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Pricing, Design and Solvency Measurement of annuity products

Pauline Milaure NGUGNIE DIFFOUO

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Jury members:

Prof. Pierre DEVOLDER (UCLouvain), Adviser
Prof. Jonathan ZIVEYI (UNSW Sydney)
Prof. Séverine ARNOLD (Université de Lausanne)
Prof. Donatien HAINAUT (UCLouvain)
Prof. Michel DENUIT (UCLouvain), President

Louvain-la-Neuve, December 2020

To my parents

Abstract

In the well-known context of the population ageing and the need for pensions funding, life annuity products occupy a very important place as natural hedging for policyholders against longevity risk. In particular, the difficulties of pay-as-you-go schemes and the need to provide funding for dependency of elderly are important factors motivating the study of life annuity as solutions to these. However, these products pose many actuarial and financial challenges, especially given their long-term nature. A precise analysis of these products indeed requires taking into account diverse risks such as interest rate, equity, longevity over periods of time that can be counted in decades depending on the nature of the product.
The new European solvency regulations require financial operators to provide a minimum capital upon their insurance commitments, called the solvency capital. Determining this capital in an adequate way is particularly difficult for long-term products. The objective of this thesis is on the one hand, to propose suitable single-risk and multi-risks models, based on investment strategies for assessing the solvency capital of an insurer selling annuities. This will thus ensures regular payment of benefits to the policyholders while they are alive, in accordance with both the regulation requirements and the type of annuity traded. We achieve this for classical annuities such as lifetime, deferred and term annuities. On the other hand, another objective is to study the valuation and actuarial design of life annuity products and to propose new methods of risk sharing between policyholders and insurers. To do so, risk-linked annuity products will be analysed, particularly the group self-annuitization with a focus on interest rate, equity and longevity risks. For that purpose, stochastic time continuous models of interest rate and mortality will be developed and applied along with in deep numerical and sensitivity studies.

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Chapter 1

Introduction

During the past decade, pension funds and insurers have faced numerous problems as consequences of the continuous increase of population life expectancy, commonly called *longevity risk*. For pension funds and insurance companies, this is materialized by the uncertain level of the future liability compared to the expected value. Many authors have proposed models to assess and hedge the longevity risk and its effect on pension funds or life insurance (Blake and Burrows, 2001; Fung et al., 2019; Antolin, 2007). This thesis is all about annuity products for which we will measure the risks, design new annuities and value these latter. In other words, we focus on the insurer point of view through the solvency measurement and then on the policyholders point of view through the design and valuation of new products. In the actuarial jargon, annuity products refer to series of payments made at equal interval of time as long as annuitants are alive (Fromenteau and Petauton, 2017). Annuity contract is made of two periods : accumulation period going from contract inception till retirement age and decumulation period going from the retirement until the end of the contract. When the affiliation age is equal to the retirement age, then the implied annuity is an immediate annuity. We consider three classical annuities (which are the most popular products for which the (equity, interest rate, longevity) risks are borne by the insurer) :

- The lifetime annuity provides annual benefits to the policyholder from her or his retirement date and as long as she or he is alive.
- The deferred annuity refers to an annuity product for which the payment stream starts few years (we denote it by d) after the retirement and until the death of the policyholder. For instance, if the retirement time is given

by T, the payment stream will start at time $T+d$, $[T, T+d]$ represents the deferred period.

- The term annuity provides a stream of payments for a predetermined number of years (we denote it by d') as long as the annuitant is alive.

The objective of this thesis is on the one hand, to propose suitable single-risk and multi-risks models, based on investment strategies for assessing the SC of an insurer selling annuities. This will thus ensures regular payment of benefits to the policyholders while they are alive, in accordance with both the regulation requirements and the type of annuity traded. We achieve this for classical annuities because determining the SC in an adequate way is particularly difficult for such long-term products. On the other hand, another objective is to study the valuation and actuarial design of life annuity products and to propose new methods of risk sharing between policyholders and insurers. To do so, risk-linked annuity products will be analysed, particularly the group self-annuitization with a focus on interest rate, equity and longevity risks. For that purpose, stochastic time continuous models for interest rate and mortality will be developed and applied along with in-deep numerical and sensitivity studies.

1.1 General perspective about solvency issue

Due to the longevity risk, insurers were obliged to increase the annuity prices in order to be more confident with regard to their solvency towards the policyholders. In order to protect the policyholders against such fluctuations of insurance product prices, the legislator has settled regulatory rules called the *Solvency II* (SII) (see EIOPA, 2014a; EIOPA, 2014b). The particularities of the SII framework are based on its risk-sensitivity and its multi-risk factors; the latter means that it recognizes the fact that insurers face different kinds of risks, such as equity, longevity and interest rate risks. SII is structured in three pillars respectively related to the quantitative requirement, qualitative requirement and market transparency. A quite important issue for pension funds and life insurances while fulfilling the first pillar is to compute the so-called *solvency capital* (*SC*) as defined in the SII regulation. This issue is a consequence of the long term structure of life insurance products such as annuities. By *solvency capital*, we refer to the amount the insurance company or the insurer has to put aside in order to be solvent until the end of the contract and in accordance with the SII regulation. Hence, high SC could imply expensive life insurance products and could increase policyholders and insurer reluctance to respectively buy and sell life insurance products. There is thus a need to find attractive products and/or strategies for both policymakers and policyholders so as to enhance life insurance market in accordance with SII.

Motivated by the difficulties of pay-as-you-go schemes as well as the need to provide funding for dependency of elderly, we study in this thesis life annuity products as solutions on both insurer and policyholder viewpoints. However, these products pose many actuarial and financial challenges, especially given their long-term nature. A precise analysis of these products indeed requires taking into account diverse risks such as interest rate, equity, longevity or even inflation risks (see Brown et al., 2000) over periods of time that can be counted in decades depending on the nature of the product.

In the first part of this work we base on the SII framework in order to value the SC on an insurer selling a classical annuity first with respect to longevity risk and with respect to equity, interest rate and longevity risks. In the literature, few authors have proposed substantiated mathematical definitions of the solvency capital requirement (SCR) (see Liebwein, 2006, Kochanski and Karnarski, 2011; Barrieu et al., 2012; Devineau and Loisel, 2009; Ohlsson and Lauzeningks, 2009). Christiansen and Niemeyer, 2014 presented some similarities and differences of the various interpretations and definitions of the SCR. Pfeifer and Strassburger, 2008a showed that some SCR methods proposed and discussed in the literature have stability problems. Devolder, 2011, has proposed an alternative way to estimate the SC based on a maturity approach for long term guarantees in life insurance. The latter is characterized by the long term horizon used to value the capital instead of the one year maturity as stated by SII. This methodology has the advantage of taking into account in the solvency level the duration of the benefits. This idea has been developed afterwards by Devolder and Lebègue (Devolder and Lebègue, 2016; Devolder and Lebègue, 2016; Devolder and Lebègue, 2017) for the case of a single cash flow as in a pure endowment insurance product, within a Brownian motion driven market. One of the purpose of this thesis is to extend this approach for classical annuities products with multiple cash flows. In other words, in the maturity approach applied to annuities, the SC is assessed upon the very last benefit (at the end of the contract); i.e by assuming the solvency of the insurer during the passed years. This will allow us to measure the risks borne by an insurer selling such products. In this regard, equity risk is represented by the random outcome or return on a risky investment and the interest rate risk is represented by a stochastic short rate. Therefore bad investment returns as well as mortality improvements are borne by the insurer which implies a high SC. For the sake of mitigating these risks, insurers tend to increase annuity prices and/or reducing benefit payout of the annuitants. A way of coping with the high level of SC, high annuity prices and low annual benefits could be to build suitable investment strategies of both the premiums and SC or to consider risk sharing annuities, or natural hedging (see Luciano et al., 2017). Several authors and insurance companies focus on the issue of SC assessment for life annuity products. Hari et al., 2008 analysed the importance of longevity risk for the solvency of portfolios of pension annuities; they distinguished two types of mortality risk: the micro-longevity risk, which quantifies the risk related to uncertainty of the time of death, and the macro-longevity risk, which is due to uncertain future survival probabilities. Olivieri and Pitacco, 2008 investigated rules for evaluating the SC by a portfolio of life annuity to meet longevity risk. We consider in this thesis the macro-longevity risk in order to measure the risks borne by an insurer selling annuities as well as to design and value risk sharing annuities.

1.2 General perspective about annuities' design and their valuation

In the policyholder point of view, even though annuities can guarantee they won't outlive their resources, people are still reluctant to buy these products. Economists refer to this problem as the "*annuity puzzle*". In this regard, researchers have been paying attention to solve this puzzle. So far, some explanations have been given to justify the annuity puzzle such as:

(i) most people prefer to invest their savings in business or account that would be accessible by their relative in case they die and buying annuity might be a bad deal for the heirs;

(ii) the fear of not living longer;

(iii) psychology reasons, in fact some retirees don't understand that annuities is a guarantee in case they live longer. Retirees seem to consider purchasing annuity as a gamble where they can die at any time and lose their premium.

Richard H. Thaler demonstrated that the explanation (i) is not always true as a retiree can decide not to annuitize all his savings and to leave aside a proportion to his heirs. The heirs could receive that legacy either immediately or at a later date[1]. Explanations (*ii*) and (*iii*) seem to be more logical and they implicitly involve the longevity risk in the sense that they raise the question of how to predict how long a person will live?

An attempt to solve the annuity puzzle could be to motivating people to purchase annuities through the design of adequate annuity products for policyholders so as to mitigate the longevity risk and other inherent risks (such as the equity or interest rate risks) they might borne. In this regard, we do not focus on the computation of optimal retirement plans (made of annuities and financial assets) as proposed by Yaari, 1965. We rather consider the annuity part in order to look for the best annuities for policyholders. Policymakers and researchers have thus far devoted substantial efforts on the longevity risk issue throughout valuing and designing risk-linked annuities, in order to protect policyholders against outliving their resources and protect insurer against possible insolvency or high SC. Even though risk-linked annuities expose the policyholder to risks, they also attempt to generate higher benefits than classical annuities. A range of such risk-linked annuities currently exists and can further be built so as to cope with the risks in both the insurer and policyholders point of view.

Basically, risk-linked annuities are obtained based on the idea of the classical annuity with benefits contingent on financial and/or mortality risks. Further annuities can be derived from the risk-linked annuities by including death benefits and/or some fixed guarantee features. In the second part of this thesis, we propose a way to design a range of financial-longevity linked annuities as well as their valuation. Among risk-contingent annuities we have inflation-linked annuities (Gong and Webb, 2010); mortality-indexed annuities (Richter and Weber, 2011); longevity-linked annuities (Denuit et al., 2011). Several annuities were developed for heterogeneous and/or homogeneous group annuitants called pooled annuities (e.g Donnelly et al., 2014). The main idea behind the pooled annuities is to mitigate the diversifiable risk within the pool. These pooled annuities are designed based on the common idea of tontines which consists of a group of annuitants providing their premiums in a pool administrated by an insurer (see Milevsky, 2014). Following the pooling mechanism, the method and products proposed in this work is designed for pool of annuitants and allow for risk sharing between the pool and insurer. These products are obtained by transferring a different proportions of both the financial and longevity risks to the policyholders. Many authors have

[1]Richard H. Thaler, The Annuity Puzzle, Economic View June 4, 2011. `https://www.nytimes.com/2011/06/05/business/economy/05view.html` accessed on November 21, 2019.

developed products in which the risks are either fully borne by a group of annuitants (see Milevsky, 2014; Piggott et al., 2005; Qiao and Sherris, 2013; Boyle et al., 2015) or shared between these latter and the insurer. Concerning risk sharing products, Chen et al., 2019 have developed a product for which the whole risk is borne by the group of annuitants during the first years following the retirement and the insurer bears the whole risk afterwards. They referred to their product as tonuity which is in fact a combination of tontine and classical annuity. Denuit et al., 2015 revisited the problem of sharing the longevity risk between a pool of annuitants and the policymakers while focusing on the deferred annuity. Denuit et al., 2011 designed longevity-index annuities allowing for longevity risk sharing between a group of annuitants and the insurer. Using an approach contrary to that of the group self-annuitization (GSA) (see Piggott et al., 2005), they proposed annuities where annuitants only bear the non-diversifiable mortality risk and the insurer bears remaining risks. Hanbali et al., 2019 focused on the systematic longevity risk in long-term insurance businesses where they proposed a framework for risk sharing between insurer and annuitants using dynamic equivalence principle. They proposed viable risk-sharing conditions that improve the trade-off between the solvency of the insurer and a fair price for policyholders for a pure endowment contract.

We proposed more general risk sharing methods that can be applied on deferred annuities. They can also be used to designed a longevity-linked, financial-linked as well as the financial-longevity-linked annuities. Our products are inspired by the group self-annuitization (GSA) introduced by Piggott et al., 2005 where they developed a formal analysis of the payout of the GSA which can be seen as a financial-longevity risk-pooling fund. The GSA is a scheme which allows annuitants to pool a part or their whole retirement fund with other annuitants with a view to afford benefits during retirement through a risk sharing arrangement within the pool. They proposed a detailed procedure of assessing the payout while considering the mortality changes of the pool. They expressed the payouts recursively and depending on the adjustment factor which depends on the ratios of survivorship. The main feature of the GSA is that unlike the classical annuity, the whole risk is borne by the group of annuitants whereas the insurer bears no risk. Some authors have proposed modified version of the GSA and developed detailed analysis of particular cases of GSA (see Qiao and Sherris, 2013; Boyle et al., 2015).

Since GSA shifts the whole risk to the group of annuitants, whereas the insurer selling classical annuity borne the whole risks; then these two products can then be seen as extreme cases. One of the question we address in Part 2 is what happens in

between these two extreme annuities. Hence, we design and value a set of annuities moving from the classical annuity to the GSA and we refer to the obtained set of annuities as the (complete) risk-sharing GSA. Our motivation of using the GSA comes from its recursive structure and from the fact that financial risk is separated from longevity risk. Following the approaches of Chen et al., 2019 and Chen and Hieber, 2016, we compare the (complete) risk-sharing GSAs with both classical annuity and GSA using the expected discounted lifetime (constant relative risk aversion) utility.
The question following the design of a product is how to value the said designed product? To answer this question, we need the following three ingredients[2]

(a) The net interest rate used to discount the annuity benefit. This could be given by the current interest rate available on the market or an interest rate guaranteed by the insurer. In the literature, it is common to do a sensitivity study with respect to the net interest rate;

(b) The base mortality table representing the company or the insurer's estimation of the population's mortality. It can be given by the best estimate of the real mortality, a mortality table proposed by the regulator or a table constructed by the company and in line with the regulator requirements;

(c) The mortality improvement assumption or the assumed trend of the real mortality of the population which represents the mortality model adopted by the insurer or the company. The mortality model is given by any existing model such as Vasicek, Hull-White, Lee-Carter, CIR model and so on.

Many authors have proposed pricing methods and formulae for annuity products. In this regard, the pricing aspects of equity-indexed annuities was explored by Tiong, 2000. Using the Esscher transform, Tiong developed a closed pricing formula of equity-indexed annuities. Bacinello et al., 2011 proposed a unifying framework for the valuation of the variable annuities using least squares Monte Carlo methods. They used three pricing approaches : the static or passive, the dynamic or active and the mixed approaches (see Milevsky and Salisbury, 2006). We proposed in the second part of this work the valuation formulae of the (complete) risk-sharing GSAs using the risk-neutral approach. Valuing these contracts allows us to value the GSA as this have not yet been valued in the literature.

[2]source: The Messenger Risk Management Newsletter written by Matthew Daitch, `http://www.scorgloballifeamericas.com/en-us/knowledgecenter/Pages/Pricing-a-Single-Premium-Immediate-Annuity.aspx` (accessed on 24 March 2020).

A common difficulty usually encountered in practice or in research while dealing with long-term contracts such as annuities is the numerical aspect; i.e. the simulation methods. Several authors have proposed efficient and robust simulation methods in this regards (see Bauer et al., 2010; Bauer et al., 2012; Hainaut et al., 2007). The simulation method used in this work is the nested Monte carlo simulations.

1.3 Summary of the main contribution and structure of the thesis

Our contribution to the literature is to measure the solvency of an insurer selling classical life annuity products using maturity approach. In Part I, we propose a profitable attractive investment strategy used to assess the SC of an insurer for a classical annuity with respect to the equity, interest rate and longevity risks. Our investment strategy comes as a modified version of those proposed by Bauer and Weber, 2008. In fact, in their paper they assessed the risk of an annuity given an investment strategy. The investment strategy they used is a fully liability hedging strategy whereas we propose here a partial (or temporal) liability hedging strategy that we called the m guaranteed cash flows strategy. The attractiveness of our strategy goes on the one hand toward the policyholder as it guarantees with a hundred percent a given number (m) of benefits. Notice that the attractiveness (for the policyholder) in terms of consumption has been studied by Hanewald et al., 2013, where they used the optimal consumption to find the optimal retirement plan in a portfolio made of zero-coupon bond (ZCB), life annuity, longevity bond and group self-annuitization. On the other hand toward the insurer in the sense that it shows that adding a SC could be seen as great investment as it gives a considerable IRR even though the SC is high. We compare the IRR using the mean-variance analysis. In line with Bauer & Weber, we find that the more benefits we guarantee the low SC we have for some annuities. We also find that there exits at least one number of guaranteed benefit that gives a lower SC as compare to the no-guaranteed (i.e $m = 0$) and fully guaranteed benefits (i.e m is equal to the total number of payouts). We further find that our strategy gives a better IRR as compare to that of Bauer & Weber in the sense that ours gives a concave IRR with respect to the number of guaranteed benefits. In the second part, our contribution is to proposed novel risk-sharing products defined as family of annuities moving from classical annuity to GSA. Furthermore, by valuing our products, we value at the same time the GSA which has not been valued yet. As result, we find that there exist (complete) risk-sharing GSA yielding better expected lifetime utility and lower value compared to both GSA and classical annuity. We further find conditions that guarantee the fair valuation of our products.

The structure of this work is as follows: the first part concerns the SC valuation and the second part concerns the valuation and design of risk-sharing annuities. In Part I, Chapter 2 present the valuation of the SC of an insurer selling one of the three classical annuities defined above, with respect to the longevity risk. We refer to the obtained model as the longevity model. A more complete model is developed in Chapter 3 where we consider equity, longevity and interest rate risks. In Part II, we design a novel risk-sharing annuities and compare it with both GSA and classical annuity in Chapter 4. The actuarial valuation of the obtained risk-sharing annuities is proposed in Chapter 5 and the conclusion follows in Chapter 6.

Part I

Solvency measurement

Chapter 2

Longevity Risk Measurement of Life Annuity Products[1]

2.1 Introduction

The main goal of this chapter is to measure the impact of the continuous increase of population life expectancy (called longevity risk) borne by an insurer selling a classical annuity (see Ngugnie Diffouo and Devolder, 2020). The longevity risk, materialized by the uncertain survival probabilities of insurer compared to the expected value is nowadays a major issue for pension funds and insurance companies. Many authors have proposed tractable stochastic mortality models with age dependent drift and volatility (see Fung et al., 2019). In this chapter we consider the macrolongevity risk, which is due to uncertain future survival probabilities which was analysed by Hari et al., 2008. This longevity risk will then be measured on the point of view of an insurer selling one of the three annuities. This longevity risk will be captured by force of mortality following the Hull–White model for which we assume a Gompertz model mean reversion level. The measurement method use consists of computing the amount an insurer has to put aside in order to be solvent until the end of the contract in accordance with the SII regulation (called SC) using the maturity approach introduced by Devolder and Lebègue, 2016. This will allow us to compare the three annuities using numerical analysis on both the insurer and the policyholders point of view. To do so, we consider a market fully hedged against any other risks differently from the longevity risk; this is called the *longevity model*. As there is no investment or financial risk considered in this

[1] Published in Risks

chapter, we assume the SC and initial premium are invested in a risk-free asset, say the discount bond with constant short rate. A common difficulty usually encountered both in practice or in research while dealing with such a problem is the numerical results; i.e., the simulation methods. The simulation method used in this thesis is the nested Monte carlo simulations and in order to be consistent with SII framework, we used the VaR as the risk measurement and considered both constant and variable confidence levels.

The goal of this chapter is to propose a deep comparative analysis of the three annuities with respect to the longevity risk generated by each, first from the policyholders point of view, and secondly from the insurer's viewpoint. From the policyholders viewpoint, we compare both the values of benefit payouts and the single premium, whereas from the insurer's viewpoint we compare the SC obtained for each product. Furthermore, we show by computing the IRR that adding a SC could be seen as an investment for shareholders, and we compare the three annuities through their IRR, based on the mean-variance approach.

In the following sections, we present the detail features of our model in Section 2.2; the theoretical results are presented in Section 2.3 where we develop the formulas of the SC for a deferred annuity. Section 2.4 presents the numerical results and the comparative remarks drawn from numerical results. These comparisons are made from the insurer, shareholder and policyholder viewpoints, with respect to the annuities, the confidence levels and the values of the IRR. Finally, Section 2.5 gives a brief conclusion.

2.2 The Model's Features

In this section, we focus on the general framework for our model. Before discussing the used products and the mortality model, we provide an overview of the financial market and the assumptions needed subsequently. In this chapter, we assume that the insurance company is fully hedged against any other risks except the longevity risk: the implied model is called the *longevity model*. We assume that the only asset available in the market is a risk-free asset: a *discount bond* $P_f(t,n),\ t \leq n$; that is, the value at time t of a financial instrument that pays a unit of currency at maturity time n. Therefore, in order to focus (or measure) only the longevity risk, we consider the discount bond to be defined using a deterministic short rate process $\{r_t\}$; i.e., $P_f(t,n) = e^{-\int_t^n r_s ds}$ (see Vasicek and Fong, 1982).

2.2.1 Mortality Model

This subsection presents in detail the mortality model used to capture the policyholders' real survival probabilities. We develop the theory behind the chosen model; this allows us to explicitly express both the liability of the insurer and the annual benefit received by the policyholders.
As extension of the approach presented in Devolder and Lebègue, 2016 (where the force of mortality is modelled using the Ornstein–Uhlenbeck process), and following Zeddouk and Devolder, 2019, we model the force of mortality of the cohort by the Hull–White process. Note that our approach can be extended to other continuous mortality models. The dynamic of the force of mortality is then defined by (see Lichters et al., 2015)

$$d\mu_t^{x_0} = (\theta(t) - a\mu_t^{x_0})dt + \sigma dW_t^{\mu}, \quad \text{for all } t \geq 0, \tag{2.1}$$

where $a > 0$ is the mean reversion rate; $\sigma > 0$ is the absolute volatility of $\mu_t^{x_0} \in \mathbb{R}$; $\{W_t^{\mu}\}$ is a Brownian motion; x_0 is the affiliation age; and $\theta(t)$ represents the mean reversion level function. We suppose that $\theta(t)$ is given by the Gompertz mortality model (see Gompertz, 1825)

$$\theta(t) = Ae^{Bt},$$

where $A > 0$ is the baseline mortality and $B > 0$ is the senescent component. Using an arbitrary starting time $t \leq s$, from short calculations we get

$$\mu_s^{x_0} = \mu_t^{x_0} e^{-a(s-t)} + \frac{A}{a+B}\left(e^{Bs} - e^{Bt-a(s-t)}\right) + \sigma e^{-as}\int_t^s e^{au} dW_u^{\mu}; \tag{2.2}$$

this represents in fact the force of mortality at time s of an individual initially of age x_0 and alive at time t (i.e., age $x_0 + t$). Let us now consider the survival index at time s of an individual initially of age x_0, alive at time t and surviving $s - t$ more years defined by the random variable

$$I_{s-t}^{x_0+t} = e^{-\int_t^s \mu_u^{x_0} du} = e^{X(t,s)} = e^{m^{x_0}(t,s)+\sigma^{x_0}(t,s)Z},$$

with Z being a normally distributed random variable with mean zero and variance one denoted by $Z \to \mathcal{N}(0,1)$. One now can show that

$$X(t,s) = -\int_t^s \mu_u^{x_0} du = m^{x_0}(t,s) - \frac{\sigma}{a}\int_t^s \left(1 - e^{-a(s-u)}\right) dW_u^{\mu} \tag{2.3}$$

$$X(t,s) \to \mathcal{N}\left(m^{x_0}(t,s), (\sigma^{x_0}(t,s))^2\right) \quad \Rightarrow \quad \frac{X(t,s) - m^{x_0}(t,s)}{\sigma^{x_0}(t,s)} \to \mathcal{N}(0,1),$$

where

$$m^{x_0}(t,s) = \frac{\mu_t^{x_0}(e^{-a(s-t)}-1)}{a} - \frac{Ae^{Bt}}{B(a+B)}\left(e^{B(s-t)}-1\right) - \frac{Ae^{Bt}}{a(a+B)}\left(e^{-a(s-t)}-1\right)$$

and

$$(\sigma^{x_0}(t,s))^2 = \frac{\sigma^2}{a^2}\left[s-t-\frac{1-e^{-a(s-t)}}{a} - \frac{\left(1-e^{-a(s-t)}\right)^2}{2a}\right].$$

We will measure only the longevity risk borne by the insurer; to that end we consider a cohort of N_0 retirees; i.e., $x_0 = 65$. Hence, the number of survivors at any time $t \in [0,n]$ is given by

$$N_t = N_0 I_t^{65},$$

Note that n is the duration of the contract. Let us denote by $p_{65}(t,s)$ the (physical) probability for a policyholder of age 65 and alive at age $65+t$, to survive until age $65+s$ for $s \geq t$

$$p_{65}(t,s) \;=\; \mathbb{E}\left[\frac{I_s^{65}}{I_t^{65}}|\mathcal{F}_t\right] = A_{65}(t,s)e^{-B_{65}(t,s)\mu_t^{65}}, \tag{2.4}$$

where $\mathcal{F}_t$ is the sigma algebra at time t, it encodes the information available at t.

$$B_{65}(t,s) = \frac{1}{a}\left(1-e^{-a(s-t)}\right)$$

and

$$A_{65}(t,s) = \exp\left(H(t,s) - \frac{\sigma^2}{2a^2}\left(B_{65}(t,s)-(s-t)\right) - \frac{\sigma^2}{4a}B_{65}(t,s)^2\right);$$

where

$$H(t,s) = \frac{Ae^{Bt}}{a+B}\left[\frac{1-e^{B(s-t)}}{B} + B_{65}(t,s)\right].$$

$p_{65}(t,s)$ is a measurable at time t. Moreover, we assume that $p_{65}(0,t)$ represents the mortality table guaranteed by the insurer at the contract inception; that is the best estimate of the survival index (from $t=0$); note that $I_0^{65} = p_{65}(0,0) = 1$.
From this, we can then value the liability of the insurer at any time of the contract. Note that if $n \geq 0$ is the duration of the contract, then a $d < n$-year deferred annuity can be seen as a generalization of both the lifetime and the term annuities. Indeed, setting $d = 0$, we obtained a lifetime annuity and for a fixed $d' < n$, substituting n by d' and setting $d = 0$ we obtained a d' term annuity. Hence, we develop in this chapter a detailed analysis for d-year deferred annuity for which we can deduce the cases of both lifetime and term annuities using the given transformations.

Remark 1. *Note that alternative mortality model could be considered as well. We would expect similar results if we consider a different continuous mortality model calibrated with the same mortality data whereas considering a discrete mortality model (e.g Lee-Carter model) could change the results we obtained with the HW model described above.*

2.2.2 Insurer's Liability

The contract consists of an initial cohort of N_0 individuals paying each an amount A_0 to an insurer at inception $t = 0$ in order to receive an annual amount of R during a period depending of the annuity purchased. We then denote the cohort's single premium by $\overline{A_0} = N_0 A_0$ and the cohort (random) annual benefit by $\overline{R_t} = N_t R$; note that the cohort initial benefit is denoted by $\overline{R} = N_0 R$.
It is important to stress that throughout this chapter in order to design the mortality model, we consider an initial cohort of retirees at inception $t = 0$ for which we model the longevity risk by a stochastic mortality model (the HW model). In other words, we measure the longevity risk only on the decumulation period (i.e., the period after the retirement) and we do not consider the accumulation period (i.e., period before retirement). The parameters of the HW mortality model are supposed to be known from inception and they are valid only for the initial cohort from which they have been computed. This statement is equivalent to the hypothesis of no model and parameter risks considered all along this chapter.
The value of the cohort's annual benefit for a d-year deferred annuity is defined based on an initially guaranteed life table $p_{65}(0, t)$ assumed to describe the cohort's mortality on the decumulation period. Moreover, we assume the value of the cohort's premium invested on the discount bond until time $t = d$ to be equal to the value at time $t = d$ of the discounted annual benefits. The insurer total liability at time d with respect to the initially guaranteed life table $p_{65}(0, t)$ for a deferred annuity is given by the expected value of the liability of the insurer. That is

$$\mathbb{E}\left[\overline{L_d}\right] = \mathbb{E}\left[\sum_{j=d}^{n} R N_j P_f(d, j)\right] = \sum_{j=d}^{n} R N_0 p_{65}(0, j) P_f(d, j), \tag{2.5}$$

$\overline{L_d}$ represents in fact the value at time d of the overall benefits invested in a discount bond that the insurer has to pay to the policyholders alive at each payment time according to the initially guaranteed survival probability $p_{65}(0, j), \quad j = 1, ..., n$ in the case of a deferred annuity, and n is the duration of the contract. Therefore, we assume the single premium $\overline{A_0}$ to be defined as $\overline{A_0} = P_f(0,d)\mathbb{E}\left[\overline{L_d}\right]$, where $\mathbb{E}\left[\overline{L_d}\right]$ is given by Formula (2.5); this implies that $\mathbb{E}\left[\overline{L_d}\right] = \frac{\overline{A_0}}{P_f(0,d)}$, and hence the individual annual benefit R is given by

$$R = \frac{A_0/P_f(0,d)}{\sum_{j=d}^{n} p_{65}(0,j)P_f(d,j)}. \tag{2.6}$$

The formula of the annual benefit of a $d' < n$-year-term annuity is obtained just by substituting n by d' and setting $d = 0$; for a lifetime annuity we only set $d = 0$. Next we compute the SC at any time $t \in [0,n]$ of the contract for the three annuities (lifetime, deferred and term annuities) in order to compare the obtained numerical results. To this end, we develop the formulae of the SC in the next section for d-year deferred annuity. As mentioned previously, we consider a cohort of N_0 retirees initially of age $x_0 = 65$.

2.3 Solvency Capital Valuation

In this section, we present the formulae of the SC for a d-year deferred annuity form which we can deduce the formulae of a d'-year term and lifetime annuities. As reminder, both the initial premium and the SC are invested in a discount bond. In order to measure the SC, we adopt in this chapter a more general view than the standard approach proposed by the SII. Taking into account the very long term horizon of this kind of product, we use a maturity approach and not a one year measure, as stated by the SII. We value the SC by the use of a risk measure applied to the final surplus so as to measure the global longevity risk borne by the insurer. This method has been used for other products, as proposed by Devolder and Lebègue, 2016. More precisely, in order to take into account the multi period character of the product, we consider the static risk measure VaR as our risk measurement tool with an annual confidence level of $\alpha = 99.5\%$ as stated by SII (see EIOPA, 2014a, EIOPA, 2014b). Then, we assume the temporal independence of events and we consider constant and equal yearly confidence level such that for a n-maturity contract, the overall confidence level is $\alpha_n = (99.5\%)^n$. We further consider a fixed constant confidence level of $\alpha = 99.5\%$. These two values of the confidence level are used for simulation purposes in order to highlight the impact of the confidence level considered on the level of SC.

The use of VaR in this chapter and in Chapter 3 is mostly motivated by the fact that it is recommanded in the SII regulation [2]. Even though some other showed stability problems within the general setup while computing the SC using the VaR (see Pfeifer and Strassburger, 2008b), this risk measurement remains widely used both in the literature (see Hejazi and Jackson, 2017; Karam and Planchet, 2013) and in practice.

We consider a policyholder buying a $0 < d < 45$-year deferred annuity at inception time $t = 0$. We will value the SC of the insurer within the deferred and the payment intervals . It is important to stress that the payment stream of a given policyholder begins if and only if he or she is alive at time $t = d$; i.e., at age $65 + d$. Additionally, the individual annual benefit for a d-year deferred annuity is given by Formula (2.6). Note that we have two computational intervals: the deferred period $[0, d)$ and the payment period $[d, n)$.

2.3.1 Deferred Period

Our approach of assessing the SC consists of measuring the risk of having a negative final surplus at time $t = n$. By final surplus, we refer to the amount of assets that exceeds the liability of the insurer at the end of the contract. Mathematically, let

- $\overline{A}(t, n) = \frac{\overline{A_t}}{P_f(t,n)}$ be the fund value at the end of the contract, where the available asset at time t is defined as

$$\overline{A_t} = \overline{A}(0, t) = \overline{A_0} P_f^{-1}(0, t); \tag{2.7}$$

- $\overline{L}(t, n, d) = \sum_{j=d}^{n} RN_j P_f^{-1}(j, n)$ be the value at time n of the liability of the insurer;

- $\mathrm{FS}(t, n, d) = \overline{A}(t, n) - \overline{L}(n, d)$ is then the final surplus of the contract.

Therefore, it follows from our maturity approach for the valuation of the SC that at the computational time t, the SC satisfies the following condition.

$$\mathbb{P}_t \left[\mathrm{FS}(t, n, d) + \frac{\overline{SC}(t, n)}{P_f(t, n)} < 0 \right] \leq 1 - \alpha_{n-t}. \tag{2.8}$$

One can show that in this period, for any $t \in [0, d)$ the definition of the SC given by Formula (2.8) is equivalent to

$$\mathbb{P}_t \left[\left(\overline{A_t} + \overline{SC}(t, n) \right) P_f^{-1}(t, n) < \overline{L}(t, n, d) \right] \leq 1 - \alpha_{n-t},$$

[2] Article 101 stipulates that the SCR corresponds to the Value-at-Risk with a confidence level of 99.5%.

where $\overline{A_t}$ is given by (2.7) and the value at term of the liability is given by

$$\begin{aligned}\overline{L}(t,n,d) &= \sum_{j=d}^{n} RN_j P_f^{-1}(j,n) = \sum_{j=d}^{n} RN_d \; I_{j-d}^{65+d} \; P_f^{-1}(j,n) \\ &= \sum_{j=d}^{n} RN_t \; I_{d-t}^{65+t} \; I_{j-d}^{65+d} \; P_f^{-1}(j,n),\end{aligned}$$

where for any $t \in [0, d)$

- N_t is the real number of survivors at time t;
- I_{d-t}^{65+t} is the survival index of an annuitant initially aged 65, alive at age $65+t$ and living at least up to age $65+d$. This term guarantees that a policyholder in the cohort must be alive at the end of the deferred period;
- I_{j-d}^{65+d} guarantees that benefits are paid if the annuitant is alive at each payment time on the payment period.

2.3.2 Payment Period

On this other interval the SC is valued with respect to the remaining liabilities at term which are the unpaid benefits yearly deduced from the asset and evaluated at term. Moreover, we have to take into account the already paid benefits; i.e., the benefits paid up to time t. This is achieved using the fact that information up to time t is known, meaning that the number of survivors at t N_t is known. Hence, the SC satisfies the following

$$\mathbb{P}_t\left[\overline{A_t} P_f^{-1}(t,n) + \overline{SC}(t,n) P_f^{-1}(t,n) < \overline{L'}(t,n,d)\right] \leq 1 - \alpha_{n-t}, \tag{2.9}$$

where $\overline{L'}(t,n,d)$ represents the total unpaid benefits at n and it is given by

$$\overline{L'}(t,n,d) = \sum_{j=t+1}^{n} RN_j P_f^{-1}(j,n) = \sum_{j=t+1}^{n} RN_t \; I_{j-t}^{65+t} \; P_f^{-1}(j,n).$$

Note that for $j \leq t$, N_j is known as the realized number of survivors; moreover, the asset is given by

$$\overline{A_t} = \overline{A_0} P_f^{-1}(0,t) - \sum_{j=d}^{t} RN_j P_f^{-1}(j,t).$$

The second term of the right hand side represents the value at t of the benefits paid up to time t.

Note that concerning a d'-year-term annuity, the payment stream begins at inception, i.e., age $x_0 = 65$, and ends after d' years, with $d' < n$. Therefore, the formulae of a d'-year-term annuity can be obtained from the previous formulae by setting $n = d'$ and $d = 0$. For a lifetime annuity, we only set $d = 0$ in the previous formulae, and for both a term and a lifetime annuities, we have only one computational period: the decumulation period, which is equal to the payment period.

2.4 Numerical Results

Some graphical representations and some numerical values of the SC with respect to the computational time t, the length of the deferred period d and the term d' in the case of lifetime, deferred and term annuities, are presented in this section.

2.4.1 Simulation Framework

The simulation of the longevity SC is made in two steps: we first calibrate the HW model (see Lichters et al., 2015), and secondly, we make use of the Monte Carlo (MC) simulations of the VaR (see Glasserman, 2013). For the calibration of the Hull–White model (2.1), we need some real data and a calibration method. To do so, we consider the unisex projected generational life table of an individual aged 65 in 2015 for an ultimate age of 110 in 2060 available on the IA|BE life table proposed by Antonio et al., 2015. Using this data, we calibrate the HW model (2.1) by the use of the mean squared error (MSE); and we obtain the following parameters in Table 2.1.

Table 2.1: Calibration parameters of the HW model using MSE.

$\boldsymbol{\mu_0^{65}}$	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{a}$	$\boldsymbol{\sigma}$	**MSE**
0.0105677	0.0005749505	0.1304207503	0.0014965354	0.0083530153	0.000303644

The obtained calibration parameters of the force of mortality fit the IA|BE data well, since the MSE= 0.000303644.

As for the MC simulations, we use the nested MC simulations (see Bauer et al., 2012). We consider a cohort of size $N_0 = 1000$ all aged 65 at the affiliation date $t_0 = 2015$. Each participant pays $A_0 = 1000\$$ to the insurer at t_0 in order to receive a constant amount R depending on the annuity bought. We further consider the following parameters

$$\begin{aligned} \alpha = 0.995; \quad & A_0 = 1000\$; \quad d = 16; \quad d' = 15 \\ N_0 = 1000; \quad & n = 45; \quad \alpha_T = \alpha^n = 0.995^n; \quad r = 0.01 \end{aligned}$$

where n represents the duration of the contract, and assuming $n = 45$ means we consider an ultimate age of 110. As for the term annuity, we suppose $d' = 15$; this means the contract will end when the policyholder is aged 80 with a maximum of 16 possible benefits. For the deferred annuity, we take $d = 16$; i.e., the payments start when the annuitants are aged $65 + d$ and ends when they are aged 110.

Moreover, as the force of mortality at a time t depends on the force of mortality at time $t - 1$, we consider for simulation purposes the correlation between two consecutive survival indexes I_j^{65} and I_{j+1}^{65}, $j \in [0, n-1]$; this implies that the liability at time n for a d-year deferred annuity computed at inception takes the form

$$\overline{L}(n) = \sum_{j=d}^{n} \overline{R}\ e^{m^{x_0}(0,d)+\sigma^{x_0}(0,d)Z^{(d)}}\ e^{m^{x_0}(d,j)+\sigma^{x_0}(d,j)Z^{(j)}}\ P_f^{-1}(j,n),$$

where by the use of Cholesky decomposition (see Cheuk-Yin and Siu-Hang, 2011), we have

$$Z^{(j)} = \rho(j-1)Z^{(j-1)} + \sqrt{1 - \rho^2(j-1)}Z_j;$$

where the Z_j's are normally distributed random variables with mean zero and variance one. Note that Z_j is independent of $Z^{(j-1)}$; $Z^0 = 0$; $Z^{(1)} = Z_1$ and $\rho(j) = \mathrm{corr}[X(0,j), X(0,j+1)]$ where $X(t,s)$ is given by Formula (2.3).

2.4.2 Results and Comparative Remarks

In order to guide their decision making, we provide the insurer as well as the shareholders with information about the impact of the longevity risk on each of the three annuities. Moreover, for annuitants, we provide a comparative study on both the benefits and the initial premium for the three annuities with respect to some parameters. This could help annuitants decide which product to buy, depending on their risk aversion. This comparison is made based on both the level of the annual benefit (for a fixed single premium) and the level of the single premium (for constant annual benefits), which are different from one annuity to another. From the insurer's viewpoint, the comparison is made using the level of the SC of each annuity and the internal rate of return on the SC invested by shareholders.

Regarding the policyholders on the one hand, considering a fixed single premium $\overline{A_0} = 10^6\$$ for each of the annuities, we obtained from some computations that the amount of annual benefit $\overline{R}$ differs from one annuity to another annuity following the relations

$$\begin{aligned} \overline{\boldsymbol{R}}(\textbf{lifetime annuity}) &< \overline{\boldsymbol{R}}(\textbf{term annuity}) \\ \overline{\boldsymbol{R}}(\textbf{lifetime annuity}) &< \overline{\boldsymbol{R}}(\textbf{deferred annuity}). \end{aligned}$$

In fact, this can be seen in Figure 2.1 and can be justified as follows.

- $\overline{\boldsymbol{R}}$**(lifetime)** $<$ $\overline{\boldsymbol{R}}$**(term)**: This comes from the fact that $n > d'$, where $\overline{A_0}$ is shared for the $n+1$ benefits for the lifetime annuity, whereas $\overline{A_0}$ is shared for the $d'+1 \leq n$ benefits for term annuity.
- $\overline{\boldsymbol{R}}$**(lifetime)** $<$ $\overline{\boldsymbol{R}}$**(deferred)**: Similarly, the unique premium $\overline{A_0}$ is distributed into $n+1$ benefits for a lifetime annuity, whereas it is distributed into $n+1-d < n$ benefits for the deferred annuity.

Moreover, one can show that $\overline{\boldsymbol{R}}$**(term annuity)** $=$ $\overline{\boldsymbol{R}}$**(deferred annuity)** if and only if

$$\sum_{j=0}^{d'} p_{65}(0,j)P_f(0,j) = \sum_{j=d}^{n} p_{65}(0,j)P_f(0,j). \tag{2.10}$$

Using the fact that

$$\sum_{j=0}^{n} = \sum_{j=0}^{d'} + \sum_{j=d'+1}^{n} = \sum_{j=0}^{d-1} + \sum_{j=d}^{n},$$

one can show that relation (2.10) can be rewritten as

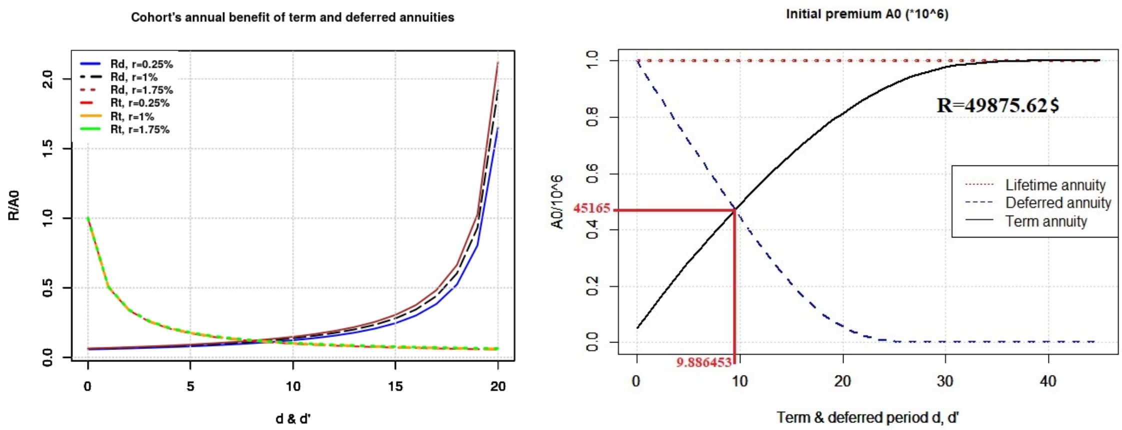
$$\sum_{j=d'+1}^{n} p_{65}(0,j)P_f(0,j) = \sum_{j=0}^{d-1} p_{65}(0,j)P_f(0,j).$$

This is shown in Figure 2.1 where we observe that there exit some values of $\boldsymbol{d}$ and $\boldsymbol{d'}$ such that the benefits of deferred and term annuities are equal. Note that the comparison of $\overline{\boldsymbol{R}}$**(deferred)** and $\overline{\boldsymbol{R}}$**(term)** depends on the values of d and d' respectively. In fact $\overline{\boldsymbol{R}}$**(deferred)** increases with d, whereas $\overline{\boldsymbol{R}}$**(term)** decreases when d' increases.

On the other hand, we consider constant annual benefits for each of the annuities assumed to be

$$\overline{R} = \frac{\overline{A_0}^L}{\sum_{j=0}^{n} p_{65}(0,j)P_f(0,j)} = 49875.62\$,$$

where $\overline{A_0}^L = 1000^2\$$ and $n = 45$. The comparison will follow from Figure 2.1 which represents both the annual benefit per unit of initial premium with respect to d and d', and the variation of the single premium of a term annuity $(\overline{A_0}^T)$ and the deferred annuity $(\overline{A_0}^D)$ for fixed benefit $\overline{R} = 49875.62\$$ with respect to d and d'.



Comparison of benefits Comparison of the single premiums

Figure 2.1: Annual benefits for equal premiums and premiums for equal annual benefits.

We observe on the right graph of Figure 2.1 that $\overline{A_0}^T$ and $\overline{A_0}^D$ have opposite behaviors; in fact $\overline{A_0}^T$ converges to $\overline{A_0}^L$, whereas $\overline{A_0}^D$ converges to 0, and they coincide at $d = d' = 9.886453$. In particular, for $d' = 15$ and $d = 16$ we have $\overline{A_0}^T(15) = 669377.1\$$ and $\overline{A_0}^D(16) = 171054.4\$$; it follows that for equal level of annual benefits, a 16-year deferred annuity is cheaper than a 15-year immediate-term annuity, which in turn is cheaper that an immediate lifetime annuity.

Concerning the insurer's viewpoint, the behavior of the SC is presented in the following graphs

- **Lifetime annuity:** Figure 2.2 shows how the SC changes with respect to the computational time $t > 0$ for different values of the confidence level and the short rate. We observe that the SC decreases when the short rate increases. We can also see that the SC decreases as t increases. Moreover the values of SC obtained with a variable confidence level are smaller than those obtained with a constant confidence level.

Note that a negative value of SC means no additional capital is required from the insurer; that is, $SC = 0$. In other words, $SC \leq 0$ means that the insurer is solvent with the considered confidence level without any additional capital. Moreover, the almost negative curves in Figure 2.2 comes from the assumption on the number of survivors at computational time t. In fact, before and up to the computational time t, we assumed the best estimate of the number of survivors and this assumption underestimates the longevity risk at each time t.

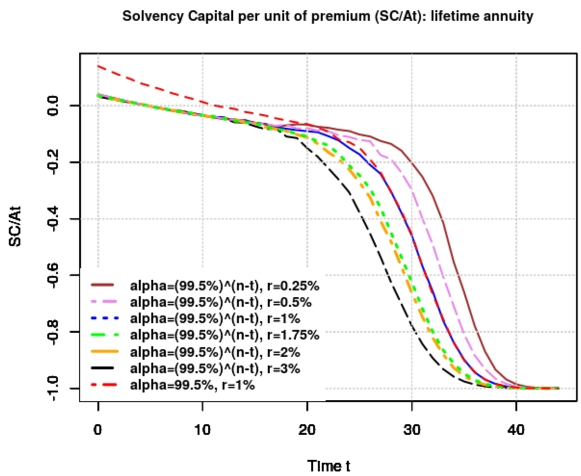


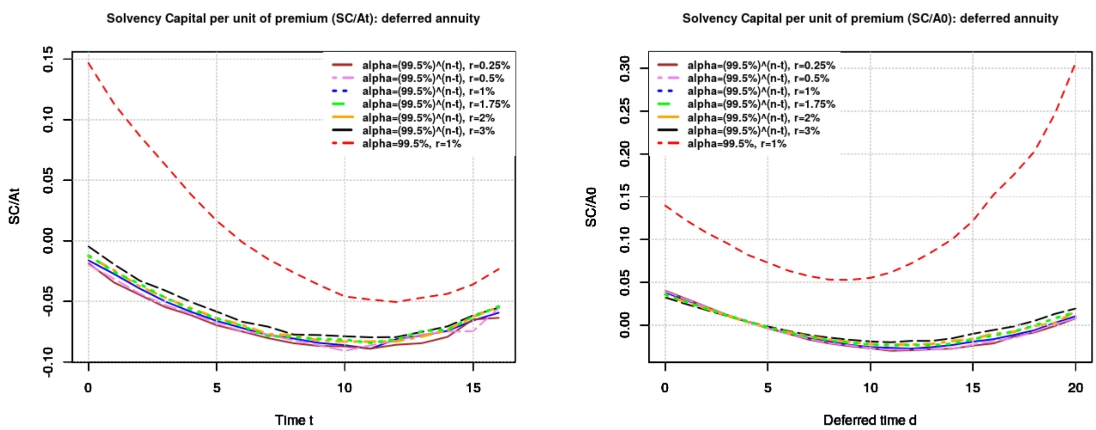
Figure 2.2: Lifetime annuity; $\overline{SC}(t)/\overline{A_t}$ for different values of the short rate r and α.

- **Deferred annuity:** Below are presented the graphs of the SC with respect to t; to d on the deferred period in Figure 2.3, and with respect to d on the payment period in Figure 2.4 respectively.

  From Figure 2.3 it comes out that the SC decreases for both values of α and slightly decreases with respect to r. Furthermore, for a constant confidence level we obtain a higher value of SC; it decreases for smaller values of d and increases for larger values of t. Moreover, the SC have a convex form with respect to d and with respect to t. This implies that there exist optimal values of d and t that minimize the SC of the insurer.

  The convexity observed in Figure 2.3, comes from two facts. (1) We take advantage of the investment during the deferred period and the low longevity risk for small values of the deferred time, which allow the SC to decreases (as the longevity risk is less important at younger ages). (2) We have high longevity risk at old ages, thus high deferred time implies high risk and high benefit; these increase the SC. Consequently, when (1) is important, the SC starts to decrease and when (2) is important, SC increases. Therefore, there exists a deferred time d for which we have a better balance between the longevity risk at old ages and the investment return during the deferred period.

Regarding Figure 2.4, it can be seen how the SC decreases with d in the payment period. Moreover, we find that the SC is strictly negative with respect to $t > d$ and for any value of r and α.



$\overline{SC}(t)/\overline{A_t}$: for d=16 $\qquad$ $\overline{SC}(d)/\overline{A_t}$: at t=0

Figure 2.3: Deferred annuity; SC(t) and SC(d) respectively for different values of the short rate r and α on the deferred period.

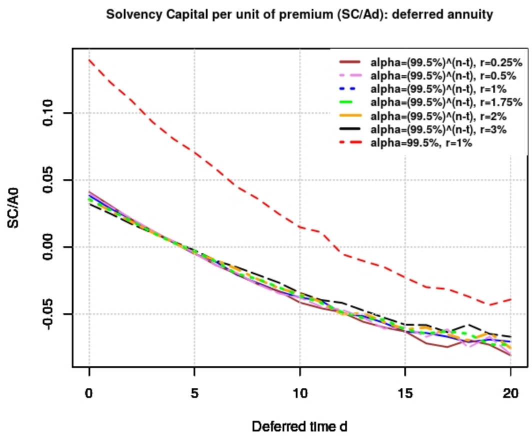
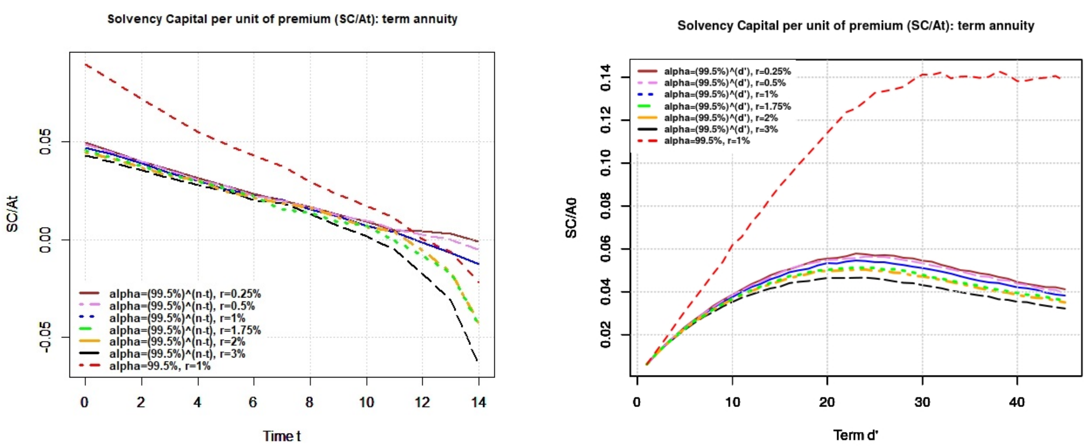


Figure 2.4: Deferred annuity; $\overline{SC}(d)/\overline{A_t}$ at t=d for different values of the short rate r and α on the payment period.

- **Term annuity:** Figure 2.5 below shows the SC with respect to t and to both d'.

  It comes out that the SC decreases with $t > 0$ and r. Moreover, the SC is strictly positive and increases with d' for constant α and takes a concave form when d' increases for variable values of α.



$\overline{SC}(t)/\overline{A_t}$: for d'=15 $\qquad$ $\overline{SC}(d')/\overline{A_t}$: at t=0

Figure 2.5: Term annuity; SC(t) and SC(d') respectively for different values of the short rate r and α.

From these figures, one could infer that the product requiring the lower amount of SC does depend on the time at which the said SC is computed. For example, for variable α and $t = 7$, $\overline{SC}/\overline{A_t}$ worth zero for the lifetime annuity, it is worth 2% for the term annuity and it is zero for the deferred annuity. Hence, one might say that the term annuity requires the most SC at that computational time. Thus it will be harsh to point a particular product to be the best for an insurer, as the comparison has to be made at same points in time. Table 2.2 below shows the values of the SC at inception (i.e., at $t = 0$) for $d = 16$ $d' = 15$ and $r = 1\%$.

Table 2.2: Comparison of the annuities at $t = 0$.

$\overline{SC}/\overline{A_0}$	Constant α	Variable α_n
Lifetime annuity	0.1403933	0.03822095
Term annuity	0.08871471	0.04748606
Deferred annuity	0.1467654	−0.01588651

It follows that the constant confidence level gives higher SC compared to the variable confidence level. This can be explained by the fact that the required probability of default with a constant α is larger than that of the variable α. Moreover Table 2.2 also shows that for variable α, the deferred annuity requires the lesser SC followed by the lifetime annuity. For constant α, the deferred annuity requires the most SC followed by the lifetime annuity. Note that the negative SC obtained for the deferred annuity means that the insurer has no additional capital to put aside in order to guarantee his solvency; i.e., $SC = 0$. Values of $\overline{SC}/\overline{A_0}$ for immediate annuity are present in Table A.1 in Appendix A.1.

Note that an alternative way to compare these products could be from the shareholder point of view: one could assess the ability of a given annuity to generate benefit from the SC invested on it. This ability can be assessed by the internal rate of return (IRR) on the SC (see Gronchi, 1986). Hence, having this rate greater than the short rate indicates an advantageous investment for shareholders, whereas a IRR less than the short rate implies an investment with loss compared to the risk-free investment. Moreover, for IRR = 0, the shareholder will just take back the initial capital invested at $t = 0$, and the worst case (i.e., loosing the whole initial capital) would be when IRR= −1. By definition, the SC satisfies

$$\mathbb{P}_t\left[\left(\overline{A_t} + \overline{SC}(t,n)\right) P_f^{-1}(t,n) < \overline{L}(t)\right] \leq 1 - \alpha_{n-t}.$$

The possible benefit at the end of the contract $t = n$ called the final surplus is a random variable given by

$$\chi(w) = \frac{\overline{A_t} + \overline{SC}(t,n)}{P_f(t,n)} - \overline{L}(t). \tag{2.11}$$

It follows that the IRR, denoted by $\tau(w)$ is a random variable satisfying the following

$$\overline{SC}(t,n)\ (1+\tau(w))^{n-t} = \chi(w); \tag{2.12}$$

thus, the IRR on the SC is given by

$$\tau(w) = \left(\frac{(\chi(w))^{+}}{\overline{SC}(t,n)}\right)^{\frac{1}{n-t}} - 1, \tag{2.13}$$

where $(x)^{+} = \max(x, 0)$.

It is important to stress that the IRR is computed if and only if the SC is strictly positive. For comparison purpose, Figure 2.6 and Table 2.3 respectively show the density and the numerical values of mean and variance of the IRR for the three annuities computed at $t = 0$, for $d = 16$ and $d' = 15$.

Based on the mean-variance criterion, it follows from Table 2.3 that at inception the deferred annuity using constant α could yield higher IRR. But considering the variable α, the comparison will depend on the level of the shareholder's risk aversion, since he could decide not to invest an SC (referring to a deferred annuity) or to invest his SC on a lifetime annuity. As regards the term annuity, it seems to be less profitable, since it has the lower mean and the highest variance for a constant α. Note that the values of the IRR for a deferred annuity using a variable confidence level cannot be computed, since the SC in this case is strictly negative.

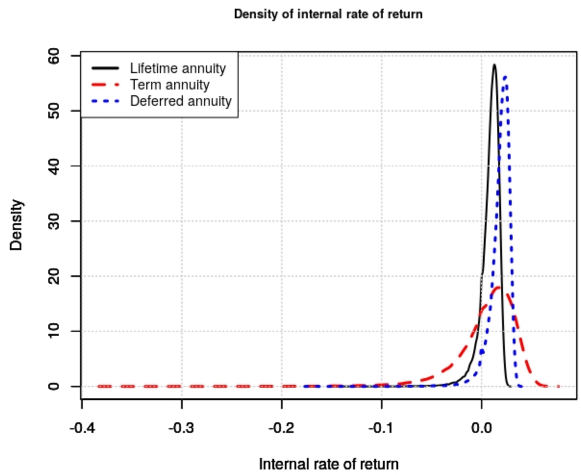


Figure 2.6: Density of the internal rate of return for $\alpha = 99.5\%$ and $r = 1\%$.

Table 2.3: Mean and Variance of the internal rates of return (IRRs) for the three annuities.

	Constant α		**Variable α**	
	Mean	**Variance**	**Mean**	**Variance**
Lifetime annuity	0.008855	9.5191×10^{-5}	0.00980413	0.00036868
Term annuity	0.00554107	0.00088871	0.0040751	0.0022854
Deferred annuity	0.0182561	0.000117404	/	/

From the observations on both the level of the annual benefits and the level of the SC, we can draw the following concluding remarks.

(i) On the policyholder side, buying a term annuity will provide the annuitant with a good level of annual benefits but he will not be fully hedged against the longevity risk, whereas buying a deferred annuity will give a better level of annual benefits until death. In these cases, the longevity risk will partly be borne by the policyholders. The lifetime annuity gives the smallest level of annual benefits and fully hedges the annuitants against the longevity risk. Furthermore, for a fixed unique premium (respectively annual benefit), we can always find a pair of deferred time and term time such that the deferred and the term annuities provide the same annual benefits (respectively, the same unique premium). Note that combining successive term annuities could be a way to fully protect the annuitant against longevity risk, but in this case, the policyholder will face a pricing risk. The latter refers to the risk of rising annuity prices as a consequence of a high survival probability.

(ii) On the insurer side, we can see that a positive SC is not always bad for an insurer, since it could yield a return even greater than a simple risk-free investment. Thus, the choice of the product to invest in depends on the risk aversion of the shareholders. Furthermore, identifying the product with the higher SC is subject to the computational time.

Even though we cannot draw up a strict comparison of these products by pointing out the best from both insurer and annuitant viewpoints, the results obtained here can be seen as a backing on which the two parties can base their choices.

2.4.3 Simulation performance

To obtain the numerical results (i.e figures and tables) in this chapter, we performed MC method with 500000 simulations.

- Figure 2.2 representing the case of lifetime annuity took approximatively 36 minutes.
- Concerning deferred annuity, the first graph of Figures 2.3 took 35 minutes whereas the second graph and 2.4 took 5 minutes to be completed.
- For the term annuity, the first graph of Figure 2.5 took approximatively 23 minutes and the second graph took 3 minutes whereas each value of Table 2.2 took less than 1 minute same as the values of Table 2.3.

2.5 Conclusions

The main goal of this chapter was first to evaluate the level of the SC of an insurer with respect to some significant parameters for three different annuities (lifetime, deferred and term annuities), and secondly to draw up some comparative remarks from the obtained numerical results. The single (longevity) risk model proposed in this chapter is characterised by the HW uncertainty.
From numerical analysis, we found that when the short rate increases, the SC decreases and the convex form obtained for the deferred annuity implies that there

exists a computational time and a deferred time that minimize the SC of the insurer. These results are some extensions of those obtained by Devolder and Lebègue, 2016 wherein they considered a lump sum at retirement instead of series of payments. From the obtained results, we found that the choice of a product on the annuitant's side strongly depends on annuitant's expectations about their life time as well as on the level of benefit they could need. On the insurer side, the choice could depend on the level of the SC, but we also showed that providing a SC could yield quite good returns. The latter has been explained based on the mean-variance criterion of the IRR of the three products. We also found that the SC computed with a constant confidence level of 99.5% is larger than the result obtained with a time-dependent confidence level. Moreover we observed that the term annuity does not fully hedge the annuitant against longevity risk, as the annuitant could still be alive after the end of the contract. In order to protect the policyholders against the longevity risk using term annuities, we could consider successive term annuities in future research.

In the following chapter, we develop a multi-risks model where we consider equity, interest rate and longevity risks. We will then value the SC of the insurer with respect to these risks and for the three annuities.

Chapter 3

Financial-Longevity risk measurement of annuity products

In this chapter we measure the SC of an insurer within a multi risks market. In fact we combine the risk measure in Chapter 2 along with the interest rate and equity risks; the obtained model is called financial-longevity model. In this case the single premiums paid by the policyholders (in order to purchase an annuity) along with the SC are invested in a financial market. The equity risk is then represented by the uncertain return on a risky investment and the interest rate risk is represented by a stochastic short rate. In order to mitigate the financial risks, insurers could think of building suitable investment strategies. Herein we assume that the premium is invested on risky assets and for simplicity purpose, we assumed the SC to be invested on a stochastic discount bond. As in the previous chapters, we follow the maturity approach proposed by Devolder and Lebègue, 2016 so as to value both SC and IRR. Concerning the longevity risk we use the mortality model defined in Section 2.2. Note that the models developed in this chapter is very helpful not only for annuity trading, but also for investment decision making of insurers. Meaning that in this work we provide the insurer with enough information regarding his risks while selling a given annuity, so as to facilitate his choice on the most solvency capital consuming product.
Our model is based on a profitable attractive investment strategy used to assess the SC of an insure for a given annuity product regarding the equity, interest rate and longevity risks. This investment strategy comes as a modified version of the one proposed by Bauer and Weber, 2008. In fact, in their paper they assessed the risk of annuity given an investment strategy. The investment strategy they used is a fully liability hedging strategy whereas we propose here a partial (or

temporal) liability hedging strategy. The attractiveness of our strategy goes on the one hand toward the policyholder as it guarantees with a hundred percent a given number of benefits; the attractiveness (for the policyholder) in terms of consumption has been studied by Hanewald et al., 2013. On the other hand toward the insurer in the sense that it gives great IRR even though the SC is high. The goal of this chapter is then to point out the most solvency capital consuming product between the three annuities, within our investment strategy. This chapter proposes a general model from which one could derived the longevity model (i.e Chapter 2), the equity model, as well as models with strategy moving from the zero to the fully guaranteed strategies.
The structure of this chapter is the following. In Section 3.1 we present the features of the equity-(interest)rate-longevity model as well as the model's settings and hypothesis. In other words we present the financial market and the investment strategy. We use the mortality model defined in Section 2.2.1 and the liability is defined in Section 2.2.2. In section 3.2, our main theoretical results are presented, i.e the formulas of the SC for a deferred annuity. Finally in Section 3.3 we compare the obtained numerical results and we draw up some remarks and the conclusion follows in the last section.

3.1 Model's features

We base our work on the Black-Scholes setting (see Black and Scholes, 1973 and Merton, 1973) with stochastic short rate within a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$. In this chapter, we assume the insurance company is exposed to the equity, interest rate and longevity risks.
In order to be consistent with the SII regulation, we consider the static risk measure VaR as the risk measurement tool . In this context, the safety level recommended by SII on a one year horizon is $\alpha = 99.5\%$ [1]. Taking into account the long-term aspect of annuity products, here we will measure the risk directly at maturity and not on a one-year time. Therefore, we have also to adapt the safety level, based now on T years instead of one year. In order to define this new confidence level function, we start with the natural overall confidence $\alpha = (99.5\%)^T$ (see for instance Devolder and Lebègue, 2016, Devolder, 2011). Moreover, we avoid having too low confidence level (which arises when T is large) by defining a minimum

[1] Article 101 stipulates that the SCR corresponds to the Value-at-Risk with a confidence level of 99.5%

confidence level α_0 such that the confidence level function never goes below α_0. Hence the confidence level function is given by

$$\alpha(T) = \max\left(\alpha^T, \alpha_0\right), \tag{3.1}$$

where $\alpha = 99.5\%$ and T is the length of the risky period. Note that if $\alpha_0 = \alpha$ then we obtained the SII safety level of $\alpha(T) = 99.5\%$. We assume independence between the equity and the longevity risks; between the interest rate and the longevity risks but dependence between the equity and the interest rate risks.
In the following subsections, we present respectively in detail the financial market framework, the mortality model, the liability of the insurer as well as the investment strategy.

3.1.1 Financial market

We assume that the financial market is made of a money market account B, a stock S and discount bonds P. We assume that there are no dividends, no transaction costs and no taxes. Following the Black-Scholes model, we define the stock by a constant drift ($\mu \in \mathbb{R}$), constant volatility ($\sigma_s \in \mathbb{R}_+^*$); regarding the money market account and the discount bonds we consider a stochastic interest rate $\{r_t\}$. As we are now facing negative interest rates and since this is a practical study, we consider a famous interest rate model: the Vasicek model. In other words, we define the interest rate as an Ornstein-Uhlenbeck process with the following dynamic under the real measure $\mathbb{P}$

$$dr_t = b(c - r_t)dt + \sigma_r dW_t^r, \tag{3.2}$$

where $b, c, \sigma_r, r_t \in \mathbb{R}$ with $b, \sigma_r > 0$ and W_t^r being a Brownian motion on the physical probability space. Note that alternative models could be considered as well. The dynamic of the money market account is then given by

$$dB(t) = r_t B(t)\ dt\ ,$$

for $t \geq 0$ and with $B(0) = 1$.
Denote by $\mathbb{F}^r = (\mathcal{F}_t^r)_{t\geq 0}$ the natural filtration of W_t^r and for a fix maturity T, we denote by $\mathbb{Q}$ the risk-neutral measure of $\mathcal{F}_T^r$ and following the Girsanov theorem (Girsanov, 1960) we have

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\lambda_r W_T^r - \frac{1}{2}\lambda_r^2 T\right), \tag{3.3}$$

where $\lambda_r \in \mathbb{R}$ represents the market price of interest rate risk. Notice that Formula (3.3) only describes the risk neutral measure with respect to the short rate and it is used to value the coupon bond. We do not consider the market price of risk for the stock as in this chapter, the used model is defined within the physical measure.

One can show that the value at time $t \geq 0$ of a zero-coupon bond (ZCB) that pays a unit of currency at maturity time $s \geq 0$ with $s \geq t$ (i.e the discount bond) is a random variable given by (see Vasicek and Fong, 1982)

$$P(t,s) = \mathbb{E}^Q \left[\frac{B(t)}{B(s)} | \mathcal{F}_t^r \right] = A_r(t,s) e^{-B_r(t,s) r_t}, \tag{3.4}$$

where $\mathbb{E}^Q$ is the expectation under measure $\mathbb{Q}$, with

$$A_r(t,s) = \exp \left[\left(c - \frac{\lambda_r \sigma_r}{b} - \frac{\sigma_r^2}{2b^2} \right) (B_r(t,s) - (s-t)) - \frac{\sigma_r^2}{4b} B_r(t,s)^2 \right]$$

and

$$B_r(t,s) = \frac{1}{b} \left(1 - e^{-b(s-t)} \right).$$

From the Ito formula, we obtain the following dynamic of the discount bond

$$dP(t,s) = (r_t - \lambda_r \sigma_r B_r(t,s)) P(t,s) dt - \sigma_r B_r(t,s) P(t,s) dW_t^r. \tag{3.5}$$

As for the stock, let $\{W_t^s\}_{t \in \mathbb{N}}$ be a Brownian motion under the physical probability measure $\mathbb{P}$; the stock satisfies the following stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma_s S_t dW_t^s,$$

with $S_0 = 1$. One can show that the value at a given time $t \in \mathbb{R}_+$ of a unit invested on such stock is given by

$$S_t = S_0 e^{\left(\mu - \frac{\sigma_s^2}{2} \right) t + \sigma_s W_t^s}.$$

We assume dependence between the stock and the interest rate, i.e W_t^s is correlated with W_t^r with $\rho = corr(W_t^s, W_t^r)$ being the correlation parameter.

3.1.2 Investment strategy

We consider that the premium received by the insurer from the policyholders is invested in the financial market described above. Our model consists of different investment strategies depending on the level of hedging. We consider two investment strategies. These strategies have no effect on the level of the benefits, but rather have effect on the assets.

Constant proportion proportion strategy without guaranteed

Denote by A_0 the single premium paid by a policyholder, then following our described financial market a constant proportion $x_s \in [0,1]$ of A_0 is assume to be invested in the stock S_t, a proportion $x_p \in [0,1]$ invested in the long term bond $P(t,T)$ with maturity $T > 0$ and the remaining proportion $(1 - x_s - x_p)$ is invested in the money market account $B(t)$ as defined in Section 3.1.1. The case of time-depending proportions and proportional rebalancing was partly studied by Devolder and Lebègue, 2016 and can be extended to our setting. Denote by A_t the value at time $t \geq 0$ of the portfolio made of x_s amount of the single premium A_0 in the stock, x_p amount in the $P(t,T)$ and $(1 - x_s - x_p)$ in the money market account, it follows that the dynamic of A_t is given by

$$\begin{aligned} dA_t &= x_s \frac{dS_t}{S_t} A_t + x_p \frac{dP(t,t+T)}{P(t,t+T)} A_t + (1 - x_s - x_p) \frac{dB(t)}{B(t)} A_t \\ &= \left((1 - x_s) r_t + x_s \mu + x_p \lambda_r \sigma_r B_r(t,t+T) \right) A_t dt + x_s \sigma_s A_t dW_t^s \\ &\quad - x_p \sigma_r B_r(t,t+T) A_t dW_t^r. \end{aligned}$$

For simplification purpose, let

$$B_r(T) = B_r(t,t+T) = \frac{1}{b}\left(1 - e^{-bT}\right),$$

one can show that the value at a given time v $(v \geq t \geq 0)$ of an investment of the portfolio during $[t,v]$ is given by

$$\begin{aligned} A(t,v) &= A_t \exp\left[\left(x_s \mu + x_p \lambda_r \sigma_r B_r(T) - \frac{1}{2} Y(T)\right)(v - t)\right] \\ &\quad \times \exp\left[(1 - x_s)\int_t^v r_u du + x_s \sigma_s W_{v-t}^s - x_p \sigma_r B_r(T) W_{v-t}^r\right]; \end{aligned} \tag{3.6}$$

where

$$Y(T) = x_s^2 \sigma_s^2 + x_p^2 \sigma_r^2 B_r^2(T) - 2\rho x_s x_p \sigma_s \sigma_r B_r(T). \tag{3.7}$$

This constant proportion allocation strategy will be used in the deferred period and we will use the notation $A_t = A(0,t)$.

Constant proportion strategy with guaranteed : the m guaranteed cash flows (mG-cf) strategy

This strategy is the combination of the constant proportion allocation strategy and a zero-coupon (ZC) strategy.

Our motivation for this strategy comes from our desire to increase the attractiveness of annuity products for the insurer, so as to obtain a product that consumes less solvency capital for the insurer or gainful for the shareholders as we will show in Section 4. Following the strategies proposed by Bauer and Weber, 2008 who considered two strategies : the constant proportion allocation strategy and the liability hedging strategy, we based on the latter to define a periodic liability hedging strategy. In fact, instead of considering the liability hedging strategy on the contract duration as they did, we consider it only at the beginning of the contract and we refer to the obtained strategy as the m guaranteed cash flows. The term *guaranteed* here refers to the fact that the first m benefits are not subject to any risk as they are invested on a risk-free asset and depends on a deterministic life table. In order to make our strategy understandable, we propose the following detailed strategy for the lifetime annuity that we generalise later on to the deferred annuity.

To this end, we first remind the structure of the contract considered in this chapter. As in the previous chapter, the contract considered here consists of a homogeneous group of retired policyholders buying either a lifetime, deferred or term annuity. The liability of the insurer is defined as in Section 2.2.2 with the cohort initial benefit denoted by $\overline{R} = N_0 R$; where the individual benefit R for a d years deferred annuity is given by Formula (2.6) and

$$\overline{R} = \frac{\overline{A_0}/P_f(0,d)}{\sum_{j=d}^{n} \ p_{65}(0,j) \ P_f(d,j)}.$$

Note that $P_f(t,s)$ is the risk-free discount bond defined in Chapter 2 whereas $P(t,s)$ is the risky bond described by the Vasicek short rate (3.2).

a) Lifetime annuity

Consider an homogeneous initial cohort of N_0 policyholders paying a total single premium of $\overline{A_0}$ in order to purchase an immediate lifetime annuity for which the payment stream ends at most at time $t = n$ (in case the policyholder is alive). Our strategy on the payment period consists of *investing the first m benefits (with $0 \leq m < n$) per survivors in a risk-free asset during the period $[0,m)$ and the remainder last $m_0 = n + 1 - m$ benefits are subject to a risky investment during the whole contract duration $[0,n)$. This strategy is called the m guaranteed cash flows (mG-cf) strategy.* Notice that those m first benefits are assumed to be equity, interest rate guaranteed and conditionally longevity-guaranteed. In this

case we have two sub-periods : the guaranteed period $[0, m)$ where the insurer is conditionally hedged against the risks and the non-guaranteed period $[m, n)$ where he is exposed to the risks. The difference between these two sub-periods is that in the guaranteed period the mortality is considered and the benefit payout does not affect the risky investment whereas on the non-guaranteed period the mortality is considered and the benefit payout does affect the risky investment. By *equity, interest rate-guaranteed benefits*, we mean the benefits which are fully hedged against both the equity and the interest rate risks; whereas by *conditionally longevity-guaranteed benefits* we refer to the benefits fully hedged against longevity risk under some predefined conditions. This implies that, if the longevity condition is satisfied, then the insurance company will be 100% solvent for the first m benefits (i.e from time $t = 0$ to time $t = m-1$) and 0.5% uncertain for the remaining period $[m, n)$ according to the SII.
The mentioned *condition* depends on both the policyholders and the insurer; it is stated at the contract inception and could be for example a given life table, a given trend such as the average trend of the mortality or the average trend of the survival and so on. In our case, we consider the best estimate of the survival index denoted by $S_{65}(0, t)$, where $x_0 = 65$ represents the retirement age of the policyholders. In other words, for all $t \in [0, m-1)$ the number of survivors at time t is

$$N_t = N_0 S^{65}(0, t) = N_0 \ \mathbb{E}\left[I_t^{65}\right],$$

where $S^{65}(0, t)$ is the survival probability of an individual initially aged 65, of living at least up to age $65 + t$ following the best estimate of the survival index. Note that $S^{65}(0, \cdot) = p_{65}(0, \cdot)$ and we use different notations just to emphasise that both could be different. Therefore, the m conditionally guaranteed benefits for a lifetime annuity evaluated at inception $t = 0$ is given by

$$\overline{L_0^{(m)}} = \sum_{j=0}^{m-1} R \ \mathbb{E}[N_j] \ P(0, j) = \sum_{j=0}^{m-1} \overline{R} \ S^{65}(0, j) \ P(0, j). \tag{3.8}$$

This amount of benefits will then be deduced from the unique premium and the remaining asset will be

$$\overline{C_0} = \overline{A_0} - \overline{L_0^{(m)}}, \tag{3.9}$$

where $\overline{A_0} = \overline{A}(0, t)$ is defined by Formula (3.6).

b) Generalization to the deferred annuity

Here we generalize the mG-cf strategy to a d years deferred annuity as follows. Considering the same initial cohort as previously, paying the same single premium for a d years deferred annuity hence we have three sub-periods :

- the deferred period $[0, d)$ where no benefit payout is made but the mortality is considered;

- the guaranteed period $[d, d+m)$ where the mortality is considered and the benefit payout does not affect the risky investment: the insurer is hedged against the financial risks in this period.

- and the non-guaranteed period $[d+m, n)$ where the mortality is considered and the benefit payout affects the risky investment: the insurer is exposed to both the financial and mortality risks in this period.

The strategy consists of *assessing at time $t = 0$ the first m benefits (with $0 \leq m < n+1-d$) per survivor invested in a risk-free asset during the guaranteed period $[d, d+m)$, let's denote it by $\overline{L_0^{(m)}}$. Note that the remaining $m_0 = n+1-d-m$ benefits will be subject to a risky investment (cf. 3.1.2) during the contract duration $[0, n)$. $\overline{L_0^{(m)}}$ will then be deduced from the unique premium $\overline{A_0}$ and the remaining premium denoted by $\overline{C_0}$ will be invested on the portfolio during $[0, n]$ in order to pay the non-guaranteed benefits.* It follows that the m conditionally guaranteed benefits for a d years deferred annuity evaluated at time $t = 0$ is given by

$$\overline{L_0^{(m)}} = \sum_{j=d}^{d+m-1} R \; \mathbb{E}[N_j \; P(0,j)] = \sum_{j=d}^{d+m-1} \overline{R} \; S^{65}(0,j) \; P(0,j). \tag{3.10}$$

The remaining asset at time $t = 0$ will be

$$\overline{C_0} = \overline{A_0} - \overline{L_0^{(m)}}. \tag{3.11}$$

Figure 3.1 represents the mG-cf strategy for a d years deferred annuity

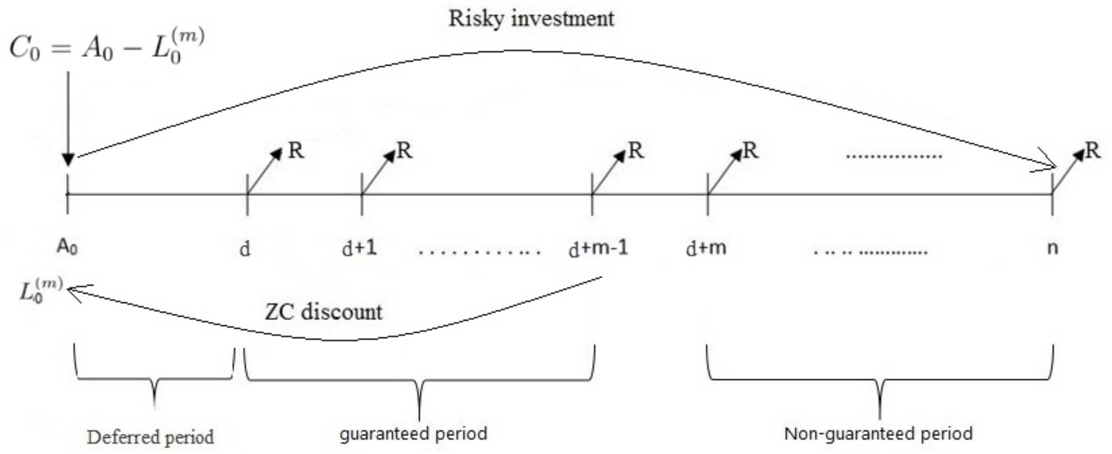


Figure 3.1: Illustration mG-cf for deferred annuity.

We observe that for $m = 0$, the mG-cf strategy is equivalent to the constant allocation strategy (*cf.* 3.1.2) since there is no benefit guaranteed. Whereas for $m = n$ we obtain the liability hedging strategy proposed by Bauer and Weber, 2008.
This formulation of the mG-cf strategy leads to the following question : what to do if the cohort's survival does not follow the average trend of the survival during the guaranteed period? In other words, how would the insurer pay the benefits of the gap of survivors during the guaranteed period? To answer to this question, we need to take into account the risk generated by a possible gap of survivors during the guaranteed period. This risk is evaluated yearly and given by the difference between the survivors under the real mortality trend (i.e using the survival index) and the survivors under the best estimate of the survival index (i.e using the guaranteed mortality table).
In the next section, we present the formulae used to measure the risk borne by an insurer selling a d years deferred annuity within the mG-cf strategy by computing the insurer SC.

3.2 Insurer's solvency Capital

This section shows the main formulae of our model, i.e the formulae of the SC. Recall that the first m guaranteed cash flows are evaluated at inception and deduced from the single premium, the remaining premium is invested on a portfolio and the SC is invested on a long term bond from the current time t till the supposed end

of the contract n. Moreover, we value the SC by the use of the static risk measure VaR with respect to the final surplus for a given annuity. The theoretical analysis below corresponds to the computation of the SC for a d years deferred annuity of length n. The corresponding analysis for a lifetime annuity will be obtained by setting $d = 0$ and that of a d' years term annuity by setting both $d = 0$ and $n = d'$. It is important to stress that the parameters of the HW mortality model of a given cohort are supposed to be known at inception, as they are calibrated from the cohort life table and these parameters are valid only for the cohort from which they have been calibrated.
We consider a $d < n$ years deferred annuity for which we have three computational intervals as described in Section 3.1.2, i.e the deferred period $[0, d)$, the guaranteed period $[d, d+m)$ and non-guaranteed $[d+m, n)$ period. Note that the following conditions should be satisfied

$$0 < m < n + 1 - d;$$

meaning the number of guaranteed benefits m should be less or equals to the maximum number of benefits $n + 1 - d$. Below is given the formula of the SC of an insurer selling a deferred annuity in each period.

3.2.1 SC during the deferred period

In the deferred period $[0, d)$, the value at inception $t = 0$ of the m guaranteed benefits $\overline{L_0^{(m)}}$ is given by

$$\overline{L_0^{(m)}} = \sum_{j=d}^{d+m-1} R\ \mathbb{E}[N_j]\ P(0,j) = \sum_{j=d}^{d+m-1} \overline{R}\ S^{65}(0,j)\ P(0,j), \tag{3.12}$$

where $\overline{R}$ is the cohort initial benefit given by Formula (2.6) and $S^{65}(0,j)$ is the best estimate of the survival index. For simplification purpose, let's denote the portfolio return by

$$\begin{aligned} Asset(t,v) &= \exp\left[\left(x_s\mu + x_p\lambda_r\sigma_r B_r(T) - \frac{1}{2}\,Y(T)\right)(v-t)\right] \\ &\quad \times \exp\left[(1-x_s)\int_t^v r_u du + x_s\sigma_s W_{v-t} - x_p\sigma_r B_r(T) W^r_{v-t}\right]; \end{aligned} \tag{3.13}$$

where $Y(T)$ is given by Formula (3.7). The value at the computational time $t \in [0, d)$ of the remaining asset available for the non-guaranteed benefits is given by

$$\overline{C_t} = \overline{C_0}\ Asset_{obs}(0,t) \tag{3.14}$$

which is known at time t and where $\overline{C_0}$ is given by Formula (3.11) and $Asset_{obs}(0,t)$ is the realised or observed return on the portfolio at time t. In the sequel of this work, variables with 'obs' subscript represent the observed variables.
The value at time n of the remaining $m_0 = n+1-d-m$ non-guaranteed benefits deduced yearly from the asset value between time $d+m$ and time n is given by

$$\begin{aligned}
\overline{L'}(t,n,d,m) &= \sum_{j=m+d}^{n} R\ N_j\ Asset(j,n) \\
&= \sum_{j=m+d}^{n} R\ N_{d+m}\ I_{j-d-m}^{65+d+m}\ Asset(j,n) \\
&= \sum_{j=m+d}^{n} R\ N_d\ I_m^{65+d}\ I_{j-d-m}^{65+d+m}\ Asset(j,n) \\
&= \sum_{j=m+d}^{n} R\ N_t\ I_{d-t}^{65+t}\ I_m^{65+d}\ I_{j-d-m}^{65+d+m}\ Asset(j,n).
\end{aligned}$$

Note that N_t^{obs} is the observed number of survivors at time t, in other words N_j^{obs} is known for $j \leq t$ and N_j is random for $j > t$. It follows that the risk of possible gap on survival during the guaranteed period is captured by

$$\overline{L'_r}(d,m) = \sum_{j=d}^{d+m-1} R\left(N_j - N_0\ S^{65}(0,j)\right) Asset(j,d+m), \tag{3.15}$$

where $N_j = N_t\ I_{d-t}^{65+t}\ I_{j-d}^{65+d}$. Hence at any time $t \in [0,d)$, the SC satisfies the following solvency condition at maturity

$$\mathbb{P}_t\left[\overline{C}(t,n) - \overline{Benef2} + \overline{SC}(t,n)P^{-1}(t,n) < \overline{L'}(t,n,d,m)\right] \leq 1-\alpha(n-t),$$

where

a) $\overline{C}(t,n)$ is the value at n of $\overline{C_t}$ (i.e Formula (3.14)) and it is given by

$$\overline{C}(t,n) = \overline{C_t}\ Asset(t,n);$$

b) $\overline{Benef2}$ is the value at n of the benefit gap occurred on the guaranteed period and added or deduced on the remaining asset. It is computed as

$$\overline{Benef2} = \overline{L'_r}(d,m)\ Asset(d+m,n),$$

where $\overline{L'_r}(d,m)$ is given by Equation(3.15);

c) $\overline{L'}(t,n,d,m)$ is the value at n of the non-guaranteed benefits computed above.

3.2.2 SC during the guaranteed period

The difference between this period $[d, d+m)$ and the previous period is that this is a payment guaranteed period as the payments are made but not from the risky investment; whereas the previous period is just the deferred period where no payments are made. Hence for a given computational time $t \in [d, d+m)$ the value at n of the non-guaranteed benefits denoted by $\overline{L''}(t,n,d,m)$ has to be adjusted based on the information obtained at time t; we have

$$\begin{aligned}\overline{L''}(t,n,d,m) &= \sum_{j=d+m}^{n} R\ N_j\ Asset(j,n)\\ &= \sum_{j=d+m}^{n} R\ N_{d+m}\ I_{j-d-m}^{65+d+m}\ Asset(j,n)\\ &= \sum_{j=d+m}^{n} R\ N_t^{obs}\ I_{d+m-t}^{65+t}\ I_{j-d-m}^{65+d+m}\ Asset(j,n)\end{aligned}$$

where $Asset(j,n)$ is given by Formula (3.13). In this period the possible benefits gap that might occur during the remaining guaranteed period i.e during the period $[t, d+m)$, is obtained according to the actual number of survivors. Therefore the future benefits gap is

$$\overline{L_r''}(t,d,m) = \sum_{j=t+1}^{d+m-1} R\left(N_j - N_0\ S^{65}(0,j)\right) Asset(j,d+m),$$

where $N_j = N_t^{obs}\ I_{j-t}^{65+t}$ with N_t^{obs} observed. The asset value at time t $\overline{C_t}$ is obtained based on both the past gaps of survival and the past investment experience and it is equal to

$$\overline{C_t} = \overline{C_0}\ Asset_{obs}(0,t) - \sum_{j=d}^{t} \overline{R}\left(I_{obs}^{65}(j) - \ S^{65}(0,j)\right) Asset_{obs}(j,t), \tag{3.16}$$

where $I_{obs}^{65}(j)$ represents the observed survival proportion at time $j \le t$. Note that the second term of the right hand side (RHS) for (3.16) is observed and represents the benefit gap observed before and up to the computational time t. Hence the SC satisfies the solvency condition

$$\mathbb{P}_t\left[\overline{C}(t,n) - \overline{Benef3} + \overline{SC}(t,n)P^{-1}(t,n) < \overline{L''}(t,n,d,m)\right] \le 1 - \alpha(n-t),$$

where

a) $\overline{C}(t,n)$ is the value at n of $\overline{C_t}$ (given by Equation(3.16)), i.e

$$\overline{C}(t,n) = \overline{C_t}\ Asset(t,n),$$

b) $\overline{Benef3}$ is the value at n of the future gap occurring on the remaining guaranteed period. It is given by

$$\overline{Benef3} = \overline{L''_r}(t,d,m)\ Asset(d+m,n);$$

c) $\overline{L''}(t,n,d,m)$ is the value at n of the non-guaranteed benefits computed above.

3.2.3 SC during the non-guaranteed period

In the non-guaranteed period $[d+m,n)$, for a given computational time $t \in [d+m,n)$, the SC satisfies the following solvency condition

$$\mathbb{P}_t\left[\overline{C}(t,n) + \overline{SC}(t,n)P^{-1}(t,n) < \overline{L'''}(t,n)\right] \leq 1-\alpha(n-t), \tag{3.17}$$

where

a) $\overline{C}(t,n)$ is the value at n of the asset t $\overline{C_t}$. The latter is obtained based on the gap of survivors during the guaranteed period, the past investment experience as well as the non-guaranteed benefits paid from time $d+m$ up to time t. More explicitly, since the information up to time t is known, it follows that $\overline{C_t}$ and N_t are known. The value at time $d+m$ of the asset used for the $n-d-m+1$ non-guaranteed benefits is known and given by

$$\overline{C'_{d+m}} = \overline{C_0}\ Asset_{obs}(0,d{+}m) - \sum_{j=d}^{d+m-1} \overline{R}\left(I^{65}_{obs}(j) - \ S^{65}(0,j)\right) Asset_{obs}(j,d{+}m),$$

where the second term of the RHS represents the gap of benefits observed and deduced from the asset during the guaranteed period. It follows that the asset at time t is given by

$$\overline{C_t} = \overline{C'_{d+m}}\ Asset_{obs}(d+m,t) - \sum_{j=d+m}^{t} \overline{R}\ I^{65}_{obs}(j)\ Asset_{obs}(j,t),$$

where the second term of the RHS represents the non-guaranteed benefits paid from time $d+m$ up to the computational time t. Therefore we have

$$\overline{C}(t,n) = \overline{C_t}\ Asset(t,n).$$

b) $\overline{L'''}(t,n)$ is the value at n of the remaining non-guaranteed benefits yearly deduced from the asset and it is given by

$$\begin{aligned}\overline{L'''}(t,n) &= \sum_{j=t+1}^{n} R\ N_j\ Asset(j,n) \\ &= \sum_{j=t+1}^{n} R\ N_t^{obs}\ I_{j-t}^{65+t} Asset(j,n).\end{aligned}$$

Note that for $m = 0$, the mG-cf strategy defined in 3.1.2 becomes the constant proportion strategy defined in 3.1.2 and in this case the Subsection 3.2.2 does not exist any more.

3.3 Comparative discussions from numerical results

For this chapter, we study the effect of some parameters on the SC. For instance, we make a sensitivity study with respect to deferred time, term time, number of guaranteed benefits, confidence level as well as ageing. Concerning the ageing effect, we consider two different cohorts initially aged 65 and 75 respectively.

3.3.1 Simulation framework

To obtain the numerical results of the SC we first calibrate both HW and Vasicek models given by Formula (2.1) and (3.2). For the HW model, we consider two generations, i.e an unisex projected generational life table of individual aged $x_0 = 65$ and $x_0 = 75$ with an ultimate age of 110 (available on the IA|BE life table proposed by Antonio et al., 2015). From this we calibrate the model using mean square error (MSE); and we obtain the following parameters

μ_0^{65} and μ_0^{75}	A	B	a	σ^μ	MSE
0.0105677	0.0005749505	0.1304207503	0.0014965354	0.0083530153	0.000303644
0.02633591	0.009698091	0.115573842	0.310552836	0.051888992	0.000145998

Table 3.1: Calibration parameters of the HW model using MSE. Second and third rows represent the parameters for $x_0 = 65$ and $x_0 = 75$ respectively.

Concerning the Vasicek models, we estimate the parameters from the European Central Bank overnight rates [2] using the MLE and we obtain

b	c	σ_r
0.6587	0.01039	0.0122

Table 3.2: Parameters of the Vasicek interest rate model.

The market price of interest rate risk is obtained from the 30 years maturity AAA-rated Euro area central government bonds in September 30, 2019 using the MSE and we have $\lambda_r = 0.0897$.

Secondly, we use the Monte Carlo (MC) simulations of the VaR (see Bauer et al., 2012). For this purpose we consider two cohorts of $N_0 = 1000$ annuitants respectively aged $x_0 = 65$ and $x_0 = 75$ at the affiliation time $t = 0$. Each participant pays $A_0 = 1000\$$ to the insurer at inception in order to receive an amount R during the payment period, then $\overline{A_0} = N_0 A_0 = 1000 * 1000\$$. Moreover, we assume $n = 45$, $r = 1\%$ and a 15 years term annuity i.e $d' = 15$. For the deferred annuity we assume that the payment starts once the annuitant is aged $x_d = 81$ (called the deferred age). In other words for the cohort initially aged $x_0 = 65$ we consider a $d = 16$ years deferred annuity and for the cohort with $x_0 = 75$ we consider a $d = 6$ years deferred annuity.

We study the sensitivity of the SC with respect to the confidence level defined in Section 3.1 and for that purpose we consider the following values of the minimal confidence level $\alpha_0 \in \{85\%, 90\%, 95\%, 99\%, 99.5\%\}$. For this values of α_0, the confidence level function (3.1) looks like

[2] `https://www.ecb.europa.eu/stats/policy_and_exchange_rates/key_ecb_interest_rates/html/index.en.html`, downloaded on the 02/09/2019

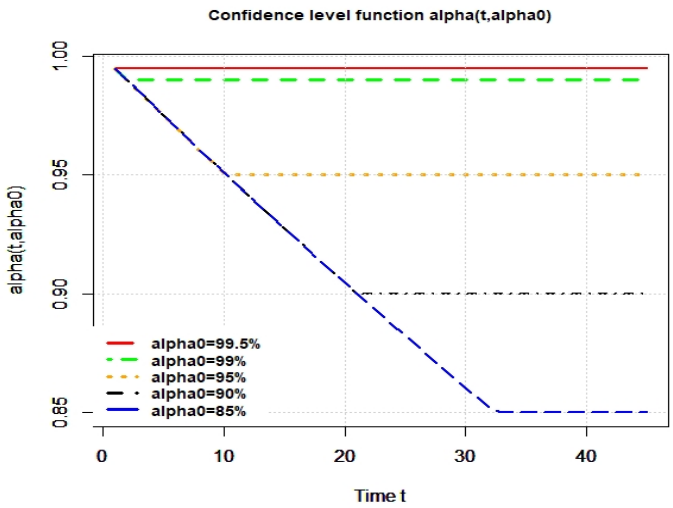


Figure 3.2: Confidence level function with respect to t and α_0.

Concerning the parameter of the stock, we consider the parameters of the geometric Brownian motion calibrated with the MLE (Phillips and Yu, 2009) using the S&P500 indexes[3] from 1871 to 2018. Hence we obtained the drift equals to $\mu = 0.058702$, the volatility equals to $\sigma_s = 0.204172$ and the correlation between the stock and the discount bond is $\rho = -60\%$.

3.3.2 Results

In this section some numerical results for each annuity are given in order to provide sufficient information to insurer so as to guide his trading decision making.

Basic case : lifetime annuity

Table 3.3 shows the sensitivity of SC for a lifetime annuity with respect to ageing, m and α_0. We observe a decreasing trend with the number of guaranteed benefits m. In others words, for a lifetime annuity, the more benefits we guarantee the less SC we have. As expected, the less minimum safety level we use, the less SC we have. Furthermore, unlike the fully guaranteed strategy (i.e $m = n$), younger cohort of annuitants leads to a higher SC. Note that when $m = n$, the obtained SC reflects both longevity and interest rate risk borne by the insurer.

[3] `http://www.multpl.com/s-p-500-historical-prices/table/by-year`, March 7 2019

	$x_0 = 65$			$x_0 = 75$		
	$m = 0$	$m = 5$	$m = n$	$m = 0$	$m = 5$	$m = n$
$\alpha_0 = 85\%$	0.267	0.132	0.0282	0.261	0.115	0.0638
$\alpha_0 = 90\%$	**0.361**	0.198	0.0443	**0.344**	0.167	0.08624
$\alpha_0 = 95\%$	0.5067	0.2987	0.067	0.471	0.241	0.121
$\alpha_0 = 99.5\%$	0.893	0.571	0.1384	0.818	0.453	0.2235

Table 3.3: Values of $\overline{SC}/\overline{A_0}$ at $t = 0$ for a lifetime annuity, with $x_s = 15\%$ and $x_p = 25\%$.

For Comparison purpose, we consider the standard case where $m = 0$ and $\alpha_0 = 90\%$. We choose the non-guaranteed cash flow strategy (i.e $m = 0$) because it is a common strategy in the literature and in practice. The choice of $\alpha_0 = 90\%$ comes from the fact that the time t for which $(99.5\%)^t = \alpha_0$ is the most closer to $n/2$ for both lifetime and deferred annuities as shown in Figure 3.2.

Deferred analysis

	$x_0 = 65$			$x_0 = 75$		
	$m = 0$	$m = 5$	$m = n - d$	$m = 0$	$m = 5$	$m = n - d$
$\alpha_0 = 85\%$	0.0282	-0.0432	0.09373	0.1674	0.0551	0.106
$\alpha_0 = 90\%$	**0.1071**	0.00553	0.1344	**0.2456**	0.0983	0.1425
$\alpha_0 = 95\%$	0.228	0.0747	0.2035	0.365	0.1632	0.2011
$\alpha_0 = 99.5\%$	0.544	0.2435	0.4015	0.681	0.337	0.3683

Table 3.4: Values of $\overline{SC}/\overline{A_0}$ at $t = 0$ for a $d = 16$ years deferred annuity, with $x_s = 15\%$ and $x_p = 25\%$.

$x_0 = 65$ $x_0 = 75$

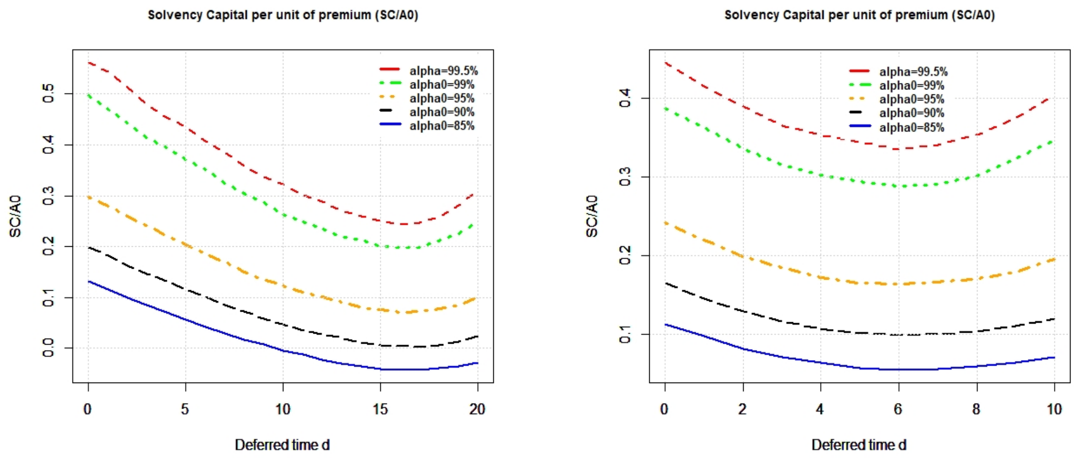


Figure 3.3: $\overline{SC}(d)/\overline{A}_0$ at $t = 0$ with $m = 5$.

Figure 3.3 illustrates the effect of the deferred time d on the SC for both cohorts with five benefits guaranteed.

- We observe for both cohorts that, the SC has a convex behaviour with respect to d; meaning that in this case there exists an optimal value of d that minimised the SC. This comes from the same justification provides in Chapter 2.

- Moreover, ageing effect increases with d and we have high SC for the older cohort than for the younger for large values of d. This comes from the fact that, the longevity risk is more important at advanced ages as also shown in Table 3.4 where we can observe that the mG-cf strategy yields lower SC compared to both non-guaranteed ($m = 0$) and fully guaranteed ($m = n - d$) strategies.

- The increasing effect with the safety level is also observed in the table and figure. Note that the negative SC obtained in Table 3.4 means that the insurer needs no extra capital to guarantee his solvency with the used confidence level, i.e $SC = 0$.

- We see that when $m = n$, lifetime annuity yields lower SC compared to the deferred annuity. This comes from the fact that when $m = n$, the value of the m guaranteed cash flows $\left(L_0^{(m)}\right)$ of the lifetime annuity is equal to the value of the m guaranteed cash flows of the deferred annuity and they are both equal to the initial premium as we assumed $S^{65}(0,j) = p_{65}(0,j)$, for $j \in [0,n]$. Moreover, benefits payments of a lifetime annuity begin before those of deferred annuity; hence insurer deals early with low unexpected survivors for lifetime annuity whereas he suddenly faces high gaps (between expected and realized survivors) at the beginning of the payment period for the deferred annuity. This highly decreases his reserve and increases his SC for the deferred annuity. In other words, this can be seen as if the insurer was spreading some benefit gaps over several years into low payments for the lifetime annuity whereas he suddenly pays high benefit gaps after the deferred period for the deferred annuity.

- From the standard case (i.e values in bold), we observe that lifetime annuity yields higher SC compared to term and deferred annuities. Deferred annuity yields higher SC than term annuity for older cohort, unlike for younger cohort.

Term analysis

	$x_0 = 65$			$x_0 = 75$		
	$m = 0$	$m = 5$	$m = d'$	$m = 0$	$m = 5$	$m = d'$
$\alpha_0 = 85\%$	0.1844	0.0825	0.031	0.2133	0.0899	0.0923
$\alpha_0 = 90\%$	**0.185**	0.0822	0.0313	**0.2132**	0.08914	0.0924
$\alpha_0 = 95\%$	0.216	0.1008	0.0374	0.2496	0.1097	0.1083
$\alpha_0 = 99.5\%$	0.3697	0.1876	0.0731	0.4302	0.211	0.1937

Table 3.5: Values of $\overline{SC}/\overline{A_0}$ at $t = 0$ for a $d' = 15$ years term annuity, with $x_s = 15\%$ and $x_p = 25\%$.

For a d' years term annuity, let's assume $m = [3 + \frac{d'-5}{8}]$, where $[x]$ represents the integer part of x.

- Figure 3.4 shows that long maturity yields high SC and ageing increases the SC as shown in Table 3.5. The non-smooth curves reflect the impact of the parameter m which changes with d'.

- Note that SC obtained for the 15 years term annuity with the older cohort in Table 3.5 are close to that of the lifetime annuity, since in that case we have an ultimate age of 90 years old.

- From Table 3.5, we observe that the values of the SC obtained for the older cohort are almost equivalent (or stable) with respect to the number of guaranteed benefits m (with $m > 1$). This is a consequence of the ageing effect in the sense that the number (or the duration) of the guarantee is less important at advanced ages, unlike younger ages.

$x_0 = 65$ $\qquad$ $x_0 = 75$

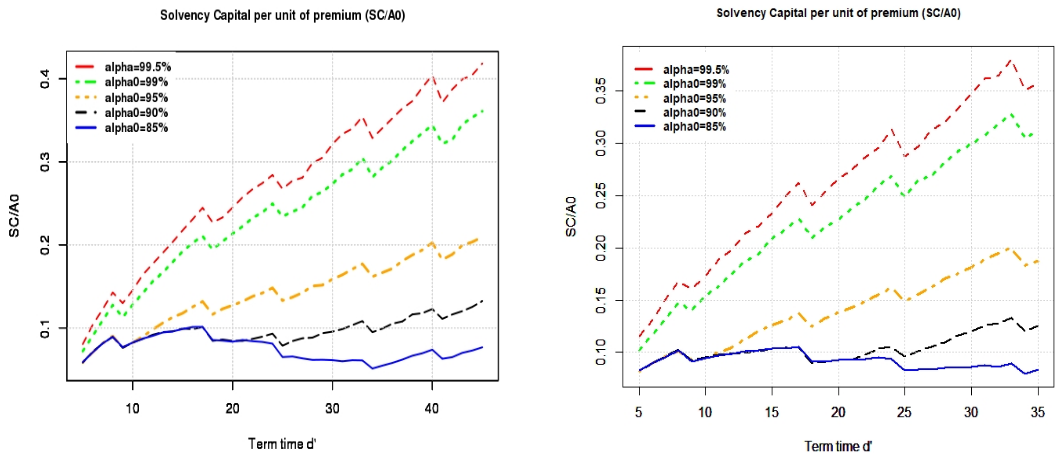


Figure 3.4: $\overline{SC}(d')/\overline{A_0}$ at $t = 0$ with $m = [3 + \frac{d'-5}{8}]$.

Guarantee effect

Below, we present changes of the SC with respect to number of guaranteed benefits m for lifetime and $d = 16$ years deferred annuities. First row of Figure 3.5 shows that for a 16 years deferred annuity, minimum SC depends on the minimum confidence level α_0. For instance, small α_0 and $m = 1$ produce the lower SC whereas for high α_0, we can find values of $m > 1$ that minimises the SC for both cohorts. Furthermore, the safety level does increase the SC of both lifetime and deferred annuities. Older cohort yields higher SC for a d years deferred annuity, unlike the lifetime annuity. For the latter annuity one can find a parameter $m \neq \{0, n\}$ that gives lower SC for both cohorts.

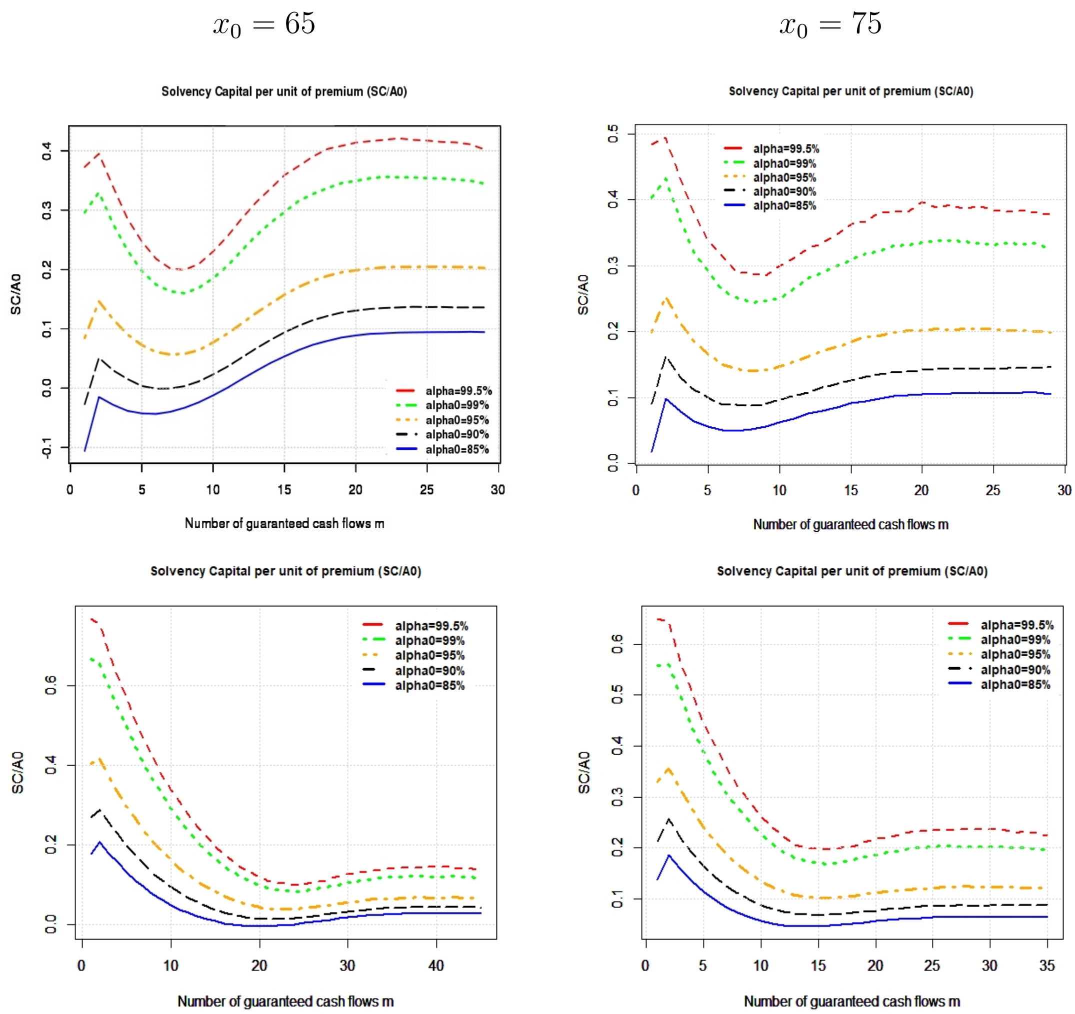


Figure 3.5: $\overline{SC}(m)/\overline{A}_0$ at $t = 0$, first row represents a 16 years deferred annuity and the second row is a lifetime annuity.

A sensitivity analysis of the SC with respect to the volatility of the force of mortality σ_μ is given in Appendix A.2.

3.4 The internal rate of return of shareholders

In this section, we discuss the possible profitability of the annuity's trading on the shareholder point of view by computing their IRR following the approach described in Section 2.4.2. We remind that the shareholder will lose all his initial capital (i.e the SC) if IRR$= -1$ and when IRR$= 0$ then the exact amount of the SC will be recovered at the end of the contract. For a strictly positive IRR, the shareholder will make a profit upon the IRR. Moreover a shareholder can define his minimum rate of return, says π, in such a way that he will invest on the annuity with an expected IRR strictly greater than π. The minimum rate π can be for example the risk-free rate available on the market at the contract inception or an expected rate of return of a given risky asset, etc. Remind that by definition, using our investment strategy the SC satisfies the following solvency condition

$$\mathbb{P}_t\left[\overline{C}(t,n) - \overline{Benef} + \overline{SC}(t+n)P^{-1}(t,n) < \overline{L'}(t,n,d)\right] \leq 1 - \alpha(n-t),$$

from this we define the total final surplus of the contract as follows

$$\chi(x,d,m;w) := \overline{C}(t,n) - \overline{Benef} + \frac{\overline{SC}(t,n)}{P(t,n)} - \overline{L'}(t,n,d), \tag{3.18}$$

where $\overline{Benef}$ represents the benefit's gap as defined in the previous section. One can show that the IRR is given by

$$\tau(x,d,m;w) = \left(\frac{(\chi(x,d,m;w))^+}{\overline{SC}(t,n)}\right)^{\frac{1}{n-t}} - 1; \tag{3.19}$$

where $(x^+) = \max(x,0)$. Note that the IRR is computed if and only if the SC is strictly positive.

Tables 3.6 and 3.7 present the mean and variance of the IRR for both cohorts with a sensitive analysis on parameters m and α_0.

	Lifetime annuity			Deferred annuity			Term annuity		
	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$
$m = 0$	1.7262% (0.04253%)	1.5327% (0.03636%)	1.327% (0.01206%)	6.1262% (0.11%)	3.793% (0.0536%)	2.086% (0.014%)	2.3311% (0.2745%)	2.3209% (0.2739%)	1.703% (0.11285%)
$m = 5$	2.48504% (0.04968%)	2.0794% (0.04%)	1.5596% (0.01274%)	/	9.454% (0.1425%)	2.61% (0.0152%)	4.1294% (0.3031%)	4.1521% (0.3018%)	2.6331% (0.1227%)
$m = n$	2.299% (0.04357%)	1.932% (0.0349%)	1.4501% ($9.557e^{-5}$)	1.888% (0.0392%)	1.63% (0.0323%)	1.33% (0.009%)	3.382% (0.261%)	3.3382% (0.261%)	2.1655% (0.09755%)

Table 3.6: Mean and variance (values in brackets) of the IRR at $t = 0$ for $d' = 15$, $xs = 15\%$, $xp = 25\%$ and a deferred age of $x_d = 81$ for cohort with $x_0 = 65$.

	Lifetime annuity			Deferred annuity			Term annuity		
	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$
$m = 0$	1.642% (0.064%)	1.407% (0.0574%)	1.245% (0.01952%)	2.065% (0.074%)	2.191% (0.06135%)	1.582% (0.0199%)	2.02% (0.2632%)	2.003% (0.26495%)	1.5333% (0.1071%)
$m = 5$	2.521% (0.07183%)	2.103% (0.061%)	1.45% (0.02025%)	4.08% (0.0945%)	3.126% (0.0697%)	1.942% (0.0216%)	4.0997% (0.29023%)	4.117% (0.2896%)	2.5613% (0.11204%)
$m = n$	2.502% (0.0734%)	2.0971% (0.06084%)	1.552% (0.02011%)	1.557% (0.05825%)	1.336% (0.04998%)	1.1977% (0.01496%)	1.5311% (0.2287%)	1.5311% (0.2287%)	1.234% (0.08425%)

Table 3.7: Mean and variance (values in brackets) of the IRR at $t = 0$ for $d' = 15$, $xs = 15\%$, $xp = 25\%$ and a deferred age of $x_d = 81$ for cohort with $x_0 = 75$.

- It comes from Tables 3.6 and 3.7 that the expected IRR is higher for the younger cohort for both deferred and term annuities. We have similar observations for the lifetime annuity except for small values of the safety level and $m > 0$.

- The IRR of a lifetime annuity decreases when α_0 increases, it decreases as well for deferred annuity with respect to α_0 except for older cohort within the non-guaranteed strategy where we observe a concave behaviour with respect to α_0. Concerning the term annuity, for both cohorts the IRR decreases with α_0 within the non-guaranteed strategy whereas it has a concave behaviour for $m = 5$ and within the fully guaranteed strategy the IRR is constant for small values of the safety level and decreases with higher α_0.

- Furthermore, the obtained values have a concave behaviour with respect to m in the sense that the expected IRR increases with smaller values of m and decreases with larger values of m except for the older cohort with high safety level where the IRR increases with m. This implies that there exists optimal value of m that maximises the expected IRR.

- The $d = 16$ years deferred annuity gives the better values of the expected IRR for the younger cohort with small values of m, whereas the $d' = 15$ years term annuity performs better for younger cohort within the fully guaranteed strategy as well as for $m = 5$ with higher value of α_0.

- For the older cohort, the deferred annuity gives higher IRR with the non-guaranteed strategy whereas the term annuity better performs for $m = 5$ and the lifetime annuity gives higher IRR within the fully guaranteed strategy.
- It is important to stress that in some cases, we obtain high expected IRR and high variance as well; meaning that the IRR is subject to an important volatility (or risk).

The sensitivity of the IRR with respect to the volatility of the force of mortality is presented in Appendix A.3.

Remark 2. *We have considered the SC as required by the SII regulation. It could be interesting (for internal models) to consider other risk measurements more linked to the tail of the distribution (extreme risk) such as the Tail VaR (TVaR) also called Average VaR by Föllmer and Schied, 2011. Hence, for comparison purposes we compute both the SC and the IRR of each product using the TVaR and the obtained results are presented in Appendix A.4.*

3.4.1 Simulation performance

To obtain the numerical results (i.e figures and tables) of this chapter, we performed MC method with 500000 simulations.

- Table 3.3 representing the case of lifetime annuity took approximatively 1 minute for each value.
- Concerning deferred annuity, each graph of Figures 3.3 and 3.5 took 17 minutes whereas each value of Table 3.4 took approximatively 40 seconds.
- For the term annuity, each value of Table 3.5 was obtained after 20 seconds and each graph of Figure 3.4 took approximatively 7 minutes.
- The values of IRR given in Tables 3.6 and 3.7 took approximatively 1 minutes for each value.

Remark 3. *Alternatively to the method used to value the number of survivors in this chapter and Chapter 2, we could consider the approximation method proposed by Gbari and Denuit, 2014. In their paper they provide accurate approximations for the present value of benefits paid by the insurer so as to avoid the problem of*

simulations within simulations (in SII calculations for instance) regardless the size of the portfolio.
In order to apply their approach in this chapter and Chapter 2, we need to consider a Lee–Carter force of mortality that we use to define the number of survivors by a family of Binomial distributions and making this family of distributions perfectly conditionally dependent leads to the approximation of the number of survivors.

3.5 Conclusion

In this chapter we proposed a triple-risk model, used to measure the equity, interest rate and longevity risks borne by an insurer selling a given classical annuity. From our model, we can deduce the equity risk-free model obtained by not investing on the risky asset, as developed in Chapter 2. We proposed a profitable investment strategy called the mG-cf strategy following the strategy proposed by Bauer and Weber, 2008 called the fully liability hedging strategy. Our proposed strategy moves from the non-guaranteed liability strategy to the fully guaranteed liability strategy, depending on the number of cash flows guaranteed. In other words, the mG-cf strategy contains (i) the constant proportion allocation strategy, obtained when no benefits are guaranteed; (ii) the fully guaranteed cash flows (or liability hedging) strategy, obtained when all the benefits are guaranteed and (iii) the alternative strategies, obtained when the number of guaranteed benefits is between one and the maximum number of benefits.
The risk measurement approach used consists of valuing insurer's SC for the three annuities within the mG-cf strategy using the maturity approach. Following SII framework with a bounded time-dependent confidence level we studied the sensitivity of SC with respect to some significant parameters.
The theoretical studies are made for a d years deferred annuity within the proposed strategy used to find numerical results for two different cohort from which we drawn up comparative observations. Numerical studies has been made for the minimum confidence level α_0 and the number of guaranteed benefits m so as to see how sensitive the SC is with respect to α_0 and m. We also studied the sensitivity of the SC with respect to the deferred period d and the term time d'.
We found a significant increase of the SC from younger to the older cohort as well as with respect to the minimal confidence level. Another finding is that the mG-cf reduces the SC of the insurer as compare to the constant proportion strategy where no benefit is guaranteed. Furthermore, the concave behaviours of the IRR

as well as convex behaviours of the SC (for a deferred annuity) show that the mG-cf strategy improves the strategy proposed by Bauer & Weber. For a 15 years term annuity, the obtained SC is less volatile with respect to ageing. Moreover, we found that the mG-cf strategy we proposed yields higher expected IRR as compare to the fully liability hedging strategy proposed by Bauer & Weber in the sense that ours gives a concave IRR with respect to the number of guaranteed benefits. Even though the ageing effect affects less the expected IRR of a term annuity, we notice that the expected IRR is higher for the younger cohort for each of the annuities. Overall, we found that the more benefits we guarantee the low SC we have for both the lifetime and term annuities (which is in line with Bauer & Weber) whereas for the deferred annuity, there exits at least one number of guaranteed benefit that gives a lower SC as compare to the non-guaranteed (i.e $m = 0$) and fully guaranteed benefits (i.e m is equal to the total number of payouts).

Further research could consist of focusing on the policyholder side by considering variable annual benefits so as to find the optimal parameters that maximise the benefit payouts of the policyholders. We present in the next chapter an approach to design risk-linked annuities (i.e with random benefits) allowing for risk sharing.

Part II

Risk-sharing annuities

Chapter 4

Design of risk sharing for risk-linked annuities

In the previous chapter, the classical annuities used are famous both in research and in practice. The mean features of these annuities is that the (financial and longevity) risks are borne by the insurer. Hence, an increase of these risks will force the insurer to increase the annuity price or to be reluctant to sell annuities. A way to solve this concern and to convince the retirees to buy annuities could be to design adequate annuity products so as to mitigate the risks linked to such products. Policymakers and researchers have thus far developed risk-linked annuities so as to protect policyholders against outliving their resources and protect insurer against possible insolvency or high solvency capital. There exits inflation; equity, morality, or multiple risk-linked annuities. In this chapter we focus on the financial-longevity risk-linked annuities.

The main product we base our work on is the GSA proposed by Piggott et al., 2005 where the authors developed a formal analysis of the payout of the GSA which can be seen as a longevity risk-pooling fund. In fact, the GSA is a scheme which allows annuitants to pool a part or their whole retirement fund with other annuitants with a view to afford benefits in retirement through a risk sharing arrangement. The results obtained in this chapter can be extend to the modified version of the GSA proposed by Qiao and Sherris, 2013 where they use the multiple-factor stochastic mortality model to show how efficient pooling can be and to quantify the limitation of pooling scheme with respect to the longevity risk. The VPA proposed by Boyle et al., 2015 is special case of the GSA, hence our model works for that as well.

Unlike the classical annuities where the financial and longevity risks are borne by the insurer, for the GSA these risks are borne by a group of annuitants. Hence, the goal in this chapter is to propose alternatives products where these risks are shared between the insurer and annuitants. In other words, we design in this chapter a new equity-mortality linked annuity that allows for the risk sharing between a group of policyholders and the insurer based on the GSA proposed by Piggott et al., 2005. One of our main result is the design of a set of annuities moving from the classical annuity to the GSA; we refer to the obtained set of annuities as the (complete) risk-sharing GSA. Our motivation of using the GSA comes from its recursive structure and from the fact that the financial risk is separated from the longevity risk.

Below, we give in Section 4.1 a brief recall on the GSA. The risk-sharing techniques are developed in Section 4.2, the obtained annuity is called the risk-sharing GSA. Section 4.3 presents the complete risk-sharing GSA which is in fact a set of annuities moving from the classical annuity to the GSA and allowing a risk-based initial benefit. Numerical analysis follows in Section 4.4 as well as a brief conclusion in Section 4.5.

4.1 Recalls on the GSA

GSA basically operates like a classical life annuity for which both the expected investment return and the policyholder's expected future mortality are captured by the benefit payout. Piggott et al., 2005 proposed a flat yield curve in order to capture the expected rate of return on the investment. It follows that if both investment and mortality actually follow these expectations, then the benefit payout stays constant.

Considering at time $t = 0$ a group of N_x policyholders aged x paying a total amount of F_0; we assume the cohort to be homogeneous (i.e annuitants with same initial age x and same initial premium F_0/N_x). The individual initial benefit payout is then given by

$$B_0 = \frac{F_0}{N_x \ddot{a}_x}, \tag{4.1}$$

with $\ddot{a}_x = \sum_{t=0}^{\infty} v^t \, \frac{N_{x+t}}{N_x}$ and $v = (1+R)^{-1}$ where R is the interest rate and N_{x+t} is the expected number of survivors at age $x+t$ for $t \geq 0$. Note that the benefit payout of a conventional annuity, (where the whole risk is borne by the insurer) is constant and is given by Formula (4.1), i.e $B_t = B_0$ for all $t > 0$.

Assuming actual survivors are different from expected number of survivors, let $N^*_{x+1},\ N^*_{x+2}, ..., N^*_{x+t}, ...$ denote the yearly actual number of survivors. With a constant and fixed interest rate, they found that the benefit payout at any time $t > 0$ is given by

$$B_t = B_{t-1}\,\frac{p_{x+t-1}}{p^*_{x+t-1}}, \tag{4.2}$$

where p_{x+t} is a deterministic one year survival probability of an individual aged $x+t$, for $t \geq 0$; the variables with the superscript '*' denote the real values of the variables and $p^*_{x+t} = \frac{N^*_{x+t}}{N^*_x}$. Similarly, considering different actual annual rate of return $R^*_1,\ R^*_2, ...$ the implied benefit at $t > 0$ is

$$B_t = B_{t-1}\,\frac{p_{x+t-1}}{p^*_{x+t-1}}\,\frac{1+R^*_t}{1+R}. \tag{4.3}$$

The two last term of the RHS respectively represent (from the right to left) the investment rate adjustment at t (IRA$_t$) and the mortality experience adjustment at t (MEA$_t$). Note that IRA$_t$ represents the financial risk whereas MEA$_t$ represents the longevity risk. From Formula (4.3) they derived the formulae of the benefit payout from an heterogeneous cohort (i.e different ages and different premiums) with identical annuity factor as well as different annuity factors with different entry time. For this latter case the formula of benefit paid at time t to the i^{th} annuitant entering the pool at age x, k period of time ago is given by

$$ {}^{k}_{x}B_{i,t} \;=\; \frac{{}^{k}_{x}\hat{F}^*_{i,t}}{\ddot{a}_{x+k,t}} \;=\; {}^{k-1}_{\;\;x}B_{i,t-1}\ \mathrm{MEA}_t\ \mathrm{CEA}_t\ \mathrm{IRA}_t, \tag{4.4}$$

where ${}^{k}_{x}\hat{F}^*_{i,t}$ is the fund value at t for the i^{th} policyholder including the inheritance of those who died during the period $[t-1, t]$; IRA$_t$ is defined as previously, MEA$_t$ is given by

$$\mathrm{MEA}_t = \frac{F^*_t}{\sum\limits_{k\geq 1}\sum\limits_{x} p^{-1}_{x+k-1,t-1}\ \sum\limits_{A_t} {}^{k}_{x}F^*_{i,t}} \tag{4.5}$$

and $\mathrm{CEA}_t = \frac{\ddot{a}_{x+k,t-1}}{\ddot{a}_{x+k,t}}$ represents the change expectation adjustment at time t. We refer the reader to Appendix A.5 for more details. CEA$_t$ is used to capture the new mortality information available at t. ${}^{k}_{x}F^*_{i,t}$ is the fund value at time t of the i^{th} policyholder who entered the pool at age x, k period ago; F^*_t is the total fund at time t; $p_{x,t}$ is the expected survival probability of an annuitant aged x at time t of living one more year and $p^*_{x,t}$ is the corresponding actual survival probability. A_t is the set of annuitants alive at time t. In this general case, the longevity risk is represented by the product of MEA$_t$ and CEA$_t$ whereas IRA$_t$ represents the financial risk. Note that Formula (4.4) is not valid for annuitants entering the pool at time t but for those who entered before time t. The benefit of those entering the pool at time t is given by

$$ {}_{x}^{0}B_{i,t} \;=\; \frac{{}_{x}^{0}F_{i,t}}{\ddot{a}_{x,t}}. \tag{4.6} $$

In the GSA, the financial and the longevity risks are borne by the pool whereas the insurer borne no risk. Moreover, these two risks are separated from each other in the formula of the benefit payout. The next section proposes two ways of sharing financial and longevity risks between both insurer and policyholders using the GSA.

4.2 Risk sharing techniques

Here we present two techniques which allow the policyholder to get rid of a proportion of risks (total or partial); the obtained product is called *risk-sharing GSA*. We first consider a sharing method based on the lower bound of the benefit payouts, secondly we present a more direct sharing method that can be viewed as a proportional sharing. These methods are used to define the annual benefits for a risk-sharing GSA.

4.2.1 Risk sharing by the mean of a lower bound threshold on benefits

Defining a lower bound threshold of the benefits, we hedge policyholders against drastic mortality or investment changes. This means they can see their benefits increase or decrease but not below a set threshold level. Let us now denote by $\varepsilon_t \in [0,1]$ the proportion at time t of the total risk (longevity and financial risks) borne by the insurer, such that the lower bound threshold of benefit is given by $B = \varepsilon_t B_0$. Hence we suggest that the benefit at a given time $t > 0$ satisfies

$$ B_t \geq B = \varepsilon_t B_0. \tag{4.7} $$

Therefore for $\varepsilon_t = 1$, the whole risk is shifted to the insurer and for $\varepsilon_t = 0$ is the case described in the previous section, i.e where the risk is borne by the pool. One could also think of an alternative situation where for all $t > 0$ the benefits rather satisfy the condition

$$ B_t \geq \varepsilon_t \; B_{t-1}. \tag{4.8} $$

Such case would be non-realistic as compared to the previous case because it could tend to zero in the case of continuously drastic mortality or investment changes. Both cases (Formulae (4.7) and (4.8)) can be roughly illustrated for a constant proportion $\varepsilon \in (0, 1)$ in Figure 4.1.

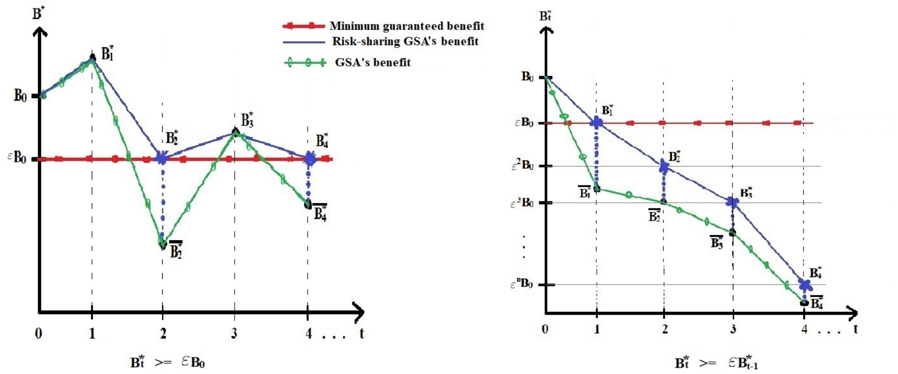


Figure 4.1: Example of benefit variation

In fact, in the first graph of Figure 4.1, we consider a volatile scenario of the GSA (green balled line) and we obtained a threshold risk-sharing GSA (blue line) which is always above the guaranteed minimum benefit (red squared line). Whereas on the second graph, we consider an extreme (strictly decreasing) scenario of the GSA (green balled line), it yields a decreasing threshold risk-sharing GSA (blue line) which goes below the guaranteed benefit (red squared line) and tends to zero.

We define the benefit payout $\overline{B_t}$ at time t of a policyholder buying a lower bound threshold risk-sharing GSA by

$$\overline{B_t} = \max\left(\overline{B_{t-1}}\ Lrisk_t\ Erisk_t, \varepsilon_t B_0\right), \tag{4.9}$$

where $Lrisk_t$ and $Erisk_t$ are respectively the longevity and the financial risk as defined in the GSA in Section 4.1. For example, considering the simple case of a close homogeneous cohort where benefit is given by Formula (4.3), we defined the benefit at $t > 0$ by

$$\overline{B_t} = \max\left(\overline{B_{t-1}}\ \frac{p_{x+t-1}}{p^*_{x+t-1}}\ \frac{1+R^*_t}{1+R},\ \varepsilon_t B_0\right). \tag{4.10}$$

In the more general case of an open heterogeneous cohort purchasing such a risk-sharing GSA (RS-GSA), we defined the lower bound threshold is

$$\overline{{}_xB_i}(t) = \varepsilon_t\ {}^0_xB_{i,t} \ = \ \varepsilon_t\ \frac{{}^0_xF_{i,t}}{\ddot{a}_{x,t}}.$$

Thus the benefit of an annuitant having survived at time t is given by

$$\overline{{}_x^k B_{i,t}} \;=\; \max\left(\overline{{}_x^{k-1} B_{i,t-1}}\ \text{Risk}_t,\ \overline{{}_x B_i}(t)\right), \tag{4.11}$$

with Risk_t being the (financial and the longevity) risks generated by the GSA of Formula (4.4).

Note that in this case, the benefit at time t is obtained from the maximum value of the previous maximums (compared to the threshold). An alternative way could consist of choosing between the threshold and the real benefit payout at t denoted by ${}_x^k B_{i,t}$. The new annual benefit is then given by

$$\overline{{}_x^k B_{i,t}} \;=\; \max\left({}_x^k B_{i,t},\ \overline{{}_x B_i}(t)\right), \tag{4.12}$$

where ${}_x^k B_{i,t}$ is given by Formula (4.4). For instance, the simple case of Formula (4.10) leads to

$$\overline{B_t} = \max\left(B_t,\ \varepsilon_t B_0\right),$$

where

$$B_t = B_{t-1}\ \frac{p_{x+t-1}}{p^*_{x+t-1}}\ \frac{1+R^*_t}{1+R}. \tag{4.13}$$

Note that considering constant benefit thresholds will be more safe for annuitants as they won't face uncertain or decreasing benefit threshold.

This sharing method allows us to share the whole (financial + longevity) risk generated by the GSA. In what follows we proposed a way of sharing different proportions of each risk.

4.2.2 Proportional risk sharing

A more straightforward risk sharing mechanism consists of proportional or direct sharing. In other words, we consider at time t a proportion $\beta_t \in [0, 1]$ of the risk to be borne by a group of annuitants and the remaining proportion $(1 - \beta_t)$ is borne by the insurer.

Considering the same annuity benefit ${}_x^k B_{i,t}$ defined by Formula (4.4), we obtain the following mathematical definition of the proportional risk sharing GSA

$${}_x^k B_{i,t}(\beta_t) \;=\; {}_x^k B_{i,t-1}(\beta_t)\left[\beta_t\ \text{Risk}_t + (1-\beta_t)\right], \tag{4.14}$$

where $\text{Risk}_t = \text{MEA}_t \times\ \text{CEA}_t \times\ \text{IRA}_t$.

Note that since β_t is the proportion of the risk borne by a group of policyholders at time t, it could represent either a proportion of the whole risk (i.e both the longevity and the financial risks as in Formula (4.14) or a proportion of a partial risk. By partial risk, we mean either the longevity risk or the financial risk. More generally, we could think of sharing at time t different proportion of longevity and financial risks. Let $\beta_t \in [0,1]$ be the proportion of the longevity risk borne by the policyholder and $\beta'_t \in [0,1]$ be the proportion of financial risk borne by the same group. The general formula of the risk sharing GSA (with different proportions) is given by

$$ {}^{k}_{x}B_{i,t}(\beta_t, \beta'_t) \;=\; {}^{k}_{x}B_{i,t-1}(\beta_t, \beta'_t)\,[\beta_t \; Lrisk_t + (1-\beta_t)]\,[\beta'_t \; Erisk_t + (1-\beta'_t)]\,. $$

For example, in this case Formula (4.14) becomes

$$ {}^{k}_{x}B_{i,t}(\beta_t, \beta'_t) \;=\; {}^{k-1}_{\;\;x}B_{i,t-1}(\beta_t, \beta')\,[\beta_t \;\mathrm{MEA}_t \;\mathrm{CEA}_t + (1-\beta_t)] \;\; [\beta'_t \;\mathrm{IRA}_t + (1-\beta'_t)]\,. \tag{4.15} $$

Note that defining the proportional RS-GSA allows us to define a set of annuities moving from the GSA (when $\beta_t = \beta'_t = 100\%$) to the classical annuity (when $\beta_t = \beta'_t = 0\%$). We observe that in both the GSA and the risk-sharing GSA, the policyholders and the insurer are not rewarded for the risk they agree to bear at the beginning of the contract. In other words, the first benefit does not depends on the risk proportions, it is equal to the benefit of a classical annuity. In the next section we propose what we called the *complete risk-sharing GSA* which defines the initial benefit depending on the risk proportions. This is done using the proportional risk-sharing method because it includes both the GSA and the classical annuity whereas the lower bound threshold method does not include the classical annuity.

4.3 On the complete risk-sharing GSA

In the Section 4.1, the individual benefit payout for a GSA is defined recursively as follows

$$ B_t = B_{t-1} \times Lrisk_t \times Erisk_t, \tag{4.16} $$

where $Lrisk_t = \mathrm{MEA}_t \times \mathrm{CEA}_t$ and $Erisk_t = (1+R_t)/(1+R)$ are respectively the longevity and financial risks at time t. In what follows, let's denote by $\beta,\ \beta' \in [0,1]$ respectively the proportion of the longevity and the financial risk borne by the pool and we denote the annuity factor depending of these risk proportions by $\ddot{a}_x(\beta, \beta')$. Our objective here is to introduce a discount rate that depends on the pool proportion of financial and longevity risks denoted by $R_\gamma(\beta, \beta')$ called risk adjusted discount rate; where γ is the policyholder's risk aversion parameter as defined in Formula (4.20). $R(\beta, \beta')$ is defined such that the risk adjusted annuity factor given by

$$\ddot{a}_x(\beta, \beta') = \sum_{s=0}^{\infty} \left(\frac{1}{(1 + R_\gamma(\beta, \beta'))} \right)^s \times \ {}_sp_x$$

could be used to annuitize individual premium. Therefore in this case the initial benefit payout of an individual aged x with an individual premium of F_0/N_x will be

$$B_0(\beta, \beta') = \frac{F_0}{N_x \ \ddot{a}_x(\beta, \beta')}. \tag{4.17}$$

Using the proportional risk-sharing method, the individual annual benefit of a complete risk-sharing GSA is defined for $t > 0$ by

$$B_t(\beta, \beta') = B_{t-1}(\beta, \beta') \times [\beta Lrisk_t + (1 - \beta)] \times [\beta' Erisk_t(\beta, \beta') + (1 - \beta')] \,, \tag{4.18}$$

The obtained annuity is referred to as the complete risk-sharing GSA. Let's compare this annuity with the classical annuity where the individual benefits is defined for all $t \geq 0$ by

$$B_t = \frac{F_0}{N_x \sum_{s=0}^{\infty} \left(\frac{1}{1+R} \right)^s \ {}_sp_x} = \frac{F_0}{N_x \ \ddot{a}_x}. \tag{4.19}$$

We use the utility approach on the policyholder point of view and we consider his risk preference; namely following Chen et al., 2019 and Chen and Hieber, 2016, we value the expected discounted lifetime utility of the policyholders. In other words, we assume that a policyholder would choose the product with a benefit B_t that maximizes the expected discounted lifetime utility as stated by Yaari, 1965. In line with the literature, we capture the policyholder's risk preferences using the constant relative risk aversion (CRRA) utility function given by

$$U(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{for } \gamma \neq 1 \\ \\ \log(x) & \text{for } \gamma = 1 \end{cases} \tag{4.20}$$

where $\gamma \geq 0$ is the relative risk aversion parameter. Our choice of the CRRA utility is motivated by the fact that it is widely used for risk aversion modelling in many domains (economic, psychology, health) (see Wakker, 2008). In particular the CRRA utility is widely used for consumption analysis (see Chen et al., 2019; Charupat and Milevsky, 2002; Vigna et al., 2009; Boyle et al., 2015). For comparison purpose, we further consider the constant absolute risk-aversion (CARA) utility following Chang and Chang, 2017 and Vigna et al., 2009.

The objective here is to find the values of $R_\gamma(\beta, \beta')$ such that the expected discounted lifetime utility of the complete risk-sharing GSA (CRS-GSA) is at least equal to the expected discounted lifetime utility of the classical annuity. It is important to stress that the values of $R(\beta, \beta')$ which give the same expected utility as the classical annuity can be seen as the indifference discount rate.
Concerning the utility of the classical annuity, let's consider a cohort of N_x policyholders paying a total premium of F_0 to purchase (classical) immediate lifetime annuity and denote by $n+1$ the maximum number of benefit payout. It follows that the policyholder's expected discounted lifetime utility for such a product is given by

$$Ucla = \sum_{t=0}^{n} U(B_0) v_\delta^t \,{}_tp_x^0; \tag{4.21}$$

where $v_\delta = \frac{1}{1+\delta}$ is the policyholder's subjective discount factor, with δ being the subjective discount rate, B_0 is given by Formula (4.19) and ${}_tp_x^0$ is a deterministic survival probability of the policyholder. In fact, ${}_tp_x^0$ represents the policyholder expected survival probability on the policyholder's view.
In what follows, we first consider single risk settings where we share only the financial risk (called the complete financial risk-sharing GSA), then the longevity risk (called the complete longevity risk-sharing GSA). Afterwards we combine the two cases to obtain the complete financial-longevity risk-sharing GSA.

4.3.1 The complete financial risk-sharing GSA (CFRS-GSA)

Let us consider a given mortality table describing the survival probability of a pool of policyholders buying an immediate lifetime annuity for a pool's premium of F_0 and let $\beta' \in [0, 1]$ be the proportion of financial risk borne by the pool. We assume that there is no longevity risk occurring in this contract, in other words the proportion of the longevity risk shifted to the pool is $\beta = 0$. Therefore $R_\gamma(\beta') = R_\gamma(0, \beta')$ only reflects the investment uncertainty and the benefit payout in this case is given by Formulae (4.17)–(4.18) where $\beta = 0$. The expected discounted lifetime utility of the CFRS-GSA is given by

$$Ucfrs(\beta') = \mathbb{E}\left[\sum_{t=0}^{n} U(B_t(\beta'))\ v_\delta^t \,{}_tp_x^0\right]. \tag{4.22}$$

Using the Equation (4.18) with $\beta = 0$, one can show that for all $t > 0$

$$B_t(\beta') = B_0(\beta') \prod_{j=1}^{t} \left[\beta' Erisk_j(\beta') + (1-\beta')\right], \tag{4.23}$$

with

$$B_0(\beta') = \frac{F_0}{N_x \ddot{a}_x(\beta')} = \frac{F_0}{N_x \sum_{j=0}^{\infty} v_R^j(\beta')\ {}_jp_x},$$

and $v_R^j(\beta') = \left(\frac{1}{1+R_\gamma(\beta')}\right)^j$. It follows that Formula (4.22) becomes

$$Ucfrs(\beta') = \begin{cases} \dfrac{\sum_{t=0}^{n} \mathbb{E}\left[\left(B_0(\beta') \prod_{j=1}^{t}\left[\beta' Erisk_j(\beta') + (1-\beta')\right]\right)^{1-\gamma}\right] v_\delta^t\ {}_tp_x^0}{1-\gamma}, & \text{for } \gamma \neq 1 \\ \sum_{t=0}^{n} \mathbb{E}\left[\log\left(B_0(\beta') \prod_{j=1}^{t}\left[\beta' Erisk_j(\beta') + (1-\beta')\right]\right)\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma = 1. \end{cases} \tag{4.24}$$

For illustration, one can show that in the simplest case of a closed homogeneous pool where the benefit payout is given by Formula (4.3), the expected utility of a policyholder buying a CFRS-GSA is given by

$$Ucfrs(\beta') = \begin{cases} \frac{1}{1-\gamma} \sum_{t=0}^{n} \mathbb{E}\left[\left(B_0(\beta')\ \prod_{j=1}^{t}\left[\beta' \frac{1+R_j}{1+R_\gamma(\beta')} + (1-\beta')\right]\right)^{1-\gamma}\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma \neq 1 \\ \sum_{t=0}^{n} \mathbb{E}\left[\log\left(B_0(\beta')\ \prod_{j=1}^{t}\left[\beta' \frac{1+R_j}{1+R_\gamma(\beta')} + (1-\beta')\right]\right)\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma = 1. \end{cases}$$

The task is then to find the set of $R_\gamma(\beta')$ such that

$$Ucfrs(\beta') \geq Ucla, \tag{4.25}$$

where $Ucla$ is given by Formula (4.21). We cannot find an explicit formula of the risk adjusted discount rate $R_\gamma(\beta')$ satisfying the Inequality (4.25), but this can be solved numerically using for example the Newton-Raphson method.

4.3.2 The complete longevity risk-sharing GSA (CLRS-GSA)

In this case we consider a financial risk-free market made of a risk-free asset (the money market account), then the proportion of financial risk shifted to the pool is $\beta' = 0$. In this case the risk adjusted discount rate given by $R_\gamma(\beta) = R_\gamma(\beta, 0)$

captures the longevity risk. It follows that the benefit payout of the CLRS-GSA is given by Formulae (4.17)–(4.18) where $\beta' = 0$ and the expected discounted lifetime utility of an annuitant buying a CLRS-GSA is

$$Uclrs(\beta) = \mathbb{E}\left[\sum_{t=0}^{n} U(B_t(\beta))\ v_\delta^t\ {}_tp_x^0\right]. \tag{4.26}$$

Using the Equation (4.18) with $\beta' = 0$, one obtains for all $t > 0$

$$B_t(\beta) = B_0(\beta) \prod_{j=1}^{t} \left[\beta Lrisk_j + (1-\beta)\right], \tag{4.27}$$

with

$$B_0(\beta) = \frac{F_0}{N_x\ \ddot{a}_x(\beta)} = \frac{F_0}{N_x \sum_{j=0}^{\infty} v_R^j(\beta)\ {}_jp_x},$$

where $v_R^j(\beta) = \left(\frac{1}{1+R_\gamma(\beta)}\right)^j$. Formula (4.26) becomes

$$Uclrs(\beta) = \begin{cases} \frac{1}{1-\gamma} \sum_{t=0}^{n} \mathbb{E}\left[\left(B_0(\beta) \prod_{j=1}^{t} \left[\beta Lrisk_j + (1-\beta)\right]\right)^{1-\gamma}\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma \neq 1 \\ \\ \sum_{t=0}^{n} \mathbb{E}\left[\log\left(B_0(\beta) \prod_{j=1}^{t} \left[\beta Lrisk_j + (1-\beta)\right]\right)\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma = 1. \end{cases} \tag{4.28}$$

In the case of a closed homogeneous pool, we find

$$Uclrs(\beta) = \begin{cases} \frac{1}{1-\gamma} \sum_{t=0}^{n} \mathbb{E}\left[\left(B_0(\beta) \prod_{j=1}^{t} \left[\beta\ \frac{p_{x+j-1}}{p^*_{x+j-1}} + (1-\beta)\right]\right)^{1-\gamma}\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma \neq 1 \\ \\ \sum_{t=0}^{n} \mathbb{E}\left[\log\left(B_0(\beta) \prod_{j=1}^{t} \left[\beta\ \frac{p_{x+j-1}}{p^*_{x+j-1}} + (1-\beta)\right]\right)\right] v_\delta^t\ {}_tp_x^0, & \text{for } \gamma = 1. \end{cases} \tag{4.29}$$

Numerical methods can be used to find the set of $R_\gamma(\beta)$ for which

$$Uclrs(\beta) \geq Ucla, \tag{4.30}$$

where $Ucla$ is given by Formula (4.21).

4.3.3 The complete financial-longevity risk-sharing GSA (CFLRS-GSA)

The idea here is to combine the previews two single-risk products into a double-risk one. Both the financial and the longevity risks are shared with different proportions, respectively $\beta,\ \beta' \in [0,1]$. One can show from Formula (4.18), that we obtain for any $t > 0$

$$B_t(\beta,\beta') = B_0(\beta,\beta') \prod_{j=1}^{t} [\beta Lrisk_j + (1-\beta)] \times [\beta' Erisk_j(\beta,\beta') + (1-\beta')] \,, \quad (4.31)$$

where $B_0(\beta,\beta')$ is given by Formula (4.17). It follows that the expected utility of an annuitant buying a CFLRS-GSA is

$$Ucflrs(\beta,\beta') \quad = \quad \sum_{t=0}^{n} \mathbb{E}\left[U\left(B_0(\beta,\beta')\prod_{j=1}^{t} Lshare_j(\beta) \times Eshare_j(\beta,\beta')\right)\right] v_\delta^t \; {}_tp_x^0, \quad (4.32)$$

where

$$Lshare_j(\beta) = [\beta \ Lrisk_j + (1-\beta)]$$

and

$$Eshare_j(\beta,\beta') = [\beta' \ Erisk_j(\beta,\beta') + (1-\beta')] \,,$$

with

$$Erisk_j(\beta,\beta') = \frac{1+R_j}{1+R_\gamma(\beta,\beta')}.$$

Note that the financial risk component depends on both the financial and longevity risk, as the risk adjusted discount rate depends on them. In particular, for a closed homogeneous group of annuitants, the longevity risk component is

$$Lrisk_j = \frac{p_{x+j-1}}{p^*_{x+j-1}}.$$

The problem here is to find a set of risk adjusted discount rates $R_\gamma(\beta,\beta')$ such that

$$Ucflrs(\beta,\beta') \geq Ucla, \quad (4.33)$$

where $Ucla$ is given by Formula (4.21). Below we solve Problem (4.33) using numerical method, for this purpose we assume ${}_tp_x^0 = {}_tp_x$.
Note that the (complete) RS-GSA we proposed could also be designed based on the extended version of the GSA proposed by Qiao and Sherris, 2013 as well as on the VPA proposed by Boyle et al., 2015.

4.4 Numerical solutions

In this section we compute some numerical solutions for both the risk-sharing GSA and the complete risk-sharing GSA. First of all, we need detailed informations about the investment strategy and the mortality model.

4.4.1 Assumptions

Let's consider the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ defined in Chapter 3 made of a stock (S_t as defined in Section 3.1.1) and a money market account ($B(t)$ with constant rate short $r \in \mathbb{R}_+^*$) with a constant proportion allocation strategy with dynamic rebalancing, i.e a constant proportion $x_s \in [0,1]$ of the initial fund is invested on the stock and the remaining proportion invested on the money market account. One can show that the value at time T of an investment on such mixed asset during the period $[t,T]$ is $A(t,T) = A_t \text{ Asset}(t,T)$, where

$$\text{Asset}(t,T) = e^{\left(\mu x_s + (1-x_s)r - \frac{x^2\sigma_s^2}{2}\right)(T-t) + \sigma_s x_s (W_T^s - W_t^s)}, \tag{4.34}$$

with W_t^s being a Brownian motion under the physical measure $\mathbb{P}$, describing the stock's uncertainty. It follows that the investment return is given by $1 + R_t = \text{Asset}(t-1,t)$.
Note that $Asset(t-1,t)$ used in the chapter and in Chapter 5 is just a notation and refers to the investment return of the reference fund used to index the liabilities of the insurer.

We model the mortality of the policyholder using the HW model defined in Section 2.2.1. The contract consists of an homogeneous group of $N_x = 1000$ initially aged $x = 65$ and paying each a unique premium $F^0 = 1000\$$ for an immediate lifetime annuity; $F_0 = N_x F^0$. The parameters of the HW model are given in Table 2.1 whereas the parameters of the risky asset is calibrated using the S&P500 indexes[1] from 1871 to 2018. Using the MLE, we get $\mu = 5.8702\%$ and $\sigma_s = 20.4172\%$. Moreover, for sensitivity analysis we consider two additional values of the drift, i.e $\mu = 2\%$ and $\mu = 4\%$. Further sensitivity analysis are made for the risk aversion parameter by considering $\gamma = 0.5,\ 1$ and we assume $\delta = r = R_0 = 2\%$.
Note that an increase in the values of the risk adjusted discount rate $R_\gamma(\beta, \beta')$ implies an increase of the initial benefit given by Formula (4.17), which could also

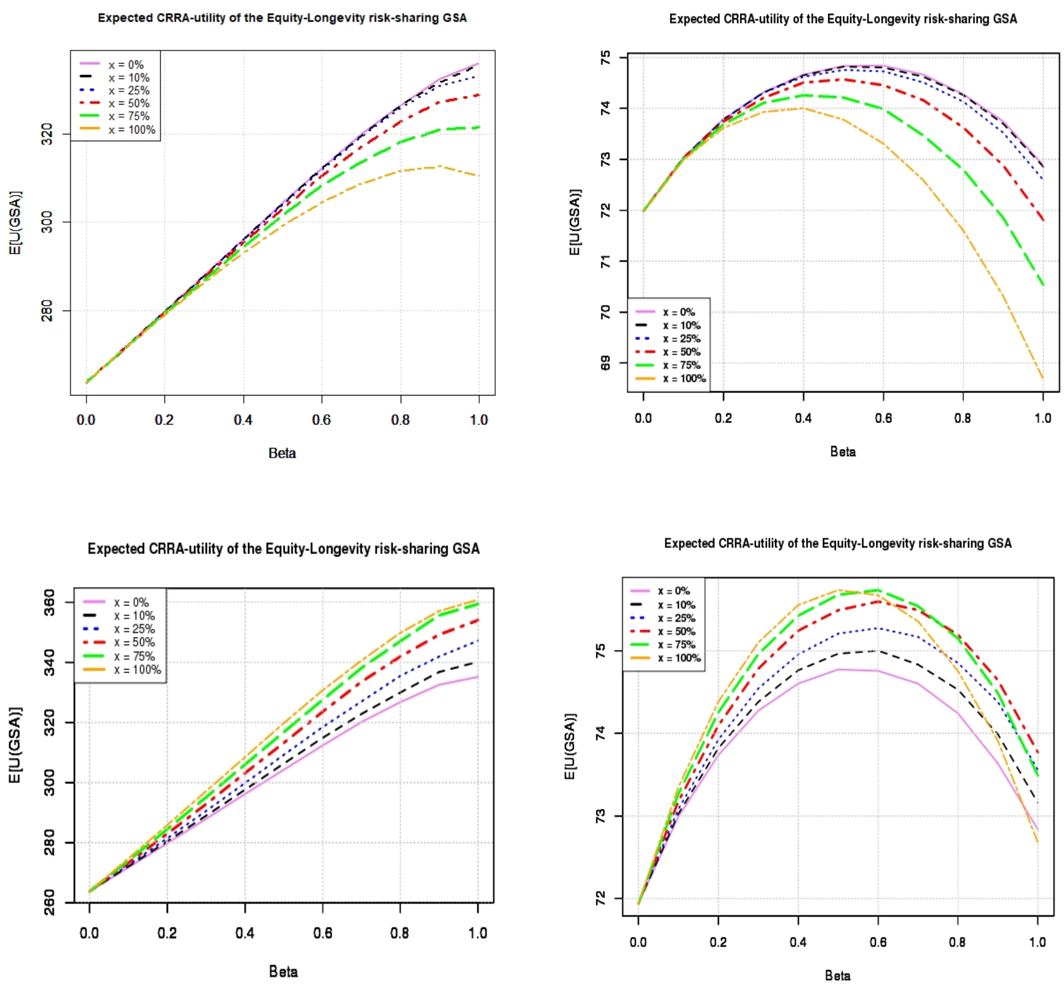
[1] `http://www.multpl.com/s-p-500-historical-prices/table/by-year`, March 7 2019

increase the upcoming benefits depending on both the investment and mortality experiences.

4.4.2 The risk-sharing GSA

Here we show how the expected utility of the FLRS-GSA changes with the annuitant's risk proportions β and β' when $\beta = \beta'$.

$\gamma = 0.5$ $\gamma = 1$



$\gamma = 0.5$ $\gamma = 1$

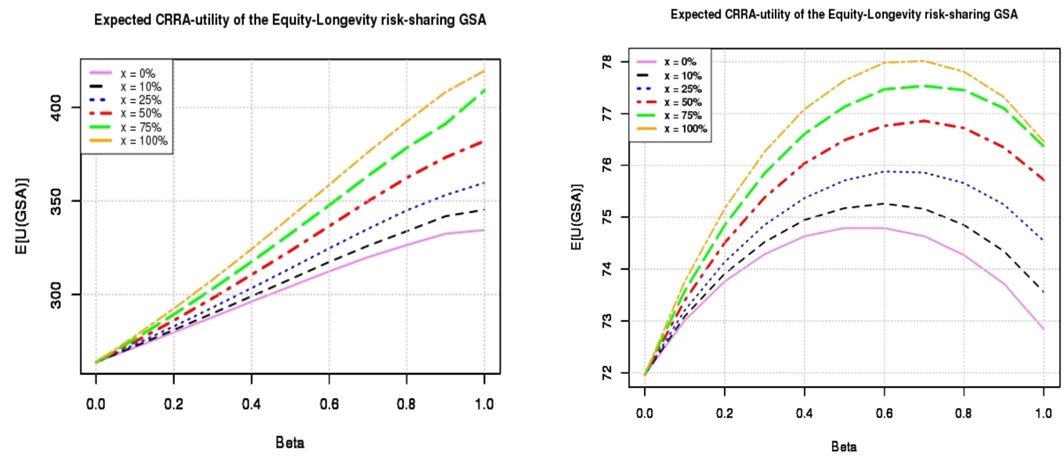


Figure 4.2: Expected utility of the FLRS-GSA for $\beta = \beta'$ — First row represents the case of $\mu = 2\%$; the second row represents the case of $\mu = 4\%$ and the third row is for $\mu = 5.8702\%$.

Moreover a sensitivity analysis is made with respect to μ, γ and x_s. Note that in this case, $\beta = \beta' = 0\%$ represents the classical annuity and $\beta = \beta' = 100\%$ represents the GSA. The case of a FRS-GSA is shown in Figure A.4 of Appendix A.6.

From Figure 4.2 we have the following observations

- The expected utility of a FLRS-GSA increases with the drift;
- When γ increases the expected utility of the classical annuity increases. In other words, for small risk aversion the GSA gives the best expected utility whereas for larger risk aversion the classical annuity gives the better expected utility compared to the GSA. Hence the concave form shown in second column of Figure 4.2 for $\gamma = 1$ implies that there exist pairs (β, x_s) for which RS-GSA gives better expected utility compared to both GSA and classical annuity;
- When the proportion invested on the risky asset x_s increases, the expected utility increases for large values of μ and decreases for small μ.

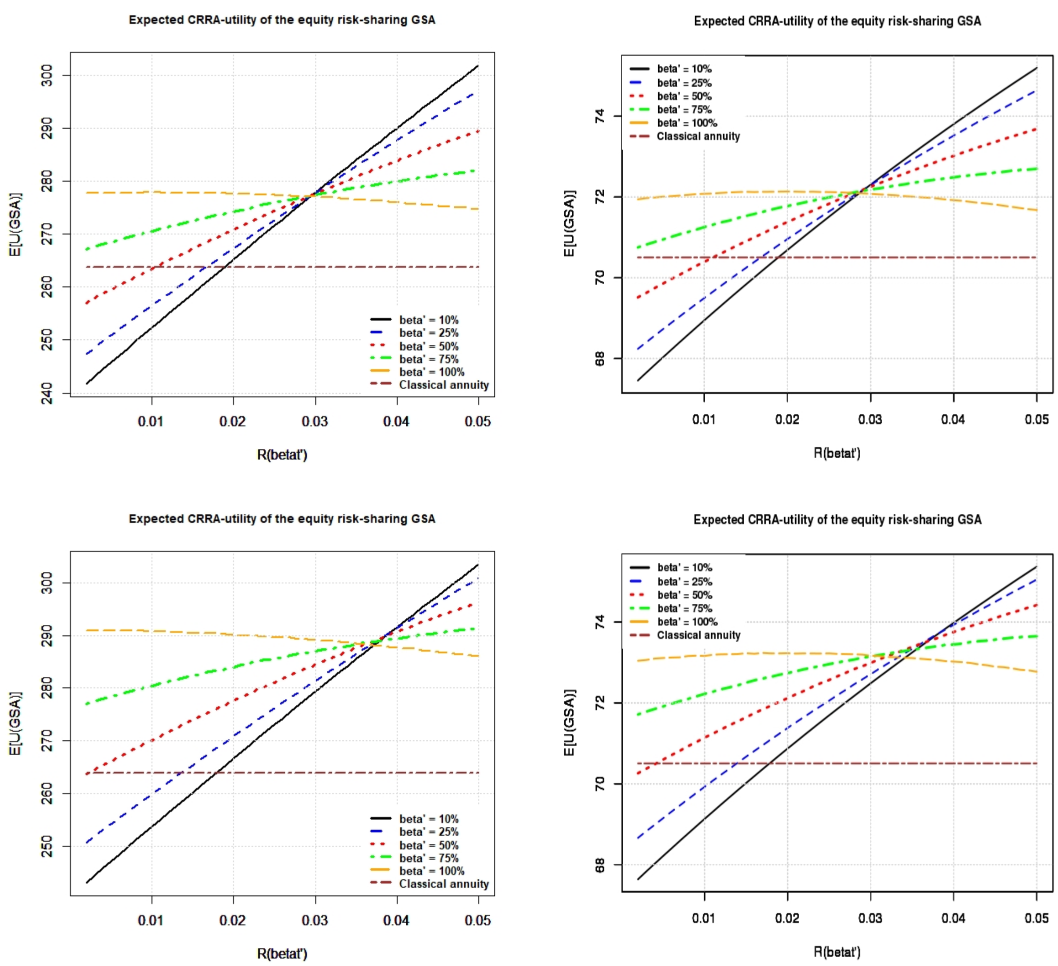
The obtained concave form (for instance when $\gamma = 1$) shows that the there exists β and x that maximise the expected utility of the policyholder. When the risk aversion increases, the product with the best expected utility moves from the GSA to RS-GSA and classical annuity. Note that in Figure 4.2 the pink lines represented the LRS-GSA.

4.4.3 The complete risk-sharing GSA

The CFRS-GSA

Similarly to the RS-GSA, Figure 4.3 illustrates changes of the expected utility of a CFRS-GSA with respect to $R_\gamma(\beta')$ for which we make a sensitivity analysis with respect to β', γ and x_s.

$\gamma = 0.5$ $\gamma = 1$



$\gamma = 0.5$ $\gamma = 1$

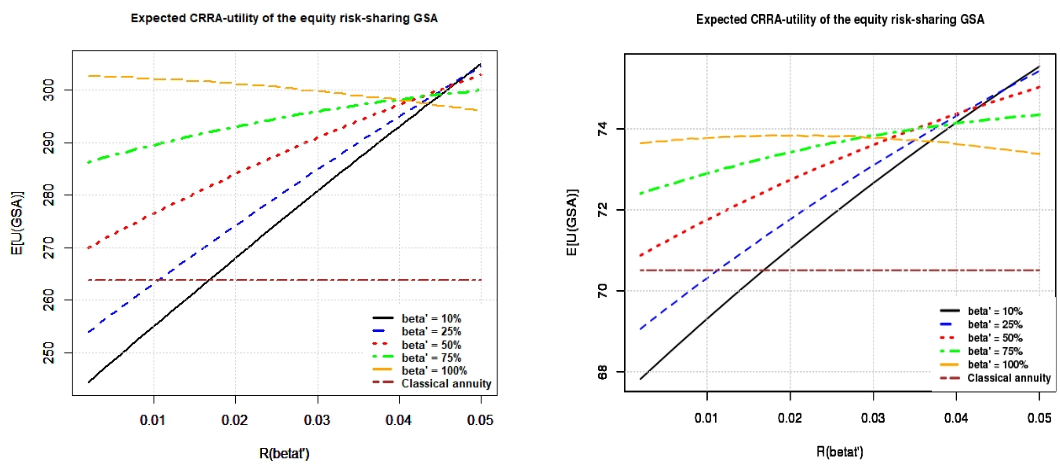


Figure 4.3: Expected utility of the CFRS-GSA for $\mu = 5.8702\%$ — First row represents the case of $x_s = 25\%$; the second row represents the case of $x_s = 50\%$ and the third row is for $x_s = 75\%$.

From this figure, we observe that there always exists pairs $(\beta', R_\gamma(\beta'))$ that improves the expected utility of a policyholder buying a CFRS-GSA compared to both GSA and classical annuity. Moreover, for all β', we can always find a $R_\gamma(\beta')$ such that GSA and CFRS-GSA give the same expected utility, unlike the classical annuity. We also find that GSA and RS-GSA do not improve only the expected utility of the consumption (as concluded by Boyle et al., 2015 for the GSA), but also the expected utility of the present value of the benefits on the policyholder viewpoint. For instance, when $R_\gamma(\beta') = 2\%$, the expected utility of a CRS-GSA is greater than that of a classical annuity whereas for $R_\gamma(\beta') < 1.5\%$ the expected utility of some CRS-GSA is less than or equal to the expected utility of a classical annuity.
We further find that for $\gamma = 0$ and for all β' the expected utilities coincide at the same $R_\gamma(\beta')$; in fact we have the following proposition.

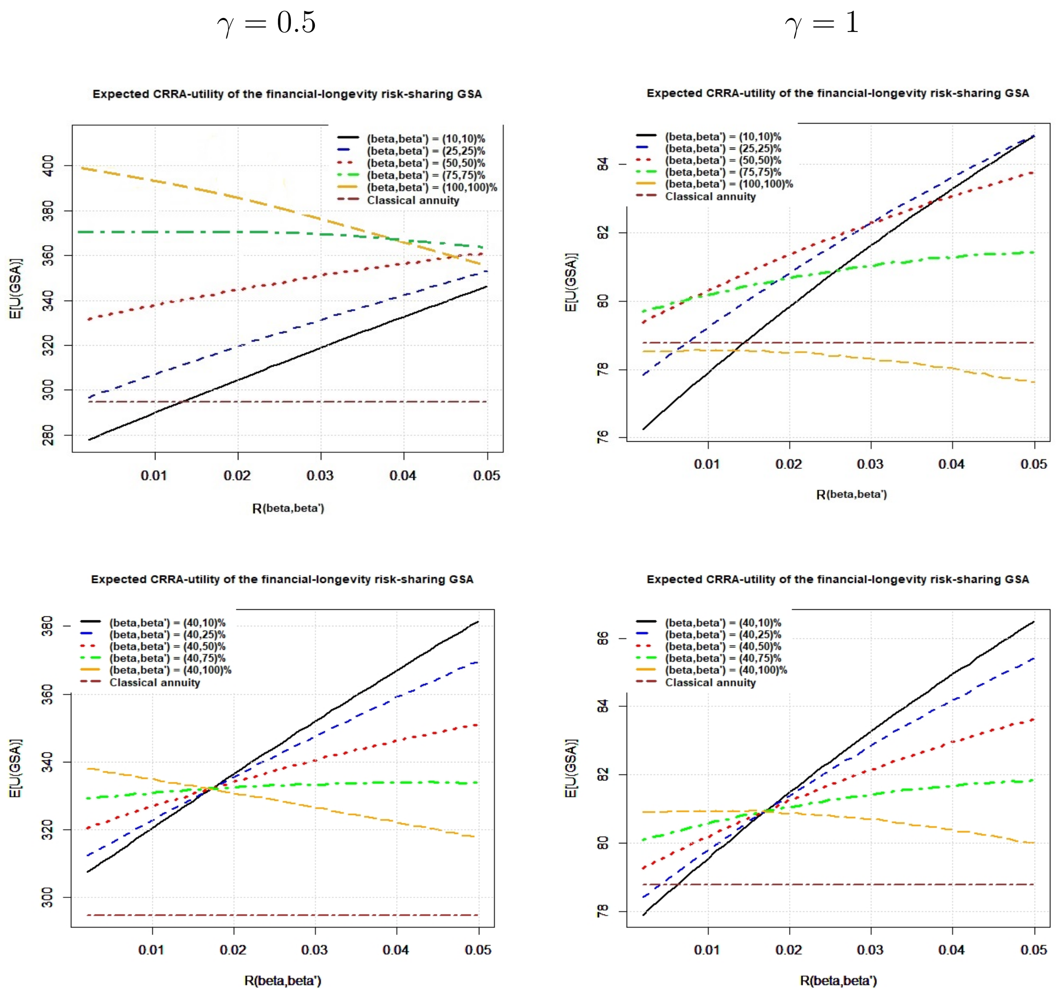
Proposition 1. *Within the financial market defined above, for a risk-neutral utility function, distinct CFRS-GSAs (with different risk proportions β's) give the same expected utilities (on the policyholder point of view) if and only if their risk adjusted discount rates are equal to the expected rate of return on the risky portfolio.*

Mathematically, for $\gamma = 0$ *and for any* $\beta'_1 \neq \beta'_2 \neq 0$ *with* $R_\gamma(\beta') := R(\beta'_1) = R(\beta'_2)$, $Ucers(\beta'_1) = Ucers(\beta'_2)$ *if and only if* $R_\gamma(\beta') = \log\left(\mathbb{E}[1 + R_t]\right) = \mu x_s + (1 - x_s)r$.

We refer the reader to Appendix A.7 for a detailed proof.

The CFLRS-GSA

The figure below illustrates the expected utility of a policyholder buying a CFLRS-GSA, for which me make a sensitivity study with respect to β, β' and $R(\beta, \beta')$.



$\gamma = 0.5$ $\gamma = 1$

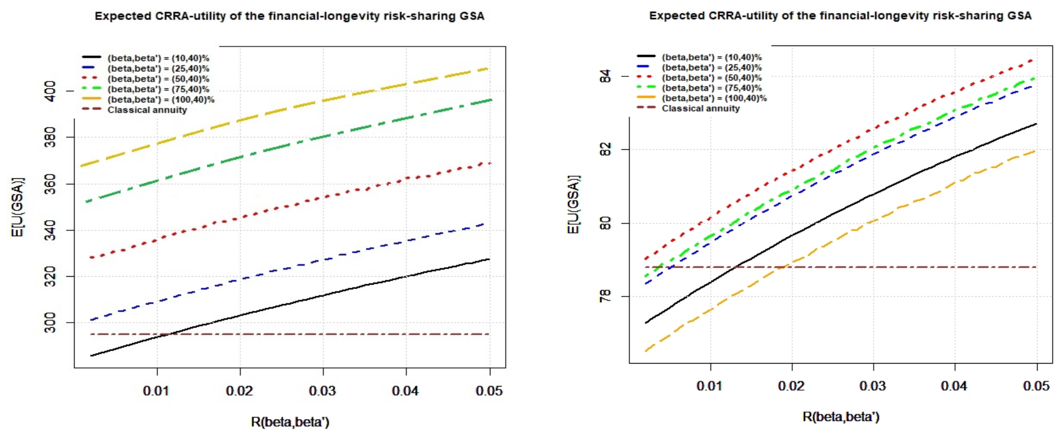


Figure 4.4: Expected utility of the CFLRS-GSA for $\mu = 5.8702\%$, $x_s = 15\%$ — First row represents the case of $\beta = \beta'$ and the second row represents the case of $\beta = 40\%$ and last row the case $\beta' = 40\%$.

We have the following observations from Figure 4.4,

- there exists some values of $R(\beta, \beta')$ that give better expected utility than classical annuity and GSA. In other words, one can always find triples $(\beta, \beta', R(\beta, \beta'))$ that yield better expected utility than both GSA and classical annuity. These are represented by the values of β, β' and $R(\beta, \beta')$ for which the expected utility is above that of the classical annuity and GSA.

- We further observe that the expected utility of the CFLRS-GSA increases with the risk adjusted rate except for $\beta = \beta'$ where the utility decreases for high values of β with $\gamma = 0.5$. The utility of the GSA decreases with respect to $R(\beta, \beta')$ except for the case $\beta = 40\%$ where it increases.

It is important to stress that for any risk adjusted rate obtained with pair (β, β'), the corresponding CRS-GSA behaves like the RS-GSA with interest rate given by the risk adjusted rate; i.e with $R = R(\beta, \beta')$. In this case we have almost similar observations and conclusions as the RS-GSA for the expected utilities. Thus the only difference will be the level of the corresponding expected utility which increases with the interest rate R. Moreover, form Figures 4.2 and 4.4, we

see that for larger risk proportions (β, β') and proportion x_s invested on the stock, RS-GSA yields better expected utility than CRS-GSA whereas we have other way round for smaller proportions of risks and stock's proportion.
Note that the pool or size effect is taken into account in both RS-GSA and the CRS-GSA, because it is taken into account in the definition of the GSA as shown in the literature (see Piggott et al., 2005; Qiao and Sherris, 2013; Boyle et al., 2015).
For comparison purpose, we consider different utility function, namely the constant absolute risk-aversion (CARA) utility function. Following Chang and Chang, 2017, we give in Appendix A.8 the numerical results obtained with the CARA utility function from which we draw comparison with respect to the CRRA utility function.

Remark 4. *Note that the obtained risk-sharing GSA and complete risk-sharing GSA allow us to define rage of annuities moving from classical annuity to GSA. The GSA is obtained when setting both proportions of longevity and financial risks to* 100%. *Therefore, our numerical studies illustrate the comparison of our proposed (complete) risk-sharing GSAs with the GSA proposed by Piggott et al., 2005 within our framework. I could be interesting as well to make such comparison within Piggott et al., 2005's framework so as to highlight the impact of the considered framework.*

4.4.4 Simulation performance

To obtain the numerical results (i.e figures) for this chapter, we performed MC method with 500000 simulations.

- For the financial-longevity RS-GSA represented in Figure 4.2, we ran the code for approximatively 53 minutes for each graph, whereas the financial RS-GSA represented in Figure A.4 took approximatively 30 minutes for each graph.

- Concerning the complete financial RS-GSA represented in Figure 4.3, the simulations took approximatively 47 minutes per graph whereas the complete financial-longevity RS-GSA presented in Figure 4.4 we spent 96 minutes per graph.

4.5 Conclusion

Aiming to design a risk-sharing annuity, we proposed in this chapter different risk-sharing methods and different products based on the GSA as defined by Piggott et al., 2005. Our motivation of using the GSA comes from its nice structure where the financial and the longevity risks are separated from each other; this allowed us to share different proportions of each of this risk between the insurer and the annuitants.

Our approach consisted of defining a modified GSA called the risk-sharing GSA aiming to share different proportions of financial and / or longevity (partial or whole) risk between the insurer and the annuitants. The two proposed sharing methods are the lower bound threshold risk sharing and the proportional risk sharing. The first consists of preventing the benefit payouts to go below a predefined threshold; in this case only the whole risk sharing is possible. For the second, we proposed the proportional risk sharing method, which is more straightforward and intuitive as it consists of splitting the (partial or whole) risk in two parts – one part in charge of the insurer and the second in charge of the policyholders. This allowed us to define not only the risk-sharing GSA but also a rage of annuity products moving from the classical annuity (where the whole risk is borne by the insurer) to the GSA (where the whole risk is borne by the annuitants) depending of the proportions of risk shifted to the annuitants. From the proportional risk-sharing GSA, we defined what we called the complete risk-sharing GSA (CRS-GSA). The CRS-GSA is obtained by defining an annuity discount rate in terms of the proportions of financial and longevity risks borne by the annuitants called the risk adjusted discount rate. We used the CRRA utility approach (on the policyholder viewpoint) in order to find sets of the risk adjusted discount rate and probability such that the CRRA expected utility of a CRS-GSA is at least equal to that of the classical annuity, for different values of the risk aversion parameter and different values of the drift.

The numerical analysis showed the classical annuity never improve the expected utility of the annuitant and depending on the level of risk aversion, we can find proportions of risk such that the PFLRS-GSA gives higher expected utility as compare to the GSA. Moreover, we found that a policyholder with low risk aversion will prefer the GSA whereas a policyholder with high risk aversion will go for a RS-GSA. This comes from the obtained concave behaviours of the RS-GSA's expected utility which implies the existence of optimal risk proportions; i.e the risk proportions of the annuitants in $(0, 1)$ that maximise their CRRA expected utility. In line with the results obtained by Boyle *et al.* (in term of the utility of consumption), we found that the RS-GSA have a CRRA expected utility higher than that of a classical annuity and for a highly risk aversion policyholder, the RS-GSA improves his expected utility compared to the GSA. Additionally, we observed that there exists some values of the risk aversion, the drift and the annuitant's risk proportions such that the classical annuity gives a better expected utility than the GSA. Numerical analysis also showed that the benefit payout of a proportional RS-GSA increases with the drift, the time and the annuitant's risk proportions. Concerning the CRS-GSA, we found that there always exists some pairs $(\beta', R_\gamma(\beta'))$ (for the CFRS-GSA) ; $(\beta, Y_\gamma(\beta))$ (for the CLRS-GSA) and $(\beta', \beta, R_\gamma(\beta'), Y_\gamma(\beta))$ (for the CFLRS-GSA) that improve the expected utility of a policyholder as compare to both the classical annuity and the GSA.

Our contribution in the literature comes from the definition of a large range of annuity products moving from the classical annuity to the GSA which in some cases improve the expected utility of a policyholder. Hence, one can think of the value of such products depending on the risk proportions.

Chapter 5

Valuation of risk-sharing group self-annitizarion

5.1 Introduction

In this chapter we will value the two contracts designed in Chapter 4. The two contracts consist of a group of policyholder buying either a proportional risk-sharing GSA defined in Section 4.2.2 or a complete risk-sharing GSA defined in Section 4.3. The question we will address in this chapter is how is the value of these contracts compared to the those of a classical annuity, GSA and the unique premium paid by annuitants? To answer this question, we need the following three ingredients[1]

(*i*) The net interest rate used to discount the annuity benefit. This could be given by the current interest rate available on the market or an interest rate guaranteed by the insurer. In the literature, it is common to do a sensitivity study with respect to the net interest rate;

(*ii*) The base mortality table representing the company or the insurer's estimation of the population's mortality. It can be given by the best estimate of the real mortality, a mortality table proposed by the regulator or a table constructed by the company and in line with the regulator requirements;

[1] source: The Messenger Risk Management Newsletter written by Matthew Daitch, `http://www.scorgloballifeamericas.com/en-us/knowledgecenter/Pages/Pricing-a-Single-Premium-Immediate-Annuity.aspx` (accessed on 24 March 2020).

(*iii*) The mortality improvement assumption or the assumed trend of the real mortality of the population, this represents the mortality model adopted by the insurer or the company. The mortality model is given by any existing model such as Vasicek, Hull-White, Lee-Carter, CIR model and so on.

For our first contract, the net interest rate and the base mortality table are equivalent to those of a classical annuity. For the second contract, the net interest rate is given by a risk-adjusted discount rate depending on both financial and longevity risks shared and the base mortality table is equal to that of a classical annuity. Valuing these contracts allows us to value the GSA as this has not yet been valued in the literature. Our valuation approach is based on the risk-neutral approach (see Zaglauer and Bauer, 2008) and is developed within the financial market developed in Section 3.1.1 along with the mortality model developed in Section 2.2.1. The risks considered in this work are the equity, interest rate and longevity risks for which we assume dependence between the equity and interest rate risks.
Using Monte Carlo methods, we will compare the value of each of the contract with annuitants' unique premium as well as with the values of classical annuity and GSA. This enables us to identify the product with lower value and to highlight the sensitivity of the contracts values with respect to some parameters. We further find the proportions of risks borne by annuitants as well as the investment strategy and the risk adjusted rate (for the second contract) that guarantee the fair valuation of each contract. Detailed analysis of these contracts with two periods of time is also made.
The remainder of this chapter is structured as follows. In Section 5.2 we present our valuation framework where the financial market, the investment strategy and the mortality model are defined. Detailed about both contracts are given in Section 5.3 and the valuation formulae are developed in Section 5.4. Numerical analysis is made in Section 5.5 using Monte Carlo method. In this section we make sensitivity analysis with respect to some significant parameters and we draw up comparison between the two contracts. Conclusion and general comments are given in Section 5.6.

5.2 Valuation framework

5.2.1 Financial setting

Within the financial market described in Section 3.1, we consider a money market account $B(t)$, a stock S_t and a long-term bond $P(t, n)$ with fixed maturity $n \geq t$.

Following the approach of Barbarin and Devolder, 2005 we assume that the stock follows a log-normal distribution with a constant drift $\mu \in \mathbb{R}$ and constant volatility $\sigma_s \in \mathbb{R}_+^*$; whereas the money market account and the long-term bonds are defined with a stochastic interest rate $\{r_t\}$. We consider a Ornstein-Uhlenbeck interest rate model defined under the physical probability measure by Formula (3.2). Let $\lambda_r \in \mathbb{R}$ be the market price of risk and $\mathbb{Q}$ be the risk-neutral measure such that $\overline{W_t^r} = W_t^r + \lambda_r t$ is a Brownian motion under $\mathbb{Q}$. The dynamic of r_t under $\mathbb{Q}$ is then

$$dr_t = b(c - \frac{\lambda_r \sigma_r}{b} - r_t)dt + \sigma_r d\overline{W_t^r}\,|_{\mathbb{Q}}\;. \tag{5.1}$$

One can show that the expression of r_t under the measure $\mathbb{Q}$ is given by

$$r_t = c_0 + (r_0 - c_0)e^{-bt} + \sigma_r \int_0^t e^{b(z-t)} d\overline{W_z^r}; \tag{5.2}$$

where $c_0 = c - \frac{\lambda_r \sigma_r}{b}$. This implies that r_t follow a normal distribution, i.e

$$r_t \rightsquigarrow \mathcal{N}^Q \left(c_0 + (r_0 - c_0)e^{-bt},\ \frac{\sigma_r^2}{2b}\left(1 - e^{-2bt}\right)\right).$$

For later convenience we shall introduce the process for the integral of the short rate under $\mathbb{Q}$

$$I_t = \int_0^t r_z dz = c_0 t + \frac{(r_0 - c_0)\left(1 - e^{-bt}\right)}{b} + \frac{\sigma_r}{b}\int_0^t \left(1 - e^{-b(t-z)}\right) d\overline{W_z^r}. \tag{5.3}$$

Hence

$$I_t \rightsquigarrow \mathcal{N}^Q \left(c_0 t + \frac{(r_0 - c_0)\left(1 - e^{-bt}\right)}{b},\ \frac{\sigma_r^2}{b^2}\left[t - \frac{\left(1 - e^{-bt}\right)}{b} - \frac{\left(1 - e^{-bt}\right)^2}{2b}\right]\right).$$

From this, one can show that for all $u > t$,

$$\begin{aligned} I_u - I_t &= \int_t^u r_z dz = c_0(u-t) + \frac{(r_t - c_0)\left(1 - e^{-b(u-t)}\right)}{b} \\ &+ \frac{\sigma_r}{b}\int_t^u \left(1 - e^{-b(u-z)}\right) d\overline{W_z^r}, \end{aligned} \tag{5.4}$$

which implies that

$$I_u - I_t \rightsquigarrow \mathcal{N}^Q \left(c_0(u-t) + \frac{(r_t - c_0)\left(1 - e^{-b(u-t)}\right)}{b},\ \frac{\sigma_r^2}{b^2}\left[(u-t) - \frac{\left(1 - e^{-b(u-t)}\right)}{b} - \frac{\left(1 - e^{-b(u-t)}\right)^2}{2b}\right]\right). \tag{5.5}$$

The value at time $t \geq 0$ of the long-term bond with maturity $n \geq t$ is given by Formula (3.4) and it has the following dynamics

$$\begin{aligned} dP(t,n) &= P(t,n)\left[(r_t - \lambda_r \sigma_r B_r(t,n))\,dt - \sigma_r B_r(t,n) dW_t^r\right]|_{\mathbb{P}} \\ &= P(t,n)\left[r_t dt - \sigma_r B_r(t,n) d\overline{W_t^r}\right]|_{\mathbb{Q}}. \end{aligned} \tag{5.6}$$

The money market account follows the dynamic

$$dB(t) = r_t B(t)\; dt,$$

with $t \geq 0$ and $B(0) = 1$.
The dynamic of the stock under the physical measure $\mathbb{P}$ is defined as

$$dS_t = \mu S_t dt + \sigma_s S_t dW_t^s|_{\mathbb{P}}; \tag{5.7}$$

where W_t^s is a Brownian motion under $\mathbb{P}$ and we set $S_0 = 1$. Assume dependence between the stock and the interest rate, such that W_t^s is correlated with W_t^r i.e $\rho = corr(W_t^s, W_t^r)$. The SDE (5.7) become

$$\begin{aligned} dS_t &= S_t\left[\mu dt + \sigma_s\ \rho\ dW_t^r + \sigma_s\ \sqrt{1-\rho^2}\ dW_t\right]|_{\mathbb{P}} \\ &= S_t\left[r_t dt + \sigma_s\ \rho\ d\overline{W_t^r} + \sigma_s\ \sqrt{1-\rho^2}\ d\overline{W_t}\right]|_{\mathbb{Q}}; \end{aligned} \tag{5.8}$$

where W_t is a Brownian motion under $\mathbb{P}$ independent of W_t^r and $\overline{W_t}$ is a Brownian motion under the measure $\mathbb{Q}$ defined as

$$d\overline{W_t} = dW_t + \frac{\mu - r_t - \lambda_r\ \rho\ \sigma_s}{\sigma_s\sqrt{1-\rho^2}}dt.$$

5.2.2 Investment strategy

We consider a portfolio A_t made of $x_s \in [0,1]$ proportion of the capital invested on the stock; $x_p \in [0,1]$ proportion of the capital invested on the bond and $(1-x_p-x_s)$ proportion of the capital invested on the money market account with dynamic rebalancing. A_t has the following dynamic

$$\begin{aligned} \frac{dA_t}{A_t} &= x_s\frac{dS_t}{S_t} + x_p\frac{dP(t,n)}{P(t,n)} + (1-x_p-x_s)\frac{dB(t)}{B(t)} \\ &= x_s r_t dt + x_s\sigma_s\ \rho\ d\overline{W_t^r} + x_s\sigma_s\ \sqrt{1-\rho^2}d\overline{W_t} + x_p r_t dt \\ &\quad - x_p\sigma_r B_r(t,n) d\overline{W_t^r} + r_t dt - x_p r_t dt - x_s r_t dt|_{\mathbb{Q}} \\ &= r_t dt + (x_s\sigma_s\ \rho - x_p\sigma_r\ B_r(t,n))\, d\overline{W_t^r} + x_s\sigma_s\ \sqrt{1-\rho^2}d\overline{W_t}|_{\mathbb{Q}} \\ &= r_t dt + \sigma(t,n) d\overline{W_t^r} + \overline{\sigma} d\overline{W_t}|_{\mathbb{Q}}; \end{aligned} \tag{5.9}$$

where

$$\sigma(t,n) = x_s\sigma_s\ \rho - x_p\sigma_r\ B_r(t,n)$$

and

$$\overline{\sigma} = x_s\sigma_s\ \sqrt{1-\rho^2}.$$

Let's denote by $\mathbb{Q}_n$ the n-forward measure, one can show that $\mathbb{Q}_n$ is generated by $\widehat{W_t^r}$ and $\widehat{W_t}$ defined as

$$d\widehat{W_t^r} = d\overline{W_t^r} + \sigma_r B_r(t,n)dt;\quad \widehat{W_t} = \overline{W_t}. \tag{5.10}$$

Formula (5.9) becomes

$$\frac{dA_t}{A_t} = \left[r_t - \sigma_r\ \sigma(t,n)B_r(t,n)\right]dt + \sigma(t,n)d\widehat{W_t^r} + \overline{\sigma}d\widehat{W_t}|_{\mathbb{Q}_n}.$$

One can show that the value at time $n > u > t \geq 0$ of this portfolio is given by

$$A(t,u) = A_t Asset(t,u); \tag{5.11}$$

where

$$\begin{aligned} Asset(t,u) &= \frac{\exp\left[\int_t^u \left(r_z - \sigma_r\ \sigma(z,n)B_r(z,n)\right)dz + \int_t^u \sigma(z,n)d\widehat{W_z^r} + \overline{\sigma}\widehat{W_{u-t}}\right]}{\exp\left[\frac{1}{2}\int_t^u \left(\sigma(z,n)^2 + \overline{\sigma}^2\right)dz\right]} \\ &= \exp\left[-\int_t^u \left(\sigma_r\ \sigma(z,n)B_r(z,n) + \frac{1}{2}\left(\sigma(z,n)^2 + \overline{\sigma}^2\right)\right)dz\right] \\ &\quad \times \exp\left[G_1 + (I_u - I_t) + G_2\right]; \end{aligned} \tag{5.12}$$

with

$$G_1 = \int_t^u \sigma(z,n)d\widehat{W_z^r};\quad G_2 = \overline{\sigma}\widehat{W_{u-t}}$$

and we also denote $A_t := Asset(0,t)$. Using Formula (5.10), we have

$$d\overline{W_t^r} = d\widehat{W_t^r} - \sigma_r B_r(t,n)dt.$$

Substituting this latter in Formula (5.2) gives the following formula of $(I_u - I_t)$ under $\mathbb{Q}_n$

$$I_u - I_t = e_r + \frac{\sigma_r}{b}\int_t^u \left(1 - e^{-b(u-z)}\right)d\widehat{W_z^r}\ |_{\mathbb{Q}_n}; \tag{5.13}$$

where

$$\begin{aligned} e_r &:= \mathbb{E}^{Q_n}\left[(I_u - I_t)\right] = c_0(u-t) + \frac{(r_t - c_0)\left(1 - e^{-b(u-t)}\right)}{b} \\ &- \frac{\sigma_r^2}{b}\int_t^u \left(1 - e^{-b(u-z)}\right)B_r(z,n)dz. \end{aligned} \tag{5.14}$$

From calculations, we obtain the following formulae and distributions of G_1 and G_2 respectively

$$\begin{cases} G_1 & = \int_t^u \left(Z + \frac{x_p \sigma_r}{b} e^{-b(n-z)} \right) d\widehat{W_z^r} \\ \\ G_1 & \rightsquigarrow \mathcal{N}^{Q_n} \left(0, \ \sigma_{G_1} \right) \end{cases} \tag{5.15}$$

$$\begin{cases} G_2 & = x_s \sigma_s \sqrt{1-\rho^2} \widehat{W_{u-t}} \\ \\ G_2 & \rightsquigarrow \mathcal{N}^{Q_n} \left(0, \ x_s^2 \sigma_s^2 (1-\rho^2)(u-t) \right); \end{cases} \tag{5.16}$$

where

$$\sigma_{G_1} = Z^2(u-t) + 2x_p\sigma_r Z \left(e^{-b(n-u)} - e^{-b(n-t)} \right) + \frac{2x_p^2\sigma_r^2}{b} \left(e^{-2b(n-u)} - e^{-2b(n-t)} \right)$$

and

$$Z = x_s \sigma_s \rho - \frac{x_p \sigma_r}{b}. \tag{5.17}$$

Set $E(t,u) := \log(Asset(t,u))$ which is normally distributed, let us find the distribution parameters (i.e the mean e_A and the variance V_A) of $E(t,u)$ under $\mathbb{Q}_n$. From Formula (5.12), we find that $e_A = \mathbb{E}^{Q_n}[E(t,u)]$ with

$$e_A = -\int_t^u \left(\sigma_r \ \sigma(z,n) B_r(z,n) + \frac{1}{2} \int_t^u \left(\sigma(z,n)^2 + \overline{\sigma}^2 \right) \right) dz + e_r; \tag{5.18}$$

where e_r is given by Formula (5.14)

$$V_A = Var\left[E(t,u)\right] = \mathbb{E}^{Q_n}\left[((I_u - I_t) + G_1 + G_2 - e_r)^2 \right]. \tag{5.19}$$

Setting $Q := (I_u - I_t) + G_1$, we have

$$\begin{aligned} Q &= e_r + \frac{\sigma_r}{b} \int_t^u \left(1 - e^{-b(u-z)} \right) d\widehat{W_z^r} + \int_t^u \sigma(z,n) d\widehat{W_z^r} \\ &= e_r + V_Q; \end{aligned} \tag{5.20}$$

where

$$V_Q = \frac{\sigma_r}{b} \int_t^u \left(1 - e^{-b(u-z)} \right) d\widehat{W_z^r} + \int_t^u \sigma(z,n) d\widehat{W_z^r}.$$

This implies that Formula (5.19) becomes

$$\begin{aligned} V_A &= \mathbb{E}^{Q_n}\left[Q^2+G_2^2+e_r^2-2e_rG_2+2QG_2-2Qe_r\right] \\ &= \mathbb{E}^{Q_n}\left[Q^2\right]+\mathbb{E}\left[G_2^2\right]+e_r^2-2e_r\mathbb{E}\left[e_r+V_Q\right] \\ &= \mathbb{E}^{Q_n}\left[V_Q^2\right]+(u-t)\overline{\sigma}^2-e_r^2. \end{aligned} \tag{5.21}$$

From the Ito isometry, we have

$$\begin{aligned} \mathbb{E}^{Q_n}\left[V_Q^2\right] &= \int_t^u \left[\frac{\sigma_r}{b}\left(1-e^{-b(u-z)}\right)+\sigma(z,n)\right]^2 dz \\ &= \frac{\sigma_r^2}{b^2}\int_t^u \left(1-e^{-b(u-z)}\right)^2 dz+\int_t^u \sigma(z,n)^2 dz \\ &\quad +2\frac{\sigma_r}{b}\int_t^u \left(1-e^{-b(u-z)}\right)\sigma(z,n)dz. \end{aligned} \tag{5.22}$$

Note that the first term of the RHS is equal to the variance of $(I_u - I_t)$ defined in Formula (5.5) and the second term of the RHS is equal to the variance of G_1 defined in Formula (5.15). Therefore, substituting Formula (5.22) in Formula (5.21) gives the variance of $E(t,u)$, such that

$$E(t,u) \rightsquigarrow \mathcal{N}^{Q_n}\left(e_A,\ V_A\right);$$

where e_A is given by Formula (5.18).
Let's denote by R_t the rate of return of the portfolio A_t, with $t>0$ such the return of A_t is defined by

$$1+R_t = Asset(t-1,t). \tag{5.23}$$

5.2.3 Mortality model

We consider the HW mortality model given in Section 2.2.1. The dynamic of the policyholders' force of mortality under the physical measure is given by Formula (2.1). Let $\mathbb{Q}_\mu$ be a longevity risk-neutral measure such that the process $\overline{W_t^\mu} = W_t^\mu + \lambda_\mu t$ is a Brownian motion under $\mathbb{Q}_\mu$. The dynamic of the HW model under the longevity risk-neutral measure is

$$d\mu_t^{x_0} = (\theta(t) - \lambda_\mu\sigma_\mu - a\mu_t^{x_0})dt + \sigma_\mu\ d\overline{W_t^\mu}|_{\mathbb{Q}_\mu}; \tag{5.24}$$

where $\lambda_\mu > 0$ is the market price of longevity risk. Assume independence between the interest rate and the longevity risks as well as independence between the financial and longevity risks. In this case the survival index is defined as

$$I_{s-t}^{x_0+t} = e^{-\int_t^s \mu_u^{x_0}du} = e^{X_q(t,s)} = e^{m_q^{x_0}(t,s)+\sigma_q^{x_0}(t,s)Z}, \tag{5.25}$$

with Z being a standard normal random variable with mean zero and variance one.
We can show that

$$X_q(t,s) = -\int_t^s \mu_u^{x_0} du = m_q^{x_0}(t,s) + \frac{\sigma_\mu}{a}\int_t^s \left(1-e^{-a(s-u)}\right) d\overline{W}_u^\mu. \tag{5.26}$$

This implies that

$$X_q(t,s) \rightsquigarrow \mathcal{N}\left(m_q^{x_0}(t,s), \left(\sigma_q^{x_0}(t,s)\right)^2\right) \quad \Rightarrow \quad \frac{X_q(t,s)-m_q^{x_0}(t,s)}{\sigma_q^{x_0}(t,s)} \rightsquigarrow \mathcal{N}(0,1),$$

with

$$\begin{aligned} m_q^{x_0}(t,s) &= \frac{\mu_t^{x_0}(e^{-a(s-t)}-1)}{a} - \frac{Ae^{Bt}}{B(a+B)}\left(e^{B(s-t)}-1\right) \\ &- \frac{Ae^{Bt}}{a(a+B)}\left(e^{-a(s-t)}-1\right) + \frac{\lambda_\mu\ \sigma_\mu}{a}\left(e^{-a(s-t)}-1+(s-t)\right) \end{aligned} \tag{5.27}$$

and

$$\left(\sigma_q^{x_0}(t,s)\right)^2 = \frac{\sigma_\mu^2}{a^2}\left[s-t-\frac{1-e^{-a(s-t)}}{a} - \frac{\left(1-e^{-a(s-t)}\right)^2}{2a}\right]. \tag{5.28}$$

Let $\rho_q(s,l)$ be the correlation factor that captures the dependecy between two survival indexes, with $0 \le t < s < l$. We define this correlation using the covariance between the $X_q(t,s)$'s as follows : for $0 \le t < s < l$, the covariance between $X_q(t,s)$ and $X(t,l)$ is defined as

$$cov[X_q(t,s), X_q(t,l)] = \frac{\sigma_\mu^2}{a^2}\left[s-t-\frac{(e^{-al}+e^{-as})(e^{as}-e^{at})}{a} + \frac{e^{-a(s+l)}\left(e^{2as}-e^{2at}\right)}{2a}\right],$$

which implies that the correlation is given by

$$\rho_q(s,l) := corr[X_q(t,s), X_q(t,l)] = \frac{cov[X_q(t,s), X_q(t,l)]}{\sigma_q^{x_0}(t,s)\ \sigma_q^{x_0}(t,l)}; \tag{5.29}$$

where $\sigma_q^{x_0}(t,s)$ is given by Formula (5.28). It follows that the survival probability of an individual initially aged x_0, alive at age x_0+t of living at least up to age x_0+s under the measure $\mathbb{Q}_\mu$ is given by

$$p_{x_0}^q(t,s) \quad = \quad \mathbb{E}^{Q_\mu}\left[\frac{I_s^{x_0}}{I_t^{x_0}}|\mathcal{F}_t^\mu\right] = A_{x_0}^q(t,s)e^{-B_{x_0}^q(t,s)\mu_t^{x_0}}; \tag{5.30}$$

where

$$B^q_{x_0}(t,s) = \frac{1}{a}\left(1 - e^{-a(s-t)}\right)$$

and

$$\begin{aligned} A^q_{x_0}(t,s) &= \exp\left(\frac{Ae^{Bt}}{a+B}\left[\frac{1-e^{B(s-t)}}{B} + B(t,s)\right] - \frac{\sigma^2}{2a^2}\left(B^q_{x_0}(t,s) - (s-t)\right)\right) \\ &\quad \times \exp\left(-\frac{\sigma^2}{4a}B^q_{x_0}(t,s)^2 + \lambda_\mu\ \sigma_\mu\left(\frac{(s-t)}{a} - B^q_{x_0}(t,s)\right)\right). \end{aligned}$$

5.3 The annuity contracts

We consider two contracts : the proportional risk-sharing GSA defined in Section 4.2.2 and the complete risk-sharing GSA defined in Section 4.3.

5.3.1 Contract 1 : the proportional risk-sharing GSA

We assume an initial cohort of N_0 policyholder paying a total single premium of F_0 to the insurer in order to buy a proportional risk-sharing GSA. The cohort is assumed to be homogeneous i.e the group members have the same initial age and the same single premium. Moreover we assume that the cohort is closed, meaning that no one can joint the cohort after the establishment of the contract. Let $\beta_t \in [0,1]$ and $\beta'_t \in [0,1]$ be respectively the proportions at time t of the longevity and the financial risks borne by the group of policyholders. For simplification purpose we assume constant risk proportions β and β'. The benefit payout of a proportional risk-sharing GSA (PRS-GSA) is given by

$$B_t(\beta,\beta') = B_{t-1}(\beta,\beta')\left[\beta\ Lrisk_t + (1-\beta)\right]\left[\beta'\ Erisk_t + (1-\beta')\right]; \quad (5.31)$$

where

$$Lrisk_t = \frac{p_{x_0+t-1}}{p^*_{x_0+t-1}} \quad \text{and} \quad Erisk_t = \frac{1+R_t}{1+R} \quad (5.32)$$

respectively represent the financial and the longevity risks at time t with $p_{x_0+t} = {}_1p_{x_0+t}$ and $p*_{x_0+t-1} = I_1^{x_0+t-1}$, where $I_1^{x_0+t-1}$ is obtained from Formula (5.25) and with $1+R_t$ defined from Formula (5.23). B_0 is the initial individual benefit and is defined by Formula (4.1), i.e

$$B_0 = \frac{F_0}{N_0\ddot{a}_{x_0}}; \quad (5.33)$$

where $\ddot{a}_{x_0} = \sum_{t=0}^{\infty} v^t \ {}_tp_{x_0}$ and $v = (1+R)^{-1}$, with R and ${}_tp_{x_0}$ being respectively the annuity technical rate and life table used by the insurer. N_t is the expected number of survivors at age $x_0 + t$ for $t \geq 0$ whereas N_t^* is it real value. Note that B_0 is constant with respect to the risk proportions β and β'. Similarly, the benefit payout of the PRS-GSA of an open heterogeneous cohort can be obtained by respectively defining $Erisk_t$ and $Lrisk_t$ as the mortality adjustment factor and the financial adjustment factor proposed by Piggott et al., 2005 or Qiao and Sherris, 2013. For a closed homogeneous cohort one can show that Formula (5.31) takes the form

$$B_t(\beta, \beta') \quad = \quad B_0 \prod_{j=1}^{t} \left[\beta \ Lrisk_j + (1-\beta)\right] \left[\beta' \ Erisk_j + (1-\beta')\right]; \tag{5.34}$$

where $Erisk_t$ and $Lrisk_t$ are given by 5.32. The annuitants will each receive $B_t(\beta, \beta')$ at $t \geq 0$ as long as they are alive.

5.3.2 Contract 2 : the complete risk-sharing GSA

We consider the same group of policyholders subscribed for Contract 1 in Section 5.3.1, buying a complete risk-sharing GSA (CRS-GSA). Let $\beta, \ \beta' \in [0,1]$ be the risk proportions defined previously. The initial benefit $B_0(\beta, \beta')$ is defined by the risk adjusted annuity factor denoted by $\ddot{a}_{x_0}(\beta, \beta')$ depending of the risk adjusted discount rate $R(\beta, \beta')$ which encodes both the financial and longevity risks shared. Hence this contract allows the insurer to propose different technical rated depending on the proportions of risks borne by annuitants. The risk adjustment annuity factor as is defined by

$$\ddot{a}_x(\beta, \beta') = \sum_{s=0}^{\infty} \left(\frac{1}{(1 + R(\beta, \beta'))} \right)^s \times \ {}_sp_x,$$

in such a way that the initial benefit of each policyholder is given by

$$B_0(\beta, \beta') = \frac{F_0}{N_0 \ \ddot{a}_x(\beta, \beta')}. \tag{5.35}$$

For all $t > 0$ the benefit at t of each policyholder subscribing to this contract is given by

$$B_t(\beta, \beta') = B_0(\beta, \beta') \prod_{j=1}^{t} \left[\beta Lrisk_j + (1-\beta)\right] \times \left[\beta' Erisk_j(\beta, \beta') + (1-\beta')\right]; \tag{5.36}$$

where $B_0(\beta,\beta')$ is given by Formula (5.35). The financial and longevity risks shared at each time $t \geq 0$ are respectively given by

$$Erisk_j(\beta,\beta') = \frac{1+R_j}{1+R(\beta,\beta')} \quad \text{and} \quad Lrisk_j = \frac{p_{x+j-1}}{p^*_{x+j-1}}. \tag{5.37}$$

From these contracts, one can derive single risk sharing GSAs, for instance setting $\beta = 0$ we have the (complete) financial risk-sharing GSA and by setting $\beta' = 0$ we have the (complete) longevity risk-sharing GSA as shown in Chapter 4.

5.4 Valuation of the annuity contracts

We remind that the maximum length of a contract bought by policyholders is denoted by $n \geq 0$ and we assume n is sufficiently large such that policyholders die at very old age, say 111 years old. This mean that each policyholder could receive at most $n+1$ cash flows as long as they are alive.

5.4.1 Contract 1

The final value (at time $t = n$) of the insurance Contract 1 defined in 5.3.1 is given by

$$\begin{aligned} V_n^{(C1)}(\beta,\beta') &= \sum_{t=0}^{n} B_t(\beta,\beta')\ {}_tp^*_{x_0}\ e^{\int_t^n r_s\ ds}, \\ &= \sum_{t=1}^{n} B_0 \prod_{j=1}^{t} \left[\beta\ Lrisk_j + (1-\beta)\right]\left[\beta'\ Erisk_j + (1-\beta')\right]\ {}_tp^*_{x_0}\ e^{\int_t^n r_s\ ds} \\ &\quad + B_0\ e^{\int_0^n r_s\ ds}. \end{aligned} \tag{5.38}$$

Hence the value at time $t = 0$ of a proportional RS-GSA is given by

$$\frac{V_0^{(C1)}(\beta,\beta')}{P(0,n)} = \mathbb{E}^{Q_0}\left[V_n^{(C1)}(\beta,\beta')\right]; \tag{5.39}$$

where $\mathbb{E}^{Q_0}$ is the expectation under the probability measure $\mathbb{Q}_0$ such that

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}} = \frac{d\mathbb{Q}_n}{d\mathbb{P}}\ \frac{d\mathbb{Q}_\mu}{d\mathbb{P}}.$$

Note that $\mathbb{Q}_0$ is a product probability risk-neutral measure made of the longevity risk-neutral measure $\mathbb{Q}_\mu$ and the forward financial risk-neutral measure $\mathbb{Q}_n$ defined in Section 5.2.2. one can show that Formula (5.39) takes the form

$$
\begin{aligned}
V_0^{(C1)}(\beta, \beta') &= B_0 + \sum_{t=1}^{n} B_0 \; P(0,n) \; \mathbb{E}^{Q_\mu} \left[{}_t p^*_{x_0} \prod_{j=1}^{t} \; [\beta \; Lrisk_j + (1-\beta)] \right] \\
&\quad \times \mathbb{E}^{Q_n} \left[P^{-1}(t,n) \prod_{j=1}^{t} [\beta' \; Erisk_j + (1-\beta')] \right]; \qquad (5.40)
\end{aligned}
$$

where $P(t,n)$ is given by Formula (3.4).

Let's set

$$
V_L^{(C1)}(\beta) \;:=\; \mathbb{E}^{Q_\mu} \left[{}_t p^*_{x_0} \prod_{j=1}^{t} \; [\beta \; Lrisk_j + (1-\beta)] \right]; \qquad (5.41)
$$

$$
V_E^{(C1)}(\beta') \;:=\; \mathbb{E}^{Q_n} \left[P^{-1}(t,n) \prod_{j=1}^{t} [\beta' \; Erisk_j + (1-\beta')] \right]. \qquad (5.42)
$$

Using Formulae (5.27)–(5.28), we have

$$
V_L^{(C1)}(\beta) \;=\; \mathbb{E}^{Q_\mu} \left[e^{m_q^{x_0}(0,t) + \sigma_q^{x_0}(0,t) Z^{(t)}} \prod_{j=1}^{t} \; \left[\beta \; p_{x_0+j-1} \; e^{-m_q^{x_0}(j-1,j) - \sigma_q^{x_0}(j-1,j) Z^{(j)}} + (1-\beta) \right] \right];
$$

where Z_j is a standard normally distributed random variable with mean zero and variance one and is dependent of Z_{j-1}. Let $\rho_q(j-1,j)$ be the correlation factor that captures the dependency between two consecutive survival indexes and is defined from Formula (5.29). From the Cholesky decomposition, we have

$$
\begin{cases}
Z^{(j)} & = \; \rho_q(j-1,j) Z^{(j-1)} + \sqrt{1-\rho^2(j-1,j)} Z_j \\
\\
Z^{(0)} = 0 \quad \& \quad Z^{(1)} = Z_1, &
\end{cases} \qquad (5.43)
$$

with $Z^{(j-1)}$ is independent of Z_j.

5.4.2 Contract 2

Concerning Contract 2 defined in 5.3.2 the value at maturity $t = n$ of such a contract is defined as

$$\begin{aligned}
V_n^{(C2)}(\beta,\beta') &= \sum_{t=0}^{n} B_t(\beta,\beta')\ {}_t p^*_{x_0}\ e^{\int_t^n r_s\ ds},\\
&= \sum_{t=1}^{n} B_0(\beta,\beta') \prod_{j=1}^{t} [\beta\ Lrisk_j + (1-\beta)] \left[\beta'_j\ Erisk_j(\beta,\beta') + (1-\beta')\right]\\
&\quad \times\ {}_t p^*_{x_0}\ e^{\int_t^n r_s\ ds} +\ B_0(\beta,\beta')\ e^{\int_0^n r_s\ ds}. \qquad (5.44)
\end{aligned}$$

It follows that the value at inception of a CRS-GSA is given by

$$\begin{aligned}
V_0^{(C2)}(\beta,\beta') &= P(0,n)\ \mathbb{E}^{Q_0}\left[V_n^{(C2)}(\beta,\beta')\right]\\
&= B_0(\beta,\beta') + \sum_{t=1}^{n} B_0(\beta,\beta')\ P(0,n)\ \mathbb{E}^{Q_\mu}\left[{}_t p^*_{x_0} \prod_{j=1}^{t}\ [\beta\ Lrisk_j + (1-\beta)]\right]\\
&\quad \times \mathbb{E}^{Q_n}\left[P^{-1}(t,n) \prod_{j=1}^{t} [\beta'\ Erisk_j(\beta,\beta') + (1-\beta')]\right]. \qquad (5.45)
\end{aligned}$$

Let's set

$$\begin{aligned}
V_L^{(C2)}(\beta) &:= \mathbb{E}^{Q_\mu}\left[{}_t p^*_{x_0} \prod_{j=1}^{t}\ [\beta\ Lrisk_j + (1-\beta)]\right]\\
&= \mathbb{E}^{Q_\mu}\left[{}_t p^*_{x_0} \prod_{j=1}^{t}\ \left(\beta\ \frac{p_{x_0+j-1}}{p^*_{x_0+j-1}} + (1-\beta)\right)\right]; \qquad (5.46)
\end{aligned}$$

$$\begin{aligned}
V_E^{(C2)}(\beta,\beta') &:= \mathbb{E}^{Q_n}\left[P^{-1}(t,n) \prod_{j=1}^{t} [\beta'\ Erisk_j(\beta,\beta') + (1-\beta')]\right]\\
&= \mathbb{E}^{Q_n}\left[P^{-1}(t,n) \prod_{j=1}^{t} \left(\beta'\ \frac{1+R_t}{1+R(\beta,\beta')} + (1-\beta')\right)\right]. \qquad (5.47)
\end{aligned}$$

Formulae (5.27)–(5.28) imply that

$$V_L^{(C2)}(\beta) = \mathbb{E}^{Q_\mu}\left[e^{m_q^{x_0}(0,t)+\sigma_q^{x_0}(0,t)Z^{(t)}} \prod_{j=1}^{t}\ \left[\beta\ p_{x_0+j-1}\ e^{-m_q^{x_0}(j-1,j)-\sigma_q^{x_0}(j-1,j)Z^{(j)}} + (1-\beta)\right]\right],$$

with Z_j's given by 5.43.
Formulas (5.40) and (5.45) represent the values of Contract 1 and 2 respectively. They are said to be fair if they are equal to the unique premium $\frac{F_0}{N_0}$. Below we present a detail study of a simple case of two period annuity.

5.4.3 Special case of two periods of time

As we could not find closed valuation formulae for the two contract (with multiple periods of time), here we make a detailed study of two period contracts; i.e when $n = 1$. In this case, the annuitants receive two payments: the first is certain (i.e risk-free) and paid right after the contract inception, whereas the second is paid if the annuitant is alive, hence depends on both the financial and longevity risks.

Contract 1

For $n = 1$, the valuation Formula (5.40) of this contract becomes

$$\begin{aligned} V_0^{(C1)}(\beta,\beta') &= B_0 + B_0\ P(0,1)\ \mathbb{E}^{Q_\mu}\left[p^*_{x_0}\left(\beta\ \frac{p_{x_0}}{p^*_{x_0}} + (1-\beta)\right)\right]\ \mathbb{E}^{Q_1}\left[\beta'\ \frac{1+R_1}{1+R} + (1-\beta')\right] \\ &= B_0 + B_0\ P(0,1)\ \left(\beta\ p_{x_0} + (1-\beta)p^q_{x_0}(0,1)\right)\ \left(\beta'\ \frac{\mathbb{E}^{Q_1}\left[Asset(0,1)\right]}{1+R} + (1-\beta')\right); \end{aligned}$$

where $p^q_{x_0}(0,1) = \mathbb{E}^{Q_\mu}\left[p^*_{x_0}\right]$ and

$$B_0 = \frac{F_0}{N_0\left(1 + \frac{1}{1+R}\ p_{x_0}\right)}.$$

By definition of the forward measure, we have $\mathbb{E}^{Q_1}\left[Asset(0,1)\right] = \frac{1}{P(0,1)}$ and it follows that the value of this contract is given by

$$\begin{aligned} V_0^{(C1)}(\beta,\beta') &= B_0 + B_0\ P(0,1)\ \left(\beta\ p_{x_0} + (1-\beta)p^q_{x_0}(0,1)\right)\ \left(\beta'\ \frac{P^{-1}(0,1)}{(1+R)} + (1-\beta')\right) \\ &= B_0 + B_0\ \left(\beta\ p_{x_0} + (1-\beta)p^q_{x_0}(0,1)\right)\ \left(\beta'\ \frac{1}{(1+R)} + (1-\beta')P(0,1)\right). \end{aligned} \tag{5.48}$$

The value of a contract is said to be fair if its value is equal to the unique premium, i.e if $\frac{V_0^{(C1)}(\beta,\beta')}{(F_0/N_0)} = 1$. Therefore, Formula (5.48) becomes

$$\frac{p_{x_0}}{1+R} = \left[\beta\ \left(p_{x_0} - p^q_{x_0}(0,1)\right) + p^q_{x_0}(0,1)\right]\ \left[\beta'\ \left(\frac{1}{(1+R)} - P(0,1)\right) + P(0,1)\right]. \tag{5.49}$$

Contract 2

Similarly to Contract 1 and with

$$B_0(\beta,\beta') = \frac{F_0}{N_0\left(1 + \frac{1}{1+R(\beta,\beta')}\ p_{x_0}\right)},$$

one can show that the value of Contract 2 is given by

$$V_0^{(C2)}(\beta,\beta') = B_0(\beta,\beta') + B_0(\beta,\beta')\left(\beta\ p_{x_0} + (1-\beta)p^q_{x_0}(0,1)\right) \times \left(\beta'\ \frac{1}{(1+R(\beta,\beta'))} + (1-\beta')P(0,1)\right). \tag{5.50}$$

It follows that the fair valuation of this contract leads to

$$\left[\beta\ \left(1 - \frac{p^q_{x_0}(0,1)}{p_{x_0}}\right) + \frac{p^q_{x_0}(0,1)}{p_{x_0}}\right]\left[\beta'\ \left(1-(1+R(\beta,\beta'))P(0,1)\right) + (1+R(\beta,\beta'))P(0,1)\right] = 1. \tag{5.51}$$

Formula (5.51) can be rewritten as

$$\frac{p_{x_0}}{1+R(\beta,\beta')} = \left[\beta\ \left(p_{x_0} - p^q_{x_0}(0,1)\right) + p^q_{x_0}(0,1)\right]\ \left[\beta'\ \left(\frac{1}{(1+R(\beta,\beta'))} - P(0,1)\right) + P(0,1)\right]. \tag{5.52}$$

Below we present the values of the proportion of financial risk borne by annuitants and the risk adjusted rates that guarantee the fair valuation in some particular cases.

a) **Assume there is no longevity risk adjustment (i.e $\beta = 0$).**

- If the risk adjusted rate $R(\beta')$ is known, one can show that the proportion of financial risk borne by annuitants satisfying fair valuation is given by

$$\beta' = \frac{\frac{p_{x_0}}{p^q_{x_0}(0,1)} - (1+R(\beta'))P(0,1)}{1-(1+R(\beta'))P(0,1)}. \tag{5.53}$$

  Note that this is also valid for Contract 1 with $R = R(\beta')$. Formula (5.53) shows that the fair valuation is guaranteed for a pure financial GSA, i.e $\beta' = 100\%$ arising when the life table consider by the insurer is equal to te best estimate of survival index.

- If the proportion of financial risk borne by annuitants is known, then the fair risk adjusted rate is given by

$$R(\beta') = \frac{\frac{p_{x_0}}{p^q_{x_0}(0,1)} - \beta'}{P(0,1)(1-\beta')} - 1. \tag{5.54}$$

  It follows that when $p_{x_0} = p^q_{x_0}(0,1)$ and $(1+R(\beta'))^{-1} = P(0,1)$, then the contract is fair for any choice of β'.

b) **Assume equal proportions of financial and longevity risks adjustment i.e $\beta = \beta'$.** Hence one can show that for a given β, the fair risk adjusted rate in this case is given by

$$R(\beta, \beta) = \frac{\frac{1}{\beta\left(1 - \frac{p^q_{x_0}(0,1)}{p_{x_0}}\right) + \frac{p^q_{x_0}(0,1)}{p_{x_0}}} - \beta}{P(0,1)(1-\beta)} - 1; \tag{5.55}$$

and we have similar observation as for Formula (5.54).

Further observations from Cases a) and b) along with alternative case will be studied numerically in Section 5.5.1.
It is important to stress that these observations made for contracts with two periods of time can not be generalized to $n > 2$ periods of time because in this latter case the long term nature of the contracts as well as the dependencies strongly affect the results as we will see below.

5.5 Numerical studies

As in the previous chapters, we consider a closed cohort made of $N_0 = 1000$ policyholders initially aged $x_0 = 65$ with an ultimate age of 110 and paying a total unique premium of $F_0 = 1000^2$\$ for one of the two contracts. Using the mean square error (MSE) method we obtain the following mortality parameters

μ_0^{65}	A	B	a	σ_μ	λ_μ	MSE
0.0105677	0.001179271	0.105780593	0.004861948	0.012706011	0.001561629	0.000170881

Table 5.1: Parameters of the mortality model using the MSE

We further assume two annuity life table used by the insurer. We first consider the unisex Belgian table calculated by Statbel [2] and then the best estimate for the real survival probabilities so as to highlight the effect of these tables on the values of the contracts. In what follows we denote the Statbel table by PsB.
The parameters of the stock and the interest rate model are given in Section 3.3.1. Below we make sensitivity analysis of each contract with respect to significant parameters.

[2] `https://data.gov.be/fr/dataset/72c1db031defb669a78ea81ddba786bc3238a78a` (accessed on June 20, 2020)

5.5.1 Contract 2 with two periods of time

Figures 5.1 and 5.2 represent Formulas (5.54) and (5.55) respectively for different annuity life table. These figures illustrate how the fair risk adjusted rate evolves with the proportions of risks borne by annuitants as well as with the annuity life table used by the insurer for case a) and b) described above.

With no longevity risk ($\beta = 0$)

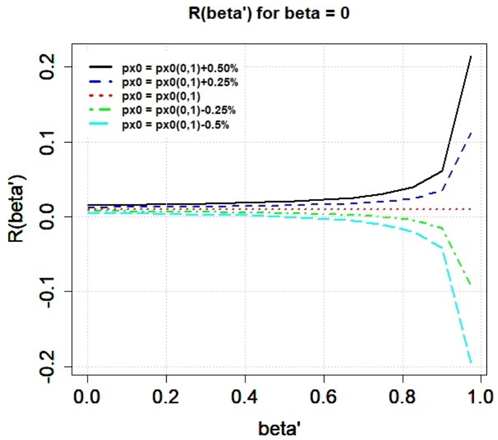


Figure 5.1: $R(\beta')$ that guarantees fair valuation with $p^q_{x_0}(0,1) = \mathbb{E}^{Q_n}[p^*_{x_0}] = 0.9885624$ for $\beta = 0$.

It follows form Figure 5.1 that

- When the insurer overestimates the longevity i.e $p_{x_0} = p^q_{x_0}(0,1) + y$, with $y > 0$, then the fair valuation will be satisfied if annuitants are granted with higher risk adjusted rate. Moreover, we observe an increasing risk adjusted rate with respect to the proportion of financial risk borne by annuitants. This means that the higher financial risk annuitants borne, the higher will be their risk adjusted rate hence higher benefits.

- When the insurer rather underestimates the longevity of the annuitants i.e $p_{x_0} = p^q_{x_0}(0,1) - y$, with $y < 0$ we have opposite observations.

- When the life table used by the insurer is equal to the best estimate of the survival index i.e $p_{x_0} = p^q_{x_0}(0,1)$, we observe a constant risk adjusted rate equal to the technical rate of the one year bond i.e $R(\beta') = P^{-1}(0,1) - 1 = 1.005017\%$.

With equal financial and longevity risk shifted to annuitants ($\beta = \beta'$)

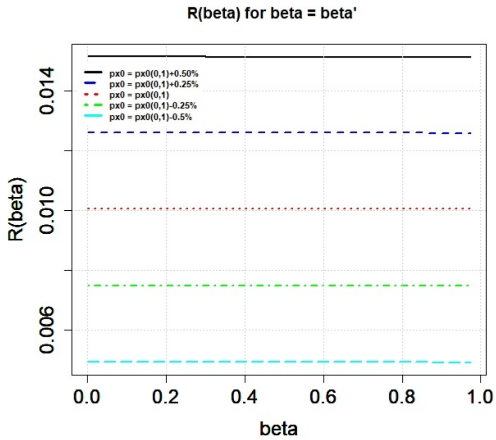


Figure 5.2: $R(\beta, \beta)$ that guarantees fair valuation with $p^q_{x_0}(0,1) = \mathbb{E}^{Q_n}[p^*_{x_0}] = 0.9885624$ for $\beta = \beta'$.

Figure 5.2 shows that

- We observe a slightly decreasing risk adjusted rate with respect to proportions of risks borne by annuitants, no matters the value of the annuity life table p_{x_0}. This figure clearly shows the high effect of longevity risk on the risk adjusted rate in the sense that we observe a compensation effect for high proportions of risks compared to Figure 5.1.

- Similarly to Figure 5.1, the more the insurer overestimates (respectively underestimates) the longevity of annuitants, the high (respectively low) risk adjusted rate will be. In other words, annuitants are granted with high rate if the insurer overestimates their longevity and they are penalised with low rate if the insurer underestimates their longevity. Moreover, when the annuity life table is equal to the best estimate of survival index, we have a constant risk adjusted rate given by the technical rate of the one year bond i.e $R(\beta') = P^{-1}(0,1) - 1 = 1.005017\%$.

5.5.2 Contract 1 with n periods of time

For this contract, in Tables 5.2 and 5.3 we show the sensitivity study of the contract's value with respect to the proportions of risks borne by annuitants as well as the proportions invested on each assets. The sensitivity study is made with respect to x_s, x_p, β, β' and ${}_tp_{x_0}$.

β'		100%	75%	50%	25%
x_s	x_p				
0%	0%	1.023416	1.053162	1.084109	1.116193
0%	50%	1.025175	1.054156	1.084268	1.11619
25%	25%	0.824781	0.891024	0.966411	1.051981
50%	0%	0.67675	0.761799	0.865673	0.993111
50%	50%	0.67417	0.759794	0.864043	0.991401

Table 5.2: Value of Contract 1 per unit of premium with $R = 1.75\%$ and ${}_tp_{x_0} = \mathbb{E}^{Q_\mu}[{}_tp^*_{x_0}]$.

β'		75%	50%	25%	100%
x_s	x_p				
0%	0%	1.059651	1.09835	1.140214	1.500616
0%	50%	1.060733	1.098695	1.139743	1.051663
25%	25%	0.89598	0.978157	1.07339	0.840644
50%	0%	0.765071	0.874212	1.011537	0.68556
50%	50%	0.76225	0.872307	1.010123	0.682931
β		25%	50%	75%	100%

Table 5.3: Value of Contract 1 per unit of premium with $R = 1.75\%$ for ${}_tp_{x_0}$ =PsB.

Note that the value of a classical annuity with $R = 1.75\%$ is given by $V_0^{(1)}(0,0)/\overline{A_0} = 1.1488996$ (when ${}_tp_{x_0} = \mathbb{E}^{Q_n}[{}_tp*_{x_0}]$) and $V_0^{(1)}(0,0)/\overline{A_0} = 0.766036$ (when ${}_tp_{x_0}$ =PsB). It follows from Tables 5.2 and 5.3, that

- The less we invest in the money market account, the less will be the value of Contract 1. rredMoreover, the value of the contract increases with proportion of longevity risk borne by annuitants whereas it decreases with proportion of financial risk as also shown in Table A.6;

- When the annuity life table is equal to the best estimate of the survival probability, we observe that the value of Contract 1 is independent from the proportion of longevity risk borne by annuitants as obtained in the case of two periods of time. Considering the annuity life table as the PsB table, we find that the contract's value increases with the annuitant's proportion of longevity risk;

- We see that the value of this contract is less sensitive with respect to the proportion invested in the discount bond, whereas the investment in stock highly decreases the value of the contact;

- Furthermore, considering the best estimate of survival yields lower contract's value compared to the PsB life table. This implies that the considered PsB table overestimates the annuitant's survival.

Overall, Tables 5.2 and 5.3 also show that there always exist couples $(\beta, \beta') \notin \{(0\%, 0\%), (100\%, 100\%)\}$ along with investment strategies for which the value of this contract is lower than that of both GSA and classical annuity. Meaning that insurer selling a risk-sharing GSA is exposed to low risk compared to classical annuity and GSA. Furthermore, one could also find such parameters for which the fair valuation is guaranteed.

5.5.3 Contract 2 with n periods of time

Similarly to Contract 1, Tables 5.4 and 5.5 illustrate the sensitivity analysis of the value of this contract with respect to $R(\beta, \beta')$, x_s, x_p, β, β' and ${}_tp_{x_0}$.

β' / $R(\beta, \beta')$	100%	75%	50%	25%	0%
0.25%	0.888838	0.907651	0.926845	0.946633	0.967336
0.75%	0.891687	0.922509	0.955385	0.990435	1.026638
1.75%	0.897214	0.952145	1.01254	1.077193	1.151059
2.75%	0.902461	0.978842	1.065564	1.164301	1.277148
4%	0.909118	1.010171	1.129279	1.272126	1.442794

Table 5.4: Value of Contract 2 per unit of premium with $x_s = 15\%$, $x_p = 25\%$ and ${}_tp_{x_0} = \mathbb{E}^{Q_\mu}[{}_tp^*_{x_0}]$.

β' / $R(\beta, \beta')$	75%	50%	25%	0%	100%
0.25%	0.913122	0.939877	0.967648	0.997007	0.91343
0.75%	0.928319	0.968268	1.011287	1.058081	0.914479
1.75%	0.957096	1.024093	1.099236	1.184534	0.917711
2.75%	0.983783	1.077507	1.187545	1.316763	0.919544
4%	1.104307	1.141776	1.296569	1.486658	0.923119
β	25%	50%	75%	100%	100%

Table 5.5: Value of Contract 2 per unit of premium with $x_s = 15\%$ and $x_p = 25\%$ for ${}_tp_{x_0}$ =PsB.

Tables 5.4 and 5.5 lead to the following observations

- An increasing contract value with the risk adjusted discount rate for both values of ${}_tp_{x_0}$. When ${}_tp_{x_0}$ = PsB, we found from simulations (presented in Table A.7) that the value of Contract 2 increases with the proportion of longevity risk borne by the annuitants whereas it is independent of this latter for ${}_tp_{x_0} = \mathbb{E}^{Q_\mu}[{}_tp^*_{x_0}]$. Furthermore, the value of the contract increases with respect to the policyholders' proportion of financial when ${}_tp_{x_0}$ is equal to the best estimate of survival whereas it will depend on the proportion of longevity considered when ${}_tp_{x_0}$=PsB;

- We observe that when ${}_tp_{x_0} = \mathbb{E}^{Q_\mu}[{}_tp^*_{x_0}]$, the pure financial GSA and GSA produce lower value whereas the classical annuity yields the higher value. When ${}_tp_{x_0}$ =PsB, the pure financial GSA yields lower value whereas the pure longevity GSA gives higher value;

- Similarly to Contract 1, this contract is less sensitive to x_p and decreases with x_s for both values of ${}_tp_{x_0}$.

It also follows that one can always find triples $(\beta, \beta', R(\beta, \beta'))$, with $(\beta, \beta') \notin \{(0\%, 0\%), (100\%, 100\%)\}$ along with investment strategies such that the contract value is lower than that of classical annuity and GSA. One can also find such parameters that guarantee the fair valuation. Moreover, the more financial risk a policyholder bears, the less value the contract yields and the more we invest on the stock, the low value we obtain as well.

The tables below give the values of risk-adjusted discount rates that guarantee the fair valuation of Contract 2; we refer to the obtained risk adjusted discount rates as the indifference discount rates. The tables further illustrate a sensitivity analysis of these rates with respect to the investment strategies and the proportions of risks borne by the group of policyholders.

β'		70%	50%	25%	5%
x_s	x_p				
0%	0%	0.02993%	0.31311%	0.45717%	0.51647%
0%	50%	−0.0091	0.29949%	0.44876%	0.51939%
15%	25%	2.90933%	1.53715%	0.86178%	0.58577%
25%	25%	4.85138%	2.3727%	1.13723%	0.62947%
50%	0%	9.73647%	4.47219%	1.83977%	0.73658%
50%	50%	9.82728%	4.51253%	1.85373%	0.74221%

Table 5.6: Fair risk adjusted discount rate with ${}_tp_{x_0} = \mathbb{E}^{Q_\mu}[{}_tp^*_{x_0}]$.

β'		25%	50%	70%	25%	50%	70%	25%	70%
x_s	x_p								
0%	0%	0.38105%	0.06383%	−0.62103%	0.45854%	0.32522%	0.01372%	0.21938%	−0.20232%
0%	50%	0.36929%	0.05577%	−0.4226%	0.45526%	0.29488%	−0.00705%	0.21551%	−0.32666%
15%	25%	0.78617%	1.31096%	2.39714%	0.86395%	1.53597%	2.91161%	0.63847%	2.73353%
25%	25%	1.0678%	2.15967%	4.44575%	1.14975%	2.38712%	4.86953%	0.92458%	4.72293%
50%	0%	1.76701%	4.29861%	9.53948%	1.83832%	4.47326%	9.72065%	1.6262%	9.66195%
50%	50%	1.7775%	4.33695%	9.6338%	1.85629%	4.51441%	9.82002%	1.64615%	9.77035%
β		25%	50%	70%	0%	0%	0%	70%	25%

Table 5.7: Fair risk adjusted discount rate with ${}_tp_{x_0}$ =PsB.

We found that there is no risk adjusted discount rate that guarantees a fair valuation when the whole financial risk is borne by annuitants (i.e for $\beta' = 100\%$). Tables 5.6 and 5.7 show that

- The indifference rates $R(\beta, \beta')$ increases with x_s whereas it is less sensitive with respect to x_p. Note that the effect of annuitants' proportions of risks strongly depends on the investment strategy;
- The indifference rate of a pure longevity GSA is $R(0, \beta') = 0.2728\%$ for ${}_tp_{x_0} = \mathbb{E}^{Q_n}[{}_tp*_{x_0}]$ and $R(0, \beta') = 0.53216\%$ for ${}_tp_{x_0}$ =PsB;
- As the contract value increases with the risk adjusted rate, then the higher the indifference risk adjusted rate we obtain, the higher value the contract will yield. Inversely, low or negative indifference rates mean that the insurer must penalise the annuitants with low rates in order to guarantee fair valuation.
- As obtained in the case of two periods of time, considering the best estimate of survival the obtained indifference rates are independent of annuitants' proportion of longevity risk. Note that in this case of n periods of time, these indifference rates strongly depend on the annuitants' proportion of financial risk along with the investment strategy.

Note that for both contracts and both annuity life tables, one can find proportions of risks borne by annuitants as well as proportion invested on each asset and risk adjusted rates (for Contract 2) that guarantee the fair valuation of these contracts. Furthermore, we can always find such parameters for which the value of each contract is lower than that of classical annuity, GSA and even lower than the initial premium. Note that the tables above shown that the PsB life table overestimates the expected survival probability as it gives higher contract values and risk adjusted rates compared to those obtained with the best estimate of survival. The effect of the proportion of longevity risk is highlighted in Figure A.8 of Appendix A.9.2.

Remark 5. *Similarly to Remark 4, the numerical studies made in this chapter illustrate the comparison of our proposed (complete) risk-sharing GSAs with the GSA proposed by Piggott et al., 2005 within our framework. It could also be interesting to make such comparison within Piggott et al., 2005's framework so as to highlight the impact of the considered framework.*

5.5.4 Simulation performance

To obtain the numerical results (i.e figures and tables) for this chapter, we performed MC method with 500000 simulations.

- Figures 5.1 and 5.2 for the case of two periods of time took less than 10 seconds each.
- For the first contract represented in Tables 5.2, 5.3 and A.6, we ran the code for approximatively 1.5 minutes for each value of the tables.
- Concerning Contract 2 represented by the tables in Section 5.5.3 and Appendix A.9.2, the simulations took approximatively 2 minutes per value in the tables.

5.6 Conclusion

We proposed in this chapter the valuation analysis of two contracts of risk-linked annuities : Contract 1 representing a risk-sharing GSA and Contract 2 being a complete risk-sharing GSA. These products are derived from the GSA proposed by Piggott et al., 2005 and allow for risk sharing between a group of policyholders and the insurer. Based on the risk-neutral approach, we valued these contracts while considering equity, interest rate and longevity risks. For comparison purpose, we have presented an in deep numerical study of these products along with a detailed sensitivity study with respect to the investment strategy, proportions of risks borne by policyholders, the risk adjusted discount rate and the annuity life table.
We made a detailed study of contracts with two periods of time for which we derived conditions that guarantee the fair valuation of both contracts. We found that the obtained conditions were not always valid for contracts with $n > 2$ periods of time because of their long-time nature and the various dependencies. Moreover, numerical studies shown a significant effect of the proportion invested on the stock and proportion of financial risk borne by policyholders on the value of both contracts. As in the case of two periods of time we found that for n periods of time, the value of both contracts and the risk adjusted rates (for Contract 2) are independent of the annuitants' proportion of longevity risk and increase with proportion of financial risk borne by annuitants when the annuity is described by the best estimate life table. The value of the contracts increases with both proportions of longevity and financial risks when the annuity life table is given by the PsB table. An interesting finding is that one can always find pairs of proportions of risk borne by annuitants along with investment strategies such that the value of the risk-sharing GSA is less than that of GSA, classical annuity and even less than the unique premium. One can also find such parameters that guarantee the fair

valuation.
Concerning Contract 2, we found that its value increases with the risk adjusted rate. Similarly to Contract 1, we observed that one can always find a triples made of proportions of financial and longevity risks and the risk adjusted rate along with investment strategy such that the obtained complete risk-sharing GSAs yield lower values than both GSA and classical annuity. We can also find such parameters for which the value of Contract 2 is less than or equal to the unique premium. We have further computed the values of the risk adjusted rates that guarantee the fair valuation of this contract, we refer to these rates as the indifference discount rates which increase with the proportion invested in the stock. Moreover, we have showed that for some values of both proportions of risks, there is no risk adjusted rates that guarantee fair valuation; meaning that in those cases the value of the contract is always less than the unique premium. Note that we have higher (receptively lower) value of the contracts (respectively indifference rates) with the PsB life table compared to the best estimate of survival. Overall, one can always find risk-sharing and complete risk-sharing GSAs with a value lower than those of GSA, classical annuity and even lower than the unique premium.
An extension of the analysis presented in this chapter could consist of analysing the impact of the dependency between financial and longevity risks on the value of each contract.

Chapter 6

Conclusion

In this dissertation, we have presented ways to enhance annuity trading on both insurer and policyholders viewpoints so as to provide them with enough information to facilitate their decision making while trading annuity products. First of all, we proposed methods to assess the SC of an insurer selling a classical annuity based on investment strategy within a Brownian driven market. Focusing on equity, longevity and interest rate risks, we made a deep comparison of the SC for lifetime, deferred and term annuities. Secondly, we have designed and valued novel annuities allowing for risk sharing between insurer and policyholders. The proposed risk-sharing annuities describes a family of products moving from classical annuity to GSA.

In Chapter 2 we developed a single risk model used to measure the longevity risk borne by an insurer selling a classical annuity following the maturity approach. We further value the profitability of such product on the shareholders viewpoint by studying the mean-variance trade-off of the corresponding IRR. We hence suggested that insurer's trading decision could be based either on the low SC or high IRR. Numerical results have shown that one can find suitable annuity (between lifetime, deferred and term annuities) that maximises the IRR or that minimises the SC. We have also highlighted the effect of the confidence level and found that a constant confidence level as required by SII strongly increases the SC. In order to take advantage of the risky investment return, we have developed a complete model in Chapter 3 taking into account interest rate, equity and longevity risks. It came out that for each of the annuity, one could find an investment strategy that minimises the SC or maximises the IRR. Our investment strategy is advantageous and attractive for both insurer and policyholders as well as for shareholders. It follows that adding a SC is not always a bad deal as it can be seen as a profitable

investment depending on the annuity sold.
Focusing on policyholder's point of view, we have presented in Chapter 4 some methods to design risk-sharing annuities. These methods allow for single-risk sharing as well as multi-risk sharing between a pool of policyholders and insurer. These approaches are based on the GSA where group of policyholders bear the whole risk. We have then developed a family of annuities moving from classical annuity to the GSA, we refer to it as the (complete) risk-sharing GSA. We shown that there exists proportions of risks borne by annuitants that give better expected lifetime utility compared to both classical annuity and GSA. It follows that the (complete) risk-sharing GSA do not only improve the utility of consumption but also the lifetime utility of policyholders compared to classical annuity and GSA. Having proposed these novel products, we then had to think about how much they cost.
In Chapter 5 we valued the (complete) risk-sharing GSA using the risk-neutral approach. Using in deep numerical studies, we found that for some proportions of risks borne by annuitants, the (complete) risk-sharing GSA is cheaper than both GSA and classical annuity. Moreover, one could find such proportions as well as interest rate (called risk adjusted rate) that guarantee the fair valuation of (complete) risk-sharing GSAs.
There are still several questions that can be addressed in future researches. A direct extension of this thesis is to study the risk proportions effect on the SC of an insurer selling a (complete) risk-sharing GSA.
As we only focused on homogeneous cohort (i.e same age, same premium and same entering time) in this thesis, it could be interesting to study the cohort's diversification effect by considering heterogeneous cohort, i.e considering a group of annuitants with different ages, different premium and different entering time. Another extension could consist of using dynamic risk measurement such as iterated VaR. One could move further by considering different mortality model such as Lee-Carter or Cox-Ingersoll-Ross as well as a Lévy driven financial market with dependency between financial and longevity risks.

Appendix A

A.1 Solvency capital with respect to the deferred and term times

The table below gives the values of the SC per unit of single premium for both immediate term and deferred annuities when both the deferred and the term times vary from 1 to 40. This is made using both constant and time-dependent confidence levels and at $t = 0$. We then observe that for the term annuity the value of the SC increases for small value of the term time d' and decreases for largest values of d'; this is observed for both values of the confidence level. As regard the deferred annuity, for variable confidence level, the SC strictly decreases for small values of the deferred time d and increases for larger value of d. Using the constant confidence level, we observe that the SC increases when d increases. Moreover, when the term time d' approaches 45, the value of the SC converges to the SC of a lifetime annuity (given by 0.1403933 for constant α and 0.03822095 for variable α). When the deferred time d goes to 0, the value of the SC converges to the SC of a lifetime annuity.

$\overline{SC}(d,d')/\overline{A_0}$	**Term annuity**		**Deferred annuity**	
Values of d and d'	**Constant α**	**Variable α**	**Constant α**	**Variable α**
1	0.006156096	0.006153251	0.12403064	0.028944735
2	0.012352836	0.011125707	0.10912764	0.020232837
3	0.018453675	0.015353574	0.09544175	0.011929853
4	0.024303704	0.019543642	0.08413130	0.003968597
5	0.030369315	0.023229284	0.07259427	-0.002910031
6	0.036514168	0.026574362	0.06354518	-0.009368481
7	0.042701852	0.029546813	0.05693445	-0.015076207
8	0.049113252	0.032413929	0.05428510	-0.019060104
9	0.054907241	0.035345726	0.05368941	-0.022470205
10	0.061024635	0.037834998	0.05721433	-0.024729544
11	0.066085821	0.040194102	0.06270525	-0.027540167
12	0.071862367	0.042155518	0.07390130	-0.026949580
13	0.078535718	0.044084758	0.08444255	-0.025193941
14	0.081923754	0.045936822	0.10052367	-0.022804985
15	0.088126534	0.047274945	0.12330174	-0.020479482
16	0.094834232	0.048906110	0.14671319	-0.017047308
17	0.099364414	0.049970686	0.17094055	-0.008042942
18	0.104767967	0.051177807	0.20153683	-0.005249659
19	0.108327306	0.052016609	0.24425237	0.003488531
20	0.114894825	0.052853503	0.28521635	0.008879507
21	0.117501784	0.053364122	0.32288946	0.022371629
22	0.124528607	0.053879230	0.38329314	0.030489628
23	0.125890879	0.054244362	0.46111143	0.043680410
24	0.129962889	0.053597210	0.56054421	0.059427455
25	0.132918217	0.054048539	0.63835185	0.074547151
26	0.135450581	0.053696949	0.77494507	0.089206222
27	0.134558526	0.053324689	0.89920010	0.106217807
28	0.138901767	0.052551723	1.04882551	0.122351220
29	0.137323859	0.051922851	1.18591028	0.145081812
30	0.139340024	0.050700867	1.59980240	0.153899525
31	0.138424839	0.050212237	1.56913581	0.178013360
32	0.140725493	0.049092220	1.87843064	0.192808945
33	0.140501927	0.048290239	2.22641035	0.220370419
34	0.139830293	0.047356370	2.51850546	0.250856412
35	0.140390117	0.046204732	2.93846784	0.244808218
36	0.139532337	0.045694537	3.52777988	0.248769619
37	0.142093689	0.044263673	4.06141286	0.267697773
38	0.140524473	0.043942408	4.23378876	0.281096107
39	0.141067107	0.043023385	5.32838507	0.283869613
40	0.139819819	0.041877271	5.84001265	0.293670850

Table A.1: Comparison of annuities for $t = 0$ and $d', d = 1, ..., 40$.

A.2 Sensitivity of the SC with respect to the volatility of the force of mortality

	$x_0 = 65$			$x_0 = 75$		
	$m = 0$	$m = 5$	$m = n$	$m = 0$	$m = 5$	$m = n$
	Lifetime annuity					
$\sigma_\mu - 0.2\%$	0.363	0.2136	0.0243	0.3447	0.1662	0.0822
$\sigma_\mu + 0.2\%$	0.362	0.1787	0.0663	0.3438	0.1635	0.0911
$\sigma_\mu + 0.5\%$	0.3592	0.1347	0.1101	0.3466	0.1634	0.0989
	Deferred annuity					
$\sigma_\mu - 0.2\%$	0.144	0.0298	0.0866	0.2468	0.0999	0.1365
$\sigma_\mu + 0.2\%$	0.0573	-0.0299	0.1912	0.2407	0.0972	0.1501
$\sigma_\mu + 0.5\%$	-0.0398	-0.1048	0.2886	0.2410	0.0959	0.1615
	Term annuity					
$\sigma_\mu - 0.2\%$	0.184	0.0842	0.0194	0.2116	0.0888	0.0873
$\sigma_\mu + 0.2\%$	0.1864	0.08	0.043	0.2148	0.0902	0.0976
$\sigma_\mu + 0.5\%$	0.1914	0.0766	0.0624	0.2177	0.0908	0.1051

Table A.2: $\overline{SC}/\overline{A_0}$ at $t = 0$, for $x_s = 15\%,\ x_p = 25\%$ and $\alpha_0 = 90\%$.

We obtain nearly similar results as in Section 3.3.2. Moreover, when $m = n$, deferred and lifetime annuities yield decreasing SC with respect to σ_μ whereas for the term annuity, we have increasing SC except for younger cohort with $m = 5$.

A.3 Sensitivity of the IRR with respect to the volatility of the force of mortality

	$x_0 = 65$			$x_0 = 75$		
	$m = 0$	$m = 5$	$m = n$	$m = 0$	$m = 5$	$m = n$
			Lifetime annuity			
$\sigma_\mu - 0.2\%$	1.5258% (0.036%)	2.933% (0.0395%)	2.4312% (0.039%)	1.4136% (0.0569%)	2.0702% (0.0621%)	1.3675% (0.0504%)
$\sigma_\mu + 0.2\%$	1.541% (0.0365%)	2.3059% (0.041%)	1.6837% (0.0323%)	1.4165% (0.0564%)	2.1052% (0.0603%)	1.3278% (0.0498%)
$\sigma_\mu + 0.5\%$	1.5987% (0.0363%)	2.8986% (0.0452%)	1.5274% (0.0287%)	1.4846% (0.0560%)	2.1647% (0.0615%)	1.3177% (0.0498%)
			Deferred annuity			
$\sigma_\mu - 0.2\%$	3.1436% (0.0479%)	5.3196% (0.0712%)	1.8993% (0.0343%)	2.1626% (0.0619%)	3.0672% (0.0699%)	1.3515% (0.0499%)
$\sigma_\mu + 0.2\%$	5.1217% (0.0671%)	/	1.4865% (0.0301%)	2.227% (0.0617%)	3.1719 (0.0699%)	1.3275% (0.0502%)
$\sigma_\mu + 0.5\%$	/	/	1.4148% (0.0271%)	2.2864% (0.0611%)	3.2816% (0.0717%)	1.302% (0.05%)
			Term annuity			
$\sigma_\mu - 0.2\%$	2.3591% (0.2732%)	4.946% (0.298%)	4.601% (0.2848%)	2.0031% (0.2639%)	4.078% (0.29%)	1.5447% (0.2369%)
$\sigma_\mu + 0.2\%$	2.3132% (0.2743%)	4.3662% (0.3063%)	2.7764% (0.2495%)	2.0193% (0.26%)	4.1412% (0.2897%)	1.5006% (0.2288%)
$\sigma_\mu + 0.5\%$	2.2753% (0.2726%)	4.8629% (0.3082%)	2.2174% (0.236%)	1.9657% (0.2696%)	4.2438% (0.287%)	1.4405% (0.2293%)

Table A.3: Mean and variance (values in brackets) of the IRR at $t = 0$, for $x_s = 15\%$, $x_p = 25\%$ and $\alpha_0 = 90\%$.

Similarly to the observations of Section 3.4, we have a concave expected IRR with respect to the number of guaranteed benefits and a decreasing IRR with ageing. Similar conclusions are obtained regarding the comparison of the three annuities. Moreover, the IRR increases with the volatility of the force of mortality model for both lifetime and deferred annuities $m < n$ whereas we have opposite behaviours for the term annuity except the case $m = 5$.

A.4 SC and IRR valuation using Tail-VaR

The Tail VaR (TVaR) is in fact the arithmetic average of the VaR and for a process $X \in L^1$, it is defined as

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\vartheta(X)d\vartheta.$$

Moreover, when X has a continuous distribution, then (see Trindade et al., 2007)

$$TVaR_\alpha(X) = CTE_\alpha(X) = \mathbb{E}[X|X \geq VaR_\alpha(X)].$$

Within our framework along with the investment strategies, we defined the SC at any time $t \in [0, n]$ using the TVaR and the confidence level function by

$$TVaR_{\alpha(n-t)}(X_t) = \mathbb{E}_t[X_t|X_t \geq VaR_{\alpha(n-t)}(X_t)];$$

where

$$X_t = \left[\overline{C}(t,n) - \overline{Benef2} - \overline{L'}(t,n,d,m)\right] P(t,n)$$

with $VaR_{\alpha(n-t)}(X_t)$ defined in Section 3.2 (within each interval and for each annuity) and $\alpha(n-t)$ given by Formula (3.1).

Within the simulation framework defined in Section 3.3.1, we give in the figures and tables below the sensitivity analysis of both the SC and IRR for each product.

A.4.1 SC sensitivity

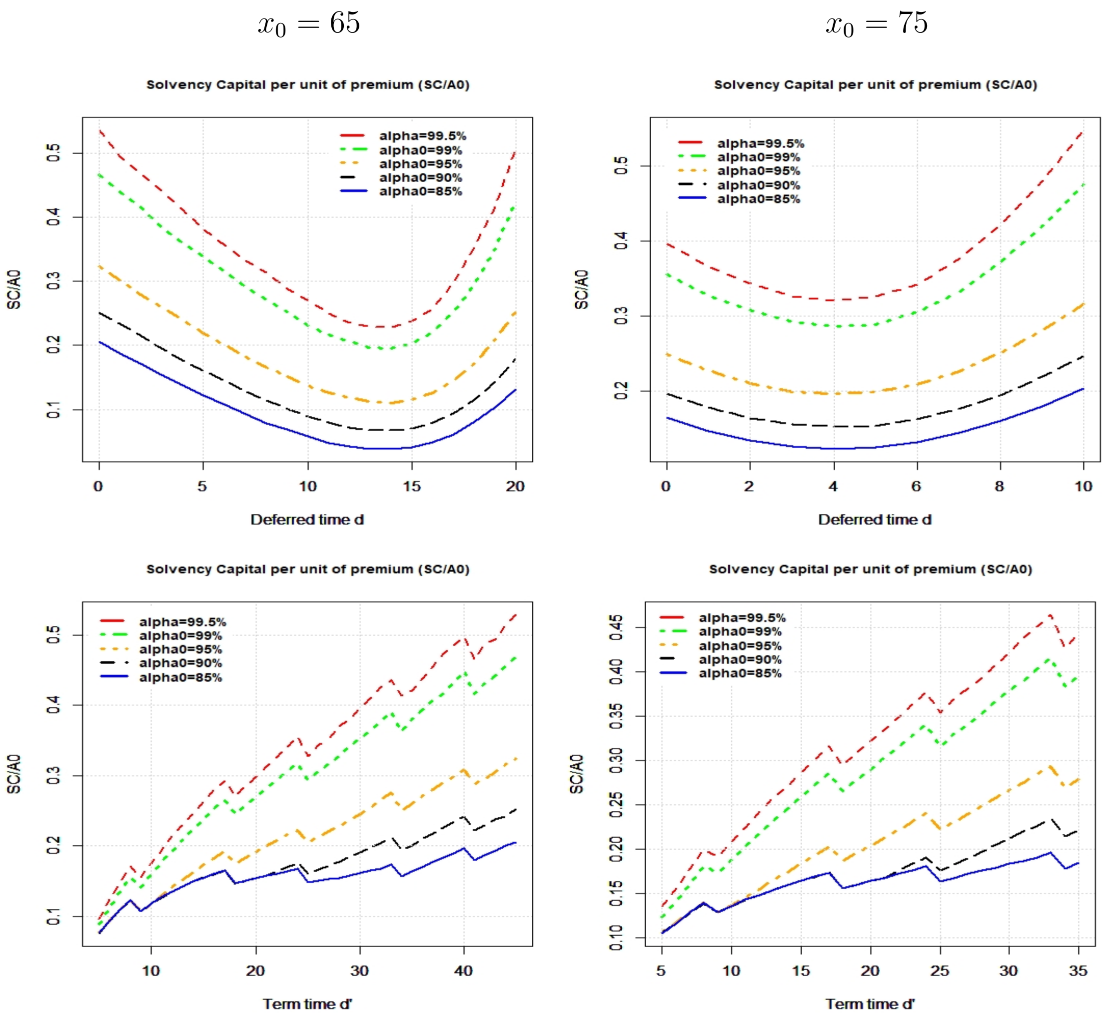


Figure A.1: TVaR $\overline{SC}(d)/\overline{A}_0$ at $t = 0$ for $x_s = 15\%$ and $x_p = 25\%$ — First row represents $\overline{SC}(d)/\overline{A}_0$ with $m = 5$ and the second row $\overline{SC}(d')/\overline{A}_0$ with $m = [3 + \frac{d'-5}{8}]$.

$x_0 = 65$ $x_0 = 75$

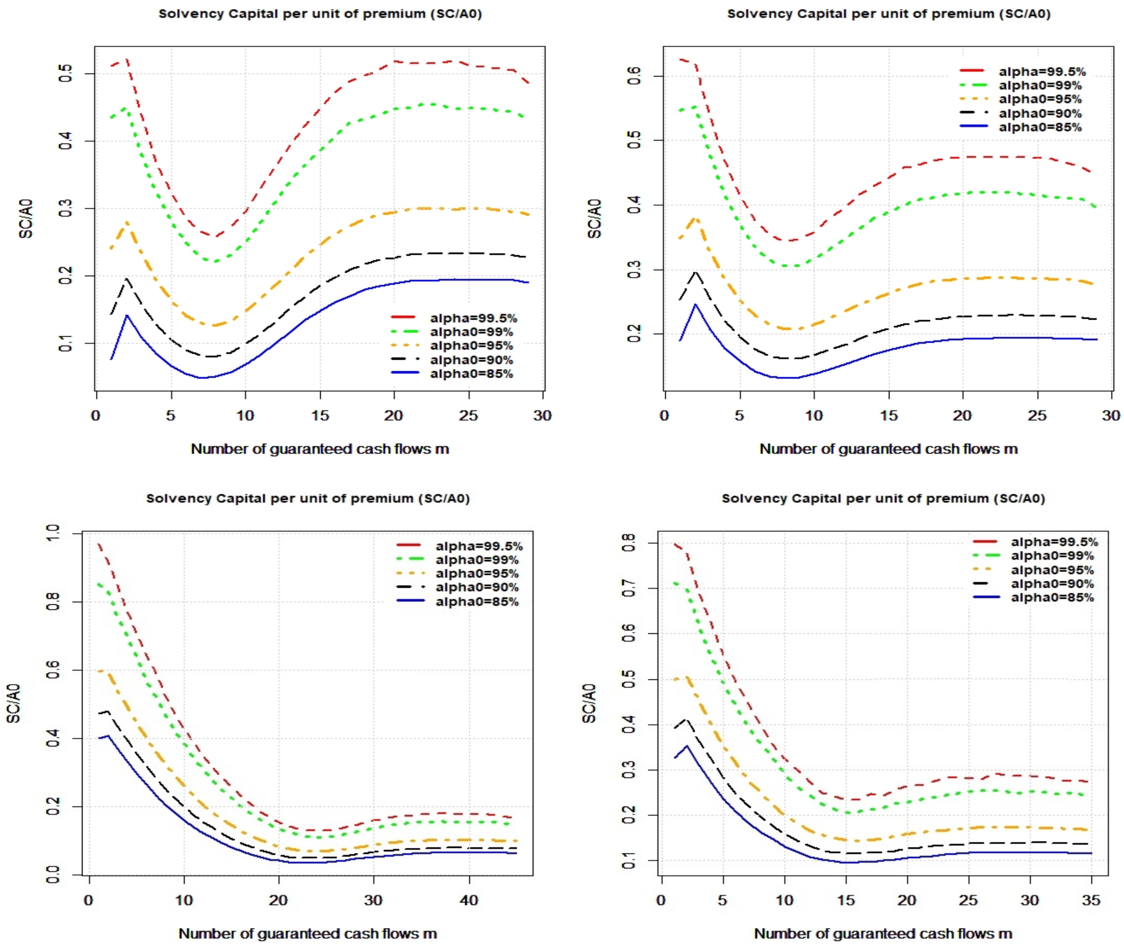


Figure A.2: TVaR $\overline{SC}(m)/\overline{A}_0$ at $t = 0$ for $x_s = 15\%$ and $x_p = 25\%$ — first row represents a 16 years deferred annuity and the second row is a lifetime annuity.

Similarly to the results obtained with the VaR, we observe the convex behaviour with respect to both deferred time d and number of guaranteed benefit m (for lifetime and deferred annuity). We also obtain similar effect of the term time d' as well as the safety level $\alpha(n)$. The difference with the VaR results is on the level of the SC; in fact, the TVaR gives higher SC except for small values of the deferred time d.

In the figure below, we compare of the SC obtained using TVaR and VaR for different values of the safety level.

$\overline{SC}(d)/\overline{A_0}$ $\overline{SC}(m)/\overline{A_0}$

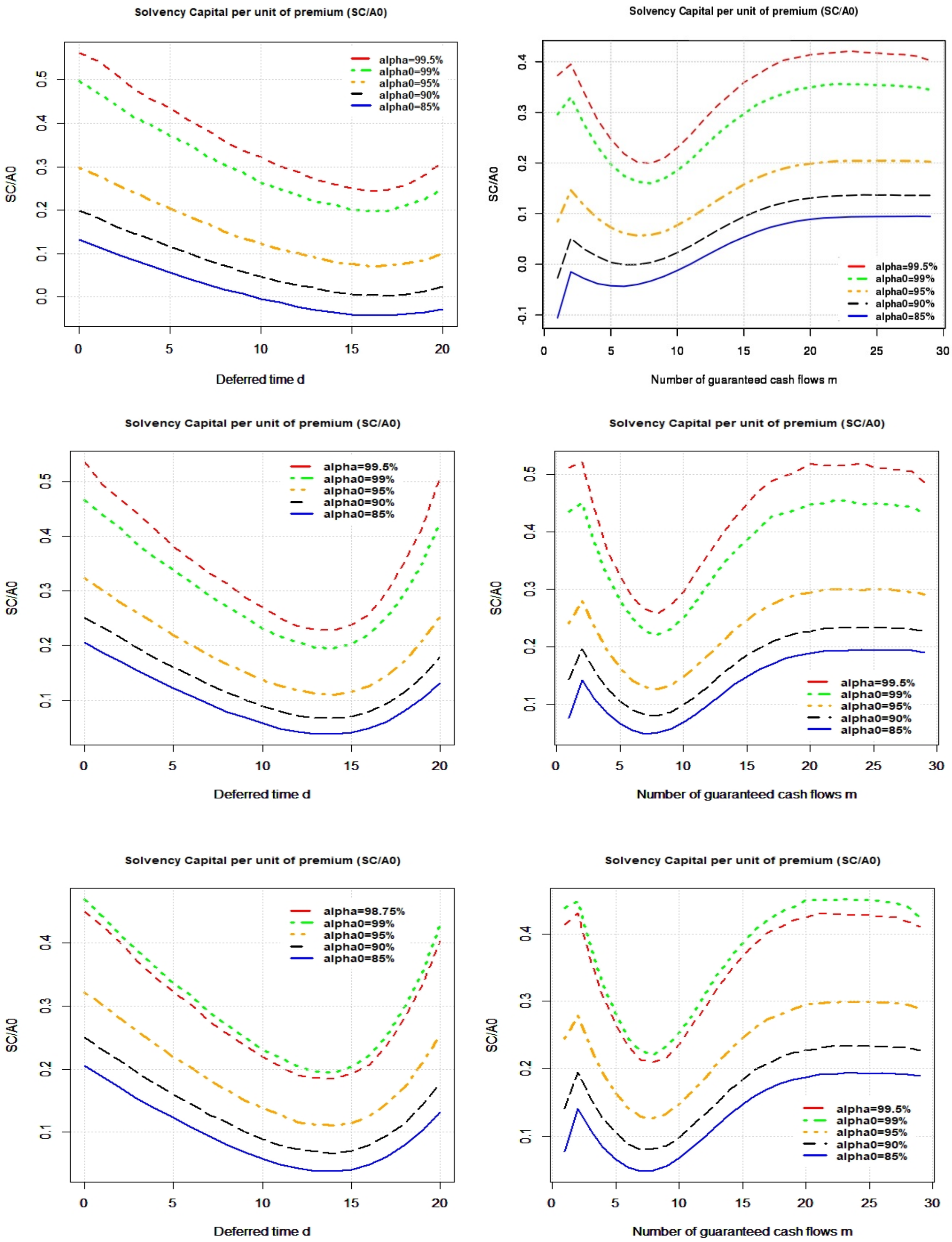


Figure A.3: $\overline{SC}/\overline{A_0}$ at $t = 0$ for $x_s = 15\%$, $x_p = 25\%$ and $x_0 = 65$ — First row represents the case VaR using $\alpha = 99.5\%$; the second row represents the case TVaR using $\alpha = 99.5\%$ and the third row is for TVaR using $\alpha = 98.75\%$.

We remind that for a normally distributed random variable $V \rightsquigarrow \mathcal{N}(\mu_V, \sigma_V)$, $\text{VaR}_{99.5\%}(V) = \text{TVaR}_{98.75\%}(V)$, where

$$\text{VaR}_{99.5\%}(V) = \mu_V + \sigma_V \Phi^{-1}(99.5\%) \quad \text{and} \quad \text{TVaR}_{98.75\%}(V) = \mu_V + \sigma_V \frac{\phi(\Phi^{-1}(98.75\%))}{(1-98.75\%)}; \tag{A.1}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are respectively the probability density function and the cumulative distribution functions of a standard normal random variable. This idea of using $\text{TVaR}_{98.75\%}$ could be interesting (for internal models) in the sense that it will allow insurer to consider extreme risk while computing their SC and to reduce their safety level compared to the $\text{VaR}_{99.5\%}$.

Figure A.3 shows that the values of the SC obtained with the $\text{TVaR}_{\max((98.75\%)^n,\alpha_0)}$ is lower than the values obtained with the $\text{TVaR}_{\max((99.5\%)^n,\alpha_0)}$ for high values of the minimum safety level. Moreover, the comparison with $\text{VaR}_{\max((99.5\%)^n,\alpha_0)}$ depends on the considered parameters (i.e α_0, d and m). Therefore, as we are not dealing with normal random variables, we cannot obtain Formula (A.1).

A.4.2 IRR sensitivity

	Lifetime annuity			**Deferred annuity**			**Term annuity**		
	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$
$m = 0$	1.3762% (0.0301%)	1.3282% (0.0259%)	1.3131% (0.0088%)	2.828% (0.0369%)	2.5566% (0.0306%)	1.9333% (0.0103%)	1.9987% (0.2052%)	1.793% (0.2031%)	1.6544% (0.0852%)
$m = 5$	1.7933% (0.032%)	1.6869% (0.0272%)	1.5124% (0.0089%)	4.2683% (0.0462%)	3.5954% (0.0352%)	2.3631% (0.0112%)	3.0167% (0.2177%)	3.0432% (0.2184%)	2.4303% (0.0955%)
$m = n$	1.6872% (0.0272%)	1.598% (0.0227%)	1.4254% (0.0065%)	1.459% (0.0253%)	1.4005% (0.0213%)	1.3237% (0.006%)	2.575% (0.1836%)	2.5629% (0.1836%)	2.0707% (0.0717%)

Table A.4: TVaR — Mean and variance (values in brackets) of the IRR at $t = 0$ for $d' = 15$, $xs = 15\%$, $xp = 25\%$ and a deferred age of $x_d = 81$ for cohort with $x_0 = 65$.

	Lifetime annuity			Deferred annuity			Term annuity		
	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$	$\alpha_0 = 85\%$	$\alpha_0 = 90\%$	$\alpha_0 = 99.5\%$
$m = 0$	1.2586% (0.0473%)	1.2236% (0.0413%)	1.2502% (0.0138%)	1.851% (0.05%)	1.7287% (0.0426%)	1.5342% (0.0142%)	1.9662% (0.1931%)	1.5448% (0.194%)	1.493% (0.0795%)
$m = 5$	1.7834% (0.0498%)	1.669% (0.0427%)	1.4001% (0.0142%)	2.5141% (0.0542%)	2.2894% (0.0465%)	1.8279% (0.0157%)	3.002% (0.2053%)	2.998% (0.2033%)	2.3763% (0.0838%)
$m = n$	1.2305% (0.0411%)	1.1863% (0.0349%)	1.2209% (0.0102%)	1.2139% (0.0407%)	1.1772% (0.0346%)	1.2135% (0.0101%)	1.243% (0.1623%)	1.2458% (0.1604%)	1.2676% (0.0597%)

Table A.5: TVaR — Mean and variance (values in brackets) of the IRR at $t = 0$ for $d' = 15$, $xs = 15\%$, $xp = 25\%$ and a deferred age of $x_d = 81$ for cohort with $x_0 = 75$.

These tables show that the expected IRRs obtained with TVaR are lower than those obtained with the VaR. Moreover, similarly to the results obtained with the VaR, younger cohort yields higher expected IRR for both deferred and term annuities as well as lifetime annuity. The concave behaviour with respect to m is also observed.

A.5 GSA

We propose here the proofs of the formulas of GSA's benefit proposed by Piggott et al., 2005 for different cohorts.

A.5.1 The case of close homogeneous cohort

For this case, let's consider a closed homogeneous cohort of N_x annuitants, the cohort fund satisfies $F_0 = N_x\ B_0 \sum_{t=0}^{\infty} \frac{N_{x+t}}{N_x} v^t$, the initial benefit B_0 given by Formula (4.1) follows from this.

Let $N^*_{x+1}, N^*_{x+2}, ..., N^*_{x+t}, ...$ and $R^*_1, R^*_2, ..., R^*_t, ...$ be respectively the realised number of survivors and the actual investment return at corresponding time. The fund at $t = 1$ is given by $F_1 = (F_0 - N_x\ B_0)(1 + R^*_1)$, using Formula (4.1), it becomes

$$F_1 = (N_x\ B_0\ \ddot{a}_x - N_x\ B_0)(1 + R^*_1) = N_x\ B_0\ (\ddot{a}_x - 1)(1 + R^*_1). \tag{A.2}$$

From Equation (4.1) and Formula (A.2), it follows that the benefit paid at $t = 1$ is given by

$$B_1 = \frac{F_1}{N^*_{x+1}\ \ddot{a}_{x+1}} = \frac{N_x}{N^*_{x+1}}\ \frac{B_0(\ddot{a}_x - 1)(1 + R^*_1)}{\ddot{a}_{x+1}}. \tag{A.3}$$

The recursive relationship between annuity factor (see Bowers and Hickman, 1986) states that

$$\ddot{a}_{x+1} = \frac{(\ddot{a}_x - 1)(1+R)}{p_x}; \tag{A.4}$$

it follows that Formula (A.3) becomes

$$B_1 = \frac{N_x}{N^*_{x+1}} \; \frac{B_0 \; \ddot{a}_{x+1} \; p_x \; (1+R_1^*)}{\ddot{a}_{x+1} \; (1+R)} = B_0 \; \frac{p_x}{p_x^*} \; \frac{1+R_1^*}{1+R}.$$

Recursively, we obtain Formula (4.3) for all $t \geq 2$ using $p_x^* = \frac{N^*_{x+1}}{N_x}$.

A.5.2 The case of open heterogeneous cohort

We assume an open (i.e new entrants are allowed) heterogeneous (i.e different premium) and we provide proofs for Formulae (4.5) and (4.4) as proposed by Piggott et al., 2005. Then at any time t, they suggest that the fund of the i$^{\text{th}}$ survivor who entered the pool k years ago is given by ${}^k_xF^*_{i,t}/p_{x+k-1}$ and the fund of the survivors (from $t-1$ to t) is

$$\mathrm{SF}^*_t = \sum_{k \geq 1} \sum_x \sum_{i \in A_t} \frac{{}^k_xF^*_{i,t}}{p_{x+k-1}}.$$

The total pool fund at time t (F^*_t) is given by

$$F^*_t = \sum_{k \geq 1} \sum_x \sum_{i \in A_t} {}^k_x\hat{F}^*_{i,t},$$

where

$${}^k_x\hat{F}^*_{i,t} = \; {}^k_xB_{i,t} \; \ddot{a}_{x+k}. \tag{A.5}$$

GSA without information updates on mortality

We assume constant information about the annuity life table and in order to find the formula of MEA_t, Piggott et al., 2005 defined the benefit at time t in the form

$${}^k_xB_{i,t} = \; {}^{k-1}_{\;\;x}B_{i,t-1} \; \mathrm{MEA}_t \; \mathrm{IRA}_t. \tag{A.6}$$

It follows that

$$\begin{aligned}
F_t^* &= \sum_{k\geq 1}\sum_x\sum_{i\in A_t} {}_x^kB_{i,t}\ \ddot{a}_{x+k} = \sum_{k\geq 1}\sum_x\sum_{i\in A_t} {}_x^{k-1}B_{i,t-1}\ \mathrm{MEA}_t\ \mathrm{IRA}_t\ \ddot{a}_{x+k} \\
&= \mathrm{MEA}_t \sum_{k\geq 1}\sum_x\sum_{i\in A_t} {}_x^{k-1}B_{i,t-1}\ \frac{1+R_t^*}{1+R}(\ddot{a}_{x+k-1}-1)\frac{1+R}{p_{x+k-1}} \\
&= \mathrm{MEA}_t \sum_{k\geq 1}\sum_x\sum_{i\in A_t} \left({}_x^{k-1}B_{i,t-1}\ \ddot{a}_{x+k-1} - {}_x^{k-1}B_{i,t-1}\right)\ \frac{1+R_t^*}{p_{x+k-1}} \\
&= \mathrm{MEA}_t \sum_{k\geq 1}\sum_x\sum_{i\in A_t} \left({}_x^{k-1}\hat{F}_{i,t-1}^* - {}_x^{k-1}B_{i,t-1}\right)\ \frac{1+R_t^*}{p_{x+k-1}}
\end{aligned}$$

.

The ith annuitant's fund is given by

$$ {}_x^kF_{i,t}^* = \left({}_x^{k-1}\hat{F}_{i,t-1}^* - {}_x^{k-1}B_{i,t-1}\right)(1+R_t^*), \tag{A.7}$$

Formula (A.7) implies that

$$F_t^* = \mathrm{MEA}_t \sum_{k\geq 1}\sum_x\sum_{i\in A_t} \frac{{}_x^kF_{i,t}^*}{p_{x+k-1}} = \mathrm{MEA}_t\ \mathrm{SF}_t^*.$$

Then Equation (4.5) follows and we have

$${}_x^kB_{i,t} = {}_x^{k-1}B_{i,t-1}\ \frac{F_t^*}{\mathrm{SF}_t^*}\ \frac{1+R_t^*}{1+R}.$$

GSA with information updates on mortality

Additionally to the previous hypothesis, Piggott et al., 2005 assumed that at any time t there are new updates about annuity table, hence a new annuity factor denoted by $\ddot{a}_{x,t}$. We provide here a proof for Formula (4.4) to do so, we consider an annuitant i joining the pool at $t>0$ with a benefit given by ${}_x^kB_{i,t} = \frac{{}_x^k\hat{F}_{i,t}^*}{\ddot{a}_{x+k,t}}$ and we note that

$$ {}_x^k\hat{F}_{i,t}^* = \frac{{}_x^kF_{i,t}^*}{p_{x+k-1,t-1}}\ \mathrm{MEA}_t. \tag{A.8}$$

Indeed, using Formulae (A.4) and (A.6), Formula (A.5) implies that

$$\begin{aligned}
{}_x^k\hat{F}_{i,t}^* &= {}_x^kB_{i,t}\ \ddot{a}_{x+k,t} = {}_x^{k-1}B_{i,t-1}\ \mathrm{MEA}_t\ \mathrm{IRA}_t\ \ddot{a}_{x+k,t} \\
&= {}_x^{k-1}B_{i,t-1}\ \mathrm{MEA}_t\ \frac{1+R_t^*}{1+R}\ \frac{(\ddot{a}_{x+k-1,t-1}-1)(1+R)}{p_{x+k-1,t-1}} \\
&= {}_x^{k-1}B_{i,t-1}\ \mathrm{MEA}_t\ (\ddot{a}_{x+k-1,t-1}-1)\ \frac{(1+R_t^*)}{p_{x+k-1,t-1}} \\
&= \mathrm{MEA}_t\left({}_x^{k-1}B_{i,t-1}\ \ddot{a}_{x+k-1,t-1} - {}_x^{k-1}B_{i,t-1}\right)\ \frac{(1+R_t^*)}{p_{x+k-1,t-1}}.
\end{aligned}$$

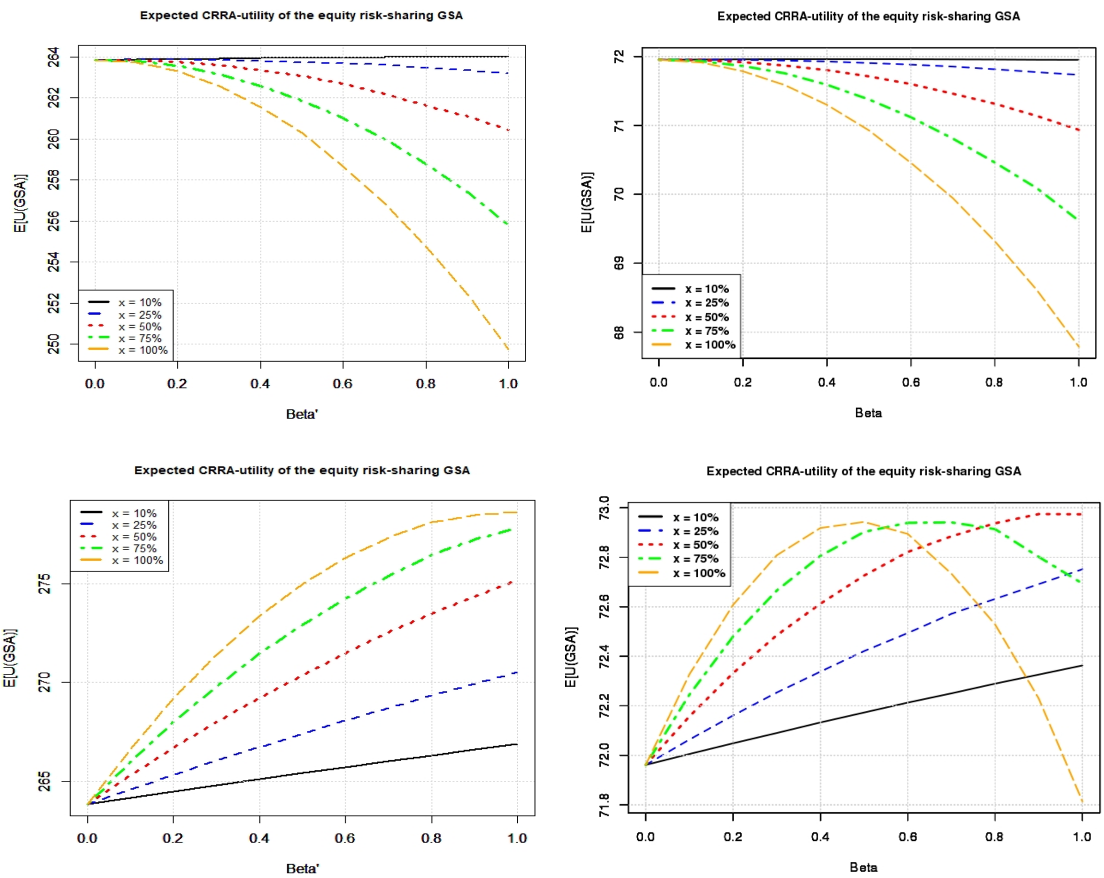
Therefore Equation (A.8) follows from Formula (A.7).
With these Formulae (A.7) and (A.8), one can express the benefit at time t as

$$\begin{aligned} {}^{k}_{x}B_{i,t} &= \frac{{}^{k}_{x}F^{*}_{i,t}\ \mathrm{MEA}_t}{p_{x+k-1,t-1}\ \ddot{a}_{x+k,t}} = \frac{\left({}^{k-1}_{x}\hat{F}^{*}_{i,t-1} - {}^{k-1}_{x}B_{i,t-1}\right)(1+R^{*}_{t})}{p_{x+k-1,t-1}\ \ddot{a}_{x+k,t}}\ \mathrm{MEA}_t \\ &= {}^{k-1}_{x}B_{i,t-1}\ \frac{(\ddot{a}_{x+k-1,t-1}-1)(1+R^{*}_{t})}{p_{x+k-1,t-1}\ \ddot{a}_{x+k,t}}\ \mathrm{MEA}_t \\ &= {}^{k-1}_{x}B_{i,t-1}\ \mathrm{MEA}_t\ \frac{1+R^{*}_{t}}{1+R}\ \frac{\ddot{a}_{x+k,t-1}}{\ddot{a}_{x+k,t}}, \end{aligned}$$

Formula (4.4) hence follows.

A.6 Expected utility of a FRS-GSA

$\gamma = 0.5$ $\qquad\qquad$ $\gamma = 1$



$\gamma = 0.5$ $\gamma = 1$

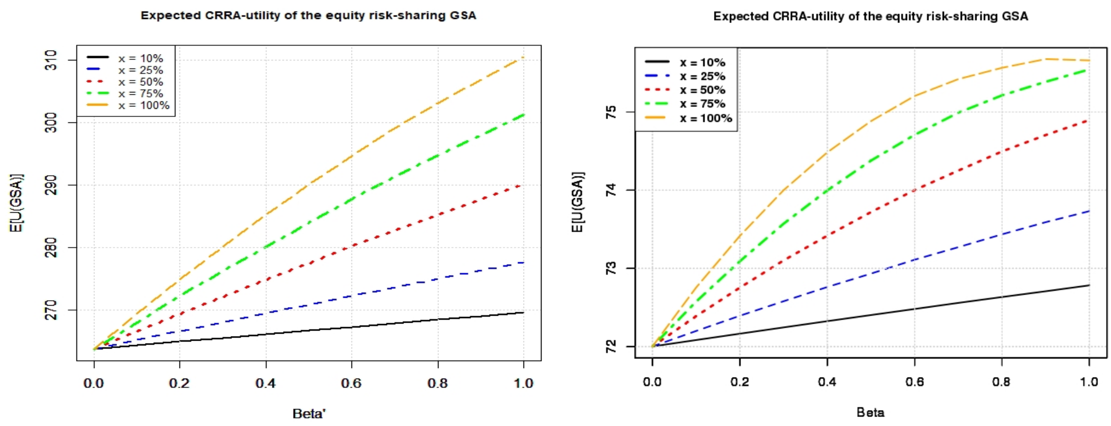


Figure A.4: Expected utility of the FRS-GSA for $\beta = \beta'$ — First row represents the case of $\mu = 2\%$; the second row represents the case of $\mu = 4\%$ and the third row is for $\mu = 5.8702\%$.

A.7 Proof of Proposition 1

Proof. For $\gamma = 0$, we have a risk-neutral utility $U(u) = u$. Let $\beta'_1 \neq \beta'_2$ and $R_\gamma(\beta') := R(\beta'_1) = R(\beta'_2)$; we want to show that $Ucfrs(\beta'_1) = Ucfrs(\beta'_2)$ iff $R_\gamma(\beta') = \log(\mathbb{E}[1 + R_t]) = \mu x + (1 - x)r$.

(i) Let assume that $Ucfrs(\beta'_1) = Ucfrs(\beta'_2)$ and show that $R_\gamma(\beta') = \log(\mathbb{E}[1 + R_t]) = \mu x + (1 - x)r$.

$$
\begin{aligned}
Ucfrs(\beta'_1) &= \sum_{t=0}^{n} \mathbb{E}\left[B_0(\beta'_1)\prod_{j=1}^{t}\left[\beta'_1\frac{1+R_j}{1+R_\gamma(\beta')} + (1-\beta'_1)\right]\right] v_\delta^t \; {}_tp_x^0 \\
&= B_0(\beta'_1) + B_0(\beta'_1)\sum_{t=1}^{n}\mathbb{E}\left[\prod_{j=1}^{t}\left[\beta'_1\frac{1+R_j}{1+R_\gamma(\beta')} + (1-\beta'_1)\right]\right] v_\delta^t \; {}_tp_x^0 \\
&= B_0(\beta'_1) + B_0(\beta'_1)\sum_{t=1}^{n}\prod_{j=1}^{t}\left[\beta'_1\frac{\mathbb{E}\left[1+R_j\right]}{1+R_\gamma(\beta')} + (1-\beta'_1)\right] v_\delta^t \; {}_tp_x^0 \\
&= B_0(\beta'_1) + B_0(\beta'_1)\sum_{t=1}^{n}\left[\beta'_1\frac{e^{\mu x_s + (1-x_s)r}}{1+R_\gamma(\beta')} + (1-\beta'_1)\right]^t v_\delta^t \; {}_tp_x^0. \qquad \text{(A.9)}
\end{aligned}
$$

Note that the Taylor expansion of $g(t) = u^t$ of order $l > 0$, is given by

$$g(t) = \sum_{j=0}^{l} \frac{\log^j(u)}{j!}\ t^j \tag{A.10}$$

Set

$$z_1 = \beta_1' \frac{e^{\mu x_s + (1-x_s)r}}{1 + R_\gamma(\beta')} + (1 - \beta_1').$$

using (A.10), Formula (A.9) becomes

$$\begin{aligned}
Ucfrs(\beta_1') &= B_0(\beta_1') + B_0(\beta_1') \sum_{t=1}^{n} \sum_{j=0}^{l} \log^j(z_1) \frac{t^j}{j!}\ v_\delta^t\ {}_tp_x^0 \\
&= B_0(\beta_1') \left[1 + \sum_{t=1}^{n} \left(\frac{\log^0(z_1)}{0!} t^0 + \frac{\log^1(z_1)}{1!} t^1 + \frac{\log^2(z_1)}{2!} t^2 + \dots + \frac{\log^l(z_1)}{l!} t^l \right) v_\delta^t\ {}_tp_x^0 \right] \\
&= B_0(\beta_1') + B_0(\beta_1') \sum_{t=1}^{n} \frac{\log^0(z_1)}{0!} t^0\ v_\delta^t\ {}_tp_x^0 + B_0(\beta_1') \sum_{t=1}^{n} \frac{\log^1(z_1)}{1!} t^1\ v_\delta^t\ {}_tp_x^0 \\
&+ B_0(\beta_1') \sum_{t=1}^{n} \frac{\log^2(z_1)}{2!} t^2\ v_\delta^t\ {}_tp_x^0 + \dots + B_0(\beta_1') \sum_{t=1}^{n} \frac{\log^l(z_1)}{l!} t^l\ v_\delta^t\ {}_tp_x^0.
\end{aligned}$$

Hence for any distinct proportions β_1' and β_2', if $R(\beta_1') = R(\beta_2')$ then $B_0(\beta_1') = B_0(\beta_2')$ and $Ucfrs(\beta_1') = Ucfrs(\beta_2')$ implies that for all $j = 0, ..., l$

$$\log^j(z_1) \sum_{t=1}^{n} \frac{t^j}{j!}\ v_\delta^t\ {}_tp_x^0 \quad = \quad \log^j(z_2) \sum_{t=1}^{n} \frac{t^j}{j!}\ v_\delta^t\ {}_tp_x^0;$$

Where

$$z_2 = \beta_2' \frac{e^{\mu x_s + (1-x_s)r}}{1 + R_\gamma(\beta')} + (1 - \beta_2').$$

This is equivalent to

$$\log \left[\beta_1' \frac{e^{\mu x_s + (1-x_s)r}}{1 + R_\gamma(\beta')} + (1 - \beta_1') \right] \quad = \quad \log \left[\beta_2' \frac{e^{\mu x_s + (1-x_s)r}}{1 + R_\gamma(\beta')} + (1 - \beta_2') \right].$$

Hence

$$R_\gamma(\beta') = e^{\mu x_s + (1-x_s)r} - 1 \tag{A.11}$$

Let $h(u) = e^u$, for $u > 0$; one can show that $e^u - 1 = \int_0^u e^s ds$. For $s \in [0, u]$, we have

$$1 \le e^s \le e^u.$$

Integrating this inequality yields

$$\int_0^u ds \le \int_0^u e^s ds \le \int_0^u e^u ds \ \Leftrightarrow\ 1 \le \frac{e^u - 1}{u} \le e^u.$$

Applying squeeze theorem at zero to this latter inequality, implies that

$$e^u - 1 = u. \tag{A.12}$$

Applying (A.12) to Formula (A.11) yields

$$R_\gamma(\beta') = \mu x_s + (1 - x_s)r.$$

(ii) Now assume $R_\gamma(\beta') = \log\left(\mathbb{E}[1 + R_t]\right) = \mu x + (1 - x)r$ then it is easy to show that $Ucfrs(\beta_1') = Ucfrs(\beta_2')$.

Thus **(i)** and **(ii)** end the proof. □

A.8 CARA utility function

Within the framework described in Chapter 4, instead of the CRRA we consider here the CARA utility function defined by (see Chang and Chang, 2017; Gao, 2010; Babcock et al., 1993)

$$U_{CARA}(x) = \frac{-e^{-\xi x}}{\xi},$$

where $\xi > 0$ it the coefficient of constant absolute risk-aversion. The formulas of the expected lifetime utility of both risk-sharing GSA and complete risk-sharing GSA can be obtained by substitution the CRRA utility by the CARA utility function. Figure A.5 below illustrates the CARA utility of (complete) RS-GSA for $\xi = 0.5$ and $\xi = 1$.

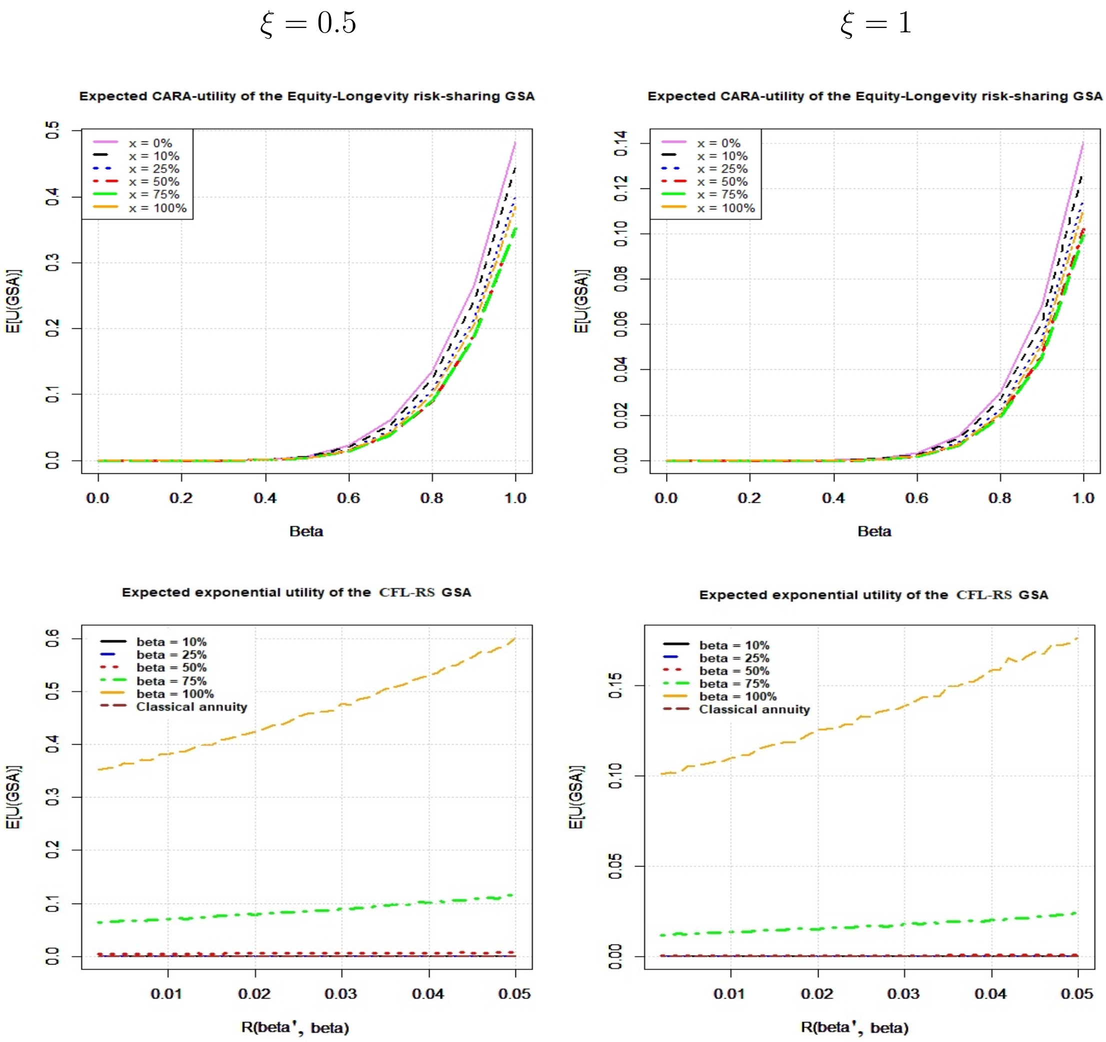


Figure A.5: Expected CARA utility of the (C)RS-GSA for $\beta = \beta'$ — First row represents the FLRS-GSA and the second row represents the CFLRS-GSA for $\mu = 5.8702\%$ and $x_s = 15\%$.

We observe that unlike the CRRA utility, the CARA utility of both RS-GSA and CRS-GSA increase with respect to risks proportions and the risk adjusted rate as well as with respect to the coefficient ξ. In his case, the GSA gives higher utility compared to both classical annuity and (complete) risk-sharing GSA.

A.9 Single risk effect

Here we make a sensitivity analysis of both the value of each contracts and the indifference rates with respect to longevity risk using the PsB life table. Note that the effect of the financial risk is similar to the case of best estimate of survival for annuities.

A.9.1 Contract 1

x_s	x_p \ β	0%	10%	25%	50%	75%	100%
0%	0%	1.134859	1.13798	1.143628	1.151947	1.161069	1.169955
0%	50%	1.135558	1.138468	1.143525	1.151823	1.1609	1.170149
25%	25%	1.108563	1.111448	1.116358	1.124135	1.13281	1.141224
50%	0%	1.082066	1.084442	1.089788	1.096917	1.104914	1.113625
50%	50%	1.081689	1.084058	1.088853	1.097145	1.105281	1.113052

Table A.6: Value of Contract 1 per unit of premium with $R = 1.75\%$ and ${}_tp_{x_0} =$ PsB for $\beta' = 10\%$.

A.9.2 Contract 2

$R(\beta, \beta')$ \ β	0%	10%	25%	50%	75%	100%
0.25%	0.958285	0.961225	0.965506	0.972957	0.980287	0.988096
0.75%	1.0111	1.014511	1.018528	1.02639	1.033589	1.042082
1.75%	1.118986	1.122706	1.127041	1.13531	1.1437	1.153188
2.75%	1.229679	1.233189	1.238187	1.247143	1.256637	1.266458
4%	1.369878	1.37355	1.379324	1.389067	1.399927	1.40982

Table A.7: Value of Contract 2 per unit of premium with $x_s = 15\%$, $x_p = 25\%$ and ${}_tp_{x_0} =$ PsB for $\beta' = 10\%$.

β (x_s)	(x_p)	0%	10%	25%	50%	75%	100%
0%	0%	0.50781%	0.48104%	0.44059%	0.36497%	0.29792%	0.21883%
0%	50%	0.50438	0.47736%	0.43948%	0.36221%	0.29122%	0.21981%
15%	25%	0.64542%	0.62144%	0.57646%	0.50670%	0.43268%	0.36233%
25%	25%	0.73448%	0.71114%	0.66759%	0.60037%	0.52307%	0.45563%
50%	0%	0.96741%	0.94264%	0.90296%	0.83551%	0.76818%	69631%
50%	50%	0.97366%	0.94847%	0.90845%	0.83551%	0.77072%	0.69767%

Table A.8: Fair risk adjusted discount rate for ${}_tp_{x_0}$ = PsB with $\beta' = 10\%$.

Bibliography

Antolin, P. (2007). *Longevity risk and private pensions*. `http://www.oecd.org/daf/fin/private-pensions/37977228.pdf`.

Antonio, K., L. Devolder, and S. Devriendt (2015). "The IA| BE 2015 mortality projection for the Belgian population". *Available at SSRN 2576167*.

Babcock, Bruce A, E Kwan Choi, and Eli Feinerman (1993). "Risk and probability premiums for CARA utility functions". *Journal of Agricultural and Resource Economics*, pp. 17–24.

Bacinello, A. R., P. Millossovich, A. Olivieri, and E. Pitacco (2011). "Variable annuities: A unifying valuation approach". *Insurance: Mathematics and Economics* 49.3, pp. 285–297.

Barbarin, Jerome and Pierre Devolder (2005). "Risk measure and fair valuation of an investment guarantee in life insurance". *Insurance: Mathematics and Economics* 37.2, pp. 297–323.

Barrieu, P., H. Bensusan, N. El Karoui, C. Hillairet, S. Loisel, C. Ravanelli, and Y. Salhi (2012). "Understanding, modelling and managing longevity risk: key issues and main challenges". *Scandinavian actuarial journal* 2012.3, pp. 203–231.

Bauer, D., D. Bergmann, and A. Reuss (2010). "Solvency II and nested simulations : a least-squares Monte Carlo approach". *Proceedings of the 2010 ICA congress*.

Bauer, D., A. Reuss, and D. Singer (2012). "On the calculation of the solvency capital requirement based on nested simulations". *ASTIN Bulletin: The Journal of the IAA* 42.2, pp. 453–499.

Bauer, D. and F. Weber (2008). "Assessing investment and longevity risks within immediate annuities". *Asia-Pacific Journal of Risk and Insurance* 3.1.

Black, F. and M. Scholes (1973). "The pricing of options and corporate liabilities". *Journal of political economy* 3.81, pp. 637–654.

Blake, D. and W. Burrows (2001). "Survivor bonds: Helping to hedge mortality risk". *Journal of Risk and Insurance, Citeseer* 68.2, pp. 339–348.

Bowers, G. and J. Hickman (1986). "Nesbitt, actuarial mathematics". *Society of Actuaries*.

Boyle, P., M. Hardy, A. Mackay, and D. Saunders (2015). *Variable payout annuities*. `https://www.soa.org/Files/Research/research-2015-variable-payout-annuities_.pdf`.

Brown, J. R., O. S. Mitchell, and J. M. Poterba (2000). *Mortality risk, inflation risk, and annuity products*. `https://www.nber.org/papers/w7812.pdf`.

Chang, Hao and Kai Chang (2017). "Optimal consumption–investment strategy under the Vasicek model: HARA utility and Legendre transform". *Insurance: Mathematics and Economics* 72, pp. 215–227.

Charupat, Narat and Moshe A Milevsky (2002). "Optimal asset allocation in life annuities: a note". *Insurance: Mathematics and Economics* 30.2, pp. 199–209.

Chen, A., P. Hieber, and J. K. Klein (2019). "Tonuity: A novel individual-oriented retirement plan". *ASTIN Bulletin: The Journal of the IAA* 49.1, pp. 5–30.

Chen, An and Peter Hieber (2016). "Optimal asset allocation in life insurance: The impact of regulation". *ASTIN Bulletin: The Journal of the IAA* 46.3, pp. 605–626.

Cheuk-Yin, N. A. and L. J. Siu-Hang (2011). "Valuing variable annuity guarantees with the multivariate Esscher transform". *Insurance: Mathematics and Economics, Elsevier* 49.3, pp. 393–400.

Christiansen, M. C. and A. Niemeyer (2014). "Fundamental definition of the solvency capital requirement in solvency II". *ASTIN Bulletin: The Journal of the IAA* 44.3, pp. 501–533.

Denuit, M., S. Haberman, and A. Renshaw (2011). "Longevity-indexed life annuities". *North American Actuarial Journal* 15.1, pp. 97–111.

Denuit, M., S. Haberman, and A. E. Renshaw (2015). "Longevity-contingent deferred life annuities". *Journal of Pension Economics & Finance* 14.3, pp. 315–327.

Devineau, L. and S. Loisel (2009). "Risk aggregation in Solvency II: How to converge the approaches of the internal models and those of the standard formula?" *Bulletin Français d'Actuariat* 9.18, pp. 107–145.

Devolder, P. (2011). "Revised version of: Solvency requirement for long term guarantee: risk measure versus probability of ruin". *European Actuarial Journal* 1.2, pp. 200–214.

Devolder, P. and A. Lebègue (2016). "Compositions of conditional risk measures and solvency capital". *Risks* 4.4, p. 49.

Devolder, P. and A. Lebègue (2016). "Risk measures versus ruin theory for the calculation of solvency capital for long-term life insurances". *Dependence Modeling, De Gruyter,* `https://www.degruyter.com/view/journals/demo/open-issue/article-10.1515-demo-2016-0018/article-10.1515-demo-2016-0018.xml`, *(accessed on 20 January 2017).*

Devolder, P. and A. Lebègue (2017). "Iterated VaR or CTE measures: A false good idea?" *Scandinavian Actuarial Journal* 2017.4, pp. 287–318.

Donnelly, C., M. Guillén, and J. P. Nielsen (2014). "Bringing cost transparency to the life annuity market". *Insurance: Mathematics and Economics* 56, pp. 14–27.

EIOPA (2014a). "Technical specification for the preparatory phase (part I)". `https://eiopa.europa.eu/Publications/`.

– (2014b). "The underlying assumptions in the standard formula for the solvency capital requirement calculation". `https://eiopa.europa.eu/Publications/`.

Föllmer, Hans and Alexander Schied (2011). *Stochastic finance: an introduction in discrete time*. Walter de Gruyter.

Fromenteau, M. and P. Petauton (2017). *Théorie et pratique de l'assurance vie-4e éd.: Cours et exercices corrigés*, pp. 113–143.

Fung, M., K. Ignatieva, and M. Sherris (2019). "Managing Systematic Mortality Risk in Life Annuities: An Application of Longevity Derivatives". *Risks* 1.

Gao, Jianwei (2010). "An extended CEV model and the Legendre transform–dual–asymptotic solutions for annuity contracts". *Insurance: Mathematics and Economics* 46.3, pp. 511–530.

Gbari, Samuel and Michel Denuit (2014). "Efficient approximations for numbers of survivors in the Lee–Carter model". *Insurance: Mathematics and Economics* 59, pp. 71–77.

Girsanov, IV (1960). "Strongly-feller processes i. general properties". *Theory of Probability and Its Applications* 5.1, pp. 5–24.

Glasserman, P. (2013). "Monte Carlo method in financial engineering". *Springer Science and Business Media* 53.7.

Gompertz, B. (1825). "XXIV. On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. In a letter to Francis Baily, Esq. FRS &c". *Philosophical transactions of the Royal Society of London* 115, pp. 513–583.

Gong, G. and A. Webb (2010). "Evaluating the Advanced Life Deferred Annuity-An annuity people might actually buy". *Insurance: Mathematics and Economics* 46.1, pp. 210–221.

Gronchi, S. (1986). “On investment criteria based on the internal rate of return”. *Oxford Economic Papers* 38.1, pp. 174–180.

Hainaut, D., P. Devolder, and A. Pelsser (2007). “Robust evaluation of SCR for participating life insurances under Solvency II”. *Insurance: Mathematics and Economics* 79, pp. 107–123.

Hanbali, H., M. Denuit, J. Dhaene, and J. Trufin (2019). “A dynamic equivalence principle for systematic longevity risk management”. *Insurance: Mathematics and Economics* 86, pp. 158–167.

Hanewald, K., J. Piggott, and M. Sherris (2013). “Individual post-retirement longevity risk management under systematic mortality risk”. *Insurance: Mathematics and Economics* 52.1, pp. 87–97.

Hari, N., A. De Waegenaere, B. Melenberg, and T. E. Nijman (2008). “Longevity risk in portfolios of pension annuities”. *Insurance: Mathematics and Economics, Elsevier* 42.2, pp. 505–519.

Hejazi, Seyed Amir and Kenneth R Jackson (2017). “Efficient valuation of SCR via a neural network approach”. *Journal of Computational and Applied Mathematics* 313, pp. 427–439.

Karam, E and F Planchet (2013). “Estimation errors and SCR calculation”. *Bulletin francais d’actuariat* 13.26, pp. 79–92.

Kochanski, M. and B. Karnarski (2011). “Solvency capital requirement for hybrid products”. *European Actuarial Journal, Springer* 42.2, pp. 173–198.

Lichters, R., R. Stamm, and Gallagher (2015). “Hull-White Model”. *Modern Derivatives Pricing and Credit Exposure Analysis, Springer*, pp. 403–422.

Liebwein, P. (2006). “Risk models for capital adequacy: Applications in the context of solvency II and beyond”. *The Geneva Papers on Risk and Insurance-Issues and Practice, Springer* 31.3, pp. 528–550.

Luciano, E., L. Regis, and E. Vigna (2017). “Single-and Cross-Generation Natural Hedging of Longevity and Financial Risk”. *Journal of Risk and Insurance, Wiley Online Library* 84.3, pp. 961–986.

Merton, Robert C. (1973). “Theory of rational option pricing”. *The Bell Journal of economics and management science, JSTOR*, pp. 141–183.

Milevsky, M. A. (2014). “Portfolio choice and longevity risk in the late seventeenth century: a re-examination of the first English tontine”. *Financial History Review* 21.3, pp. 225–258.

Milevsky, M. A. and T. S. Salisbury (2006). “Financial valuation of guaranteed minimum withdrawal benefits”. *Insurance: Mathematics and Economics* 38.1, pp. 21–38.

Ngugnie Diffouo, P. M. and P. Devolder (2020). "Longevity Risk Measurement of Life Annuity Products". *Risks* 8.1, pp. 1–16.

Ohlsson, E. and J. Lauzeningks (2009). "The one-year non-life insurance risk". *Insurance: Mathematics and Economics* 45.2, pp. 203–208.

Olivieri, A. and E. Pitacco (2008). "Solvency requirements for life annuities: some comparisons". *Available at SSRN 1266094*.

Pfeifer, D. and D. Strassburger (2008a). "Solvency II: stability problems with the SCR aggregation formula". *Scandinavian actuarial journal, Taylor & Francis* 2008.2, pp. 61–77.

Pfeifer, Dietmar and Doreen Strassburger (2008b). "Solvency II: stability problems with the SCR aggregation formula". *Scandinavian actuarial journal* 2008.1, pp. 61–77.

Phillips, P.C.B. and J. Yu (2009). "Maximum likelihood and Gaussian estimation of continuous time models in finance". *Handbook of financial time series, Springer*, pp. 497–530.

Piggott, J., E. A. Valdez, and B. Detzel (2005). "The simple analytics of a pooled annuity fund". *Journal of Risk and Insurance* 72.3, pp. 497–520.

Qiao, C. and M. Sherris (2013). "Managing Systematic Mortality Risk With Group Self-Pooling and Annuitization Schemes". *Journal of Risk and Insurance* 80.4, pp. 949–974.

Richter, A. and F. Weber (2011). "Mortality-indexed annuities managing longevity risk via product design". *North American Actuarial Journal* 15.2, pp. 212–236.

Tiong, S. (2000). "Valuing equity-indexed annuities". *North American Actuarial Journal* 4.4, pp. 149–163.

Trindade, A Alexandre, Stan Uryasev, Alexander Shapiro, and Grigory Zrazhevsky (2007). "Financial prediction with constrained tail risk". *Journal of Banking & Finance* 31.11, pp. 3524–3538.

Vasicek, O. A. and G. H. Fong (1982). "Term structure modeling using exponential splines". *The Journal of Finance, Wiley Online Library* 37.2, pp. 339–348.

Vigna, Elena et al. (2009). "Mean-variance inefficiency of CRRA and CARA utility functions for portfolio selection in defined contribution pension schemes". *Collegio Carlo Alberto Notebook 108, or CeRP wp 89* 9.

Wakker, Peter P (2008). "Explaining the characteristics of the power (CRRA) utility family". *Health economics* 17.12, pp. 1329–1344.

Yaari, Menahem E (1965). "Uncertain lifetime, life insurance, and the theory of the consumer". *The Review of Economic Studies* 32.2, pp. 137–150.

Zaglauer, K. and D. Bauer (2008). "Risk-neutral valuation of participating life insurance contracts in a stochastic interest rate environment". *Insurance: Mathematics and Economics* 43.1, pp. 29–40.

Zeddouk, F. and P. Devolder (2019). "Pricing of longevity derivatives and cost of capital". *Risks* 7.2, p. 41.