q+1)}$ as established by Kneip et al. (1998) for the DEA estimator of $\theta(x, y \mid \Psi)$, or $n^{1/(p+q)}$ as established by Park et al. (2000) for the FDH estimator. But in fact, due to Lemma 3.2, cost efficiency can be estimated at the rate $n^{2/(q+2)}$ using the DEA estimator, or $n^{1/(q+1)}$ using the FDH estimator. The result in Lemma 3.2 is not new—it is suggested by duality theory in the microeconomics literature—but it is important for purposes of statistical inference.

To simplify notation, from this point the "FDH" or "DEA" is omitted from subscripts, noting that $\widehat{\Psi}_n$ and $\widehat{\Psi}_{w_x,n}$ may refer to either the FDH or DEA estimators of Ψ and Ψ_{w_x} (respectively). Then consider the estimator $\theta(c_0, y_0 \mid \widehat{\Psi}_{w_x,n})$ of cost efficiency where $\widehat{\Psi}_{w_x,n}$ denotes either the FDH or DEA estimator of Ψ_{w_x} defined in (3.12) or (3.13) and where c_0 and $\widehat{\Psi}_{w_x,n}$ replace x_0 and $\widehat{\Psi}$ (respectively) in (3.3) as discussed in Section 3.2.

The result in Lemma 3.2 may seem obvious to some. Färe and Grosskopf (1985), Färe et al. (1988) and Staub et al. (2010) estimate cost efficiency using $\theta(c_0, y_0 \mid \widehat{\Psi}_{w_x,n})$ where $\widehat{\Psi}_{w_x,n}$ denotes the DEA estimator in (3.13), while De Borger and Kerstens (1996) estimate cost efficiency using $\theta(c_0, y_0 \mid \widehat{\Psi}_{w_x,n})$ where $\widehat{\Psi}_{w_x,n}$ denotes the FDH estimator in (3.12). Staub et al. (2010) estimate cost efficiency in $(q+1)$ dimensions with $p = q = 3$, but then regress their (DEA) cost efficiency estimates on some explanatory variables in a second-stage panel regression after citing (and then ignoring) Simar and Wilson (2007) which cautions against their approach. In their panel regression, Staub et al. rely upon conventional inference, which is invalid due to reasons given below. None of these authors specify a statistical model, nor do they mention the statistical properties of their estimators. Moreover, the far more typical approach in the literature is to estimate cost efficiency using (3.14) and (3.15) as described above.

Lemma 3.2 establishes that estimation of cost efficiency is a $(q+1)$-dimensional problem, and that the usual FDH and DEA input-oriented efficiency estimators can be used to estimate cost efficiency, where univariate cost replaces p input variables.[7] Consequently, cost efficiency is consistently estimated with rates n^{κ_x}, where $\kappa_x = 2/(q+2)$ when the DEA estimator is used and

[7]Tone (2002) makes the obvious note that in the case of $p = 1$, technical and cost efficiencies are identical.

$\kappa_x = 1/(q+1)$ when the FDH estimator is used due to the results of Kneip et al. (1998) (for the DEA case) and Park et al. (2000) and Daouia et al. (2017) (for the FDH case). Moreover, a limiting distribution exists in both cases, and hence the sub-sampling ideas of Simar and Wilson (2011a) can be used to make asymptotically valid inference about the cost efficiency of individual producers. Simar and Wilson (2011a) also discuss a method for choosing the sub-sample size, which is critical to the finite-sample performance of the sub-sampling method. Implementation of the sub-sampling method also depends critically on knowledge of the convergence rate with which cost efficiency is estimated, which is established above due to Lemma 3.2.

To be clear, different firms may face different input prices as noted earlier in Section 2. Suppose a random sample $\{(X_i, Y_i, W_{x,i}\}_{i=1}^n$ of input, output and input-price triplets is observed. When estimating cost efficiency for each firm, each firm's observed input-price vector is used to construct a transformed attainable cost-output set. Firm i has observed cost $C_i = W_{x,i}' X_i$ while firm j has observed cost $C_j = W_{x,j}' X_j$, but to estimate the cost efficiency of firm i, its cost C_i is compared not to C_j but instead to $C_j^i = W_{x,i}' X_j$ for $j = 1\ \ldots,\ n$, thereby conditioning on the observed prices $W_{x,i}$ of firm i. Of course, this similar to the case in regression settings where cost is regressed on output quantities and input prices, and cost efficiency for firm i is estimated conditionally on firm i's observed input prices. Formally, the sample $\{(X_i, Y_i, W_{x,i}\}_{i=1}^n$ is used to construct n samples $\mathcal{S}_{W_{x,i},n} = \left\{C_\ell^i, Y_\ell\right\}_{\ell=1}^n$ obtained by applying the function $h_{W_{x,i}}$ to the observations in $\{(X_i, Y_i, W_{x,i}\}_{i=1}^n$ for each $i = 1,\ \ldots,\ n$. In other words, for firm i, $W_{x,i}$ replaces w_x in (3.6) leading to the image $\Psi_{W_{x,i}}$ of Ψ under $h_{W_{x,i}}$, analogous to (3.7) where $h_{W_{x,i}}$ is defined by replacing w_x in (3.6) with $W_{x,i}$. Similarly, for firm j, $W_{x,j}$ replaces w_x in (3.6) leading to the image $\Psi_{W_{x,j}}$ of Ψ under $h_{W_{x,j}}$, again analogous to (3.7). Finally, for $i = 1,\ \ldots,\ n$, estimators (either FDH or DEA) of $\Psi_{W_{x,i}}$ are constructed from the samples $\mathcal{S}_{W_{x,i},n}$ and these are used to obtain cost efficiency estimates $\theta(C_i, Y_i \mid \widehat{\Psi}_{W_{x,i}})$.[8]

The result in Lemma 3.2 also means that properties of the first two moments of either FDH or DEA estimators of cost efficiency are established by Kneip et al. (2015), where in the notation of Kneip et al. (2015), $p = 1$. This means that standard CLTs (e.g., the Lindeberg-Feller CLT) can be

[8]Linna et al. (2006) use cost data state in their Section 4.6 that they use year-end accounting data on costs, and thus apparently do not account for different input prices faced by different producers, thereby failing to condition their estimates on input prices of each unit. Banker and Natarajan (2011, pp. 279–281) propose estimation of technical, cost, and input allocative efficiencies when only the costs of inputs, but neither their prices nor their quantities are observed. If input prices vary across firms, as is likely due to differing local market conditions, then their approach will result in failure to condition on input prices.

used for inference about mean cost efficiency if and only if $q=1$ when DEA estimators are used.[9] When FDH estimators are used, standard CLTs never hold. Unless $q=1$ and DEA estimators are used, the bias of the cost efficiency estimates becomes critical and must be dealt with as described by Kneip et al. (2015).

In order see how to make valid inference about mean cost efficiency, let $\mu_{W_x} = E\left[\theta(C,Y \mid \Psi_{W_x})\right]$ and $\sigma^2_{W_x} = \text{VAR}\left[\theta(C,Y \mid \Psi_{W_x})\right] < \infty$ denote the mean and variance of cost efficiency, where expectations are with respect to (C,Y,W_x). Let

$$\widehat{\mu}_{W_x,n} := n^{-1}\sum_{i=1}^{n}\theta\left(C_i, Y_i \mid \widehat{\Psi}_{W_x,i,n}\right). \tag{3.17}$$

Let $\kappa_x = 1/(q+1)$ for the FDH case or $\kappa_x = 2/(q+2)$ for the DEA case, and define $n_{\kappa_x} := \min(\lfloor n^{2\kappa_x}\rfloor,\ n) \leq n$ where $\lfloor a \rfloor$ denotes the largest integer less than or equal to $a \in \mathbb{R}$. Assume the observations in $\mathcal{S}_n$ and the corresponding samples $\mathcal{S}_{W_x,i,n}$ are randomly sorted. Define

$$\widehat{\mu}_{W_x,n_{\kappa_x}} := n_{\kappa_x}^{-1}\sum_{i=1}^{n_{\kappa_x}}\theta\left(C_i, Y_i \mid \widehat{\Psi}_{W_x,i,n}\right). \tag{3.18}$$

Note that the efficiency estimates under the summation sign are computed using the full sample of n observations, but the summation is over only the first n_{κ_x} estimates.

Next, let $\widetilde{B}_{W_x,n,\kappa_x}$ denote the generalized jackknife estimate of the $O\left(n^{-\kappa_x}\right)$ bias of $\theta\left(C_i, Y_i \mid \widehat{\Psi}_{W_x,i,n}\right)$, with $\widetilde{B}_{W_x,n,\kappa_x}$ computed from using $\mathcal{S}_{W_x,i,n}$ as described by Kneip et al. (2015, Section 4). Computation of this bias estimate requires splitting the sample, and as noted by Kneip et al. (2016), there are $\binom{n}{n/2}$ possible splits. To reduce the bias estimate, randomly split the sample $K << \binom{n}{n/2}$ times and compute a bias estimate $\widetilde{B}_{W_x,n,\kappa_x,k}$ after each split. Then compute the average

$$\widehat{B}_{W_x,n,\kappa_x} = K^{-1}\sum_{k=1}^{K}\widetilde{B}_{W_x,n,\kappa_x,k}. \tag{3.19}$$

The next result permits inference about mean cost efficiency for any number q of outputs.

Theorem 4.1 of Kneip et al. (2015) establishes that $\sigma^2_{W_x}$ is estimated consistently by the sample variance

$$\widehat{\sigma}^2_{W_x} := n^{-1}\sum_{i=1}^{n}\left(\theta(C_i, Y_i \mid \widehat{\Psi}_{W_x,i,n}) - \widehat{\mu}_{W_x,n}\right)^2 \tag{3.20}$$

[9] Kohersa et al. (2000), cited above, specify $q=5$ outputs, and hence their reliance on the Lindeberg-Feller CLT for inference means that their inference is invalid. Similarly, for reasons given by Kneip et al. (2015, Section 5), inference in the second-stage regression of Staub et al. (2010) is also invalid.

of the cost efficiency estimates. Then Theorem 4.3 of Kneip et al. (2015) ensures that the confidence interval

$$\left[\widehat{\mu}_{W_x,n} - \widehat{B}_{W_x,n,\kappa_x} \pm \frac{\widehat{\sigma}_{W_x}}{\sqrt{n}} z_{(1-\frac{\alpha}{2})}\right], \tag{3.21}$$

where $z_{(1-\frac{\alpha}{2})}$ is the $(1-\frac{\alpha}{2})$ quantile of the standard normal distribution function, has asymptotic coverage of $(1-\alpha) \times 100$-percent whenever $q \leq 2$ in the FDH case or $q \leq 3$ in the DEA case. Alternatively, for $q \geq 3$ in the FDH case or $q \geq 4$ in the DEA case, the asymptotically valid $(1-\alpha)$ confidence interval

$$\left[\widehat{\mu}_{W_x,n_{\kappa_x}} - \widehat{B}_{W_x,n,\kappa_x} \pm \frac{\widehat{\sigma}_{W_x}}{n^{\kappa_x}} z_{(1-\frac{\alpha}{2})}\right] \tag{3.22}$$

can be used.

As discussed by Kneip et al. (2015), when $\kappa_x < 1/2$ the randomness due to the subsample mean $\widehat{\mu}_{W_x,n_{\kappa_x}}$ appearing in (3.22) can be eliminated by replacing $\widehat{\mu}_{W_x,n_{\kappa_x}}$ with the full mean $\widehat{\mu}_{W_x,n}$, which has the effect of averaging over all the possible subsamples of size n_{κ_x}. The resulting interval has the same width as the one in (3.22), but has coverage tending to 1 as $n \to \infty$ due to the results obtained above.[10]

3.4 Input Allocative Efficiency

Estimators of input allocative efficiency defined in (2.8) can be obtained by substituting either FDH or DEA estimators of cost and technical efficiency for the true values appearing on the right-hand side of (2.8).[11] This is the approach of Cummins et al. (1999), Sharmaa et al. (1999), Hartman et al. (2001), Coelli et al. (2002) Isik and Hassan (2002), Wadud (2003), Barros and Sampaio (2004), Barros and Mascarenhas (2005), Chen et al. (2005), Havrylchyk (2006), Hsu and Petchsakulwong (2010) and Merkert and Hensher (2011), all of whom estimate cost efficiency in $(p+q)$ dimensions. Staub et al. (2010) similarly estimate input allocative efficiency, but estimate cost efficiency in only $(q+1)$ dimensions. However, the properties of these estimators are unknown until now.

[10] Simar and Zelenyuk (2018) develop CLTs for estimates of aggregate efficiencies consisting of ratios of weighted sample means. Their main focus is on output-oriented technical efficiency, but they remark (p. 140) that their results can be adapted to aggregate revenue efficiency, aggregate output allocative efficiency, aggregate cost efficiency, aggregate input-oriented technical efficiency and aggregate input allocative efficiency. However, they do not provide the peculiarities of the asymptotic theory. Clearly, by using our results in Theorem 3.2 for the cost efficiencies and in Theorem B.2 for revenue efficiencies, the asymptotic theory in Simar and Zelenyuk for aggregate technical efficiency could be adapted to aggregate revenue efficiency and aggregate cost efficiency, but the rates will be governed by κ_y and κ_x respectively, and not by κ as given in Theorems 1, 2 and 3 of Simar and Zelenyuk.

[11] Of course, due to the result in Lemma 3.1 showing that free disposability of Ψ is preserved in Ψ_{w_x}, and that also convexity of Ψ (when it exists) is preserved in Ψ_{w_x} means that estimators of the same type should be used, rather than mixing FDH and DEA estimators.

Nonetheless, a number of papers use estimates of input allocative efficiency in statistical exercises. Among these, Cummins et al. (1999), Sharmaa et al. (1999), Isik and Hassan (2002), Wadud (2003) and Hsu and Petchsakulwong (2010) report both sample means and sample standard deviations. Sharmaa et al. (1999) and Wadud (2003) report t-tests of significance, and Wadud also employs F tests. Isik and Hassan (2002) use their input allocative efficiency estimates in a Kolmogorov-Smirnov test, and Cummins et al. (1999), Isik and Hassan (2002) and Wadud (2003) report results based on conventional inference from second-stage regressions of input allocative efficiency estimates on some explanatory variables. Banker and Natarajan (2011, Section 11.2.4) propose two tests of whether allocative efficiency is present (which amounts to testing whether technical efficiency and cost efficiency are equivalent) based on restrictive distributional assumptions regarding technical and cost efficiency, as well as a Kolmogorov-Smirnov test to compare the distributions of technical and cost efficiencies. No statistical results exist that would justify these exercises, and none of these statistical procedures or results are valid due to fact that (i) the true efficiencies are unobserved, and (ii) the observed estimates are biased, which prevents use of standard CLT results on which the aforementioned papers rely. This will become clear below.

In order to develop properties of estimators of input allocative efficiency, an additional assumption is needed.

Assumption 3.1. *There exists a constant* $0 < M_x < \infty$ *such that* $\|x\| \leq M_x$ *for all* $(x, y) \in \mathcal{D}$.

Assumption 3.1 is necessary to guarantee existence of moments of $\log(\theta(X, Y \mid \widehat{\Psi}_n)$. Although moments necessarily exist for $\theta(X_i, Y_i) \in (0, 1]$, $|\log \theta(X_i, Y_i)|$ is potentially unbounded. Moreover, up to this point we have only assumed compactness of $\mathcal{D}^*$ and not necessarily of $\mathcal{D}$. As noted by Kneip et al. (2018), Assumption 3.1 could in principle be replaced by a weaker version requiring only existence of all relevant moments, but boundedness of $\|x\|$ greatly simplifies asymptotic arguments used below.

The next result establishes the existence of a limiting distribution, the rate of convergence, and the properties of the first two moments for FDH and DEA estimators of input allocative efficiency.

Theorem 3.1. *Let* $\kappa = 1/(p + q)$ *for the FDH case and* $\kappa = 2/(p + q + 1)$ *for the DEA case. Then under Assumptions 2.1–2.5, 2.8–2.9 and 3.1 for the FDH case, and under Assumptions 2.1–2.7, 2.9 and 3.1 for the DEA case, for each* $(x, y) \in \mathcal{D}$,

$$n^{\kappa} \left(\mathcal{A}_x(x, y \mid \widehat{\Psi}_n, w_x) - \mathcal{A}_x(x, y \mid \Psi, w_x) \right) \xrightarrow{\mathcal{L}} Q_{\mathcal{A}_x, x, y} \tag{3.23}$$

where $Q_{\mathcal{A}_x,x,y}$ is a non-degenerate distribution with finite variance. In addition, let $(\zeta_1,\ \zeta_2,\ \zeta_3) = (\frac{2}{p+q},\ \frac{p+q+2}{p+q},\ \frac{p+q+1}{p+q})$ *for the FDH case, and* $(\zeta_1,\ \zeta_2,\ \zeta_3) = (\frac{3}{p+q+1},\ \frac{p+q+4}{p+q+1},\ \frac{p+q+2}{p+q+1})$ *for the DEA case. Then* $\exists$ *a constant* $D_1 \in (0,\infty)$ *such that for all* $i,j \in \{1,\ \ldots,\ n\}$, $i \neq j$,

$$E\left[\mathcal{A}_x(X_i,Y_i \mid \widehat{\Psi}_n, W_{x,i}) - \mathcal{A}_x(X_i,Y_i \mid \Psi, W_{x,i})\right] = D_1 n^{-\kappa} + O\left(n^{-\zeta_1}(\log n)^{\zeta_2}\right), \tag{3.24}$$

$$VAR\left[\mathcal{A}_x(X_i,Y_i \mid \widehat{\Psi}_n, W_{x,i}) - \mathcal{A}_x(X_i,Y_i \mid \Psi, W_{x,i})\right] = O\left(n^{-\zeta_1}(\log n)^{\zeta_1}\right) \tag{3.25}$$

and

$$\begin{aligned}\Big|COV\Big[&\mathcal{A}_x(X_i,Y_i \mid \widehat{\Psi}_n, W_{x,i}) - \mathcal{A}_x(X_i,Y_i \mid \Psi, W_{x,i}),\\ &\mathcal{A}_x(X_j,Y_j \mid \widehat{\Psi}_n, W_{x,j}) - \mathcal{A}_x(X_j,Y_j \mid \Psi, W_{x,j})\Big]\Big| = O\left(n^{-\zeta_3}(\log n)^{\zeta_3}\right)\\ &= o\left(n^{-1}\right).\end{aligned} \tag{3.26}$$

where expectations are with respect to (X,Y,W_x) *and the constant* D_1 *depends on the particular estimator (FDH or DEA), the density* f_{X,Y,W_x} *and the sets* $\mathcal{D} \in \Psi$ *and* $\mathcal{D}_W \subset \mathbb{R}^p_{++} \times \mathbb{R}^q_{++}$.

For purposes of making inference about mean input allocative efficiency, more work is needed due to the bias term $D_1 n^{-\kappa}$ in (3.24). Let $\mu_{\mathcal{A}_x} = E[\mathcal{A}_x(X,Y \mid \Psi, W_x)]$ and $\sigma^2_{\mathcal{A}_x} = \text{VAR}[\mathcal{A}_x(X,Y \mid \Psi, W_x)] < \infty$ denote the mean and variance of input allocative efficiency, where again expectations are with respect to (X,Y,W_x). Let

$$\widehat{\mu}_{\mathcal{A}_x,n} := n^{-1}\sum_{i=1}^{n} \mathcal{A}_x(X_i,Y_i \mid \widehat{\Psi}_n, W_{x,i}). \tag{3.27}$$

Let $\kappa = 1/(p+q)$ for the FDH case and $\kappa = 2/(p+q+1)$ for the DEA case, and define $n_\kappa := \min\left(\lfloor n^{2\kappa} \rfloor,\ n\right) \leq n$. Assume the observations in $\mathcal{S}_n$ are randomly sorted. Define

$$\widehat{\mu}_{\mathcal{A}_x,n_\kappa} := n_\kappa^{-1}\sum_{i=1}^{n_\kappa} \mathcal{A}_x(X_i,Y_i \mid \widehat{\Psi}_n, W_{x,i}). \tag{3.28}$$

Analogous to (3.18), the estimates of input allocative efficiency under the summation sign in (3.28) are computed using the full sample of n observations, but the summation is over only the first n_κ estimates.

Finally, let $\widetilde{B}_{\mathcal{A}_x,n,\kappa}$ denote the generalized jackknife estimate of the bias term $D_1 n^{-\kappa}$ in (3.24) computed as described by Kneip et al. (2015, Section 4). Analogous to (3.19), compute the average

$$\widehat{B}_{\mathcal{A}_x,n,\kappa} = K^{-1}\sum_{k=1}^{K} \widetilde{B}_{\mathcal{A}_x,n,\kappa,k} \tag{3.29}$$

over $K << \binom{n}{n/2}$ random splits of the sample to reduce the variance of the bias estimate. The next result gives a CLT for mean input allocative efficiency.

Theorem 3.2. *Let κ, ζ_1 and ζ_2 be defined for the FDH and DEA cases as in Theorem 3.1. Then under Assumptions 2.1–2.5, 2.8–2.9 and 3.1 for the FDH case, and under Assumptions 2.1–2.7, 2.9 and 3.1 for the DEA case, for $(p+q) \leq 3$ in the FDH case or $(p+q) \leq 4$ in the DEA case,*

$$\sqrt{n}\left(\widehat{\mu}_{\mathcal{A}_x,n} - \widehat{B}_{\mathcal{A}_x,n,\kappa} - \mu_{\mathcal{A}_x} + \xi_{\mathcal{A}_x,n,\kappa}\right) \xrightarrow{\mathcal{L}} N\left(0, \sigma^2_{\mathcal{A}_x}\right) \tag{3.30}$$

where $\xi_{\mathcal{A}_x,n,\kappa} = O\left(n^{-\zeta_1}(\log n)^{\zeta_2}\right) = o\left(n^{-\kappa}\right)$. In addition, for $(p+q) > 2$ in the FDH case or $(p+q) > 3$ in the DEA case, as $n \to \infty$

$$n^{\kappa}\left(\widehat{\mu}_{\mathcal{A}_x,n_\kappa} - \widehat{B}_{\mathcal{A}_x,n,\kappa} - \mu_{\mathcal{A}_x} + \xi_{\mathcal{A}_x,n,\kappa}\right) \xrightarrow{\mathcal{L}} N\left(0, \sigma^2_{\mathcal{A}_x}\right) \tag{3.31}$$

as $n \to \infty$. In addition, as $n \to \infty$,

$$\widehat{\sigma}^2_{\mathcal{A}_x} := \sum_{i=1}^{n}\left[\mathcal{A}_x(X_i, Y_i \mid \widehat{\Psi}_n, w_{x,i}) - \widehat{\mu}_{\mathcal{A}_x,n}\right]^2 \xrightarrow{p} \sigma^2_{\mathcal{A}_x}. \tag{3.32}$$

The CLT results in Theorem 3.2 can be used to construct confidence intervals for mean input allocative efficiency or to test hypotheses about mean input allocative efficiency. Note that either (3.30) or (3.31) can be used when $(p+q) = 3$ in the FDH case or $(p+q) = 4$ in the DEA case. In the DEA case, intervals based on (3.30) neglect $\sqrt{n}\xi_{\mathcal{A}_x,n_\kappa} = O\left(n^{-1/10}\right)$, while those based on (3.31) neglect $n^{\kappa}\xi_{\mathcal{A}_x,n_\kappa} = O\left(n^{-1/5}\right)$. Hence (3.31) is expected to provide more accurate intervals than (3.30) when $(p+q) = 4$ in the DEA case. Similar reasoning applies in the FDH case when $(p+q) = 3$.

3.5 Revenue Efficiency and Output Allocative Efficiency

Extending the results from Sections 3.3–3.4 to revenue efficiency and output allocative efficiency is straightforward, but there are some subtleties. Explicit details are given in the separate Appendices B–C. One should carefully note that revenue efficiency can be estimated with convergence rates $n^{1/(p+1)}$ and $n^{2/(p+2)}$ for the FDH and DEA cases, respectively. Output allocative efficiency is estimated with rates $n^{1/(p+q)}$ and $n^{2/(p+q+1)}$ for the FDH and DEA cases. Consequently, conventional CLTs do not hold for mean revenue efficiency nor for mean output allocative efficiency when FDH estimators are used, and in the DEA case hold for revenue efficiency only when $p = 1$ and for output allocative efficiency only when $p = 1 = q$.

Examples of applications where revenue efficiency is estimated include Sharma et al. (1999), Bojnec and Latruffe (2008), Cummins et al. (2010), Hsu and Petchsakulwong (2010), Eller et al. (2011) and Al-Khasawneh et al. (2012). Sharma et al. (1999, Table 2) report sample means of revenue and output allocative efficiencies for groups of producers, and rely on conventional CLTs to determine significance of differences from 1 and to test whether means are the same across groups. Cummins et al. (2010) and Eller et al. (2011) regress their estimates of revenue efficiency on some explanatory variables in second-stage regressions, and use conventional inference to determine significance or non-significance of their results. Due to the results obtained in the separate Appendices B–C, none of these inferences are valid.

3.6 Profit Efficiency

The usual approach to estimating profit efficiency defined by (2.13) involves first estimating maximum profit $\mathcal{P}_{\max}(x_0, y_0 \mid \Psi, w_x, w_y)$ by replacing the unknown Ψ with $\widehat{\Psi}_n$ in (2.12) to obtain

$$\begin{aligned}\mathcal{P}_{\max}(x_0, y_0 \mid \widehat{\Psi}_n, w_x, w_y) &= \max_{x,y,\boldsymbol{v}} \left\{ w_y' y - w_x' x \mid \boldsymbol{Y}\boldsymbol{v} \geq y,\ \boldsymbol{X}\boldsymbol{v} \leq x,\ \boldsymbol{1}_n'\boldsymbol{v} = 1,\ \boldsymbol{v} \in \mathbb{R}_+^n \right\} \\ &= w_y' \widehat{y}_{w_x,w_y} - w_x' \widehat{x}_{w_x,w_y} \qquad (3.33)\end{aligned}$$

where $\widehat{y}_{w_x,w_y}$ and $\widehat{x}_{w_x,w_y}$ are solutions to the optimization problem in the first line of (3.33). Then profit efficiency is estimated by

$$\begin{aligned}\mathcal{P}(x_0, y_0 \mid \widehat{\Psi}_n, d_x, d_y, w_x, w_y) :=& \frac{\mathcal{P}_{\max}(x_0, y_0 \mid \widehat{\Psi}_n, w_x, w_y) - (w_y' y_0 - w_x' x_0)}{w_y' d_y + w_x' d_x} \\ =& \frac{(w_y' \widehat{y}_{w_x,w_y} - w_x' \widehat{x}_{w_x,w_y}) - (w_y' y_0 - w_x' x_0)}{w_y' d_y + w_x' d_x} \qquad (3.34)\end{aligned}$$

This is the approach of Chambers et al. (1998), Färe and Grosskopf (2006), Färe et al. (2008) and others. Unfortunately, as with the estimators of cost and revenue efficiency, the properties of the estimators of maximum profit in (3.33) and profit efficiency in (3.34) are unknown.

The next result establishes that the profit maximization problem in (2.12) can be characterized as a one-dimensional problem.

Lemma 3.3. *Under Assumption 2.1,*

$$\mathcal{P}_{max}(x_0, y_0 \mid \Psi, w_x, w_y) = \max \left\{ \pi \mid \pi \in \Psi_{w_x,w_y} \right\}. \qquad (3.35)$$

The proof is obvious and is left to the reader.

Now apply the function h_{w_x,w_y} to each observation $(X_i, Y_i) \in \mathcal{S}_n$ to transform $\mathcal{S}_n$ to a set of iid observations $\mathcal{S}_{w_x,w_y,n} = \{\pi_i\}_{i=1}^n$. Due to Lemma 3.3 it is obvious that $\mathcal{P}_{\max}(x_0, y_0 \mid \Psi, w_x, w_y)$ can be estimated by

$$\mathcal{P}_{\max}(x_0, y_0 \mid \mathcal{S}_{w_x,w_y,n}, w_x, w_y) := \max\left\{\pi \mid \pi \in \mathcal{S}_{w_x,w_y,n}\right\}. \tag{3.36}$$

This amounts to a one-dimensional version of an FDH or DEA estimator, and the two are equivalent in one dimension. From the properties of both FDH and DEA estimators, it is clear that this estimator is consistent, converges at rate n^1 and has a non-degenerate limiting distribution. Consequently, substituting $\mathcal{P}_{\max}(x_0, y_0 \mid \mathcal{S}_{w_x,w_y,n}, w_x, w_y)$ for $\mathcal{P}_{\max}(x_0, y_0 \mid \Psi, w_x, w_y)$ for in (2.13) yields a consistent estimator $\mathcal{P}(x_0, y_0 \mid \mathcal{S}_{w_x,w_y,n}, d_x, d_y, w_x, w_y)$ of the Nerlovian profit efficiency measure $\mathcal{P}(x_0, y_0 \mid \Psi, d_x, d_y, w_x, w_y)$ for given direction vectors d_x and d_y and given price vectors w_x and w_y. Moreover, the resulting estimator converges at rate n^1 and has a non-degenerate limiting distribution due to the properties of $\mathcal{P}_{\max}(x_0, y_0 \mid \mathcal{S}_{w_x,w_y,n}, w_x, w_y)$. Knowledge of the convergence rate and existence of a non-degenerate limiting distribution permit use of the subsampling methods described by Simar and Wilson (2011a) for inference about the profit efficiency $\mathcal{P}(x_0, y_0 \mid \Psi, d_x, d_y, w_x, w_y)$ of a particular firm operating at $(x_0, y_0) \in \Psi$. In addition, the classical Lindeberg-Feller CLT can be used to make inference about mean profit efficiency due to the n^1 convergence rate.

3.7 Profit Allocative Efficiency

As noted in Section 2, the results of Kneip et al. (2015) for the input-oriented efficiency estimator $\theta(x, y \mid \widehat{\Psi}_n)$ in (3.3) extend to the directional efficiency estimator $\delta(x, y \mid d_x, d_y, \widehat{\Psi}_n)$ in (3.5) using arguments similar to those of Simar and Vanhems (2012) and Simar et al. (2012). Substituting $\mathcal{P}(x_0, y_0 \mid \mathcal{S}_{w_x,w_y,n}, w_x, w_y)$ and $\delta(x, y \mid d_x, d_y, \widehat{\Psi}_n)$ for $\mathcal{P}(x_0, y_0 \mid \Psi, d_x, d_y, w_x, w_y)$ and $\delta(x, y \mid d_x, d_y, \Psi)$ in (2.14) leads to the estimator

$$\mathcal{A}_\pi(x_0, y_0 \mid \mathcal{S}_n, d_x, d_y, w_x, w_y) := \mathcal{P}(x_0, y_0 \mid \mathcal{S}_{w_x,w_y,n}, d_x, d_y, w_x, w_y) - \delta(x, y \mid d_x, d_y, \widehat{\Psi}_n). \tag{3.37}$$

The following pair of results are straightforward extensions of Simar and Vanhems (2012, Theorem 4.1), Simar et al. (2012, Theorem 3.1) and Kneip et al. (2015, Theorems 3.1 and 3.3). Consequently, proofs are left to the reader. Assumption C.1 required by both Theorems 3.3 and 3.4 appears in Appendix C.

Theorem 3.3. *Let $\kappa = 1/(p+q)$ and for the FDH case and $\kappa = 2/(p+q+1)$ for the DEA case. Under Assumptions 2.1–2.5, 2.8, 3.1 and C.1 for the FDH case and under Assumptions 2.1–2.7, 3.1 and C.1 for the DEA case, for each $(x,y) \in \mathcal{D}$,*

$$n^{\kappa}\left(\mathcal{A}_{\pi}(x,y \mid \mathcal{S}_n, d_x, d_y, w_x, w_y) - \mathcal{A}_{\pi}(x,y \mid \Psi, d_x, d_y, w_x, w_y)\right) \xrightarrow{\mathcal{L}} Q_{\mathcal{A}_{\pi},x,y} \tag{3.38}$$

where $Q_{\mathcal{A}_{\pi},x,y}$ is a non-degenerate distribution with finite variance depending on the particular estimator (i.e., FDH or DEA).

Theorem 3.3 confirms that $\mathcal{A}_{\pi}(x,y \mid \mathcal{S}_n, d_x, d_y, w_x)$ is a consistent estimator of $\mathcal{A}_{\pi}(x,y \mid \Psi, d_x, d_y, w_x)$ with estimation error of order $O_p\left(n^{-\kappa}\right)$. The n^1 convergence rate of $\mathcal{P}(x_0, y_0 \mid \mathcal{S}_{w_x,w_y,n}, d_x, d_y, w_x, w_y)$ is dominated by the n^{κ} rate of $\delta(x,y \mid d_x, d_y, \widehat{\Psi}_n)$, and hence $\mathcal{A}_{\pi}(x_0, y_0 \mid \mathcal{S}_n, d_x, d_y, w_x, w_y)$ inherits the slower convergence rate. The existence of a limiting distribution and knowledge of the convergence rate permits use of the subsampling methods of Simar and Wilson (2011a) for making inference about profit allocative efficiency.

The next result establishes properties of moments of the profit allocative efficiency estimator.

Theorem 3.4. *Let κ, ζ_1, ζ_2 and ζ_3 be defined as in Theorem 3.1 for the FDH and DEA cases. Under Assumptions 2.1–2.5, 2.8, 2.9, 3.1 and C.1 for the FDH case and under Assumptions 2.1–2.7, 2.9, 3.1 and C.1 for the DEA case, $\exists$ a constant $D_2 \in (0,\infty)$ such that for all $i,j \in \{1, \ldots, n\}$, $i \neq j$,*

$$\begin{aligned} E\Big[\mathcal{A}_{\pi}(X_i, Y_i \mid \mathcal{S}_n, d_x, d_y, W_{x,i}, W_{y,i}) - \mathcal{A}_{\pi}(X_i, Y_i \mid \Psi, d_x, d_y, W_{x,i}, W_{y_i})\Big] \\ = D_2 n^{-\kappa} + O\left(n^{-\zeta_1}(\log n)^{\zeta_2}\right), \end{aligned} \tag{3.39}$$

$$\begin{aligned} VAR\Big[\mathcal{A}_{\pi}(X_i, Y_i \mid \mathcal{S}_n, d_x, d_y, W_{x,i}, W_{y,i}) - \mathcal{A}_{\pi}(X_i, Y_i \mid \Psi, d_x, d_y, W_{x,i}, W_{y,i})\Big] \\ = O\left(n^{-\zeta_1}(\log n)^{\zeta_1}\right) \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} \Big| COV\Big[&\mathcal{A}_{\pi}(X_i, Y_i \mid \mathcal{S}_n, d_x, d_y, W_{x,i}, W_{y,i}) - \mathcal{A}_{\pi}(X_i, Y_i \mid \Psi, d_x, d_y, W_{x,i}, W_{y,i}), \\ &\mathcal{A}_{\pi}(X_j, Y_j \mid \mathcal{S}_n, d_x, d_y, W_{x,j}, W_{y,j}) - \mathcal{A}_{\pi}(X_j, Y_j \mid \Psi, d_x, d_y, W_{x,j}, W_{y,j})\Big]\Big| \\ &= O\left(n^{-\zeta_3}(\log n)^{\zeta_3}\right) \\ &= o\left(n^{-1}\right). \end{aligned} \tag{3.41}$$

The constant D_2 depends on the density f_{X,Y,W_x,W_y}, the particular estimator (FDH or DEA) and the structure of the sets $\mathcal{D} \subset \Psi$ and $\mathcal{D}_W \subset \mathbb{R}^p_{++} \times \mathbb{R}^q_{++}$.

In order to make inference about mean profit allocative efficiency, let $\mu_{\mathcal{A}_\pi} = E[\mathcal{A}_\pi(X, Y \mid \Psi, d_x, d_y, W_x, W_y)]$ and $\sigma^2_{\mathcal{A}_\pi} = \text{VAR}[\mathcal{A}_\pi(X, Y \mid \Psi, d_x, d_y, W_x, W_y)] < \infty$ denote the mean and variance of profit allocative efficiency, where again expectations are with respect to (X, Y, W_x, W_y). In studies where profit overall (i.e., cost) efficiency and profit allocative efficiency are estimated, both input and output prices typically vary across firms. Let

$$\widehat{\mu}_{\mathcal{A}_\pi,n} := n^{-1} \sum_{i=1}^{n} \mathcal{A}_\pi(X_i, Y_i \mid \mathcal{S}_n, d_x, d_y, W_{x,i}, W_{y_i}). \tag{3.42}$$

Let $\kappa = 1/(p+q)$ for the FDH case and $\kappa = 2/(p+q+1)$ for the DEA case, and define $n_\kappa := \min\left(\lfloor n^{2\kappa} \rfloor,\ n\right) \leq n$. Assume the observations in $\mathcal{S}_n$ are randomly sorted. Define

$$\widehat{\mu}_{\mathcal{A}_\pi,n_\kappa} := n_\kappa^{-1} \sum_{i=1}^{n_\kappa} \mathcal{A}_\pi(X_i, Y_i \mid \mathcal{S}_n, d_x, d_y, W_{x,i}, W_{y,i}). \tag{3.43}$$

Analogous to (3.18), the estimates of profit allocative efficiency under the summation sign in (3.43) are computed using the full sample of n observations, but the summation is over only the first n_κ estimates.

Finally, let $\widetilde{B}_{\mathcal{A}_\pi,n,\kappa}$ denote the generalized jackknife estimate of the bias term $D_2 n^{-\kappa}$ in (3.39) computed as described by Kneip et al. (2015, Section 4). Analogous to (3.19), compute the average

$$\widehat{B}_{\mathcal{A}_\pi,n,\kappa} = K^{-1} \sum_{k=1}^{K} \widetilde{B}_{\mathcal{A}_\pi,n,\kappa,k} \tag{3.44}$$

over $K << \binom{n}{n/2}$ random splits of the sample to reduce the variance of the bias estimate. The next result gives a CLT for mean profit allocative efficiency.

Theorem 3.5. *Assume the conditions of Theorem 3.4 hold for either the FDH or DEA case. For $(p+q) \leq 3$ in the FDH case or $(p+q) \leq 4$ in the DEA case, as $n \to \infty$,*

$$\sqrt{n}\left(\widehat{\mu}_{\mathcal{A}_\pi,n} - \widehat{B}_{\mathcal{A}_\pi,n,\kappa} - \mu_{\mathcal{A}_\pi} + \xi_{\mathcal{A}_\pi,n,\kappa}\right) \xrightarrow{\mathcal{L}} N\left(0, \sigma^2_{\mathcal{A}_\pi}\right) \tag{3.45}$$

where $\xi_{\mathcal{A}_\pi,n,\kappa} = O\left(n^{-\frac{3}{p+q+1}} (\log n)^{\frac{p+q+4}{p+q+1}}\right) = o\left(n^{-\kappa}\right)$. In addition, for $(p+q) \geq 2$ in the FDH case or $(p+q) > 3$ in the DEA case, as $n \to \infty$

$$n^\kappa \left(\widehat{\mu}_{\mathcal{A}_\pi,n_\kappa} - \widehat{B}_{\mathcal{A}_\pi,n,\kappa} - \mu_{\mathcal{A}_\pi} + \xi_{\mathcal{A}_\pi,n,\kappa}\right) \xrightarrow{\mathcal{L}} N\left(0, \sigma^2_{\mathcal{A}_\pi}\right). \tag{3.46}$$

Moreover, as $n \to \infty$,

$$\widehat{\sigma}^2_{\mathcal{A}_\pi} := \sum_{i=1}^{n} \left[\mathcal{A}_\pi(X_i, Y_i \mid \widehat{\mathcal{S}}_n, d_x, d_y, w_{x,i}, w_{y,i}) - \widehat{\mu}_{\mathcal{A}_\pi,n} \right]^2 \xrightarrow{p} \sigma^2_{\mathcal{A}_\pi}. \tag{3.47}$$

The CLT results in Theorem 3.5 can be used to construct confidence intervals for mean input allocative efficiency or to test hypotheses about mean input allocative efficiency. Similar to Theorem 3.2, either (3.45) or (3.46) can be used when $(p+q) = 4$. Intervals based on (3.45) neglect $\sqrt{n}\xi_{\mathcal{A}_\pi,n_\kappa} = O\left(n^{-1/10}\right)$, while those based on (3.46) neglect $n^\kappa \xi_{\mathcal{A}_\pi,n_\kappa} = O\left(n^{-1/5}\right)$. Hence (3.46) is expected to provide more accurate intervals than (3.45) when $(p+q) = 4$ and DEA estimators are used. Similar reasoning applies when $(p+q) = 3$ and FDH estimators are used.

4 Empirical Illustration

To illustrate the methods developed above, we revisit Aly et al. (1990) who examine 322 U.S. Banks operating in 1986. The authors specify $p = 3$ inputs and $q = 5$ outputs and report means of input-oriented DEA estimates of technical efficiency, cost efficiency and input allocative efficiency. Means are reported for estimates from the full sample, as well as estimates from the subsample of 212 banks allowed to operate branches and corresponding subsample of 110 banks prohibited from operating branches. In addition to means, Aly et al. also report standard deviations of the various efficiency estimates obtained with the full sample (but not for the efficiency estimates obtained from the two subsamples). The authors also report results of five tests—analysis of variance, median test, Wilcoxon test, Van der Waerden test, and Savage scores test—to examine whether the distributions of efficiency distributions differ across the two subsamples. They state (p. 216) that, "As can be seen from Table 4, for all of the efficiency measures, except allocative, the test statistics indicate that the null hypothesis cannot be rejected. As a result, it may be concluded that the differences in the distributions of the efficiency measures between the two separate samples are not significant and that they are drawn from the same population, i.e., face similar environments."

Of course, it is now known, due to the results obtained in Section 3.3 and 3.4 as well as the results of Kneip et al. (2015) that the tests used by Aly et al. (1990) are invalid due to the tests' failure to properly account for the bias of the efficiency estimators. Using the Aly et al. data, we estimate technical efficiency using $\theta(x, y \mid \widehat{\Psi}_n)$, cost efficiency using $\theta(c, y \mid \widehat{\Psi}_{\mathcal{C},n})$ and input allocative efficiency using $\mathcal{A}_x(x, y, \widehat{\Psi}_n, w_x)$.[12] Estimates of technical efficiency are computed by

[12]We are grateful to Richard Grabowski for making the data available.

solving the linear program in (3.3) n times for each observed input-output pair in the sample. Cost efficiency for the ith observation is computed by first computing costs $W'_{xi}X_j$ for $j = 1, \ldots, n$ and then computing $\theta(W'_{xi}X_i, Y_i \mid \Psi_{\mathrm{DEA},W_{xi},n})$ as described in Sections 3.2 and 3.3, noting that the set of reference costs must be computed separately for each observation. Input allocative efficiency is then estimated by dividing the input cost estimate by the input technical efficiency estimate for observation i.

Table 1 gives sample means $\widehat{\mu}_{\bullet,n}$ and $\widehat{\mu}_{\bullet,n_\kappa}$ for each of the three types of efficiencies, where "$\bullet$" represents either θ, $\mathcal{C}$ or $\mathcal{A}_x$. Estimated 95-percent confidence intervals for the true means are also reported, as well as sample standard deviations and the corresponding bias estimates. The confidence interval estimates are based on the re-centering idea discussed at the end of Section 3.3. All estimates in Table 1 are computed using R and the Wilson (2008) FEAR library. Computational details are given in the separate Appendix D.

Table 1: Efficiency Estimates for Aly et al. (1990) Data

	$\widehat{\mu}_{\bullet,n}$	$\widehat{\mu}_{\bullet,n_\kappa}$	— 95% CI —		$\widehat{\sigma}_\bullet$	$\widehat{B}_{\bullet,n,\kappa}$
Full Sample, $n = 322$						
Tech. Eff.	0.8021	0.7760	0.4573	0.6514	0.1785	0.2477
Cost Eff.	0.7078	0.6790	0.4542	0.5980	0.1906	0.1816
Alloc. Eff.	0.8819	0.8979	0.7995	0.9294	0.1195	0.0175
Subsample with no branches, $n = 110$						
Tech. Eff.	0.8690	0.8866	0.5122	0.7237	0.1526	0.2511
Cost Eff.	0.7802	0.7465	0.5094	0.7124	0.1938	0.1693
Alloc. Eff.	0.8928	0.7891	0.8031	0.9795	0.1273	0.0015
Subsample with branches, $n = 212$						
Tech. Eff.	0.8462	0.8858	0.5326	0.7450	0.1714	0.2074
Cost Eff.	0.7726	0.8135	0.5509	0.7057	0.1809	0.1443
Alloc. Eff.	0.9133	0.9564	0.8617	0.9831	0.0979	−0.0091

Note that the estimated confidence intervals for mean technical efficiency and mean cost efficiency in Table 1 lie to the left of and do not cover either of the point estimates of the means. This is due to the biases—note also that the estimated biases for technical and cost efficiency are large, ranging from about 0.14 to about 0.25. The bias estimates for cost efficiency are smaller than the bias estimates for technical efficiency, reflecting the fact that cost efficiency is estimated in a 6-dimensional space whereas technical efficiency is estimated in an 8-dimensional space. By contrast, the estimated confidence intervals for mean input allocative efficiency cover the corresponding sample means. In all three cases, the corresponding bias estimates are close to 0. The bias estimate corresponding to mean input allocative efficiency in the sample of banks with branches is negative, but close to 0. Apparently, the biases in technical and cost efficiency tend to cancel each other when allocative efficiency is computed.[13]

Aly et al. (1990) did not report estimated confidence intervals, but implicitly invite the reader to do so using the classical, Lindeberg-Feller CLT since they report both sample means and standard deviations. The classical confidence interval estimates can be obtained by adding the bias estimates reported in Table 1 to the corresponding estimated confidence bounds. For the full sample, doing so yields estimated bounds (0.7051, 0.8991) for technical efficiency, (0.6358, 0.7797) for cost efficiency and (0.8170, 0.9469) for input allocative efficiency. The classically-estimated bounds for technical and cost efficiency are quite different from the ones reported in Table 1 due to the large biases associated with the estimates of mean technical and cost efficiency. Moreover, due to the results obtained in Sections 3.3 and 3.4, it is clear that the classical confidence intervals have (even for input allocative efficiency) coverage tending to 0 as $n \to \infty$.

5 Conclusions

This paper provides results on rates of convergence and existence of limiting distributions for nonparametric FDH and DEA estimators of cost, revenue and profit efficiency as well as the corresponding allocative efficiencies. The nonparametric estimators of cost, revenue and profit efficiency are shown to have faster rates of convergence than their corresponding estimators of technical or allocative efficiency. Combined with the subsampling methods of Simar and Wilson (2011b), these results enable researchers to make inference about these efficiencies for individual firms or producers. In addition, results on moments of the various estimators are provided. These results indicate that standard CLT results (e.g., the Lindeberg-Feller CLT) cannot be used to make inference about

[13] Aly et al. (1990) assume constant returns to scale (CRS) when estimating cost efficiency and input allocative efficiency, and report separate estimates of technical efficiency assuming either constant or variable returns to scale. Using the FEAR software library (Wilson, 2008) we obtain a mean of 0.7489 for the CRS-DEA estimates on the full sample, with a variance of standard deviation of 0.1801, consistent with what Aly et al. report in the third row of their Table 2. But when using the VRS version of the DEA estimators, we obtain a mean of 0.8021 with standard deviation 0.1785, whereas Aly et al. report 0.77 and 0.19 (respectively).

mean the mean efficiencies except in very limited cases. New CLTs are developed, enabling inference about mean cost, revenue and profit efficiency as well as the corresponding allocative efficiencies for all dimensions $(p+q)$. These results enable applied researchers for the first time to estimate confidence intervals and to test hypotheses about model features in general settings.

A Technical Details

A.1 Proof of Lemma 3.1

To prove (i), recall that $(c, y) = h_{w_x}(x, y) = (w_x' x, y)$ and $(\widetilde{c}, y) = h_{w_x}(\widetilde{x}, y) = (w_x' \widetilde{x}, y)$. Now consider $(x, y) \in \Psi$. Given y, we have

$$
\begin{aligned}
\widetilde{c} \geq c &\Rightarrow w_x' \widetilde{x} \geq w_x' x \\
&\Rightarrow \widetilde{x} \geq x \qquad \text{since } w_x > 0 \\
&\Rightarrow (\widetilde{x}, y) \in \Psi \qquad \text{by Assumption 2.3} \\
&\Rightarrow h_{w_x}(\widetilde{x}, y) \in h_{w_x}(\Psi) \qquad \text{by (3.7)} \\
&\Rightarrow (\widetilde{c}, y) \in \Psi_{w_x}.
\end{aligned} \tag{A.1}
$$

Alternatively, given x we have $c = w_x' x$ and

$$
\begin{aligned}
\widetilde{y} \leq y &\Rightarrow (x, \widetilde{y}) \in \Psi \qquad \text{by Assumption 2.3} \\
&\Rightarrow h_{w_x}(x, \widetilde{y}) \in h_{w_x}(\Psi) \\
&\Rightarrow (w_x' x, \widetilde{y}) \in \Psi_{w_x} \\
&\Rightarrow (c, \widetilde{y}) \in \Psi_{w_x},
\end{aligned} \tag{A.2}
$$

establishing (i). Results (ii) and (iii) follow from similar reasoning. ■

A.2 Proof of Lemma 3.2

Define the level set

$$
\mathcal{X}(y) := \{x \mid (x, y) \in \Psi\}. \tag{A.3}
$$

Let $x_* = \underset{x}{\operatorname{argmin}}\{w_x' x \mid (x, y) \in \Psi,\ w_x, x \in \mathbb{R}_+^p\} = \underset{x}{\operatorname{argmin}}\{w_x' x \mid x \in \mathcal{X}(y_0),\ w_x, x \in \mathbb{R}_+^p\}$. The point $x_* \in \mathcal{X}(y_0)$ is *minimal* in the sense that it results in cost lower than any other point in $\mathcal{X}(y_0)$. By the Supporting Hyperplane Theorem there exists x_{**} such that $w_x' x_{**} = w_x' x_* = \mathcal{C}_{\min}(x_0, y_0 \mid \Psi, w_x)$ and

$$
x_0 = k x_{**} \tag{A.4}
$$

for some $k \in [1, \infty)$.

By definition in (2.7), $\mathcal{C}(x_0, y_0 \mid \Psi, w_x) = \frac{w_x' x_*}{w_x' x_0}$, and by (A.4) $\frac{w_x' x_*}{w_x' x_0} = \frac{w_x' x_{**}}{w_x' x_0} = \frac{w_x'(k^{-1} x_0)}{w_x' x_0} = k^{-1}$. Moreover, from (A.4) it is clear that $||x_{**}||_2 = ||k^{-1} x_0||_2 = k^{-1} ||x_0||_2$ and hence $k^{-1} = \frac{||x_{**}||_2}{||x_0||_2}$. Therefore cost efficiency is given by the ratio of lengths between three collinear points (i.e., the origin, x_{**} and x_0). It is well-known (e.g., see Byer et al., 2010, Theorem 12.7) that affine transformations such as h_{w_x} that maps Ψ to Ψ_{w_x} preserve such ratios. Moreover, the affine transformation h_{w_x} maps extreme points of Ψ to extreme points of Ψ_{w_x}. In addition, the half-space $H^+ := \{x \mid w_x' x \geq w_x' x_*\} \subset \mathbb{R}_+^p$ is mapped by h_{w_x} to the half-space $H^{++} := \{c \mid c \geq c_*\} \subset \mathbb{R}_+^1$ where $c_* = w_x' x_*$. Hence h_{w_x} maps both x_{**} and the minimal point $x_* \in \mathcal{X}(y_0)$ to the minimal point $(c_*, y_0) \in \Psi_{w_x}$, establishing the result. ■

A.3 Proof of Theorem 3.1

Before beginning the proof of Theorem 3.1, some additional, intermediate results are needed.

Lemma A.1. *Let κ, ζ_1, ζ_2 and ζ_3 be defined as in Theorem 3.1 for the FDH and DEA cases. Under Assumptions 2.1–2.5, 2.8 and 3.1 for the FDH case and under Assumptions 2.1–2.7 and 3.1 for the DEA case, for each $(x, y) \in \mathcal{D}$,*

$$n^{\kappa} \left(\log \left(\theta(x, y \mid \widehat{\Psi}_n) \right) - \log \left(\theta(x, y \mid \Psi) \right) \right) \xrightarrow{\mathcal{L}} Q_{\theta,x,y}^{log} \tag{A.5}$$

where $Q_{\theta,x,y}^{log}$ is a non-degenerate distribution with finite variance. In addition, $\exists$ a constant $D_3 \in (0, \infty)$ such that for all $i, j \in \{1, \ldots, n\}$, $i \neq j$,

$$E \left[\log \left(\theta(X_i, Y_i \mid \widehat{\Psi}_n \right) - \log \left(\theta(X_i, Y_i \mid \Psi \right) \right] = D_3 n^{-\kappa} + O \left(n^{-\zeta_1} (\log n)^{\zeta_2} \right), \tag{A.6}$$

$$VAR \left[\log \left(\theta(X_i, Y_i \mid \widehat{\Psi}_n) \right) - \log \left(\theta(X_i, Y_i \mid \Psi) \right) \right] = O \left(n^{-\zeta_1} (\log n)^{\zeta_1} \right), \tag{A.7}$$

and

$$\begin{aligned} \Big| COV \Big[& \log \left(\theta(X_i, Y_i \mid \widehat{\Psi}_n) \right) - \log \left(\theta(X_i, Y_i \mid \Psi) \right), \\ & \log \left(\theta(X_j, Y_j \mid \widehat{\Psi}_n) \right) - \log \left(\theta(X_j, Y_j \mid \Psi) \right) \Big] \Big| = O \left(n^{-\zeta_3} (\log n)^{\zeta_3} \right) \\ & \qquad = o \left(n^{-1} \right). \end{aligned} \tag{A.8}$$

The value of the constant D_3 depends on the particular estimator, the density f and the structure of the set $\mathcal{D} \subset \Psi$.

Proof. From Kneip et al. (2008, Theorem 2) we have

$$n^{\frac{2}{p+q+1}}\left(\theta(x,y\mid\widehat{\Psi}_n)-\theta(x,y\mid\Psi)\right)\xrightarrow{\mathcal{L}}Q_{\theta,x,y} \tag{A.9}$$

for the DEA case, and a similar result (with scaling factor $n^{1/(p+q)}$) for the FDH case from Daouia et al. (2017, Proposition 2) after transforming to the input-oriented case. In addition, the log function is monotonic and differentiable with nonzero derivatives on $\mathbb{R}_+$. Hence (A.5) follows by the delta method for both the FDH and DEA cases.

The results in (A.6)–(A.8) follow from arguments similar to those in the proof of Theorem 3.2 in Kneip et al. (2018) and the proof of Theorems 3.1 and 3.3 in Kneip et al. (2015). In particular, the convergence rate here is $n^{2/(p+q+1)}$ for the DEA case (and $n^{1/(p+q)}$ for the FDH case) as is the case in Theorem 3.2 of Kneip et al. (2018) where distance is measured to boundary of the conical hull of $\widehat{\mathcal{P}}$. The arguments rely again on the fact that the log function is monotonic and differentiable, permitting Taylor expansions and the delta method. ■

Now recall the definition of input-allocative efficiency in (2.8). Substituting $\theta(c_0,y_0\mid\Psi_{w_x})$ for $\mathcal{C}(x_0,y_0\mid\Psi,w_x)$ in (2.8) and then taking logs yields

$$\log\left(\mathcal{A}_x(x_0,y_0\mid\Psi,w_x)\right)=\log\left(\theta(c_0,y_0\mid\Psi_{w_x})\right)-\log\left(\theta(x_0,y_0\mid\Psi)\right). \tag{A.10}$$

A natural estimator of $\log\left(\mathcal{A}_x(x_0,y_0\mid\Psi,w_x)\right)$ is obtained by replacing $\theta(c_0,y_0\mid\Psi_{w_x})$ and $\theta(x_0,y_0\mid\Psi)$ on the right-hand side of (A.10) with the corresponding estimators $\theta(c_0,y_0\mid\widehat{\Psi}_{w_x,n})$ discussed in Section 3.3 and $\theta(x_0,y_0\mid\widehat{\Psi}_n)$ given by (3.3). The next result establishes the properties of the resulting estimator

$$\log\left(\mathcal{A}_x(x_0,y_0\mid\widehat{\Psi}_n,w_x)\right)=\log\left(\theta(c_0,y_0\mid\widehat{\Psi}_{w_x,n})\right)-\log\left(\theta(x_0,y_0\mid\widehat{\Psi}_n)\right). \tag{A.11}$$

Theorem A.1. *Let κ be defined for the FDH and DEA cases as in Lemma A.1. Under Assumptions 2.1–2.5, 2.8 and 3.1 for the FDH case and under Assumptions 2.1–2.7 and 3.1 for the DEA case, for each $(x,y)\in\mathcal{D}$,*

$$n^{\kappa}\left(\log\left(\mathcal{A}_x(x,y\mid\widehat{\Psi}_n,w_x)\right)-\log\left(\mathcal{A}_x(x,y\mid\Psi,w_x)\right)\right)\xrightarrow{\mathcal{L}}Q^{log}_{\mathcal{A}_x,x,y} \tag{A.12}$$

as $n\to\infty$ where $Q^{log}_{\mathcal{A}_x,x,y}$ is a non-degenerate distribution with finite variance.

Proof. Recall that $\mathcal{C}(x_0,y_0\mid\widehat{\Psi}_n,w_x)=\theta(c_0,y_0\mid\widehat{\Psi}_{w_x,n})$ where $\widehat{\Psi}_{w_x,n}$ is the DEA estimator of the image of Ψ under the affine transformation h_{w_x}. Then the properties of

$\log\left(\mathcal{C}(x_0, y_0 \mid \widehat{\Psi}_n, w_x)\right) = \log\left(\theta(c_0, y_0 \mid \widehat{\Psi}_{w_x,n})\right)$ are given by Lemma A.1 where the number of "inputs" p is 1. The results (A.12)–(A.15) for the DEA case follow trivially after recognizing that the rate of $\log\left(\theta(c_0, y_0 \mid \widehat{\Psi}_{w_x,n})\right)$ is dominated by the slower rate of $\log\left(\theta(c_0, y_0 \mid \widehat{\Psi}_n)\right)$.

Similar reasoning establishes the result for the FDH case. ■

Theorem A.1 establishes the existence of a non-degenerate limiting distribution and the rate of convergence for FDH and DEA estimators of the log of input allocative efficiency. Consequently, confidence intervals with asymptotically correct coverage for the log of input allocative efficiency of individual firms can be estimated using the sub-sampling methods described by Simar and Wilson (2011a) while noting that the rate of convergence is $n^{1/(p+q)}$ for the FDH case or $n^{2/(p+q+1)}$ for the DEA case as established by Theorem A.1. Since the resulting intervals are transformation-respecting, one can take exponentials of the endpoints to obtain an asymptotically valid confidence interval for $\mathcal{A}_x(x_0, y_0 \mid \Psi, w_x)$.

The next result establishes moment properties for FDH and DEA estimators of log input allocative efficiency.

Theorem A.2. *Let κ, ζ_1, ζ_2 and ζ_3 be defined for the FDH and DEA cases as in Theorem 3.1. Under Assumptions 2.1–2.5, 2.8 and 3.1 for the FDH case and under Assumptions 2.1–2.7 and 3.1 for the DEA case, $\exists$ a constant $D_4 \in (0, \infty)$ such that for all $i, j \in \{1, \ldots, n\}$, $i \neq j$,*

$$E\left[\log\left(\mathcal{A}_x(X_i, Y_i \mid \widehat{\Psi}_n, W_{x,i})\right) - \log\left(\mathcal{A}_x(X_i, Y_i \mid \Psi, W_{x,i})\right)\right] = D_4 n^{-\kappa} + O\left(n^{-\zeta_1}(\log n)^{\zeta_2}\right), \tag{A.13}$$

$$VAR\left[\log\left(\mathcal{A}_x(X_i, Y_i \mid \widehat{\Psi}_n, W_{x,i})\right) - \log\left(\mathcal{A}_x(X_i, Y_i \mid \Psi, W_{x,i})\right)\right] = O\left(n^{-\zeta_1}(\log n)^{\zeta_1}\right) \tag{A.14}$$

and

$$\begin{aligned}\Big| COV\Big[&\log\left(\mathcal{A}_x(X_i, Y_i \mid \widehat{\Psi}_n, W_{x,i})\right) - \log\left(\mathcal{A}_x(X_i, Y_i \mid \Psi, W_{x,i})\right), \\ &\log\left(\mathcal{A}_x(X_j, Y_j \mid \widehat{\Psi}_n, W_{x,j})\right) - \log\left(\mathcal{A}_x(X_j, Y_j \mid \Psi, W_{x,j})\right)\Big]\Big| = O\left(n^{-\zeta_3}(\log n)^{\zeta_3}\right) \\ &= o\left(n^{-1}\right) \end{aligned} \tag{A.15}$$

as $n \to \infty$. The constant D_4 depends on the particular estimator (FDH or DEA), the density f_{X,Y,W_x} and the structure of the sets $\mathcal{D} \subset \Psi$ and $\mathcal{D}_W \subset \mathbb{R}^p_{++} \times \mathbb{R}^q_{++}$.

Proof. The results follow due to (A.6)–(A.8), noting that the slower convergence rate in the denominator of $\log(\mathcal{A}_x(X_i, Y_i) \mid \widehat{\Psi}_n, W_{x,i})$ dominates the faster rate of the cost efficiency estimator in the numerator. ■

Due to Theorem A.1, $\log\left(\mathcal{A}_x(x, y \mid \widehat{\Psi}_n, w_x)\right)$ is a consistent estimator of $\log\left(\mathcal{A}_x(x, y \mid \Psi, w_x)\right)$ with estimation error of order $O\left(n^{-\kappa}\right)$. In addition, Theorem A.2 makes clear that standard CLT results can be used to make inference about mean input allocative efficiency only if $(p+q) < 3$ in the DEA case and not at all in the FDH case.

Theorem 3.1 can now be proved. The exponential function is monotonic and differentiable with nonzero derivatives on $\mathbb{R}_+$. Hence the result in (3.23) follows from Theorem A.1 via the delta method. Now let $\Gamma(\cdot)$ denote the log function. Due to Assumption 3.1, $\Gamma(\theta(X_i, Y_i \mid \widehat{\Psi}_n))$ as well as its derivatives $\Gamma'(\theta(X_i, Y_i \mid \widehat{\Psi}_n))$ and $\Gamma''(\theta(X_i, Y_i \mid \widehat{\Psi}_n))$ are uniformly bounded for all $(X_i, Y_i) \in \mathcal{D}$. Then the results in (3.24), (3.25) and (3.26) follow after applying Taylor expansions and arguments analogous to those used to prove results (3.17)–(3.19) in Theorem 3.2 of Kneip et al. (2018). ■

A.4 Proof of Theorem 3.2

The results in (3.30) and (3.31) follow immediately using arguments analogous to those of Kneip et al. (2015) leading to their Theorems 4.3 and 4.4. The result in (3.32) follows from (3.23) in Theorem 3.1 using arguments analogous to those used to prove (4.5) appearing in Theorem 4.2 of Simar and Wilson (2019). ■

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