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Double coverings of racks and quandles

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Abstract. In this Ph.D. thesis we lay down the foundations of a higher covering theory of racks and quandles. This project is rooted in M. Eisermann's work on quandle coverings, and the categorical perspective brought to the subject by V. Even, who characterizes quandle coverings as those surjections which are central, relatively to trivial quandles. We revisit and extend this work by applying the techniques from higher categorical Galois theory, in the sense of G. Janelidze. In particular we extend the study of quandle coverings to the more general context of racks, we consolidate the understanding of their relationship with central extensions of groups on the one hand and topological coverings on the other. We further identify and study a meaningful two-dimensional centrality condition defining our double coverings of racks and quandles. We also introduce the definition of a suitable commutator which describes the zero, one and two-dimensional concepts of centralization in this context.

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pour mon frère

Contents

Introduction

Abstract. In this Ph.D. thesis we lay down the foundations of a higher covering theory of racks and quandles. This project is rooted in M. Eisermann's work on quandle coverings, and the categorical perspective brought to the subject by V. Even, who characterizes quandle coverings as those surjections which are central, relatively to trivial quandles. We revisit and extend this work by applying the techniques from higher categorical Galois theory, in the sense of G. Janelidze. In particular we extend the study of quandle coverings to the more general context of racks, we consolidate the understanding of their relationship with central extensions of groups on the one hand and topological coverings on the other. We further identify and study a meaningful two-dimensional centrality condition defining our double coverings of racks and quandles. We also introduce the definition of a suitable commutator which describes the zero, one and two-dimensional concepts of centralization in this context.

0.1. Context. We like to describe racks as sets equipped with a self-distributive system of symmetries, each attached to a given point (element). More precisely, a *rack* is a set A equipped with a binary operation $\triangleleft\colon A \times A \to A$ such that for each $a \in A$, the function $- \triangleleft a\colon A \to A$ (which is called the *symmetry at* a) admits an inverse (denoted $- \triangleleft^{-1} a\colon A \to A$, which is the other *symmetry at* a) and it is *compatible* with the operation $\triangleleft$ (self-distributivity), i.e. for all x, a and b in A:

(R1) $(x \triangleleft a) \triangleleft^{-1} a = x = (x \triangleleft^{-1} a) \triangleleft a$;
(R2) $(x \triangleleft a) \triangleleft b = (x \triangleleft b) \triangleleft (a \triangleleft b)$.

A *morphism of racks* is a function between (the underlying sets of two) racks that preserves the operation $\triangleleft$ (see Section 2.1 for more details). The term *wrack* was introduced by J.C. Conway and G.C. Wraith, in an unpublished correspondence of 1959. Their curiosity was driven towards the algebraic structure obtained from a group, when only the operations defined by conjugation are kept, and one forgets about the multiplication of elements. Sending a group G to its so defined "wreckage" (the rack with underlying set G and operation defined by conjugation $x \triangleleft a \coloneqq a^{-1}xa$), defines the *conjugation functor* $\mathrm{Conj}\colon \mathsf{Grp} \to \mathsf{Rck}$, from the category of groups Grp to the category of racks Rck. We use the more common spelling *rack* (instead of wrack) as in [**55**] and [**100**]. Other names in the literature are *automorphic sets* by E. Brieskorn [**15**], *crossed G-sets* by

P.J. Freyd and D.N. Yetter [**58**], and *crystals* by L.H. Kauffman in [**84**]. The former is (as far as we know) the first detailed published study of these structures.

The image of the conjugation functor Conj: $\mathsf{Grp} \to \mathsf{Rck}$ actually lands in the category Qnd, which is the full subcategory of Rck whose objects are racks Q such that for each $a \in Q$,

(Q1) $a \lhd a = a$,

i.e. the symmetries $(- \lhd a)$ of Q are required to fix the point (a) which they are attached to. Such algebraic structures were introduced and extensively studied by D.E. Joyce in his Ph.D. thesis [**82**], under the name of *quandles.* Around the same time (1980's), S. Matveev was studying the same structures independently (with comparable results), under the name of *distributive groupoids* [**91**]. D.E. Joyce describes the theory of quandles as the "algebraic theory of group conjugation" since the *freely generated quandle* on a set of generators X is a *subquandle* of the image by Conj of the *freely generated group* on that set X.

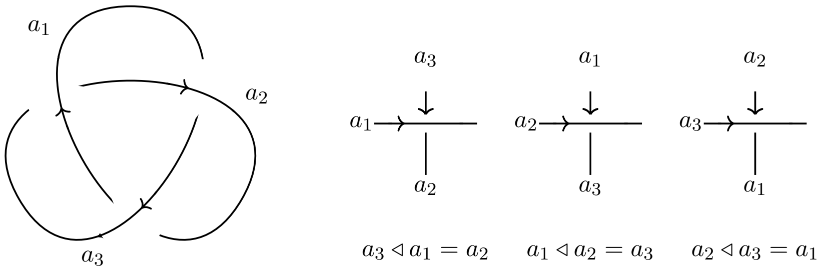


Figure 1. Oriented trefoil knot
Crossings "define" a $\lhd$-operation on the set of arcs

Most importantly for the development of this subject, which is largely due to its applications in knot theory, D.E. Joyce describes how to associate to each oriented knot diagram its *knot quandle* [**83**]; a construction which provides a *complete knot invariant for oriented knots.* The knot quandle resembles the earlier concept of *knot group* [**94**] (see Remark 3.5.7.1), but the knot quandle distinguishes itself as a *complete* invariant. In particular, computing whether two *oriented knot diagrams* represent the same oriented knot (up to isotopy) amounts (exactly) to computing whether their associated *knot quandles* coincide (up to isomorphism). As it has been done for the knot group, given an oriented knot diagram, a *presentation* of the associated knot quandle, in terms of generators and relations, is easily obtained using the arcs of the diagram as generators and using the crossings to describe relations on these generators. This construction exhibits an obvious parallel between the three axioms defining the theory of quandles ((R1), (R2) and (Q1)) and the *Reidemeister moves* [**93, 1**] from knot theory (these characterize the possible ways to modify a knot diagram without changing the knot it represents).

Given an oriented knot diagram K, such as in Figure 1, consider its set of arcs $A = \{a_1,\ a_2,\ \dots,\ a_n\}$, and identify that each crossing in K is either as on the left or as on the right of Figure 2. Each of these crossings thus gives rise to a corresponding identity as depicted in Figure 2. The knot quandle $Q(K)$ of the oriented knot diagram K is then obtained as the freely generated quandle on this set of arcs "modulo these identities" (a detailed construction for the free quandle on a set is given in Section 2.5.12). Using this translation from knot diagram crossings to algebraic identities, each Reidemeister move corresponds to an axiom from the algebraic theory of quandles as in Figure 3. Note that in order to obtain the proof that the knot quandle is an oriented knot invariant, one needs to study a few more possible choices of orientations for the strands in the Reidemeister moves of Figure 3. Each such choice of orientation similarly corresponds to an identity which is deducible from the axioms in the theory of quandles – see [**82**, Section 4.7] and Section 2.1.

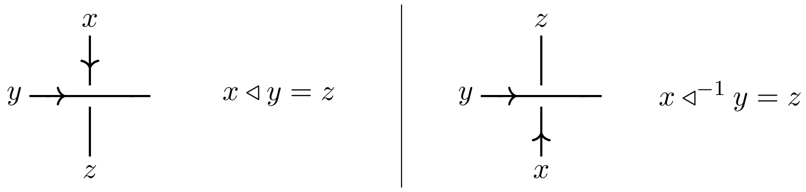


Figure 2. "Acting on x, with (the symmetry at) y, gives z"

Over the last decades, racks and quandles have been applied to knot theory and subsequently to physics in various works – see for instance [**83, 15, 55, 32, 56, 29, 28, 84, 38**] and references there. More historical remarks are made in [**55**], including references to applications in computer science. In geometry, the earlier notion of *symmetric space*, as studied by O. Loos in [**86**], gives yet another context for applications – see [**5, 63**] for up-to-date introductions to the field. This line of work goes

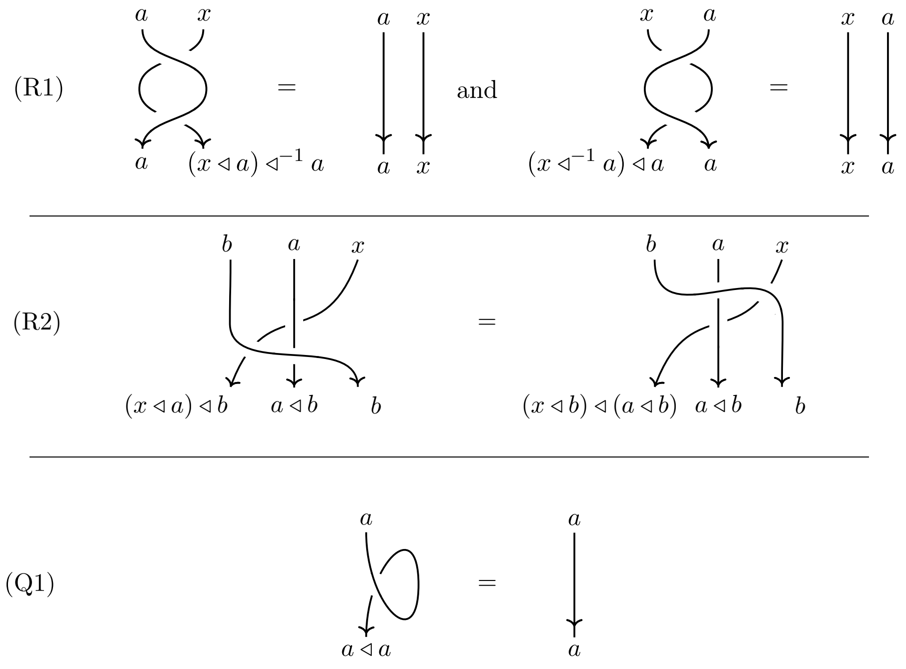
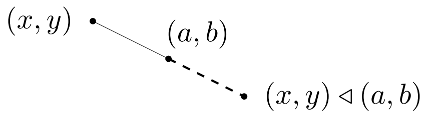


Figure 3. Reidemeister moves and axioms

back to 1943 with M. Takasaki's abstraction of a *Riemannian symmetric space*: a *kei* [**103**], which would now be called an *involutive quandle*. As a basic example consider the real plane $\mathcal{R}^2$, where each point (a, b) can act on any other point (x, y) using the central symmetry of $\mathcal{R}^2$ at (a, b).



More recently (2007) M. Eisermann worked on a covering theory for quandles (published in [**38**]). He defines quandle coverings as those surjective morphisms of quandles $c\colon A \to B$ such that for each x, a and $a' \in A$, if $c(a) = c(a')$ then $x \lhd a = x \lhd a'$, i.e. those elements a and a' that are identified by c act in the same way on any other element of A. M. Eisermann studies these coverings of quandles in analogy with topological coverings. Recall that a topological covering $c\colon X \to Y$ is a surjective and continuous function such that there exists an open cover $\{U_i\}_{i \in I}$ of Y such that for each index $i \in I$, the preimage $c^{-1}(U_i)$ of U_i is a disjoint union of opens of X, each of which is mapped homeomorphically onto

U_i by c (see for instance [**62**, Section 1.3] for an easy introduction to the subject). In particular, M. Eisermann derives several classification results for coverings of quandles, in the form of *Galois correspondences* as in topology (or Galois theory of field extensions). In order to do so, he works with some suitable constructions such as a (weakly) universal covering or a fundamental group(oid) of a quandle. Even though the link with quandle coverings is unclear a priori, these constructions use the left adjoint of the conjugation functor, which is justified a posteriori by the fact that the theory produces the aforementioned classification results.

Note that in topology, it is convenient to view the category of sets (Set) as the category of *discrete topological spaces.* The functor sending a topological space to its set of connected components is then obtained as the left adjoint to the inclusion of Set in the category of topological spaces.

$$\begin{tikzcd}
\mathsf{Top} \arrow[r, bend left, "\pi_0"] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, bend left, "I"]
\end{tikzcd}$$

As it is explained by F. Borceux and G. Janelidze in [**9**, Section 6.3], the covering theory of locally connected topological spaces arises from the study of this connected component adjunction using categorical Galois theory (in the sense of G. Janelidze [**65**], see Section 1). In particular, the concepts of covering and fundamental group(oid) of a space can be defined abstractly from the data of this adjunction. Moreover, the aforementioned classification theorem for topological coverings can be derived from the *fundamental theorem of categorical Galois theory* which describes, in this context, an equivalence between the category of coverings above a pointed space, and the *internal pre-sheaves* (think group actions) above the fundamental group of that space (see Subsection 1.0.8 below).

Now the category Set is also isomorphic to the category of *trivial quandles* where a *trivial quandle* Q is a quandle such that its binary operations simply return the first term (first product projection): $x \triangleleft y = x \triangleleft^{-1} y = x$ for all x and $y \in Q$. The inclusion of Set in Qnd has a left adjoint $\pi_0 \colon \mathsf{Qnd} \to \mathsf{Set}$ which is thus also a connected component functor (in this sense).

$$\begin{tikzcd}
\mathsf{Qnd} \arrow[r, bend left, "\pi_0"] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, bend left, "\mathrm{I}"]
\end{tikzcd} \tag{1}$$

The resulting notion of *connectedness* was first introduced by D.E. Joyce. Two elements x and y of a quandle A are said to be *connected* if there exists a *primitive path* from x to y, i.e. a sequence of elements, a_1, ..., a_n in A, and a choice of exponents $\delta_i \in \{-1,\ 1\}$ such that

$$y = (\cdots (x \triangleleft^{\delta_1} a_1) \cdots) \triangleleft^{\delta_n} a_n.$$

In his Ph.D. thesis [**40**], V. Even applies categorical Galois theory (in the sense of G. Janelidze, see for instance [**71**] or Section 1), to the context of quandles. By doing so, he establishes that M. Eisermann's coverings arise from the *admissible adjunction* between *trivial quandles* (i.e. sets) and quandles, in the same way that topological coverings arise from the admissible adjunction between discrete topological spaces (i.e. sets) and locally connected topological spaces (see Section 6.3 in [**9**] and Section 1 below). He also derives that M. Eisermann's notion of *fundamental group* of a connected, pointed quandle coincides with the corresponding notion from categorical Galois theory. This, in turn, makes the bridge with the fundamental group of a pointed, connected topological space. Thus, V. Even clarifies the analogy with topology, even though his results still rely on some constructions such as M. Eisermann's *weakly universal covers*.

Note that racks and quandles have recently received a lot of attention from experts in categorical algebra, not only in relation to quandle coverings [**39, 41, 42, 43**] but also for the development of the notion Σ-local properties [**11, 14, 12**].

In this thesis, we further develop the covering theory of quandles (and racks) from the perspective of (higher) categorical Galois theory. Our main objective is to develop the theory in higher dimensions in order to access more sophisticated information. In order to do so, we need to show that higher categorical Galois theory applies in this context. The development of the higher-dimensional theory also requires to further clarify the foundations of the covering theory in dimension zero and one.

In order to understand and motivate these developments, it is useful to make the analogy with yet another important application of categorical

Galois theory, which is the theory of central extensions of groups.

$$\begin{tikzcd}
\mathsf{Grp} \arrow[r, bend left, "\mathrm{ab}"] \arrow[r, phantom, "\perp"] & \mathsf{Ab} \arrow[l, bend left, "\mathrm{I}"]
\end{tikzcd} \tag{2}$$

In the category of groups Grp, the adjunction of interest is ab $\dashv$ I, the *abelianization adjunction* (Diagram (2)), where $\mathrm{ab}\colon \mathsf{Grp} \to \mathsf{Ab}$ is the *left adjoint* of the inclusion functor to groups $\mathrm{I}\colon \mathsf{Ab} \to \mathsf{Grp}$, and sends a group B to the abelian group $B/[B,B]_{\mathsf{Grp}}$, constructed by quotienting out the *commutator subgroup* $[B,B]_{\mathsf{Grp}}$ of B. Recall that for any normal subgroups X and Y of B, the normal subgroup $[X,Y]_{\mathsf{Grp}} \coloneqq \langle xyx^{-1}y^{-1} \mid x \in X,\ y \in Y\rangle \leq X \cap Y \leq B$ denotes the classical commutator from group theory. Note that from the point of view of *homological algebra* the abelianization of B is the first homology group with integer coefficients $H_1(B,\mathbb{Z})$.

In this context, the Galois-theoretic concept of *covering* coincides with the concept of a *central extension* from group theory. Recall that an *extension* in Grp is merely a surjective group homomorphism, and a *central extension* $c\colon A \to B$ is an extension such that the *kernel* $\mathrm{Ker}(c)$ *of* c is in the *center* $Z(A)$ of the group A, i.e. the elements $g \in A$ that are sent to the neutral element $c(g) = e \in B$ commute with any other element in A. In other words, c is a central extension if and only if $[\mathrm{Ker}(c), A]_{\mathsf{Grp}} = \{e\}$ is the trivial subgroup in A. Moreover, given any extension of groups $f\colon A \to B$, it can universally be made central by taking the quotient of A by the so-called *centralizing subgroup* $[\mathrm{Ker}(f), A]_{\mathsf{Grp}}$.

Whereas in the topological context categorical Galois theory relates to *homotopy theory*, its applications in group theory led to strong results in *homological algebra*. As we will not recall the necessary material for discussing homological algebra, we merely illustrate the arguments behind certain of these results on the simple example of a perfect group B. Recall that a *perfect group* is a group whose abelianization (first homology group) is trivial, i.e. $H_1(B,Z) \coloneqq B/[B,B]_{\mathsf{Grp}} = \{e\}$. A perfect group B admits a *universal central extension* $e\colon E \to B$, for which there is a *unique* factorization through any other central extension $c\colon A \to B$.

$$\begin{tikzcd}
E \arrow[dr, dotted, "\exists! \text{ factorization}"] \arrow[dd, "e"'] & \\
& A \arrow[dl, "c"] \\
B &
\end{tikzcd}$$

The fundamental Galois groupoid of B is then equivalent to the abelian group given by the kernel of e, which also describes the *second homology group with integer coefficients* $H_2(B,\mathbb{Z})$. The fundamental theorem of categorical Galois theory then gives an equivalence between the central extensions above B and the slice category whose objects are the morphisms with codomain $H_2(B,\mathbb{Z})$ in the category of abelian groups Ab. In other words, the *second integral homology group* of B can be *presented* as a "Galois group" (see [**65**, Remark 5.4], [**9**, Section 5.2.(10-17)] and [**69**]). For a general group B, and a *projective presentation* $p\colon F \to B$ of B with kernel $\mathrm{Ker}(p)$ (see Paragraph 1.0.11), the classical Hopf formula (Equation (3)) and its independence from the choice of presentation p can be obtained by similar arguments.

$$H_2(B,\mathbb{Z}) \cong \frac{\mathrm{Ker}(p) \cap [F,F]_{\mathsf{Grp}}}{[\mathrm{Ker}(p),F]_{\mathsf{Grp}}} \tag{3}$$

Now observe that the denominator of the Hopf formula is obtained as the aforementioned *centralizing subgroup* of the presentation p of B. Expanding on the fact that extensions of groups can be *centralized*, the inclusion of the category of central extensions of groups $\mathsf{CExtGrp}$ in the category of extensions of groups ExtGrp admits a left adjoint called the *centralization functor*. This functor $\mathrm{ab}^1\colon \mathsf{ExtGrp} \to \mathsf{CExtGrp}$ sends a surjective group homomorphism $f\colon A \to B$ to the central extension of groups

$$\mathrm{ab}^1(f)\colon A/[\mathrm{Ker}\, f, A]_{\mathsf{Grp}} \to B$$

obtained from the quotient $A/[\mathrm{Ker}\, f, A]_{\mathsf{Grp}}$ of the domain A of f.

$$\begin{tikzcd}
\mathsf{ExtGrp} \arrow[r, bend left, "\mathrm{ab}^1"] \arrow[r, phantom, "\perp"] & \mathsf{CExtGrp} \arrow[l, bend left, "\mathrm{I}"]
\end{tikzcd} \tag{4}$$

The key observation is that the resulting adjunction satisfies the conditions under which categorical Galois theory can be applied. This led G. Janelidze to the concept of *double central extension* [**67**], which is the induced notion of *covering* in this two-dimensional context. The Hopf formula for the third homology group of a group B and for a presentation of B

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F/K_2 \arrow[d] \\ F/K_1 \arrow[r] & B \end{tikzcd} \tag{5}$$

(where F, F/K_1 and F/K_2 are free groups and B is isomorphic to $F/(K_1 \cdot K_2)$) can then be obtained via categorical Galois theory as in [**69**].

$$H_3(B, \mathbb{Z}) \cong \frac{K_1 \cap K_2 \cap [F,F]_{\mathsf{Grp}}}{[K_1, K_2]_{\mathsf{Grp}} \cdot [K_1 \cap K_2, F]_{\mathsf{Grp}}} \tag{6}$$

Note that the denominator in Equation (6) then describes the *centralizing subgroup* for a *double extension* of the form of Diagram (5) (see also Paragraph 1.0.9), leading to the same developments in dimension three. By iterating this procedure, and abstracting away from the category of groups, powerful generalizations of the higher-dimensional Hopf formulae of R. Brown and G. Ellis [**18**] were found, leading to a whole new approach to homological algebra developed by M. Duckerts, A. Duvieusart, T. Everaert, M. Gran, G. Janelidze, J. Goedecke, D. Rodelo, C. Simeu, T. Van der Linden, and others [**68, 50, 61, 45, 48, 49, 99, 35, 37, 101**].

As it is the case for central extensions of groups, the category of quandle coverings $\mathsf{CExtQnd}$ is reflective in the category of extensions of quandles ExtQnd which was shown by M. Duckerts, V. Even and A. Montoli in [**36**].

$$\begin{tikzcd} \mathsf{ExtQnd} \arrow[r, bend left, "\mathrm{F}_1"] \arrow[r, phantom, "\perp"] & \mathsf{CExtQnd} \arrow[l, bend left, "\mathrm{I}"] \end{tikzcd} \tag{7}$$

This is the first requirement for obtaining the analogous higher- dimensional theory which we just decribed in the group-theoretic case. Note that higher categorical Galois theory has also been applied in topology where higher homotopical information of spaces can be studied via the Galois-theoretic higher fundamental groupoids. A detailed survey about the study of higher-dimensional homotopy group(oid)s can be found in [**16**], see also [**19**]. Some insights are given in [**21**] where higher Galois theory is used to build a homotopy double groupoid for maps of spaces (see also [**20**]).

The covering theory of racks and quandles is a particularly interesting instance of categorical Galois theory since it combines intuitive, "geometrical" interpretations, inspired by the topological example with strong connections to the group theoretic case. The study of the covering theory of racks and quandles and the development of its higher-dimensional aspects will thus be an opportunity to derive both homotopical and homological results in this context.

0.2. Content. In analogy with the aforementioned developments in Grp, we show in this thesis (Section 1) that categorical Galois theory applies to the adjunction between the category of central extensions and the category of extensions of racks and quandles (Diagram (7)). Note that, as it is the case in Grp, most instantiations of higher categorical Galois theory in the literature are such that the base category, and subsequently the higher-dimensional categories of extensions, are *Mal'tsev categories* (see [**26, 27, 25**] and Subsection 1.0.10). The categories of racks and quandles are not Mal'tsev categories. An important contribution of this thesis thus consists in the refinement and generalization of the techniques for the application of higher categorical Galois theory in new contexts. Note that the (lower- and higher-dimensional) covering theory of racks and quandles is an enlightening instantiation of Galois theory which exhibits some of its more geometric/topological aspects in an algebraic context. We are interested in studying the covering theory of racks and quandles by using categorical Galois theory, but we are also interested in what this instance of Galois theory teaches us about the general theory.

The main results of the thesis (Section 3) consist in the characterization of the induced Galois-theoretic concept of covering in this two-dimensional context, via the definition and study of *double coverings*, also called *algebraically central double extensions of racks and quandles* (Section 2). Moreover, we define a suitable and well behaved commutator which captures the zero, one and two-dimensional concepts of centralization in the category of quandles (Subsection 2.1). We keep track of the links with the corresponding concepts in the category of groups and hint at possible developments inspired by this analogy (see for instance Section 4). Note that we recall all the necessary concepts of categorical Galois theory in the first section of the thesis (Section 1).

In order to achieve these objectives, a lot of work has been put towards re-working the material about one-dimensional coverings (see for instance [**38, 39, 40, 42, 36**]). The generalization of the covering theory to higher dimensions is far from trivial and the existing literature on the lower-dimensional theory was not aimed at facilitating such a development. In the first part of this thesis (which is based on [**95**]), and after introducing the categorical Galois-theoretic prerequisites, we provide a detailed introduction to racks and quandles which is inspired by the perspective of the covering theories of interest. We then develop our refined understanding of these covering theories, and provide interesting new results, definitions and proofs in the lower-dimensional theory (leading to the higher-dimensional outcomes).

As a first step, the generalization of the covering theory of quandles to a covering theory of racks, and the study of the free/forgetful adjunction ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}} \dashv \mathrm{I}$ between racks and quandles participates in clarifying the key ingredients of both covering theories. In particular, we derive the desired results comparing the covering theories in racks and quandles from the study of the commutative diagram of admissible adjunctions below (see Section 3.3). As we will show in Section 2.5, categorical Galois theory also applies to the free/forgetful adjunction ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}} \dashv \mathrm{I}$.

$$\begin{tikzcd}
\mathsf{Rck} \arrow[rr, bend left, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}"] \arrow[rr, phantom, "\perp"] \arrow[dr, bend right, "\pi_0"'] \arrow[dr, phantom, "\dashv"] & & \mathsf{Qnd} \arrow[ll, bend left, "\mathrm{I}"] \arrow[dl, bend right, "\pi_0"'] \arrow[dl, phantom, "\dashv"] \\
& \mathsf{Set.} \arrow[ul, bend right, "\mathrm{I}"'] \arrow[ur, bend right, "\mathrm{I}"'] &
\end{tikzcd}$$

As we mentioned before, M. Eisermann makes use of the left adjoint of the conjugation functor $\mathrm{Conj}\colon \mathsf{Grp} \to \mathsf{Rck}$ between groups and racks (or quandles) in order to produce *weakly universal coverings* and a notion of *fundamental group(oid)* which are crucial to his theory and the subsequent contributions of V. Even. We identify a meaningful construction of this left adjoint, based on the axioms of racks and study its crucial role for the covering theories of interest (in Section 2.4 and beyond). We rename it as the *paths functor* Pth and explain in which sense it sends a rack A to its *group of homotopy classes of paths* $\mathrm{Pth}(A)$.

$$\begin{tikzcd}
\mathsf{Rck} \arrow[r, bend left, "\mathrm{Pth}"] \arrow[r, phantom, "\perp"] & \mathsf{Grp} \arrow[l, bend left, "\mathrm{Conj}"]
\end{tikzcd} \tag{8}$$

Recall the definition of a *primitive path* of a rack A, i.e. a signed sequence of elements a_1, …, a_n in A, which is used to link elements x and y that are *connected* in A:

$$y = (\cdots (x \triangleleft^{\delta_1} a_1) \cdots) \triangleleft^{\delta_n} a_n.$$

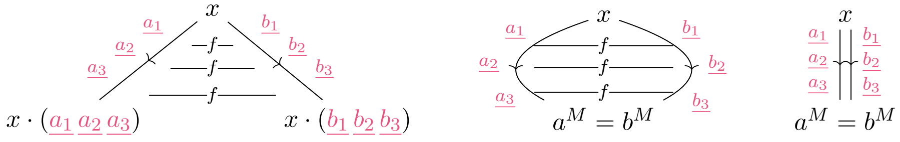
We may easily find a different primitive path with $b_1, \ldots, b_m$ in A and $\gamma_j \in \{-1, 1\}$ such that

$$y = (\cdots (x \triangleleft^{\gamma_1} b_1) \cdots) \triangleleft^{\gamma_m} b_m.$$

As in topology, this other path linking x and y may or may not provide interesting extra information about how to reach y from x. A study of the axioms of racks provides a notion of equivalence between primitive paths in a given rack A. Intuitively speaking, this equivalence identifies those ways to go from one point to another in A which "provide the same information" about the rack A. The resulting equivalence classes of primitive paths organize themselves into a group. The group of paths $\mathrm{Pth}(A)$ is then seen to be a group of representatives of these equivalence classes of primitive paths. Similarly the study of the idempotency axiom (Q1) exhibits a construction of a group of representatives of homotopy classes of primitive paths for an object of Qnd.

Based on this refined understanding of the algebraic ingredients at play, we provide new characterizations for coverings, relative centrality, and the centralization of an extension using Pth (see Section 3). Using our study of what the elements in the group of paths represent, we develop a more visual "geometrical" understanding of the whole theory, which is particularly helpful in expressing its higher-dimensional equivalents. Note that such visual representations also permitted the characterization (in full generality) of the fundamental groupoid of a rack (or quandle) and subsequently the application of the fundamental theorem of categorical Galois theory in this context (Sections 3.5 and 3.6). Most importantly, for the second part of the thesis, we were able to produce new, alternative, generalizable and visual proofs for the characterization of the Galois-theoretic coverings (or central extensions) of racks and quandles, which do not require the construction of a weakly universal covering (Section 3.2).

In short, given a surjective morphism of racks (or quandles) $f\colon A \to B$, the idea is to look at primitive paths as $(a_i, \delta_i)_{1 \leq i \leq n}$ and $(b_i, \delta_i)_{1 \leq i \leq n}$ that are identified by f, i.e. such that $f(a_i) = f(b_i)$ for each $i \in \{1, \ldots, n\}$. Think of it as a sort of (f-induced) surface linking the two given paths. We then have that f is a covering if and only if such "surfaces" (called *membranes*) actually "retract" as on the picture below. In dimension two, these *membranes* are organised into four-faced cones and the conditions of centrality expressed in terms of such cones.



As we develop this refined understanding of the subject in Part I, we lay down the ideas and results that lead to our main objective: the higher-dimensional theory developed in Part II (see also [**96, 98**]). The forthcoming generalization to arbitrary dimensions is not included in this thesis, and it will be developed in [**97**].

Part I

1. The point of view of categorical Galois theory

In this first Section we recall and motivate what we need to know about categorical Galois theory in order to study the covering theory of racks and quandles.

Categorical Galois theory (in the sense of [**65**], see also [**71**]) is a very general theory with rich and various interpretations depending on the numerous contexts of application. On a theoretical level, Galois theory exhibits strong links with, for example, factorization systems, commutator theory, homology and homotopy theory (see for instance [**73, 24, 70**]). Looking at applications, it unifies, in particular, the theory of field extensions from classical Galois theory (as well as both of its generalizations by A. Grothendieck and A. R. Magid), the theory of coverings of locally connected topological spaces, and the theory of central extensions of groups. The covering theory of racks and quandles [**38**] is yet another example [**39**], which combines intuitive interpretations inspired by the topological example with features of the group-theoretic case. A detailed historical account of the developments of Galois theory is given in [**9**] and [**70**] gives an overview of the developments of categorical Galois theory (from the perspective of universal algebra).

We consider a convenient particular instance of the general theory which was developed in [**65**]. The axiomatic framework in which categorical Galois theory takes place is that of a *Galois structure* (see [**66**]). For our purposes, a Galois structure, say Γ, mainly consists in the data of a category $\mathcal{C}$ (for instance the category of locally connected topological spaces Top), a full subcategory $\mathcal{X}$ in $\mathcal{C}$ (for instance the category of discrete topological spaces Set), together with a reflection of $\mathcal{C}$ on $\mathcal{X}$, i.e. a left adjoint $F\colon \mathcal{C} \to \mathcal{X}$ to the inclusion $\mathrm{I}\colon \mathcal{X} \to \mathcal{C}$ (e.g. $\pi_0\colon \mathsf{Top} \to \mathsf{Set}$; note that we often omit the inclusion functor I from our notation in what follows). The "bigger" context $\mathcal{C}$ is understood to be more "sophisticated", more difficult to study, whereas the "smaller" context $\mathcal{X}$ is supposedly more "primitive", or merely better understood. In order to obtain a Galois structure from such a reflection, we also need to specify a class of morphisms in $\mathcal{C}$, whose "elements" will be called *extensions*. We define Galois structures more precisely in Convention 1.0.1. For the connected component adjunction in topology, an example of a suitable Galois structure (which we denote Γ_T) is described more precisely in [**9**, Section 6.3]. The class of extensions in Γ_T is given by the class of *surjective étale maps*. Another example of Galois structure, say Γ_Q, is given by the connected component adjunction between Qnd and Set, for which the class of extensions is given by the class of surjective morphisms of Qnd.

Given a Galois structure Γ, the purpose of categorical Galois theory is to study special classes of extensions in $\mathcal{C}$ which are naturally associated to those extensions which lie in the subcategory $\mathcal{X}$. In this work, we call an extension which is a morphisms in $\mathcal{X}$ a *primitive extension*. Note that both in the example from topology, and in the example in Qnd, a primitive extension is just a surjective function in Set. These special classes of extensions in $\mathcal{C}$ which are associated to primitive extensions measure a "sphere of influence" of $\mathcal{X}$ in $\mathcal{C}$ (with respect to the chosen concept of extension). In particular, the most important special class of extensions is the class of *coverings* (sometimes called *central extensions*) defined below. Since we will be discussing different notions of coverings arising from different Galois structures, we sometimes use the terminology Γ-*covering* in order to avoid any confusion between the different contexts Γ. For the aforementioned Galois structure Γ_T, the induced concept of Γ_T-covering coincides with the classical concept of *covering* defined in topology. As we mentioned before, V. Even showed in [**40**] that in the Galois structure Γ_Q, the induced Galois-theoretic concept of Γ_Q-covering coincides, with the *coverings* defined by M. Eisermann in [**38**]. In the category of groups Grp, the *abelianization adjunction* adjunction $\mathrm{ab} \dashv \mathrm{I}$ gives rise to a Galois structure, say Γ_G. Recall that the *left adjoint* $\mathrm{ab}\colon \mathsf{Grp} \to \mathsf{Ab}$ sends a group G to the abelian group $G/[G,G]_{\mathsf{Grp}}$, constructed by quotienting out the *commutator subgroup* $[G,G]_{\mathsf{Grp}}$ of G. For any subgroups X and Y of G, the subgroup $[X,Y]_{\mathsf{Grp}} \coloneqq \langle xyx^{-1}y^{-1} \mid x \in X,\ y \in Y\rangle \leq X \cap Y \leq G$ defines the classical commutator from group theory. In this context, the *extensions* are chosen to be the *regular epimorphisms*, which are merely the surjective group homomorphisms. Given this Galois structure Γ_G, the concept of a Γ_G-covering coincides with the concept of a *central extension* from group theory.

In general, given a suitable Galois structure Γ one defines three different special classes of extensions, the simplest of which is the class of *trivial coverings* (or more explicitly *trivial* Γ-*coverings*). A trivial covering is defined as an extension t which is the pullback of a primitive extension p in $\mathcal{X}$, along the unit morphism η (see Figure 4). In a suitable Galois structure, the category of trivial coverings above an object $E \in \mathcal{C}$ is equivalent to the category of primitive extensions above $F(E)$ in $\mathcal{X}$. In topology, this Galois-theoretic definition of trivial Γ_T-covering coincides with the classical definition of trivial covering.

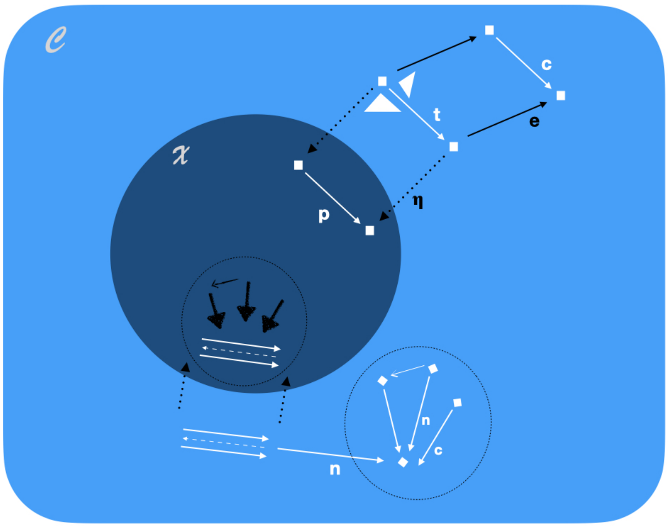


Figure 4. A kid's drawing of categorical Galois theory

The class of *coverings* (or more explicitely Γ-*coverings*) is then defined as the class of those extensions $c\colon A \to B$ in $\mathcal{C}$, for which there exists another extension $e\colon E \to B$, which is said to *split* c, i.e. such that the pullback t of c along e is a trivial covering. In certain contexts, such as in [**71**], coverings describe a relative notion of *centrality* [**59, 88**], hence the alternative terminology of *central extension*, borrowed from the aforementioned group-theoretic example. The remaining special class of extensions is the class of *normal coverings* (or more explicitly *normal* Γ-*coverings*), which are those extensions which are split by themselves. We will come back to these for the statement of the fundamental theorem of categorical Galois theory (see Paragraph 1.0.8).

CONVENTION 1.0.1. *For our purposes, a* Galois structure *(see* [**66**]*)*

$$\Gamma := (\mathcal{C}, \mathcal{X}, F, \mathrm{I}, \eta, \epsilon, \mathcal{E}),$$

is the data of an inclusion I*, of a* full *(*replete*) subcategory* $\mathcal{X}$ *in a category* $\mathcal{C}$*, with left adjoint* $F\colon \mathcal{C} \to \mathcal{X}$*, unit* η*, counit* ϵ *and a chosen class of* extensions $\mathcal{E}$ *within the morphisms of* $\mathcal{C}$*. The class* $\mathcal{E}$ *is subject to the following conditions:*

1. *$\mathcal{E}$ contains all isomorphisms, and $\mathcal{E}$ is closed under composition;*
2. *the image of an extension by the reflection F yields an extension;*
3. *pullbacks along extensions exist, and the pullback of an extension is an extension.*

For our purposes, $\mathcal{E}$ *will always be a class of* regular epimorphisms*. Moreover, we require the components of the unit* η *to be extensions, i.e. for each object* X *in* $\mathcal{C}$*,* $\eta_X\colon X \to \mathrm{I}FX$ *is an extension. Such a category* $\mathcal{X}$ *is said to be* $\mathcal{E}$*-reflective in* $\mathcal{C}$*. Finally, taking pullbacks along extensions should be a "well behaved algebraic" operation i.e. we require our extensions to be of* effective $\mathcal{E}$-descent *in* $\mathcal{C}$ *(see* [**79, 78**] *and Section 1.2 below).*

As mentioned before, we call primitive extensions*, those extensions* p *which lie in* $\mathcal{X}$*. A* trivial covering *(sometimes called* trivial extension*), is an extension* $t\colon T \to E$ *which is the pullback of a primitive extension* $p\colon X \to F(E)$ *in* $\mathcal{X}$*, along the unit morphism* $\eta_E\colon E \to F(E)$ *(see Paragraph 1.0.3 for more details).*

$$\begin{tikzcd} X \arrow[d, "p"'] & T \arrow[l] \arrow[r] \arrow[d, "t"] & A \arrow[d, "c"] \\ F(E) & E \arrow[l, "\eta_E"] \arrow[r, "e"'] & B \end{tikzcd} \tag{9}$$

A covering *(sometimes called* central extension*), is an extension* $c\colon A \to B$ *such that there is another extension* $e\colon E \to B$ *such that the pullback* t *of* c *along* e *is a trivial covering. A* normal covering *(sometimes called* normal extension*),* n*, is such that the projections of its kernel pair are trivial coverings, i.e.* n *is split by itself.*

REMARK 1.0.2. *Given such a Galois structure* Γ*, observe that* I *creates finite limits which exist in* $\mathcal{C}$*. The subcategory* $\mathcal{X}$ *is thus closed under* *finite limits in $\mathcal{C}$. Moreover since η is an extension, and thus in particular a regular-epimorphism, any subobject $i\colon A \rightarrowtail X$ in $\mathcal{C}$ of an object X in $\mathcal{X}$ factors as $i = f\eta_A$ for some f in $\mathcal{C}$, by the universal property of the unit. But then η_A is monic and thus it is an isomorphism. Since $\mathcal{X}$ is replete in $\mathcal{C}$, one concludes that $\mathcal{X}$ is then also closed under subobjects.*

These observations can be found in [**71**] where an important class of examples is studied in depth (see also Section 1.0.10). In particular G. Janelidze and M. Kelly observe that if $\mathcal{C}$ is a *variety of algebras*, and $\mathcal{X}$ is a *subvariety* of $\mathcal{C}$ in the sense of *universal algebra* [**6, 23**], the inclusion $\mathrm{I}\colon \mathcal{X} \to \mathcal{C}$ always admits a left adjoint $F\colon \mathcal{C} \to \mathcal{X}$, such that if we define $\mathcal{E}$ to be the class of regular epimorphisms in $\mathcal{C}$, then this data satisfies the conditions of Convention 1.0.1. The Galois structure for groups Γ_G and the Galois structure for quandles Γ_Q are examples of such Galois structures of varieties of algebras. In Part I, we prefer to use the terminology *trivial extension* (for trivial covering), *normal extension* (for normal covering) and *central extension* (for covering), as it is the case in [**71**], since the Galois structures we study are examples of the Galois structures considered in [**71**].

Note that not all examples of a subvariety of algebras in a variety of algebras give rise to a meaningful covering theory, or to a relative notion of central extension using categorical Galois theory. For the definitions provided in Convention 1.0.1 to be meaningful, the Galois structure Γ must satisfy the additional *admissibility* condition described below – see for instance [**71**]. Admissibility (in the sense of G. Janelidze) describes the fact that pullbacks of unit morphisms along extensions are unit morphisms. It can be understood as an exactness condition on the left adjoint F which implies in particular that trivial, central and normal extensions are pullback stable. Most importantly, this condition implies that Γ-coverings (central extensions) above a given object can be classified using data which is internal to $\mathcal{X}$ – in a form which is often called a *Galois correspondence*, as in the theory of coverings in topology. In short, admissibility is the condition on a Galois structure (or an adjunction) for Galois theory to be applicable and its fundamental theorem classifying coverings (or central extensions) to hold. We give more details about this fundamental theorem after the definition of admissibility.

1.0.3. *Admissibility.* With the definition of trivial covering in mind, let us observe that a Galois structure Γ induces an adjunction between the *slice categories* of extensions above an object in $\mathcal{C}$, which consists in taking quotients (left-adjoint) and pullbacks (right-adjoint) along the unit morphisms of the reflection $F \dashv \mathrm{I}$. Given an object E in $\mathcal{C}$, the slice category $\mathsf{Ext}(E)$ is the category whose objects are extensions $e\colon T \to E$ with codomain E. A morphism with domain e and codomain $h\colon C \to E$ in $\mathsf{Ext}(E)$ is given by a morphism $f\colon T \to C$ in $\mathcal{C}$ such that $e = hf$. Similarly define $\mathsf{PExt}(X)$ as the slice category of primitive extensions with codomain X, object of $\mathcal{X}$.

Given a Galois structure $\Gamma = (\mathcal{C}, \mathcal{X}, F, \mathrm{I}, \eta, \epsilon, \mathcal{E})$, it induces an adjunction $F^E \dashv \mathrm{I}^E$ for each object E in $\mathcal{C}$,

$$\begin{tikzcd}
\mathsf{Ext}(E) \arrow[r, bend left, "F^E"] \arrow[r, phantom, "\perp"] & \mathsf{PExt}(F(E)) \arrow[l, bend left, "\mathrm{I}^E"]
\end{tikzcd}$$

where the left adjoint $F^E\colon \mathsf{Ext}(E) \to \mathsf{PExt}(F(E))$ is defined by taking the image of objects and morphisms by F – for instance an object $e\colon T \to E$ is sent to $F(e)\colon F(T) \to F(E)$; and the right adjoint $\mathrm{I}^E\colon \mathsf{PExt}(F(E)) \to \mathsf{Ext}(E)$ is defined on $p\colon X \to F(E)$ by pulling back p along the unit $\eta_E\colon E \to F(E)$ in $\mathcal{C}$ (see Diagram (10) below).

We are interested in the components η^E_e and ϵ^E_p of the unit and counit of the adjunction $F^E \dashv \mathrm{I}^E$ for each extension $e\colon T \to E$ and each primitive extension $p\colon X \to F(E)$. Define the *reflection square at a morphism* $e\colon T \to E$ in $\mathcal{C}$ (with respect to Γ) to be the outer commutative (naturality) square of morphisms $F^E(e)\eta_T = \eta_E e$ on the left-hand side of Diagram (10). The map $\langle e, \eta_T\rangle$ induced by the universal property of the pullback $P \coloneqq E \times_{F(E)} F(T)$ of $F(e)$ and η_E is called the *comparison map* of this reflection square at e (see Paragraph 1.0.9). This comparison map is also the unit η^E_e of the adjunction $F^E \dashv \mathrm{I}^E$ which measures how far e is from being a trivial covering. Recall that we omit the inclusion $\mathrm{I}\colon \mathcal{X} \to \mathcal{C}$ from our notation.

$$\begin{tikzcd}
T \arrow[rr, "\eta_T"] \arrow[dr, dotted, "\eta^E_e:=\langle e,\eta_T\rangle"] \arrow[dd, "e"'] & & F(T) \arrow[dd, "F^E(e)"] \\
& P \arrow[ur, "\pi_2"'] \arrow[dl, "\mathrm{I}^E F^E(e):=\pi_1"] & \\
E \arrow[rr, "\eta_E"'] & & F(E)
\end{tikzcd}
\qquad
\begin{tikzcd}
P' \arrow[rr, "\pi_X"] \arrow[dr, "\eta_{P'}"] \arrow[dd, "\mathrm{I}^E(p):=\pi_E"'] & & X \arrow[dd, "p"] \\
& F(P') \arrow[ur, dotted, "\epsilon^E_p"'] \arrow[dr, "F^E\,\mathrm{I}^E(p))"'] & \\
E \arrow[rr, "\eta_E"'] & & F(E)
\end{tikzcd}
\qquad (10)$$

The counit ϵ_p^E of the adjunction $F^E \dashv \mathrm{I}^E$ at a given primitive extension $p\colon X \to F(E)$ is obtained by the universal property of $\eta_{P'}$ as in the right-hand side of Diagram (10). It essentially measures how far the pullback $P' := E \times_{F(E)} X$ of p and η_E is from being the reflection square at $\mathrm{I}^E(p)$. If ϵ_p^E is an isomorphism, then the outer square on the right-hand side of Diagram (10) is isomorphic to the reflection square at $\mathrm{I}^E(p)$, and this reflection square is then a pullback. A trivial covering is defined in most references to be an extension $t\colon T \to E$ such that the reflection square at t is a pullback. An *admissible* Galois structure is a Galois structure such that the pullback of a primitive extension in $\mathcal{X}$ along a unit morphism (see trivial covering in Convention 1.0.1) always gives a trivial covering in the usual sense (such as in [**71**]). Admissibility thus describes a form of compatibility of the reflection $F \dashv \mathrm{I}$ with pullbacks along primitive extensions – “the pullback of a unit morphism, along a primitive extension, is still a unit morphism”.

DEFINITION 1.0.4. *A Galois structure* $\Gamma := (\mathcal{C}, \mathcal{X}, F, \mathrm{I}, \eta, \epsilon, \mathcal{E})$ *as in Convention 1.0.1 is said to be* admissible *if for each* E*, object of* $\mathcal{C}$*, the induced right adjoint* $\mathrm{I}^E\colon \mathsf{PExt}(F(E)) \to \mathsf{Ext}(E)$ *is fully faithful i.e. for each primitive extension* $p\colon X \to F(E)$*, the component* ϵ_p^E *of the couinit of* $F^E \dashv \mathrm{I}^E$ *is an isomorphism.*

If we define $\mathsf{TExt}(E)$ to be the full subcategory of $\mathsf{Ext}(E)$ whose objects are trivial coverings, we may restrict the adjunction $F^E \dashv I^E$ to trivial coverings, which gives an equivalence of categories $\mathsf{TExt}(E) \simeq \mathsf{PExt}(F(E))$. Paraphrasing from [**71**] we conclude that in an admissible Galois structure Γ, “trivial extensions are nothing more than primitive extensions (but over an object in $\mathcal{C}$!) and one moves back and forth between the two concepts via (pullbacks and respectively quotients along) the reflection of $\mathcal{C}$ on $\mathcal{X}$”. Note that the image of the monad $\mathrm{I}^E F^E$ is exactly $\mathsf{TExt}(E)$, and the restriction $\mathrm{Trv}^E := \mathrm{I}^E F^E\colon \mathsf{Ext}(E) \to \mathsf{TExt}(E)$ provides a left adjoint to the inclusion of $\mathsf{TExt}(E)$ in $\mathsf{Ext}(E)$.

As it is said in [**71**], “Admissibility may be seen as a kind of exactness condition on F – the preservation by F of some pullbacks (but not of all, which would make $\mathcal{X}$ a localization of $\mathcal{C}$)”. Observe that in Diagram 10, ϵ_p^E is an isomorphism if and only if the commutative triangle $p\epsilon_p^E = \mathrm{id}_{F(E)} F^E \mathrm{I}^E(p)$ is a pullback square in $\mathcal{C}$ (the pullback of p and $\mathrm{id}_{F(E)}$). Now this triangle is the image by F of the outer square $p\pi_X = \eta_E \mathrm{I}^E(p)$ (on the right-hand side of Diagram (10)) which is the pullback of p and

η_E. Hence admissibility is characterized by the preservation of such pullback squares. It is then convenient to extend this characterization to the preservation by F of pullbacks of primitive extensions along any morphism.

Proposition 1.0.5. *Given* $\Gamma := (\mathcal{C}, \mathcal{X}, F, \mathrm{I}, \eta, \epsilon, \mathcal{E})$*, a Galois structure as in Convention 1.0.1, the following conditions are equivalent:*

(1) Γ *is admissible in the sense of Definition 1.0.4;*
(2) the reflection F *preserves pullbacks of primitive extensions along the unit morphisms;*
(3) the reflection F *preserves pullbacks of primitive extensions along any morphism.*

Proof. See **[71]**; the second condition implies the third one because any pullback of x in $\mathcal{X}$ and f in $\mathcal{C}$ can be decomposed as the composite of two pullbacks which are both preserved by F (using the fact that $\mathcal{X}$ is closed under pullbacks in $\mathcal{C}$ – Remark 1.0.2):

$$\begin{tikzcd} P \arrow[r, "\pi_X"] \arrow[d, "\pi_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "x"] \\ E \arrow[r, "f"'] & Y \end{tikzcd} \quad \text{factors into} \quad \begin{tikzcd} P \arrow[r] \arrow[d, "\pi_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bar{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "x"] \\ E \arrow[r, "\eta_E"'] & F(E) \arrow[r, "\bar{f}"'] & Y. \end{tikzcd}$$ ☐

Using this characterization, another easy consequence of admissibility is the pullback stability of trivial coverings. Given a morphism $f\colon E' \to E$, the pullback functor $f^*\colon \mathsf{Ext}(E) \to \mathsf{Ext}(E')$ is defined on an object $e\colon T \to E$ by $f^*(e) := \pi_{E'}\colon E' \times_E T \to E'$ where $\pi_{E'}$ is the pullback of e along f. The definition of f^* on morphisms follows by the universal property of these pullbacks along f.

Proposition 1.0.6. *Given a morphism* $f\colon E' \to E$*, we have a pullback functor* $f^*\colon \mathsf{TExt}(E) \to \mathsf{TExt}(E')$ *sending trivial coverings to trivial coverings.*

Proof. See **[71]**; use the naturality of η and the fact that F preserves pullbacks of primitive extensions. ☐

The pullback stability of normal coverings and coverings then follows easily. Write $\mathsf{CExt}(B)$ for the full subcategory of $\mathsf{Ext}(B)$ whose objects are coverings (sometimes called central extensions).

COROLLARY 1.0.7. *Given a morphism* $g\colon B' \to B$*, we have a pullback functor* $g^*\colon \mathsf{CExt}(B) \to \mathsf{CExt}(B')$ *sending coverings above* B *to coverings above* B'*. Similarly for normal coverings.*

PROOF. See [**71**]; use the pullback stability of extensions and trivial coverings. ☐

As we will discuss below, we are not only interested in admissible Galois structures, but in towers of admissible Galois structures of arbitrary dimension. We thus actually work with another property of Galois structures which is stronger than admissibility – see Section 1.0.10.

1.0.8. *The fundamental theorem of categorical Galois theory.* In order to state the classification theorem for coverings, we recall the concept of *internal groupoid.* In the category Set, an (internal) groupoid is the data of a small category such that all its morphisms are isomorphisms. More generally in any category $\mathcal{C}$ (with enough pullbacks), an internal groupoid $\mathcal{G}$ in $\mathcal{C}$ is given by:

$$\begin{tikzcd}
G_2 \arrow[r, shift left=3, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=3, "p_2"'] & G_1 \arrow[loop above, "-1"] \arrow[r, shift left=3, "c"] \arrow[r, shift right=3, "d"'] & G_0 \arrow[l, "i" description]
\end{tikzcd}$$

- an object *of objects* G_0 in $\mathcal{C}$;
- an object *of morphisms* G_1 in $\mathcal{C}$;
- a *domain* morphism and a *codomain* morphism $c, d\colon G_1 \rightrightarrows G_0$;
- an *identity* morphism $i\colon G_0 \to G_1$ which is the splitting of c and d (i.e. such that $di = ci = \mathrm{id}_{G_0}$), making the preceding data into a reflexive graph in $\mathcal{C}$;
- an object *of composable morphisms* $G_2 \coloneqq G_1 \times_{G_0} G_1$ which is the pullback of c and d with projections $\pi_1, \pi_2\colon G_2 \rightrightarrows G_1$ such that in particular $c\pi_1 = d\pi_2$;
- an *inverse* morphism $(-1)\colon G_1 \to G_1$ such that $c(-1) = d$, and $d(-1) = c$;
- and finally a composition morphism $m\colon G_2 \to G_1$, such that the following diagrams commute in $\mathcal{C}$:

- associativity of composition:

$$\begin{tikzcd}
G_1 \times_{G_0} G_1 \times_{G_0} G_1 \arrow[r, "\mathrm{id}_{G_1} \times_{G_0} m"] \arrow[d, "m \times_{G_0} \mathrm{id}_{G_1}"'] & G_2 = G_1 \times_{G_0} G_1 \arrow[d, "m"] \\
G_2 = G_1 \times_{G_0} G_1 \arrow[r, "m"'] & G_1
\end{tikzcd}$$

- domain and codomain of the composite of two morphisms:

$$\begin{tikzcd}
G_2 \arrow[r, "\pi_1"] \arrow[d, "m"'] & G_1 \arrow[d, "d"] \\
G_1 \arrow[r, "d"'] & G_0
\end{tikzcd}
\qquad
\begin{tikzcd}
G_2 \arrow[r, "\pi_2"] \arrow[d, "m"'] & G_1 \arrow[d, "c"] \\
G_1 \arrow[r, "c"'] & G_0
\end{tikzcd}$$

- unit laws for composition (where $G_0 \times_{G_0} G_1$ is the pullback of id_{G_0} and d; and $G_1 \times_{G_0} G_0$ is the pullback of c and id_{G_0}):

$$\begin{tikzcd}
G_0 \times_{G_0} G_1 \arrow[r, "i \times_{G_0} \mathrm{id}_{G_1}"] \arrow[dr, "\pi_{G_1}"'] & G_2 \arrow[d, "m"] & G_1 \times_{G_0} G_0 \arrow[l, "\mathrm{id}_{G_1} \times_{G_0} i"'] \arrow[dl, "\pi_{G_1}"] \\
 & G_1 &
\end{tikzcd}$$

- (-1) produces the inverse of a morphism;

$$\begin{tikzcd}
G_1 \arrow[r, "\langle (-1), \mathrm{id}_{G_1} \rangle"] \arrow[d, "c"'] & G_2 \arrow[d, "m"] \\
G_0 \arrow[r, "i"'] & G_1
\end{tikzcd}
\qquad
\begin{tikzcd}
G_1 \arrow[r, "\langle \mathrm{id}_{G_1}, (-1) \rangle"] \arrow[d, "d"'] & G_2 \arrow[d, "m"] \\
G_0 \arrow[r, "i"'] & G_1
\end{tikzcd}$$

  where the morphisms $\langle (-1), \mathrm{id}_{G_1} \rangle$ and $\langle \mathrm{id}_{G_1}, (-1) \rangle$ are the morphisms induced by (-1) and id_{G_1} from G_1 to the pullback G_2 of c and d.

Recall moreover that the kernel pair $p_1, p_2 \colon \mathrm{Eq}(f) \rightrightarrows A$ of a morphism $f \colon A \to B$ in $\mathcal{C}$ determines an internal groupoid $\mathcal{G}(f)$ in $\mathcal{C}$ (with the same underlying reflexive graph) such that A is its objects of objects, $\mathrm{Eq}(f)$ is its object of morphisms, the multiplication of internal morphisms is obtained by transitivity of $\mathrm{Eq}(f)$ and the inverse of an internal morphism $(x, y) \in Eq(f)$ is obtained by symmetry $(y, x) \in \mathrm{Eq}(f)$.

Given an admissible Galois structure Γ, a normal Γ-covering $n \colon A \to B$ (see Conventions 1.0.1) is such that the image $\mathcal{G}$ of the internal groupoid $\mathcal{G}(n)$ (in $\mathcal{C}$) by the reflector $F \colon \mathcal{C} \to \mathcal{X}$ is an internal groupoid in $\mathcal{X}$. This groupoid $\mathcal{G}$ is called the *Galois groupoid* of n. When n is not a normal extension, but merely an extension, the image $\mathcal{G}$ of the internal groupoid

$\mathcal{G}(n)$ (in $\mathcal{C}$) by the reflector $F\colon \mathcal{C} \to \mathcal{X}$ is an *internal pregroupoid* (see for instance [**71**]).

The *fundamental theorem of categorical Galois theory* then says that *internal presheaves* over that (pre)groupoid $\mathcal{G}$ (think "groupoid actions in $\mathcal{X}$") yield a category which is equivalent to the category of those extensions above B which are split by n. Internal groupoids and internal actions are well explained in [**81**], a standard reference for the use of groupoids is [**17**]. In certain contexts, the category of *internal presheaves* over $\mathcal{G}$ is easy to describe, keeping in mind that the category $\mathcal{X}$ is well understood. This theorem can then be used to easily classify those extensions which are split by n.

In order to classify all Γ-coverings above an object B, it is convenient to work with extensions $n\colon A \to B$ that split any covering above B. When *projective presentations* exist, such an extension n can be easily obtained as a projective presentations of B as in Section 1.0.11. These projective presentations are not normal coverings in general (and thus require the concept of a Galois *pre*groupoid as we mentioned earlier). In varieties of algebras, one can *centralize* a projective presentation of an object B in order to obtain a *weakly universal covering* of B as we describe in Paragraph 1.0.11. In general, and in order to obtain a normal covering which splits any covering of its codomain B, it is convenient to work with objects B above which there exists a *weakly universal covering*, i.e. a covering $e\colon E \to B$ which factors through any other covering $c\colon A \to B$ above B (see Diagram (11) and Section 1.0.11).

$$\begin{tikzcd}
E \arrow[dd, "e"'] \arrow[dr, dotted, "\exists \text{ factorization}"] & \\
& A \arrow[dl, "c"] \\
B &
\end{tikzcd} \tag{11}$$

In certain contexts, such a weakly universal covering $e\colon E \to B$ splits any other covering above B. It is the case in Γ_T [**9**, Section 6.6-7], Γ_G [**9**, Proposition 5.2.9], when coverings and normal coverings coincide [**71**, Section 4.3], but also more generally as we will see in Proposition 3.5.3. In these contexts, a weakly universal cover is split by itself, and thus it is actually a normal covering. Its Galois groupoid $\mathcal{G}$ is then called the *fundamental groupoid* of B. Since two weakly universal covers above an object B factor through each other, they induce an *equivalence* between the corresponding fundamental groupoids of B. The concept of fundamental groupoid is thus independent (up to equivalence) from the aforementioned choice of weakly universal cover. Note that the conditions – connectedness, local path-connectedness and semi-local simply-connectedness – on the space X, in the classical Galois correspondence for topological coverings [**62**, Theorem 1.38], are there to guarantee the existence of a weakly universal covering above X [**9**, Section 6.6-8].

In the context of Γ_G for instance, any weakly universal covering of an object B splits any other covering above B. The reader can check this by using the fact that central extensions and normal extensions coincide in this context. As we did in the introduction, let us consider the easier case when B is a perfect group. Recall that a *perfect group* is a group whose abelianization $B/[B,B]_{\mathsf{Grp}}$ (which from the point of view of homological algebra is the first homology group with integer coefficients $H_1(B,\mathbb{Z})$) is trivial, i.e. $H_1(B,Z) \coloneqq B/[B,B]_{\mathsf{Grp}} = \{e\}$. A perfect group B admits a *universal central extension* $e\colon E \to B$, for which there is a *unique* factorization through any other central extension $c\colon A \to B$ as in Diagram (11). The fundamental Galois groupoid of B is then equivalent to the abelian group given by the kernel of e, which also describes the *second homology group with integer coefficients* $H_2(B,\mathbb{Z})$. The fundamental theorem of categorical Galois theory then gives an equivalence between the central extensions above B and the slice category whose objects are the morphisms with codomain $H_2(B,\mathbb{Z})$ in the category of abelian groups Ab. In other words, the *second integral homology group* of B can be *presented* as a "Galois group" (see [**65**, Remark 5.4], [**9**, Section 5.2] and [**69**]).

This example hints at how the fundamental theorem of categorical Galois theory can be used to better understand the classical cohomology theories for central extensions of groups and the other classical contexts described in [**71**, Section 1.5]. A variety of homological and cohomological results in new contexts relies on this perspective brought by a *general notion of central extension* and *pure Galois theory in categories* [**65, 67, 71, 68, 50, 45, 48, 35, 49, 99, 37, 101**]. Most of these generalizations rely on *higher-dimensional* categorical Galois theory which we introduce in Paragraph 1.0.9. The main objective of this thesis is to apply this higher-dimensional perspective in the context of racks and quandles, with potential applications to cohomology and homology (and also homotopy) theories in this context ([**56, 28**], see also [**38**, Section 9]).

1.0.9. *Higher-dimensional applications.* From the example in the category of groups, and the aforementioned observation about links with homology, the development of Galois theory led for instance to a generalisation [**50**] of the Hopf formulae for the (integral) homology of groups [**18**] to other non-abelian settings, leading to a whole new approach to non-abelian homology, by the means of higher central extensions [**67, 50, 45**]. This approach is compatible with settings such as the cotriple homology of Barr and Beck [**3, 51**], including, for instance, group homology with coefficients in the cyclic groups $\mathbb{Z}_n$. In order to access the relevant higher-dimensional information, as in [**50**], one actually "iterates" categorical Galois theory. The increase in dimension consists in shifting from the context of $\mathcal{C}$ to the *category of extensions* of $\mathcal{C}$: $\mathsf{Ext}\mathcal{C}$ defined as the full subcategory of the arrow category $\mathsf{Arr}\mathcal{C}$ with objects being extensions. A morphism $\alpha\colon f_A \to f_B$ in such a category of morphisms is given by a pair of morphisms in $\mathcal{C}$, which we denote $\alpha = (\alpha_\top, \alpha_\bot)$ (the *top* and *bottom components* of α), such that these form an (oriented) commutative square (on the left).

$$\begin{tikzcd}
A_\top \arrow[r, "\alpha_\top"] \arrow[d, "f_A"'] \arrow[dr, phantom, "(\rightarrow)"] & B_\top \arrow[d, "f_B"] \\
A_\bot \arrow[r, "\alpha_\bot"'] & B_\bot
\end{tikzcd}$$

$$\begin{tikzcd}
A_\top \arrow[rr, "\alpha_\top"] \arrow[dr, "p"] \arrow[dd, "f_A"'] & & B_\top \arrow[dd, "f_B"] \\
& P \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\
A_\bot \arrow[rr, "\alpha_\bot"'] & & B_\bot
\end{tikzcd}$$

We call the *comparison map* of such a morphism (or commutative square) the unique map $p\colon A_\top \to P$ induced by the universal property of $P \coloneqq A_\bot \times_{B_\bot} B_\top$, the pullback of $\alpha_\bot$ and f_B. Now from the study of the *admissible adjunction* $F \dashv \mathrm{I}$ (within the Galois structure Γ), Galois theory produces the concept of a Γ-covering (central extension), and thus we may look at the *full subcategory* $\mathsf{CExt}\mathcal{C}$ of $\mathsf{Ext}\mathcal{C}$ whose objects are Γ-coverings. The category of Γ-coverings $\mathsf{CExt}\mathcal{C}$ is not reflective, even less so admissible, in the category of extensions $\mathsf{Ext}\mathcal{C}$ in general (see [**72**]). In groups, extensions can be universally *centralized*, along a quotient of their domain, and the category of central extensions of groups $\mathsf{CExtGrp}$ is actually a full replete *(regular epi)-reflective* subcategory of ExtGrp. The *centralization functor* $\mathrm{ab}^1\colon \mathsf{ExtGrp} \to \mathsf{CExtGrp}$ sends a surjective group homomorphism $f\colon G \to H$ to the central extension of groups $\mathrm{ab}^1(f)\colon G/[\mathrm{Ker}\, f, G]_{\mathsf{Grp}} \to H$ obtained from the quotient of the domain A of f: $G/[\mathrm{Ker}\, f, G]_{\mathsf{Grp}}$ where $\mathrm{Ker}\, f$ is the *kernel of* f.

When such a reflection exists, one may further wonder whether there is a Galois structure behind it, and whether it is admissible. What is the "sphere of influence" of central extensions in extensions, and with respect to which class of *extensions of extensions*, i.e. can we re-instantiate Galois theory in this induced (two-dimensional) context?

An appropriate class of morphisms to work with, in order to obtain an admissible Galois structure in such a two-dimensional setting, is the class of *double extensions* (see for instance [**67, 60, 52, 45**]). A *double extension* is a morphism $\alpha = (\alpha_{\top}, \alpha_{\bot})$ in $\mathsf{Ext}\mathcal{C}$ such that both $\alpha_{\top}$ and $\alpha_{\bot}$ are extensions and the comparison map of α is also an extension. Note that double extensions are indeed a subclass of regular epimorphisms in $\mathsf{Ext}\mathcal{C}$, provided $\mathcal{C}$ is a regular category (see [**4**]). *Double central extensions of groups* were described in [**67**], and higher-dimensional Galois theory developed further [**68, 50, 45**], leading to the aforementioned results in homology and cohomology.

Similarly in topology, higher homotopical information of spaces can be studied via the higher fundamental groupoids in the higher-dimensional Galois theory of locally connected topological spaces. A detailed survey about the study of higher-dimensional homotopy group(oid)s can be found in [**16**], see also [**19**]. Some insights are given in [**21**] where higher Galois theory is used to build a homotopy double groupoid for maps of spaces (see also [**20**]).

In Part I we consolidate the understanding of the 1-dimensional covering theory of racks and quandles, and introduce all the necessary ideas to start a higher-dimensional Galois theory in this context. In Part II we obtain an admissible Galois structure Γ^1 for the inclusion of the category of rack and quandle coverings in the category of extensions; we define and study double coverings of racks and quandles, which are shown to describe the Γ^1-coverings or say *double central extensions of racks and quandles* as in [**67**]. In Part III (which is not included in this thesis) we generalize to arbitrary dimensions.

1.0.10. *Admissibility via the strong Birkhoff condition, in two steps.* Note that in the literature, most instantiations of higher categorical Galois theory are such that the "base" category $\mathcal{C}$ is a *Mal'tsev category* (with the exception of [**20, 21**]), and such that moreover all the induced higher-dimensional categories of extensions ($\mathsf{Ext}\mathcal{C}$, $\mathsf{ExtExt}\mathcal{C}$, and so on) are also Mal'tsev categories (see [**26, 27, 25**]). Admissibility conditions as well as computations with higher extensions are easier to handle in such a context. The categories we are interested in are not Mal'tsev categories. Showing how higher categorical Galois theory can apply in this more general setting thus requires some refinements on the arguments which are used in the existing examples – see for instance [**40**] for the most general example of *a tower of admissible adjunctions* known to the author.

The difficulty is in the induction for higher dimensions: the study of a given Galois structure is one thing, the study of which properties of a Galois structure induce good properties of the subsequent Galois structures in higher dimensions, is another. In this thesis, we present the (≤ 2)-dimensional covering theory of racks and quandles in such a way that can be generalized to arbitrary dimensions. This generalization to arbitrary dimensions will be detailed in a separate article [**97**] (in preparation and not included in this thesis), which is the continuation of [**95, 96**]. Let us sketch here, without technical details, what are these key ingredients in lower dimensions that are generalized to higher dimensions.

In Part I, our context is that of [**71**] which we refer to for more details. We look at the inclusion $\mathrm{I}\colon \mathcal{X} \to \mathcal{C}$ of $\mathcal{X}$, a *full*, *(regular epi)-reflective subcategory* of a *finitely cocomplete Barr-exact* category $\mathcal{C}$, such that $\mathcal{X}$ is *closed under isomorphisms* and *quotients*. In short *Barr exactness* means that $\mathcal{C}$ has finite limits; every morphism factors uniquely, up to isomorphism, into a regular epimorphism, followed by a monomorphism, and these factorizations are stable under pullbacks; and, moreover, every *equivalence relation* is the kernel pair of its *coequalizer* [**2**]. Here *(regular epi)-reflectiveness* refers to the fact that the unit η of the adjunction $F \dashv \mathrm{I}$ (with left adjoint $F\colon \mathcal{C} \to \mathcal{X}$) is a regular epimorphism (surjection), as in Convention 1.0.1. As we mentioned in remark 1.0.2, $\mathcal{X}$ is thus also closed under subobjects and finite limits in $\mathcal{C}$.

The fact that $\mathcal{X}$ is closed under quotients is then the remaining condition for $\mathcal{X}$ to be called a *Birkhoff subcategory* of $\mathcal{C}$ [**23, 71**]. Given a more general Galois structure $\Gamma = (\mathcal{C}, \mathcal{X}, F, \mathrm{I}, \eta, \epsilon, \mathcal{E})$ as in Convention 1.0.1, we say that Γ is *Birkhoff* if $\mathcal{X}$ is closed in $\mathcal{C}$ under quotients along extensions. In the Galois structures of interest (see for instance [**71**]), this condition is shown to be equivalent to the fact that the *reflection squares* of extensions are *pushouts*. Given $f\colon A \to B$ in $\mathcal{C}$, *the reflection square at f* (with respect to Γ) is the morphism (η_A, η_B) with domain f and codomain $\mathrm{I}F(f)$ in $\mathsf{Arr}(\mathcal{C})$. Finally, $\mathcal{X}$ is said to be *strongly Birkhoff* in $\mathcal{C}$ if moreover these reflection squares of extensions are themselves double extensions.

$$\begin{tikzcd}
A \arrow[rr, "\eta_A"] \arrow[dd, "f"'] \arrow[dr, "p" description] & & \mathrm{I}\,F(A) \arrow[dd, "\mathrm{I}\,F(f)" description] \\
& P \arrow[ur, "\pi_2" description] \arrow[dl, "\pi_1" description] & \\
B \arrow[rr, "\eta_B"'] & & \mathrm{I}\,F(B)
\end{tikzcd} \tag{12}$$

Proposition 2.6 in [**50**] implies that if Γ is *strongly Birkhoff*, then it is in particular admissible.

Now observe that in the Barr-exact context from above, Proposition 5.4 in [**25**] implies that if Γ is Birkhoff, it is strongly Birkhoff if and only if, for any object A in $\mathcal{C}$, the kernel pair of η_A *commutes* (in the sense of the composition of relations) with any other equivalence relation on A (see [**90, 25**]). For instance, in the category of groups, any two equivalence relations commute with each other (see *Mal'tsev categories* [**25**]). Hence since Ab is a Birkhoff subcategory of Grp, it is actually strongly Birkhoff in Grp, which implies the *admissibility* of $\mathrm{ab} \dashv \mathrm{I}$ (see [**71**, Theorem 3.4]). However, working in a Mal'tsev category is not necessary, as it was already known (see for instance [**71**]), and observed again by V. Even in [**39**] and [**40**], where he uses the permutability property of the kernel pairs of unit morphisms to conclude the admissibility of his Galois structure. In Part I, we briefly re-discuss these results and illustrate the argument on a new adjunction. In higher dimensions, we shall also aim to obtain strongly Birkhoff Galois structures by splitting the work in two steps: (1) closure by quotients along higher extensions and (2) the permutability condition on the kernel pairs of (the non-trivial component of) the unit morphisms.

1.0.11. *Splitting along projective presentations and weakly universal covers.* Remember that in any category, an object E is *projective* – with respect to a given class of morphisms, which we always take to be our extensions – if for any extension $f\colon A \twoheadrightarrow B$ and any morphism $p\colon E \to B$, there exists a factorization of p through f i.e. $g\colon E \to A$ such that $f \circ g = p$. A projective presentation of an object B is then given by an extension $p\colon E \twoheadrightarrow B$ such that E is projective (with respect to extensions). For instance, in *varieties of algebras* (in the sense of universal algebra [**23**]), there are *enough projectives*, in particular each object has a canonical projective presentation given by the *counit* of the "free-forgetful" *monadic* adjunction with sets [**89**]. Given a group B in Grp for instance, the counit morphism $\epsilon^g_B\colon \mathrm{F}_{\mathrm{g}}(B) \to B$ with domain the free group $\mathrm{F}_{\mathrm{g}}(B)$ on the underlying set of B, is the canonical projective presentation of B. It sends an element $g \in \mathrm{F}_{\mathrm{g}}(B)$, represented by the formal word $b_1 b_2 \cdots b_n$ (for some elements b_1, b_2, $\ldots$, $b_n \in B$), to the corresponding product in B. The free object $\mathrm{F}_{\mathrm{g}}(B)$ is a projective object in Grp as one can easily deduce from the universal property of $\eta_B\colon B \to \mathrm{F}_{\mathrm{g}}(B)$ in Set and the fact that (by the axiom of choice) any surjective morphism of groups $f\colon A \to B$ admits a splitting $s\colon B \to A$ in the category of sets.

In the Galois structures Γ we assume that the base category $\mathcal{C}$ has *enough projectives*. Then any Γ-covering (central extension) f is in particular split by any projective presentation p of its codomain. We have the following diagram

$$\begin{tikzcd}
E \times_B A \arrow[rr, "p_A"] \arrow[dd, "p_E"'] \arrow[dr, dotted, "t"'] & & A \arrow[dd, "f"] \\
& T \times_B A \arrow[ur] \arrow[dd, "p_T"] & \\
E \arrow[rr, "p"] \arrow[dr, dashed, "p'"'] & & B \\
& T \arrow[ur] &
\end{tikzcd} \tag{13}$$

where p' is induced by E being projective, t is induced by the universal property of $T \times_B A$ and p_T is a trivial covering by assumption. Then with no assumptions on $\mathcal{C}$, the left hand face is a pullback since the back face and the right hand face are. Assuming that the Galois structure Γ is admissible, trivial coverings, normal coverings and coverings are pullback stable (see Paragraph 1.0.3), and thus p_E is a trivial covering, since it is the pullback of a trivial covering. Hence if $\mathcal{C}$ has enough projectives, then for any object B in $\mathcal{C}$ the category of coverings (central extensions) $\mathsf{CExt}(B)$ above B is the same as the category of those extensions which are split by one given morphism such as the foregoing projective presentation p of B. Such a projective presentation $p\colon E \to B$ is not a normal covering in general, however, the classification of Γ-coverings above B can still be obtained using its (fundamental) Galois pregroupoid instead of its Galois groupoid (see Paragraph 1.0.8).

In the contexts of interest, the category of coverings (central extensions) is reflective in the category of extensions, and thus a weakly universal covering of an object B can always be obtained from the *centralization* of a projective presentation of B. One can for example recover this idea from [**92**]. Consider a group B in Grp, and its canonical projective presentation $\epsilon^g_B\colon \mathrm{F}_{\mathrm{g}}(B) \to B$ described above. The image of

ϵ^g_B by the centralization functor $\mathrm{ab}^1\colon \mathsf{ExtGrp} \to \mathsf{CExtGrp}$ is the central extension of groups $\mathrm{ab}^1(\epsilon^g_B)\colon\ \mathrm{F_g}(B)/[\mathrm{Ker}\,\epsilon^g_B, \mathrm{F_g}(B)]_{\mathsf{Grp}} \to B$ obtained from the quotient $\mathrm{F_g}(B)/[\mathrm{Ker}\,\epsilon^g_B, \mathrm{F_g}(B)]_{\mathsf{Grp}}$ of the domain of ϵ^g_B. Note that (as before) the kernel of this map can be computed to be $\mathrm{Ker}\,\epsilon^g_B \cap [\mathrm{F_g}(B), \mathrm{F_g}(B)]_{\mathsf{Grp}}/[\mathrm{Ker}\,\epsilon^g_B, \mathrm{F_g}(B)]_{\mathsf{Grp}}$ which is the *second integral homology group* of B (see [**69**]).

In our "general" Galois structure Γ such that $\mathsf{CExt}\mathcal{C}$ is reflective in $\mathsf{Ext}\mathcal{C}$, consider an extension $f\colon A \to B$, and the *centralization* (i.e. the reflection in $\mathsf{CExt}\mathcal{C}$) of a projective presentation of B.

$$\begin{tikzcd}
 & E \arrow[dl, "\text{centralization unit}"'] \arrow[dd, "p"] \arrow[dr, dashed, "a"] & \\
E' \arrow[dr, "p'"'] \arrow[rr, dotted, bend right, "b"] & & A \arrow[dl, "f"] \\
 & B &
\end{tikzcd}$$

We get a since E is projective and b by the universal property of p'. We will see that in the contexts of interest, a covering (central extension) is necessarily split by each weakly universal covering of its codomain (Proposition 3.5.3). Such weakly universal covers above an object B are then split by themselves which makes them normal coverings. The reflection of the kernel pair of such is then the fundamental Galois groupoid of B, which classifies all coverings (central extensions) above B.

1.0.12. *General strategy for characterizing central extensions.* Finally we describe our general strategy, suggested by G. Janelidze, when it comes to identifying a property which characterizes the Γ-coverings (central extensions), given an admissible Galois structure Γ (as in Convention 1.0.1) such that $\mathcal{C}$ has enough projectives. Observe that if a covering f is split by a split epimorphism p, then it is a trivial covering.

$$\begin{tikzcd}
A \arrow[rrrr, equal] \arrow[drr, hook, "\bar{s}"'] \arrow[dd, "f"'] & & & & A \arrow[dd, "f"] \\
 & & E \times_B A \arrow[urr, "\bar{p}"'] \arrow[dd, "\overline{f}"'] & & \\
B \arrow[rrrr, equal] \arrow[drr, hook, "s"'] & & & & B \\
 & & E \arrow[urr, "p"'] & &
\end{tikzcd}$$

Indeed, if the pullback $\overline{f}$ of f along p is a trivial covering by assumption, then the pullback of $\overline{f}$ along the splitting s of p is again a trivial covering and isomorphic to f. As a consequence, split epimorphic normal coverings are trivial coverings. Also, those coverings that have a projective codomain are trivial coverings. Now suppose one has identified a special class of extensions, called *candidate-coverings*, such that candidate-coverings are preserved and reflected by pullbacks along extensions. Provided primitive extensions are candidate-coverings, then all trivial coverings are candidate-coverings and also Γ-coverings are. Moreover, given a candidate-covering $f\colon A \to B$, pulling back f along a projective presentation p of B yields a candidate-covering with projective codomain. Since f is a covering if and only if it is split by such a p, we see that candidate-coverings are coverings (central extensions) if and only if all candidate-coverings with projective codomains are actually trivial coverings, which is usually easier to check.

2. An introduction to racks and quandles

We introduce all the ingredients of the theory of racks and quandles needed for this work, which we describe and develop from the perspective inspired by the covering theory of interest.

2.1. Axioms and basic concepts.

2.1.1. *Racks and quandles as a system of symmetries.* Symmetry is classically modeled or studied using groups. Informally speaking: given a space X, one studies the group of automorphisms $\mathrm{Aut}(X)$ of X. In his PhD thesis [**82**], D.E. Joyce describes quandles as another algebraic approach to symmetry such that, locally, each point x in a space X would be equipped with a global symmetry S_x of the space X. Groups themselves always come with such a system of symmetries given by conjugation and the definition of inner automorphisms. Quandles, and more primitively racks, can be seen as an algebraic generalisation of such.

2.1.2. *Describing the algebraic axioms.* Consider a set X that comes equipped with two functions

$$\begin{tikzcd} X \arrow[r, shift left, "\mathrm{S}"] \arrow[r, shift right, "\mathrm{S}^{-1}"'] & X^X, \end{tikzcd}$$

which assign functions S_x and S_x^{-1} in X^X (the set of functions from X to X) to each element x in X. Each element x then acts on any other y in X via those functions S_x and S_x^{-1}. By convention we shall always write actions on the right:

$$y \cdot \mathrm{S}_x \coloneqq \mathrm{S}_x(y) \qquad\qquad y \cdot \mathrm{S}_x^{-1} \coloneqq \mathrm{S}_x^{-1}(y)$$

The functions S_x and S_x^{-1} at a given point $x \in X$ are required to be inverses of one another, in particular for all y in X we have

$$(y \cdot \mathrm{S}_x^{-1}) \cdot \mathrm{S}_x = y = (y \cdot \mathrm{S}_x) \cdot \mathrm{S}_x^{-1} .$$

Note that, under this assumption, S^{-1} and S determine each other. Now we want to call such bijections S_x *symmetries* (or *inner automorphisms*) of X. But observe that the set X is now equipped with two binary operations

$$\begin{tikzcd} X \times X \arrow[r, shift left, "\triangleleft"] \arrow[r, shift right, "\triangleleft^{-1}"'] & X, \end{tikzcd}$$

defined by $x \triangleleft y := x \cdot \mathrm{S}_y$ and $x \triangleleft^{-1} y := x \cdot \mathrm{S}_y^{-1}$ for each x and y in X. Read "y acts on x (positively or negatively)". Automorphisms of X should then preserve these operations. In particular we thus require that for each x, y and z in X:

$$(x \triangleleft y) \triangleleft z = (x \triangleleft y) \cdot \mathrm{S}_z = (x \cdot \mathrm{S}_z) \triangleleft (y \cdot \mathrm{S}_z) = (x \triangleleft z) \triangleleft (y \triangleleft z).$$

2.1.3. *Defining a rack.* Any set X equipped with such structure, i.e. two binary operations $\triangleleft$ and $\triangleleft^{-1}$ on X such that for all x, y and z in X:

(R1) $(x \triangleleft y) \triangleleft^{-1} y = x = (x \triangleleft^{-1} y) \triangleleft y$;
(R2) $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$;

is called a *rack* (as we already mentioned in the introduction). We write Rck for the category of racks with rack homomorphisms defined as usual (functions preserving the operations).

We refer to the axiom (R2) as *self-distributivity.* For each x in X, the *positive* (resp. *negative*) *symmetry at* x is the automorphism S_x (resp. S_x^{-1}) defined before. A *symmetry*, also called *right-translation*, of X is S_x or S_x^{-1} for some x in X. The *symmetries* of X refers to the set of those.

2.1.4. *Racks from group conjugation.* One crucial class of examples is given by group conjugation. D.E. Joyce describes quandles as "the algebraic theory of conjugation" [**82**]. We have the functor:

$$\mathsf{Grp} \xrightarrow{\mathrm{Conj}} \mathsf{Rck} ,$$

which sends a group G to the rack $\mathrm{Conj}(G)$ with same underlying set, and whose rack operations are defined by conjugation:

$$x \triangleleft a := a^{-1}xa \text{ and } x \triangleleft^{-1} a := axa^{-1},$$

for a and x in G. Group homomorphisms are sent to rack homomorphisms by just keeping the same underlying function. The forgetful functor $\mathrm{U}\colon \mathsf{Grp} \to \mathsf{Set}$ thus factors through $\mathrm{U}\colon \mathsf{Rck} \to \mathsf{Set}$ via Conj. However the functor Conj is not *full*, since given groups G and H, there are more rack homomorphisms between $\mathrm{Conj}(G)$ and $\mathrm{Conj}(H)$ than there are group homomorphisms between G and H.

This peculiar "inclusion" functor consists in "forgetting an operation" in comparison with subvarieties which are about "adding an equation". When forgetting an operation, an obvious question is to ask: what equations should the remaining operations satisfy? Racks form one candidate theory. We will see that quandles (Subsection 2.1.10) give another option. In which sense is one different/better than the other? Can we characterize (as a subcategory) those racks which arise from groups? An important ingredient for answering those questions and understanding the relationship between groups, racks and quandles is the left adjoint of Conj (Subsection 2.4). The thorough study and understanding of this left adjoint (first defined by D.E. Joyce who denoted it *Adconj*, referred to as *Adj* in [**38**]) is central to this piece of work, also with respect to its crucial role in the covering theory of racks and quandles.

In what follows, we often consider groups as racks without necessarily mentioning the functor Conj.

2.1.5. *Other identities.* Note that for the symmetries S_x to define automorphisms of racks, one needs distributivity of $\triangleleft$ on $\triangleleft^{-1}$, distributivity of $\triangleleft^{-1}$ on $\triangleleft$, and self-distributivity of $\triangleleft^{-1}$. All these identities are induced by the chosen axioms. Besides, it suffices for a function f to preserve one of the operations in order for it to preserve the other. We do not give a detailed survey of rack identities here. Bear in mind that in the theory of racks, the roles of $\triangleleft$ and $\triangleleft^{-1}$ are interchangeable. Swapping them in a given equation, gives again a valid equation. Finally we focus on an important characterization of (R2) using (R1):

2.1.6. *Self-distributivity.*

LEMMA 2.1.7. *Under the axiom* (R1)*, the axiom* (R2) *is equivalent to*

(R2') $x \triangleleft (y \triangleleft z) = ((x \triangleleft^{-1} z) \triangleleft y) \triangleleft z.$

PROOF. Given (R1), we formally show that

$$\begin{aligned}\text{(R2)} \Rightarrow \text{(R2')}:\quad x \triangleleft (y \triangleleft z) &= ((x \triangleleft^{-1} z) \triangleleft z) \triangleleft (y \triangleleft z) && \text{(by (R1))}\\ &= ((x \triangleleft^{-1} z) \triangleleft y) \triangleleft z && \text{(by (R2))}\end{aligned}$$

$$\begin{aligned}\text{(R2')} \Rightarrow \text{(R2)}:\ (x \triangleleft z) \triangleleft (y \triangleleft z) &= (((x \triangleleft z) \triangleleft^{-1} z) \triangleleft y) \triangleleft z && \text{(by (R2'))}\\ &= (x \triangleleft y) \triangleleft z && \text{(by (R1))}\ \square\end{aligned}$$

Similarly (R2) is also equivalent to (R2"): $x \triangleleft (y \triangleleft^{-1} z) = ((x \triangleleft z) \triangleleft y) \triangleleft^{-1} z$. From the preceding discussion we also have

$$x \triangleleft^{-1} (y \triangleleft^{-1} z) = ((x \triangleleft z) \triangleleft^{-1} y) \triangleleft^{-1} z,$$

and finally

$$x \triangleleft^{-1} (y \triangleleft z) = ((x \triangleleft z) \triangleleft y) \triangleleft^{-1} z.$$

Considering these as identities between *formal terms in the language of racks* (see for instance Chapter II, Section 10 in [**23**]), we say that the term on the right-hand side is *unfolded*, whereas the term on the left hand side isn't.

2.1.8. *Composing symmetries – inner automorphisms.* By construction (see Paragraph 2.1.5), given a rack X, the images of S and S^{-1} (defined as above) are in the group of automorphisms of X. Define the *group of inner automorphisms* as the subgroup $\mathrm{Inn}(X)$ of $\mathrm{Aut}(X)$ generated by the image of S. For each rack X, we then restrict S to the morphism

$$X \xrightarrow{\mathrm{S}} \mathrm{Inn}(X).$$

An inner automorphism is thus a composite of symmetries. Remember that we write our actions on the right, and thus we use the notation $z\cdot(\mathrm{S}_x \circ \mathrm{S}_y) := \mathrm{S}_y(\mathrm{S}_x(z))$ for x, y, and z in X, such that "$\circ$" means "before" rather than "after". We use the same notation S for different racks X and Y. Note that the construction of the group of inner automorphisms Inn does not define a functor from Rck to Grp. It does so when restricted to surjective morphisms (see for instance [**22**]).

Observe that if $z = x \triangleleft y$ in X, then $\mathrm{S}_z = \mathrm{S}_y^{-1} \circ \mathrm{S}_x \circ \mathrm{S}_y$ by self-distributivity (R2'). The function S is actually a rack homomorphism from X to $\mathrm{Conj}(\mathrm{Inn}(X))$. Again this describes a natural transformation in the restricted context of surjective morphisms.

Of course inner automorphisms of a group coincide with the inner automorphisms of the associated conjugation rack. However, observe that for a group G, a composite of symmetries is always a symmetry, whereas in a general rack, the composite of a sequence of symmetries does not always reduce to a one-step symmetry. Indeed, given a and b in a group G, then for all $x \in G$:

$$(x \lhd a) \lhd b = b^{-1}a^{-1}xab = x \lhd (ab) \quad \text{and, moreover,} \quad x \lhd^{-1} a = x \lhd a^{-1}.$$

So, given a group G, the morphism $G \xrightarrow{\mathrm{S}} \mathrm{Conj}(\mathrm{Inn}(G)) = \mathrm{Inn}(G)$ is surjective.

2.1.9. *Acting with inner automorphisms – representing sequences of symmetries.* Given a rack X, we have of course an action of $\mathrm{Inn}(X)$ on X given by evaluation. Elements of the group of inner automorphisms $\mathrm{Inn}(X)$ allow for a "representation" of successive applications of symmetries, seen as a composite of the automorphisms S_x.

More explicitly, any $g \in \mathrm{Inn}(X)$ decomposes as a product $g = \mathrm{S}_{x_1}^{\delta_n} \circ \cdots \circ \mathrm{S}_{x_n}^{\delta_1}$ for some elements $x_1, \ldots, x_n$ in X and exponents $\delta_1, \ldots, \delta_n$ in $\{-1, 1\}$. Such a decomposition is not necessarily unique, but for any x in X the action of g on x is well defined by

$$x \cdot g := x \cdot (\mathrm{S}_{x_1}^{\delta_n} \circ \cdots \circ \mathrm{S}_{x_n}^{\delta_1}) = x \lhd^{\delta_1} x_1 \lhd^{\delta_2} x_2 \cdots \lhd^{\delta_n} x_n,$$

where we omit parentheses using the convention that one should always compute the left-most operation first.

2.1.9.1. As we shall see, we need these successive applications of symmetries in order to study *connectedness* in racks. For our purposes, using the group of inner automorphisms for their study is not satisfactory. Note that given $x \neq y$ in a rack X, two symmetries S_x and S_y are identified in $\mathrm{Inn}(X)$ if they define the same automorphism. Motivated by the covering theories of interest, we study different ways to organize the *set of symmetries* $\{\mathrm{S}_x, \mathrm{S}_x^{-1}\}_{x \in X}$ into a group acting on X. Note that, for those who know the definition of *augmented quandles* or *augmented racks* (in the sense of [**82**], see also Paragraph 2.4.5), we may understand these as a tool to abstract away from "representing" sequences of symmetries via composites of such (in the sense of the group of inner automorphisms).

2.1.10. *Quandles, the idempotency axiom.* As explained in [**82**] by D.E. Joyce, it is reasonable (in reference to applications) to require that a symmetry at a given point fixes that point. If for each x in a rack X we have moreover that

(Q1) $x \triangleleft x = x$;

then X is called a *quandle*. We have the category of quandles Qnd defined as before. Again, (Q1) is equivalent to (Q1'): $x \triangleleft^{-1} x = x$, under the axiom (R1).

For the purpose of Part I, we shall mainly be working in the more general context of racks since these exhibit all the necessary features for the covering theory of interest. Actually all concepts of centrality and coverings remain the same whether one works with the category of racks or of quandles. Directions for a systematic conceptual understanding of these facts will be provided. The addition of the idempotency axiom still has certain consequences on ingredients of the theory such as the fundamental groupoid or the *homotopy classes of paths*. We shall always make explicit these differences and similarities, also using the enlightening study of the "free-forgetful" adjunction between racks and quandles.

2.1.11. *Idempotency in racks.* An essential observation to make is that, even though (Q1) does not hold in each rack, a weaker version of the idempotency axiom still holds in all racks as a consequence of self-distributivity. Indeed, racks and quandles are very close – which we shall illustrate throughout Part I. The axiom (Q1) requires the $\triangleleft$ operations to be idempotent: $x \triangleleft x = x$. Now observe that in a rack X, such identities can be deduced by self-distributivity in "the tail of a term": given any y and $x \in X$, we have

$$x \triangleleft (y \triangleleft y) = x \triangleleft^{-1} y \triangleleft y \triangleleft y = x \triangleleft y.$$

The symmetries S_y and $\mathrm{S}_{(y \triangleleft y)}$, at y and $y \triangleleft y$ are always identified in $\mathrm{Inn}(X)$, even when $y \neq (y \triangleleft y)$ in X. Similarly, for x and y in X any chain $y \triangleleft^k y$ (for $k \in \mathbb{Z}$, the action of y on y, repeated $|k|$ times – use $\triangleleft^{-1}$ when $k < 0$) is such that $x \triangleleft (y \triangleleft^k y) = x \triangleleft y$. For more details, the left adjoint ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}} \colon \mathsf{Rck} \to \mathsf{Qnd}$ to the inclusion $\mathrm{I} \colon \mathsf{Qnd} \to \mathsf{Rck}$ will be described in Section 2.5.1. In what follows, the present comment translates in several different ways, such as in Example 3.1.6 for instance.

2.2. From axioms to geometrical features.

We informally highlight two additional elementary features of the axioms which play an important role in what follows. We then illustrate them in the characterization of the free rack on a set A.

2.2.1. *Heads and tails – detachable tails.* Observe that on either side of the identities defining racks, the *head* x of each term is the same and does not play any role in the described identifications.

(R1) $x \triangleleft y \triangleleft^{-1} y = x = x \triangleleft^{-1} y \triangleleft y$ (R2') $x \triangleleft (y \triangleleft z) = x \triangleleft^{-1} z \triangleleft y \triangleleft z$

Now consider any *formal term* in the language of racks (built inductively from atomic variables and the rack operations – see Chapter II Section 10 in **[23]**), such as for instance

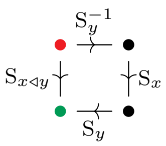
$$(x \triangleleft y) \triangleleft^{-1} (\cdots ((a \triangleleft b) \triangleleft^{-1} c) \triangleleft d) \cdots \triangleleft z. \qquad (14)$$

Remember that roughly speaking, the elements of the *free rack* on a set A can be constructed as equivalence classes of such formal terms, built inductively from the atomic variables in A, where two terms are identified if one can be obtained from the other by replacing subterms according to the axioms, or according to any provable equations derived from the axioms.

Given any term such as above, we shall distinguish the *head* x of the term from the rest of it which is called the *tail* of the term. The informal idea is that the "behaviour" of the tail is independent from the head it is attached to. It thus makes sense to consider the tails (or equivalence classes of such) separately from the heads these tails might act upon.

Observe that the idempotency axiom plays a slightly different role in that respect since, although the heads of terms are left unchanged under the use of (Q1), the identifications in the tails of terms might depend on the heads these are attached to. We shall however see that the discussion about racks still lays a clear foundation for understanding the case of quandles which we discuss in Section 2.5.

2.2.2. *Tails as sequences of symmetries.* By Paragraph 2.1.6, acting with a symmetry of the form $\mathrm{S}_{(x \triangleleft y)}$ translates into successive applications of $\mathrm{S}_y^{-1}, \mathrm{S}_x, \mathrm{S}_y$ from left to right.



Now consider any formal term such as in Equation (14) for instance. Using (R2') repeatedly, we may *unfold* the tail of a term into a string of successive actions of the form

$$x \triangleleft y \triangleleft^{-1} c \triangleleft c \triangleleft^{-1} b \triangleleft^{-1} a \triangleleft b \triangleleft^{-1} c \triangleleft c \triangleleft d \cdots \triangleleft z.$$

We can then interpret the tail as a *path* of successive actions of the symmetries which are applied to the head x. Using (R1) repeatedly again, we may also discard all possible occurrences of the successive application of a symmetry and its inverse

$$x \triangleleft y \triangleleft^{-1} b \triangleleft^{-1} a \triangleleft b \triangleleft d \cdots \triangleleft z.$$

It is then possible to show that such *unfolded* and *reduced* terms provide normal forms (unique representatives) for elements in the free rack. The elements of a free rack on a set A are thus described with this architectural feature of having a head in A and an independent tail, such that the tail is a sequence of "representatives" of the symmetries which organize themselves as the elements of the free group on A.

2.2.3. *The free rack.* The following construction can be found in [**55**]. It was also studied in [**85**].

Given a set A, the free rack on A is given by

$$\mathrm{F_r}(A) := A \rtimes \mathrm{F_g}(A) := \{(a,g) \mid g \in \mathrm{F_g}(A);\ a \in A\},$$

where $\mathrm{F_g}(A)$ is the free group on A and the operations on $\mathrm{F_r}(A)$ are defined for (a,g) and (b,h) in $A \rtimes \mathrm{F_g}(A)$ by

$$(a,g) \triangleleft (b,h) := (a, gh^{-1}\underline{b}h) \quad \text{and} \quad (a,g) \triangleleft^{-1} (b,h) := (a, gh^{-1}\underline{b}^{-1}h).$$

In order to distinguish elements x in A from their images under the injection $\eta^g_A \colon A \to \mathrm{F_g}(A)$, we shall use the convention to write

$$\underline{a} := \eta^g_A(a).$$

Looking for the unit of the adjunction, we then have the injective function sending an element in A to the trivial path starting at that element, i.e. $\eta^r_A \colon A \to \mathrm{F_r}(A) \colon a \mapsto (a,e)$, where e is the empty word (neutral element) in $\mathrm{F_g}(A)$.

Note that since any element $g \in \mathrm{F_g}(A)$ decomposes as a product $g = \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n} \in \mathrm{F_g}(A)$ for some $g_i \in A$ and exponents $\delta_i = 1$ or -1, with $1 \leq i \leq n$, we have, for any $(a,g) \in \mathrm{F_r}(A)$, a decomposition as

$$(a,g) = (a, \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) = (a,e) \triangleleft^{\delta_1} (g_1,e) \triangleleft^{\delta_2} (g_2,e) \cdots \triangleleft^{\delta_n} (g_n,e).$$

As we discussed before, if we have moreover that $g_i = g_{i+1}$ and $\delta_i = -\delta_{i+1}$ for some $1 \leq i \leq n$, then

$$\begin{aligned}(a,e) \lhd^{\delta_1} (g_1,e) \cdots \lhd^{\delta_{i-1}} (g_{i-1},e) \lhd^{\delta_i} (g_i,e)&\\ \lhd^{\delta_{i+1}} (g_{i+1},e) \lhd^{\delta_{i+2}} (g_{i+2},e) \cdots \lhd^{\delta_n} (g_n,e) &=\\ = (a, \underline{g_1}^{\delta_1} \cdots \underline{g_{i-1}}^{\delta_{i-1}} \underline{g_i}^{\delta_i} \underline{g_{i+1}}^{\delta_{i+1}} \underline{g_{i+2}}^{\delta_{i+2}} \cdots \underline{g_n}^{\delta_n})&\\ = (a, \underline{g_1}^{\delta_1} \cdots \underline{g_{i-1}}^{\delta_{i-1}} \underline{g_{i+2}}^{\delta_{i+2}} \cdots \underline{g_n}^{\delta_n})&\\ = (a,e) \lhd^{\delta_1} (g_1,e) \cdots \lhd^{\delta_{i-1}} (g_{i-1},e) \lhd^{\delta_{i+2}} (g_{i+2},e) \cdots \lhd^{\delta_n} (g_n,e)&\end{aligned}$$

which expresses the first axiom of racks, using group cancellation.

From there we derive the universal property of the unit of the adjunction: given a function $f \colon A \to X$ for some rack X, we show that f factors uniquely through η^r_A. Given an element $(a,g) \in \mathrm{F_r}(A)$, we have that for any decomposition $g = \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}$ as above, we must have

$$\begin{aligned} f(a,g) &= f(a, \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) \\ &= f\big((a,e) \lhd^{\delta_1} (g_1,e) \cdots \lhd^{\delta_n} (g_n,e)\big) \\ &= f(a) \lhd^{\delta_1} f(g_1) \cdots \lhd^{\delta_n} f(g_n) \end{aligned}$$

which uniquely defines the extension of f along η^r_A to a rack homomorphism $f \colon \mathrm{F_r}(A) \to X$. This extension is well defined since two equivalent decompositions in $\mathrm{F_r}(A)$ are equivalent after f by the first axiom of racks.

The left adjoint $\mathrm{F_r} \colon \mathsf{Set} \to \mathsf{Rck}$ of the forgetful functor $\mathrm{U} \colon \mathsf{Rck} \to \mathsf{Set}$ with unit η^r is then defined on functions $f \colon A \to B$ by

$$\mathrm{F_r}(f) \coloneqq f \times \mathrm{F_g}(f) \colon A \rtimes \mathrm{F_g}(A) \to B \rtimes \mathrm{F_g}(B).$$

This is easily seen to define a rack homomorphism. Functoriality of $\mathrm{F_r}$ and naturality of η^r are immediate.

2.2.3.1. Terminology and visual representation. In order to emphasize its visual representation, we call an element $(a,g) \in \mathrm{F_r}(A)$ a *trail*. We call g the *path* (or *tail*) component and a the *head* component of the trail (a,g). It is understood that the path g formally acts on a to produce an *endpoint* of the trail (see Paragraph 2.2.3). Formally (a,g) stands for both the trail and its endpoint:

$$a \overset{g}{\relbar\joinrel\mapstochar\joinrel\rightarrow} (a,g).$$

The action of a trail (b,h) on another trail (a,g) consists in adding, at the end of the path g, the contribution of the *symmetry associated to the* *endpoint* of (b,h) (see Subsection 2.2.4 and further). We say that a trail acts on another *by endpoint*, as in the diagram below, where composition of arrows is computed by multiplication in the path component:

$$\begin{tikzcd}
a \arrow[dd, "g"'] & & b \arrow[dd, "h"'] & & a \arrow[d, "g"] \\
 & \triangleleft & & = & (a,g) \arrow[d, "h^{-1}\underline{b}h"] \\
(a,g) & & (b,h) & & (a,gh^{-1}\underline{b}h)
\end{tikzcd} \qquad (15)$$

2.2.4. *Canonical projective presentations.* Since Rck is a variety of algebras, any object X can be canonically presented as the quotient

$$\begin{tikzcd}
\mathrm{F_r}\,\mathrm{F_r}\,X \arrow[r, shift left=2, "\mathrm{F_r}\,\epsilon^r_X"] \arrow[r, shift right=2, "\epsilon^r_{\mathrm{F_r}\,X}"'] & \mathrm{F_r}\,X \arrow[l, "\mathrm{F_r}\,\eta^r_X" description] \arrow[r, "\epsilon^r_X"] & X
\end{tikzcd}$$

where we have omitted the forgetful functor $\mathrm{U}\colon \mathsf{Rck} \to \mathsf{Set}$ (understand X alternatively as a rack or a set), and ϵ^r_X is the counit of the "free-forgetful" adjunction $\mathrm{F_r} \dashv \mathrm{U}$. This counit ϵ^r_X is the coequalizer of the reflexive graph on the left. This canonical presentation of racks allows us to capture a sense in which the geometrical features of free objects are carried through to any general rack. We shall illustrate this on the important functorial constructions of the Galois theory of interest. Let us make explicit these objects and morphisms to exhibit some of the mechanics at play. Think of what this *right-exact fork* represents for groups, where the operation is associative.

First of all we may exhibit heads and tails and rewrite this *right-exact fork* as

$$\begin{tikzcd}
(X \rtimes \mathrm{F_g}(X)) \rtimes \mathrm{F_g}(X \rtimes \mathrm{F_g}(X)) \arrow[r, shift left=2, "\epsilon^r_X \times \mathrm{F_g}[\epsilon^r_X]"] \arrow[r, shift right=2, "\epsilon^r_{\mathrm{F_r}\,X}"'] & X \rtimes \mathrm{F_g}\,X \arrow[l, "\mathrm{F_r}\,\eta^r_X" description] \arrow[r, "\epsilon^r_X"] & X
\end{tikzcd}$$

Then it is immediate from Paragraph 2.2.3 that the counit ϵ^r_X should send a pair $(x,g) = (x, \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n})$ for $g_i \in X$ to the element in the rack X given by:

$$\epsilon^r_X(x,g) = x \cdot g := x \triangleleft^{\delta_1} g_1 \cdots \triangleleft^{\delta_n} g_n.$$

Hence the canonical projective presentation ϵ^r_X of a rack X covers each element $x \in X$ by all possible formal decompositions (x_0, g) of that element x, such that x *is the endpoint of the trail* (x_0, g), i.e. the result of the action of a *path* on a *head*: $x = x_0 \cdot g$. Now this head x_0 and each "representative of a symmetry" ${g_i}^{\delta_i}$ in the path component $g = {g_1}^{\delta_1} \cdots {g_n}^{\delta_n}$ may itself be expressed as the endpoint of some trail (i.e. $x_0 = x_{00} \cdot h$, and $g_i = y_i \cdot k_i$ for h and k_i in $\mathrm{F_g}\, X$). This is what is captured by the object $\mathrm{F_r}\,\mathrm{F_r}(X)$ on the left of the fork.

Then from the definition of the counit, we may derive the two projections. These may be understood as expressing two things:

First observe that an element $t = [(a, g); e]$ in $\mathrm{F_r}\,\mathrm{F_r}(X)$ (i.e. an element which has a trivial path component, but an interesting head) is sent to $((a \cdot g), e)$ by the first projection and to (a, g) by the second projection. The two projections thus allow us to move part of the tail of a trail towards the head of that trail and part of the head towards the tail.

Then an element $[(a, e); (b, h)]$ – i.e. an element with a trivial head component and a non trivial (but simple) tail – is sent by the first projection to $(a, (b \cdot h))$, and by the second projection to $(a, h^{-1}bh)$. Coequalizing these two projections expresses self-distributivity (see Paragraphs 2.1.6 and 2.2.2). In other words it illustrates how to compute the representative of the symmetry associated to the endpoint of a trail. This is already part of the definition of the rack operation in the free rack. We have the rack homomorphism on the left

$$\begin{tikzcd} X \rtimes \mathrm{F_g}(X) \arrow[r, "i_X"] & \mathrm{F_g}(X) \\ (x, g) \arrow[r, mapsto, "i_X"] & g^{-1}xg \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[rr, "\eta^g_X"] \arrow[dr, "\eta^r_X"'] & & \mathrm{Conj}(\mathrm{F_g}(X)) \\ & \mathrm{F_r}(X) \arrow[ur, dashed, "i_X"'] & \end{tikzcd}$$

which sends a path to the symmetry associated to its endpoint. It is actually induced by the universal property of free racks as displayed in the diagram on the right.

2.3. The connected component adjunction.

2.3.1. *Trivial racks and trivializing congruence.* Another important theoretical example of racks is given by the so-called *trivial racks* (or trivial quandles) for which each symmetry at a given point is chosen to be the identity. Each point acts trivially on the rest of the rack. This may be expressed as an additional axiom:

(Triv) $x \lhd y = x$.

Since each set comes with a unique structure of trivial rack and each function between trivial racks is a homomorphism, we get an isomorphism between the category of sets (Set) and the category of trivial racks. The category of sets is thus a subvariety of algebras within racks.

The inclusion functor $\mathrm{I}\colon \mathsf{Set} \to \mathsf{Rck}$ sends a set to the trivial rack on that set. Now this inclusion functor should have a left adjoint which sends a rack to the *freely trivialized* rack. Since trivial racks are those which satisfy (Triv), a good candidate for the trivialization of a rack X is thus by quotienting out the congruence $\mathrm{C}_0\, X$ generated by the pairs

$$(x, x \triangleleft y).$$

Using the comments of Section 2.2, it is not too hard to show that it actually suffices to consider the transitive closure of the set of pairs $\{(x,x),(x,x \triangleleft y),(x,x \triangleleft^{-1} y) \mid x,y \in X\}$ which gives the congruence $\mathrm{C}_0\, X$ when endowed with the rack structure of the cartesian product. Symmetry and compatibility with rack operations are obtained for free. This further yields the concepts of *connectedness* and *primitive path* of Paragraph 2.3.3.

CONVENTION 2.3.2. *For the purpose of this work, understand sets, or trivial racks, to be the* zero-dimensional coverings *of the covering theory of racks (and quandles), in the same way that abelian groups and central extensions of groups are respectively the zero-dimensional coverings and one-dimensional coverings in groups. Similarly* C_0 *is the* centralizing relation in dimension 0*. In Section 3 we study the subsequent one-dimensional covering theory of racks and quandles.*

2.3.3. *Connectedness and primitive paths.* Two elements x and y in a rack A are said to be *connected* ($[x] = [y]$) if there exists $n \in \mathbb{N}$ and elements a_1, a_2, …, a_n in A such that

$$y = x \triangleleft^{\delta_1} a_1 \triangleleft^{\delta_2} a_2 \cdots \triangleleft^{\delta_n} a_n,$$

for some coefficients $\delta_i \in \{-1,\, 1\}$ for $1 \leq i \leq n$.

Such a sequence of elements together with the choice of coefficients is viewed as a formal sequence of symmetries (see Paragraph 2.1.9.1). Bearing in mind Paragraphs 2.2.1 and 2.2.2, we call such a formal sequence of symmetries $(a_i, \delta_i)_{1 \leq i \leq n}$ a *primitive path* of the rack A. In particular this specific primitive path *connects* x to y but may be applied to different elements in the rack. We call the data of such a pair $T = (x, (a_i, \delta_i)_{1\leq i\leq n})$ a *primitive trail* in X, where x is the head of T and y the *endpoint* of T.

We have that (x, y) is in $\mathrm{C}_0\, A$ if and only if there exists a primitive path which connects x to y. For the sake of precision, and following the point of view of [**82**], let us take this as definition for $\mathrm{C}_0\, A$.

2.3.4. *Left adjoint* π_0. Then any rack homomorphism $f\colon A \to X$ for some trivial rack X is such that $\mathrm{C}_0\, A \leq \mathrm{Eq}(f)$ since given $y = x \lhd^{\delta_1} a_1 \cdots \lhd^{\delta_n} a_n$ in A we must have in X:

$$f(y) = f(x) \lhd^{\delta_1} f(a_1) \cdots \lhd^{\delta_n} f(a_n) = f(x).$$

Hence we define the functor $\pi_0\colon \mathsf{Rck} \to \mathsf{Set}$ such that

$$\pi_0(A) := A/(\mathrm{C}_0\, A)$$

is the set of connected components of A (i.e. the set of $\mathrm{C}_0\, A$ -equivalence classes) and $\pi_0 \dashv \mathrm{I}$ with unit

$$A \xrightarrow{\eta_A} \pi_0(A),$$

sending an element $a \in A$ to its connected component $\eta_A(a)$ (also denoted $[a]$) in $\pi_0(A)$. For any $f\colon A \to X$ as before, there is a unique function $f'\colon \pi_0(A) \to X$ defined on a connected component by the image under f of any of its representatives.

2.3.5. *From free objects to all – definition as a colimit.* Observe that the composite

$$\mathsf{Set} \xrightarrow{\mathrm{I}} \mathsf{Rck} \xrightarrow{\mathrm{U}} \mathsf{Set}$$

gives the identity functor. As a consequence, the composite of left adjoints $\pi_0\, \mathrm{F_r}$ also gives the identity functor. More precisely we may deduce from the composite of adjunctions that, given a set X, the unit $\eta_{\mathrm{F_r}(X)}\colon X \rtimes \mathrm{F_g}(X) \to X$ is "projection on X", i.e. the connected component of a trail $(x, g) \in \mathrm{F_r}(X)$ is given by projection on its head x.

Since π_0 is a left adjoint, it preserves colimits, hence $\pi_0(X)$ should be the coequalizer, in Set, of the pair:

$$\begin{tikzcd} \pi_0((X \rtimes \mathrm{F_g}(X)) \rtimes \mathrm{F_g}(X \rtimes \mathrm{F_g}(X))) \arrow[r, shift left, "\pi_0(\epsilon^r_X \times \mathrm{F_g}[\epsilon^r_X])"] \arrow[r, shift right, "\pi_0(\epsilon^r_{\mathrm{F_r}\,\mathrm{U}\,X})"'] & \pi_0(X \rtimes \mathrm{F_g}\, X), \end{tikzcd}$$

which indeed reduces to being the coequalizer of

$$\begin{tikzcd} X \times \mathrm{F_g}(X) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \end{tikzcd}$$

where

$$p_1(x, {\underline{g_1}}^{\delta_1} \cdots {\underline{g_n}}^{\delta_n}) = x \triangleleft^{\delta_1} g_1 \cdots \triangleleft^{\delta_n} g_n;$$

$$p_2(x, {\underline{g_1}}^{\delta_1} \cdots {\underline{g_n}}^{\delta_n}) = x.$$

See also *adjoint triangle theorems* in [**33**].

2.3.6. *Equivalence classes of primitive paths.* The term *primitive path* is used to express the idea that it is the most unrefined way we shall use to acknowledge that two elements are connected. Literally it is just a formal sequence of symmetries.

As explained in Paragraph 2.1.9, inner automorphisms also “represent” sequences of symmetries. Again, each primitive path naturally reduces to an inner automorphism simply by composing all the symmetries in the sequence. We also have that (x, y) is in $\mathrm{C}_0\, A$ if and only if there exists $g \in \mathrm{Inn}(A)$ such that $x \cdot g = y$. In other words, $\mathrm{C}_0\, A$ is the congruence generated by the action of $\mathrm{Inn}(A)$. We call it the *orbit congruence* of $\mathrm{Inn}(A)$ (see Paragraph 2.3.9). In what follows, we like to view inner automorphisms as equivalence classes of primitive paths. As mentioned earlier we shall consider other such equivalence classes of primitive paths which lie in between formal sequences of symmetries and composites of such. Each of these represent different witnesses of how to connect elements in a rack A. All of these generate the same trivializing congruence $\mathrm{C}_0\, A$.

2.3.7. *Conjugacy classes.* Observe that for a group G, the set of connected components of $\mathrm{Conj}(G)$ is given by the set of conjugacy classes in G. In this case the congruence $\mathrm{C}_0(\mathrm{Conj}(G))$ is characterised as follows: $(a, b) \in \mathrm{C}_0(\mathrm{Conj}(G))$ if and only if there exists $c \in G$ such that $b = c^{-1}ac$. Again, any primitive path, or sequence of symmetries, can be described via a single symmetry obtained as the symmetry of the product of the elements in the sequence.

Note that if H is an abelian group, then $\mathrm{Conj}(H)$ is the trivial rack on the underlying set of H. More precisely the restriction to Ab of the functor Conj yields the forgetful functor to Set:

$$\mathsf{Ab} \xrightarrow{\mathrm{Conj\ \ restricts\ to\ \ U}} \mathsf{Set}.$$

2.3.8. *Racks and quandles have the same connected components.* The functor π_0 may be restricted to the domain Qnd and is then left adjoint to the inclusion functor $\mathrm{I}\colon \mathsf{Set} \to \mathsf{Qnd}$ by the same arguments as above. More precisely we have for any rack X that $\pi_0\ {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(X) = \pi_0(X)$, where ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(X)$ is the free quandle on the rack X.

2.3.9. *Orbit congruences permute.* In order to obtain the admissibility of Set in Qnd, V. Even shows that certain classes of congruences commute with all congruences. As for quandles, we define *orbit congruences* [**22**] as the congruences induced by the action of a normal subgroup of the group of inner automorphisms. More precisely, if X is a rack, and N a normal subgroup of $\mathrm{Inn}(X)$ we shall write $\sim_N$ for the N-*orbit congruence* defined for elements x and y in X by: $x \sim_N y$ if and only if there exists $g \in N$ such that $x \cdot g = y$. As it is explained in [**40**] (see Proposition 2.3.9), this is well defined and yields a congruence (also in Rck).

We then have the following – see [**41**] and [**40**, Lemma 3.1.2] for the proof, which also holds in Rck.

LEMMA 2.3.10. *Let X be a rack, R a reflexive (internal) relation on X and N a normal subgroup of* $\mathrm{Inn}(X)$*, then the relations $\sim_N$ and R permute:*

$$\sim_N \circ R = R \circ \sim_N .$$

2.3.11. *Admissibility for Galois theory.* Of course the kernel pair of the unit $\eta_X\colon X \to \pi_0(X)$ is an orbit congruence, since by Paragraph 2.3.6, two elements are in the same connected component if and only if they are in the same orbit under the action of $\mathrm{Inn}(X)$.

As it was recalled in Section 1.0.10 (see also [**71**]), this yields Theorem 1 of [**39**]:

PROPOSITION 2.3.12. *The subvariety* Set *is strongly Birkhoff and thus admissible in* Rck*. Similarly for* Set *in* Qnd*.*

The Galois structure $\Gamma := (\mathsf{Rck}, \mathsf{Set}, \pi_0, \mathrm{I}, \eta, \epsilon, \mathcal{E})$ (respectively $\Gamma^q := (\mathsf{Qnd}, \mathsf{Set}, \pi_0, \mathrm{I}, \eta, \epsilon, \mathcal{E})$) (see [**71**]) where $\mathcal{E}$ is the class of surjective morphisms of racks (respectively quandles), is thus admissible, i.e. the study of Galois theory is relevant in this context and gives rise, in principle, to a meaningful notion of relative centrality.

2.3.13. *Connected components are not connected.* Given an element a in a rack A, we may consider its *connected component* C_a, i.e. the elements of A which are connected to a. The set C_a is actually a *subrack* of A as it is closed under the operations in A. We may construct the rack C_a as a pullback in Rck:

$$\begin{tikzcd} \mathrm{C}_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "{[a]}"] \\ A \arrow[r, "\eta_A"] & \pi_0(A), \end{tikzcd} \tag{16}$$

where $1 = \{*\}$ is the one element set, which is the terminal object in Rck and also the free quandle on the one element set. Note that if A is connected, then by definition $\pi_0(A) = \{*\}$ and thus $\mathrm{C}_a = A$. However if $\mathrm{C}_a \subset A$, then C_a might have more than one connected component itself (i.e. $\pi_0(\mathrm{C}_a)$ has cardinality $|\pi_0(\mathrm{C}_a)| > 1$), since the existence of a primitive path between some c and b in C_a, might depend on elements which are not connected to a. The same comments apply in the context of Qnd.

EXAMPLE 2.3.14. *A rack A is called* involutive *if the two operations $\triangleleft$ and $\triangleleft^{-1}$ coincide. The subvariety of* involutive racks *is thus obtained by adding the axiom*

(Inv) $x \triangleleft y \triangleleft y = x.$

We define the involutive quandle $Q_{ab\star}$ with three elements a, b and $\star$ such that the operation $\triangleleft$ is defined by the following table (see $Q_{(2,1)}$ from [**38**, Example 1.3]*).*

$\triangleleft$	a	b	$\star$
a	a	a	b
b	b	b	a
$\star$	$\star$	$\star$	$\star$

The connected component of a is the trivial rack $\mathrm{C}_a = \{a, b\}$ which has itself two connected components $\{a\}$ and $\{b\}$.

We like to say that, for racks (and quandles) the notion of connectedness is not local. In categorical terms, we may say that the functor π_0 is not *semi-left-exact* [**30, 24**]. This property is indeed characterised, in this context, by the preservation of pullbacks such as in Equation (16) above, i.e. π_0 is semi-left-exact if and only if any such connected component (C_a)

is connected ($\pi_0(\mathrm{C}_a) = \{*\}$) (see for instance [**9**] and [**104**, Theorem 2.1]). This is an important difference with the case of topological spaces for instance, where the connected components are connected and thus the corresponding π_0 functor is semi-left-exact. See also [**41**] for further insights on connectedness.

Finally, with [**31**, Corollary 2.5] in mind, we compute the set of connected components $\pi_0(\mathrm{F_r}(1) \times \mathrm{F_r}(1)) = \mathbb{Z}$ and thus we have that $\pi_0\colon \mathsf{Rck} \to \mathsf{Set}$ does not preserve finite products; wheareas $\pi_0\colon \mathsf{Qnd} \to \mathsf{Set}$ does, as was shown in [**39**, Lemma 3.6.5].

2.3.15. *Towards covering theory.* Knowing that Γ is admissible, we may now wonder what is the "sphere of influence" of Set in Rck, with respect to surjective maps, and start to develop the covering theory. Since Set is strongly Birkhoff in Rck, trivial extensions (first step influence) are easy to characterize as those surjections which are "injective on connected components":

PROPOSITION 2.3.16. *(See also* [**39, 40**]*) Given a surjective morphism of racks $t\colon X \to Y$, the following conditions are equivalent:*

- *(i) t is a trivial extension;*
- *(ii) $\mathrm{Eq}(t) \cap \mathrm{C}_0\, X = \Delta_X$;*
- *(iii) if a and b in X are connected, then $t(a) = t(b)$ implies $a = b$.*

Recall that the construction of inner automorphisms (Inn) induces a functor on surjective morphisms: given a surjective morphism $t\colon X \to Y$, we write $\hat{t}$ or $\mathrm{Inn}(t)\colon \mathrm{Inn}(X) \to \mathrm{Inn}(Y)$ for the induced homomorphism between the inner automorphism groups (see first two sections of [**22**]).

We may then also describe a trivial extension as an extension which *reflects loops*: trivial extensions are those extensions such that for any a in A, if g in $\mathrm{Inn}(A)$ is such that $t(a) \cdot \hat{t}(g) = t(a)$, then $a \cdot g = a$.

$$\begin{tikzcd} (a \arrow[r, "g"] & a \cdot g) \arrow[r, mapsto, "t"] & t(a) = t(a \cdot g) \arrow[loop above, "\hat{t}(g)"] & \Rightarrow & a = a \cdot g \arrow[loop above, "g"] \end{tikzcd}$$

In what follows, we shall use such geometrical interpretations to make sense of the algebraic conditions of interest for the covering theory. However, the non-functoriality of Inn on general morphisms appears as a serious weakness (see for instance the need for Remark 2.4.7 in the proof of Proposition 3.2.1). It will become clear from what follows that a more suitable way to represent sequences of symmetries is needed. This is achieved by the *group of paths* which we motivate and describe in the next section. It is not a new concept, but our name for the left adjoint of the conjugation functor, which was described by D.E. Joyce and then used by M. Eisermann to construct weakly universal covers and a fundamental groupoid for quandles. However, we provide a hopefully enlightening description of the construction and the role of this functor, which naturally arises from the geometrical features described in Section 2.2.

2.4. The group of paths.

2.4.1. *Definition.* Consider a rack X and two elements x and y in X which are connected by a primitive path $\mathrm{S}_{x_1}^{\delta_1}, \ldots, \mathrm{S}_{x_n}^{\delta_n}$:

$$x \cdot (\mathrm{S}_{x_1}^{\delta_1}, \ldots, \mathrm{S}_{x_n}^{\delta_n}) \coloneqq x \triangleleft^{\delta_1} x_1 \cdots \triangleleft^{\delta_n} x_n = y.$$

Because of (R1), we discussed that it makes sense to identify such formal sequences so as to obtain elements of the free group on X. Now in the same way that we used Paragraph 2.1.6 to unfold formal terms, we still have that whenever $x_i = b \triangleleft c$ for $1 \leq i \leq n$ and b, c in X, acting with S_{x_i} amounts to successively acting with S_c^{-1}, S_b and S_c. From a rack X we may thus build the quotient:

$$\mathrm{F_g}(X) \xrightarrow{q_X} \mathrm{Pth}(X) \coloneqq \mathrm{F_g}(X)/\langle \underline{c}^{-1}\underline{a}^{-1}\underline{x}\,\underline{a} \mid a, x, c \in X \text{ and } c = x \triangleleft a\rangle$$

which is understood as a group of equivalence classes of primitive paths. Two primitive paths are identified in the group of paths if and only if one can be formally obtained from the other, using the identities induced by the graph of the rack operations (such as $c = x \triangleleft a$), as well as the axioms of racks (or more precisely the axiom-induced identities between tails of formal terms).

2.4.2. *Unit and universal property.* The composition of the function $\eta^g \colon X \to \mathrm{F_g}(X)$ with this quotient $q_X \colon \mathrm{F_g}(X) \to \mathrm{Pth}(X)$ yields a rack homomorphism

$$X \xrightarrow{\mathrm{pth}_X} \mathrm{Conj}(\mathrm{Pth}(X))$$

which sends each element x of X to $\mathrm{pth}_X(x)$ in $\mathrm{Pth}(X)$, such that $\mathrm{pth}_X(x)$ “represents” the positive symmetry at x in the same way S_x

does in $\mathrm{Inn}(X)$ (see Paragraph 2.4.5). As for the inclusion in the free group, we shall use the convention

$$\underline{x} \coloneqq \mathrm{pth}_X(x).$$

Now given a rack homomorphism $f\colon X \to \mathrm{Conj}(G)$ for some group G, there is a unique group homomorphism f' induced by the universal property of the free group, which, moreover, factors uniquely through the quotient

$$q_X\colon \mathrm{F_g}(X) \to \mathrm{F_g}(X)/\langle(\underline{x \triangleleft a})^{-1}\underline{a}^{-1}\underline{x}\,\underline{a} \mid a, x \in X\rangle,$$

since for any a and x in X, $f(x \triangleleft a) = f(a)^{-1}f(x)f(a)$ in G:

$$\begin{tikzcd}
X \arrow[r, "\eta^g_X"] \arrow[dr, "f"'] & \mathrm{F_g}(X) \arrow[r, "q_X"] \arrow[d, dotted, "\exists! f'"] & \mathrm{Pth}(X) \arrow[dl, dashed, "\exists! \bar{f}"] \\
& G &
\end{tikzcd}$$

Hence, the construction Pth uniquely defines a functor which is the left adjoint of Conj with unit $\mathrm{pth}\colon 1_{\mathsf{Rck}} \to \mathrm{Conj}\,\mathrm{Pth}$. As usual, given $f\colon X \to Y$ in Rck, there is a unique morphism $\mathrm{Pth}(f)$, such that the square

$$\begin{tikzcd}
X \arrow[r, "\mathrm{pth}_X"] \arrow[d, "f"'] & \mathrm{Conj}(\mathrm{Pth}(X)) \arrow[d, dotted, "\exists!\,\mathrm{Conj}(\mathrm{Pth}(f))"] \\
Y \arrow[r, "\mathrm{pth}_Y"] & \mathrm{Conj}(\mathrm{Pth}(Y)),
\end{tikzcd}$$

commutes and this defines the functor Pth on morphisms.

NOTATION 2.4.3. *In what follows, we write* $\vec{f}$ *for the image* $\mathrm{Pth}(f)$ *of a morphism* f *from* Rck.

2.4.4. *From free objects to all – construction as a colimit.* Again, observe that the composite $\mathrm{Pth}\,\mathrm{F_r}$ is left adjoint to the forgetful functor $\mathrm{U}\colon \mathsf{Grp} \to \mathsf{Set}$, i.e. $\mathrm{Pth}(\mathrm{F_r}(X)) = \mathrm{F_g}(X)$. More precisely, we may interpret pth as the extension to all objects of the functorial construction on free objects

$$i_X\colon X \rtimes \mathrm{F_g}(X) \to \mathrm{F_g}(X) \ :\ (x, g) \mapsto g^{-1}\underline{x}g$$

which sends a trail to the “representative of the symmetry” associated to its endpoint (Subsection 2.2.4). Indeed, by the composition of adjunctions, as before, this i is easily seen to define the restriction to free objects of the unit pth of the Pth ⊣ Conj adjunction:

$$\begin{tikzcd}
X \arrow[rr, "\eta^g_X"] \arrow[dr, "\eta^r_X"'] & & \mathrm{Conj}(\mathrm{F_g}(X)) \arrow[dd, dashed, "\exists!\,\mathrm{Conj}(f'')"] \\
& \mathrm{F_r}(X) \arrow[ur, "i_X = \mathrm{pth}_{\mathrm{F_r}(X)}"] \arrow[dr, "\forall f"'] & \\
& & \mathrm{Conj}(G)
\end{tikzcd} \tag{17}$$

where $i_X(x,e) = i_X\eta^r_X(x) = \eta^g_X(x) = \underline{x}$.

Then since Pth is a left adjoint, $q_X\colon \mathrm{F_g}(X) \to \mathrm{Pth}(X)$ should be the coequalizer of the pair

$$\begin{tikzcd}
\mathrm{Pth}((X \rtimes \mathrm{F_g}(X)) \rtimes \mathrm{F_g}(X \rtimes \mathrm{F_g}(X))) \arrow[r, shift left, "\mathrm{Pth}(\epsilon^r_X \times \mathrm{F_g}[\epsilon^r_X])"] \arrow[r, shift right, "\mathrm{Pth}(\epsilon^r_{\mathrm{F_r\,U}\,X})"'] & \mathrm{Pth}(X \rtimes \mathrm{F_g}\,X)
\end{tikzcd}$$

which, using i above, we compute to be

$$\begin{tikzcd}
\mathrm{F_g}(X \times \mathrm{F_g}(X)) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathrm{F_g}(X)
\end{tikzcd}$$

where p_1 and p_2 are defined by

$$p_1(x,g) = i_X(x \cdot g, e) = \eta^g_X(x \cdot g) = \underline{x \cdot g}$$

and

$$p_2(x,g) = i_X(x,g) = g^{-1}\underline{x}g.$$

The universal property of the unit and definition on morphisms then follows easily as before. See also *adjoint triangle theorems* in [**33**].

We insist on the tight relationship between the left adjoint Pth of the conjugation functor Conj, and the geometrical features of the free racks as described in Subsection 2.2. We also use this detailed construction of Pth as a colimit, in the proof of Proposition 2.4.16.

Finally, note that this pair p_1, p_2 is reflexive and thus from the coequalizer q_X we also get the pushout $q_X, q_X\colon \mathrm{F_g}(X) \rightrightarrows \mathrm{Pth}(X)$ of p_1 and p_2. Even though the original fork in Rck is not necessarily a *double extension*, the resulting fork in Grp is a *double extension* (because Grp is an exact Mal'tsev category [**25**]) i.e. the comparison map

$$p\colon \mathrm{F_g}(X \times \mathrm{F_g}(X)) \to \mathrm{Eq}(q_X)$$

to the kernel pair of the coequalizer q_X, is a surjection.

2.4.5. *Action by inner automorphisms.* It is already clear from the construction of Pth that the group of paths $\mathrm{Pth}(X)$ acts on the rack X "via representatives of the symmetries". For any x and y in X we have

$$x \cdot (\underline{y}) = x \triangleleft y,$$

which uniquely defines the action of any element in $\mathrm{Pth}(X)$.

Compare this action with the action by inner automorphisms: for each rack X, the universal property of pth_X on $\mathrm{S}\colon X \to \mathrm{Inn}(X)$ (defined in Subsection 2.1.8) gives

$$\begin{tikzcd} X \arrow[r, "\mathrm{pth}_X"] \arrow[dr, "\mathrm{S}"'] & \mathrm{Pth}(X) \arrow[d, dashed, "s"] \\ & \mathrm{Inn}(X), \end{tikzcd} \tag{18}$$

where we have omitted Conj, and s is the group homomorphism which relates the representatives of symmetries in $\mathrm{Pth}(X)$ to those in $\mathrm{Inn}(X)$. Then the action of $g \in \mathrm{Pth}(X)$ on X is also uniquely described by the action of the inner automorphism $s(g)$. If preferred, the reader may use this as the definition of action by the group of paths. The morphism s is called the *excess* of X in [**55**]. It is shown to be a central extension of groups in [**38**, Proposition 2.26]. Note that if $N \triangleleft \mathrm{Pth}(X)$ is a normal subgroup of $\mathrm{Pth}(X)$, then $s(N)$ is a normal subgroup of $\mathrm{Inn}(X)$. Hence the congruence $\sim_N$ induced by the action of N on X always defines an *orbit congruence* ($\sim_N = \sim_{s(N)}$) in the sense of Paragraph 2.3.9.

We extend the concept of a *trail* from Paragraph 2.2.3.1.

DEFINITION 2.4.6. *Given a rack X, a* trail *(in X) is the data of a pair (x, g) given by a* head *$x \in X$ and a* path *$g \in \mathrm{Pth}(X)$. The* endpoint *of such a trail is then the element obtained by the action $x \cdot g$, of g on x.*

In some sense, $\mathrm{Pth}(X)$ is the initial such group containing representatives of the symmetries of X and acting via those symmetries on X – whereas $\mathrm{Inn}(X)$ is the terminal such. This can be described via the notion of an *augmented rack* (see for instance [**83, 55**]). Those are given by a group G and a rack homomorphism $\iota\colon X \to G$ together with a right action of G on X such that for g, h in G and x, y in X,

(1) if e is the neutral element in G, then $x \cdot e = x$;
(2) $x \cdot (gh) = (x \cdot g) \cdot h$;
(3) $(x \triangleleft y) \cdot g = (x \cdot g) \triangleleft (y \cdot g)$;

(4) $\iota(x \cdot g) = g^{-1}\iota(x)g$.

Looking at augmented racks on a fixed rack X, a morphism between augmented racks $\iota\colon X \to G$ and $\iota'\colon X \to G'$ is given by a group homomorphism $f\colon G \to G'$ such that $f\iota = \iota'$. An example of such is given by $s\colon \mathrm{Pth}(X) \to \mathrm{Inn}(X)$ from Diagram (18). It is then easy to derive that $\mathrm{pth}_X\colon X \to \mathrm{Pth}(X)$ is initial amongst augmented racks (on X) whereas $\mathrm{S}\colon X \to \mathrm{Inn}(X)$ is terminal. This describes why Inn can be used as the reference to define such *actions by representatives of the symmetries*, described as *actions by inner automorphisms*. On the other hand, it also exhibits $\mathrm{Pth}(A)$ as the *freest way* to produce an augmented rack.

REMARK 2.4.7. *As mentioned before,* Pth *has the crucial advantage of functoriality, i.e. for any morphism of racks $f\colon X \to Y$ (including non-surjective ones), and for any $x \in Y$, $g = \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n} \in \mathrm{Pth}(X)$, we have that*

$$\begin{aligned} x \cdot (\vec{f}(g)) &= x \cdot (\vec{f}(\underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n})) = x \cdot (\underline{f(g_1)}^{\delta_1} \cdots \underline{f(g_n)}^{\delta_n}) \\ &= x \triangleleft^{\delta_1} f(g_1) \cdots \triangleleft^{\delta_n} f(g_n). \end{aligned}$$

In the next paragraph, we observe that in the case of free objects $\mathrm{F_r}(X)$, these two constructions coincide ($\mathrm{Pth}(\mathrm{F_r}(X)) = \mathrm{Inn}(\mathrm{F_r}(X))$ is $\mathrm{F_g}(X)$) and, most importantly for what follows, they act freely on $\mathrm{F_r}(X)$ (also see [**55, 85**], where these results were first discussed). Our hope is that, in view of the preceding discussion, these results do not take the reader by surprise any more.

2.4.8. *Free actions on free objects.* By Paragraph 2.4.4, and for any set X, the group of paths $\mathrm{Pth}(\mathrm{F_r}(X)) \cong \mathrm{F_g}(X)$ is freely generated by the elements

$$\mathrm{pth}_{\mathrm{F_r}(X)}[\eta^r_X(x)] = \mathrm{pth}_{\mathrm{F_r}(X)}[(x,e)] = \underline{(x,e)}$$

for $x \in X$. Using the identification $\underline{(x,e)} \leftrightarrow \underline{x}$, for any element (x,g) of $\mathrm{F_r}(X)$ and any word $h = \underline{h_1}^{\delta_1} \cdots \underline{h_n}^{\delta_n}$ in $\mathrm{Pth}(\mathrm{F_r}(X)) = \mathrm{F_g}(X)$, with $h_i \in X$ and $\delta_i \in \{-1, 1\}$ for each $1 \leq i \leq n$, we have that

$$\begin{aligned} (x,g) \cdot h = (x,g) \cdot (\underline{h_1}^{\delta_1} \cdots \underline{h_n}^{\delta_n}) &= (x,g) \triangleleft^{\delta_1} (h_1,e) \cdots \triangleleft^{\delta_n} (h_n,e) \\ &= (x,gh). \end{aligned}$$

PROPOSITION 2.4.9. *The action of* $\mathrm{F_g}(X) = \mathrm{Pth}(\mathrm{F_r}(X))$ *on* $\mathrm{F_r}(X) = X \rtimes \mathrm{F_g}(X)$ *corresponds to the usual* $\mathrm{F_g}(X)$ *right action in* Set

$$(X \times \mathrm{F_g}(X)) \times \mathrm{F_g}(X) \to X \times \mathrm{F_g}(X)\colon ((a,g),h) \mapsto (a,g)\cdot h = (a,gh),$$

given by multiplication in $\mathrm{F_g}(X)$. *Such an action is free, since if* $(a, hg) = (a, g)$, *then* $hg = g$ *and thus* $h = e$.

Observe that $\mathrm{Inn}(\mathrm{F_r}(X))$ is generated as a group by the elements in the image of $\mathrm{S}\,\eta^r_X$. Indeed for each

$$(a, g) = (a, g_1^{\delta_1} \cdots g_n^{\delta_n}) = (a, e) \triangleleft^{\delta_1} (g_1, e) \cdots \triangleleft^{\delta_n} (g_n, e) = (a, e) \cdot g,$$

in $\mathrm{F_r}(A)$, as before, we have

$$\mathrm{S}_{(a,g)} = \mathrm{S}^{-\delta_n}_{(g_n,e)} \cdots \mathrm{S}^{-\delta_1}_{(g_1,e)}\, \mathrm{S}_{(a,e)}\, \mathrm{S}^{\delta_1}_{(g_1,e)} \cdots \mathrm{S}^{\delta_n}_{(g_n,e)};$$

see identity (4) from page 53: $\mathrm{S}_{(a,e)\cdot g} = g^{-1}\, \mathrm{S}_{(a,e)}\, g$.

We conclude that $\mathrm{Inn}(\mathrm{F_r}(X))$ is actually freely generated. Indeed, the group homomorphism

$$s\colon \mathrm{Pth}(\mathrm{F_r}(X)) = \mathrm{F_g}(X) \to \mathrm{Inn}(\mathrm{F_r}(X))$$

defined in Subsection 2.4.5, is such that:

- it is surjective, since the generating set $s(X) = \{\mathrm{S}_{(x,e)} \mid x \in X\} \subset \mathrm{Inn}(\mathrm{F_r}(X))$ is the image of $X \subset \mathrm{F_g}(X)$ by s;
- it is injective, since $s(\underline{h_1}^{\delta_1} \cdots \underline{h_n}^{\delta_n}) = e$ for some $h_i \in X$ and $\delta_i \in \{-1, 1\}$ for $1 \leq i \leq n$, if and only if

  $$(x, g) = (x, g) \cdot (\mathrm{S}^{\delta_1}_{(h_1,e)} \cdots \mathrm{S}^{\delta_n}_{(h_n,e)}) = (x, g) \cdot (\underline{h_1}^{\delta_1} \cdots \underline{h_n}^{\delta_n}),$$

  for all $(x, g) \in \mathrm{F_r}(X)$, which implies that $\underline{h_1}^{\delta_1} \cdots \underline{h_n}^{\delta_n} = e$ since the action of $\mathrm{F_g}(X)$ is free.

PROPOSITION 2.4.10. *We may always identify* $\mathrm{Inn}(\mathrm{F_r}(X))$, $\mathrm{Pth}(\mathrm{F_r}(X))$ *and* $\mathrm{F_g}(X)$ *as well as their action on* $\mathrm{F_r}(X)$, *which is free. We refer to them as the* group of paths of $\mathrm{F_r}(X)$.

2.4.11. *The kernels of induced morphisms* $\vec{f}$. In this section we introduce the results which we use to describe the relationship between the group of paths Pth, and the central extensions (coverings) and centralizing relations of racks and quandles.

Our Lemma 2.4.13 is only a slight generalization of a lemma in [**7**]. We further generalize to *higher dimensions* in Part II.

DEFINITION 2.4.12. *Given a group homomorphism* $f\colon G \to H$, *and a chosen subset* $A \subseteq G$, *we define (implicitly with respect to* A*)*

*(i) two elements g_a and g_b in G are f-*symmetric (to each other) *if there exists $n \in \mathbb{N}$ and a sequence of pairs (a_1, b_1), ..., (a_n, b_n) in $(A \times A)$, such that*

$$f(a_i) = f(b_i), \quad g_a = a_1^{\delta_1} \cdots a_n^{\delta_n}, \quad \text{and} \quad g_b = b_1^{\delta_1} \cdots b_n^{\delta_n},$$

*for some $\delta_i \in \{-1, 1\}$, where $1 \leq i \leq n$. Alternatively say that g_a and g_b are an f-*symmetric pair.

(ii) K_f *is the set of f-*symmetric paths *defined as the elements $g \in G$ such that $g = g_a g_b^{-1}$ for some g_a and $g_b \in G$ which are f-symmetric to each other.*

Observe that the elements of K_f are in the kernel of f. The idea is to understand when K_f actually describes all the elements in the kernel of f. For instance if the chosen subset A is the whole group, or if it contains $\mathrm{Ker}(f)$ (the kernel of f), then we easily derive that $\mathrm{K}_f = \mathrm{Ker}(f)$. For a general $f \colon G \to H$ as in Definition 2.4.12, the condition $\mathrm{K}_f = \mathrm{Ker}(f)$ expresses the fact that the kernel of f in Grp is entirely described by the restriction $f \colon A \to f(A)$ of the underlying function f in the category Set. This is for instance the case when $f \colon G \to H = \mathrm{F_g}(h) \colon \mathrm{F_g}(A) \to \mathrm{F_g}(B)$ is the group homomorphism induced by a function $h \colon A \to B$ in Set (where the chosen subset of $G = \mathrm{F_g}(A)$ is A – see Proposition 2.4.15). Even though in the examples of interest, the chosen subset A is a generating set of G (i.e. such that the subgroup $\langle a \mid a \in A \rangle_G$ of G generated by the elements of A is equal to G), it is neither sufficient, nor necessary, for A to be such a generating set of G in general. For instance, consider the quotient map $f \colon \mathrm{F_g}(\{a\}) \to \{e,\ a\}$ where $f(a^n) = e$ if n is even and $f(a^n) = a$ if n is odd. The element a^2 is in $\mathrm{Ker}(f)$. However, if $\{a\}$ is our chosen set (of generators) of $\mathrm{F_g}(\{a\})$, the element a^2 is not an f-symmetric path. Conversely, Definition 2.4.12 and the condition $\mathrm{K}_f = \mathrm{Ker}(f)$ still make sense when A is merely a subset of G which is not generating. For instance, consider the product $\mathrm{F_g}(h) \times \mathrm{id}_{G'} \colon \mathrm{F_g}(A) \times G' \to \mathrm{F_g}(B) \times G'$ of $\mathrm{F_g}(h) \colon \mathrm{F_g}(A) \to \mathrm{F_g}(B)$ with the identity function on some other group G'. Then $\mathrm{Ker}(\mathrm{F_g}(h) \times \mathrm{id}_{G'}) = \mathrm{Ker}(\mathrm{F_g}(h)) \times \{e\} = \mathrm{K}_{\mathrm{F_g}(h) \times \mathrm{id}_{G'}}$ with chosen subset $A \times \{e\}$. However, we need A to be a generating set of G for our proof of the following lemma.

LEMMA 2.4.13. *Given the hypotheses of Definition 2.4.12 and assuming that A is a generating set of G, the set of f-symmetric paths $\mathrm{K}_f \subseteq G$ defines a normal subgroup in G. More precisely it is the normal subgroup generated by the elements of the form ab^{-1} such that a, $b \in A$, and*

$f(a) = f(b)$:

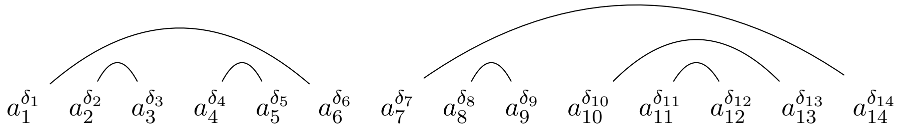
$$\mathrm{K}_f = G_f := \langle\langle ab^{-1} \mid (a,b) \in A \times A,\ f(a) = f(b) \rangle\rangle_G.$$

PROOF. First we show that K_f is a normal subgroup of G. Let g_a and g_b be f-symmetric (to each other). Observe that g_b^{-1} and g_a^{-1} are also f-symmetric, and thus K_f is closed under inverses. Moreover, if h_a and h_b are f-symmetric, and $g = g_a g_b^{-1}$, $h = h_a h_b^{-1}$, then $gh = k_a k_b^{-1}$, with $k_a = h_a h_a^{-1} g_a$ and $k_b = h_b h_a^{-1} g_b$ which are f-symmetric. Finally since A generates G by assumption, for any $k \in G$, kg_a and kg_b are f-symmetric to each other, and thus $kgk^{-1} \in \mathrm{K}_f$ is an f-symmetric path.

Since the generators of G_f are in the normal subgroup K_f, it suffices to show that $\mathrm{K}_f \leq G_f$. Given an f-symmetric pair g_a and g_b, we show that $g = g_a g_b^{-1} \in G_f$ by induction, on the minimum length n_g of the sequences $(a_i, b_i)_{1 \leq i \leq n}$ in the set $(A \times A) \cap \mathrm{Eq}(f)$ such that $g_a = a_1^{\delta_1} \cdots a_n^{\delta_n}$ and $g_b = b_1^{\delta_1} \cdots b_n^{\delta_n}$ for some $\delta_i \in \{-1, 1\}$. If $n_g = 1$, then g is a generator of G_f. Suppose that $g = g_a g_b^{-1} \in G_f$ for all such f-symmetric pair with $n_g < n$ for some fixed $n \in \mathbb{N}$. Then given a pair $g_a = a_1^{\delta_1} \cdots a_n^{\delta_n}$ and $g_b = b_1^{\delta_1} \cdots b_n^{\delta_n}$ for some (a_1, b_1), …, (a_n, b_n) in the set $(A \times A) \cap \mathrm{Eq}(f)$, and $\delta_i \in \{-1, 1\}$, we have that $h_a := a_1^{-\delta_1} g_a$ and $h_b := b_1^{-\delta_1} g_b$ are such that $h = h_a h_b^{-1} \in G_f$ by assumption. Moreover, $g = a_1^{\delta_1} h a_1^{-\delta_1} a_1^{\delta_1} b_1^{-\delta_1}$ which is a product of elements in G_f. □

OBSERVATION 2.4.14. *Consider a function $f \colon A \to B$, and a word $\nu = a_1^{\delta_1} \cdots a_n^{\delta_n}$ with $a_i \in A$ and $\delta_i \in \{-1, 1\}$, for $1 \leq i \leq n$. This word represents an element g in the free group $\mathrm{F_g}(A)$. As usual, a* reduction *of ν consists in eliminating, in the word ν, an adjacent pair $a_i^{\delta_i} a_{i+1}^{\delta_{i+1}}$ such that $\delta_i = -\delta_{i+1}$ and $a_i = a_{i+1}$. Every element $g \in \mathrm{F_g}(A)$ represented by a word ν admits a unique* normal form *i.e. a word ν' obtained from ν after a sequence of reductions, such that there is no reduction possible in ν', but ν' still represents the same element g in $\mathrm{F_g}(A)$.*

Suppose that ν represents an element g which is in the kernel $\mathrm{Ker}(\mathrm{F_g}(f))$. The normal form of the word $f[\nu] := f(a_1)^{\delta_1} \cdots f(a_n)^{\delta_n}$ (which represents $\mathrm{F_g}(f)(g) = e \in \mathrm{F_g}(B)$) is the empty word $\emptyset$, and thus there is a sequence of reductions of $f[\nu]$ such that the end result is $\emptyset$. From this sequence of reductions, we may deduce that $n = 2m$ for some $m \in \mathbb{N}$ and the letters in the word (or sequence) ν organize themselves in m pairs $(a_i^{\delta_i}, a_j^{\delta_j})$ (the pre-images of those pairs that are reduced at some *point in the aforementioned sequence of reductions) such that* $i < j$, $f(a_i) = f(a_j)$, $\delta_i = -\delta_j$, *each letter of the word* ν *appears in only one such pair and finally given any two such pairs* $(a_i^{\delta_i}, a_j^{\delta_j})$ *and* $(a_l^{\delta_l}, a_m^{\delta_m})$, *then* $l < i$ *(respectively* $l > i$*) if and only* $m > j$ *(respectively* $m < j$*), i.e. drawing lines which link those letters of the word* ν *that are identified by the pairing, none of these lines can cross.*



Given such a pairing of the letters of ν, *for each* $k \in \{1, \ldots, n\}$ *we write* $(a_{i_k}^{\delta_{i_k}}, a_{j_k}^{\delta_{j_k}})$ *for the unique pair such that either* $i_k = k$ *or* $j_k = k$. *Note that, conversely, any element* g *in* $\mathrm{F_g}(A)$ *which is represented by a word* ν *which admits such a pairing of its letters, is necessarily in* $\mathrm{Ker}(\mathrm{F_g}(f))$.

Using this observation, we characterize the kernels of maps between free groups.

Proposition 2.4.15. *Given a function* $f \colon A \to B$, *the kernel* $\mathrm{Ker}(\mathrm{F_g}(f))$ *of the induced group homomorphism*

$$\mathrm{F_g}(f) \colon \mathrm{F_g}(A) \to \mathrm{F_g}(B)$$

is given by the normal subgroup $\mathrm{K}_{\mathrm{F_g}(f)}$ *of* $\mathrm{F_g}(f)$*-symmetric paths (as in Definition 2.4.12):* $\mathrm{Ker}(\mathrm{F_g}(f)) = \mathrm{K}_{\mathrm{F_g}(f)}$.

Proof. The inclusion $\mathrm{Ker}(\mathrm{F_g}(f)) \supseteq \mathrm{K}_{\mathrm{F_g}(f)}$ is obvious. Consider a reduced word $\nu = a_1^{\delta_1} \cdots a_n^{\delta_n}$ of length $n \in \mathbb{N}$ which represents an element g in $\mathrm{F_g}(A)$ with $\delta_i \in \{-1, 1\}$, for $1 \leq i \leq n$ and suppose that $g \in \mathrm{Ker}(\mathrm{F_g}(f))$. Then the letters $a_k^{\delta_k}$ of the sequence (or word) $\nu := (a_k^{\delta_k})_{1 \leq k \leq n}$ organize themselves in pairs $(a_{i_k}^{\delta_{i_k}}, a_{j_k}^{\delta_{j_k}})$ as in Observation 2.4.14. Define the word $\nu' = b_1^{\delta_1} \cdots b_n^{\delta_n}$ such that for each $1 \leq k \leq n$, $b_k := a_{i_k}$. Then by construction ν' represents an element h which reduces to the empty word in $\mathrm{F_g}(A)$, so that $g = gh^{-1}$. Moreover, g and h form an f-symmetric pair, which shows that $g \in \mathrm{K}_{\mathrm{F_g}(f)}$. $\square$

Finally the same characterization holds for kernels of maps

$$\mathrm{Pth}(f) = \vec{f} \colon \mathrm{Pth}(X) \to \mathrm{Pth}(Y)$$

induced by a surjective morphism of racks $f \colon X \to Y$.

PROPOSITION 2.4.16. *Given* $f\colon X \twoheadrightarrow Y$*, a surjective morphism of racks, the kernel* $\mathrm{Ker}(\vec{f})$ *of the group homomorphism*

$$\vec{f}\colon \mathrm{Pth}(X) \twoheadrightarrow \mathrm{Pth}(Y)$$

is given by the normal subgroup $\mathrm{K}_{\vec{f}}$ *of* $\vec{f}$*-symmetric paths (as in Definition 2.4.12):*

$$\mathrm{Ker}(\vec{f}) = \mathrm{K}_{\vec{f}} = \langle\langle ab^{-1} \mid (a,b) \in \mathrm{Eq}(f)\rangle\rangle_{\mathrm{Pth}(X)}.$$

PROOF. From Subsection 2.4.4, we reconstruct the image $\vec{f}$ as in the following diagram, where we also draw the kernels of $\mathrm{F_g}(f)$ and $\vec{f}$:

$$\begin{tikzcd}
 & \mathrm{F_g}(X \times \mathrm{F_g}(X)) \arrow[r, two heads, "\mathrm{F_g}(f \times \mathrm{F_g}(f))"] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{F_g}(Y \times \mathrm{F_g}(Y)) \arrow[d, shift left] \arrow[d, shift right] \\
\mathrm{Ker}(\mathrm{F_g}(f)) \arrow[r, tail, "\ker(\mathrm{F_g}(f)"] \arrow[d, dashed, "k_1"'] & \mathrm{F_g}(X) \arrow[r, two heads, "\mathrm{F_g}(f)"] \arrow[d, two heads, "q_X"] \arrow[dr, phantom, "(*)"] & \mathrm{F_g}(Y) \arrow[d, two heads, "q_Y"] \\
\mathrm{Ker}(\vec{f}) \arrow[r, tail, "\ker(\mathrm{F_g}(f)"] & \mathrm{Pth}(X) \arrow[r, dashed, two heads, "\vec{f}"] & \mathrm{Pth}(Y).
\end{tikzcd}$$

Since q_X and q_Y are the coequalizers of the pairs above (see Subsection 2.4.4 for more details), and the map $\mathrm{F_g}(f \times \mathrm{F_g}(f))$ is surjective, by Lemma 1.2 in [**10**], the square $(*)$ is a double extension (regular pushout), and thus the comparison map k_1 is surjective. Then $\mathrm{Ker}(\vec{f})$ coincides with the image $\ker \mathrm{F_g}(f)$ along of q_X, by uniqueness of (regular epi)-mono factorizations in Grp. We may compute this image to be $\mathrm{K}_{\vec{f}}$. Indeed, in elementary terms, any $g \in \mathrm{Pth}(X)$ such that $\vec{f}(g) = e$ can be "covered" by an element $h \in \mathrm{F_g}(X)$ such that $q_X(h) = g$ and $\mathrm{F_g}(f)[h] = e$ as well. Then by Lemma 2.4.15, we have that $h = h_a h_b^{-1}$ for some h_a and h_b in $\mathrm{F_g}(X)$ which are $\mathrm{F_g}(f)$-symmetric to each other. The images $q_X(h_a)$ and $q_X(h_b)$ are then $\vec{f}$-symmetric to each other by commutativity of $(*)$, hence the quotient $g = q_X(h) = q_X(h_a)q_X(h_b)^{-1} \in \mathrm{K}_{\vec{f}}$ is an $\vec{f}$-symmetric path. □

NOTATION 2.4.17. *For a morphism of racks f, we often write* f-symmetric *(pair or path) instead of* $\vec{f}$*-symmetric (pair or path). An* f-symmetric trail (x, g) *is a trail with an f-symmetric path g.*

2.4.18. *The left adjoint* Pth *is not faithful.* Observe that given a set A, the morphism

$$\mathrm{F_r}(A) \xrightarrow{i_A = \mathrm{Pth}_{\mathrm{F_r}(A)}} \mathrm{F_g}(A) \ ,$$

is not injective. Indeed the elements $(a, \underline{a}g)$ and (a, g) have the same image. We shall see that the kernel pair of i_A yields the quotient producing the free quandle from the free rack. Then the free quandle $\mathrm{F_q}(A)$ on the set A embeds in the group $\mathrm{Conj}(\mathrm{F_g}(A))$, which is why D.E. Joyce calls quandles the *algebraic theory of conjugation*. Observe, though, that not all quandles embed in a group.

EXAMPLE 2.4.19. *In the involutive quandle* $Q_{ab\star}$ *defined in Example 2.3.14, the elements* $\underline{a}$ *and* $\underline{b}$ *are identified in* $\mathrm{Pth}(Q_{ab\star})$*. Indeed,* $\underline{a}$ *and* $\underline{b}$ *act trivially on* $Q_{ab\star}$*, hence they are in the center of the group* $\mathrm{Pth}(Q_{ab\star})$*. Moreover,* a *and* b *are in the same connected component, and thus they are also sent to conjugates in* $\mathrm{Pth}(Q_{ab\star})$*, which yields* $\underline{a} = \underline{b}$*. Note that from there we have* $\mathrm{Pth}(Q_{ab\star}) = \mathrm{F_g}(\{a, \star\})/\langle\langle a^{-1} \star^{-1} a\star \rangle\rangle_{\mathrm{F_g}(\{a,\star\})} = \mathrm{F_{ab}}(\{a, \star\}) = \mathbb{Z} \times \mathbb{Z}$*, where* $\mathrm{F_{ab}}$ *is the free abelian group functor, and in* $\mathbb{Z} \times \mathbb{Z}$*, we have* $\underline{a} = \underline{b} = (1, 0)$ *and* $\underline{\star} = (0, 1)$ *(also see* [**38**, Proposition 2.27]*).*

In particular, the unit of the adjuntion $\mathrm{Pth} \dashv \mathrm{Conj}$ *is not injective and* Pth *is not faithful (note that the right adjoint* Conj *is faithful, but not full). As a consequence* $Q_{ab\star}$ *is not a subquandle of a quandle in* $\mathrm{Conj}(\mathsf{Grp})$ *since this would imply that* $\mathrm{pth}_{Q_{ab\star}}$ *is injective. We may also observe that a subquandle of a conjugation quandle is such that* $(x \triangleleft y = x) \Leftrightarrow (y \triangleleft x = y)$*.*

2.4.20. *Racks and quandles have the same group of paths.* Observe that we may restrict Pth to the domain Qnd. By the same argument $\mathrm{Pth}\,\mathrm{I} \colon \mathsf{Qnd} \to \mathsf{Grp}$ (which we denote Pth) is then left adjoint to the functor $\mathrm{Conj} \colon \mathsf{Grp} \to \mathsf{Qnd}$. We may conclude by uniqueness of left adjoints that if ${}_{\mathrm{r}}\mathrm{F_q}$ is the left adjoint to the inclusion $\mathrm{I} \colon \mathsf{Qnd} \to \mathsf{Rck}$, then $\mathrm{Pth}\ {}_{\mathrm{r}}\mathrm{F_q} \cong \mathrm{Pth} \colon \mathsf{Rck} \to \mathsf{Grp}$. The adjunction between racks and groups factorizes into

$$\begin{tikzcd}
\mathsf{Rck} \arrow[rr, bend left, "{}_{\mathrm{r}}\mathrm{F_q}"] \arrow[rr, phantom, "\perp"] \arrow[dr, bend right] & & \mathsf{Qnd} \arrow[ll, bend left, "\mathrm{I}"] \arrow[dl, bend right] \\
& \mathsf{Grp} \arrow[ul, bend right] \arrow[ul, phantom, "\mathrm{Pth} \dashv \mathrm{Conj}"] \arrow[ur, bend right] \arrow[ur, phantom, "\mathrm{Pth} \dashv \mathrm{Conj}"'] &
\end{tikzcd} \tag{19}$$

in which all possible triangles of functors commute. Considering the comment of Paragraph 2.1.11 about the idempotency axiom, we may want to rephrase this as follows: for each rack X, the quotient defining $\mathrm{Pth}(X)$ always identifies generators that would be identified in the free quandle on X.

More informally, considering the way Pth, the left adjoint of Conj, is constructed from equivalence classes of tails in the theory of racks, we may wonder in which sense racks could be a better context to study group conjugation. From the perspective of their respective covering theories, we further describe the relationship between groups, racks and quandles in what follows (see for instance Section 3.7).

2.5. Working with quandles. We introduce the necessary material to make the transition from the context of racks to the context of quandles. See also the *associated quandle* in [**55**].

2.5.1. *The free quandle on a rack.* Remember from Paragraph 2.1.11 that the idempotency axiom is a consequence of the axioms of racks "for elements in the tail of a term". In order to turn a rack into a quandle the identifications that matter are thus of the form

$$x \triangleleft^{\delta_x} x \triangleleft^{\delta_x} \cdots x \triangleleft^{\delta_x} x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n = x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n,$$

where a use of the idempotency axiom cannot be avoided. Now by self-distributivity of the operations, we may write $y := (x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n)$, and then rewrite these identities as

$$y \triangleleft^{\delta_x} y \triangleleft^{\delta_x} \cdots y \triangleleft^{\delta_x} y = y.$$

DEFINITION 2.5.2. *Given a rack X, define Q_X as the relation (in Set) defined for $(x, y) \in X \times X$ by $(x, y) \in Q_X$ if and only if $x = y \triangleleft^k y$ for some integer k (see Paragraph 2.1.11), where $y \triangleleft^0 y := y$.*

LEMMA 2.5.3. *Given a rack X, the relation Q_X defines a congruence on X.*

PROOF.
(1) The relation Q_X is reflexive by definition.
(2) As aforementioned, for x and a in some rack, any chain $a \triangleleft^k a$ for some $k \in \mathbb{Z}$ is such that $x \triangleleft (a \triangleleft^k a) = x \triangleleft a$. Hence Q_X is symmetric since $b = a \triangleleft^k a$ implies that $b \triangleleft^{-k} b = b \triangleleft^{-k} a = a$.
(3) Now Q_X is transitive by self-distributivity.
(4) And finally it is internal since if $a = b \triangleleft^k b$ and $c = d \triangleleft^l d$ then $a \triangleleft c = (b \triangleleft^k b) \triangleleft (d \triangleleft^l d) = (b \triangleleft^k b) \triangleleft d = (b \triangleleft d) \triangleleft^k (b \triangleleft d)$. □

Lemma 2.5.4. *Given a rack X, then a pair of elements $(x, y) \in X \times X$ is in the kernel pair* $\mathrm{Eq}({}^r\eta^q_X)$ *of* ${}^r\eta^q_X \colon X \to {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(X)$ *if and only if* $y = x \triangleleft^n x$ *for some integer n, i.e.* $Q_X = \mathrm{Eq}({}^r\eta^q_X)$.

Proof. Since Rck is a Barr-exact category [**2**], it suffices to show that the quotient of X by the equivalence relation Q_X (on the left) is the same as the quotient of X by $\mathrm{Eq}({}^r\eta^q_X)$ (on the right):

$$X \xrightarrow{q} X/Q_X \qquad\qquad X \xrightarrow{{}^r\eta^q_X} {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(X).$$

For this we show that X/Q_X is a quandle and that q has the same universal property as ${}^r\eta^q_X$. Indeed we have that $q(a) \triangleleft q(a) = q(a \triangleleft a) = q(a)$ since $(a, a \triangleleft a) \in Q_X$ for each a. Finally observe that if $f \colon X \to Q$ is a rack homomorphism such that Q is a quandle, then we necessarily have that f coequalizes the projections $\pi_1,\ \pi_2 \colon Q_X \rightrightarrows X$ of the congruence Q_X. We then conclude by the universal property of the coequalizer. □

2.5.5. *Galois theory of quandles in racks.* We study the Galois structure ${}_r\Gamma_q :=$(Rck, Qnd, ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$, ${}^r\eta^q$, ${}^r\epsilon^q$, $\mathcal{E}$) where $\mathcal{E}$ is the class of surjective morphisms (see Section 1 and [**71**]).

Since Qnd is a Birkhoff subcategory of Rck, for ${}_r\Gamma_q$ to be admissible, it suffices to show that for each rack X the kernel pair $\mathrm{Eq}({}^r\eta^q_X)$ of the unit permutes with other congruences on X (see Section 1.0.10). Observe that this is not a consequence of Lemma 2.3.10.

Lemma 2.5.6. *Given a rack X, then the congruence* $Q_X = \mathrm{Eq}({}^r\eta^q_X)$ *commutes with any other internal relation R on X.*

Proof. We prove that a pair $(a, b) \in X \times X$ is in $\mathrm{Eq}({}^r\eta^q_X)\, R$ if and only if it is in $R\ \mathrm{Eq}({}^r\eta^q_X)$. As in Lemma 2.3.10, we show that if there is $c \in X$ such that (a, c) is in one of these relations (say for instance $\mathrm{Eq}({}^r\eta^q_X)$) and (c, b) in the other one (R), then there is a $c' \in X$ such that (a, c') is in the latter (R) and (c', b) in the former ($\mathrm{Eq}({}^r\eta^q_X)$). Now observe that if $(x, y) \in R$, then $(x, y) \triangleleft^k (x, y) = (x \triangleleft^k x, y \triangleleft^k y)$ is in R for any integer k. The result then follows from reading the following diagram for any $k \in \mathbb{Z}$, where horizontal arrows represent membership in $\mathrm{Eq}({}^r\eta^q_X)$ and vertical arrows represent membership in R. Indeed from the top right corner below we construct the bottom left corner and the other way around:

$$\begin{tikzcd}
c_1 \triangleleft^{-k} c_1 = a \arrow[r, shift left, "\mathrm{S}_a^k"] \arrow[d, dash] & c_1 = a \triangleleft^k a \arrow[l, shift left, "\mathrm{S}_{c_1}^{-k}"] \arrow[d, dash] \\
b \triangleleft^{-k} b = c_2 \arrow[r, shift left, "\mathrm{S}_{c_2}^k"] & b = c_2 \triangleleft^k c_2 \arrow[l, shift left, "\mathrm{S}_b^{-k}"]
\end{tikzcd}$$

where we use the fact that if $x = y \triangleleft^k y$ then $\mathrm{S}_x = \mathrm{S}_y$. Algebraically we read $(a, c_1) \in Q_X$ implies $c_1 \triangleleft^{-k} c_1 = a$ for some $k \in \mathbb{Z}$ and $(c_1, b) \in R$ implies $(c_1 \triangleleft^{-k} c_1, b \triangleleft^{-k} b) \in R$, thus choosing $c_2 = b \triangleleft^{-k} b$ yields one of the implications. The other direction translates similarly. □

REMARK 2.5.7. *Given a rack X, the congruence Q_X is not an orbit congruence in general. For instance, observe that $Q_{\mathrm{F_r}(\{a,b\})}$ contains the pairs $(a, a \triangleleft a)$ and $(b, b \triangleleft b)$. Suppose by contradiction that there is a normal subgroup $N \leq \mathrm{Inn}(\mathrm{F_r}(\{a,b\})) = \mathrm{F_g}(\{a,b\})$ for which $\sim_N = Q_{\mathrm{F_r}(\{a,b\})}$. Then since $\mathrm{F_g}(\{a,b\})$ acts freely on $\mathrm{F_r}(X)$, both inner automorphisms S_a and S_b need to be in N. This leads to a contradiction since $a \sim_N (a \triangleleft b)$ but $(a, a \triangleleft b) \notin Q_{\mathrm{F_r}(\{a,b\})}$. By contrast $Q_{\mathrm{F_r}(\{*\})}$ is of course an orbit congruence.*

COROLLARY 2.5.8. *Quandles form a strongly Birkhoff (and thus admissible) subcategory of* Rck.

PROOF. By Proposition 5.4 in [**25**], the reflection squares of surjective morphisms are double extensions (see Section 1.0.10). This implies the admissibility of the Galois structure ${}_r\Gamma_q$, for instance by [**50**, Proposition 2.6]. □

Note that the left adjoint ${}_\mathrm{r}\mathrm{F}_\mathrm{q}$ is actually semi-left-exact as we may deduce from the fact that "connected components are connected" (see Paragraph 2.3.13).

PROPOSITION 2.5.9. *Any pullback of the form*

$$\begin{tikzcd}
C_a \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "{[a]}"] \\
X \arrow[r, "{}^r\eta^q_X"'] & {}_\mathrm{r}\mathrm{F}_\mathrm{q}(X),
\end{tikzcd}$$

in Rck, *is preserved by the reflector ${}_\mathrm{r}\mathrm{F}_\mathrm{q}$, i.e. ${}_\mathrm{r}\mathrm{F}_\mathrm{q}(C_a) = 1$; and thus by* [**104**, Theorem 2.1], *we conclude that ${}_\mathrm{r}\mathrm{F}_\mathrm{q}$ is* semi-left-exact *in the sense of* [**30, 24**].

PROOF. Observe that $X \times 1 \cong X$ and thus elements of the pullback C_a are merely elements $x \in X$ such that that ${}^r\eta^q(x) = [a] \in {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(X)$ i.e. all elements x and y in C_a are such that there is $k \in \mathbb{Z}$ such that $x = y \triangleleft^k y$. Hence by Lemma 2.5.4 the image of this pullback by ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$ gives indeed 1, which concludes the proof. □

As a consequence we could use *absolute Galois theory* in this context [**65**]. We stick to the relative approach (see Section 1) since we are interested in the composite of adjunctions as in Diagram (19) where the other two adjunctions (of the form $\mathrm{Pth} \dashv \mathrm{Conj}$) are not semi-left exact.

Observe that there is a limit to the exactness properties satisfied by ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$: we already saw in Paragraph 2.3.13 that ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$ cannot preserve finite products, since $\pi_0 \colon \mathsf{Qnd} \to \mathsf{Set}$ does but $\pi_0\, {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}} \colon \mathsf{Rck} \to \mathsf{Set}$ does not. Moreover, since Qnd is an idempotent subvariety of Rck, Proposition 2.6 of [**31**] induces that ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$ does not have *stable units* (in the sense of [**30**]).

To conclude, we show that, besides semi-left-exactness, the ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$-covering theory is "trivial" in the sense that all surjections are ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$-central (Proposition 2.5.11). We use the general strategy which was stated in Section 1.0.12. Since the Galois structure is strongly Birkhoff, the "first step influence" is as usual:

LEMMA 2.5.10. *A surjective morphism $f \colon X \to Y$, in the category of racks, is ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$-trivial if and only if $Q_X \cap \mathrm{Eq}(f) = \Delta_X$.*

PROOF. The morphism f is trivial if and only if the reflection square at f is a pullback (see Section 1.0.10, Diagram (31)). Since this reflection square is a double extension, it suffices for the comparison map to be injective. Since the square is a pushout, the kernel pair of the comparison map is given by the intersection $Q_X \cap \mathrm{Eq}(f)$ of the kernel pairs of q_X and f respectively. □

PROPOSITION 2.5.11. *All surjections $f \colon X \to Y$ in the category of racks are ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$-central.*

PROOF. In order to show this, consider the canonical projective presentation $\epsilon^r_Y \colon \mathrm{F_r}(UY) \to Y$, and take the pullback of f along ϵ^r_Y. This yields a morphism

$$\bar{f} \colon X \times_Y \mathrm{F_r}(UY) \to \mathrm{F_r}(UY).$$

Now any morphism $g\colon X \to \mathrm{F_r}(Y)$ with free codomain is ${}_{\mathrm{r}}\mathrm{F_q}$-trivial since if $x = x \lhd^k x$ in X for some integer k and if, moreover, $f(x) = f(x) \lhd^k f(x)$ in $\mathrm{F_r}(Y)$, then $f(x)^k = e$ by the free action of $\mathrm{Pth}(\mathrm{F_r}(Y))$ on $\mathrm{F_r}(Y)$. However this can only be if $k = 0$, which implies that $Q_X \cap \mathrm{Eq}(f) = \Delta_X$. $\square$

2.5.12. *Towards the free quandle.* Given a set A, in order to develop a good candidate description for the free quandle on A (see also [**82**]), we may now consider $\mathrm{F_q}(A)$ as the free quandle on the rack $\mathrm{F_r}(A)$. As aforementioned and roughly speaking, the following identifications between terms:

$$x \lhd^{\delta_x} x \lhd^{\delta_x} \cdots x \lhd^{\delta_x} x \lhd^{\delta_1} a_1 \cdots \lhd^{\delta_k} a_k = x \lhd^{\delta_1} a_1 \cdots \lhd^{\delta_k} a_k, \tag{20}$$

define the relation $Q_{\mathrm{F_r}(A)}$ such that $\mathrm{F_q}(A) = \mathrm{F_r}(A)/Q_{\mathrm{F_r}(A)}$.

We want to select one representative $(a, g) \in A \rtimes \mathrm{F_g}(A)$ for each equivalence class determined by these identifications. Thinking in terms of trails, we observe that if (a, g) and (b, h) are identified, then they must have the same head $a = b$. We thus focus on the paths and use a clever semi-direct product decomposition of $\mathrm{F_g}(A)$.

2.5.12.1. Characteristic of a path. We have the following commutative diagram in Set,

$$\begin{tikzcd}
A \arrow[r, "\eta^g_A"] \arrow[d, "\mathrm{Cst}"'] & \mathrm{F_g}(A) \arrow[d, dotted, "\chi := \mathrm{F_g}(\mathrm{Cst})"] \\
1 \arrow[r, "\eta^g_1"'] & \mathbb{Z} = \mathrm{F_g}(1),
\end{tikzcd}$$

where $\mathbb{Z}$ is the underlying set of the additive group of integers, and the composite $\eta^g_1\,\mathrm{Cst}$ is the constant function with image $1 \in \mathbb{Z}$. Given an element $g \in \mathrm{F_g}(A)$, there exists a decomposition $g = g_1^{\delta_1} \cdots g_n^{\delta_n}$ for some $g_i \in A$ and exponents $\delta_i = \{-1, 1\}$, with $1 \leq i \leq n$. The characteristic function sums up the exponents $\chi(g) = \sum_{i=1}^{n} \delta_i$ (of course the result does not depend on the chosen decomposition of g). We may then classify paths in $\mathrm{F_g}(A)$ in terms of their characteristic (i.e. their image by χ). Looking at Equation (20), two terms with same head, and same characteristic, that are moreover identified by $Q_{\mathrm{F_r}(A)}$, must actually be equal. In other words, given a fixed head a each equivalence class $[(a, g)]$ in $\mathrm{F_q}(A)$ has only one representative (a, g') such that the path g' is of a given characteristic.

2.5.12.2. Characteristic zero and semi-direct product decomposition. The kernel of χ defines a normal subgroup $\mathrm{F}^{\circ}_{\mathrm{g}}(A) \leq \mathrm{F}_{\mathrm{g}}(A)$ which is characterized (see [**82**] and Proposition 2.4.15) by

$$\mathrm{F}^{\circ}_{\mathrm{g}}(A) = \langle ab^{-1} \mid a, b \in A \rangle_{\mathrm{F}_{\mathrm{g}}(A)}.$$

Then for each $a \in A$, we may identify $\mathbb{Z}$ with the subgroup $\langle a^n \mid n \in \mathbb{Z} \rangle \leq \mathrm{F}_{\mathrm{g}}(A)$ which may be seen as the subgroup of $\mathrm{F}_{\mathrm{g}}(A)$ which fixes $[(a,e)] \in \mathrm{F}_{\mathrm{q}}(A) := \mathrm{F}_{\mathrm{r}}(A)/Q_{\mathrm{F}_{\mathrm{r}}(A)}$. This then gives a splitting for χ, on the left, yielding the split short exact sequence on the right:

$$\iota_a \colon \mathbb{Z} \to \mathrm{F}_{\mathrm{g}}(A) \ : \ k \mapsto a^k$$

$$\begin{tikzcd}
\mathrm{F}^{\circ}_{\mathrm{g}}(A) \arrow[r, tail, "\nu_A"] & \mathrm{F}_{\mathrm{g}}(A) \arrow[r, two heads, "\chi"] & \mathbb{Z} \arrow[l, bend left, tail, "\iota_a"]
\end{tikzcd}$$

2.5.12.3. Characteristic zero representatives. Then given an element $a \in A$, any $g \in \mathrm{F}_{\mathrm{g}}(A)$ decomposes uniquely as $\underline{a}^{\chi(g)} g_0$, where $g_0 = \underline{a}^{-\chi(g)} g$. This defines a function sending equivalence classes $[(a,g)] \in \mathrm{F}_{\mathrm{q}}(A)$, to their representatives of characteristic zero (a, g_0). Note that, for two different a and b in A, the construction of g_0 will vary, however elements of $\mathrm{F}_{\mathrm{r}}(A)$ with different heads are always sent to different equivalence classes in $\mathrm{F}_{\mathrm{q}}(A)$.

2.5.12.4. Transporting structure. This function is indeed bijective, and thus we may transport the quandle structure from the quotient $\mathrm{F}_{\mathrm{r}}(A)/Q_{\mathrm{F}_{\mathrm{r}}(A)}$ to the set of representatives $A \times \mathrm{F}^{\circ}_{\mathrm{g}}(A)$. More explicitly we compute for (b,h) and (a,g) in $\mathrm{F}_{\mathrm{r}}(A)$ that

$$(a, g_0) \triangleleft (b, h_0) = (a, g_0 h_0^{-1} \underline{b} h_0),$$

where $w := g_0 h_0^{-1} \underline{b} h_0$ is not of characteristic zero. We then want to take $w_0 = \underline{a}^{-1} g_0 h_0^{-1} \underline{b} h_0$ and define in $\mathrm{F}_{\mathrm{q}}(A)$:

$$(a, g_0) \triangleleft (b, h_0) := (a, w_0).$$

2.5.13. *The free quandle.* After this analysis, we may confidently build the free quandle (first described in [**82**]) as follows.

Given a set A the free quandle on A is given by

$$\mathrm{F}_{\mathrm{q}}(A) := A \rtimes \mathrm{F}^{\circ}_{\mathrm{g}}(A) := \{(a,g) \mid g \in \mathrm{F}^{\circ}_{\mathrm{g}}(A);\ a \in A\},$$

where the operations on $\mathrm{F}_{\mathrm{q}}(A)$ are defined for (a,g) and (b,h) in $A \rtimes \mathrm{F}^{\circ}_{\mathrm{g}}(A)$ by

$$(a,g) \triangleleft (b,h) := (a, \underline{a}^{-1} g h^{-1} \underline{b} h) \text{ and } (a,g) \triangleleft^{-1} (b,h) := (a, \underline{a} g h^{-1} \underline{b}^{-1} h).$$

As before, g is the *path* component and a is the *head* component of the so-called *trail* $(a, g) \in \mathrm{F_q}(A)$ and we say that an element (b, h) *acts on an element* (a, g) *by endpoint*. These operations indeed define a quandle structure.

From there, we translate all main results from the construction of free racks. Looking for the unit of the adjunction, we have the injective function $\eta^q_A \colon A \to \mathrm{F_q}(A) : a \mapsto (a, e)$.

Moreover, since any element $g \in \mathrm{F}^\circ_\mathrm{g}(A)$ decomposes as a product $g = \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n} \in \mathrm{F_g}(A)$ for some $g_i \in A$ and exponents $\delta_i \in \{-1, 1\}$, with $1 \leq i \leq n$, and $\sum_i \delta_i = 0$, we have, for any $(a, hg) \in \mathrm{F_q}(A)$ with g and $h \in \mathrm{F}^\circ_\mathrm{g}(A)$, a decomposition as

$$\begin{aligned}
(a, hg) &= (a, h\underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) \\
&= (a, \underline{a}^{\sum_i -\delta_i} h\underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) \\
&= (a, \underline{a}^{-\delta_n} \cdots \underline{a}^{-\delta_1} h\underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) \\
&= (a, h) \triangleleft^{\delta_1} (g_1, e) \cdots \triangleleft^{\delta_n} (g_n, e).
\end{aligned}$$

Observing that if $\underline{g_i}^{-\delta_i} = \underline{g_{i+1}}^{\delta_{i+1}}$ for some

$$(a, g) = (a, \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) \in \mathrm{F_q}(A)$$

as above, then

$$\begin{aligned}
(a, e) \triangleleft^{\delta_1} (g_1, e) \cdots \triangleleft^{\delta_{i-1}} (g_{i-1}, e) \triangleleft^{\delta_{i+2}} (g_{i+2}, e) \cdots \triangleleft^{\delta_n} (g_n, e) &= \\
= (a, \underline{g_1}^{\delta_1} \cdots \underline{g_{i-1}}^{\delta_{i-1}} \underline{g_{i+2}}^{\delta_{i+2}} \cdots \underline{g_n}^{\delta_n}) & \\
= (a, \underline{g_1}^{\delta_1} \cdots \underline{g_{i-1}}^{\delta_{i-1}} \underline{g_i}^{\delta_i} \underline{g_{i+1}}^{\delta_{i+1}} \underline{g_{i+2}}^{\delta_{i+2}} \cdots \underline{g_n}^{\delta_n}) & \\
= (a, e) \triangleleft^{\delta_1} (g_1, e) \cdots \triangleleft^{\delta_n} (g_n, e), &
\end{aligned}$$

which expresses the first axiom of racks, using group cancellation, as before.

From there we derive the universal property of the unit: given a function $f \colon A \to Q$ for some quandle Q, we show that f factors uniquely through η^q_A. Given an element $(a, g) \in \mathrm{F_q}(A)$, we have that for any decomposition $g = \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}$ as above, we must have

$$\begin{aligned}
f(a, g) &= f(a, \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n}) = f((a, e) \triangleleft^{\delta_1} (g_1, e) \cdots \triangleleft^{\delta_n} (g_n, e)) \\
&= f(a) \triangleleft^{\delta_1} f(g_1) \cdots \triangleleft^{\delta_n} f(g_n)
\end{aligned}$$

which uniquely defines the extension of f along η^q_A to a quandle homomorphism $f\colon \mathrm{F_q}(A) \to Q$. This extension is well defined since equal such decompositions in $\mathrm{F_q}(A)$ are equal after f by the first axiom of racks.

Finally the left adjoint $\mathrm{F_q}\colon \mathsf{Set} \to \mathsf{Qnd}$ of the forgetful functor $\mathrm{U}\colon \mathsf{Qnd} \to \mathsf{Set}$ with unit η^q is then defined on functions $f\colon A \to B$ by

$$\mathrm{F_q}(f) \coloneqq f \times \mathrm{F_g^\circ}(f)\colon A \rtimes \mathrm{F_g^\circ}(A) \to B \rtimes \mathrm{F_g^\circ}(B),$$

where $\mathrm{F_g^\circ}(f)$ is the restriction of $\mathrm{F_g}(f)$ to the normal subgroup $\mathrm{F_g^\circ}(A) \leq \mathrm{F_g}(A)$, whose image is in $\mathrm{F_g^\circ}(B)$. This defines quandle homomorphisms. Also functoriality of $\mathrm{F_q}$ and naturality of η^q are immediate.

2.5.13.1. Free action of $\mathrm{F_g^\circ}(A)$. Now remember the action by inner automorphisms of $\mathrm{F_g}(A) = \mathrm{Pth}(\mathrm{F_q}(A))$ defined by the commutative diagram in Set:

$$\begin{tikzcd}
A \arrow[rr, "\eta^g_A"] \arrow[dr, "\eta^q_A"'] & & \mathrm{F_g}(A) \arrow[dd, dotted, "s"] \\
& \mathrm{F_q}(A) \arrow[ur, "\mathrm{pth}_{\mathrm{F_q}(A)}"'] \arrow[dr, "\mathrm{S}"'] & \\
& & \mathrm{Inn}(\mathrm{F_q}(A)),
\end{tikzcd}$$

where s is the group homomorphism induced by the universal property of η^g_A or equivalently that of $\mathrm{pth}_{\mathrm{F_q}(A)}$.

This action is *not* in general given by left multiplication in $\mathrm{F_g^\circ}(A)$, since in particular an h in $\mathrm{F_g}(A)$ is of course not always of characteristic zero. However, from Paragraph 2.5.13 we deduce that whenever $h \in \mathrm{F_g^\circ}(A)$, the action of h on an element $(a, g) \in \mathrm{F_q}(A)$ gives (a, gh) as before.

Corollary 2.5.14. *The action of* $\mathrm{F_g^\circ}(A)$ *on* $\mathrm{F_q}(A)$ *given via the restriction*

$$\mathrm{F_g^\circ}(A) \xrightarrow{s^\circ} \mathrm{Inn}^\circ(\mathrm{F_q}(A)),$$

of s thus corresponds to the usual left-action of $\mathrm{F_g^\circ}(A)$ *in* Set*:*

$$(A \times \mathrm{F_g^\circ}(A)) \times \mathrm{F_g^\circ}(A)) \to A \times \mathrm{F_g^\circ}(A),$$

given by multiplication in $\mathrm{F_g^\circ}(A)$*. Such an action is free since if* $(a, gh) = (a, g)$*, then* $gh = g$ *and thus* $h = e$*.*

2.5.15. *The group of paths of a quandle.* Observe that the construction of χ for the free group $\mathrm{F_g}(A) = \mathrm{Pth}(\mathrm{F_r}(A))$ generalizes to any rack

X. The function $\mathrm{Cst}\colon X \to 1$ is actually a rack homomorphism to the trivial rack 1. It thus induces a group homomorphism $\chi = \mathrm{Pth}(\mathrm{Cst})$:

$$\begin{tikzcd}
X \arrow[r, "\mathrm{pth}_X"] \arrow[d, "\mathrm{Cst}"'] & \mathrm{Pth}(X) \arrow[d, dotted, "\chi=\mathrm{Pth}(\mathrm{Cst})"] \\
1 \arrow[r, "\mathrm{pth}_1"'] & \mathbb{Z} = \mathrm{Pth}(1).
\end{tikzcd}$$

As in the case of the free rack, we have the short exact sequence of groups:

$$\mathrm{Pth}^{\circ}(X) \xrightarrow{\nu_X} \mathrm{Pth}(X) \xrightarrow{\chi} \mathbb{Z} = \mathrm{Pth}(1),$$

where $\nu_X\colon \mathrm{Pth}^{\circ}(X) \to \mathrm{Pth}(X)$ is the kernel of χ. This construction defines a functor $\mathrm{Pth}^{\circ}\colon \mathsf{Rck} \to \mathsf{Grp}$. Most importantly it defines a functor $\mathrm{Pth}^{\circ}\colon \mathsf{Qnd} \to \mathsf{Grp}$ which can be interpreted as sending a quandle to its group of equivalence classes of primitive paths, such that two primitive paths are identified if one can be obtained from the other with respect to the axioms defining quandles. In the same way that Pth describes homotopy classes of paths in racks, Pth° describes homotopy classes of paths in quandles, as it was already explained in [**38**] and we shall rediscover in the covering theory described below.

2.5.15.1. The transvection group. As in the case of free groups, given a rack X, Proposition 2.4.16 implies that the kernel $\mathrm{Pth}^{\circ}(X)$ of χ is characterized as the subgroup:

$$\mathrm{Pth}^{\circ}(X) = \langle \underline{a}\, \underline{b}^{-1} \mid a,\, b \in X \rangle_{\mathrm{Pth}(X)}, \tag{21}$$

which is the definition that was used by D.E. Joyce in [**82**]. Then the restriction of the quotient $s\colon \mathrm{Pth}(X) \to \mathrm{Inn}(X)$ (defined in Subsection 2.1.9) yields the normal subgroup

$$\mathrm{Inn}^{\circ}(X) \coloneqq \langle \underline{a}\, \underline{b}^{-1} \mid a,\, b \in X \rangle_{\mathrm{Inn}(X)},$$

which was called the *transvection group* of X by D.E. Joyce.

This transvection group plays an important role in the literature. In the context of this work, we understand that the construction Pth° has better properties such as functoriality, and is of more significance to the theory of coverings than its image Inn° within inner automorphisms.

2.5.15.2. The case of free quandles. Observe that for a set X (for instance by Equation (21)), $\mathrm{Pth}^{\circ}(\mathrm{F_q}(X)) = \mathrm{F}^{\circ}_{\mathrm{g}}(X)$. As in the case of free racks we get that:

Proposition 2.5.16. *Given a set A, we may identify* $\mathrm{Inn}^\circ(\mathrm{F_q}(A)) = \mathrm{Pth}^\circ(\mathrm{F_q}(A)) = \mathrm{F}^\circ_\mathrm{g}(A)$*, and their actions on* $\mathrm{F_q}(A)$*. We refer to them as the* group of paths *of* $\mathrm{F_q}(A)$*. This group acts freely on* $\mathrm{F_q}(A)$ *by Corollary 2.5.14.*

Proof. Given a set A, the morphism $s^\circ \colon \mathrm{F}^\circ_\mathrm{g}(A) \to \mathrm{Inn}^\circ(\mathrm{F_q}(A))$ is a group isomorphism:

- it is surjective, since $\mathrm{Inn}^\circ(\mathrm{F_q}(A))$ is generated by the set
$$s(A)s(A)^{-1} = \{\mathrm{S}_{(a,e)}(\mathrm{S}_{(b,e)})^{-1} \mid a, b \in A\} \subset \mathrm{Inn}^\circ(\mathrm{F_q}(A))$$
which is the image of $AA^{-1} \subset \mathrm{F}^\circ_\mathrm{g}(A)$ by s;
- it is injective, as before because of the free action of $\mathrm{F}^\circ_\mathrm{g}(A)$ via s°. □

2.5.16.1. Inner automorphism groups. In the case of quandles, the group of inner automorphisms $\mathrm{Inn}(\mathrm{F_q}(A))$ is not isomorphic to $\mathrm{F_g}(A)$ in general. However, the only counter-example is actually the case $A = \{1\}$: $\mathrm{F_q}(\{1\}) = \{1\}$ is the trivial quandle on one element and $\mathrm{Inn}(\{1\}) = \{e\}$ is the trivial group, whereas $\mathrm{F_g}(\{1\})$ is $\mathbb{Z}$. Of course we do have $\mathrm{F}^\circ_\mathrm{g}(\{1\}) = \{e\}$. Now in all the other cases $\mathrm{Inn}(\mathrm{F_q}(A)) \cong \mathrm{F_g}(A)$. The case $A = \emptyset$ is trivial. Then whenever
$$x \triangleleft^{\delta_x} x \triangleleft^{\delta_x} \cdots x \triangleleft^{\delta_x} x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_k} a_k = x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_k} a_k,$$
it suffices to pick $y \neq x \in A$ and then
$$y \triangleleft^{\delta_x} x \triangleleft^{\delta_x} x \triangleleft^{\delta_x} \cdots x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_k} a_k \neq y \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_k} a_k,$$
showing that in $\mathrm{Inn}(\mathrm{F_q}(A))$:
$$\underline{x}^{\delta_x}\underline{x}^{\delta_x}\cdots\underline{x}^{\delta_x}\underline{a_1}^{\delta_1}\cdots\underline{a_k}^{\delta_k} \neq \underline{a_1}^{\delta_1}\cdots\underline{a_k}^{\delta_k},$$
just as in $\mathrm{Inn}(\mathrm{F_r}(A))$.

3. Covering theory of racks and quandles

In this section we study the relative notion of centrality induced by the sphere of influence of Set in Rck, with respect to extensions (surjective homomorphisms). Remember that pullbacks of primitive extensions (surjections in Set) along the unit η induce the concept of trivial extensions, which we saw are those extensions which reflect *loops*. Central extensions in Rck are those from which a trivial extension can be reconstructed by pullback along another extension. Equivalently, central extensions are those extensions whose pullback, along a projective presentation of their codomain, is trivial. In Section 3.1 we thus look for a condition (C) such that, if a surjective rack homomorphism $f\colon A \to B$ satisfies (C), then the pullback t of f along $\epsilon^r_B\colon \mathrm{F_r}(B) \to B$ reflects loops (see Section 1 and references there).

3.1. One-dimensional coverings.

Quandle coverings were defined in [**38**], and shown to characterize Γ_q-central extensions of quandles in [**39**]. We give the same definition for rack coverings (already suggested in M. Eisermann's work), which we then characterize in several ways. In Section 3.2 we further show that these are exactly the central extensions of racks.

Remember that in dimension zero, a rack A is actually a set, if *zero-dimensional data*, i.e. an element $a \in A$, acts trivially on any element $x \in A$: $x \triangleleft a = x$. We saw that this may be expressed by the fact that $\mathrm{Pth}(A)$ acts trivially on A or alternatively by the fact that any two elements which are connected by a primitive path are actually equal.

Now in *dimension one*, an extension $f\colon A \twoheadrightarrow B$ is a covering if *one-dimensional data*, i.e. a pair (a, b) in the kernel pair of f, acts trivially on any element in A:

DEFINITION 3.1.1. *A morphism of racks $f\colon A \to B$ is said to be a* covering *if it is surjective and for each pair $(a, b) \in \mathrm{Eq}(f)$, and any $x \in A$ we have*

$$x \triangleleft a \triangleleft^{-1} b = x.$$

Of course a trivial example is given by surjective functions between sets (the primitive extensions). The following implies that central extensions are coverings:

LEMMA 3.1.2. *Coverings are preserved and reflected by pullbacks along surjections in* Rck.

PROOF. Same proof as in [**40**] see also [**39**]. □

3.1.3. *Coverings and the group of paths.* Observe that given data f, x, a and b, such as in Definition 3.1.1, we have in particular that $x \triangleleft^{-1} a = x \triangleleft^{-1} a \triangleleft a \triangleleft^{-1} b = x \triangleleft^{-1} b$. In fact we can easily deduce that f is a covering if and only if for all such x, a and b as before

$$x \triangleleft^{-1} a \triangleleft b = x.$$

This is to say that f is a covering if and only if any path of the form $\underline{a}\,\underline{b}^{-1}$ or $\underline{a}^{-1}\underline{b} \in \mathrm{Pth}(A)$, for a and b in A, such that $f(a) = f(b)$, acts trivially on elements in A. But then f is a covering if and only if the subgroup of $\mathrm{Pth}(A)$ generated by those elements acts trivially on elements of A. Now, given $g \in \mathrm{Pth}(A)$, if $z \cdot g = z$ for all z in A, then also $x \cdot a^{-1} \cdot g \cdot a = (x \triangleleft^{-1} a) \cdot g \cdot a = (x \triangleleft^{-1} a) \cdot a = x$ for all $a \in A$. Hence we conclude that f is a covering if and only if the normal subgroup $\langle\langle ab^{-1} \mid (a,b) \in \mathrm{Eq}(f) \rangle\rangle_{\mathrm{Pth}(A)}$ acts trivially on elements of A. Finally by Proposition 2.4.16 we get the following result which illustrates the importance of Pth in the covering theory of racks and quandles.

THEOREM 3.1.4. *Given a surjective morphism* $f\colon A \to B$ *in* Rck *(or in* Qnd*), the following conditions are equivalent:*

(1) f *is a covering;*
(2) the group of $\vec{f}$*-symmetric paths* $\mathrm{K}_{\vec{f}}$ *acts trivially on* A *(as a subgroup of* $\mathrm{Pth}(A)$*) – i.e. any* f*-symmetric trail loops in* A*;*
(3) $\mathrm{Ker}(\vec{f})$ *acts trivially on* A *(as a subgroup of* $\mathrm{Pth}(A)$*);*
(4) $\mathrm{Ker}(\vec{f})$ *is a subobject of the kernel* $\mathrm{Ker}(s)$*, where*

$$s\colon \mathrm{Pth}(A) \to \mathrm{Inn}(A)$$

is the canonical quotient described in Paragraph 2.4.5.

PROOF. The statements (1), (2) and (3) are equivalent by the previous paragraph (and thus by Proposition 2.4.16). Statement (4) is merely a way to rephrase (3) using the fact that elements of the inner automorphism groups are defined by their action. ☐

As it was observed by M. Eisermann in Qnd, we have:

COROLLARY 3.1.5. *A rack covering* $f\colon A \to B$ *induces a surjective morphism* $\bar{f}\colon \mathrm{Pth}(B) \to \mathrm{Inn}(A)$ *such that* $\vec{f}\bar{f} = s$ *and thus induces an action of* $\mathrm{Pth}(B)$ *on* A *given for* $g_B \in \mathrm{Pth}(B)$ *and* $x \in A$ *by* $x \cdot g_B \coloneqq x \cdot g_A$*, where* g_A *is any element in the pre-image* $\vec{f}^{-1}(g_b)$*.*

Observe that an easy way to obtain a rack covering is by constructing a quotient $f\colon A \twoheadrightarrow B$ such that $\vec{f}$ is an isomorphism.

EXAMPLE 3.1.6. *The components of the unit* ${}^{r}\eta^{q}$ *of the* ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$ *adjunction are rack coverings. Indeed, we discussed in Paragraph 2.4.20 that* $\mathrm{Pth}\ {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}} = \mathrm{Pth}$*, also see Paragraph 2.1.11. In particular, we look at the* *one element set* 1 *and consider the map* $f := {}^r\eta^q_{\mathrm{F_r}(1)} \colon \mathrm{F_r}(1) \to \mathrm{F_q}(1) = 1$. *We then compute that* $\vec{f} = \mathrm{Pth}({}^r\eta^q_{\mathrm{F_r}(1)})$ *and* $\mathrm{Inn}(f) = \mathrm{Inn}({}^r\eta^q_{\mathrm{F_r}(1)})$ *are respectively the morphisms*

$$\mathrm{Pth}(\mathrm{F_r}(1)) = \mathbb{Z} \xrightarrow{\mathrm{id}_{\mathbb{Z}}} \mathrm{Pth}(\mathrm{F_q}(1)) = \mathbb{Z}$$

and

$$\mathrm{Inn}(\mathrm{F_r}(1)) = \mathbb{Z}_3 \longrightarrow \mathrm{Inn}(\mathrm{F_q}(1)) = \{e\} \ ,$$

where $\mathbb{Z}$ *is the infinite cyclic group,* $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ *is the cyclic group with* 3 *elements and* $\{e\}$ *the trivial group. In this case* $\vec{f}$ *is an isomorphism, but* $\mathrm{Inn}(f)$ *is not.*

REMARK 3.1.7. *In the article* [**22**]*, Theorem 4.2 says that quandle coverings (such as in* (3) *of Proposition 3.1.4 above) should coincide with* rigid quotients *of quandles, i.e. surjective morphisms* $f \colon A \to B$ *which induce an isomorphism* $\mathrm{Inn}(f) \colon \mathrm{Inn}(A) \to \mathrm{Inn}(B)$. *Looking at the proof on page 1150, the authors assume "by construction" that the map* η *(between the* excess *of* Q *and* R [**55**]*) is surjective, which is equivalent to asking for the bottom right-hand square* c_R *Adconj*$(h) = \mathrm{Inn}(h)\ c_Q$ *to be a pushout. This does not seem to hold in the generality asked for in* [**22**]. *Note that these results are presented in such a way that they should also hold in* Rck*, since the idempotency axiom is never used. Then the example above provides a counter-example to* [**22**, Theorem 4.2] *in* Rck. *We further give a counter-example in* Qnd*, which shows that* [**22**, Theorem 4.2] *must be incorrect.*

EXAMPLE 3.1.8. *Consider the quandle* $Q_{ab\star}$ *from Example 2.3.14, which by Example 2.4.19 is such that* $\mathrm{Pth}(Q_{ab\star}) = \mathbb{Z}\times\mathbb{Z}$ *with* $\underline{a} = \underline{b} = (1,0)$ *and* $\underline{\star} = (0,1)$. *Moreover, observe that the trivial quandle with two elements* $\pi_0(Q_{ab\star})$ *is also such that* $\mathrm{Pth}(\pi_0(Q_{ab\star})) = \mathrm{F_{ab}}(\{[a],[\star]\}) = \mathbb{Z}\times\mathbb{Z}$ *where* $\underline{[a]} = (1,0)$ *and* $\underline{[\star]} = (0,1)$. *Hence the morphism of quandles* $f := \eta_{Q_{ab\star}} \colon Q_{ab\star} \to \pi_0(Q_{ab\star})$ *is such that* $\vec{f} = \mathrm{id}_{\mathbb{Z}\times\mathbb{Z}}$. *In particular* $\mathrm{Ker}(\vec{f}) = \{e\}$ *is the trivial group, but* $\mathrm{Inn}(f) \colon \mathbb{Z}/2\mathbb{Z} \to \{e\}$ *is not an isomorphism.*

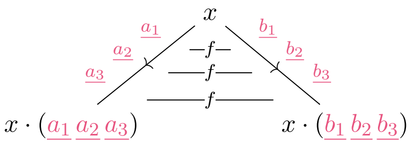
Other such examples can be built using morphisms between quandles from Example 1.3, as well as Proposition 2.27 and Remark 2.28 in [**38**].

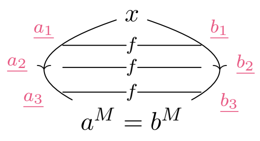
3.1.9. *Visualizing coverings.* Coverings are characterized by the trivial action of f-symmetric paths, which are the elements $g = g_a g_b^{-1} \in \mathrm{Pth}(A)$ such that g_a and g_b are f-symmetric to each other. Notice that an f-symmetric pair g_a, g_b is obtained from the projections of a primitive path in $\mathrm{Eq}(f)$. We emphasize the geometrical aspect of these 2-dimensional primitive paths by defining *membranes* and *horns*. An f-symmetric trail is a compact 1-dimensional concept which remains so when generalized to higher dimensions. The concept of f-horn allows for a more visual, geometrical and elementary description of these ingredients as well as their higher-dimensional generalizations.

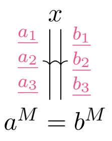
DEFINITION 3.1.10. *Given a morphism $f\colon A \to B$ in* Rck *(or* Qnd*), we define an f-membrane $M = ((a_0, b_0), ((a_i, b_i), \delta_i)_{1\leq i\leq n})$ to be the data of a primitive trail in* $\mathrm{Eq}(f)$ *(see Paragraph 2.3.3). We call such an f-membrane M an f-horn if $a_0 = b_0 =: x$ which we denote $M = (x, (a_i, b_i, \delta_i)_{1\leq i\leq n})$. The* associated f-symmetric pair *of the membrane or horn M is given by the paths $g_a^M := \underline{a_1}^{\delta_1} \cdots \underline{a_n}^{\delta_n}$ and $g_b^M := \underline{b_1}^{\delta_1} \cdots \underline{b_n}^{\delta_n}$ in* $\mathrm{Pth}(A)$*. The* top trail *is $t_a = (a_0, g_a^M)$ and the* bottom trail *is $t_b = (b_0, g_b^M)$. The* endpoints *of the membrane or horn are given by $a_M = a_0 \cdot g_a^M$ and $b_M = b_0 \cdot g_b^M$.*

Given an f-symmetric trail (x, g) for $g = g_a g_b^{-1} \in \mathrm{Ker}(\vec{f})$ as before, there is an f-horn such that its associated f-symmetric pair is given by g_a and g_b (in particular the associated f-symmetric trail is then (x, g)). Given a horn $M = (x, (a_i, b_i, \delta_i)_{1\leq i\leq n})$, we represent it (with $n = 3$ and $\delta_i = 1$ for $1 \leq i \leq 3$) as in the left-hand diagram below.

DEFINITION 3.1.11. *A horn $M = (x, (a_i, b_i, \delta_i)_{1\leq i\leq n})$ is said to* close *(into a* disk*) if its endpoints are equal $a^M = x \cdot g_a^M = x \cdot g_b^M = b^M$. The horn M is said to* retract *if for each $1 \leq k \leq n$, the truncated horn $M_{\leq k} := (x, (a_i, b_i, \delta_i)_{1\leq i\leq k})$ closes.*







COROLLARY 3.1.12. *A surjective morphism $f\colon A \twoheadrightarrow B$ in* Rck *(or* Qnd*) is a covering if and only if every f-horn retracts (or equivalently, if every f-horn closes into a disk).*

3.1.13. *Visualizing normal extensions.* The normal extensions of quandles are described by V. Even in [**39**]. The same description works in racks. We reinterpret it using our own terminology.

DEFINITION 3.1.14. *Given a surjective morphism $f\colon A \to B$ in* Rck*, together with an f-membrane $M = (a_i, b_i, \delta_i)_{0\leq i\leq n}$, we say that the membrane M* forms a cylinder *if both the top and the bottom trails of M are loops.*

PROPOSITION 3.1.15. *A surjective morphism $f\colon A \to B$ in* Rck *(or* Qnd*) is a normal extension if and only if f-membranes are* rigid*, i.e. if and only if given any f-membrane $M = (a_i, b_i, \delta_i)_{0\leq i\leq n}$, M forms a cylinder as soon as either the top or the bottom trail of M is a loop.*

PROOF. The surjection f is normal if and only if the projections $\pi_1, \pi_2\colon \mathrm{Eq}(f) \rightrightarrows A$ of the kernel pair of f are trivial. Such projections are trivial if and only if they reflect loops. The π_1 (resp. π_2) projection of a trail $t = ((a_0, b_0), h)$ in $\mathrm{Eq}(f)$ loops if and only if there is an f-membrane $M = ((a_0, b_0), ((a_i, b_i), \delta_i)_{1\leq i\leq n})$ such that $\vec{\pi}_1(h) = g_a^M$, $\vec{\pi}_2(h) = g_b^M$ and the top (resp. bottom) trail of M loops (see also [**39**, Proposition 3.2.3]). $\square$

3.2. Characterizing central extensions. V. Even's strategy to prove the characterization is to split coverings along the weakly universal covers constructed by M. Eisermann. These weakly universal covers can be understood as the centralization of the canonical projective presentations (using free objects – see Section 3.5). Their structure and properties used to show V. Even's result derive from the structure and properties of the free objects we described before. Thus even though V. Even's proof can be translated to the context of racks, we prefer to work directly with free objects in the alternative proof below. This approach then easily generalizes to higher dimensions without us having to build the weakly universal higher-dimensional coverings from scratch.

PROPOSITION 3.2.1. *Any rack-covering $f\colon A \to \mathrm{F_r}(B)$ with free codomain is a trivial extension.*

PROOF. In order to test whether f is a trivial extension, consider $x \in A$ and $g = \underline{a_1}^{\delta_1} \cdots \underline{a_n}^{\delta_n}$ in $\mathrm{Pth}(A)$ for $n \in \mathbb{N}$, $a_1, \ldots, a_n$ in A and $\delta_1, \ldots, \delta_n$ in $\{-1, 1\}$. Assume that f sends the trail (x, g) to the loop $(f(x), \vec{f}(g))$:

$$\begin{aligned} f(a) \cdot (\underline{f(a_1)}^{\delta_1} \cdots \underline{f(a_n)}^{\delta_n}) &= f(x) \triangleleft^{\delta_1} f(a_1) \cdots \triangleleft^{\delta_n} f(a_n) \\ &= f(x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n) = f(x), \end{aligned}$$

where we write $\underline{f(a_i)} := \mathrm{pth}_{\mathrm{F_r}(B)}(f(a_i))$ (which does not mean that $f(a_i)$ is in B). We have to show that (x, g) was a loop in the first place:

$$x \cdot g = x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n = x.$$

$(*)$ Since $\mathrm{F_r}(B)$ is projective (with respect to surjective morphisms) and f is surjective, there is a morphism of racks

$$s\colon \mathrm{F_r}(B) \to A$$

such that $fs = 1_{\mathrm{F_r}(B)}$. Then s induces a group homomorphism

$$\vec{s}\colon \mathrm{Pth}(\mathrm{F_r}(B)) \to \mathrm{Pth}(A)$$

such that for each $1 \leq i \leq n$,

$$\vec{s}[\underline{f(a_i)}] = \mathrm{pth}_A(sf(a_i)) =: \underline{sf(a_i)}$$

(see Paragraph 2.4.2), and thus

$$e = \vec{s}[\underline{f(a_1)}]^{\delta_1} \cdots \vec{s}[\underline{f(a_n)}]^{\delta_n} = \underline{sf(a_1)}^{\delta_1} \cdots \underline{sf(a_n)}^{\delta_n}.$$

Hence in particular we have

$$x \triangleleft^{\delta_1} sf(a_1) \cdots \triangleleft^{\delta_n} sf(a_n) = x \cdot (\underline{sf(a_1)}^{\delta_1} \cdots \underline{sf(a_n)}^{\delta_n}) = \ x \cdot e = x.$$

Finally since for each $1 \leq i \leq n$ we have $f(sf(a_i)) = f(a_i)$, $M = (x, (a_i, sf(a_i), \delta_i)_{1 \leq i \leq n})$ is an f-horn, which has to retract since f is a covering:

$$x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_n} a_n = x \triangleleft^{\delta_1} sf(a_1) \cdots \triangleleft^{\delta_n} sf(a_n) = x. \qquad \square$$

In this one-dimensional context, the characterization of coverings from Proposition 3.1.4 allows for a shorter version of this proof. Since a direct generalization of Proposition 3.1.4 in higher dimensions is not yet clear to us, we prefer to keep the previous, more visual version of the proof as our main reference. However, you may want to replace what follows $(*)$ in the previous proof by:

Proof. [...] $(*)$ Now since the action of $\mathrm{Pth}(\mathrm{F_r}(B))$ on $\mathrm{F_r}(B)$ is free, any loop in $\mathrm{Pth}(\mathrm{F_r}(B))$ must be trivial, and in particular

$$\underline{f(a_1)}^{\delta_1} \cdots \underline{f(a_n)}^{\delta_n} = e.$$

Hence $g \in \mathrm{Ker}(\vec{f})$, and thus by Proposition 3.1.4, $x \cdot g = x$, which concludes the proof. $\square$

Note finally that the exact same proofs work for quandle coverings, using the fact that if A is a quandle, we may then always choose a_i's and δ_i's such that $\sum_i \delta_i = 0$. Then $\underline{f(a_1)}^{\delta_1} \cdots \underline{f(a_n)}^{\delta_n}$ is in $\mathrm{Pth}^{\circ}(\mathrm{F}_{\mathrm{q}}(B))$ which acts freely on $\mathrm{F}_{\mathrm{q}}(B)$. The rest of both proofs remain identical.

PROPOSITION 3.2.2. *If a quandle-covering $f\colon A \to \mathrm{F}_{\mathrm{q}}(B)$ has a free codomain, then it is a trivial extension.*

By Lemma 3.1.2, and the previous propositions, the strategy of Section 1.0.12 yields Theorem 2 from [**39**], as well as:

THEOREM 3.2.3. *Rack coverings are the same as central extensions of racks.*

3.3. Comparing admissible adjunctions by factorization. The notions of trivial object and connectedness, or trivialising relation C_0, coincide in racks and quandles. These are understood as the zero-dimensional central extensions and centralizing relations. In dimension 1, the notions of central extensions in racks and quandles also coincide. Further we also have coincidence of the centralizing relations and the corresponding notions in dimension 2. Before we move on, we show how these results are no coincidence and can be studied systematically as a consequence of the tight relationship between the π_0-admissible adjunctions of interest.

Expanding on Paragraph 2.3.8 we get a factorization as in 2.4.20, where all triangles commute and all the adjunctions are admissible:

$$\begin{tikzcd}
\mathsf{Rck} \arrow[rr, bend left, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}"] \arrow[rr, phantom, "\perp"] \arrow[dr, bend right, "\pi_0"'] \arrow[dr, phantom, "\dashv"] & & \mathsf{Qnd} \arrow[ll, bend left, "\mathrm{I}"] \arrow[dl, bend right, "\pi_0"'] \arrow[dl, phantom, "\dashv"] \\
& \mathsf{Set}. \arrow[ul, bend right, "\mathrm{I}"'] \arrow[ur, bend right, "\mathrm{I}"'] &
\end{tikzcd}$$

Since we are dealing here with several different Galois structures: Γ from Rck to Set, ${}_r\Gamma_q$ from Rck to Qnd and say $\Gamma_q := (\mathsf{Qnd}, \mathsf{Set}, \pi_0, \mathrm{I}, \eta, \epsilon, \mathcal{E})$; we specify the Galois structure with respect to which the concepts of interest are discussed.

LEMMA 3.3.1. *If* $f\colon A \to B$ *is a* Γ*-trivial extension, then* f *is also* ${}_r\Gamma_q$*-trivial, and the image* ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)$ *of* f *is a* Γ_q*-trivial extension in* Qnd.

PROOF. The Γ-canonical square of f in Rck is given on the left, and factorizes into the composite of double extensions on the right:

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & \pi_0(A) \arrow[d, "\pi_0(f)"] \\
B \arrow[r, "\eta_A"'] & \pi_0(B),
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "{}^r\eta^q_A"] \arrow[d, "f"'] & {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(A) \arrow[r, "\eta_{{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(A)}"] \arrow[d, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)"'] & \pi_0(A) = \pi_0({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(A)) \arrow[d, "\pi_0(f)"] \\
B \arrow[r, "{}^r\eta^q_B"'] & {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B) \arrow[r, "\eta_{{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)}"'] & \pi_0(B) = \pi_0({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)).
\end{tikzcd}$$

Hence if f is a trivial extension, then this composite is a pullback square. The composite of two double extensions is a pullback if and only if both double extensions are pullbacks themselves (see for instance Part II, Lemma 1.1.4). ☐

LEMMA 3.3.2. *An extension* $f\colon A \to B$ *in* Qnd *is*

(i) Γ_q*-trivial in* Qnd *if and only if* $\mathrm{I}(f)$ *is* Γ*-trivial in* Rck*;*
(ii) Γ_q*-central in* Qnd *if and only if* $\mathrm{I}(f)$ *is* Γ*-central in* Rck*.*

PROOF. The first point (i) is immediate by the previous lemma, and the fact that the π_0-canonical squares of $\mathrm{I}(f)$ in Rck is the same as the image by I of the Γ_q-canonical square of f in Qnd. Note also that I preserves and reflects pullbacks.

For the second statement (ii), if f is Γ_q-central, then there is an extension $p\colon E \to B$ such that the pullback of f along p is Γ_q-trivial. We may conclude by taking the image by I of this pullback square. Now if $\mathrm{I}(f)$ is Γ-central in Rck, there exists $p\colon E \to B$ in Rck such that the pullback t of $\mathrm{I}(f)$ along p is Γ-trivial in Rck. Taking the quotient along ${}^r\eta^q$ of this pullback square (1) yields a factorization of (1):

$$\begin{tikzcd}
E \times_B A \arrow[r, "{}^r\eta^q_P"] \arrow[d, "t"'] & {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(E \times_B A) \arrow[r] \arrow[d, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(t)"'] & A \arrow[d, "f"] \\
E \arrow[r, "{}^r\eta^q_E"'] & {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(E) \arrow[r, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(p)"'] & B.
\end{tikzcd}$$

Again, since the left hand square is a double extension, and the composite is a pullback, both squares are actually pullbacks and thus f is Γ_q-central. ☐

Now since the π_0-adjunction is strongly Birkhoff (both in Rck and Qnd), central extensions are closed by quotients along double extensions in ExtRck (or ExtQnd – see also Proposition 3.4.7).

COROLLARY 3.3.3. *The image by* ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$ *of a* Γ*-central extension* $f\colon A \to B$ *in* Rck *is a* Γ_q*-central extension in* Qnd*.*

PROOF. The image ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)$ is Γ_q-central extension if and only if $\mathrm{I}({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f))$ is Γ-central. Since Set is strongly Birkhoff in Rck, $\mathrm{I}({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f))$ is the quotient of a Γ-central extension in Rck along a double extension and thus is still Γ-central in Rck. ☐

PROPOSITION 3.3.4. *If the image by* ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$ *of an* ${}_r\Gamma_q$*-trivial extension* $f\colon A \to B$ *in* Rck *is a* Γ_q*-central extension in* Qnd*, then* f *is* Γ*-central in* Rck*.*

PROOF. Consider the following commutative cube in Rck where we omit the inclusion $\mathrm{I}\colon \mathsf{Qnd} \to \mathsf{Rck}$. The back face is a pullback by construction. The right hand face is a pullback by assumption, and the left hand face is a pullback by Proposition 2.5.11. We deduce that the front face is a pullback as well.

$$\begin{tikzcd}
 & P_1 \arrow[rr] \arrow[dl, "{}^r\eta^q_{P_1}"'] \arrow[dd, "t"] & & A \arrow[dl, "{}^r\eta^q_A"'] \arrow[dd, "f"] \\
{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(P_1) \arrow[rr] \arrow[dd, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(t)"'] & & {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(A) \arrow[dd, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)"] & \\
 & \mathrm{F}_{\mathrm{r}}(B) \arrow[rr, "\epsilon^r_B"] \arrow[dl, "{}^r\eta^q_{\mathrm{F}_{\mathrm{r}}(B)}"'] & & B \arrow[dl, "{}^r\eta^q_B"] \\
\mathrm{F}_{\mathrm{q}}(B) \arrow[rr, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(\epsilon^r_B)"'] & & {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B) &
\end{tikzcd}$$

Since ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)$ is Γ_q-central by assumption, and since

$$ {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(\epsilon^r_B)\colon \mathrm{F}_{\mathrm{q}}(B) = {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(\mathrm{F}_{\mathrm{r}}(B)) \to {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B) $$

factorizes as

$$ \mathrm{F}_{\mathrm{q}}(B) \xrightarrow{\mathrm{F}_{\mathrm{q}}({}^r\eta^q_B)} \mathrm{F}_{\mathrm{q}}({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)) \xrightarrow{\mathrm{F}_{\mathrm{q}}(\epsilon^q_{{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)})} {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B), $$

both ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(t)$ and t are Γ-trivial as the pullback of a trivial extension. ☐

EXAMPLE 3.3.5. *Some extensions of racks which are not central, still have central images under* ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$. *Define the involutive rack with underlying set* $\{a, a^2, b, b^2, 1, 2\}$, *and an operation* $\triangleleft$ *such that* a, a^2, b *and* b^2 *have the same action and, moreover,*

$\triangleleft$	a	a^2	b	b^2	1	2
a	a^2	a	b^2	b	1	2
1	b	b^2	a	a^2	1	2
2	b^2	b	a^2	a	1	2

We may check by hand that the axioms (R1) *and* (R2) *are satisfied. Then define the morphism of racks* f, *with codomain the trivial rack* $\{x, 1\}$ *and which sends letters to* x *and numbers to* $\star$. *We have that* $a \triangleleft 1 = b \neq b^2 = a \triangleleft 2$, *and thus* f *is not central. However we compute the morphism* ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)\colon Q_{ab\star\star} \to \{x, \star\}$, *where* $Q_{ab\star\star}$ *is as in Example 2.3.14 but with two distinct* $\star$*'s which act in the same way. This morphism merely identifies the letters and the stars and thus it is central.*

Of course some rack homomorphisms which are not ${}_{r}\Gamma_{q}$*-trivial are still* Γ*-central: we already mentioned the important example of* ${}^{r}\eta^{q}_{A}$ *for any rack* A.

Before even studying the next steps of the covering theory, we can predict that what happens in Qnd directly follows from what happens in Rck.

COROLLARY 3.3.6. *If the full subcategory* $\mathsf{CExtRck}$ *of central extensions of racks is reflective within the category of extensions* ExtRck *(see Theorem 3.3.1 for details), then also* $\mathsf{CExtQnd}$ *is reflective in* ExtQnd *and the reflection is computed as in* ExtRck, *via the inclusion* $\mathrm{I}\colon \mathsf{Qnd} \to \mathsf{Rck}$.

PROOF. Since Qnd is closed under quotients in Rck, the centralization of an extension in $\mathsf{Qnd} \subseteq \mathsf{Rck}$ yields an extension in Qnd which is moreover central by Lemma 3.3.2. The universality in $\mathsf{CExtQnd}$ directly derives from the universality in $\mathsf{CExtRck}$ by the same arguments. □

3.4. Centralizing extensions. We adapt the result from [**36**], showing the reflectivity of quandle coverings in the category of extensions, to the context of racks. We put the emphasis on our new characterizations of the centralizing relation which works the same for racks and for quandles. We also prepare the ingredients to show the admissibility of coverings within extensions, and the forthcoming covering theory in dimension 2.

Let us define $\mathcal{E}_1$ to be the class of double extensions in ExtRck.

THEOREM 3.4.1. *The category* $\mathsf{CExtRck}$ *is an* $(\mathcal{E}_1)$*-reflective subcategory of the category* ExtRck *with left adjoint* F_1 *and unit* η^1 *defined for an object* $f\colon A \to B$ *in* ExtRck *by* $\eta^1_f := (\eta^1_A, \mathrm{id}_B)$*, where* $\eta^1_A\colon A \to A/\,\mathrm{C}_1(f)$ *is the quotient of* A *by the* centralizing *congruence* $\mathrm{C}_1(f)$*, which can be defined in the following equivalent ways:*

- *(i)* $\mathrm{C}_1(f)$ *is the equivalence relation on* A *generated by the pairs* $(x \lhd a \lhd^{-1} b, x)$ *for* x*,* a*, and* b *in* A *such that* $f(a) = f(b)$*,*
- *(ii) a pair* (a, b) *of elements from* A *is in the equivalence relation* $\mathrm{C}_1(f)$ *if and only if* a *and* b *are the endpoints of a horn, i.e. there exists a horn* $M = (x, (a_i, b_i, \delta_i)_{1\leq i\leq n})$ *such that* $x \cdot g^M_a = a$ *and* $x \cdot g^M_b = b$*,*
- *(iii)* $\mathrm{C}_1(f)$ *is the orbit relation* $\sim_{\mathrm{Ker}(\vec{f})}$ *(or equivalently* $\sim_{\mathrm{K}_{\vec{f}}}$*) induced by the action of the kernel of* $\vec{f}$ *(i.e. the group of* f*-symmetric paths).*

Observing that $\mathrm{C}_1(f) \leq \mathrm{Eq}(f)$*, the image of* f *by* F_1 *is defined as the unique factorization of* f *through this quotient:*

$$\begin{tikzcd}
A \arrow[rr, two heads, "f"] \arrow[dr, two heads, "\eta^1_A"'] & & B \\
& A/\,\mathrm{C}_1(f) \arrow[ur, two heads, "\mathrm{F}_1(f)"'] &
\end{tikzcd}$$

The definition of F_1 *on morphisms* $\alpha = (\alpha_\top, \alpha_\bot)\colon f_A \to f_B$ *decomposes into the top component* $\mathrm{F}_1^\top(\alpha)\colon A_\top/\,\mathrm{C}_1(f_A) \to B_\top/\,\mathrm{C}_1(f_B)$ *defined by the universal property of the quotients* $\eta^1_{A_\top}$ *for* $f_A\colon A_\top \to A_\bot$*; and the bottom component* $\mathrm{F}_1{}^\bot(\alpha) = \alpha_\bot$ *which simply returns the bottom component of* α*.*

PROOF. Using definition (*i*) for the centralizing relation, the proof of Theorem 5.5 in **[36]** easily translates to the context of racks. Then given an extension $f\colon A \to B$, the unit $\eta^1_f = (\eta^1_A, \mathrm{id}_B)$ is indeed a double extension since its bottom component is an isomorphism. It remains to show that the definitions (*ii*) and (*i*) are equivalent, since (*iii*) is equivalent to (*ii*) by Proposition 2.4.16.

First we show by induction on $n \in \mathbb{N}$ that $\mathrm{C}_1(f)$, defined as in (i), contains all pairs that are endpoints of a horn. Then we show that the collection of such pairs defines a congruence containing the generators of $\mathrm{C}_1(f)$. This then concludes the proof.

Step 0 is satisfied by reflexivity of $\mathrm{C}_1(f)$. Now assume that if (a,b) is a pair of elements in A, which are the endpoints of a given horn $M = (x, (a_i, b_i, \delta_i)_{1 \leq i \leq n})$ of length $n \leq k$, for some fixed natural number k, then $(a,b) \in \mathrm{C}_1(f)$. We show that the endpoints $a := x \cdot g_a^M$ and $b := x \cdot g_b^M$ of any given horn $M = (x, (a_i, b_i, \delta_i)_{1 \leq i \leq k+1})$ of length $k+1$ are in relation by $\mathrm{C}_1(f)$. Indeed, define $a' = a \triangleleft^{-\delta_{k+1}} a_{k+1}$ and $b' = b \triangleleft^{-\delta_{k+1}} b_{k+1}$. Then we have that $(a', b') \in \mathrm{C}_1(f)$ by assumption and, moreover,

$$\left(a = a' \triangleleft^{\delta_{k+1}} a_{k+1}\right) \ \mathrm{C}_1(f) \ \left(b' \triangleleft^{\delta_{k+1}} a_{k+1}\right) \ \mathrm{C}_1(f) \ \left(b' \triangleleft^{\delta_{k+1}} b_{k+1} = b\right)$$

by compatibility of $\mathrm{C}_1(f)$ with the rack operation, together with reflexivity, and further by definition (i) of $\mathrm{C}_1(f)$. We may conclude by transitivity of $\mathrm{C}_1(f)$.

Now define the symmetric set relation S as the subset of $A \times A$, given by pairs of endpoints of f-horns. Looking at horns of length 0 and 1, S defines a reflexive relation containing the generators of $\mathrm{C}_1(f)$. It is also easy to observe that it is compatible with the rack operation. Thus it remains to show transitivity. In order to do so, for k and n in $\mathbb{N}$, consider a horn $M = (x, (a_i, b_i, \delta_i)_{1 \leq i \leq k})$, and its endpoints a and b as before, as well as a horn $N = (z, (c_i, d_i, \gamma_i)_{1 \leq i \leq n})$ with endpoints $c = z \cdot g_a^N$ and $d = z \cdot g_b^N$. If $b = c$ then also (a, d) is in S since:

$$\begin{aligned} a &= x \triangleleft^{\delta_1} a_1 \cdots \triangleleft^{\delta_k} a_k \triangleleft^{-\gamma_n} c_n \cdots \triangleleft^{-\gamma_1} c_1 \triangleleft^{\gamma_1} c_1 \cdots \triangleleft^{\gamma_n} c_n, \\ d &= x \triangleleft^{\delta_1} b_1 \cdots \triangleleft^{\delta_k} b_k \triangleleft^{-\gamma_n} c_n \cdots \triangleleft^{-\gamma_1} c_1 \triangleleft^{\gamma_1} d_1 \cdots \triangleleft^{\gamma_n} d_n. \qquad \square \end{aligned}$$

By Corollary 3.3.6, what we deduced about the functor F_1 restricts to the domain $\mathsf{CExtQnd}$, and so also describes the left adjoint to the inclusion in ExtQnd from Theorem 5.5. in [**36**]. In addition to Corollary 3.3.6, we further describe how centralization behaves with respect to ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}$.

3.4.2. *Navigating between racks and quandles.* Observe that the adjunction

$$\begin{tikzcd} \mathsf{Rck} \arrow[r, bend left, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}"] \arrow[r, phantom, "\perp"] & \mathsf{Qnd} \arrow[l, bend left, "\mathrm{I}"] \end{tikzcd}$$

induces (in the obvious way) an adjunction

$$\begin{tikzcd}
\mathsf{ExtRck} \arrow[r, bend left, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1}"] \arrow[r, phantom, "\perp"] & \mathsf{ExtQnd} \arrow[l, bend left, "\mathrm{I}"]
\end{tikzcd}$$

with unit given by ${}^{r}_{1}\eta^{q} = ({}^{r}\eta^{q}, {}^{r}\eta^{q})$. Then by Corollary 3.3.3 this adjunction restricts to the full subcategories of central extensions:

$$\begin{tikzcd}
\mathsf{CExtRck} \arrow[r, bend left, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1}"] \arrow[r, phantom, "\perp"] & \mathsf{CExtQnd} \arrow[l, bend left, "\mathrm{I}"]
\end{tikzcd}$$

PROPOSITION 3.4.3. *We have the following square of adjunctions, in which all possible squares of functors commute (up to isomorphism):*

$$\begin{tikzcd}
\mathsf{ExtRck} \arrow[r, bend right, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1}"'] \arrow[r, phantom, "\top"] \arrow[d, bend right, "\mathrm{F}_1"'] \arrow[d, phantom, "\dashv"] & \mathsf{ExtQnd} \arrow[l, bend right, "\mathrm{I}"'] \arrow[d, bend right, "\mathrm{F}_1"'] \arrow[d, phantom, "\dashv"] \\
\mathsf{CExtRck} \arrow[u, bend right, "\mathrm{I}"'] \arrow[r, bend right, "{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1}"'] \arrow[r, phantom, "\top"] & \mathsf{CExtQnd}. \arrow[u, bend right, "\mathrm{I}"'] \arrow[l, bend right, "\mathrm{I}"']
\end{tikzcd}$$

PROOF. Corollary 3.4.4 gives commutativity of the square $\mathrm{F}_1\,\mathrm{I} = \mathrm{I}\,\mathrm{F}_1$ from the top right to the bottom left. In the opposite direction, $\mathrm{I}\;{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1} = {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1}\,\mathrm{I}$ by Corollary 3.3.3 again. Finally bottom-right to top-left $\mathrm{I}\,\mathrm{I} = \mathrm{I}\,\mathrm{I}$ commutes trivially, from which we can deduce, by uniqueness of left adjoints, that ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1}\,\mathrm{F}_1 = \mathrm{F}_1\;{}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}{}^{1}$. ☐

In particular we have:

COROLLARY 3.4.4. *If $f\colon A \to B$ is a morphism of racks, then the centralization*

$$\mathrm{F}_1({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f))\colon\; {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(A)/\,\mathrm{C}_1({}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)) \to {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)$$

of ${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(f)$ is equal (up to isomorphism) to the reflection

$${}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(\mathrm{F}_1(f))\colon\; {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(A/\,\mathrm{C}_1(f)) \to {}_{\mathrm{r}}\mathrm{F}_{\mathrm{q}}(B)$$

of the centralization $\mathrm{F}_1(f)$ of f.

3.4.5. *Towards admissibility in dimension 2.* A reflector such as F_1, of a subcategory of morphisms containing the identities into a larger class of morphisms can always be chosen such that the bottom component of the unit of the adjunction is the identity [**64**, Corollary 5.2]. This is important in order to obtain higher order reflections and admissibility, for we relate certain problems back to the first level context (which has the advantage of being complete, cocomplete and Barr-exact). For dimension 2, we need this reflection to be strongly Birkhoff. Below we have the results we need for the permutability condition on the kernel pair of the unit ("strongly") and for the closure by quotients of central extensions ("Birkhoff").

PROPOSITION 3.4.6. *Given a rack extension $f\colon A \to B$ (or in particular an extension in* Qnd*) as before, the kernel pair $\mathrm{C}_1(f)$ of the domain-component η^1_A of the unit $\eta^1_f := (\eta^1_A, \mathrm{id}_B)$, commutes with all congruences on A, in* Rck *(and so also in particular in* Qnd*).*

PROOF. By Theorem 3.3.1, the centralizing relation $\mathrm{C}_1(f)$ is an orbit congruence which thus commutes with any other congruence on A. ☐

As we shall see in Part II (Remark 1.3.3), the following property is a consequence of the fact that the Galois structure Γ, in dimension 0, is strongly Birkhoff. For now we show by hand:

PROPOSITION 3.4.7. *If $\alpha = (\alpha_\top, \alpha_\bot)\colon f_A \to f_B$ is a double extension of racks (or in particular quandles)*

$$\begin{tikzcd}
A_\top \arrow[rr, "\alpha_\top"] \arrow[dr, "p"] \arrow[dd, "f_A"'] & & B_\top \arrow[dd, "f_B"] \\
& A_\bot \times_{B_\bot} B_\top \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\
A_\bot \arrow[rr, "\alpha_\bot"'] & & B_\bot
\end{tikzcd}$$

then the morphism $\bar{\alpha}\colon\ \mathrm{C}_1(f_A) \to \mathrm{C}_1(f_B)$ induced between the centralizing relations $\mathrm{C}_1(f_A)$ and $\mathrm{C}_1(f_B)$ is a regular epimorphism. Moreover, if f_A is a central extension then f_B is a central extension.

PROOF. Certainly if we show that $\bar{\alpha}$ is a regular epimorphism, then assuming that f_A is central, then its centralizing relation is trivial, hence the centralizing relation of f_B is trivial, showing that f_B is central (note that in this context, it is enough to have preservation of centrality by quotients along double extensions in order to have surjectivity of $\bar{\alpha}$, see Part II and III).

We pick a pair $(x \triangleleft y, x \triangleleft z)$ amongst the generators of $\mathrm{C}_1(f_B)$ (i.e. with $f_B(y) = f_B(z)$). Since $\alpha_{\perp}$ is surjective we get $a \in A_{\perp}$ such that $\alpha_{\perp}(a) = f_B(y)$. Now both pairs (a, y) and (a, z) are in the pullback $A_{\perp} \times_{B_{\perp}} B_{\top}$ hence there exist t and s in $A_{\top}$ such that $\alpha_{\top}(t) = y$, $\alpha_{\top}(s) = z$ and $f_A(t) = f_A(s) = a$, by surjectivity of p. Now there is also $u \in A_{\top}$ such that $t(u) = x$ and the pair $(u \triangleleft t, u \triangleleft s)$ is a generator of $\mathrm{C}_1(f_A)$ by definition. It is also sent to $(x \triangleleft y, x \triangleleft z) \in \mathrm{C}_1(f_B)$ by $\bar{\alpha}$ by construction. All generators of $\mathrm{C}_1(f_B)$ are thus in the image of $\bar{\alpha}$, and this concludes the proof. ☐

COROLLARY 3.4.8. *Given a morphism* $\alpha = (\alpha_{\top}, \alpha_{\perp}) \colon f_A \to f_B$ *in* ExtRck *such that* $\alpha_{\top}$ *and* $\alpha_{\perp}$ *are surjections, then the square below (where* $P := (A_{\top}/\,\mathrm{C}_1(f_A)) \times_{(B_{\top}/\,\mathrm{C}_1(f_B))} B_{\top}$*) is a double extension of racks. Similarly in* ExtQnd.

$$\begin{tikzcd}
A_{\top} \arrow[rr, "\alpha_{\top}"] \arrow[dr, "p"] \arrow[dd, "\eta^1_{A_{\top}}"'] & & B_{\top} \arrow[dd, "\eta^1_{B_{\top}}"] \\
& P \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\
(A_{\top}/\,\mathrm{C}_1(f_A)) \arrow[rr, "\mathrm{F}_1^{\top}(\alpha)"'] & & (B_{\top}/\,\mathrm{C}_1(f_B))
\end{tikzcd}$$

PROOF. By Lemma 1.2 in [**10**], this square is a pushout as a consequence of Proposition 3.4.7. Then by Proposition 5.4 in [**25**], p is a surjection as well, making α into a double extension. ☐

In Part II we complete the proof that

$$\Gamma_1 = (\mathsf{ExtRck}, \mathsf{CExtRck}, \mathrm{F}_1, \mathrm{I}, \eta^1, \epsilon^1, \mathcal{E}^1)$$

forms an admissible Galois structure such that morphisms in $\mathcal{E}^1$ are of *effective* $\mathcal{E}^1$*-descent* [**79, 78**].

3.5. Weakly universal covers & fundamental groupoid. We insist on the importance of the new results of this section and the following, in achieving a precise theoretical understanding and expansion of M. Eisermann's covering theory of quandles (as a continuation of V. Even's contributions).

3.5.1. *Centralizing the canonical presentations.* Weakly universal covers (w.u.c.) for quandles were described by M. Eisermann. He also indicated how to adapt his theory to the case of racks. In this section, we recover his constructions from the centralization of the canonical projective presentations as explained in the introduction. Note that the difference between the w.u.c. in racks and in quandles is then due to the difference between the canonical projective presentations rather than the centralizations which are the same.

Given $\epsilon_X^r \colon \mathrm{F_r}(X) \to X$, the canonical projective presentation of a rack, we saw in Paragraph 2.4.4 that the induced morphism $\vec{\epsilon}_X^{\,r}$ is actually the quotient map $\vec{\epsilon}_X^{\,r} = q_X \colon \mathrm{F_g}(X) \to \mathrm{Pth}(X)$ from Subsection 2.4. Hence the kernel of $\vec{\epsilon}_X^{\,r}$ is given by

$$\mathrm{Ker}(\vec{\epsilon}_X^{\,r}) = \langle\langle \underline{c}^{-1}\underline{a}^{-1}\underline{x}\,\underline{a} \mid a, x, c \in X \text{ and } c = x \triangleleft a \rangle\rangle_{\mathrm{F_g}(X)}.$$

Since the action of $\mathrm{Pth}(\mathrm{F_r}(X)) = \mathrm{F_g}(X)$ is by right multiplication, two elements (a, g) and (b, h) in $\mathrm{F_r}(X)$ are identified by the centralizing relation $\mathrm{C}_1(\epsilon_X^r)$ if and only if $a = b$ and there is $k \in \mathrm{Ker}(\vec{\epsilon}_X^{\,r})$ such that $g = hk$. In other words, the domain component $\eta^1_{\mathrm{F_r}(X)}$ of the centralization unit is given by the product

$$X \rtimes \mathrm{F_g}(X) \xrightarrow{\mathrm{id}_X \times q_X} X \rtimes \mathrm{Pth}(X),$$

where the operation in $\tilde{X} \coloneqq X \rtimes \mathrm{Pth}(X)$ is defined as in Paragraph 2.2.3.1, Equation (15).

DEFINITION 3.5.2. *Given a rack X, we define the associated weakly universal cover of X to be the centralised map $\omega_X \coloneqq \mathrm{F}_1(\epsilon_X^r)$*

$$\tilde{X} \coloneqq X \rtimes \mathrm{Pth}(X) \xrightarrow{\omega_X} X,$$

where ω_X sends a trail $(a, g) \in \tilde{X}$ to its endpoint $a \cdot g$, and trails in $\tilde{X}$ "act by endpoint" as in $\mathrm{F_r}(X)$. Note that this construction is functorial in X, yielding a functor $\tilde{-} \colon \mathsf{Rck} \to \mathsf{Rck}$ which sends a morphism of racks $f \colon A \to B$ to the morphism $\tilde{f} \coloneqq f \times \vec{f} \colon \tilde{A} \to \tilde{B}$; and a natural transformation $\omega \colon \tilde{-} \to \mathrm{id}_{\mathsf{Rck}}$, whose component at X is ω_X.

Then the action of $\mathrm{Pth}(X)$ induced by the covering ω_X on $\tilde{X} = X \rtimes \mathrm{Pth}(X)$ is by right multiplication, and is thus free. Given any other covering $f \colon B \to X$, together with a splitting function $s \colon X \to B$ in Set such that $fs = \mathrm{id}_X$, a factorization $\omega_f \colon \tilde{X} \to B$ of ω_X through f is given by $\omega_f(a, e) \coloneqq s(a)$ and compatibility with the action of $\mathrm{Pth}(X)$ on $\tilde{X}$ and B (see Corollary 3.1.5).

Starting with the canonical projective presentation of a quandle

$$\epsilon^q_X \colon X \rtimes \mathrm{Pth}^\circ(X) \to X,$$

the same reasoning yields a w.u.c. with the same properties

$$\tilde{X}^\circ \coloneqq X \rtimes \mathrm{Pth}^\circ(X) \xrightarrow{\omega^q_X} X,$$

such that the quandle structure on $X \rtimes \mathrm{Pth}^\circ(X)$ is as for $\mathrm{F_q}(X)$ (Paragraph 2.5.13). As in the case of racks, this describes a functor as well as a natural transformation whose component at any quandle A is ω^q_A. Observe that Corollary 3.4.4 implies that $\tilde{X}^\circ \coloneqq X \rtimes \mathrm{Pth}^\circ(X)$ is actually the free quandle on the rack $\tilde{X} \coloneqq X \rtimes \mathrm{Pth}(X)$ and thus if X is a quandle, then ω^q_X is merely the image of ω_X by ${}_{\mathrm{r}}\mathrm{F_q}$.

As it was proved by V. Even [**40**], every covering of X is split by ω^q_X in Qnd and a similar argument shows that every covering of X is split by ω_X in Rck. This derives more generally from Corollary 3.4.8:

PROPOSITION 3.5.3. *If the extension* $c\colon A \to B$ *is split by an extension* $e\colon E \to B$*, then it is also split by the centralization of this extension* e*, namely* $\mathrm{F}_1(e)\colon E \to B$*. As a consequence,* c *must be split by any weakly universal cover above* B*.*

PROOF. Consider the reflection by F_1 of the pullback P of e and c as on the right-hand side of Diagram (22). Since the composite of two double extensions is a pullback if and only if both double extensions are pullbacks themselves, Corollary 3.4.8 implies that the commutative squares $t'\eta^1_P = \eta^1_E t$ and $c\,\mathrm{F}_1(f) = \mathrm{F}_1(e)t'$ are pullback squares, where $t' \coloneqq \mathrm{F}^{\top}_1[(t, c)]$. Hence, since $\eta_E = \eta_{(E/\,\mathrm{C}_1(e))}\eta^1_E$, and similarly $\eta_P = \eta_{(P/\,\mathrm{C}_1(f))}\eta^1_P$, the F-reflection square $\eta_E t = \pi_0(t)\eta_P$ at t (which is a pullback by assumption) factors through the F-reflection square $\pi_0(t')\eta_{(P/\,\mathrm{C}_1(f))} = \eta_{(E/\,\mathrm{C}_1(e))}t'$ at t' via the pullback square $t'\eta^1_P = \eta^1_E t$. Since the square $\pi_0(t')\eta_{(P/\,\mathrm{C}_1(f))} = \eta_{(E/\,\mathrm{C}_1(e))}t'$ is a double extension, it is actually a pullback, which shows that t' is a trivial extension.

$$\begin{tikzcd}
& \pi_0(P) \arrow[dl, equal] \arrow[dd, "\pi_0(t)"] & & P \arrow[ll, "\eta_P"'] \arrow[rr, "f"] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dl, "\eta^1_P"'] \arrow[dd, "t"] & & A \arrow[dl, equal] \arrow[dd, "c"] \\
\pi_0(P) \arrow[dd, "\pi_0(t)=\pi_0(t')"'] & & P/\,\mathrm{C}_1(f) \arrow[ll, "\eta_{(P/\,\mathrm{C}_1(f))}"'] \arrow[rr, "\mathrm{F}_1(f)"'] \arrow[dd, "t'"] & & A \arrow[dd, "c"] & \\
& \pi_0(E) \arrow[dl, equal] & & E \arrow[ll, "\eta_E"'] \arrow[rr, "e"] \arrow[dl, "\eta^1_E"'] & & B \arrow[dl, equal] \\
\pi_0(E) & & E/\,\mathrm{C}_1(e) \arrow[ll, "\eta_{(E/\,\mathrm{C}_1(e))}"] \arrow[rr, "\mathrm{F}_1(e)"'] & & B &
\end{tikzcd} \tag{22}$$

We conclude by observing that a weakly universal cover above B factors through $\mathrm{F}_1(e)$ and trivial extensions are stable by pullbacks (see also Diagram 13). □

Given any X in Rck (respectively Qnd), the covering ω_X (respectively ω^q_X) is split by itself and thus it is a normal covering. Hence its kernel pair is sent to a groupoid by the reflection π_0 (see [**9**, Lemma 5.1.22]) and thus we can construct the *fundamental groupoid* (see Section 1 in the introduction) yielding functors $\pi^r_1\colon \mathsf{Rck} \to \mathsf{Grpd}$ and $\pi^q_1\colon \mathsf{Qnd} \to \mathsf{Grpd}$, with codomain the category of ordinary groupoids Grpd (i.e. the category of internal groupoids in Set).

DEFINITION 3.5.4. *The functor* $\pi_1\colon \mathsf{Rck} \to \mathsf{Grpd}$ *is defined on objects by sending a rack* X *to* $\pi^r_1(X)$*, the image by* π_0 *of the groupoid induced by taking the kernel pair of* ω_X*. Functoriality is induced by functoriality of* ω*.*

Similarly the functor $\pi^q_1\colon \mathsf{Qnd} \to \mathsf{Grpd}$ *is defined by sending a quandle* X *to* $\pi^q_1(X)$*, the image by* π_0 *of the groupoid induced by taking the kernel pair of* ω^q_X*.*

From there, the Galois theorem yields an equivalence of categories between the category of coverings of X and the category of internal covariant presheaves over $\pi_1(X)$ (and similarly for Qnd, see Section 1 and references).

3.5.5. *The fundamental groupoid.* We show that the fundamental groupoid $\pi_1(X)$ (respectively $\pi_1^q(X)$) for an object X in the category Rck (respectively Qnd) is indeed the groupoid induced by the action of $\mathrm{Pth}(X)$ (respectively $\mathrm{Pth}^{\circ}(X)$) on X, as suggested in M. Eisermann's work (see [**38**, Section 8]). As was mentioned in the introduction, these results, and categorical Galois theory, give a positive answer to M. Eisermann's questions about the relevance of his analogies with topology. Results about the fundamental group of a connected pointed quandle were given by V. Even in [**39**]. We generalize these results to the non-connected, non-pointed context in both categories Rck and Qnd. Exploiting the analogy with the covering theory of locally connected topological spaces, this result confirms the intuition that the elements of the group $\mathrm{Pth}(X)$ (respectively $\mathrm{Pth}^{\circ}(X)$) are representatives of the classes of *homotopically equivalent paths* which connect elements in the rack (respectively quandle) X.

DEFINITION 3.5.6. *Given a set X and a group G together with an action of G on X, we build the ordinary groupoid (of elements) $\mathcal{G}_{(X,G)}$ (in* Set*)*

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & X_1 \arrow[loop above, "(-)^{-1}"] \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] & X \arrow[l, "i" description] \end{tikzcd}$$

where $X_0 := X$, $X_1 := X \times G$ and for $a \in X_0$, $(a,g) \in X_1$,

$$d(a,g) := a;\ c(a,g) := a \cdot g;\ i(a) := (a,e);\ (a,g)^{-1} := (a \cdot g, g^{-1});$$

$p_1, p_2 \colon X_2 \rightrightarrows X_1$ form the pullback of c and d; and m is the composition function defined for $\langle (a,g),(b,h)\rangle$ in X_2 by

$$m\langle (a,g),(b,h)\rangle := (a,g)\cdot(b,h) := (a,gh).$$

Note that this construction actually defines a functor from the category of group actions to the category of ordinary groupoids.

THEOREM 3.5.7. *Given an object X in* Rck *(respectively* Qnd*), the fundamental groupoid $\pi_1(X)$ (resp. $\pi_1^q(X)$) is given by the set groupoid $\mathcal{G}_{(X,\mathrm{Pth}(X))}$ (resp. $\mathcal{G}_{(X,\mathrm{Pth}^{\circ}(X))}$). Moreover, the groupoid morphisms induced by $f\colon X \to Y$ via* Pth *(resp.* Pth°*) and $\mathcal{G}$ correspond to $\pi_1(f)$ (resp. $\pi_1^q(f)$).*

PROOF. Given the kernel pair $d_1, d_2 \colon X'_1 \rightrightarrows X'$ of the weakly universal cover $\omega_X \colon \tilde{X} \to X$ (resp. $\omega_X^q \colon \tilde{X}^{\circ} \to X$), we define the groupoid $\mathcal{G}$ as in Diagram (23), where X'_2 is the pullback of d_2 and d_1, and

m' is the composition function defined by the unique factorization of $d_2 \circ p'_2, d_1 \circ p'_1 \colon X'_2 \rightrightarrows X'$ through $d_2, d_1 \colon X'_1 \rightrightarrows X'$.

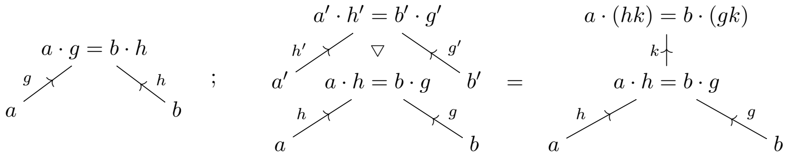
$$\begin{tikzcd} X'_2 \arrow[r, shift left=3, "p'_1"] \arrow[r, "m'" description] \arrow[r, shift right=3, "p'_2"'] & X'_1 \arrow[loop above, "(-)^{-1}"] \arrow[r, shift left=3, "d_2"] \arrow[r, shift right=3, "d_1"'] & X' \arrow[l, "u" description] \end{tikzcd} \tag{23}$$

Remember that a trail $(a, g) \in X'$ is represented as an arrow

$$g \colon\ a \nrightarrow a \cdot g$$

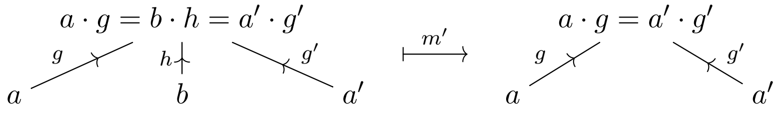
and the action of a trail on another is as in Paragraph 2.2.3.1, Equation (15), where the composition of arrows is understood by multiplication in $\mathrm{Pth}(X)$ (resp. $\mathrm{Pth}^{\circ}(X)$).

By definition, the elements in X'_1 are then pairs of trails with same endpoint (diagram on the left), and the rack (resp. quandle) operation is defined component-wise such that we have the equality on the right:



(24)

where $k := (h')^{-1}\underline{a'}h'$ (resp. $k := \underline{(a \cdot h)}^{-1}(h')^{-1}\underline{a'}h'$). Finally observe that X'_2 is composed of pairs of elements in X'_1 with one matching leg (such as represented on the left), which images by m' are given as in the right-hand diagram:



Again the operation in X'_2 is defined component-wise and behaves as in X'_1.

We compute the image $\pi_0(\mathcal{G})$ which is $\pi_1(X)$ (resp. $\pi_1^q(X)$) by definition. Working on each object separately, first observe that as for $\mathrm{F_r}(X)$ (resp. $\mathrm{F_q}(X)$), the unit $\eta_{X'} \colon\ X' \to \pi_0(X') = X$ sends a trail $(a, g) \in X \rtimes \mathrm{Pth}(X)$ (resp. in $X \rtimes \mathrm{Pth}^{\circ}(X)$) to its head $a \in X$, i.e. $\eta_{X'}$ is given by the product projection on X. Now for each pair of trails

$\alpha = \langle (a,g),(b,h)\rangle$ in X_1', we define the trail $\mu(\alpha) \coloneqq (a, gh^{-1})$ in X':

$$\begin{tikzcd}[column sep=small]
 & & a\cdot g = b\cdot h & & & & a\cdot g = b\cdot h & & \\
\alpha = & a \arrow[ur, dash, "g"] & & b \arrow[ul, dash, "h"'] & \mapsto & a \arrow[ur, dash, "g"] & & b \arrow[ul, dash, "h^{-1}"'] & \eqqcolon \mu(\alpha).
\end{tikzcd}$$

Observe that this trail $\mu(\alpha)$ is invariant under the action on α, of other pairs $\beta = \langle (a',g'),(b',h')\rangle$ in X_1', since

$$\mu(\alpha \triangleleft \beta) = (a, hkk^{-1}g^{-1}) = \mu(\alpha),$$

where $k = (h')^{-1}\underline{a'}h'$ (resp. $k = \underline{(a\cdot h)}^{-1}\,(h')^{-1}\underline{a'}h'$) is the common part of both left and right legs as in Equation (24). Conversely suppose that α, α' in X_1' have the same image by μ, we show that α and α' are connected in X_1'. Indeed, α and α' must then be of the form $\alpha = \langle (a,g),(b,h)\rangle$ and $\alpha' = \langle (a,g'),(b,h')\rangle$, such that moreover $gh^{-1} = g'h'^{-1}$. Then the path $l \coloneqq h^{-1}h' = g^{-1}g' \in \mathrm{Pth}(X)$ (resp. in $\mathrm{Pth}^{\circ}(X)$) decomposes as a product $l = \underline{x_0}^{\delta_0}\cdots\underline{x_n}^{\delta_n}$, such that all the pairs $\langle (x_i,e),(x_i,e)\rangle$ are in X_1' (and we have moreover $\sum_{i=0}^{n}\delta_i = 0$ in the context of Qnd). By acting with these pairs "$- \triangleleft^{\delta_i} \langle (x_i,e),(x_i,e)\rangle$" on α, we may obtain α' as in the diagram on the right:

$$\begin{tikzcd}[column sep=small]
 & & a\cdot g = b\cdot h & \\
\alpha \coloneqq & a \arrow[ur, dash, "g"] & & b \arrow[ul, dash, "h"']
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small]
 & & & & & & a\cdot(gl) = a\cdot(hl) & \\
 & & a\cdot g' = b\cdot h' & & & & a\cdot g = b\cdot h \arrow[u, "l"] & \\
\alpha' \coloneqq & a \arrow[ur, dash, "g'"] & & b \arrow[ul, dash, "h'"'] & = & a \arrow[ur, dash, "g"] & & b \arrow[ul, dash, "h"']
\end{tikzcd}$$

Hence we have the unit morphism $\eta_{X_1'} = \mu\colon X_1' \to \pi_0(X_1')$ where $\pi_0(X_1')$ is $\pi_0(\mathrm{Eq}(\omega_X)) = X \times \mathrm{Pth}(X)$ (resp. $\pi_0(\mathrm{Eq}(\omega_X^q)) = X \times \mathrm{Pth}^{\circ}(X)$). We may then compute $\pi_0(d_2) = c$, $\pi_0(d_1) = d$, $\pi_0(i) = u$ and $\pi_0(-1) = -1$, as displayed in the commutative diagram of plain arrows,

$$\begin{tikzcd}[column sep=huge]
X_2' \arrow[r, shift left=2, "p_1'"] \arrow[r, "m'" description] \arrow[r, shift right=2, "p_2'"'] \arrow[d, dotted, "\eta_{X_2'}=\mu\times\mu"'] & X_1' \arrow[loop above, "-1"] \arrow[r, shift left=2, "d_2"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "\eta_{X_1'}=\mu"'] & X' \arrow[l, "u" description] \arrow[r, "\omega_X\ (\text{resp. } \omega_X^q)"] \arrow[d, "\eta_{X'}=d"] & X \\
X_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & X_1 \arrow[loop below, "-1"] \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] & X \arrow[l, "i" description] &
\end{tikzcd}$$

where the bottom groupoid is the inclusion in Rck (resp. Qnd) of the groupoid $\mathcal{G}_{(X,\mathrm{Pth}(X))}$ (resp. $\mathcal{G}_{(X,\mathrm{Pth}^{\circ}(X))}$) from Set. Hence $X_1 = X \times \mathrm{Pth}(X)$ (resp. $X_1 = X \times \mathrm{Pth}^{\circ}(X)$) has the same underlying set as X', and the underlying functions of $\eta_{X'}$ and d are both given by "projection on X".

Then since ω_X (resp. ω^q_X) is a normal covering, d_1 and d_2 are trivial extensions, so that the commutative squares $dd_1 = d\mu$ and $dd_2 = c\mu$ are actually pullback squares. Hence the pullback

$$p_1', p_2' \colon X_2' \rightrightarrows X_1'$$

of d_2 and d_1 and the pullback $p_1, p_2 \colon X_2 \rightrightarrows X_1$ of c and d, induce a morphism $f \colon X_2' \to X_2$ which is thus the pullback of $\eta_{X_1'} = \mu$ and computed component-wise as $f = \mu \times \mu$. By admissibility of the Galois structure Γ (see Paragraph 2.3.11 and [**71**]), this morphism is also the unit component $f = \eta_{X_2'}$. Finally the commutativity of the square $\mu m' = m\eta_{X_2'}$ is given by construction (and easy to check by hand), which concludes the proof that $\pi_1(X) = \pi_0(\mathcal{G}) = \mathcal{G}_{(X,\mathrm{Pth}(X))}$ (resp. $\pi^q_1(X) = \mathcal{G}_{(X,\mathrm{Pth}^{\circ}(X))}$ in Qnd). □

3.5.7.1. Remarks. Remember from Paragraph 2.3.13 that the notion of connectedness is not local. Now relate this fact to the *regularity* of the fundamental groupoid of a rack, whose domain map is the projection map of a cartesian product: given a rack A, the set of homotopy classes of paths of a given domain $a \in A$ is always $\mathrm{Pth}(A)$ and thus independent of the domain a. Since every path is invertible, the same is true for the homotopy classes of paths of a given endpoint.

One of D.E. Joyce's main results is to show that the *knot quandle* [**83**] is a complete invariant for oriented knots. Now the *knot group* of an oriented knot, which is the fundamental group of the ambient space of the knot [**94**], is also computed as the group of paths of the knot quandle. In other words, the knot group is the *fundamental group* of the knot quandle, in the sense of the covering theory of racks (not in the sense of the covering theory of quandles).

Finally observe that $\pi_1(X)$ (resp. $\pi^q_1(X)$) can be equipped with a non-trivial ad hoc structure of rack (resp. quandle) making it into an internal groupoid in Rck (resp. Qnd) with internal object of objects the rack (resp. quandle) X. Given two trails (a, g) and (b, h) in X_1, define $(a, g) \triangleleft (b, h) := (a \triangleleft b, \underline{b}^{-1} g h^{-1} \underline{b} h)$ (note that if g, $h \in \mathrm{Pth}^{\circ}(X)$, then

$\underline{b}^{-1}gh^{-1}\underline{b}h \in \mathrm{Pth}^{\circ}(X)$). Unlike in $\hat{X}$ (resp. $\hat{X}^{\circ}$), trails act on each other with both their heads and end-points, which means that both projections to X are morphisms in Rck (resp. Qnd). The rest of the structure is easy to derive.

3.5.7.2. Working with skeletons. As we shall see in the next section, we are interested in the fundamental groupoid, up to equivalence. Given a rack A, we thus also describe a *skeleton* S of $\pi_1(A)$ (in the sense of [**89**, Section IV.4]). The resulting groupoid S is not regular like $\pi_1(A)$, it is totally disconnected and its vertices are the connected components of A. With the objective of interpreting the fundamental theorem of Galois theory, the homotopical information contained in $\pi_1(A)$ can be made more explicit using its skeleton.

DEFINITION 3.5.8. *Given an object A in Rck (respectively in Qnd), we call a* pointing *of A any choice of representatives $I := \{a_i\}_{i\in\pi_0(A)} \subseteq A$ such that $\eta_A(a_i) = [a_i] = i$ for each equivalence class $i \in \pi_0(A)$. Then for any element $a \in A$, define Loop_a as the group of loops $l \in \mathrm{Pth}(A)$ (resp. $l \in \mathrm{Pth}^{\circ}(A)$) such that $a \cdot l = a$. Observe that if $[a] = [b]$, for some a and b in A, then there is $g \in \mathrm{Pth}(A)$ (resp. $g \in \mathrm{Pth}^{\circ}(A)$) such that $a = b \cdot g$ and thus the subgroups Loop_a and Loop_b are isomorphic, via the automorphism of $\mathrm{Pth}(A)$ (resp. Pth°) given by conjugation with g.*

Let us fix a pointing $I := \{a_i\}_{i\in\pi_0(A)} \subseteq A$ of A, then we define the groupoid $\pi_1(A, I)$ (resp. $\pi_1^q(A, I)$) as

$$\begin{tikzcd} A_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & A_1 \arrow[loop above, "(-)^{-1}"] \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] & \pi_0(A), \arrow[l, "i" description] \end{tikzcd}$$

where $A_1 := \coprod_{i\in\pi_0(A)} \mathrm{Loop}_{a_i}$ is defined as the disjoint union, of the underlying sets of Loop_{a_i}'s indexed by $i \in \pi_0(A)$. The domain and codomain maps send a loop $l \in \mathrm{Loop}_{a_i}$ to the index $i \in \pi_0(A)$. The set A_2 is then the disjoint union of products $A_2 := \coprod_{i\in\pi_0(A)}(\mathrm{Loop}_{a_i} \times \mathrm{Loop}_{a_i})$ and m is defined by multiplication in $\mathrm{Loop}_{a_i} \leq \mathrm{Pth}(A)$ (resp. $\mathrm{Loop}_{a_i} \leq \mathrm{Pth}^{\circ}(A)$).

From the description of the skeleton of a groupoid obtained as in Definition 3.5.6, we deduce:

LEMMA 3.5.9. *For each I pointing of A object of Rck (respectively of Qnd), $\pi_1(A, I)$ (respectively $\pi_1^q(A, I)$) is a skeleton of the fundamental groupoid $\pi_1(A)$ (respectively $\pi_1^q(A)$).*

3.6. The fundamental theorem of categorical Galois theory. In sections 5, 6 and 7 of [**38**], M. Eisermann studies in detail different classification results for quandle coverings. We will not go into so much depth ourselves, however we show how to recover and extend the main theorems from these sections using categorical Galois theory.

Given an object A in Rck (respectively Qnd), the category of *internal covariant presheaves* over $\pi := \pi_1(A)$ (resp. $\pi := \pi_1^q(A)$) are externally described as the category of functors from π to Set and thus as the category of π-groupoid actions on sets Set^{π}. Given a pointing I of A, define $\pi(I) := \pi_1(A, I)$ (resp. $\pi(I) := \pi_1^q(A, I)$ and deduce from $\pi(I) \cong \pi$ that $\mathsf{Set}^{\pi} \cong \mathsf{Set}^{\pi(I)}$. Now $\pi(I)$ is totally disconnected, thus the category of $\pi(I)$-actions is equivalent to the category $\coprod_{i\in\pi_0(A)} \mathsf{Set}^{\mathrm{Loop}_{a_i}}$ whose objects are sequences of Loop_{a_i}-group actions (see Definition 3.5.8), indexed by $i \in \pi_0(A)$, and morphisms between these are π_0-indexed sums of group-action morphisms. From the fundamental theorem of categorical Galois theory (see for instance [**71**, Theorem 6.2]), classifying central extensions above an object we deduce in particular:

THEOREM 3.6.1. *Given an object A in Rck and given a pointing $I := \{a_i\}_{i\in\pi_0(A)} \subseteq A$ of A, there is a natural equivalence of categories between the category $\mathsf{CExt}(A)$ of central extensions above A and the category $\mathsf{Set}^{\pi_1(A)}$. The latter category is then also equivalent (but not naturally) to $\mathsf{Set}^{\pi_1(A,I)} \cong \coprod_{i\in\pi_0(A)} \mathsf{Set}^{\mathrm{Loop}_{a_i}}$. The same theorem holds in Qnd, using the appropriate definition of Loop_{a_i} and using $\pi_1^q(A)$ and $\pi_1^q(A, I)$ instead of $\pi_1(A)$ and $\pi_1(A, I)$.*

COROLLARY 3.6.2. *The category of central extensions above a connected rack A is equivalent to the category of Loop_a-actions (from Definition 3.5.8), for any given element $a \in A$. The same is true in Qnd.*

EXAMPLE 3.6.3. *We illustrate this result on a trivial example, to show the difference between the context of Rck and that of Qnd. Consider the one element set 1. The coverings above 1 in Qnd should all be surjective maps to 1 in Set, whereas the coverings above 1 in Rck include for instance the unit morphism ${}^r\eta^q_{\mathrm{F_r}(1)} = \eta_{\mathrm{F_r}1} \colon \mathrm{F_r}(1) \to \mathrm{F_q}(1) = 1$, whose domain is not a set. Then observe that $\mathrm{Pth}(1) = \mathbb{Z}$ and thus $\mathrm{Pth}^{\circ}(1) = \{e\}$ and since there is only one element $* \in 1$, Loop_* is the former in Rck and the latter in Qnd. Hence the category of coverings above 1 in Qnd is $\mathsf{Set}^{\{e\}}$ which is indeed equivalent to Set. The category of coverings above 1 in Rck is given by $\mathsf{Set}^{\mathbb{Z}}$, the category of $\mathbb{Z}$-actions on sets, where $\mathbb{Z}$ is the additive group of integers.*

3.7. Relationship to groups and abelianization. The following relationship between $\pi_0 \dashv \mathrm{I}$ in Rck (or Qnd) and the abelianization in groups has played an important role in the study of the present paper, and in the identification of the relevant centrality conditions in higher dimensions described in Part II and III.

Let us comment first of all that the subvariety of sets is absolutely not a *Mal'tsev category*, and the adjunction $\pi_0 \dashv \mathrm{I}$ does not arise from an *abelianization adjunction* like, for instance, in the case of *abelian sym quandles* studied in [**43**] (a quandle is *sym* if $\triangleleft$ is commutative). For instance, the distinction with the study in [**43**] is clear since the only connected *sym quandle* is $\{*\}$, also the only group whose conjugation is sym is the trivial group $\{e\}$. The relation between centrality in racks/quandles and the classical notions of centrality induced by Mal'tsev or partial Mal'tsev contexts, appears to us as more subtle than: one being merely an example of the other.

The following comments also apply to the context of Qnd, however we like to work in the more "primitive" context of Rck considering the role of Pth in the comparison with groups, and its tight relationship with the axioms of racks.

We study which squares of functors commute in the following square of adjunctions.

$$\begin{tikzcd}
\mathsf{Rck} \arrow[r, bend right, "\pi_0"'] \arrow[r, phantom, "\top"] \arrow[d, bend right, "\mathrm{Pth}"'] \arrow[d, phantom, "\dashv"] & \mathsf{Set} \arrow[l, bend right, "\mathrm{I}"'] \arrow[d, bend right, "\mathrm{F_{ab}}"'] \arrow[d, phantom, "\dashv"] \\
\mathsf{Grp} \arrow[u, bend right, "\mathrm{Conj}"'] \arrow[r, bend right, "\mathrm{ab}"'] \arrow[r, phantom, "\top"] & \mathsf{Ab} \arrow[l, bend right, "\mathrm{I}"'] \arrow[u, bend right, "\mathrm{U}"']
\end{tikzcd} \tag{25}$$

Starting with an abelian group G, conjugation in G is trivial hence $\mathrm{Conj}(\mathrm{I}(G))$ is the trivial quandle on the underlying set of G. Since also both composites send a morphism to the underlying function we have $\mathrm{Conj}\,\mathrm{I} = \mathrm{I}\,\mathrm{U}$ and thus the restriction of Conj to abelian groups gives the forgetful functor to Set. By uniqueness of left adjoints we must also have $\mathrm{F_{ab}}\,\pi_0 = \mathrm{ab}\,\mathrm{Pth}$. A direct proof easily follows from the corresponding group presentations.

Now starting with a set X in Set we may consider it as a trivial quandle by application of I. Then we compute

$$\begin{aligned}\mathrm{Pth}(\mathrm{I}(X)) &\coloneqq \mathrm{F_g}(X)/\langle (x \triangleleft a)^{-1}a^{-1}xa | a, x \in X\rangle \\ &= \mathrm{F_g}(X)/\langle x^{-1}a^{-1}xa | a, x \in X\rangle,\end{aligned}$$

which shows that for each set X we have $\mathrm{Pth}(\mathrm{I}(X)) = \mathrm{I}\,\mathrm{F_{ab}}(X)$, which then easily gives $\mathrm{Pth}\,\mathrm{I} = \mathrm{I}\,\mathrm{F_{ab}}$, i.e. the restriction of Pth to trivial racks gives the free abelian group functor.

Observe that we cannot use uniqueness of adjoints to derive that $\pi_0\,\mathrm{Conj}$ is the same as $\mathrm{U}\,\mathrm{ab}$. Indeed we compute that (π_0, Conj, U, ab) is the only square of functors that does not commute. Given a group G, the image $\pi_0(\mathrm{Conj}(G))$ is given by the set of conjugacy classes. The corresponding congruence in Qnd is given by

$$a \sim b \Leftrightarrow (\exists c \in G)(c^{-1}ac = b). \tag{26}$$

Then the abelianization $\mathrm{ab}(G)$ is the quotient of G by the congruence generated in Grp by the identities $\{c^{-1}ac = a \mid a, c \in G\}$. We may show that in general the equivalence relation defined in (26) does not define a group congruence. A counter-example is given by the group of permutations S_3. It has three conjugacy classes given by cycles, two permutations and the unit. The derived subgroup is the alternating group A_3 which is of order 2. This shows that there are less elements in the abelianization of S_3 than there are conjugacy classes in S_3.

Understand that an "image" of the covering theory in Rck, arising from the adjunction $\pi_0 \dashv \mathrm{I}$ can be studied in groups through the functor Pth and its restriction to sets $\mathrm{F_{ab}}$. Note that Pth is neither full, or faithful, nor essentially surjective. The functor $\mathrm{F_{ab}}$ is full and faithful. We will not study what information to extract from this image. Yet, again, we have been using ingredients of this image to describe centrality in Rck such as in Theorem 3.3.1. Observe moreover that any covering in racks induces a central extension between the groups of paths [**38**, Proposition 2.39]. However, certain morphisms, such as $f\colon Q_{ab\star} \to \{*\}$, which are not central in Rck (or Qnd) are sent by Pth to central extensions of groups, e.g.

$$\vec{f}\ :\ \ \mathrm{Pth}(Q_{ab\star}) = \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} = \mathrm{Pth}(\{*\})\ :\ (k, l) \mapsto k + l.$$

In the other direction an "image" of the theory of central extensions of groups can be studied in Rck via the "inclusion" Conj and its restriction

U on abelian groups. Both Conj and U are not full but faithful, U is moreover surjective. Again we shall not develop the full potential of this study. Observe nonetheless that, similarly to the fact that a group is abelian if and only if its conjugation operation is trivial, a morphism of groups is central if and only if it gives a covering in racks [**38**, Example 2.34], see also [**38**, Example 1.2] and comments below.

Part II

In this second part of the thesis, and based on the firm theoretical groundings of categorical Galois theory, we identify the *second order coverings* of racks and quandles, and the *relative* concept of *centralization*, together with the definition of a suitable *commutator* in this context.

0.1. Towards higher dimensions. Recall that in order to extend the covering theory of racks and quandles to higher dimensions, we first look at the arrow category ExtRck (or ExtQnd). Given any category $\mathcal{C}$ with a chosen class of extensions $\mathcal{E}$ (Convention 1.0.1), $\mathsf{Ext}\mathcal{C}$ refers to the full subcategory of extensions within the category of morphisms (arrow category) $\mathsf{Arr}\mathcal{C}$. A morphism $\alpha\colon f_A \to f_B$ in such a category of morphisms is given by a pair of morphisms in $\mathcal{C}$, which we denote $\alpha = (\alpha_{\top}, \alpha_{\bot})$ (the *top* and *bottom components* of α) as in the commutative Diagram (27).

$$\begin{tikzcd}
A_{\top} \arrow[rr, "\alpha_{\top}"] \arrow[dr, "p"] \arrow[dd, "f_A"'] & & B_{\top} \arrow[dd, "f_B"] \\
& A_{\bot} \times_{B_{\bot}} B_{\top} \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"] & \\
A_{\bot} \arrow[rr, "\alpha_{\bot}"'] & & B_{\bot}
\end{tikzcd} \tag{27}$$

If all morphisms in this commutative diagram are in $\mathcal{E}$, including α's so-called *comparison map* p, then α is said to be a *double extension* [**67**]. For our purposes, the class of double extensions $\mathcal{E}^1$ is the appropriate induced class of (two-dimensional) extensions in $\mathsf{Ext}\mathcal{C}$.

The inclusion $\mathrm{I}\colon \mathsf{CExtRck} \to \mathsf{ExtRck}$ (and similarly for the inclusion $\mathrm{I}\colon \mathsf{CExtQnd} \to \mathsf{ExtQnd}$), of the full subcategory of coverings in the category of extensions, admits a left adjoint $\mathrm{F}_1\colon \mathsf{ExtRck} \to \mathsf{CExtRck}$. The functor F_1 universally makes an extension into a covering (or central extension). It is said to *universally centralize* an extension (one-dimensional centralization) in the same way that π_0 *universally trivializes* objects (zero-dimensional centralization). The unit η^1 of the adjunction

$\mathrm{F}_1 \dashv \mathrm{I}$ is defined for an extension $f\colon A \to B$ by $\eta^1_f := (\eta^1_A, \mathrm{id}_B)$, where the kernel pair $\mathrm{Eq}(\eta^1_A)$ of the quotient η^1_A is denoted $\mathrm{C}_1\, f$. As mentioned before, it is generated by the pairs $(x \triangleleft a \triangleleft^{-1} b, x)$ for x, a, and b in A such that $f(a) = f(b)$. Then $\eta^1_f \in \mathcal{E}^1$ is a double extension making $\mathsf{CExtRck}$ into an $\mathcal{E}^1$-reflective subcategory of ExtRck (Convention 1.0.1). This data then fits into the square of adjunctions

$$\begin{tikzcd}
\mathsf{ExtRck} \arrow[r, bend right, "\mathrm{F}_1"'] \arrow[r, phantom, "\top"] \arrow[d, bend right, "\mathrm{Pth}^1"'] \arrow[d, phantom, "\dashv"] & \mathsf{CExtRck} \arrow[l, bend right, "\mathrm{I}"'] \arrow[d, bend right, "\mathrm{Pth}^1"'] \arrow[d, phantom, "\dashv"] \\
\mathsf{ExtGrp} \arrow[r, bend right, "\mathrm{ab}^1"'] \arrow[r, phantom, "\top"] \arrow[u, bend right, "\mathrm{Conj}^1"'] & \mathsf{CExtGrp} \arrow[l, bend right, "\mathrm{I}"'] \arrow[u, bend right, "\mathrm{Conj}^1"']
\end{tikzcd} \qquad (28)$$

where the functors Pth^1 and Conj^1 are the appropriate restrictions of the adjoint pairs induced by $\mathrm{Pth} \dashv \mathrm{Conj}$ between the categories of morphisms above Rck and Grp. The functor ab^1 is the *centralization functor* in Grp sending a surjective group homomorphism $f\colon G \to H$ to the central extension of groups $\mathrm{ab}^1(f)\colon G/[\mathrm{Ker}\, f, G]_{\mathsf{Grp}} \to H$ obtained from the quotient of the domain of f: $G/[\mathrm{Ker}\, f, G]_{\mathsf{Grp}}$. Here $\mathrm{Ker}\, f$ is the kernel of f and for any normal subgroups X and $Y \leq G$, the normal subgroup $[X, Y]_{\mathsf{Grp}} := \langle xyx^{-1}y^{-1} \mid x \in X,\ y \in Y\rangle \leq X \cap Y \leq G$ denotes the classical commutator from group theory. As before, all squares of functors in Diagram (28) commute, but for the square (F_1, Conj^1, ab^1, Conj^1) which does not (see Example 2.2.1).

0.2. Content. In Section 1, we first show that categorical Galois theory applies to the adjunction $\mathrm{F}_1 \dashv \mathrm{I}$ on the top line of Diagram (28), which we fit into a *strongly Birkhoff Galois structure* Γ^1 (Section 1.3). Alongside the results of Part I, this mainly consists in the recollection of classical properties of double extensions, including a bit of *descent theory* (see [**79, 78**] and references therein). The rest of Part II is then aimed at the characterization and "visualization" of the induced notion of Γ^1*-covering*, or *double central extension of racks and quandles*, as it was previously done for groups [**67**], leading to the developments of [**50, 45**]. Note that the study of the more technical categorical aspects of Section 1 is not necessary for the readers' understanding of what follows. Section 1.4 provides a useful transition to the rest of Part II, as we recall our general method for the characterization of coverings, and produce a first visual representation of trivial Γ^1-coverings. We define and study the concept of *double covering*, also called *algebraically central double extension of racks and quandles* in Section 2. We provide examples, and the definition of a meaningful and well-behaved notion of commutator, which captures (in the usual sense) the centralization congruence for objects, extensions and double extensions in the category of quandles. We illustrate our methods and definitions via the characterization of normal Γ^1-coverings, which leads to a better understanding of two-dimensional centrality. The concept of double covering (or algebraically central double extension of racks and quandles) and the concept of Γ^1-covering (or double central extension of racks and quandles) are then shown to coincide in Section 3. Section 3.3 is dedicated to the centralization of double extensions (i.e. the reflection of the category of double extensions on the category of double coverings) leading to the next step of the covering theory. Finally, in Section 4 we hint at further research and we adapt the concept of *Galois structure with (abstract) commutators* in such a way that fits to our context and remains compatible with the developments in [**69**].

1. An admissible Galois structure in dimension 2

In order to apply categorical Galois theory (see Part I, Section 1 and references there) to the inclusion $\mathrm{I}\colon \mathsf{CExt}\mathcal{C} \to \mathsf{Ext}\mathcal{C}$ (where $\mathcal{C}$ stands for Rck or Qnd) of the category of coverings of racks (or quandles) in the category of extensions, we first fit the reflection $\mathrm{F}_1 \dashv \mathrm{I}$ into a Galois structure $\Gamma^1 \coloneqq (\mathsf{Ext}\mathcal{C}, \mathsf{CExt}\mathcal{C}, \mathrm{F}_1, \mathrm{I}, \eta^1, \epsilon^1, \mathcal{E}^1)$ satisfying the conditions of Convention 1.0.1. In order to do so, we need an appropriate class of extensions in dimension 2. In dimension 1, the base category $\mathcal{C} = \mathsf{Rck}$ or $\mathcal{C} = \mathsf{Qnd}$ is finitely *cocomplete* and *Barr-exact* [**2**], like any variety of algebras. In short this means that $\mathcal{C}$ has finite limits and colimits, it is *regular* (i.e. every morphism factors uniquely, up to isomorphism, into a regular epimorphism, followed by a monomorphism, and these factorizations are stable under pullbacks) and, moreover, every *equivalence relation* is the kernel pair of its *coequalizer*. In such a context, a fruitful class of extensions is given by the class of *regular epimorphisms*. However, for a general Barr-exact $\mathcal{C}$, the category $\mathsf{Ext}\mathcal{C}$ is not necessarily Barr-exact (see comment preceding Definition 3.4 in [**50**]); ExtRck and ExtQnd even fail to be regular categories (see below). The class of regular epimorphisms is then not appropriate for applying Galois theory. As mentioned before, the appropriate class of extensions (in the category of extensions) is given by the class of double extensions ($\mathcal{E}^1$).

Given $\mathcal{C}$ and $\mathcal{E}$ as above, let us briefly recall some well-known basic properties of the category $\mathsf{Ext}\mathcal{C}$, full subcategory of extensions within $\mathsf{Arr}\mathcal{C}$. Limits in $\mathsf{Arr}\mathcal{C}$ are *computed component-wise.* Given a diagram D in $\mathsf{Arr}\mathcal{C}$, compute the limits $L_{\top}$ and $L_{\bot}$ in $\mathcal{C}$ of the diagrams obtained as the top component of D and the bottom component of D respectively. The limit $l\colon L_{\top} \to L_{\bot}$ of D in $\mathsf{Arr}\mathcal{C}$ is given by the induced comparison map between $L_{\top}$ and $L_{\bot}$. Using the regularity of $\mathcal{C}$, limits can be computed in $\mathsf{Ext}\mathcal{C}$ as the regular epic part e of the regular epi-mono factorization $l = me$ of the limit in $\mathsf{Arr}\mathcal{C}$ (precompose the legs of the limit $L_{\bot}$ with the mono part m to obtain the bottom legs of the limit e in $\mathsf{Ext}\mathcal{C}$). Pushouts in $\mathsf{Ext}\mathcal{C}$ are computed component-wise in $\mathcal{C}$. The initial object is the identity on the initial object of $\mathcal{C}$. The coequalizer of a parallel pair of morphisms in $\mathsf{Ext}\mathcal{C}$ is computed component-wise in $\mathcal{C}$, and the resulting commutative square is a pushout square of regular epimorphisms. Given a morphism α in $\mathsf{Ext}\mathcal{C}$ which is a pushout-square of regular epimorphisms in $\mathcal{C}$, it is the coequalizer of its kernel pair computed in $\mathsf{Ext}\mathcal{C}$. Regular epimorphisms in $\mathsf{Ext}\mathcal{C}$ are thus the same as (oriented) pushout squares of regular epimorphisms in $\mathcal{C}$. Monomorphisms are morphisms for which the top component is a monomorphism in $\mathcal{C}$. Regular epi-mono factorizations exist, and are unique in $\mathsf{Ext}\mathcal{C}$, however these might not be pullback stable in general.

Remark 1.0.1. *When $\mathcal{C}$ is Rck or Qnd, regularity of $\mathsf{Ext}\mathcal{C}$ would imply that the category of surjective functions ExtSet is regular (since ExtSet is closed under regular quotients and finite limits in $\mathsf{Ext}\mathcal{C}$). We recall that not all regular epimorphisms in ExtSet are pullback stable (see also* [**77**, Remark 3.1] *and Remark 1.2.4). Since Set is Barr-exact, ExtSet is equivalent to the category ERSet of (internal) equivalence relations over Set. Using the arguments from* [**76**, Section 2]*, a regular epimorphism in ERSet is given by a morphism $\bar{\alpha}\colon \mathrm{Eq}(f_A) \to \mathrm{Eq}(f_B)$ and a surjective morphism $\alpha_{\top}\colon A_{\top} \to B_{\top}$ that commute with the projections of the equivalence relations $\mathrm{Eq}(f_A) \rightrightarrows A_{\top}$ and $\mathrm{Eq}(f_B) \rightrightarrows B_{\top}$ (as in the top-right corner of the commutative Diagram* (44)*) and such that $(b, b') \in \mathrm{Eq}(f_B)$ if and only if there exists a finite sequence (a_1, a_1'), …, $(a_n, a_n') \in \mathrm{Eq}(f_A)$ with $b = \alpha_{\top}(a_1)$, $\alpha_{\top}(a_i') = \alpha_{\top}(a_{i+1})$ for $i \in \{1, \dots, n-1\}$ and $\alpha_{\top}(a_n') = b'$* [**76**, Proposition 2.2]*. Such a morphism is a pullback stable regular epimorphism if and only if it is a regular epimorphism such that $(b, b') \in$*

$\mathrm{Eq}(f_B)$ *if and only if there exists* $(a, a') \in \mathrm{Eq}(f_A)$ *with* $b = \alpha_\top(a_1)$ *and* $\alpha_\top(a') = b'$ [**76**, Proposition 2.3(b)]. *We adapt* [**87**, Example 2.4] *to this context: define* $A_\top = \{(0,a),(0,b),(1,a),(1,b)\}$, $B_\top = \{0,1,2\}$ *and* $\alpha_\top$ *such that* $\alpha_\top(0,a) = 0$, $\alpha_\top(1,b) = 2$ *and* $\alpha_\top(0,b) = \alpha_\top(1,a) = 1 \in B_\top$. *If* $\mathrm{Eq}(f_A)$ *is the equivalence relation generated by the pairs* $((0,a),(1,a))$ *and* $((0,b),(1,b))$*; and* $\mathrm{Eq}(f_B)$ *is* $B_\top \times B_\top$*, then the pair* $(\bar{\alpha}, \alpha_\top)$ *defines a regular epimorphism in* ERSet*, but it is not pullback stable. Indeed, its pullback along the inclusion of* $\{0,2\} \times \{0,2\} \rightrightarrows \{0,2\}$ *in* $B_\top \times B_\top \rightrightarrows B_\top$ *is not a regular epimorphism.*

It is convenient to bring back a problem or computation in $\mathsf{Ext}\mathcal{C}$ to a couple of problems and computations in $\mathcal{C}$, using the projections on the top and bottom components (this component-wise decrease in dimension is essential for the inductive approach to higher covering theory [**45**]). "From an engineering perspective", our interest in the concept of a double extension lies in the fact that pullbacks of such, and subsequently many other constructions involving double extensions, can be computed component-wise in $\mathcal{C}$.

1.1. Basic properties of double extensions. In short, we hope for the class of double extensions to have as many good properties in $\mathsf{Ext}\mathcal{C}$ as the class of regular epimorphisms has in the Barr-exact category $\mathcal{C}$. Note that since double extensions are regular epimorphisms in $\mathsf{Ext}\mathcal{C}$, a double extension which is a monomorphism in $\mathsf{Ext}\mathcal{C}$ is an isomorphism. Note that Proposition 3.5 and Lemma 3.8 from [**50**] easily generalize to our context as it was observed in [**48**] (Example 1.11, Proposition 1.6 and Remark 1.7). For any *regular* [**2**] category $\mathcal{C}$:

LEMMA 1.1.1. *(1) If a morphism* $\alpha = (\alpha_\top, \alpha_\bot)$ *in* $\mathsf{Ext}\mathcal{C}$ *is such that* $\alpha_\top$ *is an extension and* $\alpha_\bot$ *an isomorphism, then* α *is a double extension.*
(2) Double extensions are closed under composition.
(3) Pullbacks along double extensions (exist in $\mathsf{Ext}\mathcal{C}$ *and) are computed* component-wise*. Moreover the pullback of a double extension is a double extension.*

Given $\alpha = (\alpha_\top, \alpha_\bot)\colon f_A \to f_B$ and $\gamma = (\gamma_\top, \gamma_\bot)\colon f_C \to f_B$, a pair of morphisms in $\mathsf{Ext}\mathcal{C}$, their *component-wise pullback* is given by the following commutative diagram in $\mathcal{C}$ where the front and back faces are pullbacks

– i.e. it is the pullback of α and γ in the arrow category $\mathsf{Arr}\mathcal{C}$:

$$\begin{tikzcd}
& C_\top \times_{B_\top} A_\top \arrow[rr, "\pi_{A_\top}"] \arrow[dl, dotted, "f"'] \arrow[dd, "\pi_{C_\top}"] \arrow[dr, phantom, "\lrcorner", very near start] & & A_\top \arrow[dl, "f_A"] \arrow[dd, "\alpha_\top"] \\
C_\bot \times_{B_\bot} A_\bot \arrow[rr, "\pi_{A_\bot}"] \arrow[dd, "\pi_{C_\bot}"'] \arrow[dr, phantom, "\lrcorner", very near start] & & A_\bot \arrow[dd, "\alpha_\bot"] & \\
& C_\top \arrow[rr, "\gamma_\top"] \arrow[dl, "f_C"'] & & B_\top \arrow[dl, "f_B"] \\
C_\bot \arrow[rr, "\gamma_\bot"'] & & B_\bot &
\end{tikzcd}$$

Provided that α is a double extension, Lemma 1.1.1 above says that f is an extension and the pullback of α and β in $\mathsf{Ext}\mathcal{C}$ is given by f together with the projections $(\pi_{A_\top}, \pi_{A_\bot})$ and $(\pi_{C_\top}, \pi_{C_\bot})$, where the latter is actually a double extension. In particular, the kernel pair of a double extension $\alpha = (\alpha_\top, \alpha_\bot)$ exists in $\mathsf{Ext}\mathcal{C}$ and is given by the kernel pairs of $\alpha_\top$ and $\alpha_\bot$ in each component (together with the induced morphism between those – see Notation 2.3.3). Moreover, the legs of such kernel pairs are themselves double extensions (see Lemma 1.2.1).

Lemma 1.1.1 is important for what follows, if only because pullbacks along extensions appear everywhere in categorical Galois theory. As we mentioned earlier, if neither α or β is known to be a double extension, their pullback in $\mathsf{Ext}\mathcal{C}$ still exists, but it is not necessarily computed component-wise and thus it is badly behaved. As we move to higher dimensions, these general pullbacks are no longer convenient for our purposes.

Note that in the context of Barr-exact *Mal'tsev* categories [**26**], double extensions are the same as pushout squares of extensions, and, as a rule, higher extensions are easier to identify – primarily using split epimorphisms. Note that the lack of such arguments is a challenge in our more general context where categories are not Mal'tsev categories.

As we may conclude from [**48**, Proposition 3.3] and the fact that our categories are not *Mal'tsev*, the axiom (E4) of "right cancellation" considered there (see also [**50**, Lemma 3.8]) cannot hold in our context. We have the following weaker version:

LEMMA 1.1.2. *If the composite $\beta\alpha$ is a double extension in $\mathsf{Ext}\mathcal{C}$, and α is a commutative square of extensions in $\mathcal{C}$, then β is a double extension.*

PROOF. Since $\beta_\top\alpha_\top$ and $\beta_\perp\alpha_\perp$ are regular epimorphisms, β is a square of extensions in $\mathcal{C}$. Consider the following commutative diagram

$$\begin{tikzcd}
A_1 \arrow[rrr, two heads, "\alpha_\top"] \arrow[dr, "\phi_1"] \arrow[ddd, two heads, "f_A"'] & & & B_1 \arrow[rr, two heads, "\beta_\top"] \arrow[dr, "\phi_2"] \arrow[ddd, two heads, "f_B"'] & & C_1 \arrow[ddd, two heads, "f_C"] \\
& P_1 \arrow[urr, two heads, "\pi_2"] \arrow[dr, "\phi_3"] \arrow[ddl, two heads, "\pi_1"] & & & P_2 \arrow[ur, two heads, "p_2"] \arrow[ddl, two heads, "p_1"] & \\
& & P_3 \arrow[urr, "q_2"] \arrow[dll, "q_1"] & & (2) & \\
A_0 \arrow[rrr, "\alpha_\perp"'] & & & B_0 \arrow[rr, "\beta_\perp"'] & & C_0
\end{tikzcd} \tag{29}$$

where P_1 is the pullback of $\alpha_\perp$ and f_B, P_2 is the pullback of $\beta_\perp$ and f_C and P_3 is the pullback of $\alpha_\perp$ and the projection p_1. Note that since P_3 is also the pullback of $\beta_\perp\alpha_\perp$ and f_C, the comparison map of the composite square is $\phi_3\phi_1$, which is an extension. As a consequence ϕ_3 is an extension and so is the composite $f_B\phi_3 = \phi_2\pi_1$. We conclude that the comparison map ϕ_2 of β is also an extension. ☐

In particular we deduce that pullbacks along double extensions reflect double extensions.

COROLLARY 1.1.3. *Given a morphism* $\alpha = (\alpha_\top, \alpha_\perp)$ *in* $\mathsf{Ext}\mathcal{C}$*, if its pullback along a double extension* γ *yields a double extension, then* α *is itself a double extension.*

Finally we observe the following.

LEMMA 1.1.4. *If a composite of two double extensions* $\beta\alpha$ *is a pullback square, then both* α *and* β *are pullback squares.*

PROOF. Using Diagram (29), we may assume that $\phi_3\phi_1$ is an isomorphism. As a consequence, both ϕ_1 and ϕ_3 are isomorphisms. Observe that the square $q_2\phi_2 = \phi_3\pi_2$ is also a pullback square of regular epimorphisms. We conclude that ϕ_2 is also an isomorphism because isomorphisms are reflected by pullbacks along regular epimorphisms in a regular category $\mathcal{C}$ – see Lemma 1.1.5 below. ☐

The following lemma can be seen as a consequence of [**13**, Theorem 2.17].

LEMMA 1.1.5. *Given the following pullback square of regular epimorphisms in a regular category $\mathcal{C}$, if p_2 is an isomorphism, then g is also an isomorphism.*

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

PROOF. Take the kernel pairs of p_2 and g and build the commutative diagram

$$\begin{tikzcd}
\mathrm{Eq}(p_2) \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] \arrow[d, "\bar{f}"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
\mathrm{Eq}(g) \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

where both commutative squares on the left are pullback squares (which can be proved in any category by looking at the underlying commutative cube). The induced morphism $f\colon \mathrm{Eq}(p_2) \to \mathrm{Eq}(g)$ is a regular epimorphism since it is the pullback of the regular epimorphism p_1 in the regular category $\mathcal{C}$. Since p_2 is an isomorphism, the projections $d, c\colon \mathrm{Eq}(p_2) \rightrightarrows P$ are isomorphisms, and thus the projections $d', c'\colon \mathrm{Eq}(g) \rightrightarrows C$ coincide, using the fact that $\bar{f}$ is a regular epimorphism. This implies that g is a monomorphism and thus it is also an isomorphism (since it is a regular epimorphism). □

1.2. Beyond Barr exactness, effective descent along double extensions. Given a Galois structure Γ as in Convention 1.0.1, Γ-coverings, which are the key concept of study, are defined as those extensions $c\colon A \to B$ for which there is another extension $e\colon E \to B$ such that the pullback t of c along e is a trivial Γ-covering. In most references [**9, 60, 69**], e is further required to be of *effective descent* or *effective $\mathcal{E}$-descent* (see [**79, 78**] and references therein). Such extensions are sometimes also called *monadic extensions* [**66, 47**]. In the contexts of interest for this work, we shall always have that all our extensions are of effective $\mathcal{E}$-descent, which is why we use this simplified definition of covering. The idea is to ask that "pulling back along e is an *algebraic* operation", which is necessary for the "information about coverings to be *tractable* in $\mathcal{X}$" in the sense of the fundamental theorem of categorical Galois theory (see for instance [**66**, Corollary 5.4]). We didn't insist on this requirement for Part I (see also [**71**]) since the class of effective descent morphisms in a Barr-exact $\mathcal{C}$ is well known to be the class of regular epimorphisms [**79**].

Given an extension $e\colon E \to B$ in $\mathcal{C}$, if we write $\mathsf{Arr}(Y)$ for the category of morphisms with codomain Y, then there is an induced pair of adjoint functors: $e_*\colon \mathsf{Arr}(E) \to \mathsf{Arr}(B)$, left adjoint of the functor $e^*\colon \mathsf{Arr}(B) \to \mathsf{Arr}(E)$, where $f_*(k\colon X \to A) \coloneqq fk$, and $f^*(h\colon Y \to B)$ is given by the pullback of h along f. This adjunction also restricts to the categories of extensions above E and B: $e_*|\colon \mathsf{Ext}(E) \to \mathsf{Ext}(B)$, left adjoint of $e^*|\colon \mathsf{Ext}(B) \to \mathsf{Ext}(E)$ (defined similarly). We say that e is of effective (global) descent if e^* is *monadic*, and e is of effective $\mathcal{E}$-descent if $e^*|$ is *monadic* (see [**80**]). Let us add for the interested reader that, in order to prove the fundamental theorem of categorical Galois theory, G. Janelidze showed that (in an admissible Galois structure) if $e^*|$ is monadic and we write T for the *monad induced by* $e_*| \dashv e^*|$ [**89**], then the category of those extensions which are split by e is equivalent to the category of those *Eilenberg-Moore* T*-algebras* [**89**] $(f\colon X \to A, \mu\colon Tf \to f)$ such that $f\colon X \to A$ is a trivial covering (see for instance [**66**, Proposition 4.2, Theorem 5.3]).

In this section we show that double extensions of racks and quandles are of effective global and $\mathcal{E}^1$-descent in the category of extensions. From Lemma 3.2 in [**48**] and the above, we have what can be understood as local $\mathcal{E}^1$-Barr exactness:

LEMMA 1.2.1. *Assuming that* $\mathcal{C}$ *is Barr-exact, and given a commutative square of extensions together with the horizontal kernel pairs and the factorization* f *between them;*

$$\begin{tikzcd}
\mathrm{Eq}(\sigma_\top) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f"'] \arrow[dr, phantom, "(*)"] & E_\top \arrow[r, "\sigma_\top"] \arrow[d, "f_E"] & B_\top \arrow[d, "f_B"] \\
\mathrm{Eq}(\sigma_\bot) \arrow[r, shift left] \arrow[r, shift right] & E_\bot \arrow[r, "\sigma_\bot"'] & B_\bot
\end{tikzcd} \tag{30}$$

then, the right hand square is a double extension if and only if any of the two left hand (commutative) squares is a double extension.

If so, then $\sigma = (\sigma_\top, \sigma_\bot)$ *is the coequalizer in* $\mathsf{Ext}\mathcal{C}$ *of the parallel pair* $(*)$ *on the left, which is in turn the kernel pair of* σ*. Such an equivalence relation* $f = \mathrm{Eq}(\sigma)$ *in* $\mathsf{Ext}\mathcal{C}$ *is* stably effective *in the sense that it is the kernel pair of its coequalizer, and any pullback of its coequalizer is still a regular epimorphism (see for instance* [**78**, Section 2.B]*). In particular,* *double extensions are the coequalizers of their kernel pairs (computed component-wise in $\mathcal{C}$).*

PROOF. The first part is a direct consequence of Lemma 3.2 [**48**]. Since the component-wise coequalizer σ of $(*)$ is a pushout square, it coincides with the coequalizer in $\mathsf{Ext}\mathcal{C}$. Then $(*)$ is the kernel pair of σ since pullbacks along double extensions are computed component-wise. It is stably effective because everything is computed component-wise, and $\mathcal{C}$ is Barr-exact. $\square$

Note that we also have the classical result, see for instance [**2**, Example 6.10], which is called the *Barr-Kock* Theorem in [**13**, Theorem 2.17]. From there we easily obtain (as in Remark 4.7 [**50**], or Lemma 3.2 (2) [**60**]):

LEMMA 1.2.2. *Double extensions are of effective (global) descent in* $\mathsf{Ext}\mathcal{C}$.

PROOF. Let $\sigma\colon f_E \to f_B$ be a double extension. The monadicity of σ^* in each component $\sigma_\top^*$ and $\sigma_\bot^*$ (see [**79**]) easily yields the monadicity of σ^* itself. For instance, we use the characterization in terms of *discrete fibrations* [**78**, Theorem 3.7].

Consider a discrete fibration of equivalence relations $f_R\colon R_\top \to R_\bot$ above the kernel pair $\mathrm{Eq}(\sigma)\colon\ \mathrm{Eq}(\sigma_\top) \to \mathrm{Eq}(\sigma_\bot)$ of $\sigma = (\sigma_\top, \sigma_\bot)$ as in the commutative diagram of plain arrows below. Then observe that f_R is computed component-wise and consists in a comparison map between a pair of discrete fibrations of equivalence relations $R_\top$ and $R_\bot$, above the pair of kernel pairs $\mathrm{Eq}(\sigma_\top)$ and $\mathrm{Eq}(\sigma_\bot)$ with comparison map $\mathrm{Eq}(\sigma)\colon\ \mathrm{Eq}(\sigma_\top) \to \mathrm{Eq}(\sigma_\bot)$. The projections of the equivalence relation f_R are also double extensions as the pullback of the projections of $\mathrm{Eq}(\sigma)$ which are themselves double extensions by Lemma 1.2.1. We build the square $(*)$ on the right, first by taking the coequalizer $\gamma = (\gamma_\top, \gamma_\bot)$ of f_R, which is computed component-wise (see 1.2.1 again). The factorization $(\bar{\beta}_\top, \bar{\beta}_\bot)$ is then obtained by the universal property of γ.

$$\begin{tikzcd}
f_R \arrow[r, shift left, two heads] \arrow[r, shift right, two heads] \arrow[d, two heads, "\hat{\beta}"'] \arrow[dr, phantom, "\lrcorner", very near start] & f_C \arrow[r, dashed, two heads, "(\gamma_\top{,}\gamma_\bot)"] \arrow[d, two heads, "\beta"'] \arrow[dr, phantom, "(*)"] & f_D \arrow[d, dotted, two heads, "(\bar{\beta}_\top{,}\bar{\beta}_\bot)"] \\
\mathrm{Eq}(\sigma) \arrow[r, shift left, two heads] \arrow[r, shift right, two heads] & f_E \arrow[r, two heads, "\sigma"'] & f_B
\end{tikzcd}$$

By Lemma 1.2.1, f_R is the kernel pair of γ which is a double extension. Also $(*)$ is a pullback square as it is component-wise by [**13**, Theorem

2.17]. Finally γ is pullback stable as a coequalizer, since everything is computed component-wise (see Lemma 1.2.1). □

What is exactly needed in our context is not effective global descent but effective $\mathcal{E}^1$-descent. This derives from Lemma 1.2.2 because of Corollary 1.1.3, as it is explained in [**79**, Section 2.7].

COROLLARY 1.2.3. *Double extensions are of effective* $\mathcal{E}^1$*-descent in* $\mathsf{Ext}\mathcal{C}$.

REMARK 1.2.4. *It was shown in* [**46**] *that given a regular category* $\mathcal{C}$, $\mathsf{Ext}\mathcal{C}$ *is regular if and only if its effective global descent morphisms are the regular epimorphisms (i.e. the pushout squares of regular epimorphisms). As far as we know, in the categories of racks and quandles, the classes of effective global and* $\mathcal{E}^1$*-descent morphisms contain the class of double extensions and are strictly contained in the class of regular epimorphisms. We do not need to characterize these more precisely for what follows.*

1.3. Strongly Birkhoff Galois structure.

In order for categorical Galois theory (and in particular its fundamental theorem) to hold in the context of a Galois structure such as Γ from Convention 1.0.1, Γ is further required to be *admissible*, in the sense of [**71, 9**], which implies for instance that pullbacks of unit morphisms along primitive extensions are unit morphisms, or subsequently that coverings, normal coverings and trivial coverings are preserved by pullbacks along extensions. We actually work with a stronger property for our Galois structures, which we require to be *strongly Birkhoff* in the sense of [**50**, Proposition 2.6], where this condition is shown to imply the admissibility condition. The Galois structure Γ is said to be *strongly Birkhoff* if *reflection squares* at extensions are double extensions. Given $f\colon A \to B$ in $\mathcal{C}$, *the reflection square at* f (with respect to Γ) is the morphism (η_A, η_B) with domain f and codomain $\mathrm{I}\,F(f)$ in $\mathsf{Arr}(\mathcal{C})$.

$$\begin{tikzcd}
A \arrow[rr, "\eta_A"] \arrow[dd, "f"'] \arrow[dr, "p"] & & \mathrm{I}\,F(A) \arrow[dd, "\mathrm{I}\,F(f)"] \\
& B \times_{\mathrm{I}\,F(B)} \mathrm{I}\,F(A) \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\
B \arrow[rr, "\eta_B"'] & & \mathrm{I}\,F(B)
\end{tikzcd} \qquad (31)$$

Our Galois structure Γ^1 is strongly Birkhoff if the reflection squares at double extensions (as defined in $\mathcal{C}$, for $\mathcal{C} = \mathsf{Rck}$ or $\mathcal{C} = \mathsf{Qnd}$) are *double extensions in* $\mathsf{Ext}\mathcal{C}$, which defines the concept of 3*-fold extension* (see [**50, 45**]).

DEFINITION 1.3.1. *Given any regular category $\mathcal{C}$, define* the category $\mathsf{Ext}^2\mathcal{C}$ *whose objects are double extensions (as in Diagram* (27)*) and whose morphisms* $(\sigma,\beta)\colon \gamma \to \alpha$ *between two double extensions* γ *and* α *are given by the data of the (oriented) commutative diagram in* $\mathsf{Ext}\mathcal{C}$ *(on the left) or equivalently in* $\mathcal{C}$ *(on the right):*

$$\begin{tikzcd}
f_C \arrow[rr, "\sigma=(\sigma_\top,\sigma_\perp)"] \arrow[dr, "\pi=(\pi_\top,\pi_\perp)"] \arrow[dd, "\gamma=(\gamma_\top,\gamma_\perp)"'] & & f_A \arrow[dd, "\alpha=(\alpha_\top,\alpha_\perp)"] \\
& f_P \arrow[ur] \arrow[dl] & \\
f_D \arrow[rr, "\beta=(\beta_\top,\beta_\perp)"'] & & f_B
\end{tikzcd}
\qquad
\begin{tikzcd}
& C_\top \arrow[rr, "\sigma_\top"] \arrow[dl, "f_C"'] \arrow[dd, "\gamma_\top"] & & A_\top \arrow[dl, "f_A"'] \arrow[dd, "\alpha_\top"] \\
C_\perp \arrow[rr, "\sigma_\perp"] \arrow[dd, "\gamma_\perp"'] & & A_\perp \arrow[dd, "\alpha_\perp"] & \\
& D_\top \arrow[rr, "\beta_\top"] \arrow[dl, "f_D"'] & & B_\top \arrow[dl, "f_B"] \\
D_\perp \arrow[rr, "\beta_\perp"'] & & B_\perp, &
\end{tikzcd} \tag{32}$$

where f_P *is the pullback of* α *and* β*. A* 3-fold extension (σ,β) *in* $\mathcal{C}$ *is given by such a morphism in* $\mathsf{Ext}^2\mathcal{C}$ *such that* σ*,* β *and the comparison map* π *are also double extensions i.e. a* 3*-fold extension in* $\mathcal{C}$ *is the same thing as a double extension in* $\mathsf{Ext}\mathcal{C}$*. Note that most results from Sections 1.1 and 1.2 generalize to* 3*-fold extensions and* higher *extensions in our context (see Part III).*

Now since all but the "very top" component of the *centralization units* in higher dimension (such as η^1 above) are identities (see Corollary 5.2 [**64**]), we can break down the strong Birkhoff condition in two steps: first the closure by quotients of (1-fold) coverings along double extensions, or equivalently, the fact that reflection squares at double extensions are pushout squares in $\mathsf{Ext}\mathcal{C}$ (Birkhoff condition); and secondly, a *permutability condition*, in the base category $\mathcal{C}$, on the kernel pair of the non-trivial component of the centralization unit η^1. From Section 3.4.5 in Part I, we get:

THEOREM 1.3.2. *The Galois structure*

$$\Gamma^1 := (\mathsf{Ext}\mathcal{C}, \mathsf{CExt}\mathcal{C}, \mathrm{F}_1, \eta^1, \epsilon^1, \mathcal{E}^1),$$

where $\mathcal{C}$ *is either* Rck *or* Qnd*, is strongly* $\mathcal{E}^1$*-Birkhoff, i.e. given a double extension of racks or quandles* $\alpha = (\alpha_\top, \alpha_\perp)\colon f_A \to f_B$ *(as in Diagram* (27)*), the reflection square at* α *(with respect to the reflection* $\mathrm{F}_1 \dashv \mathrm{I}_1$*) is a* 3*-fold extension, i.e. the reflection square's comparison map is a double extension, and it defines a cube of double extensions in* $\mathsf{Ext}\mathcal{C}$*.*

PROOF. Since the bottom component of η^1 is an isomorphism, it suffices to show that the top component is a double extension for the whole cube to be a 3-fold extension. This was shown in Corollary 3.4.8 of Part I. □

In particular this justifies the study of Γ^1-coverings and the relative *second order centrality* in the categories of racks and quandles.

REMARK 1.3.3. *A consequence of the strong Birkhoff condition is that if γ is a morphism of $\mathsf{Ext}\mathcal{C}$, and γ factorizes as $\gamma = \alpha\beta$, where α and β are double extensions, then if γ is a trivial Γ^1-covering, by Lemma 1.1.4, both β and α are trivial Γ^1-coverings. Hence if γ is a Γ^1-covering (see Convention 1.0.1 or Section 1.4 below), then both α and β are Γ^1-coverings. From there, and by the fact that Γ^1-covering are reflected by pullbacks along double extensions (see Convention 1.0.1), it is easy to conclude that Γ^1-coverings are closed under quotients along 3-fold extensions in $\mathsf{Ext}^2\mathcal{C}$.*

1.4. Towards higher covering theory. The main aim of this thesis is to describe *what are the double central extensions of racks and quandles* as in the case of groups [**67**]. Following the more general terminology for *coverings*, this consists in characterizing the Γ^1-coverings of racks and quandles. These are defined abstractly in ExtRck (or ExtQnd) as the double extensions $\alpha\colon f_A \to f_B$ for which there exists a double extension $\sigma\colon f_E \to f_B$, such that σ *splits* α, i.e. the pullback of α along σ yields a trivial Γ^1-covering (see Convention 1.0.1).

1.4.1. *Projective presentations in dimension 2.* In Part I (see Section 1.0.11) we recalled that a double extension $\alpha\colon f_A \to f_B$ is split by some double extension $\sigma\colon f_E \to f_B$ if and only if α can be split by a projective presentation of its codomain f_B – provided such a projective presentation exists. Hence we want to recall that given any Barr-exact category $\mathcal{C}$, if we choose extensions to be the regular epimorphisms in $\mathcal{C}$, extensions in $\mathcal{C}$ with projective domain and projective codomain are projective objects in $\mathsf{Ext}\mathcal{C}$ (with respect to double extensions – see for instance [**45**, Section 5]). Note that when $\mathcal{C}$ is a variety of algebras, and $F\colon \mathsf{Set} \to \mathcal{C}$ is the left adjoint (with counit ϵ) of the forgetful functor $\mathrm{U}\colon \mathcal{C} \to \mathsf{Set}$, the canonical projective presentation of an object B in $\mathcal{C}$ is given by the counit morphism $\epsilon_B\colon F(B) \to B$ (where we omit U). Given an extension $f_B\colon B_\top \to B_\bot$ in such a $\mathcal{C}$, we define the *canonical projective presentation of* f_B to be the double extension $p_{f_B}: p_B \to f_B$, below, where $P \coloneqq F(B_\perp) \times_{B_\perp} B_\top$, is the pullback of f_B and $\epsilon_{B_\perp}$.

$$\begin{tikzcd}
F(P) \arrow[rr, "p^\top_{f_B}"] \arrow[dr, "\epsilon_P"'] \arrow[dd, "p_B"'] & & B_\top \arrow[dd, "f_B"] \\
& P \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\
F(B_\perp) \arrow[rr, "p^\perp_{f_B} \coloneqq \epsilon_{B_\perp}"] & & B_\perp
\end{tikzcd} \tag{33}$$

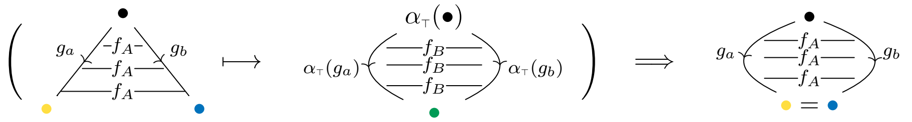
1.4.2. *Trivial* Γ^1*-coverings.* Now we want to be able to identify when the pullback of a double extension α is a trivial Γ^1-covering in $\mathsf{Ext}\mathcal{C}$ (where $\mathcal{C}$ stands for Rck or Qnd). As usual, because the Galois structure Γ^1 is strongly Birkhoff, trivial Γ^1-coverings are easy to characterize. Remember that trivial Γ^1-coverings are those double extensions in $\mathsf{Ext}\mathcal{C}$ which "behave exactly like" the primitive double extensions, i.e. those double extensions in $\mathsf{CExt}\mathcal{C}$ – see for instance [**71**, Section 1.3] and Example 1.4.5 below.

From Part I we know that trivial (1-fold) coverings of racks (or quandles) are characterized as those extensions that reflect *loops*, which are trails (x, g) whose endpoint $y = x \cdot g$ coincides with the head x. Further remember from Paragraph 3.1.9 of Part I, that given a morphism of racks (or quandles) $f \colon A \to B$, an *f-membrane* $M = ((a_0, b_0), ((a_i, b_i), \delta_i)_{1 \leq i \leq n})$ is the data of a primitive trail in $\mathrm{Eq}(f)$, whose *length* is the natural number n, whose *length* is the natural number n, and whose endpoint is denoted (a_M, b_M). We say that a_M and b_M are the *endpoints* of M in A (obtained via the projections of $\mathrm{Eq}(f)$). An *f-horn* is an f-membrane $M = ((a_0, b_0), ((a_i, b_i), \delta_i)_{1 \leq i \leq n})$ such that $x \coloneqq a_0 = b_0$. It is said to *close* (into a disk) if moreover the endpoints coincide $a_M = b_M$. It is said to *retract* if for each $1 \leq k \leq n$, the truncated horn $M_{\leq k} \coloneqq (x, (a_i, b_i, \delta_i)_{1 \leq i \leq k})$ closes. Finally, the *associated f-symmetric pair* of the membrane or horn M is given by the paths $g_a^M \coloneqq \underline{a_1}^{\delta_1} \cdots \underline{a_n}^{\delta_n}$ and $g_b^M \coloneqq \underline{b_1}^{\delta_1} \cdots \underline{b_n}^{\delta_n}$ in $\mathrm{Pth}(A)$; in general, an *f-symmetric path* is a path $g \in \mathrm{Pth}^\circ(A)$, such that $g = g_a^M (g_b^M)^{-1}$ for some membrane M as above. These definitions were used in Part I to characterize a general element in the aforementioned centralization congruence $\mathrm{C}_1\, A$ of some extension $f \colon A \to B$. We showed that $(x, y) \in \mathrm{C}_1\, A$ if and only if $x \cdot g = y$ for some *f-symmetric path g*. We repeat this approach for the two-dimensional context in Section 2. For now we observe that:

LEMMA 1.4.3. *If $\alpha \colon f_A \to f_B$ is a double extension in* Rck *(or in* Qnd*), then the following conditions are equivalent:*

(1) α is a trivial Γ^1-covering;

(2) any f_A-horn which is sent by $\alpha_\top$ to an f_B-disk in $B_\top$, actually closes into a disk in $A_\top$;



(3) $\alpha_\top$ reflects f_A-symmetric loops, in the sense that if the image by $\alpha_\top$ of an f_A-symmetric trail (x, g) loops in $B_\top$, then the trail was already a loop in $A_\top$: $x \cdot g = x$.

In what follows, we prefer to call a double extension α which satisfies these conditions a trivial double covering*. This terminology will be justified by Theorem 3.2.2 where we characterize Γ^1-coverings to be the double coverings from Definition 2.0.1 below.*

PROOF. Using the material from Section 1.3, we observe that our definition of trivial Γ-covering from Convention 1.0.1 coincides, for an admissible or strongly Birkhoff Galois structure Γ, with the more common definition: the extension t is a trivial Γ-covering if and only if the reflection square at t is a pullback. Hence α is a trivial Γ^1-covering (or trivial double covering) if and only if the reflection square at α is a pullback. Since pullbacks along double extensions are computed component-wise, and the bottom component is trivial, it suffices to check that the diagram below, where $P \coloneqq (A_\top/\,\mathrm{C}_1(f_A)) \times_{(B_\top/\,\mathrm{C}_1(f_B))} B_\top$,

$$\begin{tikzcd}
A_\top \arrow[rr, "\alpha_\top"] \arrow[dr, "p"] \arrow[dd, "\eta^1_{A_\top}"'] & & B_\top \arrow[dd, "\eta^1_{B_\top}"] \\
& P \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\
(A_\top/\,\mathrm{C}_1(f_A)) \arrow[rr, "\mathrm{F}^\top_1(\alpha)"'] & & (B_\top/\,\mathrm{C}_1(f_B))
\end{tikzcd}$$

is a pullback square, i.e. the comparison map p should be an isomorphism. Since $(\alpha_\top, \mathrm{F}^\top_1(\alpha))$ is already a double extension by Corollary 3.4.8 of Part I, it suffices to check that $\mathrm{Eq}(\alpha_\top) \cap \mathrm{C}_1(f_A) = \Delta_{A_\top}$ (the *diagonal relation* on $A_\top$). Now any element $(a, b) \in \mathrm{C}_1(f_A)$ is either such that a and b are the endpoints of an f_A-horn, or equivalently, a and b are respectively the head and endpoint of an f_A-*symmetric trail* (i.e. a trail whose path component is f_A-symmetric). □

Example 1.4.4. *As a consequence of the fact that* $\mathrm{F}_2 \dashv \mathrm{I}$ *is* $\mathcal{E}^1$*-reflective (using Lemma 1.1.4) (or simply by Lemma 1.4.3 above): if the comparison map* p *of a double extension* $\alpha\colon f_A \to f_B$ *is an isomorphism (i.e. if* α *is a pullback square), then both* α *and* (f_A, f_B) *are trivial double coverings (i.e. trivial* Γ^1*-coverings).*

Example 1.4.5. *Since any primitive double extension (i.e. a double extension whose domain and codomain are (1-fold) coverings) is a trivial double covering and coverings are closed under quotients along double extensions* **[95]***, if* $\alpha\colon f_A \to f_B$ *is a double extension and* f_A *is a covering, then* $\alpha\colon f_A \to f_B$ *is a trivial double covering (i.e. a trivial* Γ^1*-covering).*

Note that the concept of trivial double covering is not symmetric in the role of $(\alpha_{\top}, \alpha_{\perp})$ and (f_A, f_B). It is not true that in general $(\alpha_{\top}, \alpha_{\perp})$ is a trivial double covering if and only if the double extension (f_A, f_B) is one.

Example 1.4.6. *Consider the sets* $Q_2 = \{\bullet,\, \star\}$*,* $Q_3 = \{\bullet,\, \star_1,\, \star_0\}$ *and* $Q_4 \coloneqq \{\star_1,\, \star_0,\, \bullet_1,\, \bullet_0\}$ *as well as the morphisms* $t_\star\colon Q_3 \to Q_2$ *and* $t\colon Q_4 \to Q_2$*, which identify the bullets with* $\bullet$ *and the stars with* $\star$*. We write* $Q_6 \coloneqq \{\star_{11},\, \star_{10},\, \star_{01},\, \star_{00},\, \bullet_1,\, \bullet_0\}$ *for the pullback of* $t_\star$ *and* t *such that the first projection* $\pi_1\colon Q_6 \to Q_4$ *identifies* $\star_{11}$ *with* $\star_{10}$ *and* $\star_{01}$ *with* $\star_{00}$*, and symmetrically for the second projection, which moreover identifies* $\bullet_1$ *with* $\bullet_0$*.*

$$\begin{tikzcd}
Q_6 \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q_3 \arrow[d, two heads, "t_\star"] \\
Q_4 \arrow[r, two heads, "t"'] & Q_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\star_{11} \arrow[r, dash, "\pi_1"] \arrow[d, dotted, dash, "\pi_2"] & \star_{10} \arrow[d, dotted, dash, "\pi_2"] & \bullet_1 \arrow[d, dotted, dash, "\pi_2"] \\
\star_{01} \arrow[r, dash, "\pi_1"] & \star_{00} & \bullet_0
\end{tikzcd}$$

Define the involutive *(*$\triangleleft^{-1} = \triangleleft$*) quandle*

$$Q \coloneqq \{\star_{11},\, \star_{10},\, \star_{01},\, \star_{00},\, \bullet_1, \bullet_1',\, \bullet_0\}$$

such that $\bullet_1 \triangleleft \star_{11} = \bullet_1 \triangleleft \star_{01} = \bullet_1'$*,* $\bullet_1' \triangleleft \star_{11} = \bullet_1' \triangleleft \star_{01} = \bullet_1$ *and* $x \triangleleft y = x$ *for any other choice of* x *and* y *in* Q*. The function* $p\colon Q \to Q_6$ *which identifies* $\bullet_1'$ *with* $\bullet_1$ *is a surjective morphism of quandles. Then note that the double extension* $(\pi_1 p,\ t_\star)$ *is a trivial double covering since* $\pi_2 p$ *is a covering. However, the double extension* $(\pi_2 p,\ t)$ *is not a double trivial covering since* $\bullet_1 \triangleleft \star_{11} \neq \bullet_1 \triangleleft \star_{10}$ *even though their images by* $\pi_2 p$ *coincide.*

Finally, we give an example of trivial double covering which does not arise as an instance of Examples 1.4.4 and 1.4.5.

EXAMPLE 1.4.7. *As it is explained in* [**38**, Example 1.14]*, for* $n > 2$*, a* dihedral quandle D_n *is the involutive quandle obtained from the (additive) cyclic group* $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z} = \{0, \ldots, n-1\}$ *by* $x \triangleleft y := 2x - y$*, for* x *and* y *in* D_n*. We define* D_1 *and* D_2 *to be the trivial quandles (i.e. sets) with one and two elements respectively. For* $n > 2$*,* D_n *is the subquandle* $\mathbb{Z}_n \rtimes \{1\}$ *of the conjugation quandle* $\mathbb{Z}_n \rtimes \mathbb{Z}_2$*, corresponding to the* n *reflections of the regular* n*-gon. In general it injects into the* circular quandle $\mathbb{S}_1$ *defined on the unit circle in* $\mathbb{R}_2$ *by the "central symmetries along* $\mathbb{S}_1$*":* $x \triangleleft y := 2\langle x, y\rangle y - x$*, for each* x *and* y *in* $\mathbb{S}_1$*, such that* $- \triangleleft y$ *defines the unique involution which fixes* y *and sends* x *to* $-x$ *whenever* x *and* y *are orthogonal (see* [**38**, Section 3.6]*).*

Now given two natural numbers n *and* m *we have that* D_{nm} *is the product (in* Qnd*) of* D_n *and* D_m*. We consider the following double extension of dihedral quandles in* Qnd *where for* $j \in \mathbb{N}$*,* $\bar{0}\colon D_j \to D_{\perp}$ *is the terminal map to* $D_{\perp}$ *and for* $i \neq 0$ *in* $\mathbb{N}$*, the morphism* $\bar{i}\colon D_{ij} \to D_i$ *sends* $x \in D_{ij}$ *to* $x \mod i$ *in* D_i*:*

$$\begin{tikzcd}
D_{2nm} \arrow[rr, "\bar{m}"] \arrow[dr, "n\bar{m}"'] \arrow[dd, "\bar{n}"'] & & D_m \arrow[dd, "\bar{0}"] \\
& D_{nm} \arrow[ur, "\bar{m}"] \arrow[dl, "\bar{n}"] & \\
D_n \arrow[rr, "\bar{0}"'] & & \{0\},
\end{tikzcd} \tag{34}$$

Note that this double extension is symmetric in the roles of m *and* n*. By Lemma 1.4.8 below, both* $(\bar{n}, \bar{0})$ *and* $(\bar{m}, \bar{0})$ *are trivial double coverings whenever* 2*,* m *and* n *are coprime. If* $m = 2$ *and* n *are coprime, then* $(\bar{n}, \bar{0})$ *is a trivial double covering but* $(\bar{m}, \bar{0})$ *is not (indeed* $0 \triangleleft 0 = 0 \neq 2n = 0 \triangleleft n$*). See also Example 2.0.3 below.*

LEMMA 1.4.8. *Using the notation from Example 1.4.7, the double extension* $(\bar{n}, \bar{0})$ *is a trivial double covering if and only if* n *is odd and coprime with* m*.*

PROOF. Consider an $\bar{m}$-horn M of length $i > 0 \in \mathbb{N}$ which is sent to a loop by $\bar{n}$. Such a horn M is given by $x \in D_{2nm}$ together with natural numbers $y_j < m$, $a_j < n$ and $b_j < n$ for each $0 \leq j \leq i$ such that

$$\begin{aligned} x + \sum_{0 \leq j \leq i} (-1)^j y_j + 2m \sum_{0 \leq j \leq i} (-1)^j a_j = \\ = x + \sum_{0 \leq j \leq i} (-1)^j y_j + 2m \sum_{0 \leq j \leq i} (-1)^j b_j \mod n; \end{aligned} \tag{35}$$

and thus also $2m\big(\sum_{0\leq j\leq i}(-1)^j(a_j-b_j)\big)=0 \mod n$. Now if the sum $\sum_{0\leq j\leq i}(-1)^j(a_j-b_j)=0 \mod n$, then Equation (35) also holds modulo $2nm$, and the horn M closes in D_{2nm}. Conversely if Equation (35) holds modulo $2nm$, we deduce that $\sum_{0\leq j\leq i}(-1)^j(a_j-b_j)=0 \mod n$. $\square$

2. Double coverings

The concepts of covering and the relative concepts of centrality induced by the Galois theory of racks and quandles are characterized, in each dimension, via a condition involving the trivial action of certain data. In dimension zero, a rack $A_\top$ is actually a set if any element $a\in A_\top$ acts trivially on $A_\top$. In dimension 1, an extension $f_A\colon A_\top\to A_\perp$ is a covering if given elements a and $b\in A_\top$, such that $(a,b)\in \mathrm{Eq}(f_A)$ (i.e. $f_A(a)=f_A(b)$), the action of $a\, b^{-1}$ is trivial: $x\triangleleft a\triangleleft^{-1} b = x$ for all $x\in A_\top$. In dimension 2, we work with double extensions $\alpha=(\alpha_\top,\alpha_\perp)\colon f_A\to f_B$. The data we are interested in is then given by those 2×2 matrices with entries in $A_\top$, whose rows are elements in $\mathrm{Eq}(f_A)$ and whose columns are elements in $\mathrm{Eq}(\alpha_\top)$.

0-dimensional: $\cdot\ a$ 1-dimensional: $\begin{tikzcd} a \arrow[r, dash, "f_A"] & b \end{tikzcd}$ 2-dimensional:

$$\begin{tikzcd} a \arrow[r, dash, "f_A"] \arrow[d, dash, dotted, "\alpha_\top"'] & b \arrow[d, dash, dotted, "\alpha_\top"] \\ d \arrow[r, dash, "f_A"'] & c \end{tikzcd}$$

Such 2×2 matrices characterize the elements of $\mathrm{Eq}(f_A)\square\,\mathrm{Eq}(\alpha_\top)$, namely the *largest double equivalence relation above* $\mathrm{Eq}(f_A)$ *and* $\mathrm{Eq}(\alpha_\top)$ [**27, 102, 8, 75**], also called *double parallelistic relation* in [**10**, Definition 2.1, Proposition 2.1]. We sometimes write these elements as quadruples $(a,b,c,d)\in \mathrm{Eq}(f_A)\square\,\mathrm{Eq}(\alpha_\top)$ which encode the entries of the corresponding 2×2 matrix as above. Their "trivial action on the elements of $A_\top$" is the condition we are interested in. We define *double coverings of racks and quandles* and later show that these coincide with the Γ^1-coverings.

DEFINITION 2.0.1. *A double extension of racks (or quandles)* $\alpha\colon f_A\to f_B$ *(as in Diagram* (27)*) is said to be a* double covering *or an* algebraically central double extension *if any of the equivalent conditions* (*i*) - (*iv*) *below are satisfied:*

$$(i)\quad x\triangleleft a\triangleleft^{-1} b\triangleleft c\triangleleft^{-1} d = x,$$

$$(ii)\quad x\triangleleft^{-1} a\triangleleft d\triangleleft^{-1} c\triangleleft b = x,$$

$$(iii)\quad x\triangleleft^{-1} a\triangleleft b\triangleleft^{-1} c\triangleleft d = x,$$

$$(iv)\quad x\triangleleft a\triangleleft^{-1} d\triangleleft c\triangleleft^{-1} b = x,$$

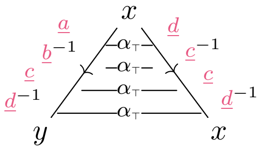
$$\text{for all } \ x \in A_\top \ \text{ and } \begin{tikzcd} a \arrow[r, dash, "f_A" description] \arrow[d, dash, dotted, "\alpha_\top" description] & b \arrow[d, dash, dotted, "\alpha_\top" description] \\ d \arrow[r, dash, "f_A" description] & c \end{tikzcd} \in \mathrm{Eq}(f_A)\square\,\mathrm{Eq}(\alpha_\top).$$

Note that, by the symmetries of quadruples (a, b, c, d) in the double equivalence relation $\mathrm{Eq}(f_A)\square\,\mathrm{Eq}(\alpha_\top)$, one could equivalently use any cyclic permutation of the letters a, b, c, and d in the equalities (i) – (iv). The equivalence between each of these (i) – (iv), is shown in Section 2.1.

REMARK 2.0.2. *In Definition 2.0.1, the roles of* f_A *and* $\alpha_\top$ *are symmetric. Hence* $(\alpha_\top, \alpha_\perp)$ *is a double covering (or algebraically central) if and only if* (f_A, f_B) *is a double covering, which can be viewed as a property of the underlying commutative square in* Rck *(or* Qnd*). Unlike trivial* Γ^1*-coverings (also called trivial double coverings), the* Γ^1*-coverings are indeed expected to be symmetric in the same sense (see* [**45**, Section 3]*).*

EXAMPLE 2.0.3. *It is easy to show that given* $\alpha\colon f_A \to f_B$*, a double extension, if either* α *or* (f_A, f_B) *is a trivial double covering, then* α *is a double covering. Note for instance that given a quadruple* $(a, b, c, d) \in \mathrm{Eq}(f_A)\square\,\mathrm{Eq}(\alpha_\top)$*, the* $\alpha_\top$*-horn* M*, displayed below, is sent to a disk by* f_A*.*

$M \quad :$



$$y := x \cdot (\underline{a}\,\underline{b}^{-1}\,\underline{c}\,\underline{d}^{-1})$$

From Example 1.4.7, when $m = 2$ *and* n *are coprime, we have that* $(\bar{m}, \bar{0})$ *is not a trivial double covering. However, it still satisfies the conditions of a double covering, which can be deduced from the fact that* $(\bar{n}, \bar{0})$ *is a trivial double covering.*

EXAMPLE 2.0.4. *Not all double coverings arise from double trivial coverings. Consider the function* $t\colon Q_4 \to Q_2$ *from Example 1.4.6 and its kernel pair* $\pi_1, \pi_2\colon Q_8 \rightrightarrows Q_2$ *where the elements of*

$$Q_8 = \{\star_{11},\ \star_{10},\ \star_{01},\ \star_{00},\ \bullet_{11},\ \bullet_{10},\ \bullet_{01},\ \bullet_{00}\}$$

organise as in the Diagram 36 below. We define the involutive quandle Q *with underlying set* $Q_8 \cup \{\bullet'_{00}\}$ *such that, for* $i \in \{0,\ 1\}$*,* $\bullet_{00} \triangleleft \star_{ii} = \bullet'_{00}$*,* $\bullet'_{00} \triangleleft \star_{ii} = \bullet_{00}$ *and* $x \triangleleft y = x$ *for any other choice of* x *and* y *in* Q*. The function* $p\colon Q \to Q_8$ *defined by* $f(\bullet'_{00}) = \bullet_{00}$ *and* $f(x) = x$ *for all*

$x \in Q_8 \subset Q$, is a morphism of quandles such that the double extension below is a double covering.

$$\begin{tikzcd} Q \arrow[rr, "\pi'_2"] \arrow[dr, "p"] \arrow[dd, "\pi'_1"'] & & Q_4 \arrow[dd, "t"] \\ & Q_8 \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\ Q_4 \arrow[rr, "t"'] & & Q_2, \end{tikzcd} \quad \Bigg| \quad \begin{tikzcd} \star_{11} \arrow[r, dash, "\pi'_1"] \arrow[d, dash, dotted, "\pi'_2"] & \star_{10} \arrow[d, dash, dotted, "\pi'_2"] \\ \star_{01} \arrow[r, dash, "\pi'_1"] & \star_{00} \end{tikzcd} \quad \begin{tikzcd} \bullet_{11} \arrow[r, dash, "\pi'_1"] \arrow[d, dash, dotted, "\pi'_2"] & \bullet_{10} \arrow[d, dash, dotted, "\pi'_2"] & \\ \bullet_{01} \arrow[r, dash, "\pi'_1"] & \bullet_{00} \arrow[r, dash, dotted, shift left] \arrow[r, dash, shift right] & \bullet'_{00} \end{tikzcd} \tag{36}$$

In anticipation of the results of Section 2.4, observe that neither (π'_1, t) nor (π'_2, t) are normal Γ^1-coverings *since $\bullet_{00} \triangleleft \star_{00} \neq \bullet_{00} \triangleleft \star_{01}$ even though $\bullet_{10} \triangleleft \star_{11} = \bullet_{10} \triangleleft \star_{10}$; and also $\bullet_{00} \triangleleft \star_{00} \neq \bullet_{00} \triangleleft \star_{10}$ even though $\bullet_{01} \triangleleft \star_{11} = \bullet_{01} \triangleleft \star_{01}$.*

OBSERVATION 2.0.5. *Finally we relate our condition (algebraic centrality of double extensions) with the existing concept of* abelian quandle *(or rack) defined in* [**82**]*. If $\alpha \colon f_A \to f_B$ is a double covering of racks (or quandles), then we have that*

$$(a \triangleleft d) \triangleleft (b \triangleleft c) = a \triangleleft a \triangleleft c = (a \triangleleft b) \triangleleft (d \triangleleft c), \tag{37}$$

for each square $\begin{tikzcd} a \arrow[r, dash] \arrow[d, dash, dotted] & b \arrow[d, dash, dotted] \\ d \arrow[r, dash] & c \end{tikzcd}$ *in* $\mathrm{Eq}(f_A) \square \mathrm{Eq}(\alpha_{\top})$ *or* $\mathrm{Eq}(\alpha_{\top}) \square \mathrm{Eq}(f_A)$, *and symmetrically in "each corner" of this square (i.e. replace the quadruple (a, b, c, d) in* (37) *by any cyclic permutation of the itself). The converse is not true in general.*

2.1. Thinking about a commutator. Let A be a rack (or quandle) and $\mathrm{ER}(A)$ be the lattice of (internal) *equivalence relations* (also called *congruences* – see for instance [**102**]), over A. We define the following binary operation on $\mathrm{ER}(A)$.

DEFINITION 2.1.1. *Given a rack A and a pair of congruences R and S in* $\mathrm{ER}(A)$*, we define $[R, S]$, element of* $\mathrm{ER}(A)$*, as the congruence generated by the set of pairs of elements of A:*

$$\{(x \triangleleft a \triangleleft^{-1} b \triangleleft c \triangleleft^{-1} d, x) \quad | \quad x \in A \;\; and \;\; \begin{tikzcd} a \arrow[r, dash, "R"] \arrow[d, dash, dotted, "S"] & b \arrow[d, dash, dotted, "S"] \\ d \arrow[r, dash, "R"] & c \end{tikzcd} \in R \square S\}.$$

Note that $[R, S]$ is in particular included in the intersection $R \cap S$. Working towards the Corollaries 2.1.4 and 2.1.5 we have that:

LEMMA 2.1.2. *Given a rack A and a pair of congruences R and S in* $\mathrm{ER}(A)$*, then $[R, S]$ is generated by the set of pairs (see also Definition 2.0.1.(ii)):*

$$\{(x \lhd^{-1} a \lhd d \lhd^{-1} c \lhd b, x) \quad \mid \quad x \in A \ \text{ and } \begin{tikzcd} a \arrow[r, dash, "R"] \arrow[d, dotted, dash, "S"'] & b \arrow[d, dotted, dash, "S"] \\ d \arrow[r, dash, "R"'] & c \end{tikzcd} \in R\square S\}.$$

PROOF. By definition, $[R, S]$ includes the pairs $(x \lhd^{-1} a \lhd a \lhd^{-1} b \lhd c \lhd^{-1} d, x \lhd^{-1} a)$ for all x, a, b, c and d as in the statement. Then by compatibility with the rack operations and reflexivity, $[R, S]$ also includes the pairs

$$(x, x \lhd^{-1} a \lhd d \lhd^{-1} c \lhd b),$$

for all such x, a, b, c and d. By symmetry this then induces that $[R, S]$ includes the congruence relation generated by the set of pairs from the statement. Now a similar argument shows that such a congruence includes the set of pairs defining $[R, S]$ as in Definition 2.1.1. ☐

COROLLARY 2.1.3. *Given a rack A and a pair of congruences R and S in* $\mathrm{ER}(A)$*, the congruence $[S, R]$ is generated by the set of pairs*

$$\{(x \lhd^{-1} a \lhd b \lhd^{-1} c \lhd d, x) \quad \mid \quad x \in A \ \text{ and } \begin{tikzcd} a \arrow[r, dash, "R"] \arrow[d, dotted, dash, "S"'] & b \arrow[d, dotted, dash, "S"] \\ d \arrow[r, dash, "R"'] & c \end{tikzcd} \in R\square S\}.$$

COROLLARY 2.1.4. *Given a rack A then for any congruences R and S in* $\mathrm{ER}(A)$*, the congruence $[R, S] = [S, R]$ is equivalently generated by any of the following sets of pairs:*

(*i*) $(x \lhd a \lhd^{-1} b \lhd c \lhd^{-1} d, x)$,

(*ii*) $(x \lhd^{-1} a \lhd d \lhd^{-1} c \lhd b, x)$,

(*iii*) $(x \lhd^{-1} a \lhd b \lhd^{-1} c \lhd d, x)$,

(*iv*) $(x \lhd a \lhd^{-1} d \lhd c \lhd^{-1} b, x)$,

$$\text{for all } x \in A \text{ and } \begin{tikzcd} a \arrow[r, dash] \arrow[d, dotted, dash] & b \arrow[d, dotted, dash] \\ d \arrow[r, dash] & c \end{tikzcd} \in R\square S.$$

PROOF. It suffices to show that $[R, S]$ contains the pairs $(x \lhd^{-1} a \lhd b \lhd^{-1} c \lhd d, x)$ for any $x \in A$ and $(a, b, c, d) \in R\square S$. Given such data, we compute that

$$\begin{tikzcd} b \arrow[r, dash] \arrow[d, dotted, dash] & a \arrow[d, dotted, dash] \\ c \arrow[r, dash] & d \end{tikzcd} \lhd \begin{tikzcd} b \arrow[r, dash] \arrow[d, dotted, dash] & b \arrow[d, dotted, dash] \\ c \arrow[r, dash] & c \end{tikzcd} = \begin{tikzcd} b \lhd b \arrow[r, dash] \arrow[d, dotted, dash] & a \lhd b \arrow[d, dotted, dash] \\ c \lhd c \arrow[r, dash] & d \lhd c \end{tikzcd} \in R\square S.$$

Then by definition $[R,S]$ contains the pair $(x \lhd (b \lhd b) \lhd^{-1} (a \lhd b) \lhd (d \lhd c) \lhd^{-1} (c \lhd c), x)$, which reduces to $(x \lhd b \lhd^{-1} b \lhd^{-1} a \lhd b \lhd^{-1} c \lhd d \lhd c \lhd^{-1} c, x)$. This concludes the proof. ☐

COROLLARY 2.1.5. *The conditions* (i)*-*(iv) *from Definition 2.0.1 are indeed all equivalent. Moreover, a double extension* $\alpha\colon f_A \to f_B$ *of racks (or quandles) is a double covering (an algebraically central double extension), if and only if* $[\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_{\top})] = \Delta_{A_{\top}}$ *(the* diagonal relation *on* $A_{\top}$*).*

Based on this result, and in anticipation of Theorems 3.2.2 and 3.3.1, we call $[\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_{\top})]$ the *centralization congruence* of the double extension $\alpha\colon f_A \to f_B$. Now observe the following:

LEMMA 2.1.6. *Given a rack* A *and a congruence* R *on* A*, the congruence* $[R, A \times A]$ *is the congruence generated by the set of pairs* $\{(x \lhd a \lhd^{-1} b, x) \mid x \in A$ *and* $(a,b) \in R\}$*.*

PROOF. Write S for the congruence generated by the set of pairs from the statement. Observe that given x, a and b such that $(a,b) \in R$ we have the square

$$\begin{tikzcd} a \arrow[r, dash, "R" description] \arrow[d, dotted, dash, "A\times A" description] & b \arrow[d, dotted, dash, "A\times A" description] \\ a \arrow[r, dash, "R" description] & a. \end{tikzcd} \qquad \in \qquad R\square(A \times A).$$

By definition we then have $S \leq [R, A \times A]$. Now observe that for any $(a,b) \in R$ and $(c,d) \in R$:

$$(x \lhd a \lhd^{-1} b) \ \text{—}S\text{—}\ x \ \text{—}S\text{—}\ (x \lhd d \lhd^{-1} c)$$

are in relation by S, and thus S also contains the generators of the congruence $[R, A \times A]$. ☐

COROLLARY 2.1.7. *Given a morphism* $f\colon A \to B$ *in* Rck *(or* Qnd*), the congruence* $\mathrm{C}_1(f)$ *can be computed as* $[\mathrm{Eq}(f), A \times A]$*, and in particular* f *is a covering (in the sense of* [**38**]*) if and only if* $[\mathrm{Eq}(f), A \times A] = \Delta_A$*.*

Recall that in the category of groups, we have the classical commutator $[-,-]_{\mathsf{Grp}}$, such that a group G is abelian if and only if its *commutator subgroup* is trivial $[G,G]_{\mathsf{Grp}} = \{e\}$ and a surjective homomorphism $f\colon G \to H$ is a central extension if and only if $[\mathrm{Ker}(f), G]_{\mathsf{Grp}} = \{e\}$.

Moreover, a double extension of groups $\gamma\colon f_G \to f_H$ is a double central extension of groups [**67**] if and only if $[\mathrm{Ker}(f_G), \mathrm{Ker}(\gamma_{\top})] = \{e\}$ and $[\mathrm{Ker}(f_G) \cap \mathrm{Ker}(\gamma_{\top}), G_{\top}] = \{e\}$ are both trivial.

For the zero-dimensional case in our context, the corresponding description of *centrality* in terms of the operation $[-,-]$ only works for quandles. Indeed, if x and a are in the quandle A, then $x \triangleleft a = x \triangleleft^{-1} x \triangleleft a$, which means that $(x \triangleleft a, x) \in [A \times A, A \times A]$. If A is a rack though, this trick does not work. In particular we compute that $[\mathrm{F_r}\, 1 \times \mathrm{F_r}\, 1, \mathrm{F_r}\, 1 \times \mathrm{F_r}\, 1] = \Delta_A \neq \mathrm{F_r}\, 1 \times \mathrm{F_r}\, 1 = \mathrm{C}_0(\mathrm{F_r}\, 1)$.

COROLLARY 2.1.8. *Given a quandle A, the congruence $\mathrm{C}_0(A)$ can be computed as $[A \times A, A \times A]$, in particular A is a trivial quandle if and only if $[A \times A, A \times A] = \Delta_A$.*

Note that in the category of groups, two-dimensional centrality is expressed using two requirements. In our context, one of the corresponding requirements entails the other (Corollary 2.1.11). First observe that our commutator is *monotone.*

LEMMA 2.1.9. *Given a rack A, as well as congruences R, S and T in $\mathrm{ER}(A)$ such that $S \leq T$, we have $[R, S] \leq [R, T]$.*

PROOF. This is a direct consequence of the fact that $R\square S \leq R\square T$. □

COROLLARY 2.1.10. *If R and S are congruences on A such that $R \leq S$ then $[R, S] = [R, A \times A]$.*

PROOF. It suffices to show that $[R, S]$ contains $T \coloneqq \langle (x \triangleleft^{-1} a, x \triangleleft^{-1} b) | aRb \rangle$. As before, observe that for any aRb, we have the quadruple $(a, b, a, a) \in R\square S$. □

COROLLARY 2.1.11. *If R and S are congruences on A then $[R \cap S, A \times A] = [R \cap S, S] \leq [R, S]$. In particular, the comparison map p of a double covering $\alpha\colon f_A \to f_B$ is a covering.*

Note that the converse of Corollary 2.1.11 is not true in general. For instance, observe that the double extension from Diagram (34) of Example 1.4.7 is such that the comparison map $\bar{mn}\colon D_{2nm} \to D_{nm}$ is always a quandle covering. However when $m = 3$ and $n = 6$, Diagram (34) is not a double covering since $0 \triangleleft 0 \triangleleft^{-1} 0 \triangleleft 0 \triangleleft^{-1} 6 = 12 \neq 0$.

In Section 2.2, where we further investigate the relationship with groups, we shall see that the converse of Corollary 2.1.11 holds for "double coverings of conjugation quandles". More comments and results about our commutator can be found in Section 4.

2.2. The case of conjugation quandles. Recall that a *conjugation quandle* is any quandle which is obtained as the image of a group by the functor $\mathrm{Conj}\colon \mathsf{Grp} \to \mathsf{Rck}$. As we recalled in the Introduction, we use the functors Conj and its left adjoint Pth to compare the covering theory of racks and quandles with the theory of central extensions of groups (see [**71, 67, 69**]). For instance, we mentioned that a surjective group homomorphism is central if and only if its image is a covering in Rck (or Qnd – [**38**, Examples 2.34;1.2]). However, as the following example shows, the *centralization* (in the sense of F_1) of a morphism between conjugation quandles does not coincide with the (image by Conj of the) centralization (in the sense of ab^1) of a group homomorphism in Grp.

EXAMPLE 2.2.1. *Indeed, consider the quotient map* $q\colon S_3 \to S_3/A_3 = \{-1, 1\}$ *in* Grp*, sending the group of permutations of the set of* 3 *elements to the (multiplicative) group* $\{-1, 1\}$ *by quotienting* S_3 *by* $A_3 = \{(), (123), (321)\}$*, the alternating subgroup of* S_3*. The morphism* q *sends* 2*-cycles to* -1*. Observe moreover that the (classical group) commutator* $[S_3, A_3]_{\mathsf{Grp}} = A_3$*. Hence the centralization of* q *in* Grp *is the identity morphism on* $\{-1, 1\}$*. Now observe that* $x \triangleleft x \triangleleft^{-1} y = z$ *for any* 2*-cycles* $x \neq y \neq z$*. Hence* 2*-cycles are also identified by the centralization of* q *in* Qnd*. However, the action of a* 2*-cycle on a* 3*-cycle always gives the other* 3*-cycle. Hence the successive action of a pair of* 2*-cycles on a* 3*-cycle does nothing. Similarly since both* 3*-cycles are inverse of each other,* 3*-cycles act trivially on each other. One easily deduces that if* $Q_{ab\star}$ *is the involutive quandle with* 3 *elements whose operation is defined in the table below, then the centralization of the morphism of quandles* q *is obtained via the quotient* $\eta^1_{S_3}\colon S_3 \to (S_3/\,\mathrm{C}_1\, q) = Q_{ab\star}$*, such that* $\eta^1_{S_3}(123) = a$*,* $\eta^1_{S_3}(321) = b$ *and all other elements of* S_3 *are sent to* $\star$*. Finally we obtain* $\mathrm{F}_1(q)\colon Q_{ab\star} \to S_3/A_3 = \{-1, 1\}$ *which takes the values* $\mathrm{F}_1(q)[a] = 1 = \mathrm{F}_1(q)[b]$ *and* $\mathrm{F}_1(q)[\star] = -1$*.*

$\lhd$	a	b	$\star$
a	a	a	b
b	b	b	a
$\star$	$\star$	$\star$	$\star$

$$\begin{tikzcd}
S_3/[S_3, A_3] \arrow[dr, "\mathrm{id}"] & \\
S_3 \arrow[u] \arrow[r, "q"] \arrow[d] & S_3/A_3 = \{-1, 1\} \\
S_3/(\mathrm{C}_1\, q) = Q_{ab\star} \arrow[ur, "\mathrm{F}_1(q)"'] &
\end{tikzcd}$$

In this section we further study how our concept of double covering behaves when applied to the image of Conj: Grp → Rck. First recall that given a group G, and given a path $g = \underline{g_1}^{\delta_1} \cdots \underline{g_n}^{\delta_n} \in \mathrm{Pth}(\mathrm{Conj}(G))$, there is always another path "of length one" $\underline{g_0}$, where $g_0 = g_1^{\delta_1} \cdots g_n^{\delta_n} \in G$, such that $x \cdot g = x \cdot \underline{g_0}$ for all $x \in \mathrm{Conj}(G)$. A primitive path in $\mathrm{Conj}(G)$ always "reduces" (as an inner automorphism, not as a homotopy class – see Paragraph 2.1.8 of Part I) to a one-step primitive path. As a consequence, our notion of double covering simplifies significantly when the quandle operations of interest are derived from the conjugation operation in groups. Note that *connectedness* in *symmetric spaces* also reduces to *strong connectedness* (i.e. connectedness in "one step") – see [**38**, Section 3.7] and references therein.

EXAMPLE 2.2.2. *Consider a group G and a pair of surjective group homomorphisms f and h with domain G in* Grp*. Let us write $R :=$* $\mathrm{Conj}(\mathrm{Eq}(f))$ *and $S :=$* $\mathrm{Conj}(\mathrm{Eq}(h))$ *(note that* Conj *preserves limits). In* Qnd *one derives easily that $[R, S] = [R \cap S, G \times G]$ since given a square $(a, b, c, d) \in R \square S$, we have $(d, (ab^{-1}c)) \in (R \cap S)$ such that moreover $x \lhd (ab^{-1}c) \lhd^{-1} d = x \lhd a \lhd^{-1} b \lhd c \lhd^{-1} d$.*

Now observe that the functor Pth: Rck → Grp preserves pushouts, and thus the image by Pth of a double extension α of racks (or quandles), yields a pushout square of extensions in Grp. Since Grp is a Mal'tsev category, $\mathrm{Pth}(\alpha)$ is a double extension as well (see [**25**] and Proposition 5.4 therein). Note however that the comparison map of $\mathrm{Pth}(\alpha)$ in Grp is not the image of the comparison map of α in Rck or Qnd.

In the other direction, the conjugation functor Conj: Grp → Rck preserves pullbacks. Hence it sends a double extension of groups $\gamma\colon f_G \to f_H$ to a double extension of quandles $\mathrm{Conj}(\gamma)$, and it sends the comparison map p of γ to the comparison map $\mathrm{Conj}(p)$ of $\mathrm{Conj}(\gamma)$. For a general double extension of racks and quandles α, the comparison map of α being a covering is necessary but not sufficient for α to be a double covering. However, by the example above and the preceding discussion we have:

PROPOSITION 2.2.3. *Given* $\gamma\colon f_G \to f_H$*, a double extension of groups, its image by* Conj *is a double covering of quandles if and only if its comparison map* $\mathrm{Conj}(p)$ *is a covering in* Qnd *or equivalently if and only if the comparison map* p *of* γ *is a central extension of groups.*

In particular, the image by Conj of a double central extension of groups yields a double covering in Qnd. However, one cannot deduce that γ is a double central extension of groups from the fact that $\mathrm{Conj}(\gamma)$ is a double covering. Finally we show that the image by Pth of a double covering of quandles is not necessarily a double central extension of groups.

EXAMPLE 2.2.4. *Consider* $\gamma\colon f_G \to f_H$*, a double extension of groups such that* $k_1 k_2^{-1} \neq k_2^{-1} k_1$ *for some* $k_1 \in \mathrm{Ker}(f_G)$ *and* $k_2 \in \mathrm{Ker}(\gamma_\top)$*, but* $ka = ak$ *for all* $k \in \mathrm{Ker}(f_G) \cap \mathrm{Ker}(\gamma_\top)$ *and* $a \in G_\top$*.*

For instance, define $\gamma_\perp\colon G_\perp \to H_\perp$ *as the surjective group homomorphism* $\mathrm{F_g}(\{a,c\}) \to \mathrm{F_g}(\{c\})$ *such that* $\gamma_\perp(c) = c$ *and* $\gamma_\perp(a) = e$*. Similarly define* $f_H\colon H_\top \to H_\perp$ *as* $\mathrm{F_g}(\{b,c\}) \to \mathrm{F_g}(\{c\})$ *such that* $f_H(c) = c$ *and* $f_H(b) = e$*. Write* $P \coloneqq G_\perp \times_{H_\perp} H_\top = \mathrm{F_g}(\{a,b,c\})$ *for their pullback, with projections* $\pi_1\colon P \to G_\perp$ *and* $\pi_2\colon P \to H_\top$*, and take the* canonical projective presentation $\epsilon^g_P\colon \mathrm{F_g}(P) \to P$*, obtained from the counit* ϵ^g *of free-forgetful adjunction* $\mathrm{F_g} \dashv \mathrm{U}$*. Compute its centralization* $\mathrm{ab}^1(\epsilon^g_P)\colon \mathrm{F_g}(P)/[\mathrm{Ker}(\epsilon^g_P), \mathrm{F_g}(P)]_{\mathsf{Grp}} \to P$*, and define* $f_G \coloneqq \pi_1\,\mathrm{ab}^1(\epsilon^g_P)$*. Similarly define* $\gamma_\top \coloneqq \pi_2\,\mathrm{ab}^1(\epsilon^g_P)$*. The resulting double extension of groups is as required.*

By Proposition 2.2.3, the double extension of quandles $\mathrm{Conj}(\gamma)$ *is a double covering. However we show that the double extension* $\mathrm{Pth}(\mathrm{Conj}(\gamma))$ *cannot be a double central extension of groups. First observe that the unit*

$$\mathrm{pth}_{\mathrm{Conj}(G_\top)}\colon \mathrm{Conj}(G_\top) \to \mathrm{Conj}(\mathrm{Pth}(\mathrm{Conj}(G_\top)))$$

is a monomorphism, since the identity morphism on $\mathrm{Conj}(G_\top)$ *factors through it. Now, if* $e_\top$ *is the neutral element in* $G_\top$*, then* $\underline{k_1}\,\underline{e_\top}^{-1} \in \mathrm{Ker}(\vec{f_G})$ *and* $\underline{e_\top}\,\underline{k_2}^{-1} \in \mathrm{Ker}(\vec{\gamma_\top})$*. Suppose by contradiction that*

$$\underline{k_1}\,\underline{e_\top}^{-1}\underline{e_\top}\,\underline{k_2}^{-1} = \underline{e_\top}\,\underline{k_2}^{-1}\,\underline{k_1}\,\underline{e_\top}^{-1}.$$

We have that

$$\underline{k_1}\,\underline{k_2}^{-1} = \underline{k_1}\,\underline{e_\top}^{-1}\underline{e_\top}\,\underline{k_2}^{-1}$$

and by the compatibility of $\mathrm{pth}_{\mathrm{Conj}(G_{\top})}$ *with* $\lhd$*, we have, moreover:*

$$\begin{aligned}\underline{k_2}^{-1}\,\underline{k_1} &= (\underline{k_2 \lhd e_{\top}})^{-1}\,\underline{k_1 \lhd e_{\top}}\\ &= (\underline{k_2} \lhd \underline{e_{\top}})^{-1}\,(\underline{k_1} \lhd \underline{e_{\top}}) = \underline{e_{\top}}\,\underline{k_2}^{-1}\,\underline{e_{\top}}^{-1}\,\underline{e_{\top}}\,\underline{k_1}\,\underline{e_{\top}}^{-1}\\ &= \underline{e_{\top}}\,\underline{k_2}^{-1}\,\underline{k_1}\,\underline{e_{\top}}^{-1}.\end{aligned}$$

Hence we must also have that $\underline{k_1}\,\underline{k_2}^{-1} = \underline{k_2}^{-1}\,\underline{k_1}$ *and thus* $\underline{k_1} \lhd^{-1} \underline{k_2} = \underline{k_1}$*, which implies* $\underline{k_1 \lhd^{-1} k_2} = \underline{k_1}$*. Since* $\mathrm{pth}_{\mathrm{Conj}(G_{\top})}$ *is injective, we must have* $k_2k_1k_2^{-1} = k_1 \lhd^{-1} k_2 = k_1 \in \mathrm{Conj}(G_{\top})$ *which is in contradiction with the hypothesis* $k_1k_2^{-1} \neq k_2^{-1}k_1$*. Hence it must also be that* $\underline{k_1}\,\underline{e_{\top}}^{-1}\underline{e_{\top}}\,\underline{k_2}^{-1} \neq \underline{e_{\top}}\,\underline{k_2}^{-1}\,\underline{k_1}\,\underline{e_{\top}}^{-1}$ *and* $\mathrm{Pth}(\gamma)$ *cannot be a double central extension of groups.*

REMARK 2.2.5. *By anticipation of Theorems 3.2.2 and 3.3.1, we cannot hope for a direct three-dimensional version of the Diagrams* (25) *and* (28) *in which the bottom adjunction's left adjoint would be the centralization of double extensions of groups.*

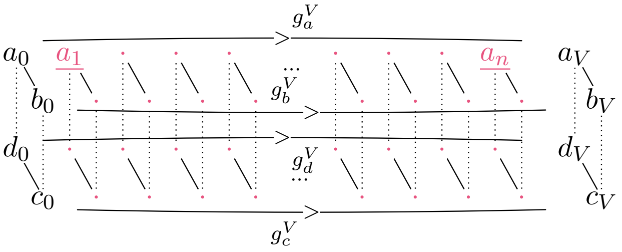
Now in order to further study double coverings (algebraically central double extensions) for general racks and quandles, and their relation to Γ^1-coverings, we need a characterization for a general element in the *centralization congruence* $[\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_{\top})]$ of a double extension $\alpha\colon f_A \to f_B$. Think about the transitive closure of the set of pairs from Definition 2.1.1. In order to identify these general pairs, we make a detour via the generalized notion of primitive trail and the characterization of normal Γ^1-coverings.

2.3. A concept of *primitive trail* in each dimension: from *membranes* to *volumes*. Similarly to what was studied in dimensions zero and one, we shall further be interested in the "action of sequences of two-dimensional data". Given a rack A in dimension zero, we have the fundamental concept of a primitive path, which is merely a sequence of elements in $A \times \{-1, 1\}$, viewed as a formal sequence of *symmetries* (Part I, Paragraph 2.3.3). Given a rack A, its *centralization* (or set of connected components) is obtained by identifying elements which are "connected by the action of a primitive path in A". In dimension one, the centralization of an extension $f\colon A \to B$ is in some sense obtained by the study of elements which are "linked by the action on A of a primitive path from $\mathrm{Eq}(f)$", leading to the concept of a membrane (see Paragraph 1.4.2 or Part I). Now given a double extension of racks α, we shall be interested in the action on $A_{\top}$ of primitive paths from $\mathrm{Eq}(f_A) \square \mathrm{Eq}(\alpha_{\top})$. We exhibit the 2-dimensional generalizations of the lower-dimensional concepts of *primitive trail*, *membrane*, and *horn*.

DEFINITION 2.3.1. *Given a pair of morphisms* $f\colon A \to B$ *and* $h\colon A \to C$ *in* Rck *(or* Qnd*), we define an* $\langle f, h\rangle$*-volume as the data*

$$V = ((a_0, b_0, c_0, d_0), ((a_i, b_i, c_i, d_i), \delta_i)_{1\leq i\leq n})$$

of a primitive trail in $\mathrm{Eq}(f)\square\,\mathrm{Eq}(h)$*. The first quadruple* (a_0, b_0, c_0, d_0) *is the* head *of* V*. We call such an* $\langle f, h\rangle$*-volume* V *an* $\langle f, h\rangle$*-horn if the head reduces to a point:* $a_0 = b_0 = c_0 = d_0 =: x$ *which we specify as* $V = (x, ((a_i, b_i, c_i, d_i), \delta_i)_{1\leq i\leq n})$*. Let us define* $a := (a_i)_{1\leq i\leq n}$ *and similarly define* b*,* c *and* d*. The* associated $\langle f, h\rangle$-symmetric quadruple *of the volume or horn* V *is given by the paths* $g_a^V := \underline{a_1}^{\delta_1} \cdots \underline{a_n}^{\delta_n}$*,* $g_b^V := \underline{b_1}^{\delta_1} \cdots \underline{b_n}^{\delta_n}$*,* $g_c^V := \underline{c_1}^{\delta_1} \cdots \underline{c_n}^{\delta_n}$ *and* $g_d^V := \underline{d_1}^{\delta_1} \cdots \underline{d_n}^{\delta_n}$ *in* $\mathrm{Pth}(A)$*. The* endpoints *of the volume or horn are given by* $a_V = a_0 \cdot g_a^V$*,* $b_V = b_0 \cdot g_b^V$*,* $c_V = c_0 \cdot g_c^V$ *and* $d_V = d_0 \cdot g_d^V$*. Finally we call* (a, b)*-membrane the* f*-membrane defined by* $M_{(a,b)}^V := ((a_0, b_0), ((a_i, b_i), \delta_i)_{1\leq i\leq n})$*. The other* f*-membrane, labelled* (c, d)*, and the two* h*-membranes, labelled by* (a, d) *and* (b, c)*, are defined similarly.*



Note that because double parallelistic relations are symmetric, both in the "vertical" and in the "horizontal" direction, Definition 2.3.1 is "symmetric" in the role of opposite membranes.

REMARK 2.3.2. *A morphism of racks* $f\colon A \to B$ *sends a primitive trail* $(x, (a_i, \delta_i)_{1\leq i\leq n})$ *in* A *to the primitive trail* $(f(x), (f(a_i), \delta_i)_{1\leq i\leq n})$ *in* B*. Similarly, a morphism* $\alpha\colon f_A \to f_B$ *in* ExtRck *sends an* f_A*-membrane to an* f_B*-membrane, and a morphism of* $\mathsf{Ext}^2\mathcal{C}$*, such as* $(\sigma, \beta)\colon \gamma \to \alpha$ *in Definition 1.3.1 sends an* $\langle f_C, \gamma_\top\rangle$*-volume to an* $\langle f_A, \alpha_\top\rangle$*-volume (via the induced morphism* $\square_{(\sigma,\beta)}$ *such as in Lemma 3.1.1).*

NOTATION 2.3.3. *Given* $\alpha\colon f_A \to f_B$*, a double extension of racks (or quandles), we can build its kernel pair in* ExtRck *component-wise, which* *we denote:*

$$\begin{tikzcd}
& \mathrm{Eq}(\alpha_\top) \arrow[rr, "\pi_2"] \arrow[dl, "\bar{f}"'] \arrow[dd, "\pi_1"] \arrow[ddrr, phantom, "\lrcorner", very near start] & & A_\top \arrow[dl, "f_A"'] \arrow[dd, "\alpha_\top"] \\
\mathrm{Eq}(\alpha_\bot) \arrow[rr, "p_2"] \arrow[dd, "p_1"'] \arrow[ddrr, phantom, "\lrcorner", very near start] & & A_\bot \arrow[dd, "\alpha_\bot"] & \\
& A_\top \arrow[rr, "\alpha_\top"] \arrow[dl, "f_A"] & & B_\top \arrow[dl, "f_B"] \\
A_\bot \arrow[rr, "\alpha_\bot"'] & & B_\bot &
\end{tikzcd}$$

REMARK 2.3.4. *Using Notation 2.3.3, the* $\langle f_A, \alpha_\top\rangle$*-volumes* V *(see Definition 2.3.1) correspond bijectively to the* $\bar{f}$*-membranes* M *in* $\mathrm{Eq}(\alpha_\top)$*, since such an* M *is defined as the data*

$$(((a_0, d_0), (b_0, c_0)), (((a_i, d_i), (b_i, c_i)), \delta_i)_{1\leq i\leq n})$$

for a certain sequence of elements (a_i, b_i, c_i, d_i) *in* $\mathrm{Eq}(f_A)\square\,\mathrm{Eq}(\alpha_\top)$*, where* $0 \leq i \leq n$*. Under the appropriate bijective correspondence, the* (a, b)*-membrane (and* (c, d)*-membrane) of an* $\langle f_A, \alpha_\top\rangle$*-volume are obtained from the corresponding* $\bar{f}$*-membrane via the projections* π_1 *and* π_2 *respectively. A* $\bar{f}$*-horn then corresponds to an* $\langle f_A, \alpha_\top\rangle$*-volume whose head* (a_0, b_0, c_0, d_0) *is such that* $a_0 = b_0$ *and* $c_0 = d_0$*.*

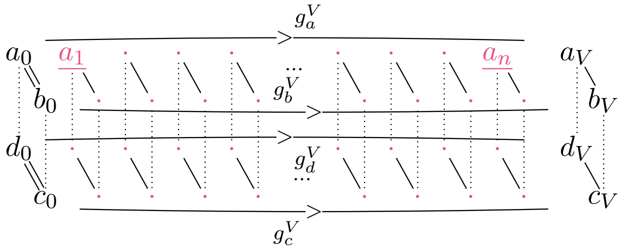
Similarly, the $\langle f_A, \alpha_\top\rangle$*-volumes correspond bijectively to* $\bar{\alpha}$*-membranes in* $\mathrm{Eq}(f_A)$*, where* $\bar{\alpha}$ *is the kernel pair of* (f_A, f_B) *in* ExtRck*. A* $\bar{\alpha}$*-horn then corresponds bijectively to an* $\langle f_A, \alpha_\top\rangle$*-volume whose head* (a_0, b_0, c_0, d_0) *is such that* $a_0 = d_0$ *and* $b_0 = c_0$*.*

2.4. Normal Γ^1-coverings and rigid horns. We illustrate these definitions in the characterization of normal Γ^1-coverings, which we subsequently refer to as *normal double coverings.*

PROPOSITION 2.4.1. *Given a double extension of racks (or quandles)* $\alpha\colon f_A \to f_B$*, it is a normal* Γ^1*-covering if and only if, given an* $\langle f_A, \alpha_\top\rangle$*-volume (as in Definition 2.3.1)*

$$V = ((a_0, b_0, c_0, d_0), ((a_i, b_i, c_i, d_i), \delta_i)_{1\leq i\leq n}),$$

if its f_A*-membranes are horns (i.e.* $a_0 = b_0$ *and* $c_0 = d_0$*) then* the $\alpha_\top$-membranes of V are rigid *in the sense that its* (d, c)*-horn closes if and only if its* (a, b)*-horn closes. We call a double extension satisfying this* *condition* a normal double covering.



$$a_V = b_V$$
$$\Updownarrow$$
$$d_V = c_V$$

Observe that by the "symmetries" of $\langle f_A, \alpha_\top\rangle$-volumes (in the role of f_A-membranes), it suffices to show that in any such volume V, a closing (a, b)-horn implies a closing (c, d)-horn in order to deduce that in any such volume V, a closing (c, d)-horn implies a closing (a, b)-horn (and conversely the latter implies the former). We relate this to the fact that (π_1, p_1) (from Notation 2.3.3) is a trivial double covering if and only if (π_2, p_2) is one.

PROOF OF PROPOSITION 2.4.1. By definition, α is a normal Γ^1-covering if and only if, in Notation 2.3.3, the left face (π_1, p_1) (or equivalently the top face (π_2, p_2)) is a trivial double covering. Then by Lemma 1.4.3, the double extension (π_1, p_1) is a trivial double covering if and only if given any $\bar{f}$-horn M, such that $\pi_1(M)$ closes in $A_\top$, then M closes in $\mathrm{Eq}(\alpha_\top)$, i.e. $\pi_2(M)$ also has to close. By Remark 2.3.4 the preceding translates into the statement: (π_1, p_1) is a trivial double covering if and only if given any $\langle f_A, \alpha_\top\rangle$-volume V such that $a_0 = b_0$ and $c_0 = d_0$, if the (a, b)-horn of V closes then the (c, d)-horn of V has to close. Similarly (π_2, p_2) is a trivial double covering if and only if given any volume V such that $a_0 = b_0$ and $c_0 = d_0$, a closing (c, d)-horn implies a closing (a, b)-horn. □

Of course trivial Γ^1-coverings (i.e. trivial double coverings) are examples of normal Γ^1-coverings (i.e. normal double coverings). However, these two concepts do not coincide.

EXAMPLE 2.4.2. *Consider the set* $\{\star, \bullet\}$*, seen as a trivial quandle, as well as two copies* $f\colon Q^\diamond \to \{\star, \bullet\}$ *and* $f\colon Q_\diamond \to \{\star, \bullet\}$ *of the same morphism where* $Q^\diamond := \{\star^\diamond, \star, \bullet^1, \bullet^0\}$*, and* $Q_\diamond := \{\star, \star_\diamond, \bullet_1, \bullet_0\}$ *are such that* $\star_\diamond$ *(respectively* $\star^\diamond$*) acts on* $\bullet_1$ *and* $\bullet_0$ *(respectively* $\bullet^1$ *and* $\bullet^0$*) by interchanging* 1 *and* 0*, and all the other actions are trivial (see also Example 2.3.14 in Part I). We then denote the kernel pair of* f *by* $Q_\diamond^\diamond$ *with* *underlying set* $\{\star^\diamond,\ \star,\ \star^\diamond_\diamond,\ \star_\diamond, \bullet^1_1,\ \bullet^1_0,\ \bullet^0_1,\ \bullet^0_0\}$, *such that the element* $\star^\diamond$ *acts on bullets by interchanging the exponents* 1 *and* 0 *and similarly with* $\star_\diamond$ *for the indices. Then* $\star^\diamond_\diamond$ *interchanges both indices and exponents of the bullets, whereas* $x \triangleleft y = x$ *for any other choice of* x *and* y *in* $Q^\diamond_\diamond$*.*

$$\begin{tikzcd}
Q^\diamond_\diamond \arrow[r, two heads, "\pi_\diamond"] \arrow[d, two heads, "\pi^\diamond"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q_\diamond \arrow[d, two heads, "f_\diamond"] & {} \arrow[d, dash] & \bullet^1_1 \arrow[r, dash, "\pi^\diamond" description] \arrow[d, dash, dotted, "\pi_\diamond" description] & \bullet^1_0 \arrow[d, dash, dotted, "\pi_\diamond" description] & \star^\diamond_\diamond \arrow[r, dash, "\pi^\diamond" description] \arrow[d, dash, dotted, "\pi_\diamond" description] & \star^\diamond \arrow[d, dash, dotted, "\pi_\diamond" description] \\
Q^\diamond \arrow[r, two heads, "f^\diamond"'] & \{\star,\ \bullet\} & {} & \bullet^0_1 \arrow[r, dash, "\pi^\diamond" description] & \bullet^0_0 & \star_\diamond \arrow[r, dash, "\pi^\diamond" description] & \star
\end{tikzcd} \tag{38}$$

The projection $\pi_\diamond$ *identifies all the elements that have the same indices (including blanks), and similarly* $\pi^\diamond$ *identifies elements with the same exponents.*

Observe that none of the morphisms above are quandle coverings. Moreover, both double extensions $(\pi_\diamond, f^\diamond)$ *and* $(\pi^\diamond, f_\diamond)$ *are such that the conditions of Lemma 1.4.3 are not satisfied. However, the conditions of Proposition 2.4.1 are easily seen to be satisfied by both* $(\pi_\diamond, f^\diamond)$ *and* $(\pi^\diamond, f_\diamond)$*. In order to check this, observe that the only "non-trivial" element in* $\mathrm{Eq}(\pi_\diamond)\square\,\mathrm{Eq}(\pi^\diamond)$ *is the square on the right of* (38) *(or any symmetric equivalent) and for any* $g,\ h \in \mathrm{Pth}(Q^\diamond_\diamond)$ *and for any* $i,\ j,\ k,\ l \in \{0,\ 1\}$, *we have that* $\bullet^i_j \cdot g = \bullet^i_j \cdot h$ *if and only if* $\bullet^k_l \cdot g = \bullet^k_l \cdot h$*.*

Even if 2.4.2 is symmetric in the sense that both $(\pi_\diamond, f^\diamond)$ and $(\pi^\diamond, f_\diamond)$ are double normal coverings, Proposition 2.4.1, does not seem to be symmetric in the role of $(\alpha_\top, \alpha_\perp)$ and (f_A, f_B). Observe that in Example 1.4.6, the double extension $(\pi_1 p, t_\star)$ is a trivial double covering and thus also a normal double covering. However, the double extension $(\pi_2 p, t)$ is neither a trivial double covering nor a normal double covering since $\bullet_1 \triangleleft \star_{11} \neq \bullet_1 \triangleleft \star_{10}$ even though $\bullet_0 \triangleleft \star_{01} = \bullet_0 \triangleleft \star_{00}$.

Recall that any normal Γ^1-covering (normal double covering) α is in particular a Γ^1-covering, since α is split by α. Now unlike trivial double coverings and normal double coverings, Γ^1-coverings are expected to be symmetric in the same way that double coverings are (see Remark 2.0.2). If we were to weaken the condition characterizing normal Γ^1-coverings to obtain a candidate condition for the characterization of Γ^1-coverings, we would look for a way to make it symmetric in the roles of f_A and $\alpha_\top$.

Now observe that an obvious asymmetrical feature of the characterization in Proposition 2.4.1 is the fact that we look at properties of $\bar{f}$-horns in

$\mathrm{Eq}(\alpha_\top)$, some of which cannot be expressed as $\bar{\alpha}$-horns in $\mathrm{Eq}(f_A)$. In the spirit of the discussions on pages 116 and 125, we are looking at the "successive action" of "two-dimensional data" on some "one-dimensional data" (in a fixed privileged direction). What we are aiming for is the "successive action" of "two-dimensional data" on some "zero-dimensional data".

We get rid of the asymmetry in Proposition 2.4.1 by collapsing the one-dimensional head of the volumes we study. Looking at $\langle f_A, \alpha_\top \rangle$-horns in $A_\top$, these can be described both as $\bar{f}$-horns in $\mathrm{Eq}(\alpha_\top)$ and as $\bar{\alpha}$-horns in $\mathrm{Eq}(f_A)$. From Proposition 2.4.1 we produce the concept of a double extension with *rigid horns*.

DEFINITION 2.4.3. *A double extension of racks (or quandles) α is said to* have rigid horns *if any $\langle f_A, \alpha_\top \rangle$-horn V in $A_\top$ has rigid $\alpha_\top$-membranes in the sense of Proposition 2.4.1: if*

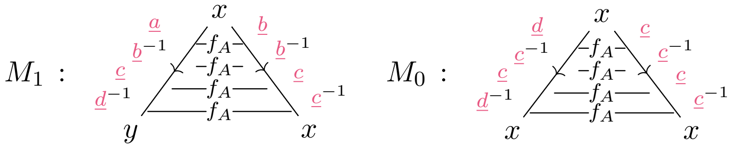
$$V = ((a_0, b_0, c_0, d_0), ((a_i, b_i, c_i, d_i), \delta_i)_{1 \leq i \leq n}),$$

as in Definition 2.3.1, its (d, c)-horn closes if and only if its (a, b)-horn closes.

Even though Definition 2.4.3 still seems asymmetric at first, it is actually not so anymore. Indeed we use the terminology *rigid horns* because we may show that given a double extension $\alpha\colon f_A \to f_B$, any $\langle f_A, \alpha_\top \rangle$-horn V in $A_\top$ has rigid $\alpha_\top$-membranes if and only if any $\langle f_A, \alpha_\top \rangle$-horn V in $A_\top$ *has rigid f_A-membranes* (its (a, d)-horn closes if and only if its (b, c)-horn closes). Observe that by Definition 2.4.3, the double extension (f_A, f_B) has rigid horns if and only if any $\langle f_A, \alpha_\top \rangle$-horn has rigid f_A-membranes. Again we may show that $(\alpha_\top, \alpha_\perp)$ has rigid horns (in the sense of Definition 2.4.3) if and only if the double extension (f_A, f_B) has rigid horns, as it is the case for double coverings. We skip this (rather elementary) step as it can be deduced from the fact that the concepts of double covering and double extension with rigid horns coincide.

PROPOSITION 2.4.4. *A double extension of racks $\alpha\colon f_A \to f_B$ is a double covering if and only if α has rigid horns (Definition 2.4.3).*

PROOF. Suppose that $\alpha\colon f_A \to f_B$ has rigid horns in the sense of Definition 2.4.3. Then given an element $(a, b, c, d) \in \mathrm{Eq}(f_A) \square \mathrm{Eq}(\alpha_\top)$ and an element $x \in X$ we build an $\langle f_A, \alpha_\top \rangle$-horn V described by superposition of the two f_A-membranes M_1 and M_0 below (the so-obtained "left-hand side" $\alpha_\top$-membrane of V is as in Example 2.0.3). Since the $\alpha_\top$-membranes of V are rigid and M_0 closes into a disk, we conclude that $y := x \cdot (\underline{a}\,\underline{b}^{-1}\,\underline{c}\,\underline{d}^{-1})$ is equal to x.



(39)

Conversely suppose that α is a double covering and consider an $\langle f_A, \alpha_\top\rangle$-horn V given by $V = (x, ((a_i, b_i, c_i, d_i), \delta_i)_{1\leq i\leq n})$ as in Definition 2.3.1. Suppose that the (c,d)-membrane of V closes into a disk (i.e. $c_V = d_V$), we have to show that the (a,b)-membrane closes into a disk (the converse is then given by symmetry of V in the role of the f_A-membrane).

More generally, and without assumption on the double extension α, we show that the endpoints a_V and b_V of such a horn V are in relation by $[\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_\top)]$, which we temporarily denote by $\approx$. Observe that for all $z \in A_\top$ we have that $z \triangleleft^{-\delta_n} d_n \triangleleft^{\delta_n} a_n \approx z \triangleleft^{-\delta_n} c_n \triangleleft^{\delta_n} b_n$ (replace $\approx$ by $=$ when α is a double covering). By taking $z = d_V := x \triangleleft^{\delta_1} d_1 \cdots \triangleleft^{\delta_n} d_n$ (and by reflexivity of $\approx$, its compatibility with the operation $\triangleleft$, and using the fact that $c_V = d_V$) we derive

$$x \triangleleft^{\delta_1} d_1 \cdots \triangleleft^{\delta_{n-1}} d_{n-1} \triangleleft^{\delta_n} a_n \approx x \triangleleft^{\delta_1} c_1 \cdots \triangleleft^{\delta_{n-1}} c_{n-1} \triangleleft^{\delta_n} b_n. \tag{40}$$

Then consider the square

$$\begin{tikzcd} a_{n-1} \triangleleft^{\delta_n} a_n \arrow[r, dash] \arrow[d, dotted, dash] & d_{n-1} \triangleleft^{\delta_n} a_n \arrow[d, dotted, dash] \\ b_{n-1} \triangleleft^{\delta_n} b_n \arrow[r, dash] & c_{n-1} \triangleleft^{\delta_n} b_n \end{tikzcd} \quad \in \ \mathrm{Eq}(f_A)\square\,\mathrm{Eq}(\alpha_\top),$$

and derive that for each $z \in A_\top$:

$$\begin{aligned} z \triangleleft^{-\delta_{n-1}} (d_{n-1} \triangleleft^{\delta_n} a_n) \triangleleft^{\delta_{n-1}} (a_{n-1} \triangleleft^{\delta_n} a_n) \\ \approx z \triangleleft^{-\delta_{n-1}} (c_{n-1} \triangleleft^{\delta_n} b_n) \triangleleft^{\delta_{n-1}} (b_{n-1} \triangleleft^{\delta_n} b_n); \end{aligned}$$

$$\begin{aligned} z \triangleleft^{-\delta_n} a_n \triangleleft^{-\delta_{n-1}} d_{n-1} \triangleleft^{\delta_{n-1}} a_{n-1} \triangleleft^{\delta_n} a_n \\ \approx z \triangleleft^{-\delta_n} b_n \triangleleft^{-\delta_{n-1}} c_{n-1} \triangleleft^{\delta_{n-1}} b_{n-1} \triangleleft^{\delta_n} b_n. \end{aligned}$$

Applying this to Equation (40) we obtain

$$\begin{aligned} x \triangleleft^{\delta_1} d_1 \cdots \triangleleft^{\delta_{n-2}} d_{n-2} \triangleleft^{\delta_{n-1}} a_{n-1} \triangleleft^{\delta_n} a_n \\ \approx x \triangleleft^{\delta_1} c_1 \cdots \triangleleft^{\delta_{n-2}} c_{n-2} \triangleleft^{\delta_{n-1}} b_{n-1} \triangleleft^{\delta_n} b_n. \end{aligned}$$

We repeat the argument with the quadruple

$$\begin{tikzcd}
a_{n-2} \lhd^{\delta_{n-1}} a_{n-1} \lhd^{\delta_n} a_n \arrow[r, dash] \arrow[d, dotted, dash] & d_{n-2} \lhd^{\delta_{n-1}} a_{n-1} \lhd^{\delta_n} a_n \arrow[d, dotted, dash] \\
b_{n-2} \lhd^{\delta_{n-1}} b_{n-1} \lhd^{\delta_n} b_n \arrow[r, dash] & c_{n-2} \lhd^{\delta_{n-1}} b_{n-1} \lhd^{\delta_n} b_n
\end{tikzcd}$$

from $\mathrm{Eq}(f_A)\square\,\mathrm{Eq}(\alpha_{\top})$ and we conclude by induction that also

$$x \lhd^{\delta_1} a_1 \cdots \lhd^{\delta_n} a_n \approx x \lhd^{\delta_1} b_1 \cdots \lhd^{\delta_n} b_n.$$

$\square$

Given a double extension $\alpha\colon f_A \to f_B$, the *rigid horns* condition from Definition 2.4.3, or more precisely Definition 2.4.5 below, make sense of what it means for two elements of $A_{\top}$ to be "linked under the action of a primitive path from $\mathrm{Eq}(f_A)\square\,\mathrm{Eq}(\alpha_{\top})$" (see page 125).

DEFINITION 2.4.5. *Given a double extension $\alpha\colon f_A \to f_B$, we define the set X_α to be the set of those pairs (x, y) in $A_{\top} \times A_{\top}$ such that there exists an $\langle f_A, \alpha_{\top}\rangle$-horn V as in Definition 2.3.1 such that x and y are the endpoints of one of the membranes M_1^V of V, such that moreover the membrane M_0^V, which is opposite to M_1^V, closes into a disk.*

These pairs in X_α are the pairs of elements which *would be identified if α had rigid horns.* We just saw that X_α contains the generators of $[\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_{\top})]$ and moreover $X_\alpha \subseteq [\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_{\top})]$. Hence if we can show that X_α defines a congruence on $A_{\top}$, we can deduce that X_α is the centralizing congruence $[\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_{\top})]$ (see Corollary 2.5.5 below).

Now recall from Part I that coverings are equivalently described via membranes or via symmetric paths. Proposition 2.4.4 corresponds to the description via membranes. In the following section, we adapt the idea of a symmetric path to the two-dimensional context. Equipped with this concept and that of a rigid horn, we provide a full description of a general element in $[\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_{\top})]$.

2.5. Symmetric paths for double extensions.

We describe *symmetric paths* in a slightly more general context than expected, because of Proposition 2.5.7 below.

DEFINITION 2.5.1. *Given $f\colon G \to H$ and $h\colon G \to K$, a pair of morphisms in Grp, and given a generating set $A \subseteq G$ (i.e. such that $G = \langle a \mid a \in A\rangle_G$), we define (implicitly* with respect to *A)*

(i) four elements g_a*,* g_b*,* g_c *and* g_d *in* G *are* $\langle f, h\rangle$-symmetric (to each other) *if there exists* $n \in \mathbb{N}$ *and a sequence of quadruples* (a_1, b_1, c_1, d_1)*, …,* (a_n, b_n, c_n, d_n) *in the intersection* $A^4 \cap (\mathrm{Eq}(f) \square \mathrm{Eq}(h))$*, i.e. such that for each index* i*,* $f(a_i) = f(b_i)$*,* $f(d_i) = f(c_i)$*,* $h(a_i) = h(d_i)$*, and* $h(b_i) = h(c_i)$*:*

$$\begin{tikzcd} a_i \arrow[r, dash, "f"] \arrow[d, dotted, no head, "h"] & b_i \arrow[d, dotted, no head, "h"] \\ d_i \arrow[r, dash, "f"] & c_i \end{tikzcd}$$

and such that, moreover, for each $1 \leq i \leq n$*, there is* $\delta_i \in \{-1, 1\}$ *such that:*

$$\begin{aligned} g_a &= a_1^{\delta_1} \cdots a_n^{\delta_n}, & g_b &= b_1^{\delta_1} \cdots b_n^{\delta_n}, \\ g_c &= c_1^{\delta_1} \cdots c_n^{\delta_n}, & g_d &= d_1^{\delta_1} \cdots d_n^{\delta_n}. \end{aligned} \tag{41}$$

We often call such g_a*,* g_b*,* g_c *and* g_d *an* $\langle f, h\rangle$-symmetric quadruple.

(ii) $\mathrm{K}_{\langle f,h\rangle}$ *is the set of* $\langle f, h\rangle$-symmetric paths*, i.e. the elements* $g \in G$ *such that* $g = g_a g_b^{-1} g_c g_d^{-1}$ *for some* $\langle f, h\rangle$*-symmetric quadruple* g_a*,* g_b*,* g_c *and* $g_d \in G$*.*

LEMMA 2.5.2. *Given the hypotheses of Definition 2.5.1, the set of* $\langle f, h\rangle$*-symmetric paths* $\mathrm{K}_{\langle f,h\rangle}$ *defines a normal subgroup of* G*.*

PROOF. Let g_a, g_b, g_c and g_d be $\langle f, h\rangle$-symmetric (to each other). Observe that g_d, g_c, g_b and g_a are also $\langle f, h\rangle$-symmetric, and thus $\mathrm{K}_{\langle f,h\rangle}$ is closed under inverses. Moreover, if h_a, h_b, h_c and h_d are $\langle f, h\rangle$-symmetric, and $g = g_a g_b^{-1} g_c g_d^{-1}$, $h = h_a h_b^{-1} h_c h_d^{-1}$, then

$$gh = k_a k_b^{-1} k_c k_d^{-1},$$

with $k_a = h_a h_b^{-1} h_b h_a^{-1} g_a$, $k_b = h_a h_a^{-1} h_b h_a^{-1} g_b$, $k_c = h_d h_d^{-1} h_b h_a^{-1} g_c$ and $k_d = h_d h_c^{-1} h_b h_a^{-1} g_d$ which are $\langle f, h\rangle$-symmetric. Finally since A generates G, for any $k \in G$, the elements kg_a, kg_b, kg_c and kg_d are $\langle f, h\rangle$-symmetric to each other, and thus

$$kgk^{-1} = kg_a g_b^{-1} k^{-1} k g_c g_d^{-1} k^{-1} \in \mathrm{K}_{\langle f,h\rangle}$$

is an $\langle f, h\rangle$-symmetric path. □

NOTATION 2.5.3. *If* $\alpha \colon f_A \to f_B$ *is a double extension of racks (or quandles), we often write* $\langle f_A, \alpha_\top\rangle$-symmetric *(quadruple or path) instead of* $\langle \vec{f}, \vec{\alpha_\top}\rangle$*-symmetric (quadruple or path – see for instance Definition 2.3.1).*

An $\langle f_A, \alpha_\top \rangle$-symmetric trail (x, g) *in* $A_\top$ *is a trail where* g *is an* $\langle f_A, \alpha_\top \rangle$*-symmetric path.*

LEMMA 2.5.4. *Given* $\alpha \colon f_A \to f_B$*, a double extension in* Rck *(or* Qnd*), the set* X_α *(Defintion 2.4.5) is the underlying set of the congruence* $\sim_{\mathrm{K}_{\langle f_A, \alpha_\top \rangle}}$ *induced by the action of* $\langle f_A, \alpha_\top \rangle$*-symmetric paths on* $A_\top$*.*

PROOF. Given x and $y \in A_\top$ such that $x \sim_{\mathrm{K}_{\langle f_A, \alpha_\top \rangle}} y$, i.e. such that $y = x \cdot (g_a g_b^{-1} g_c g_d^{-1})$ for some $\langle f_A, \alpha_\top \rangle$-symmetric quadruple as in Definition 2.5.1. The pair (x, y) is in X_α as one can deduce from the construction of V as in Equation (39) from the proof of Proposition 2.4.4, where one replaces every occurrence of $\underline{a}$ by $\underline{a_1}^{\delta_1} \cdots \underline{a_n}^{\delta_n}$ and also for $\underline{b}$ by $\underline{b_1}^{\delta_1} \cdots \underline{b_n}^{\delta_n}$, and similarly $\underline{c}$ by $\underline{c_1}^{\delta_1} \cdots \underline{c_n}^{\delta_n}$ and $\underline{d}$ by $\underline{d_1}^{\delta_1} \cdots \underline{d_n}^{\delta_n}$.

Conversely, and without loss of generality, consider an $\langle f_A, \alpha_\top \rangle$-horn V given by the data $V = (x, ((a_i, b_i, c_i, d_i), \delta_i)_{1 \leq i \leq n})$ as in Definition 2.3.1, such that moreover the endpoints c_V and d_V are equal. Observe that the endpoint $b_V = a_V \cdot \big((g_a^V)^{-1} \, g_d^V \, (g_c^V)^{-1} \, g_b^V\big)$ is obtained from the endpoint a_V by the action of an $\langle f_A, \alpha_\top \rangle$-symmetric path. □

As a conclusion to the discussion below Definition 2.4.5, we give a characterization of a general element in $[\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_\top)]$ which we show to be an *orbit congruence* (see Part I and reference therein).

COROLLARY 2.5.5. *If* $\alpha \colon f_A \to f_B$ *is a double extension of racks (or quandles), then the centralization congruence* $[\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_\top)]$ *coincides with the congruence* $\sim_{\mathrm{K}_{\langle f_A, \alpha_\top \rangle}}$ *generated by the action of* $\langle f_A, \alpha_\top \rangle$*-symmetric paths, also described by the set of pairs in* X_α *(Definition 2.4.5, i.e. those pairs of elements of* $A_\top$ *which would be identified if* α *had rigid horns).*

2.5.6. *Describing symmetric paths differently?* Given a morphism f in Rck (or Qnd), f-symmetric paths are described as the elements in the kernel $\mathrm{Ker}(\vec{f})$ of $\vec{f}$ (which is our notation for $\mathrm{Pth}(f)$). It is unclear to us whether this result generalizes in higher dimensions. Our understanding is that the question should be: given a double extension α, do the normal subgroups $\mathrm{Ker}(\vec{f_A}) \cap \mathrm{Ker}(\vec{\alpha_\top})$ and $\mathrm{K}_{\langle f_A, \alpha_\top \rangle}$ coincide ? Whether the answer is negative or positive, this would help to specify more precisely how to understand these $\langle f_A,\ \alpha_\top \rangle$-symmetric paths algebraically. Following the strategy from Section 2.4.11 of Part I, we were able to show that:

PROPOSITION 2.5.7. **[98]** *Given* $f \colon A \to B$ *and* $h \colon A \to C$*, two surjective functions such that* $\mathrm{Eq}(f) \circ \mathrm{Eq}(h) = \mathrm{Eq}(h) \circ \mathrm{Eq}(f)$*, the intersection*

$\mathrm{Ker}(\mathrm{F_g}(f)) \cap \mathrm{Ker}(\mathrm{F_g}(h))$ *of the kernels of the induced group homomorphisms*

$$\mathrm{F_g}(f)\colon \mathrm{F_g}(A) \to \mathrm{F_g}(B) \text{ and } \mathrm{F_g}(h)\colon \mathrm{F_g}(A) \to \mathrm{F_g}(C)$$

is given by $\mathrm{K}_{\langle \mathrm{F_g}(f), \mathrm{F_g}(h)\rangle}$ *(with respect to A) as in Definition 2.5.1.*

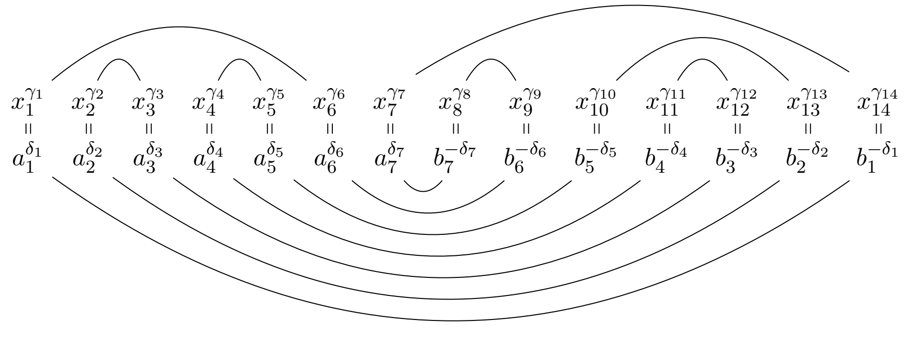
PROOF. Given an element $g \in \mathrm{Ker}(\mathrm{F_g}(f)) \cap \mathrm{Ker}(\mathrm{F_g}(h)) \subseteq \mathrm{F_g}(A)$, and following the proof of Proposition 2.4.15 based on Observation 2.4.14, we may identify an $\mathrm{F_g}(f)$-symmetric pair $g_a = a_1^{\delta_1} \cdots a_n^{\delta_n}$, $g_b = b_1^{\delta_1} \cdots b_n^{\delta_n}$ in $\mathrm{F_g}(A)$, such that $g = g_a g_b^{-1}$. Moreover, since $g \in \mathrm{Ker}(\mathrm{F_g}(h))$, by Observation 2.4.14, the elements in the sequence (or word)

$$(x_i^{\gamma_i})_{1 \leq i \leq 2n} := a_1^{\delta_1},\ \ldots,\ a_n^{\delta_n},\ b_n^{-\delta_n},\ \ldots,\ b_1^{-\delta_1}$$

organize themselves in n pairs $(x_i^{\gamma_i}, x_j^{\gamma_j})$ such that $i < j$, $(x_i, x_j) \in \mathrm{Eq}(h)$, $\gamma_i = -\gamma_j$, each element of the sequence $(x_i^{\gamma_i})_{1 \leq i \leq 2n}$ appears in only one such pair, and given any two such pairs $(x_i^{\gamma_i}, x_j^{\gamma_j})$ and $(x_l^{\gamma_l}, x_m^{\gamma_l})$, $l < i$ (respectively $l > i$) if and only if $m > j$ (respectively $m < j$). Let us fix such a choice of pairs. For each $k \in \{1,\ \ldots,\ 2n\}$, we write $(x_{i_k}^{\gamma_{i_k}}, x_{j_k}^{\gamma_{j_k}})$ for the unique pair such that either $i_k = k$ or $j_k = k$.

For what follows, consider the "paired index" operation p defined by $p(i_k) := j_k$ and $p(j_k) := i_k$ for any $k \in \{1,\ \ldots,\ 2n\}$ – of course $p(p(k)) = k$ for all such k. Define the operation "opposite index" $o(i) = 2n + 1 - i$ for $1 \leq i \leq 2n$.

We give an example of such a word representing $g \in \mathrm{F_g}(A)$ below, where $n = 7$, the lower lines link elements which are sent to opposites by f, and the upper lines link the elements which are paired – and thus sent to opposites by h.

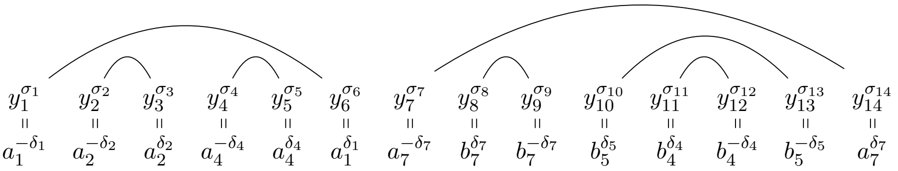


We have for instance $(x_{i_6}^{\gamma_{i_6}}, x_{j_6}^{\gamma_{j_6}}) = (x_1^{\gamma_1}, x_6^{\gamma_6}) = (x_{i_1}^{\gamma_{i_1}}, x_{j_1}^{\gamma_{j_1}})$ and similarly $(x_{i_7}^{\gamma_{i_7}}, x_{j_7}^{\gamma_{j_7}}) = (x_7^{\gamma_7}, x_{14}^{\gamma_{14}}) = (x_{i_{14}}^{\gamma_{i_{14}}}, x_{j_{14}}^{\gamma_{j_{14}}})$, moreover $o(n) = 8$, $p(n) = 14$ and $p(o(n)) = 9$, $o(p(n)) = 1$.

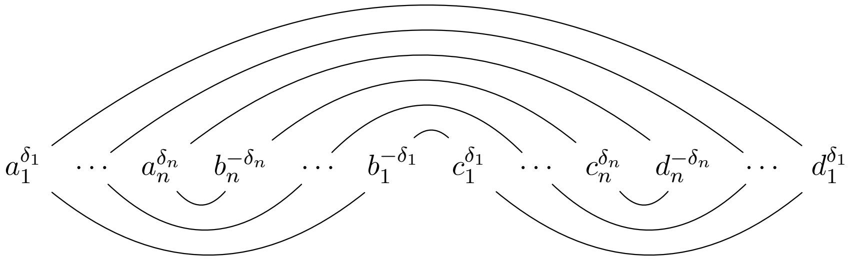
We may rewrite the word representing g as the word

$$a_1^{\delta_1} \cdots a_n^{\delta_n} b_n^{-\delta_n} \cdots b_1^{-\delta_1} y_{2n}^{\delta_1} \cdots y_{n+1}^{\delta_n} y_n^{-\delta_n} \cdots y_1^{-\delta_1},$$

where the y_k are systematically chosen to be x_{i_k} as in the pair $(x_{i_k}^{\gamma_{i_k}}, x_{j_k}^{\gamma_{j_k}})$. If we define $\sigma_k := -\delta_k$ for $1 \leq k \leq n$ and $\sigma_k := \delta_{2n+1-k}$ for $n + 1 \leq k \leq 2n$, then $y_{p(k)} = y_k$ and $\sigma_{p(k)} = -\sigma_k$ for all $k \in \{1, \ldots, 2n\}$, and the sequence (or word) $(y_k^{-\sigma_k})_{1 \leq k \leq 2n}$ reduces to the empty word, i.e. it represents the neutral element e in $\mathrm{F_g}(A)$. In our example



Observe moreover that for all $i \in \{1, \ldots, 2n\}$, $(x_i, y_i) \in \mathrm{Eq}(h)$, $\gamma_i = -\sigma_i$ and $\sigma_i = -\sigma_{2n+1-i}$. In order for g to be in $\mathrm{K}_2(\mathrm{F_g}(f), \mathrm{F_g}(h))$, it then suffices to check that $(y_i, y_{2n+1-i}) \in \mathrm{Eq}(f)$ for each $i \in \{1, \ldots, n\}$. Since this is not the case in general, we describe how to "algorithmically" replace the value of each y_i ($y_i \mapsto d_i$ for $1 \leq i \leq n$ and $y_i \mapsto c_{2n+1-i}$ for $n + 1 \leq i \leq 2n$) in such a way that for each index $1 \leq i \leq 2n$, it is still the case that $y_{p(i)} = y_i$, moreover, the previous and new value of y_i are identified by h and finally $(c_k, d_k) \in \mathrm{Eq}(f)$ for each $k \in \{1, \ldots, n\}$. The resulting word below still represents g which thus satisfies all the conditions for being an element of $\mathrm{K}_2(\mathrm{F_g}(f), \mathrm{F_g}(h))$.



Let us illustrate this rewriting method on our example. The general method is then described below in Algorithm 2.5.8. We start by looking at the pair $(y_7, y_8) = (a_7, b_7)$ and we observe that it is in $\mathrm{Eq}(f)$. Hence we do not change the values of y_7 and y_8, i.e. define $d_7 := a_7$ and $c_7 := b_7$ respectively. Now once we have set the values of d_7 and c_7, we do not want to modify these anymore. But remember that $y_{p(7)} = y_{14}$ has to be equal to $y_7 = d_7$ and similarly $y_{p(o(7))} = y_{p(8)} = y_9$ has to be equal to $y_8 = c_7$. We thus fix $y_9 = c_6 := c_7 = b_7$ and $y_{14} = c_1 := d_7 = a_7$. We then proceed by looking at the pair $(y_{p(o(7))}, y_{o(p(o(7)))}) = (y_9, y_6) = (b_7, a_1)$ which is not known to be in $\mathrm{Eq}(f)$. We are then going to modify the value of y_6 accordingly, since the value of y_9 is set. Observe that since

$$\begin{tikzcd}
y_9 = c_6 = b_7 \arrow[r, dash, bend left, "h"] & x_9 = b_6 \arrow[r, dash, bend right, "f"'] & x_6 = a_6 \arrow[r, dash, bend left, "h"] & y_6 = a_1,
\end{tikzcd}$$

there is $z \in A$ such that

$$\begin{tikzcd}
y_9 \arrow[r, dash, bend right, "f"'] & z \arrow[r, dash, bend left, "h"] & x_6 \arrow[r, dash, bend left, "h"] & y_6,
\end{tikzcd} \tag{42}$$

where we link two elements with a line labelled by h if these are identified by h; similarly for f.

We then let $d_6 := z$ be the new value of y_6. Then $y_{p(6)} = y_1$ has to be set to $d_1 := d_6$. We should then look at the pair (y_1, y_{14}), but both values d_1 and c_1 have been defined already. This is not a problem since $(y_1, y_{14}) \in \mathrm{Eq}(f)$ by construction ($f(y_1) = f(d_6) = f(c_6) = f(c_7) = f(c_1) = f(y_{14})$).

We have just completed an *Inner loop* in our method. We may then start over by choosing any of the remaining indices i, such that the value of y_i has not been set yet. Take for instance $i = 2$, we look at the pair $(y_2, y_{13}) = (a_2, b_5)$ which is not known to be in $\mathrm{Eq}(f)$. We thus modify the value of y_{13} accordingly: since

$$\begin{tikzcd}
y_2 = a_2 \arrow[r, dash, bend left, "h"] & x_2 = a_2 \arrow[r, dash, bend right, "f"'] & x_{13} = b_2 \arrow[r, dash, bend left, "h"] & y_{13} = b_5,
\end{tikzcd}$$

there is $z' \in A$ such that

$$\begin{tikzcd}
y_2 \arrow[r, dash, bend right, "f"'] & z' \arrow[r, dash, bend left, "h"] & x_{13} \arrow[r, dash, bend left, "h"] & y_{13},
\end{tikzcd}$$

we choose $y_{13} = c_2 := z'$ and keep $y_2 = d_2 := a_2$. As a consequence we define $y_{p(2)} = y_3 = d_3 := a_2$ and $y_{p(13)} = y_{10} = c_5 := c_2$. We must then look at $(y_{10}, y_5) = (c_2, a_4)$ which is not known to be in $\mathrm{Eq}(f)$. Then since

$$\begin{tikzcd}
y_{10} = c_2 \arrow[r, dash, bend left, "h"] & x_{10} = b_5 \arrow[r, dash, bend right, "f"'] & x_5 = a_5 \arrow[r, dash, bend left, "h"] & y_5 = a_4,
\end{tikzcd}$$

there is $z'' \in A$ such that

$$\begin{tikzcd}
y_{10} \arrow[r, dash, bend right, "f"'] & z'' \arrow[r, dash, bend left, "h"] & x_5 \arrow[r, dash, bend left, "h"] & y_5,
\end{tikzcd}$$

we define $y_5 = d_5 := z''$ and thus $y_{p(5)} = y_4 = d_4 := d_5$. Similarly, we define $y_{11} = c_4$ such that $(y_4, y_{11}) \in \mathrm{Eq}(f)$ and let $y_{12} = c_3 := c_4$. Now observe that $y_3 = d_3 = a_2$ was defined in such a way that $(y_{12}, y_3) \in \mathrm{Eq}(f)$. This ends a second *Inner loop* in the method. The values of all the y_i have been set in the appropriate way, which ends the *Outer loop* of our method. In general this method can be implemented as in Algorithm 2.5.8 below.

ALGORITHM 2.5.8. *Declare two variables m and l, which range over the indices $\{1, \ldots, 2n\}$, and which we use to run the two embedded loops of the algorithm. Define the variable I which is the set of "indices not yet visited". We use the symbol $:=$ to change the value of these variables.*

Start by setting $m := n$ and $I := \{1, \ldots, 2n\}$.

Outer loop. Define $I := I \setminus \{n, o(m)\}$ and start by running the method TestPair(m) *defined below. Then:*

(1) *If $p(m) \neq o(m)$, replace the values of $y_{p(m)}$ and $y_{p(o(m))}$ by those of y_m and $y_{o(m)}$ respectively. Define $l := p(o(m))$ and proceed to the Inner loop.*
(2) *Otherwise (if $p(m) = o(m)$), proceed to Switch.*

Inner loop. Define $I := I \setminus \{l, o(l)\}$ and then:

(1) *If $o(l) \neq p(m)$, then the value of $y_{o(l)}$ has not been modified yet. Run* TestPair(l)*, and replace the value of $y_{p(o(l))}$ by that of $y_{o(l)}$. Observe that $y_{p(m)} = y_m$, $y_{o(m)} = y_{p(o(m))}$, y_l, $y_{o(l)} = y_{p(o(l))}$ are all identified by f by construction. Finally, redefine $l := p(o(l))$ and proceed to the Inner loop.*
(2) *Otherwise ($o(l) = p(m)$), proceed to Switch.*

Switch.

1. *If I is not empty, choose any $i \in I$ and define $m := i$. Proceed to the Outer loop.*
2. *Otherwise stop the algorithm.*

TestPair. Given an index $j \in \{1, \ldots, 2n\}$, define the method denoted by $\mathrm{TestPair}(j)$ *as follows.*

1. *If $(y_j, y_{o(j)}) \in \mathrm{Eq}(f)$, keep these as they are;*
2. *otherwise, since*

$$\begin{tikzcd}
y_j \arrow[r, dash, bend left, "h"] & x_j \arrow[r, dash, bend right, "f"'] & x_{o(j)} \arrow[r, dash, bend left, "h"] & y_{o(j)},
\end{tikzcd}$$

there exists $z \in A$ such that

$$\begin{tikzcd}
y_j \arrow[r, dash, bend right, "f"'] & z \arrow[r, dash, bend left, "h"] & x_{o(j)} \arrow[r, dash, bend left, "h"] & y_{o(j)}.
\end{tikzcd}$$

Then replace the value of $y_{o(j)}$ by z.

Note that l and m keep visiting new indices, which have not yet been visited by l, $o(l)$, m or $o(m)$. Since there is a finite amount of such indices and $o(p(m))$ is one of those, the *Inner loop* always reaches an end: $o(l) = p(m)$. When this happens, by construction y_l and $y_{p(m)} = y_{o(l)}$ are in relation by $\mathrm{Eq}(f)$. We can thus proceed to the *Outer loop* after redefining m to be any of the remaining indices in I. No value of any y_i is changed twice, but all values are visited once so that the resulting sequence is as required. ☐

REMARK 2.5.9. *Note that the existence of a z as in Equation* (42) *does not mean that there is a workable algorithm to find this z. However, assuming that such a procedure exists, one can implement the whole method on a computer.*

Now given a double extension of racks (or quandles) α, it is easy to obtain $\mathrm{K}_{\langle f_A, \alpha_\top \rangle} \leq \mathrm{Ker}(\vec{f_A}) \cap \mathrm{Ker}(\vec{\alpha_\top})$ as the image of

$$\mathrm{K}_{\langle \mathrm{F_g}(f_A), \mathrm{F_g}(\alpha_\top) \rangle} = \mathrm{Ker}(\mathrm{F_g}(f_A)) \cap \mathrm{Ker}(\mathrm{F_g}(\alpha_\top))$$

by $q_{A_\top} \colon \mathrm{F_g}(\mathrm{U}(A_\top)) \to \mathrm{Pth}(A_\top)$ (defined as in Section 2.4 above). Hence $\mathrm{Ker}(\vec{f_A}) \cap \mathrm{Ker}(\vec{\alpha_\top}) = \mathrm{K}_{\langle \vec{f_A}, \vec{\alpha_\top} \rangle}$ if and only if the induced morphism

$\bar{q}\colon \mathrm{Ker}(\mathrm{F_g}(f_A)) \cap \mathrm{Ker}(\mathrm{F_g}(\alpha_{\top})) \to \mathrm{Ker}(\vec{f_A}) \cap \mathrm{Ker}(\vec{\alpha_{\top}})$ is a surjection. We were unfortunately not able to identify a reason why this should be true in general (see Observation 2.5.10 for alternative descriptions).

Besides, we note that even if the two groups do not coincide, it might still be that the action of $\mathrm{K}_{\langle f_A, \alpha_{\top}\rangle}$ on $A_{\top}$ and the action of $\mathrm{Ker}(\vec{f_A}) \cap \mathrm{Ker}(\vec{\alpha_{\top}})$ on $A_{\top}$ define the same congruence in Rck (or Qnd). Finally, we ask the "even weaker" question: is $\mathrm{Ker}(\vec{f_A}) \cap \mathrm{Ker}(\vec{\alpha_{\top}})$ in the center of $\mathrm{Pth}(A_{\top})$? This would imply that the image by $\mathrm{Conj}\,\mathrm{Pth}$ of a double covering is still a double covering (see Section 2.2).

OBSERVATION 2.5.10. *More precisely, observe that a double extension of racks or quandles α is sent to a double extension $\alpha' = \mathrm{Pth}(\alpha)$ in groups since* Pth *preserves pushouts of surjections and* Grp *is a Mal'tsev category. Call $c'\colon \mathrm{Pth}(A_{\top}) \twoheadrightarrow P'$ the surjective comparison map of α'. The double extension α is also sent to a double extension in* Set *by the forgetful functor, as pullbacks and surjections are preserved. This double extension in* Set *is then sent by* $\mathrm{F_g}$ *to a double extension α'' in* Grp*, since pushouts of surjections are preserved by left-adjoints. Write $c''\colon \mathrm{F_g}(A_{\top}) \twoheadrightarrow P''$ for the comparison map of α''. Finally α'' is sent by* Conj *to a double extension in* Rck *again, which is sent to a double extension α''' by* Pth*. Write $c'''\colon \mathrm{F_g}(A_{\top} \times \mathrm{F_g}(A_{\top})) \twoheadrightarrow P'''$ for the comparison map of α'''. We have thus three layers α''', α'' and α' of double extensions in* Grp *fitting into a fork $\alpha''' \rightrightarrows \alpha'' \to \alpha'$ of 3-dimensional arrows, such that each arrow is a square of double extensions, and the top pair is a reflexive graph whose legs are 3-fold extensions.*

By the universal property of the pullbacks P''', P'' and P', there is an induced reflexive graph $p_1, p_2\colon P''' \rightrightarrows P''$, as well as a surjection $q\colon P'' \to P'$ which coequalizes p_1 and p_2, such that the whole fork fits into the commutative diagrams of Figure 5. By Lemma 1.2 in [**10**]*, $(*)$ is a double extension if and only if q is the coequalizer of p_1 and p_2, which is also equivalent to the fork being a double extension. These three equivalent conditions are satisfied if and only if the aforementioned morphism*

$$\bar{q}\colon \mathrm{Ker}(\mathrm{F_g}(f_A)) \cap \mathrm{Ker}(\mathrm{F_g}(\alpha_{\top})) \to \mathrm{Ker}(\vec{f_A}) \cap \mathrm{Ker}(\vec{\alpha_{\top}})$$

is a surjection.

$$\begin{tikzcd}
 & \mathrm{F}_{\mathrm{g}}(A_{\top} \times \mathrm{F}_{\mathrm{g}}(A_{\top})) \arrow[rr, "\mathrm{F}_{\mathrm{g}}(f_A \times \mathrm{F}_{\mathrm{g}}(f_A))"] \arrow[dl, "\mathrm{F}_{\mathrm{g}}(\alpha_{\top} \times \mathrm{F}_{\mathrm{g}}(\alpha_{\top}))"'] \arrow[dd, shift right=1.5ex, "t"'] \arrow[dd, shift left=1.5ex, "t"] & & \mathrm{F}_{\mathrm{g}}(A_{\bot} \times \mathrm{F}_{\mathrm{g}}(A_{\bot})) \arrow[dl] \arrow[dd, shift right=1.5ex, "t"'] \arrow[dd, shift left=1.5ex, "t"] \\
\mathrm{F}_{\mathrm{g}}(B_{\top} \times \mathrm{F}_{\mathrm{g}}(B_{\top})) \arrow[rr] \arrow[dd, shift right=1.5ex, "t"'] \arrow[dd, shift left=1.5ex, "t"] & & \mathrm{F}_{\mathrm{g}}(B_{\bot} \times \mathrm{F}_{\mathrm{g}}(B_{\bot})) \arrow[dd, shift right=1.5ex, "t"'] \arrow[dd, shift left=1.5ex, "t"] & \\
 & \mathrm{F}_{\mathrm{g}}(A_{\top}) \arrow[uu, dashed] \arrow[rr, "\mathrm{F}_{\mathrm{g}}(f_A)"'] \arrow[dl, "\mathrm{F}_{\mathrm{g}}(\alpha_{\top})"'] \arrow[dd, "q_{A_{\top}}"] & & \mathrm{F}_{\mathrm{g}}(A_{\bot}) \arrow[uu, dashed] \arrow[dl] \arrow[dd, "q_{A_{\bot}}"] \\
\mathrm{F}_{\mathrm{g}}(B_{\top}) \arrow[uu, dashed] \arrow[rr] \arrow[dd, "q_{B_{\top}}"'] & & \mathrm{F}_{\mathrm{g}}(B_{\bot}) \arrow[uu, dashed] \arrow[dd, "q_{B_{\bot}}"] & \\
 & \mathrm{Pth}(A_{\top}) \arrow[rr, "\vec{f_A}"] \arrow[dl, "\vec{\alpha_{\top}}"'] & & \mathrm{Pth}(A_{\bot}) \arrow[dl] \\
\mathrm{Pth}(B_{\top}) \arrow[rr] & & \mathrm{Pth}(B_{\bot}) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{F}_{\mathrm{g}}(A_{\top} \times \mathrm{F}_{\mathrm{g}}(A_{\top})) \arrow[r, two heads, "c'''"] \arrow[d, shift right=1.5ex, "p^1_{A_{\top}}"'] \arrow[d, shift left=1.5ex, "p^2_{A_{\top}}"] & P''' \arrow[d, shift right=1.5ex, "p_1"'] \arrow[d, shift left=1.5ex, "p_2"] \\
\mathrm{F}_{\mathrm{g}}(A_{\top}) \arrow[u, dashed] \arrow[r, two heads, "c''"] \arrow[d, two heads, "q_{A_{\top}}"] \arrow[dr, phantom, "(*)"] & P'' \arrow[u, dashed] \arrow[d, two heads, "q"] \\
\mathrm{Pth}(A_{\top}) \arrow[r, dashed, two heads, "c'"] & P'
\end{tikzcd}$$

Figure 5. The fork $\alpha''' \rightrightarrows \alpha'' \to \alpha'$ and its comparison map

3. The Γ^1-coverings (or double central extensions of racks and quandles)

In this section, we show that the concept of double covering of racks and quandles (or algebraically central double extension) and the concept of Γ^1-covering (or double central extension of racks and quandles) coincide. In order to do so, we first show that double coverings are reflected and preserved by pullbacks along double extensions. Since trivial Γ^1-coverings are double coverings, this implies that Γ^1-coverings are also double coverings.

3.1. Double coverings are reflected and preserved by pullbacks. We first show a general result about morphisms induced by 3-fold extensions (see Definition 1.3.1). Observe that given the hypothesis of Lemma 1.2.1, we deduce from [**10**, Lemma 2.1] that if the right hand square of Diagram (30) is a double extension, then f is an extension even if $\mathcal{C}$ is merely regular (and not Barr-exact).

LEMMA 3.1.1. *Consider a 3-fold extension* $(\sigma, \beta)\colon \gamma \to \alpha$ *in a regular category* $\mathcal{C}$. *The morphism*

$$\square_{(\sigma,\beta)}\colon \operatorname{Eq}(f_C)\square\operatorname{Eq}(\gamma_\top) \to \operatorname{Eq}(f_A)\square\operatorname{Eq}(\alpha_\top)$$

induced by (σ, β) *between the parallelistic double equivalence relations is a regular epimorphism.*

PROOF. First we recall how to build the double parallelistic relations of interest. By taking kernel pairs horizontally and then vertically, we build the Diagrams (43) and (44), where the induced pairs $(p_1, p_2)\colon \operatorname{Eq}(f_\gamma) \rightrightarrows \operatorname{Eq}(f_C)$ and $(\pi_1, \pi_2)\colon \operatorname{Eq}(f_\alpha) \rightrightarrows \operatorname{Eq}(f_A)$ on the top rows, are the kernel pairs of $\bar{\gamma}$ and $\bar{\alpha}$ by a local version of the denormalized 3×3 Lemma (see [**10**] and Lemma 3.1.2 below). As a consequence, all the rows and columns of Diagrams (43) and (44) are exact forks.

$$\begin{tikzcd}
\operatorname{Eq}(f_\gamma) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift left] \arrow[d, shift right] & \operatorname{Eq}(f_C) \arrow[r, "\bar{\gamma}"] \arrow[d, shift left] \arrow[d, shift right] & \operatorname{Eq}(f_D) \arrow[d, shift left] \arrow[d, shift right] \\
\operatorname{Eq}(\gamma_\top) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_\gamma"'] & C_\top \arrow[r, "\gamma_\top"] \arrow[d, "f_C"] & D_\top \arrow[d, "f_D"] \\
\operatorname{Eq}(\gamma_\bot) \arrow[r, shift left] \arrow[r, shift right] & C_\bot \arrow[r, "\gamma_\bot"'] & D_\bot
\end{tikzcd} \tag{43}$$

$$\begin{tikzcd}
\operatorname{Eq}(f_\alpha) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, shift left] \arrow[d, shift right] & \operatorname{Eq}(f_A) \arrow[r, "\bar{\alpha}"] \arrow[d, shift left] \arrow[d, shift right] & \operatorname{Eq}(f_B) \arrow[d, shift left] \arrow[d, shift right] \\
\operatorname{Eq}(\alpha_\top) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_\alpha"'] & A_\top \arrow[r, "\alpha_\top"] \arrow[d, "f_A"] & B_\top \arrow[d, "f_B"] \\
\operatorname{Eq}(\alpha_\bot) \arrow[r, shift left] \arrow[r, shift right] & A_\bot \arrow[r, "\alpha_\bot"'] & B_\bot
\end{tikzcd} \tag{44}$$

Then by Proposition 2.1 from [**10**], the objects $\operatorname{Eq}(f_\gamma) = \operatorname{Eq}(f_C)\square\operatorname{Eq}(\gamma_\top)$ and $\operatorname{Eq}(f_\alpha) = \operatorname{Eq}(f_A)\square\operatorname{Eq}(\alpha_\top)$ yield the double parallelistic relations of interest.

Now the 3-fold extension (σ, β) induces morphisms between the Diagrams (43) and (44), such that on the top row we have

$$\begin{tikzcd}
\mathrm{Eq}(f_C)\square\,\mathrm{Eq}(\gamma_{\top}) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "\square_{(\sigma,\beta)}"'] & \mathrm{Eq}(f_C) \arrow[r, "\bar{\gamma}"] \arrow[d, "\bar{\sigma}"] & \mathrm{Eq}(f_D) \arrow[d, "\bar{\beta}"] \\
\mathrm{Eq}(f_A)\square\,\mathrm{Eq}(\alpha_{\top}) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathrm{Eq}(f_A) \arrow[r, "\bar{\alpha}"'] & \mathrm{Eq}(f_B).
\end{tikzcd}$$

Hence, by [**10**, Lemma 2.1], it suffices to prove that the right hand commutative square $(\bar{\sigma}, \bar{\beta})$ is a double extension. This can be deduced from the fact that (σ, β) is a 3-fold extension. When $\mathcal{C}$ is Barr-exact category, we may use [**48**, Lemma 3.2]. However, for a general regular category $\mathcal{C}$, we consider the "fork of comparison maps":

$$\begin{tikzcd}
\mathrm{Eq}(f_C) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "p"'] & C_{\top} \arrow[r, "f_C"] \arrow[d] & C_{\perp} \arrow[d] \\
\mathrm{Eq}(f_A) \times_{\mathrm{Eq}(f_B)} \mathrm{Eq}(f_D) \arrow[r, shift left] \arrow[r, shift right] & A_{\top} \times_{B_{\top}} D_{\top} \arrow[r, "f_P"'] & A_{\perp} \times_{B_{\perp}} D_{\perp}.
\end{tikzcd} \tag{45}$$

where the bottom row is exact by Lemma 3.1.2 (as for the top rows in Diagrams (43) and (44) above). Moreover, the right hand square (f_C, f_P) is a double extension since (σ, β) is a 3-fold extension, and thus the morphism p is a regular epimorphism. Since p is also the comparison map of $(\bar{\sigma}, \bar{\beta})$, this concludes the proof. □

Using the study of the denormalized 3×3 Lemma from [**10**], we obtain the following result, where, as usual, we locally use double extensions instead of working globally in a Mal'tsev category.

LEMMA 3.1.2. *Given a regular category $\mathcal{C}$ as well as a 3×3 diagram such as any of the two Diagrams* (43) *and* (44)*, where all columns are exact, the middle row and the bottom row are exact, and the bottom right-hand square is a double extension, then the top row is also exact.*

PROOF. The top row is left-exact by [**10**, Theorem 2.2]. Then the top right morphism is a regular epimorphism by [**10**, Lemma 2.1]. We conclude by the fact that in any category with pullbacks, regular epimorphisms are the coequalizers of their kernel pairs. □

Working in the categories Rck and Qnd again we obtain the following.

COROLLARY 3.1.3. *Double coverings are stable by pullbacks along double extensions and reflected along* 3*-fold extensions. In particular,* Γ^1*-coverings are double coverings.*

PROOF. Consider a 3-fold extension $(\sigma,\beta)\colon \gamma \to \alpha$ in Rck (or Qnd) such as in Definition 1.3.1.

Assume that γ is a double covering. Given $x \in A_\top$ and (a,b,c,d) in $\mathrm{Eq}(f_A)\square\,\mathrm{Eq}(\alpha_\top)$, the surjectivity of $\sigma_\top$ and $\square_{(\sigma,\beta)}$ (from Lemma 3.1.1) yields $x' \in C_\top$ and $(a',b',c',d') \in \mathrm{Eq}(f_C)\square\,\mathrm{Eq}(\gamma_\top)$ such that $\sigma_\top(x') = x$, $\sigma_\top(a') = a$, $\sigma_\top(b') = b$, $\sigma_\top(c') = c$, and $\sigma_\top(d') = d$. Since $x' {\triangleleft} a' {\triangleleft^{-1}} b' {\triangleleft} c' {\triangleleft^{-1}} d' = x'$ in $C_\top$, the image of this equation by $\sigma_\top$ yields $x {\triangleleft} a {\triangleleft^{-1}} b {\triangleleft} c {\triangleleft^{-1}} d = x$ in $A_\top$. Hence α is a double covering.

Conversely assume that α is a double covering and suppose that (σ,β) describes the pullback of α and β, i.e. suppose that the comparison map π of (σ,β) is an isomorphism (see Definition 1.3.1). Then we consider $x \in C_\top$ and $(a,b,c,d) \in \mathrm{Eq}(f_C)\square\,\mathrm{Eq}(\gamma_\top)$, and we have to show that $y := x \triangleleft a \triangleleft^{-1} b \triangleleft c \triangleleft^{-1} d$ is equal to x. It suffices to check the equality in both components of the pullback $C_\top$, via the projections $\gamma_\top$ and $\sigma_\top$. We have indeed $\gamma_\top(y) = \gamma_\top(x)$ and since α is a double covering, we have also $\sigma_\top(y) = \sigma_\top(x)$. Hence γ is a double covering. $\square$

3.2. Double coverings are Γ^1-coverings. As described in Section 1.4, given a double covering $\alpha\colon f_A \to f_B$, we build the canonical double projective presentation of its codomain f_B:

$$p_{f_B} := (p^\top_{f_B}, p^\perp_{f_B})\colon p_B \to f_B \quad \text{(see Diagram (33)).}$$

We then consider the pullback of our double covering α along our projective presentation p_{f_B} (see Diagram (46) in ExtRck or ExtQnd). This yields a double covering $\gamma\colon f_C \to p_B$ which has a projective codomain $p_B\colon \mathrm{F_r}(P) \to \mathrm{F_r}(B_\perp)$ (or $p_B\colon \mathrm{F_q}(P) \to \mathrm{F_q}(B_\perp)$ if we work in Qnd).

$$\begin{tikzcd}
f_C \arrow[r] \arrow[d, "(\gamma^\top{,}\gamma_\perp)"'] \arrow[dr, phantom, "\lrcorner", very near start] & f_A \arrow[d, "(\alpha^\top{,}\alpha_\perp)"] \\
p_B \arrow[r, "(p^\top_{f_B}{,}p^\perp_{f_B})"'] & f_B
\end{tikzcd} \tag{46}$$

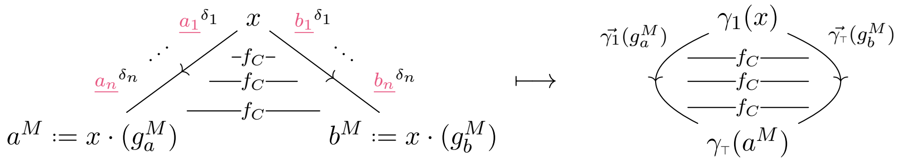
We show that such double coverings are always trivial double coverings, which implies that the double covering α is a Γ^1-covering.

PROPOSITION 3.2.1. *If a double covering of racks* $\gamma = (\gamma_\top, \gamma_\bot)$ *has a projective codomain of the form* $p\colon \mathrm{F_r}(P) \to \mathrm{F_r}(B_\bot)$ *for some sets* P *and* $B_\bot$*:*

$$\begin{tikzcd}
C_\top \arrow[rr, "f_C"] \arrow[dr, "\langle \gamma_\top, f_C \rangle"'] \arrow[dd, "\gamma_\top"'] & & C_\bot \arrow[dd, "\gamma_\bot"] \\
& Q \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\
\mathrm{F_r}(P) \arrow[rr, "p"'] & & \mathrm{F_r}(B_\bot),
\end{tikzcd}$$

where $Q \coloneqq \mathrm{F_r}(P) \times_{\mathrm{F_r}(B_\bot)} C_\bot$*, then* γ *is a trivial double covering. The result holds similarly in* Qnd*, for a double covering* γ *with codomain of the form* $p\colon \mathrm{F_q}(P) \to \mathrm{F_q}(B_\bot)$.

PROOF. Consider an f_C-horn $M = (x, (a_i, b_i, \delta_i)_{1\leq i \leq n})$ in $C_\top$ such that the image of M by $\gamma_\top$ closes into a disk in $\mathrm{F_r}(P)$, i.e. $\gamma_\top(x) \cdot \big(\vec{\gamma_\top}(\underline{a_1}^{\delta_1} \cdots \underline{a_n}^{\delta_n} \underline{b_n}^{-\delta_n} \cdots \underline{b_1}^{-\delta_1})\big) = \gamma_\top(x)$.



Let us write $y_i \coloneqq \gamma_\top(a_i)$ and $x_i \coloneqq \gamma_\top(b_i)$ for each $1 \leq i \leq n$. Then

$$h \coloneqq \vec{\gamma_\top}(\underline{a_1}^{\delta_1} \cdots \underline{a_n}^{\delta_n} \underline{b_n}^{-\delta_n} \cdots \underline{b_1}^{-\delta_1}) = \underline{y_1}^{\delta_1} \cdots \underline{y_n}^{\delta_n} \underline{x_n}^{-\delta_n} \cdots \underline{x_1}^{-\delta_1},$$

in $\mathrm{Pth}(\mathrm{F_r}(C_\top))$ yields the neutral element e since the action of the group $\mathrm{Pth}(\mathrm{F_r}(C_\top))$ on $\mathrm{F_r}(C_\top)$ is free – note that in the context of Qnd, this path h is in the group $\mathrm{Pth}^\circ(\mathrm{F_q}(C_\top))$ which acts freely on $\mathrm{F_q}(C_\top)$.

Since p is projective (with respect to double extensions), there is a splitting $s \coloneqq (s_\top, s_\bot)$ of γ such that $\gamma s = (1_{\mathrm{F_r}(P)}, 1_{\mathrm{F_r}(B_\bot)})$ is the identity morphism. If we define $d_i \coloneqq s_\top(\gamma_\top(a_i))$ and $c_i \coloneqq s_\top(\gamma_\top(b_i))$ for each $1 \leq i \leq n$, then we have that

- $\underline{c_1}^{\delta_1} \cdots \underline{c_n}^{\delta_n} \underline{d_n}^{-\delta_n} \cdots \underline{d_1}^{-\delta_1} = \vec{s_\top}(h^{-1}) = e \in \mathrm{Pth}(C_\top)$ is trivial;
- moreover $\gamma_\top(d_i) = \gamma_\top(a_i)$ and $\gamma_\top(c_i) = \gamma_\top(b_i)$ for each $1 \leq i \leq n$;
- and thus the product $g_a^M (g_b^M)^{-1}$ in $\mathrm{Pth}(C_\top)$ defines an $\langle f_C, \gamma_\top \rangle$-symmetric path in $\mathrm{Pth}(C_\top)$ since we have that the product $g_a^M (g_b^M)^{-1} = g_a^M (g_b^M)^{-1} \vec{s_\top}(h^{-1})$ which is equal to
  $\underline{a_1}^{\delta_1} \cdots \underline{a_n}^{\delta_n} \underline{b_n}^{-\delta_n} \cdots \underline{b_1}^{-\delta_1} \underline{c_1}^{\delta_1} \cdots \underline{c_n}^{\delta_n} \underline{d_n}^{-\delta_n} \cdots \underline{d_1}^{-\delta_1}$.

Since γ is a double covering, we conclude that

$$x = x \cdot (g_a^M (g_b^M)^{-1}) = x \cdot (\underline{a_1}^{\delta_1} \cdots \underline{a_n}^{\delta_n} \underline{b_n}^{-\delta_n} \cdots \underline{b_1}^{-\delta_1}),$$

which shows that γ is a trivial double covering. $\square$

THEOREM 3.2.2. *A double extension of racks (or quandles) is a Γ^1-covering (also called double central extension of racks and quandles), if and only if it is a double covering (also called algebraically central double extension of racks and quandles). The category of double coverings and the category of Γ^1-coverings above an extension of racks (or quandles) are isomorphic.*

3.3. Centralizing double extensions. Consider a double extension of racks (or quandles) $\alpha\colon f_A \to f_B$. We may universally *centralize it* (i.e. make it into a double covering) by a quotient of its *initial object* $A_\top$. We studied the reflection of $\mathsf{CExtRck}$ in ExtRck. Now in $\mathsf{Ext}^2\mathsf{Rck}$ (from Definition 1.3.1), we may identify the full subcategory $\mathsf{CExt}^2\mathsf{Rck}$ whose objects are the double coverings (or equivalently the Γ^1-coverings, also called double central extensions). The following result is the 2-dimensional equivalent of Theorem 3.3.1 from Part I. We define $\mathcal{E}_2$ as the class of 3-*fold extensions* from Definition 1.3.1.

THEOREM 3.3.1. *The category $\mathsf{CExt}^2\mathsf{Rck}$ is a ($\mathcal{E}_2$)-reflective subcategory of the category $\mathsf{Ext}^2\mathsf{Rck}$ with left adjoint F_2 and unit η^2 defined for an object $\alpha\colon f_A \to f_B$ in $\mathsf{Ext}^2\mathsf{Rck}$ by*

$$\eta^2_\alpha := (\eta^2_{\alpha_\top}, \eta^2_{\alpha_\perp}) := ((\eta^2_{A_\top}, \mathrm{id}_{A_\perp}), (\mathrm{id}_{B_\top}, \mathrm{id}_{B_\perp}))\colon \alpha \longrightarrow \mathrm{F}_2(\alpha),$$

where $\eta^2_{A_\top}\colon A_\top \to A_\top/\, \mathrm{C}_2(\alpha)$ is defined as the quotient of $A_\top$ by the centralizing relation $\mathrm{C}_2(\alpha) := [\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_\top)]$, *and its equivalent descriptions from Corollary 2.5.5. Observing that $\mathrm{C}_2(\alpha) \leq \mathrm{Eq}(f_A) \cap \mathrm{Eq}(\alpha_\top)$, the image $\mathrm{F}_2(\alpha) := (\mathrm{F}_{\hat{2}}(\alpha_\top), \alpha_\perp)\colon \mathrm{F}_{\hat{2}}(f_A) \to f_B$ is defined via the unique* *factorization of the comparison map* $p\colon A_\top \to A_\bot \times_{B_\bot} B_\top$ *through the quotient* $\eta^2_{A_\top}$*:*

$$\begin{tikzcd}
A_\top \arrow[rr, "p"] \arrow[dr, "\eta^2_{A_\top}"'] & & A_\bot \times_{B_\bot} B_\top \\
& A_\top/\,\mathrm{C}_2(\alpha) \arrow[ur, "p'"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& A_\top \arrow[rr, "\eta^2_{A_\top}"] \arrow[dl, "f_A"'] \arrow[dd, "\alpha_\top" near start] & & A_\top/\,\mathrm{C}_2(\alpha) \arrow[dl, "\mathrm{F}_{\hat{2}}(f_A) := \pi_1 p'"'] \arrow[dd, "\mathrm{F}_{\hat{2}}(\alpha_\top) := \pi_2 p'"] \\
A_\bot \arrow[rr, equal] \arrow[dd, "\alpha_\bot"'] & & A_\bot \arrow[dd, "\alpha_\bot" near start] & \\
& B_\top \arrow[rr, equal] \arrow[dl, "f_B"'] & & B_\top \arrow[dl, "f_B"] \\
B_\bot \arrow[rr, equal] & & B_\bot &
\end{tikzcd}$$

where π_1 *and* π_2 *are the projections of* $A_\bot \times_{B_\bot} B_\top$*, as in Equation 27.*

The image by F_2 *of a morphism* $(\sigma, \beta)\colon \gamma \to \alpha$ *is then given by the identity in all but the initial component:*

$$\mathrm{F}_2(\sigma, \beta) = ((\hat{\sigma}_\top, \sigma_\bot), (\beta_\top, \beta_\bot)),$$

where $\hat{\sigma}_\top$ *is defined by the unique factorization of* $\eta^2_{A_\top}\sigma_\top$ *through* $\eta^2_{C_\top}$*, as displayed below, where* $P := (C_\top/\,\mathrm{C}_2(\gamma)) \times_{(A_\top/\,\mathrm{C}_2(\alpha))} A_\top$*:*

$$\begin{tikzcd}
C_\top \arrow[rr, "\sigma_\top"] \arrow[dr, "p"] \arrow[dd, "\eta^2_{C_\top}"'] & & A_\top \arrow[dd, "\eta^2_{A_\top}"] \\
& P \arrow[ur, "\pi_2"] \arrow[dl, "\pi_1"] & \\
C_\top/\,\mathrm{C}_2(\gamma) \arrow[rr, dashed, "\hat{\sigma}_\top"'] & & (A_\top/\,\mathrm{C}_2(\alpha))
\end{tikzcd} \tag{47}$$

PROOF. First observe that the double extension $\mathrm{F}_2(\alpha)$ is indeed a double covering by construction. As was already mentioned in the proof of Theorem 1.3.2, η^2 is easily seen to be a 3-fold extension since its bottom component is an isomorphism. Hence given a quadruple $(a, b, c, d) \in \mathrm{Eq}(\mathrm{F}_{\hat{2}}(f_A)) \square \mathrm{Eq}(\mathrm{F}_{\hat{2}}(\alpha_\top))$, it is the quotient of some quadruple $(a', b', c', d') \in \mathrm{Eq}(f_A) \square \mathrm{Eq}(\alpha_\top)$ by Lemma 3.1.1, and thus for any $x \in (A_\top/\,\mathrm{C}_2(\alpha))$, the elements x and $x \cdot (\underline{a}\,\underline{b}^{-1}\,\underline{c}\,\underline{d}^{-1})$ have $\eta^2_{A_\top}$-pre-images x' and $x' \cdot (\underline{a'}\,(\underline{b'})^{-1}\,\underline{c'}\,(\underline{d'})^{-1})$ which are in relation by $\mathrm{C}_2(\alpha)$.

Then we show that η^2_α has the right universal property. We first show the universality of η^2_α in the subcategory $\mathsf{Ext}^2(f_B)$ of double extensions over f_B. Consider a double covering $\gamma\colon f_C \to f_B$ and a morphism $\theta \in \mathsf{Ext}^2(f_B)$ between α and γ, yielding the following commutative diagram of plain arrows in ExtRck, whose top and bottom components in $\mathcal{C}$ are given below.

$$\begin{tikzcd}
\mathrm{F}_{\hat{2}}(f_A) \arrow[drr, "\mathrm{F}_2(\alpha)"] \arrow[dd, dotted, bend right=60, "{\vartheta:=(\vartheta_\top,\theta_\bot)}"'] & & \\
f_A \arrow[u, "\eta^2_{\alpha_\top}"] \arrow[rr, "\alpha"] \arrow[d, "\theta"'] & & f_B \\
f_C \arrow[urr, "\gamma"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
(A_\top/\,\mathrm{C}_2(\alpha)) \arrow[drr, "\mathrm{F}_{\hat{2}}(\alpha_\top)"] \arrow[dd, dotted, bend right=60, "\vartheta_\top"'] & & \\
A_\top \arrow[u, "\eta^2_{A_\top}"] \arrow[rr, "\alpha_\top"] \arrow[d, "\theta_\top"'] & & B_\top \\
C_\top \arrow[urr, "\gamma_\top"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A_\bot \arrow[drr, "\alpha_\bot"] \arrow[d, equal] \arrow[dd, dotted, bend right=60, "\theta_\bot"'] & & \\
A_\bot \arrow[rr, "\alpha_\bot"] \arrow[d, "\theta_\bot"'] & & B_\bot \\
C_\bot \arrow[urr, "\gamma_\bot"'] & &
\end{tikzcd}$$

Given any pair $(x \cdot g, x) \in \mathrm{C}_2(\alpha)$, where g is some $\langle f_A, \alpha_\top \rangle$-symmetric path, $\vec{\theta}(g)$ is an $\langle f_C, \gamma_\top \rangle$-symmetric path and thus we have that $\theta_\top(x) = \theta_\top(x) \cdot \vec{\theta}_\top(g)$ since γ is a double covering. As a consequence, $\mathrm{C}_2(\alpha) \leq \mathrm{Eq}(\theta_\top)$ and thus there exists a unique factorization $\vartheta_\top$ of $\theta_\top$ through $\eta^2_{A_\top}$. Since $f_C \vartheta_\top \eta^2_{A_\top} = \theta_\bot \, \mathrm{F}_{\hat{2}}(f_A) \eta^2_{A_\top}$, and $\eta^2_{A_\top}$ is an epimorphism, we may define the morphism $\vartheta := (\vartheta_\top, \theta_\bot) \colon \mathrm{F}_{\hat{2}}(f_A) \to f_C$, which is moreover a double extension by Lemma 1.1.2. This shows the existence of a factorization of θ through $\eta^2_{\alpha_\top}$. The uniqueness of ϑ is easily deduced from the uniqueness in each component.

Now working in the category $\mathsf{Ext}^2\mathsf{Rck}$, we consider a double covering $\gamma \colon f_C \to f_D$ and a morphism $(\tau, \iota) \colon \alpha \to \gamma$ in $\mathsf{Ext}^2\mathsf{Rck}$. We compute the pullback ρ of γ along ι and the induced comparison map π of the underlying square of (τ, ι) in ExtRck:

$$\begin{tikzcd}
\mathrm{F}_{\hat{2}}(f_A) \arrow[drrr, "\mathrm{F}_2(\alpha)"] & & & \\
& f_A \arrow[ul, "\eta^2_{\alpha_\top}"] \arrow[rr, "\alpha"'] \arrow[dr, dashed, "\pi"] \arrow[dd, "\tau"'] & & f_B \arrow[dd, "\iota"] \\
& & f_C \times_{f_D} f_B \arrow[ur, "\rho"'] \arrow[dl, "\varrho"] & \\
& f_C \arrow[rr, "\gamma"'] & & f_D,
\end{tikzcd}$$

Since ρ is a double covering (by pullback-preservation), we obtain

$$\vartheta \colon \mathrm{F}_{\hat{2}}(f_A) \to f_C \times_{f_D} f_B$$

by the preceding discussion. Then the morphism $(\varrho\vartheta, \iota)$ is a factorization of (τ, ι) through $\eta^2(\alpha)$ which is easily shown to be unique. $\square$

Note that, as usual, the monadicity of I implies that $\mathsf{CExt}^2\mathcal{C}$, is closed under finite limits computed in $\mathsf{Ext}^2\mathcal{C}$. Also since η^2 has regular epimorphic components, double coverings are closed under subobjects in $\mathsf{Ext}^2\mathcal{C}$ (see for instance [**71**, Section 3.1]; note that the same comments hold for the adjunction $\mathrm{F}_1 \dashv \mathrm{I}$). Closure by quotients along 3-fold extensions was discussed in Remark 1.3.3. We conclude the proof that $\mathrm{F}_2 \dashv \mathrm{I}$ fits into a strongly Birkhoff Galois structure Γ^2 in the forthcoming article, *Part III* [**97**], where we study higher coverings of arbitrary dimensions.

4. Further developments

Besides the following theoretical developments, more explicit examples of double coverings should be studied, for instance in the known contexts of application cited in Part I. Now from the perspective of categorical Galois theory, future developments should also include the description of a *weakly universal double covering* above an extension, and subsequently the characterization of the *fundamental double groupoid* of an extension (see for instance [**21**]). From there, the *fundamental theorem of categorical Galois theory* should be applied in order to "classify" the double coverings above an extension.

Another obvious line of work concerns the commutator defined in Section 2.1. A review of the links between commutators and Galois theory can be found in [**70**]. For instance, it should be checked whether or not our commutator coincides with (or compares to) what was defined for general varieties in terms of internal pregroupoids [**75**], or other theories such as the classical approach of [**57**] which was already applied in this context [**7**]. In the last paragraphs below, we suggest to apply the developments of [**69**] to our context, with the objective to investigate the links between categorical Galois theory and *homology* [**56, 28, 34**] within racks and quandles (see also [**38**, Section 9]).

4.1. Galois structure with abstract commutator. As it was suggested in Part I, one of the important outcomes for the application of higher categorical Galois theory in groups was the elegant generalization of the *higher Hopf formulae* from [**18**] to *semi-abelian categories* [**74, 50, 45, 34**]. In [**69**], G. Janelidze shares his perspective on how to understand the mechanics behind the Hopf formulae from the viewpoint of categorical Galois theory, and in particular via the description and understanding of what an *abstract Galois group* is. He introduces the definition of a *Galois structure with (abstract) commutators*, which is suggested as another starting point (more general than that from [**50**]) for applying the methodology that he illustrates in the context of groups. In this section, we adapt this definition in order to include the covering theory of quandles as an example, in a way which is compatible with the aims and developments from [**69**]. Further details about the application of the ideas from [**69**] to the covering theory of quandles is left for future work.

Our definition of Galois structure with (abstract) commutators is not aimed at being the most general possible. Our main point is the use of higher extensions to clarify the conditions which are displayed in [**69**].

DEFINITION 4.1.1. *A Galois structure with commutators is a system of the form* $\Gamma = (\mathcal{C}, \mathcal{X}, F, \mathrm{I}, \eta, \epsilon, \mathcal{E}, [-,-])$, *in which:*

(1) $\Gamma = (\mathcal{C}, \mathcal{X}, F, \mathrm{I}, \eta, \epsilon, \mathcal{E})$ *is an* admissible *Galois structure (see Convention 1.0.1 and Subsection 1.3);*
(2) if $f = pq$ *in* $\mathcal{C}$ *and* f *and* q *are in* $\mathcal{E}$, *then* p *is in* $\mathcal{E}$;
(3) $[-,-]$ *is a family of binary operations*

$$\mathrm{ER}_{\mathcal{E}}(A) \times \mathrm{ER}_{\mathcal{E}}(A) \to \mathrm{ER}_{\mathcal{E}}(A)$$

defined for each A *in* $\mathcal{C}$ *and all written as* $(S,T) \mapsto [S,T]$; *here* $\mathrm{ER}_{\mathcal{E}}(A)$ *denotes the class of* $\mathcal{E}$*-congruences on* $\mathcal{C}$, *i.e. the class of subobjects of* $A \times A$ *that are kernel pairs of morphisms from* $\mathcal{E}$;
(4) for S *and* T *in* $\mathrm{ER}_{\mathcal{E}}(A)$, *we always have* $[S,T] \leq S \cap T$;
(5) if $(\sigma, \beta)\colon \gamma \to \alpha$ *is a morphism between the double extensions* γ *and* β, *then* (σ,β) *induces a morphism*

$$[\sigma_{\scriptscriptstyle\top}]\colon [\mathrm{Eq}(f_C), \mathrm{Eq}(\gamma_{\scriptscriptstyle\top})] \to [\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_{\scriptscriptstyle\top})];$$

(6) if $(\sigma, \beta)\colon \gamma \to \alpha$ *above is a* 3*-fold extension, then the morphism* $[\sigma_{\scriptscriptstyle\top}]\colon [\mathrm{Eq}(f_C), \mathrm{Eq}(\gamma_{\scriptscriptstyle\top})] \to [\mathrm{Eq}(f_A), \mathrm{Eq}(\alpha_{\scriptscriptstyle\top})]$ *is in* $\mathcal{E}$;
(7) for each A *in* $\mathcal{C}$, $F(A) = A/[A \times A, A \times A]$ *and* η_A *is the canonical morphism* $A \to A/[A \times A, A \times A]$;
(8) for a morphism $p\colon E \to B$ *from* $\mathcal{E}$, p *is a* Γ*-covering if and only if* $[E \times E, \mathrm{Eq}(p)] = \Delta_E$, *i.e.* $[E \times E, \mathrm{Eq}(p)]$ *is the smallest congruence on* E.

Observe that conditions (5) and (6) differ from G. Janelidze's presentation (see 4.4(i) and (j) in [**69**]). We explain how to translate from his context to ours. In order to avoid confusion, let us point out a small typo in [**69**]: the conclusions of Condition (g) in Definitions 4.1 and Condition (f) in Definition 4.4 should be that p is in F (rather than in C – see Condition (2) below). Also in Condition (i) of Definition 4.4 we should read $[E \times E, \mathrm{Eq}(p)]$ instead of $[E \times E, \mathrm{Ker}(p)]$.

Now if we translate the notation in [**69**] to ours, G. Janelidze considers the data of $\sigma_{\top} \colon C_{\top} \to A_{\top}$, $S := \mathrm{Eq}(f_C)$, $T := \mathrm{Eq}(\gamma_{\top})$, $S' := \mathrm{Eq}(f_A)$ and $T' := \mathrm{Eq}(\alpha_{\top})$ as well as *induced morphisms* $s \colon S \to S'$ and $t \colon T \to T'$. From this data, we easily build the entire morphism (σ, β) with no further assumptions. The only difference is then the assumption that α and γ are not merely pushout squares of extensions, but also double extensions. Whenever $\mathcal{C}$ is a Mal'tsev category, which is the case in the examples considered by G. Janelidze and others [**44, 50, 53, 54, 46**], α and γ are automatically double extensions. In our context, this is the "natural" extra requirement to work with. We work locally with congruences which commute since in our context, pairs of congruences above a given object do not commute in general. Now when s and t are further required to be extensions (such as in 4.4(j)), by the same reasoning, the natural generalization from Mal'tsev categories consists in requesting (σ, β) to be a square of double extensions. Finally observe that 4.4(j) was already challenged in Remark 4.6 of [**69**]. Observe that under the restrictions suggested by T. Everaert or G. Janelidze (in this same remark), our square of double extensions (σ, β) becomes a 3-fold extension. Hence our choice of presentation is arguably an adequate and elegant variation from [**69**], which is coherent with the example we are interested in, as well as the examples considered in [**69**] and related works.

EXAMPLE 4.1.2. *From the results of Section 2.1, and Lemma 3.1.1, we deduce that the Galois structure from Theorem 1.3.2 together with the operation* $[-,-]$ *from Definition 2.1.1 satisfies the conditions of Definition 4.1.1.*

Since compatibility with unions is understood to be an important property for commutators, we show the following result which may be used to study Example 4.1.2 from the perspective of [**69**]. Note that our hypotheses might not be optimal; we deduce the *modular law* locally from the less general permutability conditions on our congruences. These are arguably more suitable for this context in which (repeatedly) we have been using, locally, some properties which are globally satisfied in Mal'tsev categories.

PROPOSITION 4.1.3. *Let A be a quandle, R, S and T congruences on A such that $S \leq R$. If R, S and T commute pairwise, and moreover $R \cap T$ commutes with S (for instance when S commutes with all congruences), then $[R, S \cup T] = [R, S] \cup [R, T] = [S, A \times A] \cup [R, T]$ (see Corollary 2.1.10).*

PROOF. First observe that $[R, S] \cup [R, T] \leq [R, S \cup T]$ is an easy consequence of Lemma 2.1.9. Then consider a generator

$$(x \lhd a \lhd^{-1} b \lhd c \lhd^{-1} d, x) \; \in \; [R, S \cup T]$$

for some $x \in A$ and

$$\begin{tikzcd} a \arrow[r, dash, "R"] \arrow[d, dash, dotted, "S \cup T"'] & b \arrow[d, dash, dotted, "S \cup T"] \\ d \arrow[r, dash, "R"'] & c \end{tikzcd} \; \in \; R\square(S \cup T).$$

Since S and T commute, there is $b_0 \in A$ such that $(b, b_0) \in S$ and $(b_0, c) \in T$. Moreover since R commutes with S and T, there are a_0 and respectively d_0 such that $(a, a_0) \in S$, $(a_0, b_0) \in R$, $(d, d_0) \in T$ and $(d_0, b_0) \in R$. Hence $(a_0, d_0) \in R \cap (S \cup T)$. Using the modular law $(a_0, d_0) \in S \cup (R \cap T)$ and thus there is $a_1 \in A$ such that $(a_0, a_1) \in S$ and $(a_1, d_0) \in (R \cap T)$. From there observe that $(a_1, b_0) \in R$ and $(a_1, d) \in T$ such that we obtain:

$$\begin{tikzcd} a \arrow[r, dash, dotted, "S"] \arrow[d, dash, "R"] & a_0 \arrow[r, dash, dotted, "S"] \arrow[d, dash, "R"] & a_1 \arrow[r, dash, dotted, "T"] \arrow[d, dash, "R"] & d \arrow[d, dash, "R"] \\ b \arrow[r, dash, dotted, "S"] & b_0 \arrow[r, dash, dotted, "S"] & b_0 \arrow[r, dash, dotted, "T"] & c \end{tikzcd}$$

Considering each of these three squares separately, by definition of $[R, S]$ and $[R, T]$ we derive that x is in relation by $[R, S] \cup [R, T]$ with the element

$$x \lhd a \lhd^{-1} b \lhd b_0 \lhd^{-1} a_0 \lhd a_0 \lhd^{-1} b_0 \lhd b_0 \lhd^{-1} a_1 \lhd a_1 \lhd^{-1} b_0 \lhd c \lhd^{-1} d,$$

which reduces to $x \lhd a \lhd^{-1} b \lhd c \lhd^{-1} d$. $\square$

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