EPSILON-REGULARITY FOR p-HARMONIC MAPS AT A FREE BOUNDARY ON A SPHERE

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ABSTRACT. We prove an ϵ-regularity theorem for vector-valued p-harmonic maps, which are critical with respect to a partially free boundary condition, namely that they map the boundary into a round sphere.

This does not seem to follow from the reflection method that Scheven used for harmonic maps with free boundary (i.e., the case $p = 2$): the reflected equation can be interpreted as a p-harmonic map equation into a manifold, but the regularity theory for such equations is only known for round targets.

Instead, we follow the spirit of the last-named author's recent work on free boundary harmonic maps and choose a good frame directly at the free boundary. This leads to growth estimates, which, in the critical regime $p = n$, imply Hölder regularity of solutions. In the supercritical regime, $p < n$, we combine the growth estimate with the geometric reflection argument: the reflected equation is super-critical, but, under the assumption of growth estimates, solutions are regular.

In the case $p < n$, for stationary p-harmonic maps with free boundary, as a consequence of a monotonicity formula we obtain partial regularity up to the boundary away from a set of $(n-p)$-dimensional Hausdorff measure.

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1. Introduction

Over the last few years the theory of half-harmonic maps received a lot of attention, beginning with the pioneering work of Da Lio and Rivière [8, 7], see also the subsequent [45, 4, 36, 48]. Half-harmonic maps appear in nature as free boundary problems — e.g., they are connected to critical points of the energy

$$\|\nabla u\|^2_{L^2(D,\mathbb{R}^N)} \quad \text{s.t. } u(\partial D) \subset \mathcal{N} \text{ in the a.e. trace sense.}$$

Here, $D \subset \mathbb{R}^n$ is an open set and $\mathcal{N} \subset \mathbb{R}^N$ is a smooth closed manifold. The Euler-Lagrange equations of the latter problem are

$$\begin{cases} \Delta u = 0 & \text{in } D \\ \partial_\nu u \perp T_u\mathcal{N} & \text{on } \partial D, \end{cases} \tag{1.1}$$

where ν denotes the outer normal vector.

For $D = \mathbb{R}^n_+$ and $\partial D = \mathbb{R}^{n-1} \times \{0\}$ the equation (1.1) is equivalent to

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ (-\Delta)^{\frac{1}{2}}_{\mathbb{R}^{n-1}} u \perp T_u\mathcal{N} & \text{on } \mathbb{R}^{n-1} \times \{0\}. \end{cases} \tag{1.2}$$

Here, $(-\Delta)^{\frac{1}{2}}_{\mathbb{R}^{n-1}}$ denotes the half-Laplacian acting on functions defined on $\mathbb{R}^{n-1} \times \{0\}$. The equation $(-\Delta)^{\frac{1}{2}}_{\mathbb{R}^{n-1}} u \perp T_u\mathcal{N}$ is the half-harmonic map equation, for an overview see [8].

The equivalence of (1.1) and (1.2) is crucially related to the fact that we are considering critical points of an L^2-energy. Several notions of fractional p-harmonic maps have been proposed. In [9, 10] Da Lio and the third-named author considered $H^{s,p}$-harmonic maps, i.e., critical points of

$$\|(-\Delta)^{\frac{s}{2}} u\|^p_{L^p(\mathbb{R}^{n-1},\mathbb{R}^N)} \quad \text{s.t. } u(x) \in \mathcal{N} \text{ for a.e. } x \in \mathbb{R}^{n-1}. \tag{1.3}$$

In [47] energies with a gradient-type structure were studied, namely

$$\|D^s u\|^p_{L^p(\mathbb{R}^{n-1},\mathbb{R}^N)} \quad \text{s.t. } u(x) \in \mathcal{N} \text{ for a.e. } x \in \mathbb{R}^{n-1}, \tag{1.4}$$

where $D^s = DI^{1-s}$ is the Riesz-fractional gradient, see also [52, 53]. Finally, $W^{s,p}$-harmonic maps were studied in [46], that is critical points of the energy

$$\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \quad \text{s.t. } u(x) \in \mathcal{N} \text{ for a.e. } x \in \mathbb{R}^{n-1}, \tag{1.5}$$

see also [35]. All these versions of fractional p-harmonic maps have one thing in common: they do not seem related to a free boundary equation (1.1). For (1.3) and (1.4) this is clear, since the energies are defined on the "wrong" function space $H^{s,p}$. Indeed, a map in

$W^{1,p}(D)$ has a trace in $W^{1-\frac{1}{p},p}(\partial D)$, but $W^{1-\frac{1}{p},p}(\partial D) \neq H^{1-\frac{1}{p},p}(\partial D)$ for $p \neq 2$. For the $W^{s,p}$-energy (1.5) it is an interesting open problem if it is possible to find a p-harmonic extension that interprets this problem as a free boundary problem.

In this work we concentrate on free boundary problems. We focus on smooth bounded domains, so in the sequel D is such a domain. We prove regularity at the free boundary for critical points $u : D \to \mathbb{R}^N$ of the energy

$$\|\nabla u\|^p_{L^p(D,\mathbb{R}^N)} \quad \text{s.t. } u(\partial D) \subset \mathcal{N} \text{ in the a.e. trace sense.} \tag{1.6}$$

It is not clear that the space $\mathcal{A} := \{u \in W^{1,p}(D,\mathbb{R}^N) : \ u(\partial D) \subset \mathcal{N}\}$ possesses a natural structure of a smooth Banach manifold. That is why we shall define what we mean by critical point.

Definition 1.1. We say that u is a critical point of $\int_D |\nabla u|^p$ in the space $\mathcal{A}$ if u satisfies

$$\int_D |\nabla u|^{p-2} \nabla u \cdot \nabla \phi = 0 \tag{1.7}$$

for all ϕ in $W^{1,p}(D,\mathbb{R}^N)$ s.t. its trace $\phi(x)|_{\partial D}$ is in $T_{u(x)}\mathcal{N}$ a.e. Such a critical point is called a *p-harmonic map with free boundary.*

Equation (1.7) is obtained by requiring that for every C^1-path $\gamma : (-1,1) \to \mathcal{A}$ such that $\gamma(0) = u$ we have

$$\frac{d}{dt}\bigg|_{t=0} \int_D |\nabla \gamma(t)|^p = 0. \tag{1.8}$$

Remark. *Although this is not relevant for our purpose, let us remark that equation (1.7) can be interpreted as u satisfying in a distributional sense*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } D \\ |\nabla u|^{p-2}\partial_\nu u \perp T_u\mathcal{N} & \text{on } \partial D. \end{cases} \tag{1.9}$$

Note that, by definition, u is a solution of (1.9) in the sense of distributions if and only if

$$\int_D |\nabla u|^{p-2} \nabla u \cdot \nabla \phi = 0 \tag{1.10}$$

for all $\phi \in C^\infty(\overline{D},\mathbb{R}^N)$ with $\phi(x) \in T_{u(x)}\mathcal{N}$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial D$. Indeed, taking $\phi \in C_c^\infty(D,\mathbb{R}^N)$ we obtain the interior equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \text{ in } D.$$

As for the boundary equation, we can see that if u is smooth enough and satisfies (1.10) then after an integration by parts we find

$$\int_{\partial D} |\nabla u|^{p-2}\partial_\nu u \cdot \varphi = 0. \tag{1.11}$$

Since any $\varphi \in C^\infty(\partial D, \mathbb{R}^N)$ with $\varphi(x) \in T_{u(x)}\mathcal{N}$ can be extended in a function $\phi \in C^\infty(\overline{D}, \mathbb{R}^N)$, thus (1.11) implies

$$|\nabla u|^{p-2}\partial_\nu u \perp T_u\mathcal{N} \text{ on } \partial D.$$

The equivalence between being a solution of (1.9) in the sense of distributions and being a critical point of the p-energy in the space $\mathcal{A}$ is true if u is smooth enough, for example $u \in C^1(\overline{D}, \mathbb{R}^n)$ is sufficient. Indeed, in this case we can see that we have density of $\{\phi \in C^\infty(\overline{D}, \mathbb{R}^N) \colon \phi \in T_u\mathcal{N}\}$ in $\{\phi \in W^{1,p}(D, \mathbb{R}^N) \colon \phi\big|_{\partial D} \in T_u\mathcal{N}\}$.

The natural starting point, when studying equations of the form (1.9), is the regularity theory. The interior regularity is known and follows from the interior equation and results of [60, 58], see also the recent [30]. Hence, the main difficulty is the regularity up to the boundary. For an arbitrary manifold $\mathcal{N}$ a regularity theory for a solution (1.9) is out of reach: even the regularity theory for the interior problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \perp T_u\mathcal{N}$$

is known only for homogeneous targets $\mathcal{N}$, see Fuchs [20], Takeuchi [57], Toro and Wang [59], Strzelecki [55, 56], and also the recent survey [50]. For this reason we shall restrict our attention to the sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$. In the rest of the paper we consider the problem:

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } D \\ |\nabla u|^{p-2}\partial_\nu u \perp T_u\mathbb{S}^{N-1} & \text{on } \partial D \\ u(\partial D) \subset \mathbb{S}^{N-1}. \end{cases} \tag{1.12}$$

We remark that the free boundary conditions can be viewed as boundary conditions mixed between Dirichlet and homogeneous Neumann boundary conditions. Indeed, in the sphere case we have a Dirichlet boundary condition for the norm of u: $|u| = 1$ on ∂D and homogeneous Neumann condition for the "phase" $\partial_\nu \left(\frac{u}{|u|}\right) = 0$. To see that in the case of a general manifold we can use Fermi coordinates near some points of $\mathcal{N}$ as explained in [18, p.938-939] in the context of minimal surfaces with free boundaries (for more on minimal surfaces with free boundaries see also [17] and the references therein).

Our main theorem is the following ϵ-regularity type theorem.

Theorem 1.2 (ϵ-regularity)**.** *Let $D \subset \mathbb{R}^n$ be a smooth, bounded domain and $p \geq 2$. Then there exist $\epsilon = \epsilon(p, n, D) > 0$ and $\alpha = \alpha(p, n, D) > 0$, such that for any $u \in W^{1,p}(D, \mathbb{R}^N)$ solution to (1.12) the following holds: If for some $R > 0$ and for some $x_0 \in \overline{D}$*

$$\sup_{|y_0 - x_0| < R} \sup_{\rho < R} \rho^{p-n} \int_{B(y_0,\rho)\cap D} |\nabla u|^p < \epsilon, \tag{1.13}$$

then u and ∇u are Hölder continuous in $B(x_0, R/2) \cap \overline{D}$. Moreover, we have the following estimates:

$$\sup_{x,y \in B(x_0,R/2)} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \precsim R^{-\alpha} \left(\sup_{|y_0-x_0|<R} \sup_{\rho<R} \rho^{p-n} \int_{B(y_0,\rho)\cap D} |\nabla u|^p \right)^{\frac{1}{p}}$$

and

$$\sup_{x,y \in B(x_0,R/2)} \frac{|\nabla u(x)-\nabla u(y)|}{|x-y|^\alpha} \precsim R^{-\alpha-1} \left(\sup_{|y_0-x_0|<R} \sup_{\rho<R} \rho^{p-n} \int_{B(y_0,\rho)\cap D} |\nabla u|^p \right)^{\frac{1}{p}}.$$

When $p = n$ this ϵ-regularity implies directly (from the absolute continuity of the Lebesgue integral) that n-harmonic maps with free boundary and their gradients are Hölder continuous.

Corollary 1.3. *Let u and α be as in Theorem 1.2 with $p = n$ then u is in $C^{1,\alpha}(\overline{D}, \mathbb{R}^N)$.*

As usual, an ϵ-regularity result such as Theorem 1.2 implies partial regularity for *stationary* p-harmonic maps with free boundary (cf. (6.1) for the definition).

Theorem 1.4 (partial regularity)**.** *Let $D \subset \mathbb{R}^n$ be a smooth, bounded domain, $p \geq 2$, and assume that $u \in W^{1,p}(D, \mathbb{R}^N)$, with trace $u \in W^{1-\frac{1}{p},p}(\partial D, \mathbb{S}^{N-1})$, is a* stationary *point of the energy (1.6) with free boundary. Then there exists a closed set $\Sigma \subset \overline{D}$ such that $\mathcal{H}^{n-p}(\Sigma) = 0$ and $u \in C^{1,\alpha}(\overline{D} \backslash \Sigma)$, where $\alpha > 0$ is from Theorem 1.2.*

Remark. *Although some of our results work for unbounded domains we note that finite energy, stationary p-harmonic maps with with free boundary satisfy a Liouville type theorem, cf. Proposition 6.3. This is why we focus on bounded domains.*

Moreover, besides giving regularity in the case $p = n$ and partial regularity in the case $p < n$, an ϵ-regularity could be useful to describe the possible loss of compactness of sequences of n-harmonic maps with free boundaries and an energy decomposition theorem. In the case $p = n = 2$, i.e., for harmonic maps with free boundaries such a result was proven in [5, 31]. Our case requires completely different methods, due to the nonlinearity of the p-Laplacian for $p \neq 2$.

Let us comment on our strategy for the proof of Theorem 1.2. The natural first attempt to prove a result like Theorem 1.2 is to adapt the beautiful geometric reflection method that Scheven used in [43] to obtain an ϵ-regularity result up to the free boundary for harmonic maps, i.e., for the case $p = 2$ (see also [1] where the authors also devised a reflection technique to prove regularity up to the boundary of solutions of some Ginzburg-Landau equations with free boundary conditions). This way, one would hope to be able to rewrite the Neumann condition at the boundary into an interior equation. For $p = 2$ the reflected equation has again the structure of a harmonic map (with a new metric in the reflected domain). Thus, the regularity theory for harmonic maps with a free boundary follows from the interior regularity for harmonic maps developed by Hélein [26], see also [40]. For $p > 2$ there is a major drawback to that strategy: as mentioned above, the regularity theory for the interior p-harmonic map equation is only understood for round targets. It was not clear to us, how to interpret the reflected equation as a map into such a round target. The reflection, which generates a somewhat "unnatural metric" seems to destroy our boundary sphere-structure. Indeed, up to now, only the regularity theory for *minimizing* p-harmonic maps with free boundary was understood, see [12, 38] where it is shown that such a map is in $C^{1,\alpha}$, for some α, outside a singular set $\mathcal{S}$ with $\dim_{\mathcal{H}}(\mathcal{S}) = n - \lfloor p \rfloor - 1$ and $\mathcal{S}$ is discrete if $n - 1 \leq p < n$. For $p = 2$ free boundary problems for *minimizing* harmonic maps were studied in [15, 24].

In this work we follow in spirit the recent work of the third-named author [49] which does not use a reflection technique, but rather computes an equation along the free boundary and applies a moving frame technique to this free boundary part of the equation itself. This strategy leads to *growth* estimates, Proposition 2.1, which for the critical case $n = p$ implies directly Hölder regularity of solutions. Once the growth estimates are established we can apply the reflection. Since the reflection is explicit, it is easy to see that the growth estimates still hold for the reflected solution, which we shall call v. Now v solves a critical or super-critical equation of the form

$$|\operatorname{div}(|\nabla v|^{p-2}\nabla v)| \precsim |\nabla v|^p.$$

In principle, solutions to this equation may be singular, e.g., $x/|x|$ or $\log\log 1/|x|$. But with the growth estimates from Proposition 2.1, which transfers to v, one can employ a blow-up argument due to [22, 23] and then bootstrap for higher regularity.

The outline of the paper is as follows: In Section 2 we state and prove the crucial growth estimate for solutions to (1.12). In Section 3 we show how this implies Hölder continuity of solutions for the case $p = n$. For $p < n$ we show in Section 4 how a generic super-critical system implies Hölder regularity of solutions once the growth estimates from Proposition 2.1 are guaranteed. Combining this with Scheven's reflection argument, we give in Section 5 the proof of Theorem 1.2. Finally, in Section 6, we prove the partial regularity of solutions, i.e., Theorem 1.4.

Notation. We denote by $B(x,r)$ the ball of radius r centered at $x \in \mathbb{R}^n$. We write $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0,\infty)$, $\mathbb{R}^n_- = \mathbb{R}^n \times (-\infty, 0)$, and $B^+(x,r) = B(x,r) \cap \mathbb{R}^n_+$. By $(u)_\Omega$ we denote the mean value of a map u on a set Ω, i.e., $(u)_\Omega = \frac{1}{|\Omega|}\int_\Omega u$.

2. The growth estimates

Recall that we assume that D is a bounded set with a smooth boundary. In view of the Lemma A.1 we know that $|u| \leq 1$ holds for any solution to (1.12). The arguments can be also extended to unbounded domains like $\mathbb{R}^n_+$ under the assumption that $u \in L^\infty_{loc}(\mathbb{R}^n_+)$, cf. Lemma A.2. Note that in principle, the constants may depend on the L^∞-norm of u.

The main result in this section, and the crucial argument in this work, is the following growth estimate that one could interpret as a kind of Caccioppoli type estimate. We were not able to obtain such an estimate by a geometric reflection argument, since that reflection changes the metric, and only in the case of round targets, such as the sphere, regularity theory (and in particular the related growth estimates) are known.

Proposition 2.1 (Growth estimates)**.** *Let $p \geq 2$. There exists a radius R_0 depending only on ∂D such that for any $u \in W^{1,p}(D, \mathbb{R}^N)$ satisfying (1.12) the following holds:*

Whenever $B(x_0, R) \subset \mathbb{R}^n$, $R \in (0, R_0)$ is such that for some $\lambda \in (0, \infty)$ it holds

$$\sup_{B(y_0,r)\subset B(x_0,R)} r^{p-n} \int_{B(y_0,r)\cap D} |\nabla u|^p < \lambda^p, \tag{2.1}$$

then for any $B(y_0, 4r) \subset B(x_0, R)$ and any $\mu > 0$,

$$\int_{B(y_0,r)\cap D} |\nabla u|^p \leq C \ \left(\lambda + \mu^{p-1}\right) \int_{B(y_0,4r)\cap D} |\nabla u|^p + C\mu^{-1} \int_{(B(y_0,4r)\backslash B(y_0,r))\cap D} |\nabla u|^p. \tag{2.2}$$

Alternatively, we have the following estimates:

If $B(y_0, 2r)\backslash D = \emptyset$, then

$$\int_{B(y_0,r)} |\nabla u|^p \leq C\lambda \int_{B(y_0,4r)\cap D} |\nabla u|^p + C\lambda^{1-p} r^{-p} \int_{B(y_0,4r)\cap D} |u - (u)_{B(y_0,4r)\cap D}|^p. \tag{2.3}$$

If $B(y_0, 2r)\backslash D \neq \emptyset$, then

$$\begin{aligned}\int_{B(y_0,r)\cap D} |\nabla u|^p \leq C\lambda \int_{B(y_0,4r)\cap D} |\nabla u|^p &+ C\lambda^{1-p} r^{-p} \int_{B(y_0,4r)\cap D} |u - (u)_{B(y_0,4r)\cap D}|^p \\ &+ C\lambda^{1-p} r^{-p} \int_{B(y_0,4r)\cap D} ||u|^2 - 1|^p,\end{aligned} \tag{2.4}$$

for a constant $C = C(n, p, D)$.

Our strategy, in principle, is to adapt the method for harmonic maps into spheres developed by Hélein [25], see Strzelecki's [55] for the n-harmonic case. To motivate our approach, we briefly outline their strategy for a p-harmonic map $w \in W^{1,p}(D, \mathbb{S}^{N-1})$, i.e., a solution to

$$\operatorname{div}(|\nabla w|^{p-2}\nabla w) \perp T_w\mathbb{S}^{N-1}. \tag{2.5}$$

The first step is to rewrite this equation. Since $w \in \mathbb{S}^{N-1}$ we have $w \in \left(T_w\mathbb{S}^{N-1}\right)^\perp$. Consequently, (2.5) can be rewritten in distributional sense as

$$\int_D |\nabla w|^{p-2}\nabla w^i \cdot \nabla\varphi = \int_D |\nabla w|^{p-2}\nabla w^k \cdot \nabla(w^k w^i \varphi), \tag{2.6}$$

which holds for all $\varphi \in C_c^\infty(D)$ and $i = 1, \ldots, N$. Here and henceforth, we use the summation convention.

Next, from $|w| \equiv 1$, we get $w^k \nabla w^k \equiv \frac{1}{2}\nabla |w|^2 = 0$. Consequently, (2.6) can be written as

$$\int_D |\nabla w|^{p-2} \nabla w^i \cdot \nabla \varphi = \int_D |\nabla w|^{p-2} \nabla w^k \cdot \left(\nabla w^k\ w^i - \nabla w^i\ w^k\right) \varphi. \tag{2.7}$$

Now one observes that from (2.6) a conservation law follows, a fact that for $p = n = 2$ was discovered by Shatah [51],

$$\operatorname{div}\left(|\nabla w|^{p-2}\left(\nabla w^k\ w^i - \nabla w^i\ w^k\right)\right) = 0 \quad \text{in } D. \tag{2.8}$$

Thus, $|\nabla w|^{p-2}\nabla w^k \cdot \left(\nabla w^k\ w^i - \nabla w^i\ w^k\right)$ is a div-curl term and with the help of the celebrated result of Coifman, Lions, Meyer, and Semmes, [3], one obtains a growth estimate.

The above argument heavily relied on the fact that $w^k \nabla w^k \equiv 0$. It is important to observe that this trick will not work in the situation from Theorem 1.2: if we only know that $u\Big|_{\partial D} \subset \mathbb{S}^{N-1}$, then there is no reason that $u \cdot \nabla u = 0$ in D. Nevertheless, we will stubbornly follow the strategy outlined above, just along the boundary ∂D, keeping the extra terms that involve $u^k \nabla u^k$. Firstly, we find:

Lemma 2.2. *For $u \in W^{1,p}(D, \mathbb{R}^N)$ satisfying (1.12) we have*

$$\int_D |\nabla u|^{p-2} \nabla u^i \cdot \nabla \varphi = \int_D |\nabla u|^{p-2} \nabla u^k \cdot \nabla(u^k u^i \varphi),$$

for any $\varphi \in W^{1,p}(D)$.

Let us stress that the test function φ above does not need to vanish at the boundary.

Proof. Let $\Phi = (0, \ldots, \varphi, \ldots, 0)$ (only the i-th coordinate is nonzero and equal to φ). Observe that

$$\Phi - u\langle u, \Phi\rangle_{\mathbb{R}^N} \in T_u \mathbb{S}^{N-1} \quad \text{a.e. on } \partial D.$$

The claim follows now from the definition of p-harmonic maps with free boundary (1.7). □

Also we have the following conservation law.

Lemma 2.3. *Let $u \in W^{1,p}(D, \mathbb{R}^N)$ satisfying (1.12). Then, for*

$$\Omega_{ij} := \left(u^i \nabla u^j - u^j \nabla u^i\right),$$

we have

$$\operatorname{div}(|\nabla u|^{p-2}\Omega_{ij}) = 0 \quad \text{in } D$$

up to the boundary. That is, for any $\varphi \in C^\infty(\overline{D})$ and any $i, j = 1, \ldots, N$,

$$\int_D |\nabla u|^{p-2} \Omega_{ij} \cdot \nabla \varphi = 0. \tag{2.9}$$

Besides, equation (2.9) is also satisfied for every φ in $W^{1,p} \cap L^\infty(D)$.

Proof. By the product rule,

$$\begin{aligned}&\int_D \nabla\varphi \cdot |\nabla u|^{p-2}\left(u^i\nabla u^j - u^j\nabla u^i\right)\\&\quad= \int_D \left(\nabla(\varphi u^i)\cdot|\nabla u|^{p-2}\nabla u^j - \nabla(\varphi u^j)\cdot|\nabla u|^{p-2}\nabla u^i\right).\end{aligned}$$

Therefore, by Lemma 2.2, we find

$$\begin{aligned}&\int_D |\nabla u|^{p-2}\Omega_{ij}\cdot\nabla\varphi\\&\quad= \int_D |\nabla u|^{p-2}\nabla u^k\cdot\nabla(u^k u^i u^j\varphi) - \int_D |\nabla u|^{p-2}\nabla u^k\cdot\nabla(u^k u^j u^i\varphi)\\&\quad= 0.\end{aligned}$$

□

We combine Lemma 2.3 and Lemma 2.2. In contrast to the argument for the p-harmonic map w, we find additional terms. Namely, instead of having $w^k\nabla w^k \equiv 0$ we merely have $u^k\nabla u^k = \frac{1}{2}\nabla(|u|^2-1)$. However, it is an improvement, because $|u|^2-1 \in W^{1,p}_0(D)$.

Lemma 2.4. *Let $u \in W^{1,p}(D,\mathbb{R}^N)$ satisfying (1.12). Then for any $\varphi \in W^{1,p}(D)$ we have*

$$\begin{aligned}&\int_D |\nabla u|^{p-2}\nabla u^i\cdot\nabla\varphi\\&\quad= \int_D |\nabla u|^{p-2}\nabla u^k\cdot\Omega_{ik}\,\varphi + \int_D |\nabla u|^{p-2}\nabla u^i\cdot\nabla\left(|u|^2-1\right)\ \varphi\\&\qquad+\frac{1}{2}\int_D |\nabla u|^{p-2}\nabla\varphi\cdot\nabla\left(|u|^2-1\right)\ u^i.\end{aligned}$$

It is important to observe that in particular we do not obtain an equation of the form $|\operatorname{div}(|\nabla u|^{p-2}\nabla u)| \precsim |\nabla u|^p$ as it is the case for p-harmonic maps (i.e., the interior situation). This is why for $p < n$ we are forced to combine our growth estimate with the geometric reflection argument, see Proposition 5.3.

Proof of Lemma 2.4. By Lemma 2.2 we have for any $\varphi \in C^\infty(\overline{D})$,

$$\begin{aligned}&\int_D |\nabla u|^{p-2}\nabla u^i\cdot\nabla\varphi\\&\quad= \int_D |\nabla u|^{p-2}\nabla u^k\cdot\nabla u^k\ u^i\varphi + \int_D |\nabla u|^{p-2}\nabla u^k\cdot u^k\nabla(u^i\varphi).\end{aligned}$$

Using the definition of Ω_{ik} from Lemma 2.3 we write

$$\begin{aligned}
&\int_D |\nabla u|^{p-2} \nabla u^i \cdot \nabla \varphi \\
&\quad = \int_D |\nabla u|^{p-2} \nabla u^k \cdot \Omega_{ik}\, \varphi + 2\int_D |\nabla u|^{p-2} \nabla u^i \cdot \nabla u^k u^k\ \varphi \\
&\qquad + \int_D |\nabla u|^{p-2} \nabla u^k u^i u^k \cdot \nabla \varphi.
\end{aligned}$$

Since $u^k \nabla u^k = \frac{1}{2}\nabla\left(|u|^2-1\right)$, we have shown that

$$\begin{aligned}
&\int_D |\nabla u|^{p-2} \nabla u^i \cdot \nabla \varphi \\
&\quad = \int_D |\nabla u|^{p-2} \nabla u^k \cdot \Omega_{ik}\, \varphi + \int_D |\nabla u|^{p-2} \nabla u^i \cdot \nabla\left(|u|^2-1\right)\ \varphi \\
&\quad + \frac{1}{2}\int_D |\nabla u|^{p-2} \nabla \varphi \cdot \nabla\left(|u|^2-1\right)\ u^i \cdot
\end{aligned}$$

□

For the second and third term on the right-hand side of the equation in Lemma 2.4 we observe that $|u|^2-1$ has zero boundary values on ∂D. In addition, and this is another crucial ingredient here, we can choose u or (its coordinates) as a test function in Lemma 2.2, 2.3, and 2.4 since u is in $W^{1,p} \cap L^\infty(D, \mathbb{R}^N)$ from Lemma A.1.

Moreover, in view of the interior equation for u, (1.9),

$$\int_D |\nabla u|^{p-2} \nabla u^i \cdot \nabla(|u|^2-1) = 0.$$

Proof of Proposition 2.1. For notational simplicity we prove the growth estimates when the boundary is flat. More precisely we treat the case where $B^+(0,R) \subset D \subset \mathbb{R}^n_+$ for some $R>0$, and $\partial D \cap B(0,R) = \partial \mathbb{R}^n_+ \cap B(0,R)$. The following argument can be easily adapted to general D — here is where one has to choose $R_0 = R_0(D)$ for flattening the boundary. We leave the details to the reader. We also recall that, since we work in a smooth bounded domain, from Lemma A.1 we have that $\|u\|_{L^\infty(D)} \leq 1$.

Let $\eta \in C_c^\infty(B(0,2))$ be the typical bump function constantly one in $B(0,1)$. Let $y_0 \in \mathbb{R}^n, r>0$ be such that $B(y_0,4r) \subset B(0,R)$. Denote by

$$\eta_{B(y_0,r)}(x) := \eta((x-y_0)/r).$$

Set

$$\tilde{u} := \eta_{B(y_0,r)}(u - (u)_{B^+(y_0,2r)})$$

and

$$\hat{u} := (1-\eta_{B(y_0,r)})\eta_{B(y_0,r)}(u-(u)_{B^+(y_0,2r)}).$$

Since $\eta_{B(y_0,r)} \equiv 1$ on $B(y_0,r)$ we have

$$\int_{B^+(y_0,r)} |\nabla u|^p \leq \int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla \tilde{u} \cdot \nabla \tilde{u}.$$

We compute

$$\begin{aligned} \nabla \tilde{u} \cdot \nabla \tilde{u} = {} & \nabla u \cdot \nabla \tilde{u} - \nabla u \cdot \nabla \hat{u} \\ & - \nabla \eta_{B(y_0,r)} \cdot \nabla u \; \tilde{u} + \nabla \eta_{B(y_0,r)} \left(u - (u)_{B^+(y_0,2r)}\right) \cdot \nabla \tilde{u}. \end{aligned} \tag{2.10}$$

Since $|\nabla \eta_{B(y_0,r)}| \precsim r^{-1}$,

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} (\nabla \eta_{B(y_0,r)} \tilde{u}) \cdot \nabla u \; \precsim r^{-1} \int_{B^+(y_0,2r) \backslash B^+(y_0,r)} |\nabla u|^{p-1} |\tilde{u}|. \tag{2.11}$$

This can be further estimated in two ways. For the estimate (2.2), by Young and Poincaré inequalities, we have for any $\mu > 0$

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} (\nabla \eta_{B(y_0,r)} \tilde{u}) \cdot \nabla u \; \precsim \frac{1}{\mu} \int_{B^+(y_0,2r) \backslash B^+(y_0,r)} |\nabla u|^p + \mu^{p-1} \int_{B^+(y_0,2r)} |\nabla u|^p.$$

For the estimates (2.3) and (2.4), by Young's inequality we have for any $\lambda > 0$

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla \eta_{B(y_0,r)} \cdot \nabla u \; \tilde{u} \precsim \lambda \int_{B^+(y_0,2r)} |\nabla u|^p + \lambda^{1-p} r^{-p} \int_{B^+(y_0,2r)} |u - (u)_{B^+(y_0,2r)}|^p.$$

For the last term of (2.10)

$$\begin{aligned} \int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla \eta_{B(y_0,r)} \left(u - (u)_{B^+(y_0,2r)}\right) \cdot \nabla \tilde{u} \precsim {} & r^{-2} \int_{B^+(y_0,2r) \backslash B^+(y_0,r)} |\nabla u|^{p-2} |u - (u)_{B^+(y_0,2r)}|^2 \\ & + r^{-1} \int_{B^+(y_0,2r) \backslash B^+(y_0,r)} |\nabla u|^{p-1} |u - (u)_{B^+(y_0,2r)}|. \end{aligned}$$

By a similar estimate, we easily get for any $\mu > 0$

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla \eta_{B(y_0,2r)} \left(u - (u)_{B^+(y_0,2r)}\right) \cdot \nabla \tilde{u} \precsim \frac{1}{\mu} \int_{B^+(y_0,2r) \backslash B^+(y_0,r)} |\nabla u|^p + \mu^{p-1} \int_{B^+(y_0,2r)} |\nabla u|^p$$

and for any $\lambda > 0$

$$\begin{aligned} \int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla \eta_{B(y_0,2r)} \left(u - (u)_{B^+(y_0,2r)}\right) \cdot \nabla \tilde{u} \precsim {} & \lambda \int_{B^+(y_0,2r)} |\nabla u|^p \\ & + \lambda^{1-p} r^{-p} \int_{B^+(y_0,2r)} |u - (u)_{B^+(y_0,2r)}|^p. \end{aligned}$$

Consequently, we found

$$\begin{aligned}\int_{B^+(y_0,r)} |\nabla u|^p \precsim & \left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u\cdot\nabla\tilde{u}\right| + \left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u\cdot\nabla\hat{u}\right| \\ & + \frac{1}{\mu}\int_{B^+(y_0,2r)\setminus B^+(y_0,r)} |\nabla u|^p + \mu^{p-1}\int_{B^+(y_0,2r)} |\nabla u|^p\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\int_{B^+(y_0,r)} |\nabla u|^p \precsim & \left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u\cdot\nabla\tilde{u}\right| + \left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u\cdot\nabla\hat{u}\right| \\ & + \lambda\int_{B^+(y_0,2r)} |\nabla u|^p + \lambda^{1-p}r^{-p}\int_{B^+(y_0,2r)} |u-(u)_{B^+(y_0,2r)}|^p.\end{aligned} \tag{2.13}$$

If we are in the interior case, i.e., $B(y_0,2r)\subset B^+(0,R)$, then $\operatorname{supp}\tilde{u}\cup\operatorname{supp}\hat{u}\subset B^+(0,R)$ and thus $\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0$ in $B^+(0,R)$ implies

$$\left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u\cdot\nabla\tilde{u}\right| + \left|\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u\cdot\nabla\hat{u}\right| = 0.$$

Thus, for $B(y_0,2r)\subset B^+(0,R)$ the claim is proven.

From now on we assume that the ball $B(y_0,r)$ is close to the boundary, i.e, $B(y_0,2r)\cap\{\mathbb{R}^{n-1}\times\{0\}\}\neq\emptyset$. By Lemma 2.4,

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u^i\cdot\nabla\tilde{u}^i = I + II + \frac{1}{2}III,$$

where

$$\begin{aligned}I &:= \int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u^k\cdot\Omega_{ik}\,\tilde{u}^i,\\ II &:= \int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u^i\cdot\nabla\left(|u|^2-1\right)\ \tilde{u}^i,\\ III &:= \int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla\tilde{u}^i\cdot\nabla\left(|u|^2-1\right)\ u^i.\end{aligned}$$

Since u is p-harmonic and by Lemma 2.3 all three terms above contain products of divergence-free and rotation-free quantities. However, the div-curl estimate by Coifman, Lions, Meyer, Semmes [3] is only applicable when at least one term vanishes at the boundary, otherwise there are counterexamples, see [6, 27].

We investigate the <u>first term I</u>. Let $\tilde{B}\subset B^+(0,R)$ be a smooth, bounded, open, and convex set, such that $B^+(y_0,2r)\subset\tilde{B}\subset B(y_0,3r)$ and $\partial\tilde{B}\cap\partial\mathbb{R}^n_+ = B(y_0,2r)\cap\partial\mathbb{R}^n_+$.

By Hodge decomposition[1] (see [28, (10.4)]) we find $\xi_{ik} \in W^{1,p'}(\tilde{B})$, with $p' = \frac{p}{p-1}$, and $\zeta_{ik} \in W_0^{1,p'}(\tilde{B}, \bigwedge^2 \mathbb{R}^n)$ such that

$$|\nabla u|^{p-2}\Omega_{ik} = \nabla \xi_{ik} + \operatorname{Curl} \zeta_{ik} \quad \text{in } \tilde{B}. \tag{2.14}$$

Moreover, we have

$$\|\zeta_{ik}\|_{W^{1,p'}(\tilde{B})} \precsim \||\nabla u|^{p-2}\Omega_{ik}\|_{L^{p'}(B(y_0,3r))}. \tag{2.15}$$

The boundary data of ζ and Lemma 2.3 imply that

$$\int_{\tilde{B}} \nabla \xi_{ik} \cdot \nabla \varphi = \int_{\tilde{B}} |\nabla u|^{p-2}\Omega_{ik} \cdot \nabla \varphi - \int_{\tilde{B}} \operatorname{Curl} \zeta_{ik} \cdot \nabla \varphi = 0 \quad \text{for any } \varphi \in C^\infty(\overline{\tilde{B}}).$$

That is, ξ_{ik} is harmonic with trivial Neumann data, and thus ξ_{ik} is constant. In particular, (2.14) simplifies to

$$|\nabla u|^{p-2}\Omega_{ik} = \operatorname{Curl} \zeta_{ik} \quad \text{in } \tilde{B}. \tag{2.16}$$

Consequently,

$$I = \int_{\mathbb{R}^n_+} \operatorname{Curl} \zeta_{ik} \cdot \nabla u^k \, \tilde{u}^i = \int_{\mathbb{R}^n} \operatorname{Curl} \zeta_{ik} \cdot \nabla u^k \, \tilde{u}^i.$$

The last equality is true, since ζ_{ik} vanishes on $\partial\mathbb{R}^n_+ \cap B(0,R)$ and we can extend it by zero to $\mathbb{R}^n_- \cap B(0,R)$. Now we use the div-curl structure and apply the result by Coifman, Lions, Meyer, Semmes [3]. Recall that BMO is the space of functions f with finite seminorm $[f]_{BMO} < \infty$. Here,

$$[f]_{BMO} := \sup_B |B|^{-1} \int_B |f - (f)_B|,$$

where the supremum is taken over all balls B. Observe that by Poincaré inequality,

$$[f]_{BMO} \precsim \sup_{x_0 \in \mathbb{R}^n,\ \rho > 0} \left(\rho^{p-n} \int_{B(x_0,\rho)} |\nabla f|^p \right)^{\frac{1}{p}}. \tag{2.17}$$

Coifman, Lions, Meyer, Semmes showed in [3] that the following inequality holds

$$\int_{\mathbb{R}^n} F \cdot G \, \varphi \precsim \|F\|_{L^p(\mathbb{R}^n)} \, \|G\|_{L^{p'}(\mathbb{R}^n)} \, [\varphi]_{BMO}$$

---

[1] More precisely, one argues, e.g., as in [44, (3.6), (3.7)]: One solves

$$\begin{cases} \Delta \zeta_{ik} = \operatorname{curl}\left(|\nabla u|^{p-2}\Omega_{ik}\right) & \text{in } \tilde{B} \\ \zeta_{ik} = 0 & \text{on } \partial\tilde{B}. \end{cases}$$

such that (2.15) is satisfied. Then one sets

$$H := |\nabla u|^{p-2}\Omega_{ik} - \operatorname{Curl}\zeta_{ik}.$$

By Poincaré lemma we can write $H = \nabla \xi$.

whenever F and G are vector fields such that $\operatorname{div} F = 0$ and $\operatorname{curl} G = 0$. See also [32] for a different proof. In our situation this inequality implies[2]

$$
\begin{aligned}
|I| &\precsim \||\nabla u|^{p-2}\Omega_{ik}\|_{L^{p'}(B^+(y_0,4r))}\ \|\nabla u\|_{L^p(B^+(y_0,4r))}\ [\tilde{u}]_{BMO} \\
&\precsim \|\nabla u\|^p_{L^p(B^+(y_0,4r))}\ [\tilde{u}]_{BMO}.
\end{aligned}
\tag{2.18}
$$

The last estimate follows readily from the definition of Ω in Lemma 2.3. Thus, for the λ from (2.1) we obtain

$$
|I| \precsim \lambda \int_{B^+(y_0,4r)} |\nabla u|^p.
$$

<u>As for II</u>, since $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in $B^+(0,R)$, there exists $\zeta_i \in W^{1,p}(B^+(y_0,2r), \bigwedge^2 \mathbb{R}^n)$ such that

$$
|\nabla u|^{p-2}\nabla u^i = \operatorname{Curl}\zeta_i \quad \text{in } B^+(y_0,2r).
$$

We can extend ζ to all of $\mathbb{R}^n$ so that

$$
\|\zeta\|_{W^{1,p'}(\mathbb{R}^n)} \precsim \|\nabla u\|^{p-1}_{L^p(B^+(y_0,2r))}.
$$

Also, since u is assumed to be bounded we have $|u|^2 \in W^{1,p}(B^+(0,R))$, and in the sense of traces $|u|^2 \equiv 1$ on $B(0,R) \cap \{\mathbb{R}^{n-1} \times \{0\}\}$. This is equivalent to saying that the extension of $|u|^2 - 1$ by zero to $B(0,R) \cap \mathbb{R}^n_-$ belongs to $W^{1,p}(B(0,R))$, that is we have, $(|u|^2-1)\chi_{\mathbb{R}^n_+} \in W^{1,p}(B(0,R))$ and the distributional gradient satisfies

$$
\nabla\left((|u|^2-1)\chi_{\mathbb{R}^n_+}\right) = \chi_{\mathbb{R}^n_+}\nabla|u|^2 \quad \text{a.e. in } B(0,R).
$$

In particular, since $(|u|^2-1)\chi_{\mathbb{R}^n_+}$ is zero on $B(y_0,2r) \cap \mathbb{R}^n_-$ we can use Poincaré inequality to get

$$
\||u|^2 - 1\|_{L^p(B^+(y_0,2r))} \precsim r\, \|u\|_{L^\infty(B^+(y_0,4r))}\, \|\nabla u\|_{L^p(B^+(y_0,4r))}.
\tag{2.19}
$$

In particular, by using that $|\nabla \eta_{B(y_0,2r)}| \precsim r^{-1}$, (2.17), the triangle inequality in L^p and (2.19), for the λ from (2.1),

$$
[\left(|u|^2-1\right)\chi_{\mathbb{R}^n_+}\eta_{B(y_0,2r)}]_{BMO} \precsim \lambda.
$$

We also observe that $\nabla\tilde{u} \equiv \eta_{B(y_0,2r)}\nabla\tilde{u}$. Thus, integrating by parts we obtain

$$
II = -\int_{\mathbb{R}^n} \operatorname{Curl}\zeta \cdot \nabla\tilde{u}^i\ \left(|u|^2-1\right)\chi_{\mathbb{R}^n_+}\eta_{B(y_0,2r)}.
$$

Hence, with the div-curl theorem from [3], see also the localized version [55, Corollary 3], we find

$$
|II| \precsim \lambda\|\nabla u\|^p_{L^p(B^+(y_0,4r))}.
$$

[2]Here, $\tilde{u}$ is extended into the whole space $\mathbb{R}^n$ in such a way that $[\tilde{u}]_{BMO} \precsim \lambda$. This can be done by an appropriate reflection of u outside of $B^+(y_0,3r)$.

It remains to treat III. Observe that

$$
\begin{aligned}
&\nabla \tilde{u}^i \cdot \nabla \left(|u|^2-1\right)\ u^i\\
&\quad= \nabla u^i \cdot \nabla \left(|u|^2-1\right)\ \eta_{B(y_0,r)} u^i + \nabla \eta_{B(y_0,r)} \left(u^i - (u^i)_{B^+(y_0,2r)}\right) \cdot \nabla \left(|u|^2-1\right)\ u^i\\
&\quad= \nabla u^i \cdot \nabla \left(|u|^2-1\right)\ \tilde{u}^i\\
&\qquad+ \nabla u^i \cdot \nabla \left(|u|^2-1\right)\ \eta_{B(y_0,r)} (u^i)_{B^+(y_0,2r)}\\
&\qquad+ \nabla \eta_{B(y_0,r)} \left(u^i - (u^i)_{B^+(y_0,2r)}\right) \cdot \nabla \left(|u|^2-1\right)\ u^i.
\end{aligned}
$$

By integration by parts, using that $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in $B^+(0,R)$, $|u|^2-1$ is zero on $\partial\mathbb{R}^n_+ \cap B(0,R)$ and then arguing as in the argument for II,

$$
\begin{aligned}
\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla u^i \cdot \nabla \left(|u|^2-1\right)\ \tilde{u}^i &= -\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla u^i \cdot \nabla \tilde{u}^i\ \left(|u|^2-1\right)\\
&\precsim \lambda \|\nabla u\|^p_{L^p(B^+(y_0,4r))}.
\end{aligned}
$$

Moreover, again since $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in $B^+(0,R)$ and $|u|^2-1$ is zero on $\partial\mathbb{R}^n_+ \cap B(0,R)$,

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla u^i \cdot \nabla \left(|u|^2-1\right)\ \eta_{B(y_0,r)} (u^i)_{B^+(y_0,2r)} \right|\\
&\qquad= \left| \int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla u^i \cdot \left(|u|^2-1\right)\ \nabla \eta_{B(y_0,r)} (u^i)_{B^+(y_0,2r)} \right|\\
&\qquad\precsim r^{-1} \|u\|_{L^\infty(B^+(0,R))} \|\nabla u\|^{p-1}_{L^p(B^+(y_0,2r)\backslash B^+(y_0,r))} \||u|^2-1\|_{L^p(B^+(y_0,2r))}.
\end{aligned}
$$

This leads to two estimates. Firstly, if we want to find (2.4), by Young's inequality,

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla u^i \cdot \nabla \left(|u|^2-1\right)\ \eta_{B(y_0,r)} (u^i)_{B^+(y_0,r)}\\
&\qquad\precsim \lambda \|\nabla u\|^p_{L^p(B^+(y_0,2r))} + \lambda^{1-p} r^{-p} \||u|^2-1\|^p_{L^p(B^+(y_0,2r))}.
\end{aligned}
$$

Secondly, for (2.2) by (2.19) and by Young's inequality we have for any $\mu > 0$

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla u^i \cdot \nabla \left(|u|^2-1\right)\ \eta_{B(y_0,r)} (u^i)_{B^+(y_0,2r)}\\
&\qquad\precsim \mu^{-1} \|\nabla u\|^p_{L^p(B^+(y_0,2r)\backslash B^+(y_0,r))} + \mu^{p-1} \|\nabla u\|^p_{L^p(B^+(y_0,2r))}.
\end{aligned}
$$

The last remaining term can be treated in a similar way and we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla \eta_{B(y_0,r)} \left(u^i - (u^i)_{B^+(y_0,2r)}\right) \cdot \nabla \left(|u|^2-1\right)\ u^i\\
&\qquad\precsim \mu^{-1} \|\nabla u\|^p_{L^p(B^+(y_0,2r)\backslash B^+(y_0,r))} + \mu^{p-1} \|\nabla u\|^p_{L^p(B^+(y_0,2r))}
\end{aligned}
$$

and

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2} \nabla \eta_{B(y_0,r)} \left(u^i - (u^i)_{B^+(y_0,2r)}\right) \cdot \nabla \left(|u|^2 - 1\right) \; u^i$$
$$\precsim \lambda \|\nabla u\|^p_{L^p(B^+(y_0,2r))} + \lambda^{1-p}\, r^{-p} \|u - (u)_{B^+(y_0,2r)}\|^p_{L^p(B^+(y_0,2r))}.$$

Combining the estimates of I, II, and III and plugging them into estimates (2.12) and (2.13), we conclude. □

3. Hölder regularity for the case $p = n$

For the case $p = n$ Hölder continuity of the solution u from Theorem 1.2 follows from Proposition 2.1 by a standard iteration argument. For higher regularity, and for $p < n$, we need to combine the growth estimates from Proposition 2.1 with the reflection method.

Proposition 3.1 (ϵ-regularity for $p = n$: Hölder continuity)**.** *Let $D \subset \mathbb{R}^n$ be a smooth, bounded domain, then there are positive constants $\epsilon = \epsilon(n, D)$, $\alpha = \alpha(n, D)$ such that the following holds for $p = n$:*

Any solution $u \in W^{1,n}(D, \mathbb{R}^N)$ to (1.12) that satisfies for an $R > 0$ and for an $x_0 \in \overline{D}$

$$\int_{B(x_0,R)\cap D} |\nabla u|^n < \epsilon$$

is Hölder continuous in $B(x_0, R/2) \cap \overline{D}$. Moreover, we have the estimate

$$\sup_{x,y\in B(x_0,R/2)\cap\overline{D}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \precsim R^{-\alpha} \|\nabla u\|_{L^n(B(x_0,R)\cap D)}.$$

Proof. Let $\lambda := \epsilon^{\frac{1}{n}}$ and apply Proposition 2.1 to any $B(y_0, 4r) \subset B(x_0, R/2)$, for $\mu > 0$ to be chosen below. We add

$$C\mu^{-1} \int_{B(y_0,r)\cap D} |\nabla u|^n$$

to both sides of (2.2). Then we find

$$\left(1 + C\mu^{-1}\right) \int_{B(y_0,r)\cap D} |\nabla u|^n \leq C\; \left(\epsilon^{\frac{1}{n}} + \mu^{n-1} + \mu^{-1}\right) \int_{B(y_0,4r)\cap D} |\nabla u|^n.$$

We choose $\epsilon, \mu > 0$ small enough so that $\tau < 1$, where

$$\tau := \left(\frac{C\; \left(\epsilon^{\frac{1}{n}} + \mu^{n-1} + \mu^{-1}\right)}{1 + C\mu^{-1}} \right)^{\frac{1}{n}}.$$

We have for any $B(y_0, 4r) \subset B(x_0, R/2)$

$$\|\nabla u\|_{L^n(B(y_0,r)\cap D)} \leq \tau \|\nabla u\|_{L^n(B(y_0,4r)\cap D)}.$$

Iterating this on successively smaller balls, cf. e.g. [21, Chapter III, Lemma 2.1], we find that for a uniform $\alpha = \alpha(\tau) > 0$ and for any $B(y_0, 4r) \subset B(x_0, R/2)$,

$$\|\nabla u\|_{L^n(B(y_0,r)\cap D)} \precsim \left(\frac{r}{R}\right)^{\alpha} \|\nabla u\|_{L^n(B(x_0,R)\cap D)}.$$

In particular, we have by Poincaré inequality

$$\sup_{B(y_0,4r)\subset B(x_0,R/2)} r^{-\alpha-1}\|u-(u)_{B(y_0,r)\cap D}\|_{L^n(B(y_0,r)\cap D)} \precsim R^{-\alpha}\,\|\nabla u\|_{L^n(B(x_0,R)\cap D)}.$$

By the characterization of Campanato spaces and Hölder spaces, e.g. see [21, Chapter III, p.75], this implies

$$\sup_{x,y\in B(x_0,R/2)\cap\overline{D}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \precsim R^{-\alpha}\|\nabla u\|_{L^n(B(x_0,R)\cap D)}.$$

□

4. Hölder-continuity for solutions to a supercritical system

In Proposition 2.1 we showed that solutions from Theorem 1.2 satisfy certain growth estimates. For $p = n$ these growth estimates imply Hölder continuity by an iteration argument, as we have seen in Proposition 3.1.

For $p < n$ more work is needed. The following Proposition shows that under a smallness assumption solutions to systems satisfying

$$|\operatorname{div}(|\nabla u|^{p-2}\nabla u)| \precsim |\nabla u|^p \tag{4.1}$$

are Hölder continuous once the growth conditions from Proposition 2.1 are satisfied, that is when (4.5) and (4.6) below are assumed *a priori*. Observe that without assuming *a priori* the growth conditions (4.5) and (4.6) below on the solution u, there is no hope for proving *any* regularity for solutions to a systems that have a structure of (4.1). Indeed, it is easy to check that $\log\log\frac{2}{|x|}$ and $\sin\log\log\frac{2}{|x|}$ satisfy (4.1) for $p = n$.

In the next section, in order to prove Theorem 1.2, we use the reflection method from Scheven's [43] to obtain an equation of the form (4.2). Since we already obtained the necessary growth estimates in Proposition 2.1, the following proposition then leads to regularity.

Proposition 4.1. *Let $D \subset \mathbb{R}^n$ be a smooth, bounded domain and let $\mathcal{M}$ be a smooth, compact $(n-1)$-dimensional manifold. Assume that $u \in W^{1,p}(D,\mathbb{R}^N)$ is a solution to*

$$\operatorname{div}(|G(x)\nabla u(x)|^{p-2}G(x)\nabla u(x)) = f_u(x), \tag{4.2}$$

where $f_u \in L^1(D,\mathbb{R}^N)$ satisfies the following estimate

$$|f_u(x)| \leq C\;|\nabla u(x)|^p \tag{4.3}$$

and $G \in C^\infty(\overline{D}, GL(n))$.

Moreover, assume a priori *that for every $B(x_0,R) \subset D$, $\lambda > 0$ such that*

$$\sup_{B(y_0,r)\subset B(x_0,R)} r^{p-n} \int_{B(y_0,r)} |\nabla u|^p < \lambda^p \tag{4.4}$$

the solution u already satisfies the following growth condition on any $B(y_0,4r) \subset B(x_0,R)$: If $B(y_0,2r)\cap \mathcal{M} = \emptyset$, then

$$\int_{B(y_0,r)} |\nabla u|^p \le C\lambda \int_{B(y_0,4r)} |\nabla u|^p + C\lambda^{1-p} r^{-p} \int_{B(y_0,4r)} |u-(u)_{B(y_0,4r)}|^p \tag{4.5}$$

and, if $B(y_0,2r)\cap \mathcal{M} \neq \emptyset$, then

$$\begin{aligned}\int_{B(y_0,r)} |\nabla u|^p \le C\lambda \int_{B(y_0,4r)} |\nabla u|^p &+ C\lambda^{1-p} r^{-p} \int_{B(y_0,4r)} |u-(u)_{B(y_0,4r)}|^p\\ &+ C\lambda^{1-p} r^{-p} \int_{B(y_0,4r)} |u-(u)_{B(y_0,4r)\cap\mathcal{M}}|^p\\ &+ C\lambda^{1-p} r^{1-p} \int_{B(y_0,4r)\cap\mathcal{M}} |u-(u)_{B(y_0,4r)\cap\mathcal{M}}|^p.\end{aligned} \tag{4.6}$$

Then there exist constants $\alpha = \alpha(G,p,n,C,D)$, $\epsilon > 0$ such that if (4.4) holds on some $B(x_0,R) \subset D$ for $\lambda < \epsilon$, then $u \in C^\alpha(B(x_0,R/2),\mathbb{R}^N)$. Moreover, we have the estimate

$$\sup_{x,y\in B(x_0,R/2)} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \le C_0\, R^{-\alpha} \left(\sup_{B(y_0,r)\subset B(x_0,R)} r^{p-n} \int_{B(y_0,r)} |\nabla u|^p \right)^{\frac{1}{p}}.$$

The constant C_0 depends on $\mathcal{M}$, D, C, and G.

To prove Proposition 4.1 we follow the strategy developed in [22] and [23, Theorem 2.4]. The crucial result is that the equation for u together with the growth assumptions (4.5) and (4.6) on u imply the following decay estimate.

Proposition 4.2. *There are uniform constants $\epsilon, \theta \in (0,1)$ and $\overline{R} = \overline{R}(\mathcal{M}) \in (0,1)$ so that the following holds:*

Let u and D be as in Proposition 4.1 and assume that for a ball $B(x_0,R) \subset D$ and $R \in (0,\overline{R})$ it holds

$$E(x_0,R)(u) := \sup_{B(y_0,r)\subset B(x_0,R)} r^{p-n} \int_{B(y_0,r)} |\nabla u|^p < \epsilon^p. \tag{4.7}$$

Then

$$E(x_0,\theta R)(u) \le \frac{1}{2} E(x_0,R)(u). \tag{4.8}$$

Proof. It suffices to prove

$$(\theta R)^{p-n}\int_{B(y_0,\theta R)}|\nabla u|^p \leq \frac{1}{2}E(x_0,R)(u) \quad \text{for any } B(y_0,4\theta R)\subset B(x_0,R/2). \tag{4.9}$$

Indeed, (4.8) follows from (4.9) by taking smaller θ and observing that $B(x_1,R_1)\subset B(x_2,R_2)$ implies $E(x_1,R_1)(u)\leq E(x_2,R_2)(u)$.

Assume the claim (4.9) is false. Then, for any $\theta\in(0,1)$ we have a sequence of balls with $B(y_i,4\theta R_i)\subset B(x_i,R_i/2)\subset D$, a sequence $(\epsilon_i)_{i=1}^{\infty}$ satisfying $\lim_{i\to\infty}\epsilon_i=0$, and a sequence $(u_i)_{i=1}^{\infty}\subset W^{1,p}(D,\mathbb{R}^N)$ of solutions to (4.2) satisfying the growth assumptions of Proposition 4.1, such that

$$\sup_{B(y,r)\subset B(x_i,R_i)} r^{p-n}\int_{B(y,r)}|\nabla u_i|^p=\epsilon_i^p, \tag{4.10}$$

but

$$(\theta R_i)^{p-n}\int_{B(y_i,\theta R_i)}|\nabla u_i|^p>\frac{1}{2}\epsilon_i^p. \tag{4.11}$$

For simplicity, we assume that $R_i\equiv R_0$ and $x_i\equiv x_0$ for some $R_0>0$ and $x_0\in\mathbb{R}^n$.

This is no loss of generality, since we can rescale the maps u by the factor R_0/R_i. Observe that this rescales the manifold $\mathcal{M}$, but in a way that (4.6) still holds. Set

$$w_i:=\frac{1}{\epsilon_i}(u_i-(u_i)_{B(x_0,R_0)}).$$

Clearly,

$$(w_i)_{B(x_0,R_0)}=0 \quad \text{for all } i\in\mathbb{N}.$$

Thus, we can apply Poincaré inequality and have by (4.10),

$$\sup_{i\in\mathbb{N}}\|\nabla w_i\|^p_{L^p(B(x_0,R_0))}\precsim R_0^{n-p} \quad \text{and} \quad \sup_{i\in\mathbb{N}}\|w_i\|^p_{L^p(B(x_0,R_0))}\precsim R_0^{n-p+1}.$$

Thus, up to a subsequence denoted again by w_i, we find $w\in W^{1,p}(B(x_0,R_0),\mathbb{R}^N)$ such that as $i\to\infty$,

$$\begin{aligned}
&w_i\rightharpoonup w && \text{weakly in } W^{1,p}(B(x_0,R_0)),\\
&w_i\to w && \text{strongly in } L^p(B(x_0,R_0)),\\
&w_i\to w && \text{strongly in } L^p(B(x_0,R_0)\cap\mathcal{M},d\mathcal{H}^{n-1}),\\
&w_i\to w && \mathcal{H}^n\text{-a.e. on } B(x_0,R_0) \text{ and } \mathcal{H}^{n-1}\text{ -a.e. on } B(x_0,R_0)\cap\mathcal{M}.
\end{aligned}$$

In particular,

$$(w)_{B(x_0,R_0)}=0, \tag{4.12}$$

also

$$\|\nabla w\|^p_{L^p(B(x_0,R_0))}\precsim R_0^{n-p} \quad \text{and} \quad \|w\|^p_{L^p(B(x_0,R_0))}\precsim R_0^{n-p+1}.$$

Moreover, for any $\varphi \in C_c^\infty(B(x_0,R_0))$,

$$\int_{B(x_0,R_0)} |G\nabla w_i|^{p-2} G\nabla w_i \cdot \nabla \varphi = (\epsilon_i)^{1-p} \int_{B(x_0,R_0)} |G\nabla u_i|^{p-2} G\nabla u_i \cdot \nabla \varphi.$$

Now, by (4.2) and (4.3),

$$\left| \int_{B(x_0,R_0)} |G\nabla w_i|^{p-2} G\nabla w_i \cdot \nabla \varphi \right| \precsim (\epsilon_i)^{1-p} \|\varphi\|_{L^\infty(B(x_0,R_0))} \, \|\nabla u_i\|^p_{L^p(B(x_0,R_0))}.$$

That is, by (4.10)

$$\left| \int_{B(x_0,R_0)} |G\nabla w_i|^{p-2} G\nabla w_i \cdot \nabla \varphi \right| \precsim \|\varphi\|_{L^\infty(B(x_0,R_0))} R_0^{n-p} \epsilon_i \leq \epsilon_i \|\varphi\|_{L^\infty(B(x_0,R_0))}.$$

Now as in [11, Section 4]

$$\operatorname{div}(|G\nabla w|^{p-2} G\nabla w) = 0 \quad \text{in } B(x_0,R_0). \tag{4.13}$$

From (4.12) and the Lipschitz estimates for solutions to (4.13), see [60] as well as [37, 14] and in particular [29, (1.7)], we have for any $B(z,r) \subset B(x_0,R_0/2)$,

$$r^{-n} \int_{B(z,r)} |w - (w)_{B(z,r)}|^p \precsim r^p,$$

and if additionally $B(z,r) \cap \mathcal{M} \neq \emptyset$ and $r < \overline{R}$ for $\overline{R} = \overline{R}(\mathcal{M})$ small enough, then

$$r^{1-n} \int_{\mathcal{M} \cap B(z,r)} |w - (w)_{\mathcal{M} \cap B(z,r)}|^p + r^{-n} \int_{B(z,r)} |w - (w)_{\mathcal{M} \cap B(z,r)}|^p \precsim r^p.$$

On the other hand, by strong L^p-convergence of w_i to w, we find $i(\theta) \in \mathbb{N}$ so that for $i \geq i(\theta)$ and for any $r \in (\theta R_0, R_0)$ such that $B(z,r) \subset B(x_0,R_0)$,

$$r^{1-n} \int_{B(z,r)\cap \mathcal{M}} |w_i - w|^p + r^{-n} \int_{B(z,r)} |w_i - w|^p \leq \theta^p.$$

Combining these estimates we get for any $i \geq i(\theta)$ and for any $r \in (\theta R_0, R_0)$ such that $B(z,r) \subset B(x_0,R_0/2)$,

$$r^{-n} \int_{B(z,r)} |u_i - (u_i)_{B(z,r)}|^p = \epsilon_i^p r^{-n} \int_{B(z,r)} |w_i - (w_i)_{B(z,r)}|^p \precsim \epsilon_i^p \left(r^p + \theta^p\right).$$

If additionally $B(z,r) \cap \mathcal{M} \neq \emptyset$, then

$$r^{-n} \int_{B(z,r)} |u_i - (u_i)_{B(z,r)\cap \mathcal{M}}|^p = \epsilon_i^p \, r^{-n} \int_{B(z,r)} |w_i - (w_i)_{B(z,r)\cap \mathcal{M}}|^p \precsim \epsilon_i^p \left(r^p + \theta^p\right)$$

and

$$r^{1-n} \int_{B(z,r)\cap \mathcal{M}} |u_i - (u_i)_{B(z,r)\cap \mathcal{M}}|^p \precsim \epsilon_i^p \left(r^p + \theta^p\right).$$

We now apply the growth estimates (4.5) and (4.6) with $\lambda = \epsilon_0 \geq \epsilon_i$ of the solutions u_i to find

$$(\theta R_0)^{p-n} \int_{B(y_i,\theta R_0)} |\nabla u_i|^p \leq C\ \epsilon_i^p\ \left(\epsilon_0 + \epsilon_0^{1-p}\theta^p\right).$$

By choosing ϵ_0 and θ sufficiently small so that $\epsilon_0 + \epsilon_0^{1-p}\theta^p < 1/2$ we arrive at a contradiction with (4.11). □

Proof of Proposition 4.1. We argue as in the proof of Proposition 3.1: Assume that (4.4) is satisfied on $B(x_0, R)$ for some $\lambda < \epsilon$. Iterating the estimate from Proposition 4.2 on successively smaller balls, cf. [21, Chapter III, Lemma 2.1], we find a small $\alpha > 0$ such that for all $r < R$ and $B(y_0, r) \subset B(x_0, R/2)$,

$$r^{p-n} \int_{B(y_0,r)} |\nabla u|^p \precsim \left(\frac{r}{R}\right)^{\alpha p} E(x_0, R).$$

In particular, for all $r < R$ and $B(y_0, r) \subset B(x_0, R/2)$,

$$r^{-\alpha p-n} \int_{B(y_0,r)} |u - (u)_{B(y_0,r)}|^p \precsim r^{p-\alpha p-n} \int_{B(y_0,r)} |\nabla u|^p \precsim R^{-\alpha p} E(x_0, R).$$

We conclude by the identification of Campanato and Hölder spaces, see [21, Chapter III, p.75]. □

5. ϵ-regularity: Proof of Theorem 1.2

The proof of Theorem 1.2 is a combination of the growth estimate for solutions, Proposition 2.1, the reflection method as in Scheven's [43], and Proposition 4.1. More precisely, we use the reflection method to find a solution to (4.2) from Proposition 4.1. The growth estimates (4.5) and (4.6) required in Proposition 4.1 come from Proposition 2.1: They hold for the unreflected solution and by an easy argument hold also for the reflection. To set up the reflection method we first gather some standard results.

Lemma 5.1. *Let D be a smooth, bounded domain in $\mathbb{R}^n$. There exists some $R_0 = R_0(D)$ such that the following holds for any $R \in (0, R_0)$. Let $u \in W^{1,p}(D, \mathbb{R}^N)$ be a solution to (1.12) and $\epsilon \in (0, 1)$. If*

$$\sup_{B(y_0,r)\subset B(x_0,R)} r^{p-n} \int_{B(y_0,r)\cap D} |\nabla u|^p < \epsilon^p \tag{5.1}$$

and $B(x_0, R/2) \cap \partial D \neq \emptyset$, then

$$\sup_{x\in B(x_0,R/2)\cap D} \operatorname{dist}\left(u(x), \mathbb{S}^{N-1}\right) \leq C\epsilon.$$

Here C is a constant depending on ∂D.

Proof. Fix $x \in B(x_0, R/2) \cap D$. Let $r := \frac{1}{10} \operatorname{dist}(x, \partial D)$, then by (5.1) and the interior Lipschitz regularity for the p-Laplace equation, see [29, (1.7)],

$$|u(x) - (u)_{B(x,r)}|^p \precsim r^{p-n} \int_{B(x,5r)} |\nabla u|^p \leq \epsilon^p.$$

Denote by $z_1 \in \partial D \cap B(x_0, R/2)$ the projection of x onto $\partial D \cap B(x_0, R/2)$. Here we assume that $R < R_0$ for $R_0 = R_0(D)$ small enough such that z_1 is well-defined.

Let $y_0, y_1, \ldots, y_{10}$ be pairwise equidistant points on the line $[x, z_1]$ where $y_0 = x$ and $y_{10} = z_1$. That is, $|y_i - y_{i+1}| = r$.

By triangle inequality, Poincaré inequality and again by (5.1),

$$\begin{aligned}
&|(u)_{B(x,r)} - (u)_{B(z_1,r)\cap D}|^p \\
&\quad \precsim \sum_{i=0}^{10} |(u)_{B(y_i,r)\cap D} - (u)_{B(y_{i+1},r)\cap D}|^p \\
&\quad \precsim \sum_{i=0}^{10} r^{p-n} \int_{B(y_i,4r)\cap D} |\nabla u|^p \\
&\quad \precsim \epsilon^p.
\end{aligned}$$

From the second to third line, before applying Poincaré inequality, we also used that $|y_i - y_{i+1}| = r$, and thus (cf. footnote 3)

$$|(u)_{B(y_i,r)\cap D} - (u)_{B(y_{i+1},r)\cap D}|^p \precsim \fint_{B(y_i,4r)\cap D} |u - (u)_{B(y_i,4r)\cap D}|^p$$

Now for any $z_2 \in \partial D$

$$\operatorname{dist}\left((u)_{B(z_1,r)\cap D}, \mathbb{S}^{N-1}\right) \precsim r^{-n} \int_{B(z_1,r)\cap D} |u(z_3) - u(z_2)| \, dz_3.$$

Integrating z_2 over $\partial D \cap B(z_1, r)$ we find

$$\begin{aligned}
\operatorname{dist}\left((u)_{B(z_1,r)\cap D}, \mathbb{S}^{N-1}\right) \precsim\ & r^{-n} \int_{B(z_1,r)\cap D} |u(z_3) - (u)_{B(z_1,r)\cap \partial D}| \, dz_3 \\
& + r^{1-n} \int_{B(z_1,r)\cap \partial D} |u(z_2) - (u)_{B(z_1,r)\cap \partial D}| \, dz_2.
\end{aligned}$$

By Poincaré inequality, trace theorem, and (5.1)

$$\operatorname{dist}\left((u)_{B(z_1,r)\cap D}, \mathbb{S}^{N-1}\right) \precsim \epsilon.$$

Now the claim follows by triangle inequality for the distance,

$$\begin{aligned}
\operatorname{dist}\left(u(x), \mathbb{S}^{N-1}\right) \leq\ & |u(x) - (u)_{B(x,r)}| + |(u)_{B(x,r)} - (u)_{B(z_1,r)\cap D}| \\
& + \operatorname{dist}\left((u)_{B(z_1,r)\cap D}, \mathbb{S}^{N-1}\right).
\end{aligned}$$

□

As an immediate corollary we obtain.

Corollary 5.2. *Let u and D be as in Theorem 1.2. There exists $\epsilon_0 > 0$ such that if $B(x_0, R/2) \cap \partial D \neq \emptyset$ and (5.1) holds for some $\epsilon < \epsilon_0$, then $|u| > \frac{1}{2}$ in $B(x_0, R/2) \cap D$.*

As a consequence, when we reflect the maps from Theorem 1.2, we obtain a critical equation with the growth estimates such that Proposition 4.1 is applicable.

Proposition 5.3. *Let u and D be as in Theorem 1.2. There exists $\epsilon_0 = \epsilon_0(D) > 0$ such that for any $B(x_0, 4R) \subset \mathbb{R}^n$ on which u satisfies (5.1) for some $\epsilon < \epsilon_0$ there exists $v \in W^{1,p}(B(x_0, R), \mathbb{R}^N)$ such that*

$$v = u \quad \text{in } B(x_0, R) \cap D,$$

$$|\operatorname{div}(|\nabla v|^{p-2}\nabla v)| \precsim |\nabla v|^p \quad \text{in } B(x_0, R). \tag{5.2}$$

Moreover, the v satisfies the growth conditions from Proposition 4.1.

Proof. The main point is to prove that v satisfies the growth conditions. The estimate (5.2) follows from the geometric reflection, more precisely [42, Lemma 2.5]. But for reader's convenience we state the argument in full in the case where the boundary is flat. This means that we work in a ball $B(x_0, 4R)$ such that $B^+(x_0, 4R) \subset D \subset \mathbb{R}^n_+$ and $\partial D \cap B(x_0, 4R) = \partial \mathbb{R}^n_+ \cap B(x_0, 4R)$.

If $B(x_0, R) \subset \mathbb{R}^n_+$ then we can just take $v \equiv u$. So assume that $B(x_0, R) \cap \partial \mathbb{R}^n_+ \neq \emptyset$, then for ϵ_0 small enough we have $|u| > \frac{1}{2}$ in $B^+(x_0, R)$ by Corollary 5.2.

Denote by $\tilde{u}$ the even reflection, i.e.,

$$\tilde{u}(x', x_n) := u(x', |x_n|).$$

Moreover, set

$$\sigma(q) := \frac{q}{|q|^2}, \quad q \in \mathbb{R}^n \backslash \{0\}.$$

Now we define the geometric reflection v as

$$v(x) := \begin{cases} u(x) & x \in B^+(x_0, R) \\ \sigma(\tilde{u}(x)) & x \in B(x_0, R) \backslash \mathbb{R}^n_+. \end{cases}$$

Since $|u| > \frac{1}{2}$ and u is uniformly bounded by Lemma A.1, v is well-defined in $B(x_0, R)$.

We also set

$$\Sigma_{ij}(q) := \partial_i \sigma^j(q) = \frac{\delta_{ij} - 2\frac{q^i q^j}{|q|^2}}{|q|^2}.$$

That is, for $x \in B(x_0, R) \backslash \mathbb{R}^n_+$,

$$\nabla v(x) = \Sigma(\tilde{u}(x))\ \nabla \tilde{u}(x). \tag{5.3}$$

Observe that Σ is symmetric, and

$$\Sigma(q) = \frac{1}{|q|^2}\left(I - 2\frac{q}{|q|} \otimes \frac{q}{|q|}\right)$$

and that $\frac{q}{|q|}$ is an eigenfunction to the eigenvalue $-\frac{1}{|q|^2}$, and any orthonormal basis of $\left(\frac{q}{|q|}\right)^{\perp}$ is the basis of the eigenspace of the eigenvalue $\frac{1}{|q|^2}$. In particular,

$$|\Sigma(q)w| = \frac{1}{|q|^2}|w| \quad \forall w \in \mathbb{R}^N.$$

Thus,

$$|\nabla v(x)| = \begin{cases} |\nabla \tilde{u}(x)| & x \in B^+(x_0, R) \\ \frac{1}{|\tilde{u}(x)|^2}|\nabla \tilde{u}(x)| & x \in B(x_0, R) \setminus \mathbb{R}^n_+. \end{cases} \tag{5.4}$$

Also observe that for $|q| = 1$,

$$\Sigma(q)v = \Pi(q)v - \Pi^{\perp}(q)v \quad \text{for all } v \in \mathbb{R}^N,$$

where $\Pi(q) := I - q \otimes q$ is the orthogonal projection onto $T_q\mathbb{S}^{N-1} = q^{\perp}$ and $\Pi^{\perp}(q) := q \otimes q$ is the orthogonal projection onto $(T_q\mathbb{S}^{N-1})^{\perp} = \operatorname{span}\{q\}$.

Therefore, for $\varphi \in C_c^{\infty}(B(x_0, R), \mathbb{R}^N)$, since $\partial_\nu u \perp T_u\mathbb{S}^{N-1}$,

$$\int_{B^+(x_0,R)} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \int_{B(x_0,R)\setminus\mathbb{R}^n_+} |\nabla \tilde{u}|^{p-2}\nabla\tilde{u} \cdot \nabla(\Sigma(\tilde{u})\varphi) = 0.$$

In particular,

$$\int_{B(x_0,R)} |\nabla \tilde{u}|^{p-2}\nabla v \cdot \nabla\varphi = -\int_{B(x_0,R)\setminus\mathbb{R}^n_+} |\nabla \tilde{u}|^{p-2}\nabla\tilde{u} \cdot \nabla(\Sigma(\tilde{u}))\ \varphi.$$

Combining this with (5.4),

$$\begin{aligned}\int_{B(x_0,R)} |\nabla v|^{p-2}\nabla v \cdot \nabla(m\varphi) = &-\int_{B(x_0,R)\setminus\mathbb{R}^n_+} |\nabla \tilde{u}|^{p-2}\nabla\tilde{u} \cdot \nabla(\Sigma(\tilde{u}))\ \varphi \\ &+ \int_{B(x_0,R)} |\nabla v|^{p-2}\nabla v \cdot \nabla m\ \varphi,\end{aligned}$$

where

$$m(x) = \begin{cases} 1 & \text{in } B^+(x_0, R) \\ |\tilde{u}(x)|^{2(p-2)} & \text{in } B(x_0, R) \setminus \mathbb{R}^n_+. \end{cases}$$

Observe that $m(x)$ and $m(x)^{-1} \in L^\infty \cap W^{1,p}(B(x_0, R))$. Now (5.2) follows from (5.4).

It remains to establish the growth estimates from Proposition 4.1 which follow from Proposition 2.1. Indeed, set $\mathcal{M} := B(x_0, R) \cap \partial\mathbb{R}^n_+$.

To obtain (4.5) let $B(y_0,4r) \subset B(x_0,R)$ and $B(y_0,2r) \cap \mathcal{M}=\emptyset$. Let us consider first $B(y_0,2r) \subset \mathbb{R}^n_-$. Then we observe that by (5.4) combined with the fact that $|u| > \frac{1}{2}$ on $B^+(x_0,R)$ we have $\int_{B(y_0,r)} |\nabla v|^p \precsim \int_{B(\tilde{y}_0,r)} |\nabla u|^p$, where $\tilde{y}_0$ is the point $y_0 = (y_0^1,\dots,y_0^n)$ reflected along the hyperplane $\partial\mathbb{R}^n_+$, i.e., $\tilde{y}_0 = (y_0^1,\dots,-y_0^n)$. Now applying (2.3) to u, we obtain

$$
\begin{aligned}
\int_{B(y_0,r)} |\nabla v|^p &\precsim C\lambda \int_{B^+(\tilde{y}_0,4r)} |\nabla u|^p + C\lambda^{1-p} r^{-p} \int_{B^+(\tilde{y}_0,4r)} |u - (u)_{B^+(\tilde{y}_0,4r)}|^p \\
&\le C\lambda \int_{B(y_0,4r)} |\nabla v|^p + C\lambda^{1-p} r^{-p} \int_{B^-(y_0,4r)} \left|\tilde{u} - (\tilde{u})_{B^-(y_0,4r)}\right|^p .
\end{aligned} \tag{5.5}
$$

To estimate the remaining part we note that since $v = \frac{\tilde{u}}{|\tilde{u}|^2}$ we have $\tilde{u} = \frac{v}{|v|^2}$ in $\mathbb{R}^n_-$ and for any $A \subset B(x_0,R) \setminus \mathbb{R}^n_+$:

$$
\begin{aligned}
&\fint_A \left|\frac{v}{|v|^2} - \left(\frac{v}{|v|^2}\right)_A\right|^p \precsim \fint_A\fint_A \left|\frac{v(x)}{|v(x)|^2} - \frac{v(y)}{|v(x)|^2}\right|^p + \fint_A\fint_A \left|\frac{v(y)}{|v(x)|^2} - \frac{v(y)}{|v(y)|^2}\right|^p \\
&\qquad \precsim \|v^{-1}\|^{2p}_{L^\infty} \fint_A\fint_A |v(x)-v(y)|^p + \|v^{-1}\|^{3p}_{L^\infty} \fint_A\fint_A \left||v(x)|^2 - |v(y)|^2\right|^p .
\end{aligned} \tag{5.6}
$$

Now, since for any a, b,

$$
|a|^2 - |b|^2 = (|a|+|b|)(|a|-|b|) \le (|a|+|b|)|a-b|
$$

we have

$$
\begin{aligned}
\fint_A\fint_A \left||v(x)|^2 - |v(y)|^2\right|^p &\precsim \|v\|^p_{L^\infty(A)} \fint_A\fint_A |v(x)-v(y)|^p \\
&\precsim \|v\|^p_{L^\infty(A)} \fint_A |v-(v)_A|^p,
\end{aligned} \tag{5.7}
$$

where the last inequality was obtained by adding and subtracting $(v)_A$ and by the triangle inequality. We deduce from (5.6) and (5.7) that

$$
\fint_A \left|\frac{v}{|v|^2} - \left(\frac{v}{|v|^2}\right)_A\right|^p \precsim \|v^{-1}\|^{2p}_{L^\infty(A)} (1 + \|v\|^p_{L^\infty(A)} \|v^{-1}\|^p_{L^\infty(A)}) \fint_A |v-(v)_A|^p.
$$

Due to the fact that $|u| > \frac{1}{2}$ and u is uniformly bounded we get

$$
\fint_A |\tilde{u} - (\tilde{u})_A|^p \precsim \fint_A |v-(v)_A|^p \quad \text{for any } A \subset B(x_0,R) \setminus \mathbb{R}^n_+. \tag{5.8}
$$

To conclude, we note[3] that since $B(y_0, 2r) \subset \mathbb{R}^n_-$ we have $\frac{|B(y_0,4r)|}{|B^-(y_0,4r)|} \approx 1$, thus

$$\fint_{B^-(y_0,4r)} |v - (v)_{B^-(y_0,4r)}|^p \precsim \fint_{B(y_0,4r)} |v - (v)_{B(y_0,4r)}|^p. \tag{5.9}$$

Combining estimates (5.5), (5.8), and (5.9) we obtain (4.5). The second case $B(y_0, 2r) \subset \mathbb{R}^n_+$ is easier and we leave it to the reader.

Finally, for (4.6) we apply (2.4) and observe that $|u|^2 \equiv 1$ on $\mathcal{I} := B(y_0, 4r) \cap \partial\mathbb{R}^n_+$. Thus,

$$\int_{B^+(y_0,4r)} \left||u|^2 - 1\right|^p \precsim (\|u\|_{L^\infty} + 1) \int_{B^+(y_0,4r)} \left||u| - (|u|)_{\mathcal{I}}\right|^p.$$

Now

$$\left||u(z)| - (|u|)_{\mathcal{I}}\right| \leq \fint_{\mathcal{I}} \left||u(z)| - |u(z_2)|\right| \, dz_2 \leq \fint_{\mathcal{I}} \left|u(z) - u(z_2)\right| \, dz_2$$

and thus

$$\fint_{B^+(y_0,4r)} \left||u| - (|u|)_{\mathcal{I}}\right|^p \precsim \fint_{B^+(y_0,4r)} \left|u - (u)_{\mathcal{I}}\right|^p + \fint_{\mathcal{I}} \left|u - (u)_{\mathcal{I}}\right|^p.$$

Proposition 5.3 is now established. □

Proof of Theorem 1.2. For $p = n$ Hölder continuity for u follows from Proposition 3.1. For $p < n$ it follows from the combination of Proposition 5.3 and Proposition 4.1. Now $C^{1,\alpha}$-regularity follows from the reflection, Proposition 5.3, and the fact that a Hölder continuous solution to the reflected system is $C^{1,\alpha}$ for some $\alpha > 0$, see [23, Theorem 3.1.] (which is stated for minimizers but actually only uses the continuity of the solution and the equation). See also [41, Theorem 1.2.].

Note that for $p = n$ there is also a more elegant argument to pass from C^α regularity to $C^{1,\alpha}$. Testing the equation (1.12) in x and $x + h$ with $\varphi(x) := \eta(x)(v(x + h) - v(x))$ for a suitable cutoff function η one obtains from the Hölder continuity of u that for some $\sigma > 0$ we have $\nabla v \in W^{1+\sigma,n}$. In particular, by Sobolev embedding $\nabla v \in L^{(n,1)}_{loc}$, and from Duzaar-Mingione's work [13] we get a Lipschitz bound for v. Now, $C^{1,\alpha}$-regularity is a consequence of the potential estimates for p-Laplace equations, see [29, 30]. We leave the details to the reader. □

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[3] Indeed, for any $\tilde{A} \subset A$ we have by enlarging the domain of integration and applying Jensen's inequality

$$\fint_{\tilde{A}} |w - (w)_{\tilde{A}}|^p \precsim \frac{|A|}{|\tilde{A}|} \fint_A |w - (w)_A|^p.$$

6. Partial Regularity: Proof of Theorem 1.4

For simplicity we assume in this section that $B^+(0,R) \subset D \subset \mathbb{R}^n_+$ and $\partial D \cap B(0,R) = \partial\mathbb{R}^n_+ \cap B(0,R)$. We begin with recalling that a map $u \in W^{1,p}(B^+(0,R),\mathbb{R}^N)$ is said to be *stationary p-harmonic* with respect to the free boundary condition $u(\partial D \cap B(0,R)) \subset \mathbb{S}^{N-1}$ if in addition to (1.9) it is a critical point of the energy with respect to variations in the domain. The latter is equivalent to u satisfying

$$\int_{B^+(0,R)} |\nabla u|^{p-2} \left(|\nabla u|^2 \delta_{ij} - p\, \partial_i u\, \partial_j u\right) \partial_i \xi^j = 0 \tag{6.1}$$

for $\xi = (\xi^1, \ldots, \xi^n) \in C^\infty_c(\overline{\mathbb{R}^n_+} \cap B(0,R), \mathbb{R}^n)$ with $\xi(\partial\mathbb{R}^n_+) \subset \partial\mathbb{R}^n_+$.

By choosing the test function as $\xi(x) := \psi(x)(x_0 - x)$ in (6.1), where $\psi \in C^\infty_c(\overline{\mathbb{R}^n_+} \cap B(0,R), [0,1])$ is a suitable bump function, one obtains the following.

Lemma 6.1 (monotonicity formula)**.** *Let $u \in W^{1,p}(B^+(0,R),\mathbb{R}^N)$ be a stationary p-harmonic map with respect to the free boundary condition $u(B^+(0,R) \cap \{x_n = 0\}) \subset \mathbb{S}^{N-1}$ and let $x_0 \in B^+(0,R) \cap \{x_n = 0\}$. Then, the normalized p-energy is monotone. In particular,*

$$r^{p-n} \int_{B^+(x_0,r)} |\nabla u|^p - \rho^{p-n} \int_{B^+(x_0,\rho)} |\nabla u|^p = p \int_{B^+(x_0,r)\setminus B^+(x_0,\rho)} |x-x_0|^{p-n} |\nabla u|^{p-2} \left|\frac{\partial u}{\partial \nu}\right|^2 \tag{6.2}$$

for all $0 < \rho < r < R - |x_0|$, where ν is the outward pointing unit normal for $\partial B(x_0,r)$, $\nu(x) := \frac{x-x_0}{|x-x_0|}$. For $x_0 \in B^+(0,R) \setminus \partial\mathbb{R}^n_+$ the same holds if r is such that $B^+(x_0,r) = B(x_0,r) \subset \mathbb{R}^n_+$.

This well-known fact was proved for Yang–Mills fields and stationary harmonic maps by Price [39], see [16, 2] and also [54, Section 2.4]. Fuchs [19] observed that (6.2) holds for stationary p-harmonic maps. As pointed out by Scheven [43, p.137] the proof holds true in the case of free boundary condition.

We will need the following lemma (see, e.g., [62, Corollary 3.2.3.]).

Lemma 6.2 (Frostman's lemma)**.** *If $f \in L^p(\mathbb{R}^n)$, $p \geq 1$, and $0 \leq \alpha < n$, then for*

$$E := \left\{ x \in \mathbb{R}^n : \limsup_{r\to 0} r^{-\alpha} \int_{B(x,r)} |f(y)|^p > 0 \right\},$$

we have $\mathcal{H}^\alpha(E) = 0$.

We shall show, using monotonicity formula (6.2) and Frostman's Lemma 6.2, that the set outside which the condition (1.13) is satisfied is of zero $(n-p)$-Hausdorff measure. We then obtain Theorem 1.4 from Theorem 1.2.

Proof of Theorem 1.4. Let

$$S := \left\{ x \in \overline{\mathbb{R}^n_+} : \limsup_{r\to 0} r^{p-n} \int_{B^+(x,r)} |\nabla u|^p > 0 \right\},$$

by Lemma 6.2, we have $\mathcal{H}^{n-p}(S) = 0$.

We define for ϵ as in Theorem 1.2

$$\Sigma_\epsilon := \left\{ x \in \overline{\mathbb{R}^n_+} : \forall R > 0 \sup_{|y_0 - x| < R} \sup_{\rho < R} \rho^{p-n} \int_{B^+(y_0,\rho)} |\nabla u|^p \geq \epsilon \right\},$$

clearly Σ_ϵ is a closed set. We will prove that $\mathcal{H}^{n-p}(\Sigma_\epsilon) = 0$. Then Theorem 1.4 is a consequence of Theorem 1.2.

Let A_ϵ be the set on which the condition (1.13) is satisfied for ϵ, i.e.,

$$A_\epsilon := \overline{\mathbb{R}^n_+} \setminus \Sigma_\epsilon = \left\{ x \in \overline{\mathbb{R}^n_+} : \exists R > 0 \text{ such that } \sup_{|y_0 - x| < R} \sup_{\rho < R} \rho^{p-n} \int_{B^+(y_0,\rho)} |\nabla u|^p < \epsilon \right\}.$$

In order to prove the theorem it suffices to show that $\left(\overline{\mathbb{R}^n_+} \setminus S\right) \subseteq A_\epsilon$.

Let $x_0 \in \left(\overline{\mathbb{R}^n_+} \setminus S\right)$, i.e., be such that $\limsup_{r\to 0} r^{p-n} \int_{B^+(x_0,r)} |\nabla u|^p = 0$. There exists an $R > 0$ such that

$$R^{p-n} \int_{B^+(x_0,R)} |\nabla u|^p < 4^{p-n}\epsilon.$$

We shall show that

$$\sup_{|y_0 - x_0| < R/4} \sup_{\rho < R/4} \rho^{p-n} \int_{B^+(y_0,\rho)} |\nabla u|^p < \epsilon.$$

Choose any y_0 such that $|y_0 - x_0| < R/4$ and any radius $\rho < R/4$. First observe that we may take $y_0 \in \overline{\mathbb{R}^n_+}$. Indeed, suppose that $y_1 \in B(x_0, R/4) \cap \mathbb{R}^n_-$, then for any $\rho < R/4$ we can choose $y_0 \in B(x_0, R/4) \cap \overline{\mathbb{R}^n_+}$ such that $B(y_1, \rho) \cap \overline{\mathbb{R}^n_+} \subset B(y_0, \rho) \cap \overline{\mathbb{R}^n_+}$ thus

$$\sup_{|y_1 - x_0| < R/4} \sup_{\rho < R/4} \rho^{p-n} \int_{B^+(y_1,\rho)} |\nabla u|^p = \sup_{y_0 \in B(x_0,R/4)\cap\overline{\mathbb{R}^n_+}} \sup_{\rho < R/4} \rho^{p-n} \int_{B^+(y_0,\rho)} |\nabla u|^p.$$

Now assume that $y_0 \in \partial\mathbb{R}^n_+$. We have $B^+(y_0, \rho) \subset B^+(y_0, R/4) \subset B^+(x_0, R)$. Thus

$$\rho^{p-n} \int_{B^+(y_0,\rho)} |\nabla u|^p \leq \left(\frac{R}{4}\right)^{p-n} \int_{B^+(y_0,R/4)} |\nabla u|^p \leq 4^{n-p} R^{p-n} \int_{B^+(x_0,R)} |\nabla u|^p < \epsilon,$$

where the first inequality is a consequence of the monotonicity formula (6.2).

Now, let us assume that $y_0 \notin \partial\mathbb{R}^n_+$. Let $\overline{\rho} = \operatorname{dist}(y_0, \partial\mathbb{R}^n_+)$ and $\overline{y}_0$ be the projection of y_0 onto $\partial\mathbb{R}^n_+$. We can assume that $\rho < \overline{\rho}$. Indeed, if not we would have

$$\begin{aligned}\rho^{p-n}\int_{B^+(y_0,\rho)}|\nabla u|^p &\le \rho^{p-n}\int_{B^+(\overline{y}_0,2\rho)}|\nabla u|^p = 2^{n-p}(2\rho)^{p-n}\int_{B^+(\overline{y}_0,2\rho)}|\nabla u|^p\\ &\le 2^{n-p}\left(\frac{R}{2}\right)^{p-n}\int_{B^+(\overline{y}_0,R/2)}|\nabla u|^p \le 4^{n-p}R^{p-n}\int_{B^+(x_0,R)}|\nabla u|^p < \epsilon.\end{aligned}$$

Next, we note that $\overline{\rho} < R/4$ and observe the following inclusions

$$B(y_0,\rho) \subset B(y_0,\overline{\rho}) \subset B^+(\overline{y}_0, 2\overline{\rho}) \subset B^+(\overline{y}_0, R/2) \subset B^+(x_0,R)$$

and the following inequalities which are consequences of the monotonicity formula (6.2):

$$\begin{aligned}\rho^{p-n}\int_{B(y_0,\rho)}|\nabla u|^p &\le (\overline{\rho})^{p-n}\int_{B(y_0,\overline{\rho})}|\nabla u|^p,\\ (2\overline{\rho})^{p-n}\int_{B^+(\overline{y}_0,2\overline{\rho})}|\nabla u|^p &\le \left(\frac{R}{2}\right)^{p-n}\int_{B^+(\overline{y}_0,R/2)}|\nabla u|^p.\end{aligned}$$

Thus

$$\begin{aligned}\rho^{p-n}\int_{B(y_0,\rho)}|\nabla u|^p &\le (\overline{\rho})^{p-n}\int_{B(y_0,\overline{\rho})}|\nabla u|^p \le 2^{n-p}(2\overline{\rho})^{p-n}\int_{B^+(\overline{y}_0,2\overline{\rho})}|\nabla u|^p\\ &\le 2^{n-p}\left(\frac{R}{2}\right)^{p-n}\int_{B^+(\overline{y}_0,R/2)}|\nabla u|^p \le 4^{n-p}R^{p-n}\int_{B^+(x_0,R)}|\nabla u|^p < \epsilon,\end{aligned}$$

which gives $x_0 \in A_\epsilon$.

We conclude $\Sigma_\epsilon \subset S$ and thus $\mathcal{H}^{n-p}(\Sigma_\epsilon) = 0$. □

6.1. **A Liouville type result.** We note that the monotonicity formula in Lemma 6.1 can be used to prove partial regularity but also Liouville type results in the spirit of [34]. Indeed, if we work in $\mathbb{R}^n_+$, for $u \in \dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N)$ we can say that u is stationary p-harmonic with respect to the free boundary condition $u(\partial\mathbb{R}^n_+) \subset \mathbb{S}^{N-1}$ if u satisfies (1.9) and

$$\int_{\mathbb{R}^n_+}|\nabla u|^{p-2}\left(|\nabla u|^2\delta_{ij} - p\,\partial_i u\,\partial_j u\right)\partial_i\xi^j = 0 \tag{6.3}$$

for $\xi = (\xi^1, \dots, \xi^n) \in C^\infty_c(\overline{\mathbb{R}^n_+}, \mathbb{R}^n)$ with $\xi(\partial\mathbb{R}^n_+) \subset \partial\mathbb{R}^n_+$. We then have

Proposition 6.3. *Let $2 \le p < n$ and $u \in \dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N)$ be such that u is a finite energy, stationary p-harmonic map with respect to the free boundary condition $u(\partial\mathbb{R}^n_+) \subset \mathbb{S}^{N-1}$, then u is constant.*

Proof. By contradiction, assume that u is not a constant. Then there exists $R_0 > 0$ such that $\int_{B^+(0,R_0)}|\nabla u|^p \ge c > 0$. Now by the monotonicity formula 6.1 we have that for any

$R > R_0$

$$\int_{B^+(0,R)} |\nabla u|^p \geq \left(\frac{R}{R_0}\right)^{n-p} \int_{B^+(0,R_0)} |\nabla u|^p \geq \left(\frac{R}{R_0}\right)^{n-p} c. \tag{6.4}$$

We can then let R go to $+\infty$ and we obtain that the p-energy of u in $\mathbb{R}^n_+$ is infinite. This is a contradiction since we assumed that $u \in \dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N)$. □

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Appendix A. On boundedness of p-harmonic maps

The following lemma is well-known. However, we could not find it explicitly in the literature, so we state it here for the convenience of the reader.

Lemma A.1. *Let $D \subset \mathbb{R}^n$ be a smooth, bounded domain. Assume that $u \in W^{1,p}(D, \mathbb{R}^N)$ is a solution to*

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in } D.$$

If $u\Big|_{\partial D} \in L^\infty(\partial D)$, then $\|u\|_{L^\infty(D)} \leq \|u\|_{L^\infty(\partial D)}$.

Proof. For scalar functions this is a consequence of the weak maximum principle for the p-Laplacian, see [33, Theorem 2.15.]. However, here we work with a system. For $\varepsilon \in (0,1)$ we find smooth solutions $u_\varepsilon \in W^{1,p} \cap C^\infty(D, \mathbb{R}^N)$ of the uniformly elliptic system

$$\begin{cases} \operatorname{div}((\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon) = 0 & \text{in } D \\ u_\varepsilon = u & \text{on } \partial D \end{cases} \tag{A.1}$$

The solution is smooth in the interior, and a direct computation shows that

$$\operatorname{div}(\big(\varepsilon + |\nabla u_\varepsilon|^2\big)^{\frac{p-2}{2}} \nabla |u_\varepsilon|^2) \geq 0. \tag{A.2}$$

Thus the weak maximum principle for scalar solutions of uniformly elliptic operators in divergence form implies

$$\sup_{\varepsilon \in (0,1)} \|u_\varepsilon\|_{L^\infty(D)} \leq \|u\|_{L^\infty(\partial D)}, \tag{A.3}$$

Moreover, we can test (A.1) with $u_\varepsilon - u$, which is trivial on ∂D, and thus

$$\int_D |\nabla u_\varepsilon|^p \leq \int_D \left(\varepsilon + |\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 = \int_D \left(\varepsilon + |\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}} \nabla u_\varepsilon \ \cdot \nabla u,$$

consequently, with Young's inequality,

$$\int_D |\nabla u_\varepsilon|^p \leq \frac{1}{2} \int_D |\nabla u_\varepsilon|^p + C \int_D |\nabla u|^p + C(|D|, p)$$

Thus, u_ε is uniformly bounded in $W^{1,p}$,

$$\sup_{\varepsilon \in (0,1)} \int_D |\nabla u_\varepsilon|^p < \infty. \tag{A.4}$$

On the other hand,

$$\int_D \left(\left(\varepsilon + |\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}} \nabla u_\varepsilon - |\nabla u|^{p-2} \nabla u \right) \cdot (\nabla u_\varepsilon - \nabla u) = 0.$$

Applying then the well-known inequality

$$|a - b|^p \precsim \left(|a|^{p-2} a - |b|^{p-2} b\right)(a - b),$$

we find that as $\varepsilon \to 0$,

$$\int_D |\nabla u - \nabla u_\varepsilon|^p \precsim o(1) \int_D \left(|\nabla u|^{p-1} + |\nabla u_\varepsilon|^{p-1}\right)$$

Therefore, in view of (A.4) and the boundedness of D,

$$u_\varepsilon \xrightarrow{\varepsilon \to 0} u \quad \text{in } W^{1,p}(D).$$

In particular, up to a subsequence, we have pointwise almost everywhere convergence, and from (A.3) we have

$$\|u\|_{L^\infty(D)} \leq \|u\|_{L^\infty(\partial D)}.$$

$\square$

Lemma A.2. *Let $D \subset \mathbb{R}^n$ be a possibly unbounded domain with smooth boundary ∂D. Assume that $p > n - 1$, $u \in \dot{W}^{1,p}(D, \mathbb{R}^N)$ is a solution to*

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in } D.$$

If $u\Big|_{\partial D} \in L^\infty(\partial D)$, then for every compact set $K \subset \overline{D}$ we have

$$\|u\|_{L^\infty(K)} < \infty.$$

Proof. For compact K we find by Fubini's theorem a smooth, bounded domain $\tilde{D} \supset K$ such that

$$u\Big|_{\partial \tilde{D} \cap D} \in W^{1,p},$$

Since $p > n - 1$ we conclude that, by Morrey-Sobolev embedding, u is continuous on $\partial \tilde{D} \cap D$, and in particular $u \in L^\infty(\partial \tilde{D})$. Now we can apply Lemma A.1 to $\tilde{D}$ to obtain the result. $\square$

We now prove a maximum principle analog of Lemma A.1 but for maps defined in the half-space $\mathbb{R}^n_+$. We work with maps with finite energy, i.e., we work with $\dot{W}^{1,p}(\mathbb{R}^n_+,\mathbb{R}^N) := \{v \in \mathcal{D}'(\mathbb{R}^n_+,\mathbb{R}^N); \nabla v \in L^p(\mathbb{R}^n_+,\mathbb{R}^N)\}$. We remark that a map in $\dot{W}^{1,p}(\mathbb{R}^n_+,\mathbb{R}^N)$ is also in $L^p_{\mathrm{loc}}(\mathbb{R}^n_+,\mathbb{R}^N)$ and hence has a trace on $\partial\mathbb{R}^n_+ := \mathbb{R}^{n-1}\times\{0\}$ which is well-defined.

Proposition A.3. *Let $u \in \dot{W}^{1,p}(\mathbb{R}^n_+,\mathbb{R}^N)$ be a solution to*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \mathbb{R}^n_+,$$

that is

$$\int_{\mathbb{R}^n_+} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi = 0 \quad \text{for all } \varphi \in C^\infty_c(\overline{\mathbb{R}^n_+}).$$

Assume that $u\big|_{\mathbb{R}^{n-1}\times\{0\}} \in L^\infty(\mathbb{R}^{n-1}\times\{0\})$, then $u \in L^\infty(\mathbb{R}^n_+)$ and

$$\|u\|_{L^\infty(\mathbb{R}^n_+)} \le \|u\|_{L^\infty(\partial\mathbb{R}^n_+)}.$$

Proof. We denote by $g := u\big|_{\mathbb{R}^{n-1}\times\{0\}}$ and $M := \|g\|_{L^\infty(\partial\mathbb{R}^n_+)}$. From Proposition A.4 below we know that u is the unique minimizer of the energy $\int_{\mathbb{R}^n_+}|\nabla v|^p$ in $X := \big\{v \in \dot{W}^{1,p}(\mathbb{R}^n_+,\mathbb{R}^N)\colon v\big|_{\mathbb{R}^{n-1}\times 0} = g \text{ in the trace sense}\big\}$. Now we define

$$\tilde{u} := \begin{cases} u & \text{if } |u| \le M,\\ \frac{Mu}{|u|} & \text{if } |u| > M.\end{cases}$$

By a direct computation we can see

$$\int_{\mathbb{R}^n_+}|\nabla\tilde{u}|^p \le \int_{\mathbb{R}^n_+}|\nabla u|^p.$$

Besides we have $\tilde{u}\big|_{\partial R^n_+} = g$. Thus by uniqueness we deduce that $\tilde{u} = u$ and $|u| \le M$ in $\mathbb{R}^n_+$. This concludes the proof. $\square$

It remains to prove:

Proposition A.4. *Let $u \in \dot{W}^{1,p}(\mathbb{R}^n_+,\mathbb{R}^N)$ be as in Proposition A.3 a solution to*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \mathbb{R}^n_+.$$

Let us denote by $g = u\big|_{\mathbb{R}^{n-1}\times 0}$ the trace of u. Then u is the unique minimizer of the energy $\int_{\mathbb{R}^n_+}|\nabla v|^p$ in

$$X := \left\{v \in \dot{W}^{1,p}(\mathbb{R}^n_+,\mathbb{R}^N)\colon v\big|_{\mathbb{R}^{n-1}\times 0} = g \text{ in the trace sense}\right\}.$$

Proof. By the direct method of calculus of variations we can prove that there exists a minimizer u_0 of $\int_{\mathbb{R}^n_+}|\nabla u|^p$ in X. Besides, by strict convexity of the p-energy we have that this minimizer is unique and it is the unique critical point of the p-energy in X. That is there is at most one map with a trace equal to g which satisfies

$$\int_{\mathbb{R}^n_+} |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\phi = 0, \quad \forall \phi \in \dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N), \quad \phi\big|_{\mathbb{R}^{n-1}\times\{0\}} = 0. \tag{A.5}$$

Observe that $C_c^\infty(\mathbb{R}^n_+, \mathbb{R}^N)$ is dense in the space

$$Y := \left\{\phi \in \dot{W}^{1,p}(\mathbb{R}^n_+, \mathbb{R}^N) \colon \phi\big|_{\mathbb{R}^{n-1}\times\{0\}} = 0\right\},$$

which can be proven as in, e.g., [61, Proposition 6.2.5]. We conclude that there is at most one map with a trace equal to g which satisfies

$$\int_{\mathbb{R}^n_+} |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\phi = 0, \quad \forall \phi \in C_c^\infty(\mathbb{R}^n_+). \tag{A.6}$$

This implies the claim. □

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