STOP-LOSS PROTECTION FOR A LARGE P2P INSURANCE POOL

M. Denuit and C. Y. Robert

DISCUSSION PAPER | 2020 / 28

UCLouvain

LiDAM
ISBA | Louvain Institute of Data Analysis and Modeling in economics and statistics

STOP-LOSS PROTECTION FOR A LARGE P2P INSURANCE POOL

Michel Denuit
Institute of Statistics, Biostatistics and Actuarial Science - ISBA
Louvain Institute of Data Analysis and Modeling - LIDAM
UCLouvain
Louvain-la-Neuve, Belgium
michel.denuit@uclouvain.be

Christian Y. Robert
Laboratory in Finance and Insurance - LFA
CREST - Center for Research in Economics and Statistics
ENSAE
Paris, France
chrobert@ensae.fr

September 25, 2020

Abstract

This paper considers a peer-to-peer (P2P) insurance scheme where the higher layer is transferred to a (re-)insurer and retained losses are distributed among participants according to the conditional mean risk sharing rule proposed by Denuit and Dhaene (2012). The global retention level of the pool of participants grows proportionally with their number. We study the asymptotic behavior of the individual retention levels, as well as individual cash-backs and stop-loss premiums, as the number of participants increases. The probability that the total loss hits the upper layer protected by the stop-loss treaty is also considered. The results depend on the proportional rate of increase of the global retention level with the number of participants, as well as on the existence of the Esscher transform of the losses brought to the pool.

Keywords: conditional expectation, risk pooling, comonotonicity, Esscher transform, regularly varying tails.

1 Introduction and motivation

The progressive demutualization of the insurance sector has broken the link with the ancestral compensation mechanism consisting in using the contributions of the many to balance the misfortunes of the few. As a consequence, the feeling to belong to a community helping its unlucky members has disappeared to a large extent. In order to remedy this situation, an offer started to develop in line with the collaborative economy by organizing peer-to-peer (P2P) insurance communities, complementing traditional business. See, e.g., Eling and Lehmann (2018), Abdikerimova and Feng (2019) and Clemente and Marano (2020). Collaborative, P2P insurance schemes may well become an additional sales channel, complementing classical ones (brokers, direct, bank).

P2P insurance builds on the fundamental principle of mutuality at the origin of insurance without the equity buffer provided by insurer's capital. Precisely, participants to a P2P insurance scheme agree to pool the first layer of the risks they face, whereas higher losses are still covered by a third party, typically an insurance or reinsurance company, with the help of a stop-loss protection. Participants to a P2P insurance scheme can thus access higher amounts of retention compared to deductibles included in standard insurance covers, thanks to the risk-reducing effect of pooling.

Unclaimed money can be paid back, given to a charity or used to fund a common project to which members of the community adhere. In the case of a cash-back mechanism, the contract operates as a participating policy. Surplus, that is, that part of the contributions that has not been used to cover the losses, must thus be fairly shared, in an understandable and transparent way, among participants. As everybody is well aware that insurance risks are heterogeneous, equally sharing the total losses among participants may well appear to be unfair. To be successful, P2P insurance schemes thus require an appropriate risk sharing mechanism recognizing the different distributions of the risks brought to the pool. Denuit (2019, 2020) and Denuit and Robert (2020d) proposed a simple and mathematically correct solution based on the conditional mean risk sharing rule introduced by Denuit and Dhaene (2012). Being based on the concept of mean value, this sharing rule turns out to be quite intuitive as most participants have at least a vague idea about averaging and its risk-reducing effect. Since participants can be informed when they enter the pool about the amount they will have to contribute as a function of the total realized loss, this approach ensures full transparency. Numerous attractive properties of the conditional mean risk sharing rule have been obtained by Denuit and Robert (2020a,b,c).

The present paper studies individual retention levels, cash-backs and stop-loss premiums within a large P2P insurance community. We assume that individual losses are mutually independent and that the total losses of the pool obey a law of large numbers so that the average loss per participant converges to some constant, μ say, as the size n of the community tends to infinity. Then, we let the pool retention grow proportionally to the number of participants and we denote as α its growth rate.

If $\alpha < \mu$ then it turns out that the limiting individual retention is given by the expected value of the Esscher transformed loss (with negative order related to the rate of increase α of the global retention level) and that the limiting individual cash-back is null since the individual retention is not sufficient to cover the expected value of the loss. As a consequence, the individual stop-loss premium for the upper layer protection is asymptotically equal to the difference between these values.

If $\alpha > \mu$ then we must distinguish two cases. Either the individual loss distributions possess moment generating functions and the limiting retentions are still given by the expected values of the Esscher transformed losses, now with positive order related to α. Or individual loss distributions have a heavier tail and we cannot use moment generating functions anymore. For losses with regularly varying tail distributions, it is shown that limiting retentions are equal to the expected losses supplemented with a quantity proportional to the difference $\alpha - \mu$. In this case, the individual retention is larger than the expected value of the loss, and a cash-back mechanism is necessary to restore fairness in the individual participations to the pool.

We also study the probability that the stop-loss protection is activated, that is, the probability that the total loss experienced by the pool exceeds the retention $n\alpha$. Again, the solution depends on the position of α with respect to asymptotic average loss μ per participant. If $\alpha < \mu$, that is, if the community ultimately retains less than expected losses, then this probability tends to 1. In a sufficiently large pool, the stop-loss protection is thus almost surely activated. On the contrary, if $\alpha > \mu$ then this probability tends to 0 so that the pool is able to cover the entire losses in that case, provided it can attract infinitely many participants. The rate of convergence of this probability to 0 is established. It is shown that this rate is exponential when losses possess moment generating functions whereas it may be much slower when the tails of individual loss distributions are heavier (with regularly varying tail distributions, here).

The remainder of this paper is organized as follows. Section 2 describes the P2P insurance scheme where the lower layer is shared among participants whereas the upper layer is transferred to a (re-)insurer by means of a stop-loss cover. In Section 3 we provide the limit of the individual retentions as the size of the P2P insurance community tends to infinity. The asymptotic behavior of the probability that the total loss attains the stop-loss layer is also studied there. In Section 4, we replace ex-post contributions by participants with ex-ante payments combined with end-of-period cash-back mechanism restoring fairness. The asymptotic values for individual ex-ante contributions and ex-post cash-backs are derived under this alternative P2P insurance system. The proofs are gathered in appendix. For two positive functions g_1 and g_2 defined in a neighborhood of infinity, we write $g_1 \sim g_2$ provided $\lim_{x\to\infty} g_1(x)/g_2(x) = 1$ in the remainder of the text. Also, we write $g_1 = o(g_2)$ provided $\lim_{x\to\infty} g_1(x)/g_2(x) = 0$.

2 P2P insurance scheme with stop-loss protection

2.1 Assumptions

Consider n participants to an insurance pool, numbered $i = 1, 2, \ldots, n$. Each of them faces a risk X_i. By risk, we mean a non-negative random variable representing monetary losses caused by some insurable peril over one period (a calendar year, say). Throughout the paper, we assume that X_1, X_2, X_3, ... are mutually independent, valued in $[0, \infty)$ and obey zero-augmented absolutely continuous distributions, that is, $\mathrm{P}[X_i = 0] > 0$ and X_i possesses a probability density function $f_{X_i|X_i>0}$ over $(0,\infty)$. We denote

$$\mu_i = \mathrm{E}[X_i] > 0 \text{ and } \sigma_i^2 = \mathrm{Var}[X_i] > 0$$

the mean and the variance of X_i, respectively. Both μ_i and σ_i^2 are assumed to be finite throughout the paper. We voluntarily exclude the cases where no randomness is present, that is, $\mu_i = 0 \Leftrightarrow \sigma_i^2 = 0 \Leftrightarrow X_i = 0$ with probability 1.

Henceforth, we use the notation

$$S_n = \sum_{i=1}^{n} X_i$$

for the total loss of the pool. The mean and variance of S_n are denoted as

$$m_n = \mathrm{E}[S_n] = \sum_{i=1}^{n} \mu_i \text{ and } s_n^2 = \mathrm{Var}[S_n] = \sum_{i=1}^{n} \sigma_i^2,$$

respectively. Throughout this paper, we assume that the conditional expectations $\mathrm{E}[X_i|S_n]$ are continuously increasing in S_n for all $i \in \{1, 2, \ldots, n\}$ so that the functions $s \mapsto \mathrm{E}[X_i|S_n = s]$ are one-to-one. As established by Denuit and Robert (2020c), this is generally the case when n is sufficiently large under the assumptions retained in this paper. Thus, the random variables $\mathrm{E}[X_i|S_n]$ are comonotonic. We refer the reader to Dhaene et al. (2002a,b) for an overview of comonotonicity and of its applications in actuarial science.

2.2 Conditional mean risk sharing

In a risk pooling scheme, each participant contributes ex-post an amount $h_{i,n}(s)$ where $s = \sum_{i=1}^{n} x_i$ is the sum of the realizations $x_1, x_2, \ldots, x_n$ of $X_1, X_2, \ldots, X_n$ and $\sum_{i=1}^{n} h_{i,n}(s) = s$. In the design of a scheme, it is important that the sharing rule represented by the functions $h_{i,n}$ is both intuitively acceptable and transparent. In that respect, the conditional mean risk sharing (or allocation) $h_{i,n}^{\star}$ proposed by Denuit and Dhaene (2012) is particularly attractive. Recall that this allocation is defined as

$$h_{i,n}^{\star}(S_n) = \mathrm{E}[X_i|S_n], \;\; i = 1, 2, \ldots, n. \tag{2.1}$$

Clearly, the conditional mean risk sharing (2.1) allocates the full risk S_n as we obviously have

$$\sum_{i=1}^{n} h_{i,n}^{\star}(S_n) = \sum_{i=1}^{n} \mathrm{E}[X_i|S_n] = S_n.$$

In words, participant i must contribute the expected value of the loss X_i he or she brings to the pool, given the total loss S_n experienced by the entire community. Stated differently, the contribution of each participant is the average part of the total loss S_n that can be attributed to the loss he or she brings to the pool. Explained in this way, the risk sharing $h_{i,n}^{\star}$ appears to be very intuitive and easily understandable by the members of the P2P insurance community.

If the random variables X_1, X_2, …, X_n are assumed to be independent and identically distributed, it is well known that

$$h^\star_{i,n}(s) = \mathrm{E}[X_i|S_n = s] = \frac{s}{n}. \tag{2.2}$$

The total loss S_n is thus uniformly allocated to all participants in that particular case, each of them contributing the same proportion $1/n$ of the total losses. The homogeneity assumption is however very restrictive for applications and the conditional mean risk sharing (2.1) extends (2.2) to heterogeneous losses X_i so that each participant supports the right share of the total loss S_n.

2.3 P2P insurance scheme

The P2P insurance community has only limited risk-bearing capacity and must thus be protected by a stop-loss arrangement with priority w_n. This means that the community covers the first layer $(0, w_n)$ of the total losses S_n experienced by all participants whereas the upper layer (w_n, ∞) is transferred to a (re-)insurer. Here, the community only retains $\min\{S_n, w_n\}$ and w_n acts as a pooled deductible.

To determine the contribution of each participant to the price of the stop-loss protection, $\mathrm{E}[(S_n - w_n)_+]$ (excluding here for simplicity any loading), we consider the system proposed by Denuit (2020). Since we have assumed that the functions $h^\star_{i,n}$ were continuous and increasing, there exist $w_{1,n}, \ldots, w_{n,n}$ such that $\sum_{i=1}^n w_{i,n} = w_n$ and the identities

$$(w_n - S_n)_+ = \sum_{i=1}^n \big(w_{i,n} - h^\star_{i,n}(S_n)\big)_+ \text{ and } (S_n - w_n)_+ = \sum_{i=1}^n \big(h^\star_{i,n}(S_n) - w_{i,n}\big)_+ \tag{2.3}$$

both hold true for any w_n such that $0 < F_{S_n}(w_n) < 1$. Under the assumptions retained throughout this paper as listed in Section 2.1, we can define the individual retention levels $w_{i,n}$ appearing in (2.3) by

$$w_{i,n} = F^{-1}_{h^\star_{i,n}(S_n)}\big(F_{S_n}(w_n)\big) = h^\star_{i,n}(w_n) = \mathrm{E}[X_i|S_n = w_n], \tag{2.4}$$

such that the individual stop-loss premium is given by $\mathrm{E}[(h^\star_{i,n}(S_n) - w_{i,n})_+]$.

Identities (2.3) also allow us to compute the respective contributions for each participant to the retained loss $\min\{S_n, w_n\}$. Specifically, we get

$$\min\{S_n, w_n\} = w_n - (w_n - S_n)_+ = \sum_{i=1}^n \min\{h^\star_{i,n}(S_n), w_{i,n}\}. \tag{2.5}$$

The loss $\min\{S_n, w_n\}$ retained by the pool is thus distributed among its members according to formula (2.5). Precisely, participant i contributes to the lower layer $(0, w_n)$ an amount

$$\min\{h^\star_{i,n}(S_n), w_{i,n}\} = \min\{\mathrm{E}[X_i|S_n], \mathrm{E}[X_i|S_n = w_n]\}$$

payable at the end of the period, where $w_{i,n}$ is given in (2.4). It is clear from the formulas given here that $w_{i,n}$ can be seen as the individual retention level for participant i in a pool of size n.

In this pooling system, contributions are paid ex-post by participants, once the realized loss S_n is known. The P2P insurance operation can thus be decomposed into

- a random ex-post contribution $\min\{h^\star_{i,n}(S_n), w_{i,n}\}$ to the lower layer $(0, w_n)$,

- a deterministic contribution $\mathrm{E}[(h^\star_{i,n}(S_n) - w_{i,n})_+]$ to the upper layer, excluding the loading,

- in exchange of the reimbursement of the loss X_i.

3 Retention capacity within a large P2P insurance community

In this section, we are interested in the individual retention levels $w_{i,n}$ given in (2.4) when the number n of participants gets large. It is natural to let w_n increase with n. In the remainder of the paper, we will assume that the retention level w_n is proportional to the number n of participants, that is, $w_n = \alpha n$ with $\alpha > 0$.

We proceed as follows. First, we consider the particular case of losses X_i for which conditional mean risk sharing reduces to proportional mean risk sharing and we derive the limit of $w_{i,n}$ in that case. The results obtained in that particular case can be extended to general risks with the help of the Esscher transform, whose definition is recalled below. Two separate cases must be distinguished, according to whether w_n exceeds the limiting average loss of the pool or not. In the latter situation, the limit always exists whereas in the former case, it depends on the thickness of the tails of the loss distribution. As an illustration, we consider light-tailed losses with moment generating function and heavy-tailed ones (with regularly varying tails).

3.1 Risks supporting a proportional allocation

As an introductory example, let us consider risks such that the conditional mean risk sharing rule $h^\star_{i,n}$ coincides with the proportional mean sharing rule defined as

$$h^{\mathrm{prop}}_{i,n}(S_n) = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} S_n, \quad i = 1, 2, \ldots, n.$$

This is for instance the case in the semi-homogeneous collective risk model, where $X_1, X_2, \ldots, X_n$ obey compound Poisson distributions with identically distributed severities, or when $X_1, X_2, \ldots, X_n$ are independent and identically distributed as it can be seen from (2.2). If we moreover assume that there exists $\mu > 0$ such that $\lim_{n\to\infty} m_n/n = \mu$, then

$$h^\star_{i,n}(\alpha n) = \mathrm{E}[X_i | S_n = \alpha n] = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} \alpha n = \frac{\alpha \mathrm{E}[X_i]}{m_n/n} \underset{n\to\infty}{\to} \frac{\alpha}{\mu} \mathrm{E}[X_i].$$

Let us try to provide another expression of this limit exploiting the fact that $h^\star_{i,n}(S_n)$ is proportional to S_n. Let us define the size-biased transform of a risk which appears to be useful to study the conditional mean risk sharing rule, as pointed out in Denuit (2019). Given a non-negative random variable X_i with distribution function F_{X_i} and strictly positive expected value $\mathrm{E}[X_i]$, define $\widetilde{X}_i$ with distribution function

$$\mathrm{P}[\widetilde{X}_i \leq t] = \frac{\mathrm{E}\big[X_i \mathrm{I}[X_i \leq t]\big]}{\mathrm{E}[X_i]}$$

where $\mathrm{I}[\cdot]$ is the indicator function, equal to 1 if the event appearing within the brackets is realized and to 0 otherwise. The risk $\widetilde{X}_i$ is called a size-biased version of X_i. Assume that the risks X_i are absolutely continuous with density function f_{X_i}. When the conditional mean risk sharing rule coincides with the proportional mean sharing rule, we can derive from Proposition 2.2 in Denuit (2019) that

$$f_{S_n - X_i + \widetilde{X}_i}(s) = f_{\widetilde{S}_n}(s) \tag{3.1}$$

where $\widetilde{X}_i$ is independent of $X_1, \ldots, X_n$. Let us denote by $\mathrm{m}_{X_i}(h) = \mathrm{E}[\mathrm{e}^{hX_i}]$ the moment generating function of X_i. If the argument h is negative then m_{X_i} coincides with the Laplace transform, which always exist since X_i is non-negative. Define

$$\mathcal{H}_i = \{h \in \mathbb{R} | \mathrm{m}_{X_i}(h) < \infty\},\ \mathcal{H}_{1,n} = \bigcap_{i=1}^{n} \mathcal{H}_i, \text{ and } \mathcal{H}_\infty = \bigcap_{i=1}^{\infty} \mathcal{H}_i.$$

From (3.1), we deduce that, for $h \in \mathcal{H}_{1,n}$,

$$\mathrm{m}_{S_n - X_i + \widetilde{X}_i}(h) = \frac{\mathrm{m}_{S_n}(h)\, \mathrm{m}_{\widetilde{X}_i}(h)}{\mathrm{m}_{X_i}(h)} = \mathrm{m}_{\widetilde{S}_n}(h) \;\Leftrightarrow\; \frac{\mathrm{m}_{\widetilde{X}_i}(h)}{\mathrm{m}_{X_i}(h)} = \frac{\mathrm{m}_{\widetilde{S}_n}(h)}{\mathrm{m}_{S_n}(h)}.$$

Since $\mathrm{m}_{\widetilde{X}_i}(h) = \mathrm{m}'_{X_i}(t)/\mathrm{E}[X_i]$, this is also equivalent to

$$\frac{1}{\mathrm{E}[X_i]} \frac{\mathrm{m}'_{X_i}(h)}{\mathrm{m}_{X_i}(h)} = \frac{1}{\mathrm{E}[S_n]} \sum_{i=1}^{n} \frac{\mathrm{m}'_{X_i}(h)}{\mathrm{m}_{X_i}(h)}.$$

Now assume that, there exists a moment generating function m_∞ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{\mathrm{m}'_{X_i}(h)}{\mathrm{m}_{X_i}(h)} = \frac{\mathrm{m}'_\infty(h)}{\mathrm{m}_\infty(h)} \text{ uniformly on } \mathcal{H}_\infty,$$

then it follows that

$$\mathrm{E}[X_i | S_n = \alpha n] = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]} \alpha n = \alpha \frac{\mathrm{m}'_{X_i}(h)}{\mathrm{m}_{X_i}(h)} \frac{1}{\frac{1}{n}\sum_{i=1}^{n} \frac{\mathrm{m}'_{X_i}(h)}{\mathrm{m}_{X_i}(h)}} \underset{n\to\infty}{\to} \alpha \frac{\mathrm{m}'_{X_i}(h)}{\mathrm{m}_{X_i}(h)} \frac{1}{\frac{\mathrm{m}'_\infty(h)}{\mathrm{m}_\infty(h)}},$$

uniformly on $\mathcal{H}_\infty$. Finally, if there exists $h_\alpha \in \mathcal{H}_\infty$ such that

$$\frac{\mathrm{m}'_\infty(h_\alpha)}{\mathrm{m}_\infty(h_\alpha)} = \alpha, \tag{3.2}$$

then we deduce that

$$\lim_{n\to\infty} \mathrm{E}[X_i | S_n = \alpha n] = \frac{\mathrm{m}'_{X_i}(h_\alpha)}{\mathrm{m}_{X_i}(h_\alpha)}. \tag{3.3}$$

It turns out that the limit in (3.3) is the expected value of the Esscher transform of X_i, of order h_α. The next section recalls the definition of this actuarial tool. It is then shown that representation (3.3) is generally valid, not only for risks supporting proportional mean risk sharing.

3.2 Esscher transform

For $h \in \mathcal{H}_i$, let us associate to the individual loss X_i its Esscher transformed version $X_i^{(h)}$ of order h with distribution function $F_{X_i^{(h)}}$ defined as

$$\mathrm{d}F_{X_i^{(h)}}(x) = \frac{\mathrm{e}^{hx}}{\mathrm{m}_{X_i}(h)} \mathrm{d}F_{X_i}(x).$$

The operator mapping the distribution function F_{X_i} of X_i to the distribution function $F_{X_i^{(h)}}$ of $X_i^{(h)}$ is called the Esscher transform. The Esscher transform is a powerful tool in actuarial science where it has been used to approximate the distribution of the aggregate claims of an insurance portfolio, for premium calculation as well as option pricing. Outside actuarial circles, it is also known as the exponential tilting of a distribution. We refer the reader to Denuit et al. (2005) for an introduction to Esscher transform.

Compared to the initial loss X_i, the Esscher transformed loss $X_i^{(h)}$ with $h > 0$ has the same support but the probabilities assigned to small values are reduced in favor of the probabilities assigned to large values. This makes $X_i^{(h)}$ "larger" compared to X_i. The opposite conclusion is reached if $h < 0$. In fact, it is easy to see that the ratio $\mathrm{d}F_{X_i^{(h_1)}}/\mathrm{d}F_{X_i^{(h_2)}}$ is increasing for $h_1 > h_2$ so that the Esscher transformed loss $X_i^{(h)}$ stochastically increases with h. We refer the reader to Denuit et al. (2005, Chapter 3) for more details concerning stochastic order relations expressing the idea of "being larger than" for random variables. In particular, the expected value of $X_i^{(h)}$ given by

$$\mathrm{E}[X_i^{(h)}] = \int_0^\infty \frac{x\mathrm{e}^{hx}}{\mathrm{m}_{X_i}(h)} \mathrm{d}F_{X_i}(x) = \frac{\mathrm{m}'_{X_i}(h)}{\mathrm{m}_{X_i}(h)}$$

indeed appears in (3.3). Since $\mathrm{E}[X_i^{(h)}]$ increases with h, we have $\mathrm{E}[X_i^{(h)}] < \mathrm{E}[X_i]$ for $h < 0$ while $\mathrm{E}[X_i] > \mathrm{E}[X_i^{(h)}]$ holds for $h > 0$.

3.3 Individual retentions and Esscher transforms

As in Section 3.1, we assume that there exists a moment generating function m_∞ defined on $\mathcal{H}_\infty$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln\left(\mathrm{m}_{X_i}(h)\right) = \ln\left(\mathrm{m}_\infty(h)\right) \text{ for } h \in \mathcal{H}_\infty \tag{3.4}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{\mathrm{m}'_{X_i}(h)}{\mathrm{m}_{X_i}(h)} = \frac{\mathrm{m}'_\infty(h)}{\mathrm{m}_\infty(h)} \text{ uniformly on } \mathcal{H}_\infty. \tag{3.5}$$

Let h_α such that (3.2) holds true. We denote

$$\mu_{i,\alpha} = \mathrm{E}[X_i^{(h_\alpha)}] \text{ and } \sigma_{i,\alpha}^2 = \mathrm{Var}[X_i^{(h_\alpha)}],$$

and let

$$m_{n,\alpha} = \sum_{i=1}^{n} \mu_{i,\alpha} \text{ and } s_{n,\alpha}^2 = \sum_{i=1}^{n} \sigma_{i,\alpha}^2.$$

We need to assume that there exist $\gamma > 0$ and $0 < \beta < \infty$ such that for all $n \geq 1$,

$$s_{n,\alpha}^2 \geq n\gamma, \; \mathrm{E}[|X_n^{(h_\alpha)} - \mu_{n,\alpha}|^{2+\delta}] \leq \beta \text{ for some } 0 < \delta \leq 1, \tag{3.6}$$

and that for every $T > 0$

$$\int_{|t|>T} \prod_{j=1}^{n} \left| \mathrm{E}[\mathrm{e}^{\mathrm{i}tX_j^{(h_\alpha)}}] \right| \mathrm{d}t = O\left(s_{n,\alpha}^{-(2+\delta)}\right). \tag{3.7}$$

The last assumption is about the rate of convergence of $m_{n,\alpha}/n$ to α: we need the condition

$$|\alpha n - m_{n,\alpha}| = o\left(s_{n,\alpha}^2\right). \tag{3.8}$$

Let us now briefly comment on these conditions. Conditions (3.4) and (3.5) have been seen as to be necessary to link α with the Esscher transforms of individual losses. Conditions (3.6), (3.7) and (3.8) are given in Theorem 4 in Zabell (1993) for establishing a rate of convergence of the conditional expectation of a random variable to its expectation given the value of the sum of other random variables as the number of terms of the sums tends to infinity. Note that Theorem 4 in Zabell (1993) needs the finiteness of at least the first second moments of $X_i^{(h_\alpha)}$ (and also the third moment for $X_i^{(h_\alpha)}$ when considering $w_i(\alpha)$), but in our case the existence of the Esscher transforms is sufficient for the existence of such moments. Finally condition (3.8) says that the heterogeneity within the expectations of $X_i^{(h_\alpha)}$ can not be too strong.

The next result establishes that the limiting value (3.3) of $w_{i,n}$ when n tends to infinity is generally valid when risks X_i possess Esscher transforms.

Proposition 3.1. *Under the assumptions listed in Section 2.1, we suppose that conditions* (3.4) *and* (3.5) *are also satisfied. If there exists h_α such that* (3.2), (3.6), (3.7) *and* (3.8) *hold true, then* (3.3) *is valid, that is,*

$$w_i(\alpha) = \lim_{n\to\infty} \mathrm{E}[X_i | S_n = \alpha n] = \mathrm{E}[X_i^{(h_\alpha)}]. \tag{3.9}$$

The proof of Proposition 3.1 is given in Appendix A. The key point of the proof is that

$$\mathrm{E}[X_i | S_n = \alpha n] = \mathrm{E}[X_i^{(h)} | S_n^{(h)} = \alpha n] \text{ for any } h \in \mathcal{H}_{1,n}.$$

This identity extends to the heterogeneous case previous results obtained by Van Campenhout and Cover (1981) in the homogeneous case. In fact, we can also show that

$$\lim_{n\to\infty} \mathrm{P}[X_i \leq x | S_n = \alpha n] = \mathrm{P}[X_i^{(h_\alpha)} \leq x]$$

which corresponds to Theorem 2 in Van Campenhout and Cover (1981) in the particular case where losses X_i are identically distributed.

Assumptions (3.4)-(3.5) involved in Proposition 3.1 imply that there exists $\mu > 0$ such that

$$\lim_{n\to\infty} \frac{m_n}{n} = \mu.$$

If $\alpha < \mu$ then h_α solving (3.2) is negative and the result stated under Proposition 3.1 is generally valid, whatever the tail of individual losses X_i. If $\alpha > \mu$ then $h_\alpha > 0$ and the limiting result established in Proposition 3.1 requires the existence of a moment generating function. There is thus a need for another approach when losses have heavier tails, as discussed next.

3.4 Regularly varying tail distributions

Let us now consider the case $\alpha > \mu$ for losses with heavier tails so that there is no h_α solving (3.2). In this section, we assume that losses X_i have regularly varying tail distributions with common index $\gamma > 3$. More specifically we assume that

$$\overline{F}_{X_i|X_i>0}(x) = \mathrm{P}[X_i > x|X_i > 0] = x^{-\gamma} L_i(x) \tag{3.10}$$

where L_i are slowly varying functions. Moreover we assume that there exist a survival function $\overline{F}$ and constants $\delta_i > 0$, $i = 1, 2, \ldots$ such that

$$\overline{F}_{X_i}(x) \sim \delta_i \overline{F}(x) \tag{3.11}$$

and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \overline{F}_{X_i}(x)}{n\overline{F}(x)} = 1 \tag{3.12}$$

uniformly for $x \geq x_0$, for some $x_0 > 0$. We will also need that

$$\lim_{n\to\infty} \frac{m_n}{n} = \mu > 0 \tag{3.13}$$

and for some $1 < q \leq 2$

$$\sum_{n=1}^{\infty} n^{-q} \mathrm{E}[|X_n - \mu_n|^q] < \infty. \tag{3.14}$$

Conditions (3.11), (3.12), (3.13) and (3.14) are used in Theorem 3.1 in Lu et al. (2013) to prove that, for any fixed $\kappa > 0$

$$\mathrm{P}[S_n - m_n > x] \sim n\overline{F}(x)$$

uniformly for $x \geq \kappa n$, when the non-negative random variables X_i are independent with consistently varying tails. It is known that distributions with regularly varying tails are also distributions with consistently varying tails. But we need the assumptions of regularly varying tail distribution because we have to prove the convergence of the ratio of two density functions (see the proof of Proposition 3.2) which is more intricate than for the survival functions and necessitates stronger conditions. Section 4 in Lu et al. (2013) provides an example of Pareto-type distribution functions for which (3.12) holds true. It is easy to extend this example when $F_{X_i|X_i>0}$ are Pareto-type distribution functions and $\mathrm{P}[X_i = 0] > 0$.

The next result summarizes the limiting behavior of $w_{i,n}$ for such losses.

Proposition 3.2. *Assume that individual losses X_i have distribution functions F_{X_i} such that* (3.11) *and* (3.12) *hold true and that the conditions* (3.13) *and* (3.14) *are satisfied. Then, for any $\alpha > \mu$*

$$w_i(\alpha) = \mathrm{E}[X_i] + (\alpha - \mu)\frac{\delta_i}{1 - \gamma^{-1}}. \tag{3.15}$$

The proof of Proposition 3.2 is given in Appendix B. It is noteworthy that $w_i(\alpha)$ now depends linearly on α.

3.5 Probability of activating the stop-loss protection

Let us consider the probability that S_n exceeds the retention level $w_n = n\alpha$ in a large pool. If $\alpha < \mu$ then this probability tends to 1 whereas it tends to 0 if $\alpha > \mu$. As in Section 3.3, we assume that (3.4) and (3.5) hold true, and that there exists a constant h_α such that (3.2) holds true, but we will also need the condition that

$$\lim n\Big(n^{-1}\ln \mathrm{m}_{S_n}(h_\alpha) - \ln \mathrm{m}_\infty(h_\alpha)\Big) = 0. \tag{3.16}$$

Moreover we suppose that there exists a constant $\sigma_\alpha^2 > 0$ such that

$$\sigma_\alpha^2 = \lim_{n\to\infty} \frac{1}{n} s_{n,\alpha}^2. \tag{3.17}$$

Finally we introduce conditions similar to the conditions of Theorem 2 in Petrov (1956) to obtain uniform approximations of the probability density functions of sums of independent random variables: there exists a positive constant G such that, for all n, we have

$$\sum_{i=1}^{n} \mathrm{E}\big[|X_i^{(h_\alpha)} - \mu_{i,\alpha}|^3\big] \le nG, \tag{3.18}$$

and there exists a constant $\varepsilon \in (0, \sigma_\alpha^2/24G)$ such that the characteristic functions $t \mapsto \mathrm{E}[\mathrm{e}^{\mathrm{i}tX_j^{(h_\alpha)}}]$ of $X_1^{(h_\alpha)}, X_2^{(h_\alpha)}, \ldots$ satisfy

$$\int_{|t|>\varepsilon} \prod_{j=1}^{n} \Big|\mathrm{E}[\mathrm{e}^{\mathrm{i}tX_j^{(h_\alpha)}}\Big| \mathrm{d}t = O\left(\frac{1}{n}\right). \tag{3.19}$$

The next result establishes the asymptotic behavior of tail probabilities $\mathrm{P}[S_n \ge \alpha n]$ for $\alpha > \mu$ when the moment generating function exists.

Proposition 3.3. *Let $\alpha > \mu$. Assume that conditions* (3.4), (3.5), (3.2), (3.16), (3.17), (3.18) *and* (3.19), *hold true. If* $\lim_{n\to\infty} n\,(\alpha - m_{n,\alpha}/n) = 0$, *then*

$$\mathrm{P}[S_n \ge \alpha n] \sim \frac{1}{\sqrt{2\pi n}} \frac{1}{h_\alpha \sigma_\alpha} \mathrm{e}^{-\eta_\alpha n},$$

where

$$\eta_\alpha = \alpha h_\alpha - \ln\left(\mathrm{m}_\infty(h_\alpha)\right). \tag{3.20}$$

The proof of Proposition 3.3 is given in Appendix C. We can see that the probability that the total losses S_n hit the upper layer (w_n, ∞) tends to 0 at exponential rate.

Let us now consider losses with heavier tails. Precisely, we consider the case of regularly varying tail loss distributions as in Section 3.4.

Proposition 3.4. *Let $\alpha > \mu$. Under the assumptions of Proposition 3.2,*

$$\mathrm{P}[S_n \ge \alpha n] \sim n\overline{F}\big((\alpha - \mu)n\big),$$

where the tail function $\overline{F}$ is defined in (3.12).

The proof of Proposition 3.4 is actually included in the proof of Proposition 3.2, as it can be seen from equation (B.1). We note that the probability that the total losses S_n hit the upper layer (w_n, ∞) now tends to 0 at hyperbolic rate.

4 Individual stop-loss premiums and cash-backs within a large P2P insurance community

The P2P insurance system discussed so far operates ex-post. It seems nevertheless reasonable to ask participants to pay a provision $\pi_{i,n}$ ex-ante and to fairly distribute the possible surplus among them, ex-post. The total amount of provision $\pi_{i,n}$ paid ex-ante by participant i is decomposed into

$$\pi_{i,n} = \pi_{i,n}^{\mathrm{P2P}} + \pi_{i,n}^{\mathrm{SL}}$$

where

$$\pi_{i,n}^{\mathrm{P2P}} = w_{i,n} = h_{i,n}^{\star}(w_n) = \mathrm{E}[X_i|S_n = w_n]$$

is that part of the total contribution paid by participant i covering the first layer $(0, w_n)$ shared within the P2P community and

$$\pi_{i,n}^{\mathrm{SL}} = \mathrm{E}\left[\left(h_{i,n}^{\star}(S_n) - w_{i,n}\right)_+\right]$$

is the contribution to the pure premium of the stop-loss protection. This ensures that the community has enough financial resources to cover the lower layer, since

$$\sum_{i=1}^{n} \pi_{i,n}^{\mathrm{P2P}} = w_n \text{ and } \sum_{i=1}^{n} \pi_{i,n}^{\mathrm{SL}} = \mathrm{E}[(S_n - w_n)_+].$$

To restore fairness, the surplus $(w_n - S_n)_+$ is shared ex-post among participants who may decide to give it to a charity. Notice that allocating the surplus is important in case a cash-back mechanism operates (the amount being reimbursed to the participant by the end of the coverage period or used to reduce next-year's contributions) but also if it is given to a charity because participants often have the choice among several projects to fund. If the community decides to distribute $(w_n - S_n)_+$ among participants then this amount must be split among participants according to formula (2.3), that is, participant i receives a bonus equal to

$$B_{i,n} = \left(\pi_{i,n}^{\mathrm{P2P}} - h_{i,n}^{\star}(S_n)\right)\mathrm{I}[S_n \leq w_n] = \left(w_{i,n} - h_{i,n}^{\star}(S_n)\right)_+.$$

To sum up, the operation can be decomposed into

- a deterministic, ex-ante payment $\pi_{i,n}^{\mathrm{P2P}} = w_{i,n}$,
- a random, ex-post cashback/benefit $B_{i,n}$,
- a deterministic contribution $\pi_{i,n}^{\mathrm{SL}}$ to the upper layer, excluding the loading,
- in exchange of the reimbursement of the loss X_i.

The next result establishes the asymptotic behavior of individual ex-ante contributions $\pi_{i,n}^{\mathrm{P2P}}$ and $\pi_{i,n}^{\mathrm{SL}}$, as well as of ex-post cash-backs $B_{i,n}$ and of the probability that the total loss hits the upper layer protected by the stop-loss treaty as n is large. As before, the discussion is with respect to the rate of increase α of the the retention level $w_n = \alpha n$ and the asymptotic average loss μ per participant. Recall that $\lim_{n\to\infty} \pi_{i,n}^{\mathrm{P2P}} = w_i(\alpha)$.

Proposition 4.1. *We assume that conditions A', B and C of Proposition 3.6 in Denuit and Robert (2020b) are satisfied. For a retention $w_n = \alpha n$, we can face two situations*

Case 1: *if $\alpha < \mu$, under the assumptions of Proposition 3.1, then*

$$\begin{aligned}\lim_{n\to\infty} \mathrm{P}\left[S_n \geq w_n\right] &= 1 \\ \lim_{n\to\infty} \pi_{i,n}^{\mathrm{P2P}} &= \mathrm{E}[X_i^{(h_\alpha)}] < \mathrm{E}[X_i] \\ \lim_{n\to\infty} \pi_{i,n}^{\mathrm{SL}} &= \mathrm{E}[X_i] - w_i\left(\alpha\right) \\ \lim_{n\to\infty} B_{i,n} &= 0 \textit{ with probability } 1.\end{aligned}$$

Case 2: *if $\alpha > \mu$, under the assumptions of Propositions 3.1, 3.2 and that $\sup_n \mathrm{E}[|\mathrm{E}[X_i|S_n]|^{1+\varepsilon}] < \infty$ for some $\varepsilon > 0$, then*

$$\begin{aligned}\lim_{n\to\infty} \mathrm{P}\left[S_n \geq w_n\right] &= 0 \\ \lim_{n\to\infty} \pi_{i,n}^{\mathrm{P2P}} &= \begin{cases} \mathrm{E}[X_i^{(h_\alpha)}] \\ \mathrm{E}[X_i] + (\alpha - \mu)\frac{\delta_i}{1-\gamma} \end{cases} \\ &> \mathrm{E}[X_i] \\ \lim_{n\to\infty} \pi_{i,n}^{\mathrm{SL}} &= 0 \\ \lim_{n\to\infty} B_{i,n} &= w_i\left(\alpha\right) - \mathrm{E}[X_i] \textit{ with probability } 1.\end{aligned}$$

The proof of Proposition 4.1 is given in Appendix D. This result describes the P2P insurance operation within an infinitely large pool. Depending on the growth rate α of the retention level w_n, we see that either the limit of $\pi_{i,n}^{\mathrm{SL}}$ is strictly positive and there is no ex-post cash-back when $\alpha < \mu$ or the limit of $\pi_{i,n}^{\mathrm{SL}}$ is zero and there is a positive ex-post cash back when $\alpha > \mu$.

Remark 4.2. If $\alpha = \mu$ and $|m_n - \mu n| = o\left(\sqrt{n}\right)$, then we would have, under conditions such that a central-limit theorem holds, that $\lim_{n\to\infty} \pi_{i,n}^{\mathrm{P2P}} = \mathrm{E}[X_i]$, $\lim_{n\to\infty} \mathrm{P}\left[S_n \geq w_n\right] = 1/2$ and $\lim_{n\to\infty} \pi_{i,n}^{\mathrm{SL}} = 0$. We would also deduce that $\lim_{n\to\infty} B_{i,n} = 0$ with probability 1. This case can therefore be viewed as an intermediary case.

References

- Abdikerimova, S., Feng, R. (2019). Peer-to-Peer multi-risk insurance and mutual aid. Available at SSRN: https://ssrn.com/abstract=3505646

- Clemente, G. P., Marano, P. (2020). The broker model for peer-to-peer insurance: An analysis of its value. The Geneva Papers on Risk and Insurance – Issues and Practice, in press.

- Denuit, M. (2019). Size-biased transform and conditional mean risk sharing, with application to P2P insurance and tontines. ASTIN Bulletin 49, 591-617.

- Denuit, M. (2020). Investing in your own and peers' risks: The simple analytics of P2P insurance. European Actuarial Journal, in press.

- Denuit, M., Dhaene, J. (2012). Convex order and comonotonic conditional mean risk sharing. Insurance: Mathematics and Economics 51, 265-270.

- Denuit, M., Dhaene, J., Goovaerts, M.J., Kaas, R. (2005). Actuarial Theory for Dependent Risks: Measures, Orders and Models. Wiley, New York.

- Denuit, M., Robert, C.Y. (2020a). Large-loss behavior of conditional mean risk sharing. ASTIN Bulletin, in press.

- Denuit, M., Robert, C.Y. (2020b). From risk sharing to pure premium for a large number of heterogeneous losses. Submitted for publication.

- Denuit, M., Robert, C.Y. (2020c). Efron's asymptotic monotonicity property in the Gaussian stable domain of attraction. Submitted for publication.

- Denuit, M., Robert, C.Y. (2020d). Collaborative insurance with stop-loss protection and team partitioning. Submitted for publication.

- Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., Vyncke, D. (2002a). The concept of comonotonicity in actuarial science and finance: Theory. Insurance: Mathematics and Economics 31, 3-33.

- Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., Vyncke, D. (2002b). The concept of comonotonicity in actuarial science and finance: Applications. Insurance: Mathematics and Economics 31, 133-161.

- Eling, M., Lehmann, M. (2018). The impact of digitalization on the insurance value chain and the insurability of risks. The Geneva Papers 43, 359-396.

- Lu, D., Song, L., Xu, Y. (2013). Precise large deviations for sums of independent random variables with consistently varying tails. Communications in Statistics - Theory and Methods 43, 28-43.

- Petrov, V.V. (1956). A local theorem for densities of sums of independent random variables. Theory of Probability and Its Applications 1, 316-322.

- Van Campenhout, J., Cover, T. (1981). Maximum entropy and conditional probability. IEEE Transactions on Information Theory 27, 483-489.

- Zabell, S.L. (1993). A limit theorem for expectations conditional on a sum. Journal of Theoretical Probability 6, 267-283.

Appendix: Proofs of the main results

A Proof of Proposition 3.1

Without loss of generality, we only focus on the case where $i=1$. Let us first show that, for $h\in\mathcal{H}_{1,n}$,

$$\mathrm{E}[X_1|S_n=\alpha n]=\mathrm{E}[X_1^{(h)}|S_n^{(h)}=\alpha n].$$

First note that $X_i^{(h)}$ has a zero-augmented absolutely continuous distributions, that is, $\mathrm{P}[X_i^{(h)}=0]=\mathrm{P}[X_i=0]/\mathrm{m}_{X_i}(h)>0$ where

$$\mathrm{m}_{X_i}(h)=\mathrm{P}[X_i=0]+\mathrm{P}[X_i>0]\mathrm{m}_{X_i|X_i>0}(h)$$

and $X_i^{(h)}$ possesses a probability density function $f_{X_i^{(h)}|X_i^{(h)}>0}$ over $(0,\infty)$ such that

$$f_{X_i^{(h)}|X_i^{(h)}>0}(x)=\frac{\mathrm{e}^{hx}}{\mathrm{m}_{X_i|X_i>0}(h)}f_{X_i|X_i>0}(x).$$

Then, we have

$$\begin{aligned}
&\mathrm{E}\left[X_1|S_n=\alpha n\right]\\
=&\int_0^{\alpha n}x\mathrm{d}F_{X_1|S_n=\alpha n}(x)\\
=&\frac{\mathrm{P}[X_1>0]\mathrm{P}\left[S_n-X_1>0\right]}{\mathrm{P}\left[S_n>0\right]}\int_0^{\alpha n}x\frac{f_{X_1|X_1>0}(x)\,f_{S_n-X_1|S_n-X_1>0}(\alpha n-x)}{f_{S_n|S_n>0}(\alpha n)}\mathrm{d}x\\
=&\frac{\mathrm{P}[X_1>0]\mathrm{m}_{X_1|X_1>0}(h)\,\mathrm{P}\left[S_n-X_1>0\right]\mathrm{m}_{S_n-X_1}(h)}{\mathrm{P}\left[S_n>0\right]\mathrm{m}_{S_n|S_n>0}(h)}\\
&\times\int_0^{\alpha n}x\frac{(\mathrm{e}^{hx}f_{X_1}(x)/\mathrm{m}_{X_1|X_1>0}(h))(\mathrm{e}^{h(\alpha n-x)}f_{S_n-X_1}(\alpha n-x)/\mathrm{m}_{S_n-X_1}(h))}{\mathrm{e}^{h\alpha n}f_{S_n}(\alpha n)/\mathrm{m}_{S_n|S_n>0}(h)}\mathrm{d}x\\
=&\frac{(\mathrm{m}_{X_1}(h)-\mathrm{P}[X_1=0])(\mathrm{m}_{S_n-X_1}(h)-\mathrm{P}[S_n-X_1=0])}{(\mathrm{m}_{S_n}(h)-\mathrm{P}[S_n=0])}\\
&\times\int_0^{\alpha n}x\frac{f_{X_1^{(h)}|X_1^{(h)}>0}(x)\,f_{S_n^{(h)}-X_1^{(h)}|S_n^{(h)}-X_1^{(h)}>0}(\alpha n-x)}{f_{S_n^{(h)}|S_n^{(h)}>0}(\alpha n)}\mathrm{d}x\\
=&\frac{\mathrm{m}_{X_1}(h)\,\mathrm{P}[X_1^{(h)}>0]\mathrm{m}_{n-X_1}(h)\,\mathrm{P}[S_n^{(h)}-X_1^{(h)}>0]}{\mathrm{m}_{S_n}(h)\,\mathrm{P}[S_n^{(h)}>0]}\\
&\times\int_0^{\alpha n}x\frac{f_{X_1^{(h)}|X_1^{(h)}>0}(x)\,f_{S_n^{(h)}-X_1^{(h)}|S_n^{(h)}-X_1^{(h)}>0}(\alpha n-x)}{f_{S_n^{(h)}|S_n^{(h)}>0}(\alpha n)}\mathrm{d}x\\
=&\frac{\mathrm{P}[X_1^{(h)}>0]\mathrm{P}[S_n^{(h)}-X_1^{(h)}>0]}{\mathrm{P}[S_n^{(h)}>0]}\int_0^{\alpha n}x\frac{f_{X_1^{(h)}|X_1^{(h)}>0}(x)\,f_{S_n^{(h)}-X_1^{(h)}|S_n^{(h)}-X_1^{(h)}>0}(\alpha n-x)}{f_{S_n^{(h)}|S_n^{(h)}>0}(\alpha n)}\mathrm{d}x\\
=&\mathrm{E}[X_1^{(h)}|S_n^{(h)}=\alpha n],
\end{aligned}$$

as announced.

Let h_α be such that (3.2) holds true. We have

$$\mathrm{E}\left[X_1|S_n=\alpha n\right]=\mathrm{E}[X_1^{(h_\alpha)}|S_n^{(h_\alpha)}=\alpha n]=\mathrm{E}[X_1^{(h_\alpha)}|S_n^{(h_\alpha)}=m_{n,\alpha}+(\alpha n-m_{n,\alpha})].$$

Since (3.6), (3.7) and (3.8) hold true, we deduce from Theorem 4 in Zabell (1993) that

$$\lim_{n\to\infty}\mathrm{E}[X_1|S_n=\alpha n]=\mathrm{E}[X_1^{(h_\alpha)}],$$

as announced. This ends the proof.

B Proof of Proposition 3.2

Again, we only focus on the case where $i=1$. We split the proof into three parts.

i) Following the proof of Proposition 3.1 in Lu et al. (2013) (taking $x=an-m_n$), we have, for $a>\mu$, and $0<\theta<1<\lambda$

$$\lambda^{-\gamma}=\liminf_{n\to\infty}\frac{\overline{F}\left(\lambda(a-\mu)n\right)}{\overline{F}\left((a-\mu)n\right)}\leq\liminf_{n\to\infty}\frac{\mathrm{P}\left[S_n\geq an\right]}{n\overline{F}\left((a-\mu)n\right)}$$

and

$$\limsup_{n\to\infty}\frac{\mathrm{P}\left[S_n\geq an\right]}{n\overline{F}\left((a-\mu)n\right)}\leq\limsup_{n\to\infty}\frac{\overline{F}\left(\theta(a-\mu)n\right)}{\overline{F}\left((a-\mu)nx\right)}=\theta^{-\gamma}$$

It follows in particular that

$$\lim_{n\to\infty}\frac{\mathrm{P}\left[S_n\geq an\right]}{n\overline{F}\left((a-\mu)n\right)}=1, \tag{B.1}$$

letting λ and θ tend to 1.

Let $T_{1,n}=S_n-X_1+\widetilde{X}_1$. Following the same reasoning as in the proof of Proposition 3.1 in Lu et al. (2013), we can show that for any $\kappa_1>0$, there exists $N_0>0$, such that for $n>N_0$

$$\begin{aligned}\mathrm{P}\left[T_{1,n}\geq an\right] \;\geq\; & \left(\overline{F}_{\widetilde{X}_1}\left(\lambda(a-\mu)n\right)+\sum_{k=2}^{n}\overline{F}_{X_i}\left(\lambda(a-\mu)n\right)\right)\\ & \times\left((1-\kappa_1)-\left(\overline{F}_{\widetilde{X}_1}\left(\lambda(a-\mu)n\right)+\sum_{k=2}^{n}\overline{F}_{X_i}\left(\lambda(a-\mu)n\right)\right)\right).\end{aligned}$$

Since (by Karamata theorem)

$$\overline{F}_{\widetilde{X}_1}\left(x\right)=\frac{1}{\mathrm{E}\left[X_1\right]}\int_x^{\infty}u\mathrm{d}F_{X_1}\left(u\right)\sim\frac{x}{\mathrm{E}\left[X_1\right](1-\gamma^{-1})}\overline{F}_{X_1}\left(x\right)\sim\frac{x\delta_1}{\mathrm{E}\left[X_1\right](1-\gamma^{-1})}\overline{F}\left(x\right),$$

we have

$$\begin{aligned}\liminf_{n\to\infty}\frac{\mathrm{P}\left[T_{1,n}\geq an\right]}{n\overline{F}\left((a-\mu)n\right)} &\geq \left(1+\frac{(a-\mu)\delta_1}{\mathrm{E}\left[X_1\right]\left(1-\gamma^{-1}\right)}\right)\liminf_{n\to\infty}\frac{\overline{F}\left(\lambda(a-\mu)n\right)}{\overline{F}\left((a-\mu)n\right)}\\ &= \left(1+\frac{(a-\mu)\delta_1}{\mathrm{E}\left[X_1\right]\left(1-\gamma^{-1}\right)}\right)\lambda^{-\gamma}.\end{aligned}$$

For the upper bound, we can also derive using the same arguments as in the proof of Proposition 3.1 in Lu et al. (2013) that

$$\begin{aligned}\limsup_{n\to\infty}\frac{\mathrm{P}\left[T_{1,n}\geq an\right]}{n\overline{F}\left((a-\mu)n\right)} &\leq \left(1+\frac{(a-\mu)\delta_1}{\mathrm{E}\left[X_1\right]\left(1-\gamma^{-1}\right)}\right)\limsup_{n\to\infty}\frac{\overline{F}\left(\theta(a-\mu)n\right)}{\overline{F}\left((a-\mu)nx\right)}\\ &= \left(1+\frac{(a-\mu)\delta_1}{\mathrm{E}\left[X_1\right]\left(1-\gamma^{-1}\right)}\right)\theta^{-\gamma}.\end{aligned}$$

It follows in particular that

$$\lim_{n\to\infty}\frac{\mathrm{P}\left[T_{1,n}\geq an\right]}{n\overline{F}\left((a-\mu)n\right)}=\left(1+\frac{(a-\mu)\delta_1}{\mathrm{E}\left[X_1\right]\left(1-\gamma^{-1}\right)}\right), \tag{B.2}$$

letting λ and θ tend to 1.

ii) We assume that $\mathrm{P}\left[X_i=0\right]=0$ for all $i\geq 1$. Let $0<\kappa<1<\beta$ such that $\kappa\lambda<\beta\theta$. We consider the ratio

$$\frac{\mathrm{P}\left[T_{1,n}\in an[\kappa,\beta]\right]}{\mathrm{P}\left[S_n\in an[\kappa,\beta]\right]}.$$

By Cauchy's mean value theorem, there exists a constant $\delta_n\in[\kappa,\beta]$ such that

$$\frac{\mathrm{P}\left[T_{1,n}\in an[\kappa,\beta]\right]}{\mathrm{P}\left[S_n\in an[\kappa,\beta]\right]}=\frac{f_{T_{1,n}}\left(\delta_n an\right)}{f_{S_n}\left(\delta_n an\right)}.$$

We have

$$\frac{\mathrm{P}\left[T_{1,n}\in an[\kappa,\beta]\right]}{\mathrm{P}\left[S_n\in an[\kappa,\beta]\right]}=\frac{\mathrm{P}\left[T_{1,n}\geq\kappa an\right]-\mathrm{P}\left[T_{1,n}\geq\beta an\right]}{n\overline{F}\left((a-\mu)n\right)}\frac{n\overline{F}\left((a-\mu)n\right)}{\mathrm{P}\left[S_n\geq\kappa an\right]-\mathrm{P}\left[S_n\geq\beta an\right]}.$$

Let

$$B\left(\kappa,\beta,\theta,\lambda\right)=\frac{(\kappa\lambda)^{-\gamma}-(\beta\theta)^{-\gamma}}{(\kappa\theta)^{-\gamma}-(\beta\lambda)^{-\gamma}}$$

and note that $0<B\left(\kappa,\beta,\theta,\lambda\right)<1$. Using Step i), we have

$$\liminf_{n\to\infty}\frac{\mathrm{P}\left[T_{1,n}\in an[\kappa,\beta]\right]}{\mathrm{P}\left[S_n\in an[\kappa,\beta]\right]}\geq\left(1+\frac{(a-\mu)\delta_1}{\mathrm{E}\left[X_1\right]\left(1-\gamma\right)}\right)B\left(\kappa,\beta,\theta,\lambda\right)$$

and

$$\limsup_{n\to\infty}\frac{\mathrm{P}\left[T_{1,n}\in an[\kappa,\beta]\right]}{\mathrm{P}\left[S_n\in an[\kappa,\beta]\right]}\leq\left(1+\frac{(a-\mu)\delta_1}{\mathrm{E}\left[X_1\right]\left(1-\gamma\right)}\right)B^{-1}\left(\kappa,\beta,\theta,\lambda\right).$$

Let us choose $\beta = 1/\kappa$ and $\lambda = 1/\theta$ to simplify the functional form of B, we have

$$B\left(\kappa, \kappa^{-1}, \theta, \theta^{-1}\right) = \frac{1 - \kappa^{2\gamma}\theta^{-2\gamma}}{1 - \kappa^{2\gamma}\theta^{2\gamma}}.$$

If now, we reparametrize with $\kappa = 1 - u$ and $\theta = 1 - v$ such that $0 < v < u < 1$, we have, if $u \to 0$ and $v \to 0$ such that $v = o\left(u\right)$

$$B\left(1 - u, (1 - u)^{-1}, 1 - v, (1 - v)^{-1}\right) \to 1.$$

We can therefore conclude that

$$\lim_{n\to\infty} \frac{f_{T_{1,n}}\left(an\right)}{f_{S_n}\left(an\right)} = \left(1 + \frac{(a - \mu)\delta_1}{\mathrm{E}\left[X_1\right]\left(1 - \gamma^{-1}\right)}\right).$$

Using Proposition 2.2 i) in Denuit (2019), we finally deduce that

$$\lim_{n\to\infty} \mathrm{E}\left[X_1 | S_n = an\right] = \mathrm{E}\left[X_1\right] + \frac{\delta_1}{\left(1 - \gamma^{-1}\right)}(a - \mu).$$

iii) We assume that $0 < \mathrm{P}\left[X_i = 0\right] < 1$. Note that $T_{1,n}$ is strictly postive and absolutely continuous, and that S_n has a zero-augmented absolutely continuous distributions such that

$$\mathrm{P}\left[S_n = 0\right] = \prod_{i=1}^{n} \mathrm{P}\left[X_i = 0\right] < 1.$$

Cauchy's mean value theorem provides the existence of a constant $\delta_n \in [\kappa, \beta]$ such that

$$\frac{\mathrm{P}\left[T_{1,n} \in an[\kappa, \beta]\right]}{\mathrm{P}\left[S_n \in an[\kappa, \beta]\right]} = \frac{f_{T_{1,n}}\left(\delta_n an\right)}{\mathrm{P}\left[S_n > 0\right] f_{S_n|S_n>0}\left(\delta_n an\right)}$$

and Proposition 2.2 iii) in Denuit (2019) that, for $s > 0$,

$$\mathrm{E}\left[X_1 | S_n = s\right] = \mathrm{E}\left[X_1\right] \frac{f_{T_{1,n}}\left(s\right)}{\mathrm{P}\left[S_n > 0\right] f_{S_n|S_n>0}\left(s\right)}. \tag{B.3}$$

Therefore was can conclude in the same way as in Step ii).

C Proof of Proposition 3.3

Let h_α and η_α be such that (3.2) and (3.20) hold true. We have

$$\mathrm{P}\left[S_n \geq \alpha n\right] = \mathrm{e}^{-\eta_{\alpha,n} n} \mathrm{E}\Big[\exp\big(-h_\alpha(S_n^{(h_\alpha)} - \alpha n)\big)\mathrm{I}[S_n^{(h_\alpha)} \geq \alpha n]\Big]$$

where

$$\eta_{\alpha,n} = \alpha h_\alpha - n^{-1} \log \mathrm{m}_{S_n}\left(h_\alpha\right).$$

Note that $\lim_{n\to\infty} \eta_{\alpha,n} = \eta_\alpha$ and that $\left(|\eta_{\alpha,n} - \eta_\alpha|\right) = o\left(n^{-1}\right)$ by condition (3.16).

Recall that $m_{n,\alpha} = \sum_{i=1}^n \mu_{i,\alpha}$ and $s_{n,\alpha}^2 = \sum_{i=1}^n \sigma_{i,\alpha}^2$ where $\mu_{i,\alpha} = \mathrm{E}[X_i^{(h_\alpha)}]$ and $\sigma_{i,\alpha}^2 = \mathrm{Var}[X_i^{(h_\alpha)}]$. Let

$$Z_{n,\alpha} = \frac{S_n^{(h_\alpha)} - m_{n,\alpha}}{s_{n,\alpha}}$$

and

$$H_{n,\alpha}(x) = \mathrm{P}[Z_{n,\alpha} \le x].$$

We have

$$\begin{aligned} & \mathrm{E}\big[\exp\big(-h_\alpha(S_n^{(h_\alpha)} - \alpha n)\big)\mathrm{I}[S_n^{(h_\alpha)} \ge \alpha n]\big] \\ = & \; \mathrm{e}^{h_\alpha n(\alpha - m_{n,\alpha}/n)}\mathrm{E}\big[\exp\big(-h_\alpha s_{n,\alpha} Z_{n,\alpha}\big)\mathrm{I}[Z_{n,\alpha} \ge (\alpha n - m_{n,\alpha})/s_{n,\alpha}]\big] \\ = & \; \mathrm{e}^{h_\alpha n(\alpha - m_{n,\alpha}/n)} \int_{(n\alpha - m_{n,\alpha})/s_{n,\alpha}}^{\infty} \mathrm{e}^{-h_\alpha s_{n,\alpha} x} \mathrm{d}H_{n,\alpha}(x) \end{aligned}$$

by (3.8). By Theorem 2 in Petrov (1956), for all n sufficiently large the derivative of $H_{n,\alpha}$ exists and

$$H'_{n,\alpha}(x) = \varphi(x) + O\left(n^{-1/2}\right)$$

for some constant η where φ is the probability density function of the standard Gaussian distribution. Thus with $g_n(x) = e^{-h_\alpha s_{n,\alpha} x/\sqrt{n}}$ we have

$$\begin{aligned} & \mathrm{E}\big[\exp\big(-h_\alpha(S_n^{(h_\alpha)} - \alpha n)\big)\mathrm{I}[S_n^{(h_\alpha)} \ge \alpha n]\big] \\ = & \; \mathrm{e}^{h_\alpha n(\alpha - m_{n,\alpha}/n)} \int_{(n\alpha - m_{n,\alpha})/s_{n,\alpha}}^{\infty} g_n\left(n^{1/2}x\right) \mathrm{d}H_{n,\alpha}(x) \\ = & \; \mathrm{e}^{h_\alpha n(\alpha - m_{n,\alpha}/n)} \int_{(n\alpha - m_{n,\alpha})/s_{n,\alpha}}^{\infty} g_n\left(n^{1/2}x\right) \varphi(x)\, \mathrm{d}x + o\left(n^{-1/2}\right) \\ = & \; \mathrm{e}^{h_\alpha n(\alpha - m_{n,\alpha}/n)} \frac{1}{\sqrt{n}} \int_{n^{1/2}(n\alpha - m_{n,\alpha})/s_{n,\alpha}}^{\infty} g_n(x)\, \varphi\left(x/\sqrt{n}\right) \mathrm{d}x + o\left(n^{-1/2}\right) \\ \sim & \; \frac{1}{\sqrt{2\pi n}} \int_0^\infty \mathrm{e}^{-h_\alpha \sigma_\alpha x} \mathrm{d}x \sim \frac{1}{\sqrt{2\pi n}} \frac{1}{h_\alpha \sigma_\alpha} \end{aligned}$$

and therefore

$$\mathrm{P}[S_n \ge \alpha n] \sim \frac{1}{\sqrt{2\pi n}} \frac{1}{h_\alpha \sigma_\alpha} \mathrm{e}^{-\eta_\alpha n},$$

as announced.

D Proof of Proposition 4.1

Under conditions A', B and C of Proposition 3.6 in Denuit and Robert (2020b), we have

$$\lim_{n\to\infty} \mathrm{E}[X_i|S_n] = \mathrm{E}[X_i] \text{ with probability 1.}$$

Moreover

$$\mathrm{E}[X_i|S_n] = w_{i,n} - \big(w_{i,n} - \mathrm{E}[X_i|S_n]\big)\mathrm{I}[S_n \le w_n] + \big(\mathrm{E}[X_i|S_n] - w_{i,n}\big)\mathrm{I}[S_n \ge w_n]$$

and therefore

$$\mathrm{E}[X_i] = w_{i,n} - \mathrm{E}[(w_{i,n} - \mathrm{E}[X_i|S_n])\mathrm{I}[S_n \le w_n]] + \pi_{i,n}^{\mathrm{SL}}.$$

Case 1: if $\alpha < \mu$, we have from Proposition 3.1

$$\lim_{n\to\infty} \pi_{i,n}^{\mathrm{P2P}} = \lim_{n\to\infty} w_{i,n} = \mathrm{E}[X_i^{(h_\alpha)}] = w_i(\alpha).$$

Since

$$B_{i,n} = \big(\pi_{i,n}^{\mathrm{P2P}} - \mathrm{E}[X_i|S_n]\big)\mathrm{I}[S_n \le w_n],$$

and $\mathrm{P}[S_n \le w_n]$ converges to 0 with exponential rate, we deduce that

$$\lim_{n\to\infty} B_{i,n} = 0 \text{ with probability } 1,$$

and that

$$\lim_{n\to\infty} \pi_{i,n}^{\mathrm{SL}} = \mathrm{E}[X_i] - w_i(\alpha).$$

Case 2: if $\alpha > \mu$, we have from Proposition 3.1 and Proposition 3.2

$$\lim_{n\to\infty} \pi_{i,n}^{\mathrm{P2P}} = \begin{cases} \mathrm{E}[X_i^{(h_\alpha)}] \\ \mathrm{E}[X_i] + (\alpha-\mu)\frac{\delta_i}{1-\gamma} \end{cases}$$

and from Proposition 3.3 and Proposition 3.4

$$\lim_{n\to\infty} \mathrm{P}[S_n \ge w_n] = 0.$$

Since $\sup_n \mathrm{E}[|\mathrm{E}[X_i|S_n]|^{1+\varepsilon}] < \infty$, we deduce that $\sup_n \mathrm{E}[|w_{i,n} - \mathrm{E}[X_i|S_n]|^{1+\varepsilon}] < \infty$, and that $(w_{i,n} - \mathrm{E}[X_i|S_n])_n$ are uniformy integrable. It follows that

$$\lim_{n\to\infty} \mathrm{E}[(w_{i,n} - \mathrm{E}[X_i|S_n])\mathrm{I}[S_n \le w_n]] = w_i(\alpha) - \mathrm{E}[X_i]$$

and that $\lim_{n\to\infty} \pi_{i,n}^{\mathrm{SL}} = 0$. Finally

$$\lim_{n\to\infty} B_{i,n} = w_i(\alpha) - \mathrm{E}[X_i] \text{ with probability } 1$$

since $\lim_{n\to\infty} \mathrm{I}[S_n \le w_n] = 1$ with probability 1 because $\sum_{n=0}^{\infty} \mathrm{P}[S_n \ge w_n] < \infty$.