# POLYNOMIAL SERIES EXPANSIONS AND MOMENT APPROXIMATIONS FOR CONDITIONAL MEAN RISK SHARING OF INSURANCE LOSSES

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# POLYNOMIAL SERIES EXPANSIONS AND MOMENT APPROXIMATIONS FOR CONDITIONAL MEAN RISK SHARING OF INSURANCE LOSSES

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**Abstract**

This paper exploits the representation of the conditional mean risk sharing allocations in terms of size-biased transforms to derive effective approximations within insurance pools of limited size. Precisely, the probability density functions involved in this representation are expanded with respect to the Gamma density and its associated Laguerre orthonormal polynomials, or with respect to the Normal density and its associated Hermite polynomials when the size of the pool gets larger. Depending on the thickness of the tails of the loss distributions, the latter may be replaced with their Esscher transform (or exponential tilting) of negative order. The numerical method then consists in truncating the series expansions to a limited number of terms. This results in an approximation in terms of the first moments of the individual loss distributions. Compound Panjer-Katz sums are considered as an application. The proposed method is compared with the well-established Panjer recursive algorithm. It appears to provide the analyst with reliable approximations that can be used to tune system parameters, before performing exact calculations.

**Keywords**: conditional expectation, size-biased transform, Esscher transform, exponential tilting, Laguerre polynomials, Hermite polynomials.

# 1 Introduction

In this paper, we consider the conditional mean risk allocation of independent losses, as defined by Denuit and Dhaene (2012). According to this rule, each participant to an insurance pool contributes the conditional expectation of the loss brought to the pool, given the total loss experienced by the entire pool. The properties of this risk allocation rule have been studied by Denuit (2019, 2020b) and Denuit and Robert (2020a,b,2021a,b,c), including applications to peer-to-peer insurance schemes. These papers demonstrate the strong potential of the conditional mean risk sharing rule in these emerging insurance markets. It is thus important that individual contributions can be effectively computed. Large-pool approximations have been obtained by Denuit and Robert (2021a). The present paper proposes a fast approximation method based on orthonormal polynomials that can be used as a first evaluation within small to moderately large pools in order to tune the parameters of the system (individual deductibles, upper layer limit, etc.) before performing exact calculation once their optimal values have been selected.

Orthonormal polynomials expansion for probability density functions consist in expressing an intricate probability density function of interest as a series involving special polynomials that are orthonormal with respect to some reference measure. If the tails of the density under considerations are too heavy, its Esscher transform (or exponential tilting) with negative order can be used instead. Using the Gamma distribution as reference measure, the target probability density function is expressed as a series involving Laguerre polynomials. Alternatively, the Normal distribution can be used with its associated Hermite polynomials. This approach has been successfully applied in actuarial applications e.g. by Jin et al. (2014), Goffard et al. (2016), Nadarajah et al. (2016), Asmussen et al. (2018) and Goffard and Laub (2020).

This paper applies orthonormal expansions to evaluate individual allocations of independent losses within an insurance pool. The representation formula for the conditional expectation of an individual risk given the aggregate loss established by Denuit (2019) is exploited here to derive approximations for the individual contributions to the pool. Precisely, the probability density functions appearing in this formula are expanded in terms of Laguerre polynomials and Gamma density. When the number of participants increases, the Normal approximation and its associated Hermite polynomials can be used instead. The series can then be truncated for numerical evaluation. The reference density no more appears in the approximation that only involves a limited number of moments and is therefore computationally effective. As an application, we consider losses modeled by compound Panjer-Katz sums, to which Panjer recursion applies. The numerical illustrations demonstrate that the proposed method is both reasonably accurate and fast, making it a good candidate for approximating individual contributions within insurance pools at an early stage of the analysis. These approximations can be used to set the values of different parameters involved in the design of the collaborative insurance scheme (selection of individual deductibles, upper layer limit, etc.). Exact calculations of individual allocations can then be carried out in the final stage, once optimal parameter values have been selected.

The remainder of this paper is organized as follows. Section 2 recalls the definition of the conditional mean risk sharing rule for individual losses. After having presented the expansion of a probability density function in terms of a reference distribution and its associated orthonormal polynomials, Section 3 applies this tool to evaluate the conditional mean risk allocations in the light-tailed case. Section 4 then replaces the loss distribution with its Esscher transform (or exponential tilting) of negative order to deal with the heavier-tailed case. As an application, compound Panjer-Katz sums are considered in Section 5, where numerical illustrations demonstrate the accuracy of the proposed approach. The final Section 6 discusses the results.

# 2 Conditional mean risk sharing rule

Consider $n$ participants to an insurance pool, numbered $i = 1, 2, \ldots, n$. Each of them faces a risk $X_i$. By risk, we mean a non-negative random variable representing monetary losses caused by some insurable peril over one period (a calendar year, say). Throughout the paper, we assume that $X_1$, $X_2$, $X_3$, ... are mutually independent, valued in $[0, \infty)$ and obey zero-augmented absolutely continuous distributions. Unless stated otherwise, we assume that the probability mass at zero is strictly positive, that is, $\mathrm{P}[X_i = 0] > 0$ and we denote as $f_{X_i|X_i>0}$ the probability density function of $X_i$ over $(0, \infty)$. This representation corresponds to the individual risk model and is widely applicable in property and casualty insurance studies.

Let $S = \sum_{i=1}^{n} X_i$ be the total loss of the pool, to be distributed ex-post among the $n$ participants. According to the conditional mean risk sharing (or allocation) $h_i^\star$ proposed by Denuit and Dhaene (2012), participant $i$ must contribute an amount

$$h_i^\star(S) = \mathrm{E}[X_i|S], \ \ i = 1, 2, \ldots, n,$$

to the total loss $S$. In words, participant $i$ must contribute the expected value of the loss $X_i$ brought to the pool, given the total loss $S$ experienced by the pool.

The conditional mean risk sharing rule decomposes $X_i$ into an aleatory part $X_i - \mathrm{E}[X_i|S]$ shared among participants by virtue of the mutuality (or random solidarity) principle and a structural part $\mathrm{E}[X_i|S]$ to be supported by participant $i$, individually. Formally, the risk $X_i$ brought by participant $i$ is decomposed into

$$X_i = \underbrace{\mathrm{E}[X_i|S]}_{=\text{structural part}} + \underbrace{X_i - \mathrm{E}[X_i|S]}_{=\text{random departure } \mathcal{E}_i}.$$

Both terms entering this split are uncorrelated and random departures $\mathcal{E}_i$ have zero means and always sum to 0. With collaborative insurance, each participant must contribute ex-post the amount $\mathrm{E}[X_i|S]$ to the pool whereas $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ are re-allocated among them according to the mutuality principle.

In order to design an attractive collaborative insurance scheme, the distribution of the individual contribution $h_i^\star(S)$ is needed. This allows the actuary to evaluate the impact of introducing individual deductibles or to determine the optimal reinsurance program protecting the community from large losses that cannot be retained within the pool, for instance. A direct calculation of $h_i^\star(s)$ requires the distributions of $S$, which can be time consuming. Denuit and Robert (2021a) derived accurate approximations within larger pools. In the next sections, we derive effective approximations based on the first few moments of the individual losses within small to moderately large insurance pools. The light-tailed case is first discussed. Then, an extension to the heavier-tailed case is proposed using the Esscher transform (or exponential tilting) of negative order.

# 3 Polynomial expansion to conditional mean risk allocations in the light-tailed case

## 3.1 Orthonormal polynomial expansion to probability density functions

Let $Z$ be a random variable with density function $f_Z$ (Radon-Nikodym derivative) with respect to some dominating measure $\zeta$ (typically Lebesgue measure on an interval or counting measure on a subset of integers). Assume that no explicit expression is available for the density $f_Z$, for instance because $Z$ is the sum of independent random variables and direct convolution is computationally expensive. If the distribution of $Z$ is expected to be close to some probability measure $\nu$ absolutely continuous with respect to $\zeta$ with density $f_\nu$ then $f_Z$ could be approximated by $f_\nu$ corrected in an appropriate way. This can be achieved using an expansion with the help of polynomials that are orhononormal with respect to $\nu$, as explained next.

Assume that all moments of $\nu$ are finite and denote as $\mathcal{L}^2_\nu$ the space of all square-integrable functions with respect to $\nu$. The usual inner product is $\langle g, h\rangle_\nu = \int gh\mathrm{d}\nu$ and the corresponding norm is $\|g\|^2_\nu = \langle g, g\rangle_\nu$. Let $\{p_k, k = 0, 1, 2, \ldots\}$ be a sequence of orthonormal polynomials with respect to $\nu$, that is, such that

$$\langle p_k, p_l\rangle_\nu = \begin{cases} 1 \text{ if } k = l \\ 0 \text{ otherwise,} \end{cases}$$

for any positive integers $k$ and $l$.

If there exists $\eta > 0$ such that

$$\int \mathrm{e}^{\eta|x|}\mathrm{d}\nu\left(x\right) < \infty,$$

then the sequence of polynomials $\{p_k, k = 0, 1, 2, \ldots\}$ forms an orthonormal basis of $\mathcal{L}^2_\nu$. Therefore, if $f_Z/f_\nu \in \mathcal{L}^2_\nu$, that is, if

$$\int \frac{f_Z^2}{f_\nu^2}\mathrm{d}\nu = \int \frac{f_Z^2}{f_\nu}\mathrm{d}\zeta < \infty$$

then the polynomial representation of the density of $Z$ with respect to $\nu$ follows from orthogonal projection so that we have

$$f_Z\left(x\right)/f_\nu\left(x\right) = \sum_{k=0}^{\infty} \langle f_Z/f_\nu, p_k\rangle_\nu\, p_k\left(x\right)$$

which gives the series expansion

$$f_Z\left(x\right) = \sum_{k=0}^{\infty} c_k p_k\left(x\right) f_\nu\left(x\right) \tag{3.1}$$

where the coefficients $c_k$ are given by

$$c_k = \langle f_Z/f_\nu, p_k \rangle_\nu = \int f_Z p_k \mathrm{d}\zeta = \mathrm{E}[p_k(Z)].$$

If $p_k(x) = \sum_{l=0}^{k} a_{lk} x^l$ then $c_k = \sum_{l=0}^{k} a_{lk} \mathrm{E}[Z^l]$. This shows that expansion (3.1) only requires the knowledge of moments of $Z$.

Expansion (3.1) is useful for the numerical evaluation of $f_Z$ if the coefficients $c_k$ tend to 0 fast enough (so that the series can be truncated to its first terms with sufficient accuracy). The Parseval relationship

$$\sum_{k=0}^{\infty} c_k^2 = \|f_Z/f_\nu\|_\nu^2$$

ensures that the coefficients $c_k$ tend to 0 as $k$ tends to infinity. Expansion (3.1) thus lends itself to numerical approximation by truncating the series to its first terms. The accuracy of the approximation for a given order of truncation depends on how rapidly these coefficients decay.

Typical choices of reference distributions are ones that belong to the Natural Exponential Family with Quadratic Variance Function (NEF-QVF) which includes Normal (for absolutely continuous random variables $Z$), Gamma (for positive absolutely continuous random variables $Z$), Hyperbolic, Poisson, Binomial, and Negative Binomial (for counting random variables $Z$). See Morris (1982). The associated orthonormal polynomials are known explicitly, avoiding a time-consuming Gram-Schmidt orthogonalization procedure.

## 3.2 Application to conditional mean risk allocation

The functions $h_i^\star$ defining the conditional mean risk sharing rule can be expressed in terms of the size-biased transform. Recall that the size-biased transform of $X_i$ obeying a zero-augmented absolutely continuous distribution is a strictly positive random variable $\widetilde{X}_i$ with probability density function given by

$$f_{\widetilde{X}_i}(x) = \frac{x f_{X_i|X_i>0}(x)}{\mathrm{E}[X_i|X_i>0]}, \quad x > 0. \tag{3.2}$$

We then have the following proposition.

**Proposition 3.1.** *Define $T_i = S - X_i + \widetilde{X}_i = \sum_{j\neq i} X_j + \widetilde{X}_i$, $i = 1, 2, \ldots, n$. If $f_{T_i}/f_\nu \in \mathcal{L}_\nu^2$ for every $i \in \{1, \ldots, n\}$ then*

$$h_i^\star(s) \;=\; \frac{\mathrm{E}[X_i] \sum_{k=0}^{\infty} \mathrm{E}[p_k(T_i)] p_k(s)}{\sum_{j=1}^{n} \mathrm{E}[X_j] \sum_{k=0}^{\infty} \mathrm{E}[p_k(T_j)] p_k(s)} s \text{ for any } s > 0.$$

*Proof.* It is proved in Denuit (2019, Proposition 2.3) that if $X_1, X_2, \ldots, X_n$ obey zero-augmented absolutely continuous distributions then

$$h_i^\star(s) = \frac{\mathrm{E}[X_i] f_{T_i}(s)}{\sum_{j=1}^{n} \mathrm{E}[X_j] f_{T_j}(s)} s \text{ for any } s > 0. \tag{3.3}$$

The announced result then follows by applying expansion (3.1) to the densities $f_{T_1}, f_{T_2}, \ldots, f_{T_n}$ appearing in the numerator and denominator of (3.3). □

The densities $f_{T_j}$ appearing in (3.3) may be difficult to compute. This calculation is avoided with the approximation in terms of orthonormal polynomials with respect to $\nu$ that only requires the knowledge of the moments of $T_j$. Also, notice that $f_\nu$ disappears from the expression of $h_i^\star$ as a ratio of two integer series in Proposition 3.1.

Truncating the series to a limited number of terms, Proposition 3.1 can be used to obtain a first approximation of the functions $h_i^\star$ at an early stage of the analysis, to figure out the performances of the conditional mean risk allocation of the losses $X_i$ for given parameter values (individual deductibles, retention levels, ...). The analyst can then decide whether deductibles should be included or whether an upper layer should be transferred to a reinsurer, for instance. Once optimal parameter values have been selected, the exact calculation of $h_i^\star$ can be performed in the final implementation stage.

# 4 Polynomial expansion to conditional mean risk allocations in the heavier-tailed case

## 4.1 Esscher transform

If $f_Z/f_\nu \notin \mathcal{L}^2_\nu$, because $Z$ has a too heavy-tailed distribution, the idea is to replace $Z$ with its Esscher transform of negative order whose definition is recalled next.

Let $\mathrm{m}_Z(h) = \mathrm{E}[\mathrm{e}^{hZ}]$ be the moment generating function of $Z$ and define $\mathcal{H} = \{h \in \mathbb{R} | \mathrm{m}_Z(h) < \infty\}$. For $h \in \mathcal{H}$, the Esscher transformed version $Z^{(h)}$ of order $h$ of $Z$ has probability density function $f_Z^{(h)}$ defined as

$$f_{Z^{(h)}}(x) = \frac{\mathrm{e}^{hx}}{\mathrm{m}_Z(h)} f_Z(x).$$

The operator mapping the distribution of $Z$ to the distribution of $Z^{(h)}$ is called the Esscher transform. The Esscher transform is a powerful tool in actuarial science where it has been used to approximate the distribution of the aggregate claims of an insurance portfolio, for premium calculation as well as option pricing. Outside actuarial circles, it is also known as the exponential tilting of a distribution. We refer the reader to Denuit et al. (2005) for an introduction to Esscher transform.

Compared to $Z$, the Esscher transformed $Z^{(h)}$ with $h < 0$ has the same support but the probabilities assigned to large values are reduced in favor of the probabilities assigned to small values. This makes $Z^{(h)}$ "smaller" compared to $Z$ when $h < 0$. Hence, the condition $f_{Z^{(h)}}/f_\nu \in \mathcal{L}^2_\nu$ might be fulfilled for an appropriate value of $h$.

If $f_Z/f_\nu \notin \mathcal{L}^2_\nu$ then the idea is to replace $Z$ with $Z^{(h)}$ for some $h < 0$ such that $f_{Z^{(h)}}/f_\nu \in \mathcal{L}^2_\nu$ and to apply expansion (3.1) to $f_{Z^{(h)}}$. Precisely, starting from the expansion

$$f_{Z^{(h)}}(x) = \frac{\mathrm{e}^{hx}}{\mathrm{m}_Z(h)} f_Z(x) = \sum_{k=0}^{\infty} \mathrm{E}[p_k\left(Z^{(h)}\right)] p_k(x) f_\nu(x)$$

where $h < 0$ is chosen such that $f_{Z^{(h)}}/f_\nu \in \mathcal{L}^2_\nu$, we obtain the following expansion

$$f_Z(x) = \mathrm{e}^{-hx} \mathrm{m}_Z(h) \sum_{k=0}^{\infty} \mathrm{E}[p_k\left(Z^{(h)}\right)] p_k(x) f_\nu(x) \tag{4.1}$$

for the density of the random variable $Z$ under consideration.

The following property of the Esscher transform will be useful in the next section. Given two positive, independent random variables $Z$ and $Y$, with respective probability density functions $f_Z$ and $f_Y$, we have

$$(Y+Z)^{(h)} \stackrel{\mathrm{d}}{=} Y^{(h)} + Z^{(h)} \tag{4.2}$$

where the Esscher transformed version $Y^{(h)}$ and $Z^{(h)}$ of $Y$ and $Z$ are mutually independent. This simply follows from

$$f_{(Y+Z)^{(h)}}(x) = \frac{\mathrm{e}^{hx}}{\mathrm{m}_{Y+Z}(h)} f_{Y+Z}(x) = \int_0^x \frac{\mathrm{e}^{hy}}{\mathrm{m}_Y(h)} f_Y(y) \frac{\mathrm{e}^{h(x-y)}}{\mathrm{m}_Z(h)} f_Z(x-y)\mathrm{d}y$$

which corresponds to the convolution product of $f_{Y^{(h)}}$ and $f_{Z^{(h)}}$.

## 4.2 Application to conditional mean risk allocation

We are now in a position to state the following result, which extends Proposition 3.1 (that is recovered for $h = 0$) to the heavier-tailed case.

**Proposition 4.1.** *Define $T_i^{(h)} = \sum_{j\neq i} X_j^{(h)} + \widetilde{X}_i^{(h)}$, $i = 1, 2, \ldots, n$, where the random variables entering the sum are mutually independent. Let $h \leq 0$ be such that $f_{T_i^{(h)}}/f_\nu \in \mathcal{L}_\nu^2$ for every $i \in \{1, \ldots, n\}$. Then,*

$$h_i^\star(s) = \frac{\mathrm{E}[X_i]\mathrm{m}_{T_i}(h)\sum_{k=0}^\infty \mathrm{E}[p_k(T_i^{(h)})]p_k(s)}{\sum_{j=1}^n \mathrm{E}[X_j]\mathrm{m}_{T_j}(h)\sum_{k=0}^\infty \mathrm{E}[p_k(T_j^{(h)})]p_k(s)} s \text{ for any } s > 0.$$

*Proof.* The announced result follows by applying expansion (4.1) to the densities $f_{T_j}$ appearing in the numerator and denominator of (3.3) and by (4.2). □

**Remark 4.2.** The Esscher transform is useful in connection with the conditional mean risk sharing rule. Denuit and Robert (2021a) established that the following identity holds true for any $s > 0$:

$$\mathrm{E}[X_i|S=s] = \mathrm{E}[X_i^{(h)}|S^{(h)}=s] \text{ where } h \text{ is such that } \mathrm{m}_{X_i}(h) < \infty \text{ for all } i,$$

where $S^{(h)} = X_1^{(h)} + \ldots + X_n^{(h)}$. This provides an alternative proof for Proposition 4.1. Indeed, it suffices to apply Proposition 3.1 to $X_1^{(h)}$, ..., $X_n^{(h)}$ to get

$$\begin{aligned} h_i^\star(s) &= \mathrm{E}[X_i^{(h)}|S^{(h)}=s] \\ &= \frac{\mathrm{E}[X_i^{(h)}]\sum_{k=0}^\infty \mathrm{E}[p_k(T_i^{(h)})]p_k(s)}{\sum_{j=1}^n \mathrm{E}[X_j^{(h)}]\sum_{k=0}^\infty \mathrm{E}[p_k(T_j^{(h)})]p_k(s)} s \text{ for any } s > 0. \end{aligned}$$

This shows that $h_i^\star$ again appears to be the ratio of two integer series when the expected value of the Esscher transformed versions of the individual losses $X_i$ are available in closed form (as it will be the case when individual losses are modeled by compound Panjer-Katz sums). Now, since

$$\mathrm{E}[X_i^{(h)}] = \frac{\mathrm{m}'_{X_i}(h)}{\mathrm{m}_{X_i}(h)} \text{ and } \mathrm{m}_{\widetilde{X}_i}(h) = \frac{\mathrm{m}'_{X_i}(h)}{\mathrm{E}[X_i]},$$

we get

$$\begin{aligned}
\mathrm{E}[X_i^{(h)}] &= \frac{\mathrm{m}_{\widetilde{X}_i}(h)\mathrm{E}[X_i]}{\mathrm{m}_{X_i}(h)} \\
&= \mathrm{E}[X_i]\frac{\mathrm{m}_{\widetilde{X}_i}(h)\prod_{j\neq i}\mathrm{m}_{X_j}(h)}{\mathrm{m}_S(h)} \\
&= \mathrm{E}[X_i]\frac{\mathrm{m}_{T_i}(h)}{\mathrm{m}_S(h)}
\end{aligned}$$

so that we end up with the representation stated in Proposition 4.1.

# 5 Applications to compound Panjer-Katz sums

## 5.1 Compound Panjer-Katz losses

Denuit (2019, 2020a) and Denuit and Robert (2020a) studied size-biasing and conditional mean risk allocation when individual losses are modeled as compound Panjer-Katz sums consisting of compound Binomial, compound Poisson, and compound Negative Binomial sums. This class of distributions is central to actuarial mathematics so that the results derived in this section are of wide applicability in insurance studies. We assume that

$$X_i = \sum_{k=1}^{N_i} C_{ik} \text{ for } i = 1, 2, \ldots, n, \tag{5.1}$$

where

- the integer-valued random variables $N_i$ belongs to the Panjer-Katz class, also known as the $(a, b, 0)$ class of distributions which fulfill the following recurrence relation:

  $$\mathrm{P}[N_i = k] = \left(a_i + \frac{b_i}{k}\right)\mathrm{P}[N_i = k-1], \;\; k = 1, 2, \ldots,$$

  for some $a_i$ and $b_i$ such that $a_i + b_i \geq 0$.

- the claim severities $C_{ik}$ are positive, absolutely continuous, independent and identically distributed. Henceforth, we denote as $C_i$ a generic random variable distributed as $C_{i1}$.

## 5.2 Esscher transformation of compound Panjer-Katz sums

In the heavier-tailed case, there is a need to work with Esscher transformed distributions. The following result gives the Esscher transform of compound Panjer-Katz sums, which remains in the same family of distributions.

**Proposition 5.1.** *The Esscher transformed version $X_i^{(h)}$ of the compound sum $X_i$ in (5.1) is a compound sum $X_i^{(h)} = \sum_{k=1}^{N_i'} C_{ik}^{(h)}$ where the random variables $C_{ik}^{(h)}$ are mutually independent, distributed as $C_i^{(h)}$ and independent of the counting random variable $N_i'$ defined as follows:*

*(i) if $N_i \sim \text{Binomial}\,(m_i, q_i)$ then $N_i' \sim \text{Binomial}(m_i, q_i^{(h)})$ with*

$$q_i^{(h)} = q_i \frac{\mathrm{m}_{C_i}(h)}{1 - q_i + q_i \mathrm{m}_{C_i}(h)}.$$

*(ii) if $N_i \sim \text{Poisson}\,(\lambda_i)$ then $N_i' \sim \text{Poisson}(\lambda_i^{(h)})$ with $\lambda_i^{(h)} = \lambda_i \mathrm{m}_{C_i}(h)$.*

*(iii) if $N_i \sim$ Negative Binomial$(\alpha_i, q_i)$ then $N_i' \sim$ Negative Binomial$(\alpha_i, q_i^{(h)})$, with $q_i^{(h)} = q_i \mathrm{m}_{C_i}(h)$ where $h$ is such that $q_i \mathrm{m}_{C_i}(h) < 1$.*

*Proof.* (i) If $N_i \sim \text{Binomial}(m_i, q_i)$ then

$$\mathrm{m}_{X_i}(t) = (1 - q_i + q_i \mathrm{m}_{C_i}(t))^{m_i}$$

and

$$\mathrm{m}_{X_i^{(h)}}(t) = \frac{(1 - q_i + q_i \mathrm{m}_{C_i}(t+h))^{m_i}}{(1 - q_i + q_i \mathrm{m}_{C_i}(h))^{m_i}} = \left(1 - q_i^{(h)} + q_i^{(h)} \frac{\mathrm{m}_{C_i}(t+h)}{\mathrm{m}_{C_i}(h)}\right)^{m_i}$$

so that we find the announced result since $\mathrm{m}_{C_i^{(h)}}(t) = \frac{\mathrm{m}_{C_i}(t+h)}{\mathrm{m}_{C_i}(h)}$.

(ii) If $N_i \sim \text{Poisson}(\lambda_i)$ then

$$\mathrm{m}_{X_i}(t) = \exp\left(\lambda_i \left(\mathrm{m}_{C_i}(t) - 1\right)\right)$$

and

$$\mathrm{m}_{X_i^{(h)}}(t) = \frac{\exp\left(\lambda_i \left(\mathrm{m}_{C_i}(t+h) - 1\right)\right)}{\exp\left(\lambda_i \left(\mathrm{m}_{C_i}(h) - 1\right)\right)} = \exp\left(\lambda_i \mathrm{m}_{C_i}(h) \left(\frac{\mathrm{m}_{C_i}(t+h)}{\mathrm{m}_{C_i}(h)} - 1\right)\right)$$

which gives the announced result.

(iii) If $N_i \sim$Negative Binomial$(\alpha_i, q_i)$ then

$$\mathrm{m}_{X_i}(t) = \left(\frac{1 - q_i}{1 - q_i \mathrm{m}_{C_i}(t)}\right)^{\alpha_i}$$

and

$$\mathrm{m}_{X_i^{(h)}}(t) = \left(\frac{1 - q_i \mathrm{m}_{C_i}(h)}{1 - q_i \mathrm{m}_{C_i}(t+h)}\right)^{\alpha_i} = \left(\frac{1 - q_i^{(h)}}{1 - q_i^{(h)} \mathrm{m}_{C_i}(t+h) / \mathrm{m}_{C_i}(h)}\right)^{\alpha_i}$$

with $q_i^{(h)} = q_i \mathrm{m}_{C_i}(h)$. This ends the proof. □

## 5.3 Moments

The formulas derived in Propositions 3.1 and 4.1 only require the moments of $T_i$ and $T_i^{(h)}$ to be implemented. Let us now provide effective ways to obtain these moments for compound Panjer-Katz sums. To this end, let us introduce the following notation:

$$\mu_{i,j} = \mathrm{E}[X_i^j],\ \ \widetilde{\mu}_{i,j} = \mathrm{E}[\widetilde{X}_i^j] \text{ and } \mu_{i,j}^{(h)} = \mathrm{E}[(X_i^{(h)})^j].$$

We know from Sundt (2003) that the moments of compound Panjer-Katz sums $X_i$ can be obtained recursively from

$$\mu_{i,j} = \frac{1}{1-a_i}\sum_{k=1}^{j}\binom{j-1}{k-1}\left(a_i\frac{j}{k}+b_i\right)\mathrm{E}[C_i^k]\mu_{i,j-k} \text{ for } j = 2, 3, \ldots,$$

starting from the well-known formula

$$\mu_{i,1} = \mathrm{E}[C_i]\mathrm{E}[N_i] = \mathrm{E}[C_i]\frac{a_i+b_i}{1-a_i}.$$

We can therefore compute iteratively the moments $\mu_{i,j}$ of $X_i$. The moments $\mu_{i,j}^{(h)}$ of $X_i^{(h)}$ are obtained in a similar way thanks to the representations obtained in Proposition 5.1. Also, it is easy to see that

$$\widetilde{\mu}_{i,j} = \frac{\mu_{i,j+1}}{\mu_{i,1}}.$$

In order to get the moments of the sums $T_i$ and $T_i^{(h)} = \sum_{j\neq i} X_j^{(h)} + \widetilde{X}_i^{(h)}$ for any $i \in \{1, \ldots, n\}$, we use the formula

$$\mathrm{E}[(T_i^{(h)})^j] = \sum_{k_1+\ldots+k_n=j}\binom{j}{k_1, \ldots, k_n}\frac{\mu_{i,k_i+1}^{(h)}}{\mu_{i,1}^{(h)}}\prod_{l\neq i}\mu_{l,k_l}^{(h)}.$$

Hence, we are able to compute all the moments involved in the expression of $h_i^\star$.

## 5.4 Comparison with Panjer algorithm for discretized severities

We assume in this section that the claim severity $C_i$ has integer values (the formulas are easily adapted to the case where $C_i$ is expressed in multiples of a suitable discretization step). As a consequence $X_i$ only takes integer values as well. The size-biased transform of $X_i$ is denoted by $\widetilde{X}_i$ and is defined by

$$\mathrm{P}[\widetilde{X}_i = k] = \frac{k\mathrm{P}[X_i = k]}{\mathrm{E}[X_i]},\ \ k = 1, 2, \ldots.$$

It is proved in Denuit (2019, Proposition 2.3) that if $X_1, X_2, \ldots, X_n$ are valued in $\{0, 1, 2, \ldots\}$ then

$$h_i^\star(s) = \frac{\mathrm{E}[X_i]\mathrm{P}[S - X_i + \widetilde{X}_i = s]}{\sum_{j=1}^{n}\mathrm{E}[X_j]\mathrm{P}[S - X_j + \widetilde{X}_j = s]}s \text{ for any } s \in \{0, 1, 2, \ldots\}.$$

We have the following results from Denuit and Robert (2020a):

- the size-biased version of the compound sum $X_i$ in (5.1) with $N_i$ ∼Binomial$(m_i, q_i)$ is given by $\widetilde{X}_i \stackrel{\mathrm{d}}{=} \sum_{k=1}^{N_i'} C_{ik} + \widetilde{C}_i$ where $N_i'$ ∼Binomial$(m_i - 1, q_i)$, $C_{i1}$, ..., $C_{i,m_i-1}$ and $\widetilde{C}_i$ are mutually independent.

- the size-biased version of the compound sum $X_i$ in (5.1) with $N_i \sim$ Poisson $(\lambda_i)$ is given by $\widetilde{X}_i \stackrel{\mathrm{d}}{=} X_i + \widetilde{C}_i$ where $X_i$ and $\widetilde{C}_i$ are mutually independent.

- the size-biased version of the compound sum $X_i$ in (5.1) with $N_i$ ∼Negative Binomial$(\alpha_i, q_i)$ is given by $\widetilde{X}_i \stackrel{\mathrm{d}}{=} X_i + \widetilde{C}_i + Z_i$ where $Z_i$ is a compound Negative Binomial sum $\sum_{k=1}^{M_i} C_{ik}'$ with $M_i$ ∼Negative Binomial$(1, q_i)$ and $C_{ik}'$ distributed as $C_{ik}$, all these random variables being independent and where $X_i$ and $\widetilde{C}_i$ are mutually independent.

As a consequence, for any $i \in \{1, \ldots, n\}$, Panjer algorithm can be used to compute the probability mass functions of $X_i$ and $\widetilde{X}_i$, and therefore to compute through convolutions $h_i^\star(s)$ for any $s \in \{0, 1, 2, \ldots\}$.

## 5.5 Numerical illustrations

In this section, we first show that approximations derived from Propositions 3.1 and 4.1 deliver the exact values for $h_i^\star$ in the semi-homogeneous compound Poisson risk model. Then, we consider a situation where the approximation should work properly: compound Poisson losses with Gamma-distributed severities. We continue with the compound Binomial case where Panjer algorithm is known to suffer from numerical instability. In both cases, we keep the number of participants voluntarily low. Increasing the number of participants, Central-Limit theorem suggests to replace the Gamma reference density with the Normal one and its associated Hermite polynomials. We show that individual contributions are accurately recovered from a limited number of moments, even within moderately large pools. Since every compound Negative Binomial sum can be rewritten as a compound Poisson one (see e.g. Example 3.3.2 in Kaas et al. (2008) for more details), we restrict our numerical illustrations to the compound Poisson and compound Binomial cases.

### 5.5.1 Semi-homogeneous compound Poisson risk model

This model corresponds to individual losses $X_i$ of the form (5.1) with $N_i$ ∼Poisson$(\lambda_i)$ and $C_1, C_2, \ldots, C_n$ identically distributed, as $C$, say. Then, $\mathrm{E}[X_i] = \lambda_i \mathrm{E}[C]$ and $T_i \stackrel{\mathrm{d}}{=} S + \widetilde{C}$ so that $T_1, T_2, \ldots, T_n$ are identically distributed and $\mathrm{E}[p_k(T_i)] = \mathrm{E}[p_k(T_j)]$ for all $i$ and $j$. The formula of Proposition 3.1 then gives $h_i^\star(s) = s\lambda_i/(\lambda_1 + \ldots + \lambda_n)$ that is obtained from (3.3) as well. Truncating the series appearing in the numerator and in the denominator to any number of terms still gives the exact value in that particular case.

### 5.5.2 Gamma reference distribution with Laguerre polynomials in the light-tailed case

The natural candidate for the reference density $f_\nu$ when random variables are positive is the Gamma density with its associated Laguerre orthonormal polynomials. It has been shown by Provost (2005) that the recovery of unknown densities supported on $(0,\infty)$ from the knowledge of their moments naturally leads to approximation in terms of the Gamma density and Laguerre polynomials. See also Jin et al. (2014) and Goffard and Laub (2020) for further evidence supporting the choice of Gamma as reference distribution.

The Gamma$(r,m)$ distribution with the shape parameter $r$ and scale parameter $m$ has probability density function

$$f_\nu(x) = \frac{x^{r-1}}{\Gamma(r)\, m^r} \mathrm{e}^{-x/m},\ x \geq 0,$$

where $\Gamma(\cdot)$ denotes the Gamma function. The associated orthonormal polynomials are given by

$$p_k(x) = (-1)^k \binom{k+r-1}{k}^{-1/2} \ell_k^{r-1}\left(\frac{x}{m}\right)$$

where $\{\ell_k^{r-1}, k = 0,1,2,\ldots\}$ are the generalized Laguerre polynomials defined as

$$\ell_k^{r-1}(x) = \sum_{l=0}^{k} \binom{k+r-1}{k-l} \frac{(-x)^l}{l!}.$$

These polynomials satisfy recurrence relations but the use of this recursion to compute Laguerre polynomials of high degrees is not a good strategy. More effective methods are available, including the use of asymptotic expansions, as documented by Gil et al. (2017, 2020).

A sufficient condition for $f_Z/f_\nu \in \mathcal{L}^2_\nu$ is

$$f_Z(x) = \begin{cases} O(\mathrm{e}^{-\delta x}) \text{ as } x \to \infty \text{ with } m > \delta/2 \\ O(x^\beta) \text{ as } x \to 0 \text{ with } r < 2\beta + 1. \end{cases}$$

When $Z$ possesses a moment generating function, one can select the Gamma parameters $r$ and $m$ so that these conditions are fulfilled, as shown by Proposition 1 in Goffard and Laub (2020). These authors also establish that expansion (3.1) consists in a linear combination of Gamma densities. The coefficients are not necessarily positive so that this cannot be interpreted as a probabilistic mixture but it nevertheless eases numerical evaluations of excess probabilities or stop-loss transforms for instance.

The function `dapx_gca` of the `R` package `PDQutils` implements Gram-Charlier expansion for probability density functions and distribution functions. It supports several reference distributions, including the Gamma and its associated Laguerre polynomials. The shape and scale parameters are inferred from the first two moments of the distribution under consideration.

We choose $n = 3$ and we first consider compound Poisson losses $X_i$ with Gamma-distributed severities. Precisely, we perform calculations in the following setting:

**Participant 1:** $N_1 \sim$Poisson(0.5), $C_1 \sim$Gamma$(1,1)$;

**Participant 2:** $N_2 \sim$Poisson(1), $C_3 \sim$Gamma$(1.5,1)$;

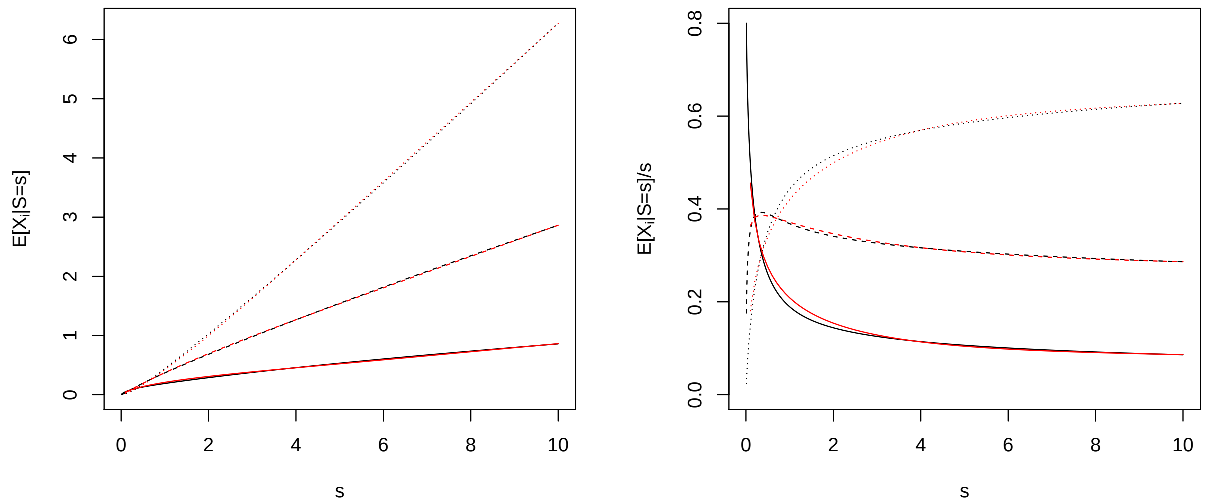


Figure 1: Left panel: Respective contributions $s \mapsto \mathrm{E}[X_i|S=s]$ for participant 1 (solid line appearing at the bottom), participant 2 (broken line appearing in the middle), and participant 3 (dotted line appearing at the top). Black curves correspond to values obtained with Panjer algorithm. Red curves correspond to values obtained with the proposed approximation based on the first 3 moments (mean, variance and skewness). Right panel: Respective relative contributions.

**Participant 3:** $N_3$ ∼Poisson$(1.5)$, $C_3$ ∼Gamma$(2,1)$.

Panjer recursive algorithm is performed with the help of the `aggregateDist` function of the `R` package `actuar` on discretized severities obtained with discretization step equal to 0.01 and local moment matching (method `unbiased` in the `discretize` function).

The functions $s \mapsto \mathrm{E}[X_i|S=s]$ are displayed in the left panel of Figure 1 for $i \in \{1,2,3\}$ and $s \in (0,10)$. The relative contributions $s \mapsto \mathrm{E}[X_i|S=s]/s$ for each individual to the total realized loss $S=s$ are displayed in the right panel of Figure 1. Black curves correspond to values obtained with Panjer algorithm, which almost exactly match exact values in that case. Red curves correspond to the proposed approximation using only 3 moments. We can see that including just the mean, variance and skewness in the approximation produces reliable values for $h_i^\star$ in this example. Moderate departures are visible in the relative contributions displayed in the right panel but the proposed moment approximation turns out to be accurate enough for practical purposes, that is, to design the collaborative insurance scheme in a preliminary stage of the analysis.

Let us now turn to compound Binomial losses with homogeneous claim severities. The expression for the conditional mean risk allocations can easily be obtained in that case (functions $h_1^\star$, $h_2^\star$, and $h_3^\star$ are equal to ratios of polynomials). Hence, we can compare approximations based on Panjer algorithm and polynomial expansions to exact values to compare their respective merits. Calculations are performed under the following assumptions:

**Participant 1:** $N_1$ ∼Binomial$(2,0.5)$, $C_1$ ∼Gamma$(1,1)$;

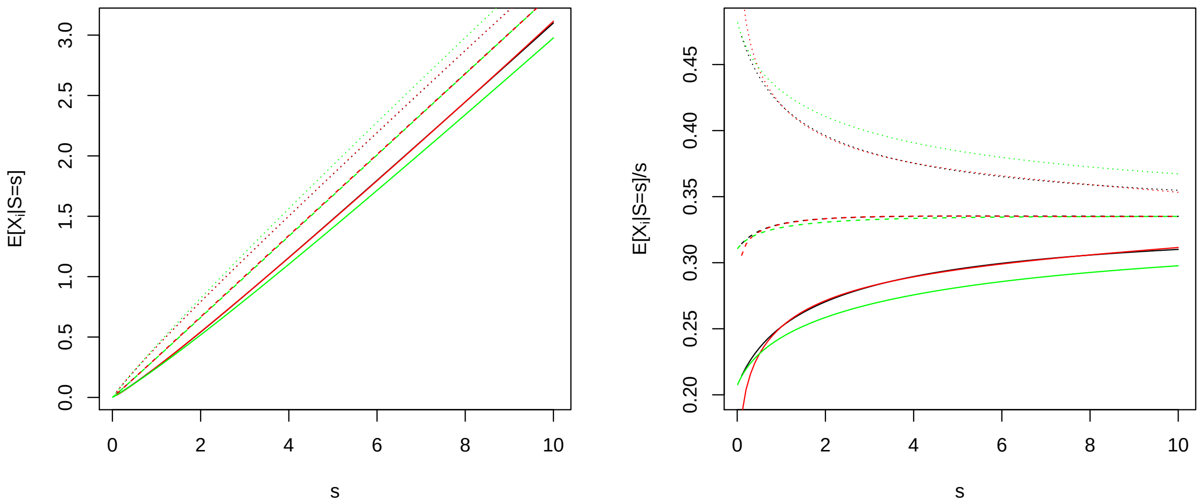


Figure 2: Left panel: Respective contributions $s \mapsto \mathrm{E}[X_i|S = s]$ for participant 1 (solid line appearing at the bottom), participant 2 (broken line appearing in the middle), and participant 3 (dotted line appearing at the top). Black curves correspond to exact values. Red curves correspond to values obtained with the proposed approximation based on the first 3 moments. Green curves correspond to approximations obtained with Panjer algorithm. Right panel: Respective relative contributions.

**Participant 2:** $N_2$ ∼Binomial$(2, 0.6)$, $C_2$ ∼Gamma$(1, 1)$;

**Participant 3:** $N_3$ ∼Binomial$(2, 0.7)$, $C_3$ ∼Gamma$(1, 1)$.

Again, we only use 3 moments in the approximations.

The functions $s \mapsto \mathrm{E}[X_i|S = s]$ are displayed in the left panel of Figure 2 for $i \in \{1, 2, 3\}$ and $s \in (0, 10)$. The relative contributions $s \mapsto \mathrm{E}[X_i|S = s]/s$ for each individual to the total realized loss $S = s$ are displayed in the right panel of Figure 2. We can see there that Panjer recursion departs from exact values whereas the proposed approximation delivers accurate values for the conditional mean risk allocations for each of the three individuals. Recall that Panjer recursion is known to suffer from numerical instability in the compound Binomial case, making the proposed approximation attractive. The latter is able to capture the behavior of the conditional mean risk sharing rule with only the mean, variance and skewness of the loss distribution.

### 5.5.3 Gamma reference distribution with Laguerre polynomials in the heavier-tailed case

Let us now consider severities with heavier tails, making Esscher transform needed. Precisely, we consider compound Binomial losses with either Gamma (light tail) or Lomax (heavier tail) severities. Recall that the Lomax distribution is a heavy-tail probability distribution of

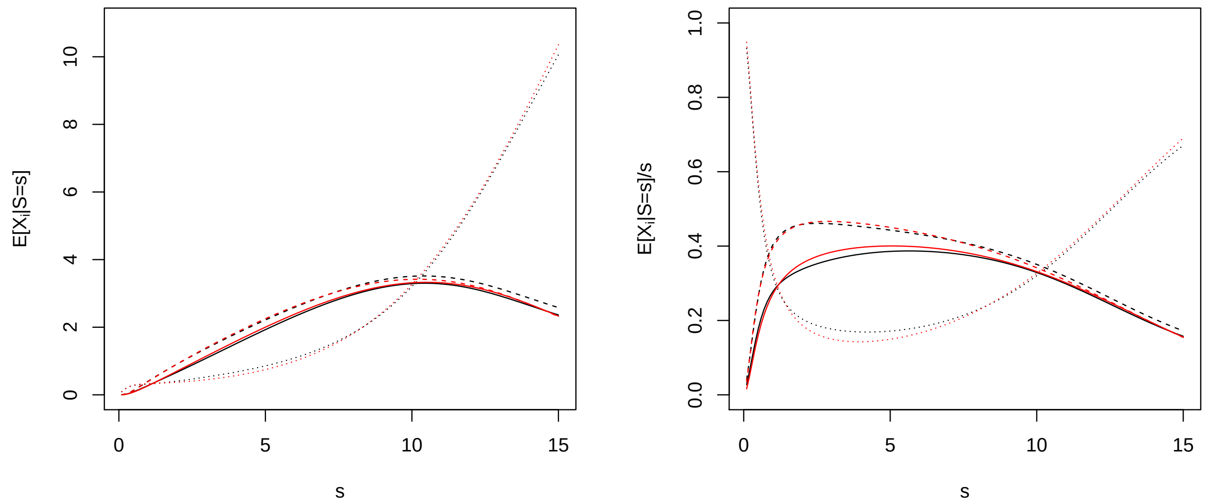


Figure 3: Left panel: Respective contributions $s \mapsto \mathrm{E}[X_i|S = s]$ for participant 1 (solid line appearing at the bottom), participant 2 (broken line appearing in the middle), and participant 3 (dotted line appearing at the top). Black curves correspond to exact values. Red curves correspond to values obtained with the proposed approximation based on the first 3 moments, with $h = -0.15$. Right panel: Respective relative contributions.

Pareto-type, with support equal to $(0, \infty)$. The corresponding probability density function is $\alpha\theta^\alpha/(x + \theta)^{\alpha+1}$. Consider

**Participant 1:** $N_1$ ∼Binomial$(2, 0.5)$, $C_1$ ∼Gamma$(1, 1)$;

**Participant 2:** $N_2$ ∼Binomial$(2, 0.6)$, $C_2$ ∼Gamma$(1, 1)$;

**Participant 3:** $N_3$ ∼Binomial$(2, 0.7)$, $C_3$ ∼Lomax$(1, 2)$.

We use 3 moments for the polynomial expansions and select $h = -0.15$.

The functions $s \mapsto \mathrm{E}[X_i|S = s]$ are displayed in the left panel of Figure 3 for $i \in \{1, 2, 3\}$ and $s \in (0, 15)$. The relative contributions $s \mapsto \mathrm{E}[X_i|S = s]/s$ for each individual to the total realized loss $S = s$ are displayed in the right panel of Figure 3. We can see there that participant 3 whose loss distribution is heavy-tailed absorbs the largest portion of the losses when $s$ gets large, in accordance with the results established by Denuit and Robert (2020a). We can also see on the basis of Figure 3 that the proposed approximations provide the analyst with a sufficiently accurate tool to select optimal parameter values, such as deductibles or retention levels, for instance. In particular, the approximations reveal that the functions $s \mapsto \mathrm{E}[X_i|S = s]$ are not monotonically increasing for participants 1 et 2, which invalidates the collaborative insurance system proposed in Denuit (2020b) and Denuit and Robert (2021b) since it requires that all conditional expectations are increasing in the total losses $S$.

### 5.5.4 Normal reference distribution with Hermite polynomials

The approximation of sums involving sufficiently many terms with the help of Normal probability density function is expected to work well near their expected value. Therefore, using the Normal distribution as the reference one and the associated family of Hermite polynomials can be an alternative in that case.

The orthonormal polynomials associated to the Normal reference distribution (with mean $\mu$ and variance $\sigma^2$) are given by

$$p_k(x) = \frac{1}{k!2^{k/2}} h_k\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)$$

where $h_k$, $k = 1, 2, \ldots$, are the Hermite polynomials obtained from successive derivatives of the function $\mathrm{e}^{-x^2/2}$ multiplied with $(-1)^k \mathrm{e}^{x^2/2}$. A sufficient (and close to necessary, as explained in Asmussen et al. (2018), see formula (1.2.6) in that paper) condition for $f_Z/f_\nu \in \mathcal{L}^2_\nu$ is that

$$f(x) = O(\mathrm{e}^{-ax^2}) \text{ as } x \to \pm\infty \text{ with } a > (4\sigma^2)^{-1}.$$

The function `dapx_gca` of the `R` package `PDQutils` also supports Normal reference distributions and its associated Hermite polynomials so that we can use it for calculating the approximation of $h_i^\star$.

As an example, let us consider again the second example above, with compound Binomial losses and Gamma severities. We keep a unique participant of type 1 but we increase the number of participants of types 2 and 3 to 10 in each case (so that we end up with 21 participants). We voluntarily keep a single participant of type 1 to have an unbalanced situation where two classes are more populated than the remaining one. The exact calculation of the conditional expectations $\mathrm{E}[X_i|S = s]$ defining the individual risk allocations is still possible since the aggregate losses for types 2 and 3 remain compound Binomial sums and can easily be obtained as finite Gamma mixtures by conditioning on the number of claims.

The functions $s \mapsto \mathrm{E}[X_i|S = s]$ are displayed in the left panel of Figure 4 for each type of participant and $s \in (5, 30)$. The relative contributions $s \mapsto \mathrm{E}[X_i|S = s]/s$ to the total realized loss $S = s$ are displayed in the right panel of Figure 4. The approximation for $\mathrm{E}[X_i|S = s]$ only uses 3 moments and appears to be accurate for small to moderate values of $s$ for each type of individuals, but deteriorates in the tails. The Normal reference distribution thus delivers reliable approximation in the center of the distribution, on the interval $(\mathrm{E}[S] \pm 2\sqrt{\mathrm{Var}[S]})$, even with moderate pool size and unbalanced classes.

# 6 Discussion

This paper proposes an approximation for the conditional mean risk allocations obtained from polynomial expansions (3.1)-(4.1) of the probability density functions entering representation formula (3.3) in terms of size-biased transforms. This approximation only requires the knowledge of the moments of individual losses. It applies both in the light-tailed and heavier-tailed cases, thanks to the Esscher transform of negative order. Considering compound Panjer-Katz sums, it performs favorably compared to Panjer recursion and even appears to be preferable in the compound Binomial case. The proposed approximation is

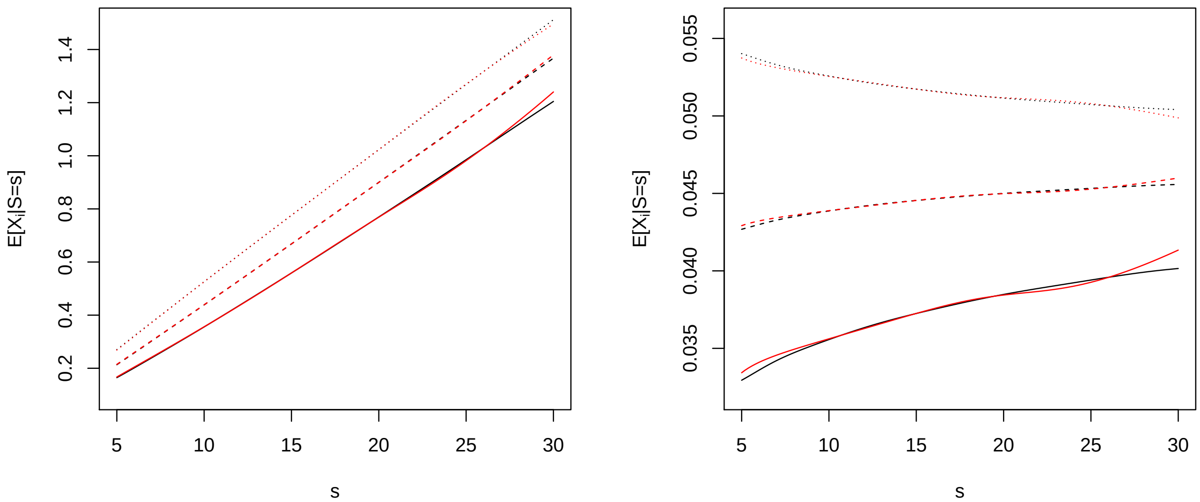


Figure 4: Upper panel: Respective contributions $s \mapsto \mathrm{E}[X_i|S = s]$ for participant 1 (solid line appearing at the bottom), for a participant of type 2 (broken line appearing in the middle), and a participant of type 3 (dotted line appearing at the top). Black curves correspond to exact values. Red curves correspond to values obtained with the proposed approximation based on the first 3 moments. Lower panel: Respective relative contributions.

thus effective in the design of the collaborative insurance scheme, to determine the value of some parameters such as individual deductibles, retention levels or upper layer limit. Once these values have been selected to make the system attractive, individual contributions can be calculated exactly in the final implementation stage.

The examples worked out in Section 5.5 show that sufficiently accurate approximations can be obtained with just the mean, variance and skewness. Instead of 3 participants, the results readily extend to 3 homogeneous risk classes with equal size $n/3$ (the number of participants within each group can be increased without limit, since the risk allocation is uniform within homogenous groups). Increasing the number of homogeneous groups slows down the calculation of the moments of the different sums $T_j$ and $\widetilde{T}_j$ involved in the formulas. The approximation using the Normal distribution as the reference one, and the associated family of Hermite polynomials appears to work well when more participants are involved. Linear approximations to the individual contributions $h_i^\star$ are also available when the number of participants becomes larger, as established in Denuit and Robert (2021a).

Throughout the paper, we have assumed that individual losses $X_1, X_2, \ldots, X_n$ were mutually independent. This is often a reasonable assumption in insurance studies, making the results derived in the preceding sections widely applicable. Sometimes, this assumption is nevertheless questionable. This is for instance the case for the cover against natural catastrophes where geographic proximity rules out independence. It is possible to extend the approach when the individual losses $X_1, X_2, \ldots, X_n$ obey a common mixture model, that is, when they are conditionally independent given some common risk factor $\Lambda$. The kind of dependence induced by this construction is comprehensively studied in Denuit et al. (2005, Chapter 7). The intuition behind this modeling approach is as follows: an external mechanism, described by the positive random variable $\Lambda$, influences the vector of risks $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ with $X_i \stackrel{\mathrm{d}}{=} X_i(\Lambda)$. Given the environmental parameter $\Lambda$, the individual risks are independent so that the joint distribution function of the random vector $\boldsymbol{X}$ can be written as

$$\begin{aligned} F_{\boldsymbol{X}}(t_1,\ldots,t_n) &= \mathrm{E}\big[\mathrm{P}[X_1 \le t_1,\ldots,X_n \le t_n|\Lambda]\big] \\ &= \int_0^\infty \left(\prod_{i=1}^n \mathrm{P}[X_i(\lambda) \le t_i]\right) \mathrm{d}F_\Lambda(\lambda). \end{aligned} \tag{6.1}$$

Notice that this construction is rather general and covers for instance the case of the common shock model, where each risk $X_i$ is obtained as the sum of two independent random variables, with the second one common to all risks. This common shock then plays the role of $\Lambda$ in the common mixture model (6.1).

Let $\boldsymbol{Y}^{[i]} = (Y_1^{[i]},\ldots,Y_n^{[i]})$ be a random vector with joint distribution function $F_{\boldsymbol{Y}^{[i]}}$ given by

$$F_{\boldsymbol{Y}^{[i]}}(t_1,\ldots,t_n) = \int_0^\infty \left(\prod_{j\neq i} \mathrm{P}[X_j(\lambda) \le t_j]\right) \mathrm{P}[\widetilde{X}_i(\lambda) \le t_i] \mathrm{d}F_{\Lambda_i^\star}(\lambda)$$

where the distribution function of $\Lambda_i^\star$ is given by

$$\mathrm{d}F_{\Lambda_i^\star}(\lambda) = \frac{\mathrm{E}[X_i(\lambda)]}{\mathrm{E}[X_i(\Lambda)]}\mathrm{d}F_\Lambda(\lambda).$$

It is proved in Denuit and Robert (2020c) that

$$h_i^\star(s) = \frac{\mathrm{E}\left[X_i\right] f_{Y_1^{[i]}+\ldots+Y_n^{[i]}}(s)}{\sum_{j=1}^n \mathrm{E}\left[X_j\right] f_{Y_1^{[j]}+\ldots+Y_n^{[j]}}(s)}.$$

Therefore, if $f_{Y_1^{[i]}+\ldots+Y_n^{[i]}}/f_\nu \in \mathcal{L}_\nu^2$ for every $i \in \{1,\ldots,n\}$ then $h_i^\star$ can also be expressed as

$$h_i^\star(s) = \frac{\mathrm{E}[X_i(\Lambda)]\sum_{k=0}^\infty \mathrm{E}[p_k(Y_1^{[i]}+\ldots+Y_n^{[i]})]p_k\left(s\right)}{\sum_{j=1}^n \mathrm{E}[X_j(\Lambda)]\sum_{k=0}^\infty \mathrm{E}[p_k(Y_1^{[j]}+\ldots+Y_n^{[j]})]p_k\left(s\right)} s \text{ for any } s>0.$$

We deduce that an approximation of $h_i^\star$ would only require the knowledge of the moments of $Y_1^{[i]}+\ldots+Y_n^{[i]}$. These moments can be rewritten as

$$\begin{aligned} \mathrm{E}[(Y_1^{[i]}+\ldots+Y_n^{[i]})^j] &= \mathrm{E}\left[\left(\sum_{h\neq i} X_h(\Lambda_i^\star) + \widetilde{X}_i(\Lambda_i^\star)\right)^j\right] \\ &= \int_0^\infty \mathrm{E}\left[\left(\sum_{h\neq i} X_h(\lambda) + \widetilde{X}_i(\lambda)\right)^j\right] \frac{\mathrm{E}[X_i(\lambda)]}{\mathrm{E}[X_i(\Lambda)]}\mathrm{d}F_\Lambda(\lambda). \end{aligned}$$

The conditional moment $\mathrm{E}[(\sum_{h\neq i} X_h(\lambda) + \widetilde{X}_i(\lambda))^j]$ can be computed as explained in the preceding sections, so that only one numerical integration is needed to obtain the unconditional moment $\mathrm{E}[(Y_1^{[i]}+\ldots+Y_n^{[i]})^j]$. This makes the proposed approximation attractive for conditionally independent losses, also.

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