Université catholique de Louvain

Institut de recherche en
mathématique et physique

# Integrable lattice models and supersymmetry

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Thesis submitted in partial fulfilment of the requirements for
the degree of Docteur en Sciences

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September 2020

# Remerciements

Sont remerciées ici, en ces quelques lignes, les personnes qui ont contribué, activement ou non, à me permettre de réaliser cette thèse de doctorat.

En premier lieu, je voudrais diriger mes plus vifs remerciements envers mon promoteur, Christian Hagendorf. Après avoir réalisé avec lui un mémoire de master, il m'a donné l'opportunité de faire un doctorat sous sa supervision. Pendant ces cinq années de doctorat, il a fait preuve de pédagogie, de patience, de disponibilité, de rigueur et d'enthousiasme. Ce fut aussi un plaisir de collaborer avec lui pour diverses tâches d'assistanat et de discuter ensemble de physique, de musique classique, et bien d'autres choses !

Nonobstant les difficultés liées à la crise sanitaire, les professeurs Philippe Ruelle, Tom Claeys, Marino Gran, Robert Weston, ainsi que Miłosz Panfil ont accepté d'être membres de mon jury de thèse. Ainsi, je tiens à les remercier d'avoir consacré du temps à la lecture de mon manuscrit et pour les améliorations qu'ils ont apportées. Je salue aussi les membres de mon comité d'accompagnement, les professeurs Philippe Ruelle, Marino Gran et Christophe Ringeval.

Découvrir de nouveaux horizons faisant partie intégrante de la vie d'un doctorant, je suis reconnaissant envers le Laboratoire de Physique Théorique et Modèles Statistiques – LPTMS – de m'avoir accueilli pour un séjour de recherche, et plus particulièrement le professeur Raoul Santachiara. Je salue aussi tous les scientifiques que j'ai pu rencontrer lors des conférences et voyages au cours de ma thèse en France, au Brésil, en Italie. Je tiens à remercier en particulier Alexandra E., qui m'a transmis de nombreux articles intéressants.

Nombreux sont les étudiants et étudiantes que j'ai eu la chance d'encadrer durant ces années. J'ai beaucoup appris à leur côté tant au niveau scientifique, pédagogique que humain. Je les salue tous et toutes, ainsi que les professeurs Jean Bricmont, Hugues Goosse avec lesquels j'ai eu le plaisir de collaborer. Je remercie l'école polytechnique de Louvain de m'avoir donné la possibilité de terminer cette thèse en tant qu'assistant, en particulier les professeurs Kouider Ben-Naoum, François Glineur, Raphaël Jungers et Vincent Wertz ainsi que Laurent Fourny.

Une thèse n'est réalisable que dans un cadre de travail propice et agréable. Je remercie donc tous mes autres collègues du groupe de physique statistique : Alexandre, pour les cookies partagés pendant les pauses, Alexi, pour les soirées de danse et notre collaboration, Adrien, pour les nombreuses discussions et les parties de badminton, Gary, pour son calme, Gilles, pour son humour particulier et notre collaboration, Philippe, pour les repas partagés à la cafétéria et Sandrine « Sansan », pour sa bonne humeur et son anglais. Je remercie aussi le doctorant Bryan Debin. Je n'ai pas la place pour citer tous les membres de l'institut, mais je remercie tous mes collègues, professeurs et professeures, doctorants et doctorantes, post-docs et secrétaires.

Durant cette thèse, j'ai aussi pu compter sur la présence de proches et amis pour me changer les idées : discuter, faire du sport, aller boire un verre ou même partir en vacances. Je remercie donc tous mes camarades de cours, de sports aquatiques et de badminton, ainsi que tous ceux et celles que j'ai rencontrés durant ces années, en particulier au sein de l'IRMP et des instituts ICTEAM, ELI et ISPOLE.

Enfin, je remercie ma famille : Papa, Chantal, merci pour votre interêt pour mon travail et le soutien inconditionnel. Que la physique est belle ! Simon et Thomas, vous aurez été comme des frères pour moi ! Merci pour les broünions, mais aussi les poèmes et haïkus, arts pour lesquels je n'ai pas votre talent.

Si cette thèse a finalement abouti, c'est aussi grâce au soutien et aux encouragements de Florence. Merci pour ta présence à mes côtés !

## List of publications

This thesis is based on the following publications:

[1] C. Hagendorf and J. Liénardy, *Open spin chains with dynamic lattice supersymmetry*, J. Phys. A. (2017) 185202,

[2] C. Hagendorf and J. Liénardy, *On the transfer matrix of the supersymmetric eight-vertex model. I. Periodic boundary conditions*, J. Stat. Mech. (2018) 033106,

[3] C. Hagendorf and J. Liénardy, *On the transfer matrix of the supersymmetric eight-vertex model. II. Open boundary conditions*, J. Stat. Mech. (2020) 033104,

[4] C. Hagendorf and J. Liénardy, *The open XXZ chain at* $\Delta = -1/2$ *and the boundary quantum Knizhnik-Zamolodchikov equations*, arXiv:2008.03220, (2020).

Other publications by the author of this thesis:

[5] A. Morin-Duchesne, G. Parez, and J. Liénardy, *Bipartite fidelity for models with periodic boundary conditions*, arXiv:2008.08952, (2020).

# Contents

# Introduction

Statistical physics aims to study macroscopic systems from the microscopic properties of its constituents. It originates from the kinetic theory of gases, a theory that was born in the nineteenth century and has explained the thermodynamic properties of gases (at equilibrium) from the movement of its constituting atoms.

One of the goals of statistical physics is to understand the magnetic properties of the matter from the atomic properties and interactions. However, classical mechanics alone cannot account for all macroscopic magnetic effects. This result is usually referred to as the Bohr-van Leeuwen theorem (which treats the case of diamagnetism) [6]. Hence, one must consider quantum effects to explain the macroscopic magnetism.

Quantum mechanics was developed in the twentieth century. This theory describes the phenomena of nature at the scale of the atom, where Newton's laws are no longer valid. One purely quantum property of a particle is its intrinsic angular momentum, called spin. The goal of this dissertation is to study specific models of spins in interactions on a lattice.

### The Heisenberg model

Among the quantum models of interacting spins, the Heisenberg model is arguably the most important and studied one. It is a model aiming to describe the magnetism in the matter due to the electrons that interact via the so-called exchange interaction [7]. (We refer the reader to [8, 9] for a derivation of the model from first principles.) It is a lattice model, meaning that its degrees of freedoms, the spins, are located on the sites of a given lattice. The system is described by a Hamiltonian.

In this dissertation, we are interested in spin-1/2 chains that are generalisations of the Heisenberg model defined on a one-dimensional lattice. In the models that we consider, the spins on the lattice only interact with their nearest neighbours, and interactions with spins that are further away are neglected.

For the sake of concreteness, let us introduce the XYZ spin-chain Hamiltonian with periodic boundary conditions:

$$H_{\mathrm{XYZ}} = -\frac{1}{2}\sum_{j=1}^{L}\left(J_1\sigma_j^1\sigma_{j+1}^1 + J_2\sigma_j^2\sigma_{j+1}^2 + J_3\sigma_j^3\sigma_{j+1}^3\right). \tag{0.1}$$

Here, $J_1, J_2$ and $J_3$ are the *anisotropy parameters*. (We postpone the precise definition of all the involved objects to subsequent chapters.) The XYZ spin chain is a generalisation of the Heisenberg model, in which $J_1 = J_2 = J_3 = J$. The latter is also called the XXX model. Moreover, if $J_1 = J_2 \neq J_3$, then the model is referred to as an XXZ spin chain and one usually normalises $J_1 = J_2 = 1$, $J_3 = \Delta$.

### The Bethe ansatz

The Heisenberg model is related to a variety of physically interesting problems, most notably the theory of quantum magnetism [10, 11]. Additionally, it has inspired the development of many modern techniques of mathematical physics.

In the history of the resolution of Heisenberg spin chains and other models of statistical physics, the work of Hans Bethe is seminal. In 1931, he proposed an *ansatz*, (*i.e.* educated guess) to find the exact eigenvalues and eigenvectors of the XXX spin chain [12]. Nowadays referred to as the *coordinate Bethe ansatz*, this solution provides the eigenvectors in terms of a linear combination of plane waves built upon a reference state. Furthermore, the *Bethe equations* are a set of coupled equations that fix the ratios of coefficients. We refer to the monograph of Gaudin for an overview of Bethe's method [13].

In 1966, Yang and Yang successfully applied the coordinate Bethe ansatz to solve the XXZ model with periodic boundary conditions. They furthermore analysed the properties of the ground state energy in the large system size limit [14, 15, 16].

One can modify the Hamiltonian (0.1) so that the boundary conditions are not periodic. In this case, we replace the last term in the sum by so-called boundary fields, and we say that the spin chain has open boundary conditions. The first application of the Bethe ansatz to an open system is due to Gaudin who solved the XXZ spin chain with vanishing boundary fields [13]. Non-vanishing boundary fields were later considered by Alcaraz *et al.* [17].

### Vertex models and integrability

In the late sixties, a relation between one-dimensional quantum systems and two-dimensional statistical models was found. Specifically, let us consider the six- and eight-vertex models on a square lattice (their origin and precise definition are given later). They are classical, *i.e.* non-quantum, lattice models in which each vertex of the lattice is in one configuration among the six or eight admissible ones [18].

Their resolution relies on the *transfer-matrix* method, which reduces the evaluation of the partition function for large systems to the computation of an eigenvalue of this matrix [19, 20, 21]. The first link between vertex models and spin chains was found by Lieb. He observed that the relations satisfied by the transfer-matrix eigenvectors were the Bethe equations of an XXZ spin chain [22].

The next milestone was reached by Baxter. In 1971, he announced that he had obtained equations for the eigenvalues of the transfer matrix of the eight-vertex model, and the XYZ Hamiltonian [23, 24]. This latter result follows from his observation that one can extract the Hamiltonian (0.1) from the transfer matrix. Baxter's results are based on the $T-Q$ equation, a matrix relation which contains the Bethe ansatz equations [25, 26]. Moreover, Baxter found expressions for the transfer-matrix eigenvectors, which are also spin-chain eigenstates [27, 28, 29].

A crucial point in Baxter's analysis is the requirement that the family of transfer matrices depending on a spectral parameter is mutually commuting. This is a consequence of the building blocks of the transfer matrices, the *R-matrices*, satisfying the nowadays famous *Yang-Baxter equation*. We refer to models where such a relation holds as *integrable*.

The work of Baxter was reformulated into an algebraic framework, the quantum-inverse scattering method, developed in the late seventies by the Leningrad school [30, 31]. (We refer the reader to the surveys [32, 33, 34] and references therein.) This method is nowadays known as the *algebraic Bethe ansatz*. It provides a construction of the eigenvectors by means of the action of matrices on a reference state and allows for the evaluation of correlation functions [34]. Sklyanin has extended the method to encompass models with boundaries using the so-called *$K$-matrices* [35].

Up to this day, Bethe's methods are still widely used in mathematical physics and are powerful tools allowing one to solve a variety of models. To provide an overview, let us discuss the different Bethe ansatz's generalisations and their respective utility.

The *thermodynamic Bethe ansatz* treats the Bethe equations and their solutions in the large system size limit [36]. It is the natural framework to extract thermodynamic properties of a model, such as the specific heat or the magnetic susceptibility. The *asymptotic Bethe ansatz* allows for the treatment of long-range potentials. It was introduced by Sutherland [37, 38]. The *nested algebraic Bethe ansatz* provides a recursion formula for Bethe vectors for higher spin models [39].

In some models, such as in the XXZ spin chain with generic boundary fields, there may not exist a reference state (for spin chains, this usually is a consequence of the total magnetisation not being conserved). The *off-diagonal Bethe ansatz* circumvents this absence and allows one to retrieve Bethe states by adding a term in the Bethe equations [40, 41]. Likewise, the *modified algebraic Bethe ansatz* developed by Crampé et Belliard provides a way to construct Bethe vectors, even when the magnetisation is not conserved [42, 43].

Finally, the *functional Bethe ansatz* is a method created by Sklyanin [44]. It amounts to separate the spectral problem of the transfer matrix into one-dimensional equations. For this reason, it is known today as the method of the quantum separation of variables. It has been applied to many models by the Lyon group and is still an active field of research (see [45, 46, 47] and references therein).

At this point, the reader may think that these problems are entirely resolved. In fact, *solving* a model may have different meanings. Usually, a model is said to be solved if there exists a technique to generate the spectrum and the eigenvectors that does not require the explicit diagonalisation of the Hamiltonian. While the Bethe ansatz (and its descendants) are useful to analyse the spectrum and extract large-system-size properties, these methods are not meant for the explicit computation of the ground states' components, which are in some cases, related to enumerative combinatorics problems.

## Spin chains and combinatorics

The study of the XXZ and XYZ spin chains received a new impetus at the turn of the century. In a series of articles, Stroganov and Razumov discovered a tight relationship between these quantum models and apparently unrelated combinatorial problems.

To be precise, they studied the spectrum and eigenvectors of the XXZ Hamiltonian with periodic boundary conditions, an odd number of sites and the anisotropy parameter set to the value $\Delta = -1/2$. Among other observations, they pointed out that the ratio of the maximal component of the ground state of a chain of length $2n+1$ by the smallest non-zero one was equal to $A_n$, with

$$A_n = 1, 2, 7, 42, 429, 7436, 218348\dots \tag{0.2}$$

for $n = 1, 2, 3\dots$ [48]. The sequence of numbers $A_n$ is well-known in the mathematics literature as it enumerates *alternating sign matrices* (ASMs): $A_n$ counts the number of $n \times n$ matrices whose entries are 0 or $\pm 1$ such that the sum of each row and column is 1 and the signs of the entries alternate along each row and each column. Mills, Robbins and Rumsey invented these combinatorial objects in a generalisation of the Dodgson's condensation formula, a method developed by Charles Dodgson (better known as Lewis Carroll) to compute determinants [49, 50]. The same authors conjectured an exact formula for the enumeration of the ASMs.

The proof of the alternating sign matrices conjecture was achieved by Zeilberger in an 84-page-long tour de force [51]. A few years later, Kuperberg provided a shorter and simpler proof, using a relation between ASMs and six-vertex models with specific boundary conditions [52]. The genesis of ASMs and the story of their enumeration are narrated in the book of Bressoud [53].

Razumov and Stroganov did not restrict their investigations to the XXZ spin chain with periodic boundary condition but also conjectured similar results for open or XYZ spin chains [54, 55, 56, 57]. Following their discoveries, a variety of conjectures *à la* Razumov and Stroganov were formulated for the XXZ spin chains and the related $O(1)$ loop model; some of which have already been proved [58, 59, 60, 61, 62].

### Supersymmetry

The combinatorial properties of the XXZ and XYZ spin chain ground states arise when their parameters take a precise value, for example, $\Delta = -1/2$ for the XXZ model. In this dissertation, we investigate a feature of spin chains (and their related vertex models) that arises for the same specific value of their parameter: supersymmetry. To be more specific, we study these models in the framework of supersymmetric quantum mechanics.

The concept of supersymmetry originates from the particle physics community. It appeared in the early seventies as a solution to combine external (space-time) symmetries and internal (gauge) symmetries of elementary particles [63, 64]. It has received massive interest in high-energy physics as supersymmetry theories beyond the standard model of particles may give an answer to unsolved problems (for example, provide particle candidates for dark matter or solve the so-called hierarchy problem) [65].

The main property of a supersymmetric theory is the existence of a symmetry between two types of degree of freedom (usually called bosons and fermions). The generators of this symmetry are called the *supercharges*. A consequence of this in particle physics is that every particle possesses a superpartner, its supersymmetric image [65].

We will not discuss further the supersymmetry from the point of view of high-energy or nuclear physics. Instead, we focus on supersymmetric quantum mechanics. The essential feature of this framework is that the Hamiltonian can be written in terms of the supercharges.

The first application of the supersymmetric techniques to spin systems is due to Nicolai, through the eponymous lattice model [66]. It contains two nilpotent supercharges whose anticommutator is a spin chain Hamiltonian. A few years later, Witten popularised the concept and techniques of supersymmetric quantum mechanics, invented the so-called *Witten* index, and investigated the phenomenon of supersymmetry breaking [67].

Due to the extra requirement that is the existence of this additional symmetry, supersymmetric quantum mechanics may appear as uncommon and as of limited use. This is not the case. In fact, one can recast various well-known models into the supersymmetric formalism, among which the notorious harmonic oscillator [68] and the Coulomb Hamiltonian of the non-relativistic hydrogen atom [69].

## Outline

The main goal of this dissertation is to study XXZ and XYZ spin chains that possess a lattice supersymmetry and related supersymmetric models. In Chapter 1, we introduce some fundamental objects pertaining to supersymmetric quantum mechanics that we use in subsequent chapters. In particular, we define the supersymmetry singlets, and their relations with, on the one hand, the ground-state eigenvectors of the model and, on the other hand, the (co)homology of the supersymmetry generator.

We present the first spin model that we investigate, the XYZ spin chain with periodic boundary conditions in Chapter 2. We exhibit a dynamic supersymmetry and construct the corresponding cohomology. Chapter 3 is devoted to the supersymmetric eight vertex domain with periodic boundary conditions along the horizontal direction. We introduce this model, its transfer matrix and show its relation with the XYZ spin chain. We use the supersymmetry singlets to compute a remarkable transfer-matrix eigenvalue, therefore proving a twenty-year-old conjecture by Stroganov.

The layout of Chapters 4 and 5 is similar to the two preceding ones. In the former, we introduce the XYZ spin chain with open boundary conditions and treat the existence of supersymmetry singlets in the case where the Hamiltonian is supersymmetric. In the latter, we focus on the supersymmetric eight-vertex model on a strip. As for the periodic case, we manage to prove the existence of a remarkably simple eigenvalue of the corresponding transfer matrix.

From Chapter 6, we only focus on XXZ spin chains. We characterise the space of the ground states and find representatives of supersymmetry singlets for both the open and periodic spin chains. The simplicity of the scalar products between supersymmetry singlets, which is a consequence of supersymmetry, allows us to define and compute *multipartite fidelities*. We compute various cases of fidelities and evaluate their large-system-size behaviour in Chapter 7.

The content of the last two chapters is different as we do not treat supersymmetric spin models. We focus on a generalisation of the open spin chain of Chapter 6 with different boundary conditions depending on a parameter $x$. To this end, we find in Chapter 8 a solution to the so-called boundary quantum Knizhnik-Zamolodchikov equations. This solution is given in terms of multiple contour integrals and depends on inhomogeneity parameters as well as a deformation parameter, $q$. We show that the solution, with $q = \mathrm{e}^{2\mathrm{i}\pi/3}$, is an eigenvector of the transfer matrix of the six-vertex model on a strip and that the homogeneous limit is, for $x > 0$, a ground-state of the corresponding Hamiltonian.

In Chapter 9, we use the homogeneous limit found in the preceding chapter to prove combinatorial identities satisfied by the ground-state (similar to the ones found by Razumov and Stroganov). Furthermore, we provide new conjectures on the enumeration of alternating sign matrices with all the symmetries of the square.

Finally, in the conclusion chapter, we give a summary of the results obtained. For each chapter, we point out a few problems that remain open, highlight the conjectures that are still to be proven, give possible ways to pursue those tasks and discuss interesting generalisations.

# Chapter 1

# Supersymmetry

In this first chapter, we recall the concept and formalism of supersymmetry for quantum-mechanical systems. Our objective is to introduce concepts, set notation, recall some general definitions, and state the results that we use in subsequent chapters.

The layout of this chapter is as follows: in Section 1.1, we review the basic definition of a quantum system as well as the generic setting of supersymmetric quantum mechanics. We focus on the case of $\mathcal{N} = 2$ supersymmetric systems in Section 1.2. We investigate the structure of the spectrum and the set of eigenvectors of a supersymmetric Hamiltonian in Section 1.3. The existence of zero-energy ground states is related to a (co)homological problem that we define in Section 1.4.

## 1.1 Supersymmetric quantum mechanics

In this section, we recall standard definitions of quantum mechanics and introduce the concepts of supersymmetric quantum mechanics.

**Quantum Mechanics.** A quantum-mechanical system is given by $(\mathscr{H}, H)$. Here, $\mathscr{H}$ is a complex Hilbert space, *i.e.* a vector space on $\mathbb{C}$ endowed with a Hermitian inner product:

$$\langle \cdot | \cdot \rangle : \mathscr{H} \times \mathscr{H} \to \mathbb{C}, \tag{1.1}$$

also called *scalar product*. The vector space is complete for this scalar product. The latter justifies the so-called bra-ket notation: we denote by $|\psi\rangle$ the elements of $\mathscr{H}$. We call them interchangeably vector or states. We write $\langle\psi|\phi\rangle$ for the scalar product between two states $|\psi\rangle$ and $|\phi\rangle$.

The dynamics of the system is determined by the Hamiltonian $H$ which is an operator on the Hilbert space, $H : \mathscr{H} \to \mathscr{H}$. The Hamiltonian is Hermitian, $H^\dagger = H$. Hence, it is a diagonalisable operator and its eigenvalues, called *energies*, are real numbers.

A *symmetry operator* of the model is an operator $S$ that is Hermitian and commutes with the Hamiltonian,

$$[H, S] = HS - SH = 0. \tag{1.2}$$

We now introduce the supercharges. They are symmetry operators that furthermore allow one to compute the Hamiltonian of the model. This defines the supersymmetric quantum mechanics [67].

**Supersymmetry.** The general setting for a supersymmetric model is an Hilbert space $\mathscr{H}$, as well as $\mathcal{N}$ operators $\mathfrak{Q}_{(i)}$, $i = 1, \dots, \mathcal{N}$ that we call *supercharges* and which obey the following algebraic relations

$$(\mathfrak{Q}_{(i)})^\dagger = \mathfrak{Q}_{(i)}, \quad \{\mathfrak{Q}_{(i)}, \mathfrak{Q}_{(j)}\} = 2\delta_{ij} \cdot H \tag{1.3}$$

for each $i, j = 1, \dots, \mathcal{N}$. The first set of equations states that the supercharges are Hermitian operators. The second allows one to compute the Hamiltonian through the anticommutator of the supercharges:

$$\{\mathfrak{Q}_{(i)}, \mathfrak{Q}_{(j)}\} = \mathfrak{Q}_{(i)}\mathfrak{Q}_{(j)} + \mathfrak{Q}_{(j)}\mathfrak{Q}_{(i)}. \tag{1.4}$$

In this setting, $(\mathscr{H}, H)$ is a supersymmetric quantum-mechanical system and we simply say that the Hamiltonian $H$ is supersymmetric.

A direct consequence of the equations (1.3) is that the operators $\mathfrak{Q}_{(i)}, i = 1, \dots, \mathcal{N}$, are symmetry operators as

$$[\mathfrak{Q}_{(i)}, H] = 0, \quad \text{for all } i = 1, \dots, \mathcal{N}, \tag{1.5}$$

hence the name supercharge.

**Parity and Grading.** We complete this algebraic picture by the addition of a *parity operator*, $W$, also called *Witten operator*. It is a Hermitian involution that anticommutes with the supercharges and therefore commutes with the Hamiltonian:

$$\{W, \mathfrak{Q}_{(i)}\} = 0, \quad [W, H] = 0, \quad W^2 = \mathbf{1}, \tag{1.6}$$

for each $i = 1, \dots, \mathcal{N}$. Here, $\mathbf{1}$ is the identity operator on $\mathscr{H}$. The name parity operator of $W$ comes from its involutive nature: its eigenvalues are $\pm 1$. The parity operator was initially introduced by Witten [67], in the form of $(-1)^F$. Here, $F$ has integer eigenvalues and is called the *grading operator* or *fermion number* operator. According to this denomination, the eigenstates of $W$ with eigenvalue 1 are called bosons, while those with eigenvalue $-1$ are referred to as fermions [70, 71].

However, this nomenclature of fermion-boson is a remnant of the history of the supersymmetry and can be misleading. In particular, one should not consider the bosons and fermions as particles (or states) with integer and half-integer spin, respectively, but as a name given to the states pertaining to the eigenspaces of $W$.

## 1.2 $\mathcal{N} = 2$ supersymmetry

In the remainder of this dissertation, we focus on the case $\mathcal{N} = 2$. Instead of the general setting given in the previous section, we define the *supercharge* $\mathfrak{Q}$ and its adjoint $\mathfrak{Q}^\dagger$ as

$$\mathfrak{Q} = \frac{1}{\sqrt{2}} \left( \mathfrak{Q}_{(1)} + \mathrm{i}\mathfrak{Q}_{(2)} \right), \quad \mathfrak{Q}^\dagger = \frac{1}{\sqrt{2}} \left( \mathfrak{Q}_{(1)} - \mathrm{i}\mathfrak{Q}_{(2)} \right). \tag{1.7}$$

The supercharge and its adjoint are nilpotent operators,

$$\mathfrak{Q}^2 = 0, \quad (\mathfrak{Q}^\dagger)^2 = 0, \tag{1.8}$$

and generate the Hamiltonian through their anticommutator:

$$\mathfrak{Q}\mathfrak{Q}^\dagger + \mathfrak{Q}^\dagger\mathfrak{Q} = H. \tag{1.9}$$

As a consequence of the two preceding formulas, the supercharge and its adjoint commute with the Hamiltonian: $[H, \mathfrak{Q}] = [H, \mathfrak{Q}^\dagger] = 0$. We stress that the supercharge is not a Hermitian operator.

**$\mathcal{N} = 2$ superalgebra.** We further assume that there exists a grading operator $F$ satisfying the algebraic relations[1]

$$[F, \mathfrak{Q}] = \mathfrak{Q}, \quad [F, \mathfrak{Q}^\dagger] = -\mathfrak{Q}^\dagger, \quad [F, H] = 0. \tag{1.10}$$

Defining the operator $W = (-1)^F$ leads to the following relations:

$$\{W, \mathfrak{Q}\} = 0, \quad \{W, \mathfrak{Q}^\dagger\} = 0, \quad [W, H] = 0. \tag{1.11}$$

In order for $W$ to be a parity operator, $F$ must have integer eigenvalues, that we denote by $f : F|\psi\rangle = f|\psi\rangle$.

We denote the spectrum of $F$ by $\mathbb{F}$. Without loss of generality, there are three possible cases for $\mathbb{F}$. First, we say that $F$ is unbounded if $\mathbb{F} = \mathbb{Z}$. Second if $\mathbb{F}$ is bounded from below but has no maximal value, we can shift $F$ in order to its spectrum be $\mathbb{F} = \mathbb{N}$. The opposite case, $F$ bounded from above, is similar to the previous one, up to the redefinition of the grading operator by $-F$ and the interchange of the supercharge and its adjoint. Third, if the spectrum of $F$ is bounded, then we can map it to $\{0, \ldots, f_{max}\}$. Slightly abusing the denomination, we say that, in these three cases, $F$ is unbounded, bounded from below, and bounded, respectively.

**Grading.** As $F$ and the Hamiltonian commute, they are simultaneously diagonalisable, and we can decompose the Hilbert space $\mathscr{H}$ into the eigenspaces of $F$:

$$\mathscr{H} = \bigoplus_{f \in \mathbb{F}} \mathscr{H}^f. \tag{1.12}$$

We make here an important remark. We defined the Hamiltonian, the supercharge and its adjoint as endomorphisms in $\mathscr{H}$. Due to the commutation relations with the grading operator (1.10), we can introduce the restrictions of these operators on the eigenspaces of $F$. Starting with $H$, we have

$$H = \bigoplus_{f \in \mathbb{F}} H_f, \tag{1.13}$$

[1] Written in terms of the initial supercharges $\mathfrak{Q}_{(1)}$, $\mathfrak{Q}_{(2)}$, these relations read $[F, \mathfrak{Q}_{(1)}] = \mathrm{i}\mathfrak{Q}_{(2)}$, $[F, \mathfrak{Q}_{(2)}] = -\mathrm{i}\mathfrak{Q}_{(1)}$. This is one of the reasons to work with non-Hermitian supercharges.

where each $H_f : \mathscr{H}^f \to \mathscr{H}^f$ is an endomorphism. The supercharge and its adjoint do not commute with $F$. We nevertheless define for each $f \in \mathbb{F}$ the set of operators $\mathfrak{Q}_f$ and their adjoint. We keep the denomination supercharge and adjoint supercharge for each $\mathfrak{Q}_f$ and $\mathfrak{Q}_f^\dagger$, respectively. They are operators that map an eigenspace of $F$, $\mathscr{H}^f$, into another eigenspace whose index differs by one:

$$\mathfrak{Q}_f : \mathscr{H}^f \to \mathscr{H}^{f+1}, \quad \mathfrak{Q}_f^\dagger : \mathscr{H}^f \to \mathscr{H}^{f-1}. \tag{1.14}$$

We need to take some precautions if $f+1$ (or $f-1$) does not belong to the spectrum of $F$, in which case we set $\mathfrak{Q}_f|\psi\rangle = 0$ (or $\mathfrak{Q}_f^\dagger|\psi\rangle = 0$) for all $|\psi\rangle \in \mathscr{H}^f$.

In this framework, the nilpotency condition (1.8) and the supersymmetry of the Hamiltonian (1.9) read

$$\mathfrak{Q}_{f+1}\mathfrak{Q}_f = 0, \quad \mathfrak{Q}_{f-1}^\dagger\mathfrak{Q}_f^\dagger = 0, \tag{1.15a}$$

$$\mathfrak{Q}_{f-1}\mathfrak{Q}_f^\dagger + \mathfrak{Q}_{f+1}^\dagger\mathfrak{Q}_f = H_f. \tag{1.15b}$$

Here both relation holds for each $f \in \mathbb{F}$. Accordingly, the commutation relation between the supercharge and the Hamiltonian is, for each $f \in \mathbb{F}$,

$$H_{f+1}\mathfrak{Q}_f - \mathfrak{Q}_f H_f = 0 \tag{1.16}$$

and similarly for its adjoint.

The supercharge $\mathfrak{Q}_f$ and its adjoint are not endomorphisms [72]. Nevertheless, we say that each Hamiltonian $H_f$ is supersymmetric if there exist $\mathfrak{Q}_f$ and $\mathfrak{Q}_f^\dagger$ such that the relations (1.15) are satisfied.

To lighten the notation, we omit in the following the subscript indicating the space each operator is acting on. Hence, we write $H = H_f, \mathfrak{Q} = \mathfrak{Q}_f$ and $\mathfrak{Q}^\dagger = \mathfrak{Q}_f^\dagger$, and similarly for other operators. If necessary, we explicitly indicate which space the operator acts on. Furthermore, unless stated explicitly, we prove each result for each $f \in \mathbb{F}$.

## 1.3 Spectrum and structure of eigenvectors

The supersymmetry of the Hamiltonian leads to special properties of its eigenvalues and its eigenvectors.

**Spectrum.** The construction of the Hamiltonian (1.9) implies that its spectrum is real and non-negative. To see this, let $|\psi\rangle$ be a solution to the Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle. \tag{1.17}$$

The projection of this equation on $\langle\psi|$ yields

$$\|\mathfrak{Q}|\psi\rangle\|^2 + \|\mathfrak{Q}^\dagger|\psi\rangle\|^2 = E\||\psi\rangle\|^2, \tag{1.18}$$

hence $E \geqslant 0$. We call the solutions of the Schrödinger equation with $E > 0$ and $E = 0$ positive-energy states and zero-energy states, respectively.

The supersymmetry fixes the structure of the eigenvectors of the Hamiltonian. As we shall see, the positive-energy states are organised in supersymmetry doublets, whereas the zero-energy states are so-called supersymmetry singlets.

**Supersymmetry doublets.** Let $|\psi_1\rangle$ be a non-zero positive-energy state $H|\psi_1\rangle = E|\psi_1\rangle$, $E > 0$; we define a *supersymmetry doublet* as the couple of states $(|\psi_1\rangle, |\psi_2\rangle)$ satisfying

$$\mathfrak{Q}|\psi_1\rangle = 0, \qquad \mathfrak{Q}^\dagger|\psi_1\rangle = \sqrt{E}|\psi_2\rangle, \tag{1.19}$$

$$\mathfrak{Q}|\psi_2\rangle = \sqrt{E}|\psi_1\rangle, \qquad \mathfrak{Q}^\dagger|\psi_2\rangle = 0. \tag{1.20}$$

As the supercharge commutes with the Hamiltonian, $|\psi_2\rangle$ is also eigenstate of $H$ with the same energy.

Given a non-zero eigenstate $|\psi\rangle$ with $E > 0$, we can generate through the action of the supercharge the states $|\psi\rangle$, $\mathfrak{Q}|\psi\rangle$, $\mathfrak{Q}^\dagger|\psi\rangle$ and $\mathfrak{Q}\mathfrak{Q}^\dagger|\psi\rangle$. Further applications of the supercharge or its adjoint yield linear combinations of these four states. The members of this multiplet are eigenvectors of $H$ with the same energy. It decomposes into two independent pairs of states:

**Proposition 1.3.1.** *Let $|\psi\rangle$ be a non-zero positive-energy state with parity $\varepsilon = \pm 1$: $W|\psi\rangle = \varepsilon|\psi\rangle$. We define the linear combinations*

$$|\psi'\rangle = \frac{1}{E}\mathfrak{Q}\mathfrak{Q}^\dagger|\psi\rangle, \quad |\psi''\rangle = \frac{1}{E}\mathfrak{Q}^\dagger\mathfrak{Q}|\psi\rangle. \tag{1.21}$$

*Then the two couples of states*

$$(|\psi'\rangle, \frac{1}{\sqrt{E}}\mathfrak{Q}^\dagger|\psi'\rangle), \quad (\frac{1}{\sqrt{E}}\mathfrak{Q}|\psi''\rangle, |\psi''\rangle) \tag{1.22}$$

*are two supersymmetry doublets. The subspaces that they span are orthogonal to each other. Furthermore, they do not both identically vanish.*

*Proof.* First, it is straightforward to verify that $(|\psi'\rangle, \frac{1}{\sqrt{E}}\mathfrak{Q}^\dagger|\psi'\rangle)$ forms a doublet. The second equation of (1.19) is satisfied by construction. We verify that

$$\mathfrak{Q}|\psi'\rangle = \mathfrak{Q}\left(\frac{1}{E}\mathfrak{Q}\mathfrak{Q}^\dagger|\psi\rangle\right) = 0, \quad \mathfrak{Q}^\dagger\left(\frac{1}{\sqrt{E}}\mathfrak{Q}^\dagger|\psi'\rangle\right) = 0 \tag{1.23}$$

by the nilpotency of the supercharge; and we use the definition of $H$ to find

$$\mathfrak{Q}\left(\frac{1}{\sqrt{E}}\mathfrak{Q}^\dagger|\psi'\rangle\right) = \frac{1}{\sqrt{E}}(H - \mathfrak{Q}^\dagger\mathfrak{Q})|\psi'\rangle = \sqrt{E}|\psi'\rangle. \tag{1.24}$$

The proof for the second couple is similar.

Second, we check that the vectors spaces spanned by the two couples are orthogonal. Indeed, we show that the four possible scalar products between different states of each couple vanish. Two of them are readily computed as a consequence of the nilpotency,

$$\langle\psi'|\psi''\rangle = \frac{1}{E^2}\langle\psi|\mathfrak{Q}\mathfrak{Q}^\dagger\ \mathfrak{Q}^\dagger\mathfrak{Q}|\psi\rangle = 0, \quad \frac{1}{E}\langle\psi'|\mathfrak{Q}\mathfrak{Q}|\psi''\rangle = 0. \tag{1.25}$$

It remains to prove that $\langle\psi'|\mathfrak{Q}|\psi''\rangle$ vanishes. We verify that the states $|\psi'\rangle$ and $|\psi''\rangle$ have the parity $\varepsilon$. We then check, using the commutation relations (1.11), that

$$\langle\psi'|\mathfrak{Q}|\psi''\rangle = \langle\psi'|W^2\mathfrak{Q}|\psi''\rangle = -\varepsilon\langle\psi'|\mathfrak{Q}W|\psi''\rangle = -\langle\psi'|\mathfrak{Q}|\psi''\rangle, \tag{1.26}$$

hence $\langle\psi'|\mathfrak{Q}|\psi''\rangle = 0 = \left(\langle\psi''|\mathfrak{Q}^\dagger|\psi'\rangle\right)^*$.

Finally, we note that the two couples cannot identically vanish as $|\psi'\rangle + |\psi''\rangle = |\psi\rangle$, which is non-zero by assumption. □

This proposition tells us that the positive-energy states are organised in doublets of states. The two states in the doublet are called supersymmetry partners [67]. However, Proposition 1.3.1 does not imply that all positive-energy levels are four times degenerate. Indeed, the states $|\psi'\rangle$ and $|\psi''\rangle$ defined in (1.21), and their corresponding partners, may vanish.

As for the states with energy zero, they do not form doublets but instead so-called supersymmetry singlets.

**Supersymmetry singlets.** According to the relation (1.18), an eigenstate $|\psi\rangle$ of energy $E = 0$ is a solution to the equations

$$\mathfrak{Q}|\psi\rangle = 0, \quad \mathfrak{Q}^\dagger|\psi\rangle = 0. \tag{1.27}$$

We call it a *supersymmetry singlet* as we cannot create another $E = 0$ eigenvector through the action of the supercharge nor its adjoint. If such a state exists, it is a ground state of the Hamiltonian. In the following, we interchangeably use the terms "zero-energy state" and "supersymmetry singlet".

The eigenstates of the Hamiltonian are organised in supersymmetry doublets or singlets. We would like to characterise the state themselves. The following result gives a decomposition of any state in terms of supersymmetry singlets and vectors in the image of the supercharge and its adjoint. It will be of great importance in the following.

**Proposition 1.3.2** (Hodge decomposition). *Any vector $|\psi\rangle \in \mathscr{H}^f$ can be written as*

$$|\psi\rangle = |\psi_0\rangle + \mathfrak{Q}|\psi_1\rangle + \mathfrak{Q}^\dagger|\psi_2\rangle, \tag{1.28}$$

*for certain states $|\psi_1\rangle \in \mathscr{H}^{f-1}$, $|\psi_2\rangle \in \mathscr{H}^{f+1}$ and $|\psi_0\rangle \in \mathscr{H}^f$ a zero-energy state.*

*Proof.* First, let $\mathscr{G}$ be a subspace of the vector space $\mathscr{H}^f$, $\mathscr{G} \subseteq \mathscr{H}^f$. Then for each $|\psi\rangle \in \mathscr{H}^f$, there exists a unique $|\phi\rangle \in \mathscr{G}$ that minimises the function

$$m : \mathscr{G} \to \mathbb{R} : |\phi\rangle \mapsto \||\psi\rangle - |\phi\rangle\|^2. \tag{1.29}$$

Indeed, let $\{|u_i\rangle\}$ be an orthonormal basis of $\mathscr{G}$, we construct $|\phi\rangle$ as the orthogonal projection of $|\psi\rangle$ into $\mathscr{G}$:

$$|\phi\rangle = \sum_i \langle u_i|\psi\rangle|u_i\rangle. \tag{1.30}$$

This construction ensures the existence and uniqueness of the state $|\phi\rangle$.

Second, we consider the space

$$\mathscr{G} = \mathrm{im}\{\mathfrak{Q} : \mathscr{H}^{f-1} \to \mathscr{H}^f\} \oplus \mathrm{im}\{\mathfrak{Q}^\dagger : \mathscr{H}^{f+1} \to \mathscr{H}^f\}, \tag{1.31}$$

where the two terms in the direct sum have a trivial intersection. As $\mathscr{G}$ is a subspace of $\mathscr{H}^f$, there exist $|\psi_1\rangle$ and $|\psi_2\rangle$ that minimise $\||\psi_0\rangle\|^2$, with

$$|\psi_0\rangle = |\psi\rangle - \mathfrak{Q}|\psi_1\rangle - \mathfrak{Q}^\dagger|\psi_2\rangle. \tag{1.32}$$

This implies that the function

$$F(\varepsilon_1, \varepsilon_2) = \||\psi_0\rangle - \varepsilon_1\mathfrak{Q}|\phi_1\rangle - \varepsilon_2\mathfrak{Q}^\dagger|\phi_2\rangle\|^2, \tag{1.33}$$

of real variables $\varepsilon_1, \varepsilon_2$, has a minimum at $\varepsilon_1 = \varepsilon_2 = 0$ for arbitrary vectors $|\phi_1\rangle$, $|\phi_2\rangle$. The calculation of the partial derivatives of $F$ with respect to $\varepsilon_1$, $\varepsilon_2$ leads to

$$\mathrm{Re}\langle\psi_0|\mathfrak{Q}|\phi_1\rangle = 0, \quad \mathrm{Re}\langle\psi_0|\mathfrak{Q}^\dagger|\phi_2\rangle = 0. \tag{1.34}$$

Replacing $|\phi_1\rangle$ with $\mathrm{i}|\phi_1\rangle$ and $|\phi_2\rangle$ with $\mathrm{i}|\phi_2\rangle$, we obtain

$$\mathrm{Im}\langle\psi_0|\mathfrak{Q}|\phi_1\rangle = 0, \quad \mathrm{Im}\langle\psi_0|\mathfrak{Q}^\dagger|\phi_2\rangle = 0, \tag{1.35}$$

and thus

$$\langle\psi_0|\mathfrak{Q}|\phi_1\rangle = 0, \quad \langle\psi_0|\mathfrak{Q}^\dagger|\phi_2\rangle = 0. \tag{1.36}$$

As $|\phi_1\rangle$ and $|\phi_2\rangle$ are arbitrary states, $|\psi_0\rangle$ is a supersymmetry singlet:

$$\mathfrak{Q}|\psi_0\rangle = 0, \quad \mathfrak{Q}^\dagger|\psi_0\rangle = 0. \tag{1.37}$$

We conclude that we can decompose any vector as (1.28) with $|\psi_0\rangle$ a zero-energy state. □

This proposition allows us to decompose any state $|\psi\rangle$ uniquely as

$$|\psi\rangle = |\psi_0\rangle + \mathfrak{Q}|\psi_1\rangle + \mathfrak{Q}^\dagger|\psi_2\rangle, \tag{1.38}$$

with $|\psi_0\rangle$ a zero-energy state. Despite this decomposition being unique, the vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ are not. Indeed, we can replace $|\psi_1\rangle$ by $|\psi_1\rangle + \mathfrak{Q}|\phi_1\rangle$, for any $|\phi_1\rangle$ and this does not affect the decomposition (1.38) as the supercharge is nilpotent. Similarly, modifying $|\psi_2\rangle$ to $|\psi_2\rangle + \mathfrak{Q}^\dagger|\phi_2\rangle$ leaves (1.38) unchanged.

## 1.4 Zero-energy states and cohomology

In this section, we focus on the eigenstates of the Hamiltonian with zero energy. As we shall see, their existence is related to a cohomological problem.

The first equation that defines a supersymmetry singlet (1.27) requires that a zero-energy state be in the kernel of the supercharge. We call its elements *cocycles*. Since $\mathfrak{Q}^2 = 0$, the kernel contains all states that are in the image of $\mathfrak{Q}$. We call the elements of the image of $\mathfrak{Q}$ *coboundaries*. The second equation of (1.27) leads to the following property of zero-energy states [67]:

**Lemma 1.4.1.** *A non-zero zero-energy state is not a coboundary.*

*Proof.* Suppose that we have a zero-energy state $|\psi\rangle \neq 0$ written as $|\psi\rangle = \mathfrak{Q}|\phi\rangle$ for some vector $|\phi\rangle$. Then $\mathfrak{Q}^\dagger|\psi\rangle = \mathfrak{Q}^\dagger\mathfrak{Q}|\phi\rangle = 0$, and we have

$$0 = \langle\phi|\mathfrak{Q}^\dagger\mathfrak{Q}|\phi\rangle = \|\mathfrak{Q}|\phi\rangle\|^2. \tag{1.39}$$

Hence, $|\psi\rangle = \mathfrak{Q}|\phi\rangle = 0$, which is a contradiction. □

A zero-energy state is a cocycle which is not in the image of $\mathfrak{Q}$. This suggests that the space of supersymmetry singlets could be related to the quotient of the kernel of the supercharge by its image. This is indeed the case. We introduce a few concepts from cohomology theory in order to explain this relation [73, 74].

The sequence of vector spaces $\mathscr{H}^f$ together with the set of supercharges $\mathfrak{Q} : \mathscr{H}^f \to \mathscr{H}^{f+1}$ defines a *cochain complex*, or *ascending complex*:

$$\cdots \longrightarrow \mathscr{H}^{f-1} \xrightarrow{\mathfrak{Q}} \mathscr{H}^f \xrightarrow{\mathfrak{Q}} \mathscr{H}^{f+1} \xrightarrow{\mathfrak{Q}} \mathscr{H}^{f+2} \longrightarrow \cdots, \tag{1.40}$$

where each $f$ takes its value in $\mathbb{F}$, the spectrum of F. We denote the cochain complex by $(\mathscr{H}^\bullet, \mathfrak{Q})$. If $F$ is bounded from below, then the vector spaces below $f = 0$ are 0, and the cochain complex is bounded from below:

$$0 \longrightarrow \mathscr{H}^0 \xrightarrow{\mathfrak{Q}} \mathscr{H}^1 \xrightarrow{\mathfrak{Q}} \cdots \xrightarrow{\mathfrak{Q}} \mathscr{H}^f \xrightarrow{\mathfrak{Q}} \mathscr{H}^{f+1} \xrightarrow{\mathfrak{Q}} \cdots. \tag{1.41}$$

If $F$ is bounded, then the cochain complex terminates: the vector spaces are 0 when their index is larger than $f_{max}$:

$$0 \longrightarrow \mathscr{H}^0 \xrightarrow{\mathfrak{Q}} \dots \xrightarrow{\mathfrak{Q}} \mathscr{H}^f \xrightarrow{\mathfrak{Q}} \mathscr{H}^{f+1} \xrightarrow{\mathfrak{Q}} \dots \xrightarrow{\mathfrak{Q}} \mathscr{H}^{f_{max}} \longrightarrow 0. \tag{1.42}$$

The cochain complex is said to be bounded.

**Quotient space.** We define for each $f$ the quotient space

$$\mathcal{H}^f = \frac{\ker\{\mathfrak{Q} : \mathscr{H}^f \to \mathscr{H}^{f+1}\}}{\mathrm{im}\{\mathfrak{Q} : \mathscr{H}^{f-1} \to \mathscr{H}^f\}}. \tag{1.43}$$

If $F$ is bounded from below, this definition holds for $f \geqslant 1$ and we define

$$\mathcal{H}^0 = \ker\{\mathfrak{Q} : \mathscr{H}^0 \to \mathscr{H}^1\}. \tag{1.44}$$

We note that if $F$ is bounded, $\mathcal{H}^{f_{max}}$ is well defined, as we imposed $\mathfrak{Q} : \mathscr{H}^{f_{max}} \to \mathscr{H}^{f_{max}+1} = 0$. The *cohomology* of the cochain complex $(\mathscr{H}^\bullet, \mathfrak{Q})$ is the direct sum

$$\mathcal{H}^\bullet = \bigoplus_{f \in \mathbb{F}} \mathcal{H}^f. \tag{1.45}$$

The elements of $\mathcal{H}^f(\mathfrak{Q})$ are equivalence classes of states belonging to the kernel of $\mathfrak{Q}$. We denote the equivalence class of $|\psi\rangle \in \ker(\mathfrak{Q})$ by $[|\psi\rangle]$, the state $|\psi\rangle$ is called a *representative* of $[|\psi\rangle]$. Two vectors are in the same equivalence class if they differ by a coboundary: we have $[|\psi\rangle] = [|\psi'\rangle]$ if and only if the relation

$$|\psi\rangle = |\psi'\rangle + \mathfrak{Q}|\phi\rangle \tag{1.46}$$

holds for a state $|\phi\rangle$.

These definitions allow us to elucidate the relation between the subspace of $\mathscr{H}^f$ spanned by the supersymmetry singlets and the quotient space $\mathcal{H}^f$.

**Proposition 1.4.2.** *The space of zero-energy states of $\mathscr{H}^f$ is isomorphic to the quotient space $\mathcal{H}^f$.*

*Proof.* We consider the quotient map restricted to the subspace of zero-energy states

$$[\,\cdot\,] : \{|\psi\rangle \in \mathscr{H}^f | H|\psi\rangle = 0\} \to \mathcal{H}^f, \quad |\psi\rangle \mapsto [|\psi\rangle]. \tag{1.47}$$

First, we observe that this map is surjective by construction. Indeed, for each equivalence class $[|\psi\rangle]$, there exists a pre-image $|\psi\rangle$ which is a zero-energy state.

Second, we prove that it is injective. Let $|\psi\rangle$ be a zero-energy state that is in the kernel of the mapping:

$$[|\psi\rangle] = [0]. \tag{1.48}$$

Then there exists a state $|\phi\rangle$ such that $|\psi\rangle = \mathfrak{Q}|\phi\rangle$. Since $|\psi\rangle$ is a zero-energy state, we have a contradiction due to Lemma 1.4.1, unless $|\psi\rangle = 0$.

The mapping is thus a bijection. □

This proposition teaches us that linearly independent singlets have distinct equivalence classes. As a direct corollary, the degeneracy of the zero eigenvalue of the Hamiltonian acting on $\mathscr{H}^f$ is the dimension of the quotient space $\mathcal{H}^f$. We use the Hodge decomposition to give a representation of a zero-energy state:

**Proposition 1.4.3.** *If the state $|\psi\rangle$ is a representative of a non-trivial element of $\mathcal{H}^f$, then there exists a state $|\phi\rangle$ such that*

$$|\psi_0\rangle = |\psi\rangle + \mathfrak{Q}|\phi\rangle \tag{1.49}$$

*is a zero-energy state.*

*Proof.* The proof follows from the Hodge decomposition of $|\psi\rangle$:

$$|\psi\rangle = |\psi_0\rangle + \mathfrak{Q}|\psi_1\rangle + \mathfrak{Q}^\dagger|\psi_2\rangle, \tag{1.50}$$

with $|\psi_0\rangle$ a zero-energy state. As $|\psi\rangle$ is a representative of a non-trivial element of $\mathcal{H}^f$, it is in the kernel of the supercharge. The application of $\mathfrak{Q}$ on $|\psi\rangle$ leads to

$$0 = \mathfrak{Q}|\psi\rangle = \mathfrak{Q}\mathfrak{Q}^\dagger|\psi_2\rangle. \tag{1.51}$$

Hence, by projection on $\langle\psi_2|$, we obtain $\|\mathfrak{Q}^\dagger|\psi_2\rangle\| = 0$ and $|\psi_0\rangle = |\psi\rangle + \mathfrak{Q}(-|\psi_1\rangle)$ is a zero-energy state. □

In the following, we refer to (1.49) as a cohomology decomposition of a zero-energy state. It means that if a zero-energy state exists, then we can write it as the sum of a representative of its equivalence class and a element in the image of the supercharge. We recall that the representative is an state which is in the kernel of the supercharge, but does not belong to its image. It is clear that the cohomology decomposition is not unique as it depends on the representative. Furthermore one can replace, for a given representative $|\psi\rangle$, the state $|\phi\rangle$ by $|\phi\rangle + \mathfrak{Q}|\phi'\rangle$ and leave the decomposition unchanged.

**Conjugation.** If we have two cochain complexes denoted by $(\mathscr{H}^\bullet, \mathfrak{Q})$ and $(\tilde{\mathscr{H}}^\bullet, \tilde{\mathfrak{Q}})$ and a set of invertible transformations, $C : \mathscr{H}^f \to \tilde{\mathscr{H}}^f$ such that the relation

$$C\mathfrak{Q} = \tilde{\mathfrak{Q}}C, \tag{1.52}$$

holds on $\mathscr{H}^f$ for each $f$, then we say that $\mathfrak{Q}$ and $\tilde{\mathfrak{Q}}$ are conjugate. The map $C$ is called a *conjugation* or a morphism of cochain complexes. Two conjugate supercharges have isomorphic cohomologies that we write $\mathcal{H}^f(\mathfrak{Q})$ and $\mathcal{H}^f(\tilde{\mathfrak{Q}})$ in order to stress their dependence on the supercharge [67]:

**Proposition 1.4.4.** *Let $\mathfrak{Q}$ and $\tilde{\mathfrak{Q}}$ be two conjugate supercharges, then $\mathcal{H}^f(\mathfrak{Q})$ is isomorphic to $\mathcal{H}^f(\tilde{\mathfrak{Q}})$. The conjugation $C$ induces the isomorphism $C^\sharp$:*

$$C^\sharp : \mathcal{H}^f(\mathfrak{Q}) \to \mathcal{H}^f(\tilde{\mathfrak{Q}}), C^\sharp[|\psi\rangle] = [C|\psi\rangle] \tag{1.53}$$

*Proof.* Let $|\psi\rangle \in \mathscr{H}^f$ be a representative of a zero-energy state. Then $C|\psi\rangle$ is a representative of a zero-energy state for $\tilde{\mathfrak{Q}}$: it is annihilated by $\tilde{\mathfrak{Q}}$. Furthermore, it cannot be in the image of $\tilde{\mathfrak{Q}}$. Otherwise, we have $\tilde{\mathfrak{Q}}|\phi\rangle = C|\psi\rangle$ for a certain $|\phi\rangle$, implying $|\psi\rangle = \mathfrak{Q}C^{-1}|\phi\rangle$. This contradicts the fact that $|\psi\rangle$ is a representative of a zero-energy state. □

If $C$ is a Hermitian involution $C^\dagger = C = C^{-1}$ then the conjugation of the supercharge and its adjoint form a new set of supercharges

$$\bar{\mathfrak{Q}} = C^{-1}\mathfrak{Q}C, \quad \bar{\mathfrak{Q}}^\dagger = C^{-1}\mathfrak{Q}^\dagger C$$

generating the Hamiltonian $\{\bar{\mathfrak{Q}}, \bar{\mathfrak{Q}}^\dagger\} = \bar{H} = C^{-1}HC$. If $C$ is a symmetry of the model, $[H, C] = 0$, then the two generated Hamiltonian are identical, $\bar{H} = H$.

**Homology.** In the definition of a supersymmetric Hamiltonian (1.9), there is an apparent symmetry between the supercharge and its adjoint. It suggests that we can reformulate the results of this section using $\mathfrak{Q}^\dagger$ instead of $\mathfrak{Q}$.

Starting from the equations satisfied by a supersymmetry singlet (1.27), we prove the following lemma similarly to Lemma 1.4.1.

**Lemma 1.4.5.** *A non-zero zero-energy state is not in the image of* $\mathfrak{Q}^\dagger$.

The preceding lemma suggests that the space of zero-energy states is related to the quotient of the kernel of $\mathfrak{Q}$ by its image. This observation naturally leads us to introduce concepts from homology theory. We define the *chain complex* or *descending complex*, $(\mathscr{H}_\bullet, \mathfrak{Q}^\dagger)$ as given by the sequence of vector spaces $\mathscr{H}^f$ and the set of adjoint supercharges $\mathfrak{Q}^\dagger : \mathscr{H}^f \to \mathscr{H}^{f-1}$,

$$\cdots \longleftarrow \mathscr{H}^{f-2} \xleftarrow{\mathfrak{Q}^\dagger} \mathscr{H}^{f-1} \xleftarrow{\mathfrak{Q}^\dagger} \mathscr{H}^f \xleftarrow{\mathfrak{Q}^\dagger} \mathscr{H}^{f+1} \longleftarrow \cdots, \tag{1.54}$$

with $f \in \mathbb{F}$. Similarly to the cochain complex case, if $F$ is bounded from below, then the vector spaces below $f = 0$ are 0, and the chain complex is also said to be bounded from below:

$$0 \longleftarrow \mathscr{H}^0 \xleftarrow{\mathfrak{Q}^\dagger} \mathscr{H}^1 \xleftarrow{\mathfrak{Q}^\dagger} \ldots \xleftarrow{\mathfrak{Q}^\dagger} \mathscr{H}^{f-1} \xleftarrow{\mathfrak{Q}^\dagger} \mathscr{H}^f \xleftarrow{\mathfrak{Q}^\dagger} \cdots. \tag{1.55}$$

If $F$ is bounded, the chain complex is also bounded, the vector spaces are 0 when their index is lower than 0 or larger than $f_{max}$:

$$0 \longleftarrow \mathscr{H}^0 \xleftarrow{\mathfrak{Q}^\dagger} \ldots \xleftarrow{\mathfrak{Q}^\dagger} \mathscr{H}^{f-1} \xleftarrow{\mathfrak{Q}^\dagger} \mathscr{H}^f \xleftarrow{\mathfrak{Q}^\dagger} \cdots \xleftarrow{\mathfrak{Q}^\dagger} \mathscr{H}^{f_{max}} \longleftarrow 0. \tag{1.56}$$

We define for each $f$ the quotient space

$$\mathcal{H}_f = \frac{\ker\{\mathfrak{Q}^\dagger : \mathscr{H}^f \to \mathscr{H}^{f-1}\}}{\operatorname{im}\{\mathfrak{Q}^\dagger : \mathscr{H}^{f+1} \to \mathscr{H}^f\}}. \tag{1.57}$$

With this definition, if $F$ is bounded from below, $\mathcal{H}_0$ is given by

$$\mathcal{H}_0 = \mathscr{H}^0 / \operatorname{im}\{\mathfrak{Q}^\dagger : \mathscr{H}^1 \to \mathscr{H}^0\} \tag{1.58}$$

whereas if $F$ is bounded above, we define

$$\mathcal{H}_{f_{max}} = \ker\{\mathfrak{Q}^\dagger : \mathscr{H}^{f_{max}} \to \mathscr{H}^{f_{max}-1}\}. \tag{1.59}$$

We call *homology* of the chain complex $(\mathscr{H}_\bullet, \mathfrak{Q}^\dagger)$ the direct sum

$$\mathcal{H}_\bullet = \bigoplus_{f\in\mathbb{F}} \mathcal{H}_f. \tag{1.60}$$

The elements of $\mathcal{H}_f$ are equivalence classes of states belonging to the kernel of $\mathfrak{Q}^\dagger$. They are represented by states $|\psi\rangle \in \ker(\mathfrak{Q}^\dagger)$. We keep the same notation for the elements of $\mathcal{H}^f$ and $\mathcal{H}_f$ and denote by $[|\psi\rangle]$ the equivalence class of such a state. Two vectors are in the same equivalence class if they differ by an element of the image of $\mathfrak{Q}^\dagger$: $[|\psi\rangle + \mathfrak{Q}^\dagger|\phi\rangle] = [|\psi\rangle]$.

As above, the space of supersymmetry singlets is isomorphic to the equivalence classes with respect to the image of the adjoint supercharge. We omit the proof, which is similar to the case for $\mathcal{H}^f$.

**Proposition 1.4.6.** *For each $f$, the space of zero-energy states of $\mathscr{H}^f$ is isomorphic to the quotient space $\mathcal{H}_f$.*

**Corollary 1.4.7.** *For each $f$, $\mathcal{H}^f$ is isomorphic to $\mathcal{H}_f$.*

We obtain a second representation of a zero-energy state, in terms of a representative of its equivalence class with respect to the image of $\mathfrak{Q}^\dagger$:

**Proposition 1.4.8.** *If the state $|\psi\rangle$ is a representative of a non-trivial element of $\mathcal{H}_f$, then there exists a state $|\phi\rangle$ such that*

$$|\psi_0\rangle = |\psi\rangle + \mathfrak{Q}^\dagger|\phi\rangle \tag{1.61}$$

*is a zero-energy state.*

*Proof.* The proof follows from the Hodge decomposition of $|\psi\rangle$:

$$|\psi\rangle = |\psi_0\rangle + \mathfrak{Q}|\psi_1\rangle + \mathfrak{Q}^\dagger|\psi_2\rangle, \tag{1.62}$$

with $|\psi_0\rangle$ a zero-energy state. As $|\psi\rangle$ is a representative of a non-trivial element of $\mathcal{H}_f$, it is in the kernel of the adjoint supercharge. The application of $\mathfrak{Q}^\dagger$ on $|\psi\rangle$ leads to

$$0 = \mathfrak{Q}^\dagger|\psi\rangle = \mathfrak{Q}^\dagger\mathfrak{Q}|\psi_1\rangle. \tag{1.63}$$

Hence, by projection on $\langle\psi_1|$, we obtain $\|\mathfrak{Q}|\psi_1\rangle\| = 0$ and $|\psi_0\rangle = |\psi\rangle + \mathfrak{Q}^\dagger(-|\psi_2\rangle)$ is a zero-energy state. □

We refer to (1.61) as the homology decomposition. As mentioned previously, this decomposition is not unique. In the following, we use the (co)homology decompositions to characterise the ground states and investigate their properties. In essence, the following proposition states that the knowledge of homology and cohomology representatives of a supersymmetry singlet allows us to compute its matrix elements with respect to operators with which the supercharge commutes:

**Proposition 1.4.9.** *Let $\mathcal{B}$ be an endomorphism of $\mathscr{H}^f$, for each $f$ that commutes with the supercharge according to*

$$\mathcal{B}\mathfrak{Q} = \lambda\mathfrak{Q}\mathcal{B}, \tag{1.64}$$

*where $\lambda$ is a non-zero complex number. Furthermore, let $|\psi_0\rangle$ be a zero-energy state whose decompositions (1.49) and (1.61) are $|\psi_0\rangle = |\psi\rangle + \mathfrak{Q}|\phi\rangle$ and $|\psi_0\rangle = |\psi'\rangle + \mathfrak{Q}^\dagger|\phi'\rangle$, respectively. Then we have*

$$\langle\psi_0|\mathcal{B}|\psi_0\rangle = \langle\psi'|\mathcal{B}|\psi\rangle. \tag{1.65}$$

*Proof.* The proof consists of a straightforward computation. First, we use (1.61) to write

$$\langle\psi_0|\mathcal{B}|\psi_0\rangle = \langle\psi'|\mathcal{B}|\psi_0\rangle + (\langle\phi'|\mathfrak{Q})\mathcal{B}|\psi_0\rangle = \langle\psi'|\mathcal{B}|\psi_0\rangle + \lambda^{-1}\langle\phi'|\mathcal{B}(\mathfrak{Q}|\psi_0\rangle). \tag{1.66}$$

The last term on the right-hand side vanishes, as $|\psi_0\rangle$ is annihilated by the supercharge.

Second, with the help of (1.49), we find

$$\begin{aligned} \langle\psi_0|\mathcal{B}|\psi_0\rangle = \langle\psi'|\mathcal{B}|\psi_0\rangle &= \langle\psi'|\mathcal{B}|\psi\rangle + \langle\psi'|\mathcal{B}\mathfrak{Q}|\phi\rangle & (1.67)\\ &= \langle\psi'|\mathcal{B}|\psi\rangle + \lambda\left(\langle\psi'|\mathfrak{Q}\mathcal{B}\right)|\phi\rangle. & (1.68)\end{aligned}$$

Because of $\mathfrak{Q}^\dagger|\psi'\rangle = 0$, we conclude that the second term on the right-hand side equals zero. This leads to (1.65). □

# Chapter 2

# XYZ spin chain: periodic boundary conditions

In this chapter, we consider the XYZ spin chain Hamiltonian with periodic boundary conditions. As explained in the introduction, it is a model of spins interacting on a one-dimensional lattice that we call a chain. This model is characterised by three so-called anisotropy parameters, denoted $J_1, J_2$ and $J_3$. We focus on the case where they obey the relation

$$J_1 J_2 + J_2 J_3 + J_1 J_3 = 0. \tag{2.1}$$

The XYZ model with anisotropy parameter satisfying (2.1) has been studied because of the remarkable combinatorial properties of its ground state [54, 56, 57, 75]. Furthermore, for this choice of parameters, the Hamiltonian possesses a lattice supersymmetry [76]. We revisit this supersymmetry and apply the (co)homology methods presented in Chapter 1 to characterise the space of the ground state of this Hamiltonian [77].

The layout of this chapter is as follows. In Section 2.1, we introduce the XYZ spin-chain Hamiltonian and recall a few of its symmetries. We show in Section 2.2 that, for a specific choice of anisotropy parameters, this Hamiltonian is supersymmetric and provide the construction of the corresponding supercharges. We discuss the existence of the supersymmetry singlets of the Hamiltonian in Section 2.3 by analysing the (co)homology of the supercharges. In Section 2.4, we characterise the supersymmetry singlets and prove that they span the space of the spin chain's ground states.

## 2.1 The XYZ Hamiltonian and its symmetries

**Hilbert Space.** The XYZ spin chain that we describe is a model of interacting spins 1/2. We use the notation $V = \mathbb{C}^2$ for the Hilbert space of a single spin. The Hilbert space of the spin chain with $L$ sites is given by

$$V^L = V_1 \otimes V_2 \otimes \cdots \otimes V_L, \tag{2.2}$$

where $V_j = V$ is a copy of the single-spin Hilbert space associated to the site $j$. We refer to the number of sites of the chain, $L$, as the length of the chain. In the vocabulary of Chapter 1, the length of the chain plays the role of the grading operator eigenvalue that we denoted by $f$. It is bounded from below, with minimal value $L = 1$.

A basis of the Hilbert space $V$ is

$$|{\uparrow}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |{\downarrow}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.3}$$

The canonical orthonormal basis of $V^L$ is given by the set of all states

$$|s_1 s_2 \cdots s_L\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_L\rangle, \tag{2.4}$$

where each $s_j$ is either $\uparrow$ (spin up) or $\downarrow$ (spin down).

The spin operators on $V^2$ are given by the standard Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

We denote by $\sigma_j^\alpha$, $\alpha = 1, 2, 3$ and $j = 1, \ldots, L$, the matrix $\sigma^\alpha$ acting on the $j$-th factor of the tensor product (2.2):

$$\sigma_j^\alpha = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{j-1} \otimes \sigma^\alpha \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{L-j}. \tag{2.6}$$

**XYZ Hamiltonian.** For $L \geqslant 2$ sites and periodic boundary conditions, the spin chain Hamiltonian is given by

$$H_{\text{XYZ}} = \sum_{j=1}^{L} h_{jj+1}^{\text{XYZ}}, \tag{2.7a}$$

where $h_{jj+1}^{\text{XYZ}}$ is the local XYZ Hamiltonian, defined by

$$h_{jj+1}^{\text{XYZ}} = -\frac{1}{2}\left(J_1\sigma_j^1\sigma_{j+1}^1 + J_2\sigma_j^2\sigma_{j+1}^2 + J_3\sigma_j^3\sigma_{j+1}^3\right). \tag{2.7b}$$

The periodic boundary conditions impose

$$\sigma_{L+1}^{\alpha} = \sigma_1^{\alpha}, \quad \alpha = 1, 2, 3. \tag{2.8}$$

The real constants $J_1, J_2, J_3$ are the spin chain's anisotropy parameters. This Hamiltonian is Hermitian and therefore, diagonalisable. Below, we focus on certain special eigenstates. The analysis of these eigenstates uses a few simple symmetries of the Hamiltonian that we discuss now.

**Symmetry operators.** We start this discussion by considering its invariance under translations. The translation operator $\mathcal{S}$ acts on the basis of $V^L$ according to

$$\mathcal{S}|s_1 \cdots s_{L-1}s_L\rangle = |s_L s_1 \cdots s_{L-1}\rangle. \tag{2.9}$$

The translation invariance of the Hamiltonian is expressed through the commutation relation

$$[H_{\text{XYZ}}, \mathcal{S}] = 0. \tag{2.10}$$

This follows from the relation

$$h_{jj+1}^{\text{XYZ}} = \mathcal{S} h_{j-1j}^{\text{XYZ}} \mathcal{S}^{-1}, \tag{2.11}$$

which holds for each $j = 2, \ldots, L$. The operator $\mathcal{S}$ is unitary. Therefore, it is diagonalisable. The Hilbert space $V^L$ is the direct sum of the corresponding eigenspaces. In the following, we will be particularly interested in the eigenstates of $\mathcal{S}$ with eigenvalue $(-1)^{L+1}$. We follow the terminology of [78] and call them *alternate-cyclic states*. We denote by $W^L$ the corresponding eigenspace.

Furthermore, we note that the Hamiltonian preserves the spin parity:

$$[H_{\text{XYZ}}, \mathcal{P}] = 0, \quad \mathcal{P} = (-1)^L \sigma_1^3 \sigma_2^3 \cdots \sigma_L^3. \tag{2.12}$$

Each basis state (2.4) is an eigenstate of the spin-parity operator $\mathcal{P}$. The corresponding eigenvalue is the parity of the number of spins up. The spin-parity invariance of the Hamiltonian allows one to look for eigenstates of $H_{\text{XYZ}}$ in sectors of $V^L$ where this parity is fixed to $+1$ or $-1$.

Finally, the Hamiltonian is invariant under spin reversal:

$$[H_{\text{XYZ}}, \mathcal{R}] = 0, \quad \mathcal{R} = \sigma_1^1 \sigma_2^1 \cdots \sigma_L^1. \tag{2.13}$$

The spin-parity and spin-reversal operators have the commutation relation $\mathcal{R}\mathcal{P} = (-1)^L \mathcal{P}\mathcal{R}$. For odd $L$, this anticommutation relation implies that each eigenvalue of $H_{\text{XYZ}}$ has an even degeneracy. Indeed, if $|\psi\rangle$ satisfies $H_{\text{XYZ}}|\psi\rangle = E|\psi\rangle$ and is of parity $+1$, $\mathcal{P}|\psi\rangle = |\psi\rangle$, then the state $\mathcal{R}|\psi\rangle$ is also an eigenstate of $H_{\text{XYZ}}$ with the same eigenvalue $E$ but has parity $(-1)^L = -1$ if $L$ is odd. As they pertain to eigenspaces of $\mathcal{R}$ with different eigenvalue, they are orthogonal to each other.

## 2.2 Lattice supersymmetry

From now on, we focus on the case where the anisotropy parameters are given by

$$J_1 = 1 + \zeta, \quad J_2 = 1 - \zeta, \quad J_3 = \frac{1}{2}(\zeta^2 - 1), \tag{2.14}$$

where $\zeta$ is a non-zero real parameter. Up to an irrelevant multiplicative factor, this is the most general solution to the relation (2.1). We show that for this choice, the XYZ Hamiltonian possesses a lattice supersymmetry on the subspace of alternate-cyclic states [76, 77]. We denote the corresponding supercharges and its adjoint by $\mathfrak{Q}$ and $\mathfrak{Q}^\dagger$, respectively.

**Supercharges.** We construct the supercharge $\mathfrak{Q}$ from an operator $\mathfrak{q}$ that we call the *local supercharge*. Its action on the basis states of the single-spin Hilbert space is given by [76]

$$\mathfrak{q}|\uparrow\rangle = 0, \quad \mathfrak{q}|\downarrow\rangle = |\uparrow\uparrow\rangle - \zeta|\downarrow\downarrow\rangle. \tag{2.15}$$

Using $\mathfrak{q}$, we define local operators $\mathfrak{q}_0, \mathfrak{q}_1, \dots, \mathfrak{q}_L$ that map the Hilbert space of a chain of length $L$ to the Hilbert space of a chain of length $L+1$. For $j = 1, \dots, L$, we set

$$\mathfrak{q}_j = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{j-1} \otimes \mathfrak{q} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{L-j}. \tag{2.16a}$$

Furthermore, we define

$$\mathfrak{q}_0 = \mathcal{S}^{-1} \mathfrak{q}_1 \mathcal{S} = \mathcal{S} \mathfrak{q}_L. \tag{2.16b}$$

Notice that in $\mathcal{S}^{-1}\mathfrak{q}_1\mathcal{S}$, the translation operators to the right and the left of $\mathfrak{q}_1$ act on $V^L$ and $V^{L+1}$, respectively.

The supercharge $\mathfrak{Q}$ is a length-increasing operator that maps $V^L$ to $V^{L+1}$ for each $L \geqslant 1$. We define it through its action on the eigenspaces of the translation operator $\mathcal{S}$ in $V^L$. On the eigenspace of alternate-cyclic states $W^L$, the supercharge acts as the alternating sum

$$\mathfrak{Q} = \sqrt{\frac{L}{L+1}} \sum_{j=0}^{L} (-1)^j \mathfrak{q}_j. \tag{2.17}$$

On every other eigenspace of the translation operator, we define the supercharge to be zero. One checks [77] that the supercharge maps $W^L$ to $W^{L+1}$ as the following anticommutation relation holds:

$$\mathcal{S}\mathfrak{Q} = -\mathfrak{Q}\mathcal{S}. \tag{2.18}$$

We define the adjoint of the supercharge $\mathfrak{Q}^\dagger$ by means of the scalar product of the spin-chain Hilbert space. It satisfies

$$\langle \psi | \mathfrak{Q}^\dagger | \phi \rangle = \langle \phi | \mathfrak{Q} | \psi \rangle^* \tag{2.19}$$

for all $|\phi\rangle \in V^L$, $|\psi\rangle \in V^{L-1}$, $L \geqslant 2$. It follows from this definition that the action of the adjoint supercharge on the eigenspaces of the translation operator $\mathcal{S}$ in $V^L$ is non-zero only on $W^L$. Furthermore, $\mathfrak{Q}^\dagger$ maps $W^L$ to $W^{L-1}$.

The key feature of this supersymmetry is that $\mathfrak{Q}$ and $\mathfrak{Q}^\dagger$ map $V^L$ into $V^{L+1}$ and $V^{L-1}$, respectively. As it does not preserve the length of the chain $L$, the supersymmetry is sometimes referred to as *dynamic* [79, 77].

**Nilpotency.** One can show that the supercharge and its adjoint are nilpotent operators [76]:

**Proposition 2.2.1.** *The supercharge and its adjoint satisfy*

$$\mathfrak{Q}^2 = 0, \quad (\mathfrak{Q}^\dagger)^2 = 0. \tag{2.20}$$

*This means that the operators* $\mathfrak{Q}^2 : V^L \to V^{L+2}$, $L \geqslant 1$, *and* $(\mathfrak{Q}^\dagger)^2 : V^L \to V^{L-2}$, $L \geqslant 3$, *yield zero on every state of* $V^L$.

*Proof.* From the definition of the adjoint, it is sufficient to prove the statement for the supercharge. The nilpotency is trivial on the subspace of $V^L$ spanned by the states that are not alternate-cyclic. Hence, we focus on the action of $\mathfrak{Q}^2$ on $W^L$. From the action of the local supercharge $\mathfrak{q}_j$ (2.16), we have

$$\mathfrak{q}_i \mathfrak{q}_j = \mathfrak{q}_{j+1}\mathfrak{q}_i \tag{2.21}$$

for each $0 \leqslant i < j \leqslant N$, apart from the case $i = 0, j = N$. This observation leads to

$$\sqrt{\frac{L+2}{L}}\mathfrak{Q}^2 = (\mathfrak{q}_0 - \mathfrak{q}_1)\mathfrak{q}_0 + \sum_{j=1}^{L}(\mathfrak{q}_j - \mathfrak{q}_{j+1})\mathfrak{q}_j + (-1)^L(\mathfrak{q}_0\mathfrak{q}_L - \mathfrak{q}_{L+1}\mathfrak{q}_0). \tag{2.22}$$

We rewrite the third term in the right-hand side of the equation using the definition of $\mathfrak{q}_0$ (2.16):

$$(-1)^L(\mathfrak{q}_0\mathfrak{q}_L - \mathfrak{q}_{L+1}\mathfrak{q}_0) = (-1)^{L+1}\mathcal{S}(\mathfrak{q}_L - \mathfrak{q}_{L+1})\mathfrak{q}_L. \tag{2.23}$$

We observe that the local supercharge has the following property: for each $|\psi\rangle \in V$,

$$(\mathfrak{q} \otimes \mathbf{1} - \mathbf{1} \otimes \mathfrak{q})\mathfrak{q}|\psi\rangle = |\chi\rangle \otimes |\psi\rangle - |\psi\rangle \otimes |\chi\rangle, \tag{2.24}$$

where $|\chi\rangle = -\zeta|\uparrow\uparrow\rangle$ belongs to $V^2$. We now apply (2.22) on a state $|\Psi\rangle \in W^L$. We separately consider the action of each term in the right-hand side of (2.22) on $|\Psi\rangle$ and find

$$\begin{aligned}
(\mathfrak{q}_0 - \mathfrak{q}_1)\mathfrak{q}_0|\Psi\rangle &= (-1)^{L+1}\mathcal{S}^{-1}\left(|\chi\rangle \otimes |\Psi\rangle\right) - |\chi\rangle \otimes |\Psi\rangle,\\
\sum_{j=1}^{L}(\mathfrak{q}_j - \mathfrak{q}_{j+1})\mathfrak{q}_j|\Psi\rangle &= |\chi\rangle \otimes |\Psi\rangle - |\Psi\rangle \otimes |\chi\rangle,\\
(-1)^{L+1}\mathcal{S}(\mathfrak{q}_L - \mathfrak{q}_{L+1})\mathfrak{q}_L|\Psi\rangle &= |\Psi\rangle \otimes |\chi\rangle - (-1)^{L+1}\mathcal{S}\left(|\Psi\rangle \otimes |\chi\rangle\right).
\end{aligned}$$

Here, the second equation follows from a telescopic cancellation, and we used $\mathcal{S}|\Psi\rangle = (-1)^{L+1}|\Psi\rangle$ to simplify the first and third equations. The sum of these three contributions is zero. This proves the nilpotency of $\mathfrak{Q}$. □

**Hamiltonian.** The supercharge and its adjoint allow us to define the Hamiltonian

$$H = \mathfrak{Q}\mathfrak{Q}^\dagger + \mathfrak{Q}^\dagger\mathfrak{Q}. \tag{2.25}$$

This Hamiltonian is a length-preserving operator, unlike $\mathfrak{Q}$ and $\mathfrak{Q}^\dagger$.

It follows from their definition that the action of $H$ yields zero on all eigenspaces of the translation operator in $V^L$ that are not equal to the subspace of alternate-cyclic states $W^L$. However, the restriction of $H$ to $W^L$ is non-trivial [76].

Using the supercharge (2.17) we find that the Hamiltonian (2.25) is a sum of terms describing nearest-neighbour interactions:

$$H = \sum_{j=1}^{L} h_{jj+1}. \tag{2.26}$$

Here, $h_{jj+1}$ denotes the Hamiltonian density $h : V^2 \to V^2$, acting on the sites $j$ and $j+1$. In terms of the local supercharge $\mathfrak{q}$ it is given by

$$h = -(\mathbf{1}\otimes\mathfrak{q}^\dagger)(\mathfrak{q}\otimes\mathbf{1}) - (\mathfrak{q}^\dagger\otimes\mathbf{1})(\mathbf{1}\otimes\mathfrak{q}) + \mathfrak{q}\mathfrak{q}^\dagger + \frac{1}{2}\left(\mathfrak{q}^\dagger\mathfrak{q}\otimes\mathbf{1} + \mathbf{1}\otimes\mathfrak{q}^\dagger\mathfrak{q}\right). \tag{2.27}$$

Up to a multiple of the identity matrix, the anticommutator (2.25) is equal to the Hamiltonian of the XYZ spin chain (2.7) with particular anisotropy parameters.

**Proposition 2.2.2.** *For each $L \geqslant 2$, we have*

$$H = H_{\text{XYZ}} - E_0 \quad \text{on } W^L, \tag{2.28}$$

*provided that $J_1, J_2, J_3$ are given by (2.14) and $E_0$ is set to*

$$E_0 = -\frac{L}{4}(3+\zeta^2). \tag{2.29}$$

*Proof.* The proof follows from the calculation of the Hamiltonian density $h$. On the subspace $W^L$, we have

$$h = h^{\text{XYZ}} + \frac{3+\zeta^2}{4}, \tag{2.30}$$

where $J_1, J_2, J_3$ satisfy (2.14). We perform the sum (2.26) and obtain (2.28). $\square$

The relation (2.28) between $H$ and the XYZ Hamiltonian implies that $H$ commutes with the spin-parity and spin-reversal operators $[H, \mathcal{P}] = [H, \mathcal{R}] = 0$. Because of (2.28), we conclude that the restriction of the XYZ Hamiltonian to $W^L$ with the anisotropy parameters (2.14) has a dynamic lattice supersymmetry.[1]

## 2.3 (Co)homology and conjugation

In the following, we are interested in the supersymmetry singlets of the Hamiltonian (2.25). As seen in Chapter 1, they are the solutions of the two equations:

$$\mathfrak{Q}|\Psi\rangle = 0, \quad \mathfrak{Q}^\dagger|\Psi\rangle = 0. \tag{2.31}$$

It follows from the definition of $\mathfrak{Q}$ and $\mathfrak{Q}^\dagger$ that these equations have many trivial solutions. Indeed, all eigenstates of the translation operator $\mathcal{S}$ that are not alternate-cyclic are zero-energy states. In the following, we focus on the alternate-cyclic zero-energy states. If they exist, then they are the ground states of the XYZ Hamiltonian with the anisotropy parameters (2.14), restricted to $W^L$. The corresponding eigenvalue is $E_0 = -L(3+\zeta^2)/4$.

Proving the absence or the existence of alternate-cyclic zero-energy states is a non-trivial problem except for a few special values of the parameter $\zeta$. One such special value is $\zeta = 1$, where the Hamiltonian reduces to

$$H = \sum_{j=1}^{L} (\mathbf{1} - \sigma_j^1 \sigma_{j+1}^1) \quad \text{on} \quad W^L. \tag{2.32}$$

[1]Previous numerical investigations and a Bethe-ansatz analysis [76, 78] suggest that there is no other eigenspace of the translation operator on which the spin-chain Hamiltonian is supersymmetric.

Its diagonalisation is elementary. It reveals that $H$ possesses no alternate-cyclic zero-energy states for even $L = 2n$. For odd $L = 2n+1$, however, the subspace of zero-energy states in $W^L$ is two-dimensional. One basis of this eigenspace is given by

$$|\Phi_n\rangle = \frac{1}{2}(\mathbf{1} + \mathcal{P}) \sum_{s_1=\uparrow,\downarrow} \cdots \sum_{s_{2n+1}=\uparrow,\downarrow} |s_1 \cdots s_{2n+1}\rangle, \tag{2.33a}$$

$$|\bar{\Phi}_n\rangle = \frac{1}{2}(\mathbf{1} - \mathcal{P}) \sum_{s_1=\uparrow,\downarrow} \cdots \sum_{s_{2n+1}=\uparrow,\downarrow} |s_1 \cdots s_{2n+1}\rangle. \tag{2.33b}$$

These basis states have a definite spin parity and can be mapped onto each other through spin reversal:

$$\mathcal{P}|\Phi_n\rangle = |\Phi_n\rangle, \quad \mathcal{P}|\bar{\Phi}_n\rangle = -|\bar{\Phi}_n\rangle, \quad \mathcal{R}|\Phi_n\rangle = |\bar{\Phi}_n\rangle. \tag{2.34}$$

For generic values of $\zeta$, the explicit diagonalisation of the Hamiltonian $H$ is non-trivial. Nonetheless, it is possible to prove the absence or the existence of alternate-cyclic zero-energy states by means of the supersymmetry. A proof was already given in [77]. For completeness, we revisit this proof and extend it here below.

**(Co)homology.** As we have seen in Chapter 1, the zero-energy states are in bijection with the elements of the kernel of the supercharge or its adjoint modulo their respective image. The quotient space (1.43) is defined for each $L \geqslant 2$, by

$$\mathcal{H}^L(\zeta) = \frac{\ker\{\mathfrak{Q}(\zeta) : W^L \to W^{L+1}\}}{\operatorname{im}\{\mathfrak{Q}(\zeta) : W^{L-1} \to W^L\}}, \tag{2.35}$$

where we wrote $\mathfrak{Q}(\zeta), \mathcal{H}^L(\zeta)$ for the supercharge and the quotient space in order to stress their dependence on the parameter $\zeta$. We recall that the elements of $\mathcal{H}^L(\zeta)$ are equivalence classes that we denote by $[|\psi\rangle]$, with $|\psi\rangle \in \ker\{\mathfrak{Q}(\zeta) : W^L \to W^{L+1}\}$ a representative, $[|\psi\rangle + \mathfrak{Q}(\zeta)|\phi\rangle] = [|\psi\rangle]$. Accordingly, the quotient space (1.57) is

$$\mathcal{H}_L(\zeta) = \frac{\ker\{\mathfrak{Q}(\zeta)^\dagger : W^L \to W^{L-1}\}}{\operatorname{im}\{\mathfrak{Q}(\zeta)^\dagger : W^{L+1} \to W^L\}} \tag{2.36}$$

for each $L \geqslant 2$. A state $|\psi\rangle \in \ker\{\mathfrak{Q}(\zeta)^\dagger : W^L \to W^{L-1}\}$ is a representative of the equivalence class $[|\psi\rangle]$.

To prove the (non-)existence of alternate-cyclic zero-energy states of the Hamiltonian $H$ for $L$ sites, it is sufficient to find $\mathcal{H}^L(\zeta)$ or $\mathcal{H}_L(\zeta)$. We now compute these spaces for each $L \geqslant 2$.

Let us consider the case where $\zeta = 1$. The explicit diagonalisation of the Hamiltonian shows that

$$\mathcal{H}^{2n}(1) = 0, \quad \mathcal{H}^{2n+1}(1) = \mathbb{C}[|\Phi_n\rangle] \oplus \mathbb{C}[|\bar{\Phi}_n\rangle], \tag{2.37a}$$
$$\mathcal{H}_{2n}(1) = 0, \quad \mathcal{H}_{2n+1}(1) = \mathbb{C}[|\Phi_n\rangle] \oplus \mathbb{C}[|\bar{\Phi}_n\rangle], \tag{2.37b}$$

for each $n \geqslant 1$, where $|\Phi_n\rangle$, $|\bar{\Phi}_n\rangle$ are the states defined in (2.33).

**Conjugation.** The corresponding results for non-zero values of $\zeta$ can be inferred from (2.37). To this end, we introduce an operator $m(\lambda)$ whose action on the basis states of the single-spin Hilbert space is given by

$$m(\lambda)|{\uparrow}\rangle = \lambda|{\uparrow}\rangle, \quad m(\lambda)|{\downarrow}\rangle = \lambda^2|{\downarrow}\rangle. \tag{2.38}$$

The operator $m(\lambda)$ and the local supercharge $\mathfrak{q} = \mathfrak{q}(\zeta)$ satisfy the relation

$$(m(\lambda) \otimes m(\lambda))\, \mathfrak{q}(\lambda^{-2}\zeta) = \mathfrak{q}(\zeta)m(\lambda). \tag{2.39}$$

On $V^L$, we define the operator $\mathcal{M}(\lambda) = m_1(\lambda)m_2(\lambda)\cdots m_L(\lambda)$ where $m_j(\lambda)$ is $m(\lambda)$ acting on the $j$-th factor of the tensor product (2.4). $\mathcal{M}(\lambda)$ preserves $W^L$ and is invertible for $\lambda \neq 0$: $\mathcal{M}(\lambda)^{-1} = \mathcal{M}(\lambda^{-1})$. For non-zero $\lambda$, the operator $\mathcal{M}(\lambda)$ is a conjugation between $\mathfrak{Q}(\zeta)$ and $\mathfrak{Q}(\lambda^{-2}\zeta)$. Indeed, it follows from (2.39) that

$$\mathcal{M}(\lambda)\mathfrak{Q}(\lambda^{-2}\zeta) = \mathfrak{Q}(\zeta)\mathcal{M}(\lambda), \tag{2.40a}$$
$$\mathcal{M}(\lambda^{-1})\mathfrak{Q}(\lambda^{-2}\zeta)^\dagger = \mathfrak{Q}(\zeta)^\dagger\mathcal{M}(\lambda^{-1}). \tag{2.40b}$$

By Proposition 1.4.4, this conjugation property implies that the following mappings are bijections:

$$\mathcal{M}^\sharp(\lambda) : \mathcal{H}^L(\lambda^{-2}\zeta) \to \mathcal{H}^L(\zeta), \quad \mathcal{M}^\sharp(\lambda)[|\Phi\rangle] = [\mathcal{M}(\lambda)|\Phi\rangle], \tag{2.41a}$$
$$\mathcal{M}_\sharp(\lambda) : \mathcal{H}_L(\lambda^{-2}\zeta) \to \mathcal{H}_L(\zeta), \quad \mathcal{M}_\sharp(\lambda)[|\Phi'\rangle] = [\mathcal{M}(\lambda^{-1})|\Phi'\rangle]. \tag{2.41b}$$

The existence of these bijections was observed in [77]. It implies that $\dim \mathcal{H}^L(\zeta) = \dim \mathcal{H}^L(\lambda^{-2}\zeta)$ and $\dim \mathcal{H}_L(\zeta) = \dim \mathcal{H}_L(\lambda^{-2}\zeta)$ for each $L \geqslant 1$. This allows one to compute the dimension of the space of alternate-cyclic zero-energy states as a function of the number of sites. Here, we extend the work of [77] and use the bijections to explicitly compute a basis of $\mathcal{H}^L(\zeta)$ and $\mathcal{H}_L(\zeta)$ for non-zero $\zeta$. For $\zeta > 0$, we introduce the states

$$|\Phi_n(\zeta)\rangle = \zeta^{-(n+1)}\mathcal{M}(\zeta^{1/2})|\Phi_n\rangle, \quad |\bar{\Phi}_n(\zeta)\rangle = \zeta^{-(n+1/2)}\mathcal{M}(\zeta^{1/2})|\bar{\Phi}_n\rangle. \tag{2.42}$$

These states are polynomials in $\zeta$. Furthermore, we infer from (2.40) that they satisfy

$$\mathfrak{Q}(\zeta)|\Phi_n(\zeta)\rangle = 0, \qquad \mathfrak{Q}(\zeta)|\bar{\Phi}_n(\zeta)\rangle = 0, \tag{2.43a}$$
$$\mathfrak{Q}^\dagger(\zeta)|\Phi_n(\zeta^{-1})\rangle = 0, \quad \mathfrak{Q}^\dagger(\zeta)|\bar{\Phi}_n(\zeta^{-1})\rangle = 0. \tag{2.43b}$$

It follows from (2.41) that for $\zeta > 0$ we have

$$\mathcal{H}^{2n}(\zeta) = 0, \quad \mathcal{H}^{2n+1}(\zeta) = \mathbb{C}[|\Phi_n(\zeta)\rangle] \oplus \mathbb{C}[|\bar{\Phi}_n(\zeta)\rangle], \tag{2.44a}$$
$$\mathcal{H}_{2n}(\zeta) = 0, \quad \mathcal{H}_{2n+1}(\zeta) = \mathbb{C}[|\Phi_n(\zeta^{-1})\rangle] \oplus \mathbb{C}[|\bar{\Phi}_n(\zeta^{-1})\rangle]. \tag{2.44b}$$

The polynomiality of the states defined in (2.42) allows us to extend these relations to non-zero values of $\zeta$.

Our construction of $\mathcal{H}^L(\zeta)$ and $\mathcal{H}_L(\zeta)$ clearly fails if $\zeta = 0$ (which is the reason for requiring that $\zeta$ be non-zero). Indeed, in this case, the conjugation relation (2.40) implies that the supercharges commute with the operator $\mathcal{M}(\lambda)$ for any $\lambda$. However, the commutation relation does not allow us to establish a relation between $\mathcal{H}^L(0)$ and $\mathcal{H}^L(1)$, nor between $\mathcal{H}_L(0)$ and $\mathcal{H}_L(1)$. We discuss the case $\zeta = 0$ at the end of this chapter.

## 2.4 Zero-energy states

We now use (2.44) in order to characterise the space of alternate-cyclic zero-energy states of the Hamiltonian $H$.

**Theorem 2.4.1.** *For each $n \geqslant 1$, the Hamiltonian (2.25) with $L = 2n$ does not possess alternate-cyclic zero-energy states. If $L = 2n + 1$, then the space of alternate-cyclic zero-energy states is spanned by*

$$|\Psi_n\rangle = |\Phi_n(\zeta)\rangle + \mathfrak{Q}|\alpha_n\rangle, \quad |\bar{\Psi}_n\rangle = |\bar{\Phi}_n(\zeta)\rangle + \mathfrak{Q}|\bar{\alpha}_n\rangle, \tag{2.45}$$

*where $|\alpha_n\rangle, |\bar{\alpha}_n\rangle \in W^{2n}$.*

*Proof.* The absence and existence of the alternate-cyclic zero-energy states in $W^{2n}$ and $W^{2n+1}$, respectively, follow from (2.44). If $L = 2n+1$, then the decompositions (2.45) are a consequence of the cohomology decomposition (1.49) of a supersymmetry singlet. □

In the following, whenever we write $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$, we refer to the two states defined in (2.45). We now show that for non-zero $\zeta$ these states have the same spin parity and transformation behaviour under spin reversal (2.34) as for $\zeta = 1$.

**Proposition 2.4.2.** *For each $n \geqslant 1$, the alternate-cyclic zero-energy states defined in (2.45) satisfy*

$$\mathcal{P}|\Psi_n\rangle = +|\Psi_n\rangle, \quad \mathcal{P}|\bar{\Psi}_n\rangle = -|\bar{\Psi}_n\rangle. \tag{2.46}$$

*Furthermore, there exists a non-zero complex number $\rho_n$ such that the relation $\mathcal{R}|\Psi_n\rangle = \rho_n|\bar{\Psi}_n\rangle$ holds.*

*Proof.* First, we consider the action of the spin-parity operator on the zero-energy states. To this end, we notice that this operator anticommutes with the supercharge

$$\mathfrak{Q}\mathcal{P} + \mathcal{P}\mathfrak{Q} = 0. \tag{2.47}$$

This follows from the definition of the local supercharge (2.15). We use this relation to show that $\mathcal{P}|\Psi_n\rangle = +|\Psi_n\rangle$. A short calculation leads to

$$\mathcal{P}|\Psi_n\rangle - |\Psi_n\rangle = -\mathfrak{Q}(\mathcal{P} + \mathbf{1})|\alpha_n\rangle, \tag{2.48}$$

where we used that $\mathcal{P}|\Phi_n(\zeta)\rangle = +|\Phi_n(\zeta)\rangle$. Since the Hamiltonian $H$ commutes with the spin-parity operator $\mathcal{P}$, the left-hand side of this equality is a zero-energy state. If it does not vanish, then it is a non-zero zero-energy state which is in the image of $\mathfrak{Q}$. Lemma 1.4.1 states that this is not possible. Hence, both sides have to vanish. This leads to the desired result. The proof of $\mathcal{P}|\bar{\Psi}_n\rangle = -|\bar{\Psi}_n\rangle$ is similar.

Second, the states $|\Psi_n\rangle$, $|\bar{\Psi}_n\rangle$ have thus opposite spin parity and span a two-dimensional eigenspace of the Hamiltonian. The Hamiltonian commutes with the spin-reversal operator. We conclude that $\mathcal{R}|\Psi_n\rangle = \rho_n|\bar{\Psi}_n\rangle$ and $\mathcal{R}|\bar{\Psi}_n\rangle = \rho_n^{-1}|\Psi_n\rangle$ for a non-vanishing complex number $\rho_n$. □

**Theorem 2.4.3.** *For each $n \geqslant 1$, the alternate-cyclic zero-energy states $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$ can be written as*

$$|\Psi_n\rangle = \mu_n|\Phi_n(\zeta^{-1})\rangle + \mathfrak{Q}^\dagger|\beta_n\rangle, \quad |\bar{\Psi}_n\rangle = \bar{\mu}_n|\bar{\Phi}_n(\zeta^{-1})\rangle + \mathfrak{Q}^\dagger|\bar{\beta}_n\rangle, \tag{2.49}$$

*where $|\beta_n\rangle, |\bar{\beta}_n\rangle \in W^{2(n+1)}$. The constants $\mu_n$ and $\bar{\mu}_n$ are non-zero and given by*

$$\mu_n = \frac{1}{4^n}\langle\Phi_n(\zeta)|\Psi_n\rangle, \quad \bar{\mu}_n = \frac{1}{4^n}\langle\bar{\Phi}_n(\zeta)|\bar{\Psi}_n\rangle. \tag{2.50}$$

*Proof.* We focus on the state $|\Psi_n\rangle$. It follows from the homology decomposition (1.61) and from (2.44) that there are constants $\mu_n$, $\nu_n$ and a state $|\beta_n\rangle \in W^{2(n+1)}$ such that

$$|\Psi_n\rangle = \mu_n|\Phi_n(\zeta^{-1})\rangle + \nu_n|\bar{\Phi}_n(\zeta^{-1})\rangle + \mathfrak{Q}^\dagger|\beta_n\rangle. \tag{2.51}$$

We act on both sides of this equality with the spin-parity operator and find

$$|\Psi_n\rangle = \mu_n|\Phi_n(\zeta^{-1})\rangle - \nu_n|\bar{\Phi}_n(\zeta^{-1})\rangle - \mathfrak{Q}^\dagger\mathcal{P}|\beta_n\rangle. \tag{2.52}$$

The difference of these two equalities leads to

$$2\nu_n|\bar{\Phi}_n(\zeta^{-1})\rangle = -\mathfrak{Q}^\dagger(\mathbf{1}+\mathcal{P})|\beta_n\rangle. \tag{2.53}$$

We take the scalar product of both sides of this equality with $|\bar{\Phi}_n(\zeta)\rangle$. The scalar product with the right-hand side vanishes because of (2.43). On the left-hand side, we find $2^{2n+1}\nu_n$ and therefore have $\nu_n = 0$. Finally, we determine the value of $\mu_n$ by taking the scalar product of both sides of (2.51) with $|\Phi_n(\zeta)\rangle$.

The reasoning for $|\bar{\Psi}_n\rangle$ is similar. □

As stated in Chapter 1, the (co)homology decomposition of a zero-energy state is not unique. We now determine an alternative cohomology decomposition for $|\bar{\Psi}_n\rangle$, which will be useful in Chapter 3.

**Proposition 2.4.4.** *For each $n \geqslant 1$, the state $|\bar{\Psi}_n\rangle$ can be written*

$$|\bar{\Psi}_n\rangle = 4^n|\uparrow\cdots\uparrow\rangle + \mathfrak{Q}|\gamma_n\rangle \tag{2.54}$$

*for some state $|\gamma_n\rangle \in W^{2n}$.*

*Proof.* We show that for each $n \geqslant 1$, there is a linear combination of $|\uparrow \cdots \uparrow\rangle \in W^{2n+1}$ and $|\Phi_n(\zeta)\rangle$ that is in the image of the supercharge.

To see this, we notice that for each $n \geqslant 1$ the state $|\uparrow \cdots \uparrow\rangle \in W^{2n+1}$ is annihilated by $\mathfrak{Q}$. This can be seen from the definition of the local supercharge (2.15). It follows from (2.44) that there are constants $\eta_n, \bar{\eta}_n$ and a state $|\delta_n\rangle \in W^{2n}$ such that

$$|\uparrow \cdots \uparrow\rangle = \eta_n|\Phi_n(\zeta)\rangle + \bar{\eta}_n|\bar{\Phi}_n(\zeta)\rangle + \mathfrak{Q}|\delta_n\rangle. \tag{2.55}$$

We act on both sides of this equality with the spin-parity operator $\mathcal{P}$, which leads to

$$-|\uparrow \cdots \uparrow\rangle = \eta_n|\Phi_n(\zeta)\rangle - \bar{\eta}_n|\bar{\Phi}_n(\zeta)\rangle - \mathfrak{Q}\mathcal{P}|\delta_n\rangle. \tag{2.56}$$

We take the sum of (2.55) and (2.56) and find

$$2\eta_n|\Phi_n(\zeta)\rangle + \mathfrak{Q}(\mathbf{1} - \mathcal{P})|\delta_n\rangle = 0. \tag{2.57}$$

The projection of this equality onto the state $|\Phi_n(\zeta^{-1})\rangle$ leads to $\eta_n = 0$ (and therefore $\mathfrak{Q}(\mathbf{1} - \mathcal{P})|\delta_n\rangle = 0$). Hence, (2.55) becomes

$$|\uparrow \cdots \uparrow\rangle = \bar{\eta}_n|\bar{\Phi}_n(\zeta)\rangle + \mathfrak{Q}|\delta_n\rangle. \tag{2.58}$$

We take the scalar product of both sides of this equality with $|\bar{\Phi}_n(\zeta^{-1})\rangle$ and find $\bar{\eta}_n = \frac{1}{4^n}$.

Finally, we combine this result with (2.45) and find

$$|\bar{\Psi}_n\rangle = 4^n|\uparrow \cdots \uparrow\rangle + \mathfrak{Q}(|\bar{\alpha}_n\rangle - 4^n|\delta_n\rangle). \tag{2.59}$$

This leads to (2.54) with $|\gamma_n\rangle = |\bar{\alpha}_n\rangle - 4^n|\delta_n\rangle$. □

As was pointed out above, the states $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$ span the space of the ground states of $H_{\text{XYZ}}$ with $L = 2n + 1$ sites and the anisotropy parameters (2.14), *restricted to* $W^{2n+1}$. The next theorem shows that they are in fact the ground states on the full Hilbert space $V^{2n+1}$. This follows from a generalisation of a classical result [14] by Yang and Yang on the ground state of the XXZ chain and an argument by Yang and Fendley [80].

**Theorem 2.4.5.** *For each $L = 2n+1$, $n \geqslant 1$, and non-zero $\zeta$, the states $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$ span the space of the ground states of $H_{\text{XYZ}}$ with the anisotropy parameters defined in (2.14). The corresponding ground-state eigenvalue is $E_0 = -(2n+1)(3+\zeta^2)/4$.*

*Proof.* We divide the proof into four steps.

First, we notice that it is sufficient to prove the statement for $\zeta > 0$. The reason is that the Hamiltonians for $\zeta > 0$ and $\zeta < 0$ can be related by a unitary transformation. Indeed, writing $H_{\text{XYZ}} = H_{\text{XYZ}}(\zeta)$, we have

$$H_{\text{XYZ}}(-\zeta) = \mathcal{M}(\mathrm{i})H_{\text{XYZ}}(\zeta)\mathcal{M}(\mathrm{i})^\dagger, \tag{2.60}$$

where $\mathcal{M}(\mathrm{i})$ is the operator introduced in Section 2.3. Furthermore, let us write $|\Psi_n(\zeta)\rangle$ and $|\bar{\Psi}_n(\zeta)\rangle$ for the alternate-cyclic zero-energy states. Using Theorem 2.4.1 and Proposition 2.4.2, one can show that $\mathcal{M}(\mathrm{i})|\Psi_n(\zeta)\rangle = \gamma_n|\Psi_n(-\zeta)\rangle$ and $\mathcal{M}(\mathrm{i})|\bar{\Psi}_n(\zeta)\rangle = \bar{\gamma}_n|\bar{\Psi}_n(-\zeta)\rangle$ where $\gamma_n$, $\bar{\gamma}_n$ are non-zero complex numbers. Hence, if $|\Psi_n(\zeta)\rangle$ and $|\bar{\Psi}_n(\zeta)\rangle$ span the space of the ground states of $H_{\text{XYZ}}(\zeta)$, then $|\Psi_n(-\zeta)\rangle$ and $|\bar{\Psi}_n(-\zeta)\rangle$ will span the space of ground states of $H_{\text{XYZ}}(-\zeta)$ as well.

Second, for $\zeta > 0$ the off-diagonal entries of $H_{\text{XYZ}}$ are zero or negative. Hence, there is a constant $\lambda$ such that $\lambda - H_{\text{XYZ}}$ is a non-negative matrix with positive diagonal entries. We consider the restriction $H_\pm$ of $\lambda - H_{\text{XYZ}}$ to the eigenspace of the spin-parity operator $\mathcal{P}$ associated to the eigenvalue $\pm 1$. The matrix $H_\pm$ is Hermitian and thus has real eigenvalues. Furthermore, the repeated action of $H_\pm$ on any basis state $|s_1 s_2 \cdots s_L\rangle$ with spin parity $\pm 1$ leads to linear combinations of basis states that have the same spin parity. The coefficients of these linear combinations are positive. Any other basis state $|s_1' s_2' \cdots s_L'\rangle$ with this spin parity can be found in one of these linear combinations. Following [14], we conclude that there exists an integer $m \geqslant 1$ such that $H_\pm^m$ is a positive matrix. Hence, $H_\pm$ is *irreducible and non-negative* [81]. We may thus apply the *Perron-Frobenius theorem for irreducible non-negative matrices*. It implies that the largest eigenvalue $\lambda_\pm$ of $H_\pm$ is non-degenerate. Furthermore, there is a unique state $|\Psi_\pm\rangle$ with positive components and norm one such that

$$H_\pm|\Psi_\pm\rangle = \lambda_\pm|\Psi_\pm\rangle. \tag{2.61}$$

Considered as a vector of $V^{2n+1}$, $|\Psi_\pm\rangle$ has non-negative components. It spans the one-dimensional space of the ground states of $H_{\text{XYZ}}$ in the subsector where the spin parity is fixed to $\pm 1$.

Third, following [80] we prove that $|\Psi_\pm\rangle$ is invariant under translations. Indeed, because of $[H_{\text{XYZ}}, \mathcal{S}] = 0$ and $[\mathcal{P}, \mathcal{S}] = 0$, we have

$$\mathcal{S}|\Psi_\pm\rangle = t_\pm|\Psi_\pm\rangle, \quad \text{with} \quad t_\pm^L = 1. \tag{2.62}$$

We now take the complex conjugate of this equation. Since the components of $|\Psi_\pm\rangle$ are real, we immediately find $t_\pm^* = t_\pm^{-1} = t_\pm$. Hence, as $L = 2n+1$ is odd, $t_\pm = 1$. The state $|\Psi_\pm\rangle$ is therefore alternate-cyclic.

Finally, it follows from Theorem 2.4.1 and Proposition 2.4.2 that $|\Psi_+\rangle$ and $|\Psi_-\rangle$ are proportional to $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$, respectively. The ground-state eigenvalue follows from (2.28) and therefore is doubly degenerate. □

Before turning to the next chapter, we briefly discuss the case $\zeta = 0$ that we did not address. If $\zeta = 0$, then the XYZ Hamiltonian (2.7) with anisotropy parameters (2.14) becomes a XXZ Hamiltonian as $J_1 = J_2 = 1$. This system has been treated extensively in the literature. In particular, its ground state energy is doubly degenerate, and we can compute the components of the corresponding eigenvector explicitly [61]. However, to gain new insight into the characterisation of this state, it is interesting to find simple (co)homology representatives as we obtained for generic $\zeta$. We postpone this discussion to Chapter 6 devoted to the XXZ spin chains.

# Chapter 3

# The supersymmetric eight-vertex model with periodic boundary conditions

In this chapter, we introduce the eight-vertex model. It is a generalisation of the six-vertex model, a two-dimensional classical vertex model developped by Pauling in 1935 [18]. As stated in the introduction, one can extract some properties of the system from the so-called transfer matrix, that we investigate. As we shall see, this two-dimensional model is closely related to a one-dimensional quantum spin chain. Specifically, it was understood by Lieb that the transfer matrix of eight-vertex model possesses the same eigenvectors as an XXZ Hamiltonian [22].

We study the eight-vertex model with specifically chosen parameters, that we give later, so that the corresponding Hamiltonian is the supersymmetric Hamiltonian discussed in the preceding chapter. Because of this relation, we follow Rosengren [82] and refer to this special case as the *supersymmetric eight-vertex model*. It is connected to a variety of topics such as families of solutions to the Lamé and Painlevé III equations [83], the Painlevé VI equation [84, 85, 86, 87, 82] as well as combinatorics [57, 75, 88].

Many of the results on the supersymmetric eight-vertex model rely on a conjecture made by Stroganov [54, 56] in 2001. He conjectured that the transfer matrix of the supersymmetric eight-vertex model possesses a remarkably simple eigenvalue $\Theta_n$, that we specify later. The goal of this chapter is to prove Stroganov's conjecture.

The layout of this chapter is as follows: in Section 3.1, we define the eight-vertex model. We recall the relation between its transfer matrix and the XYZ Hamiltonian in Section 3.2. In Section 3.3, we focus on the supersymmetric eight-vertex model and prove a commutation relation between the supercharge and the transfer matrix. We use it to compute the action of the transfer matrix on the space of alternate-cyclic zero-energy states. This result allows us to prove Stroganov's conjecture, which reduces the computation of $\Theta_n$ to a combinatorial problem, done in Section 3.4. Finally, we show that $\Theta_n$ is the largest eigenvalue of the transfer matrix if the weights are positive.

## 3.1 Lattice formulation

The eight-vertex model is a generalisation of the six-vertex model developped by Pauling in 1935 to calculate the residual entropy of ice water [18]. In ice $I_h$, the ordinary type of frozen water, the oxygen atoms of the water molecules arrange themselves on a (nearly perfect) hexagonal crystal.

Each oxygen atom is covalently bound to two hydrogen atoms, these bonds are strong and the hydrogen atoms are close to the oxygen. Furthermore, each hydrogen lies on the segment between two oxygen atoms, creating a hydrogen bond. The crystalline structure being tetravalent, there is exactly one hydrogen atom on each oxygen-oxygen axis. This hydrogen is strongly bound to one oxygen (and is close to it), and weakly bound to the other one (and is further away from it). This is the ice rule, as formulated by Pauling [18].

We now consider a vertex model, which reproduces the main features of the physical crystal.

**Domain and configurations.** The domain of the model consists of a square lattice with $L$ columns and $N$ rows. We call *vertex* the intersection of two lines on the square lattice and *edge* the segment connecting two adjacent vertices. On each edge resides an arrow. In the ice water picture, this arrow points towards the vertex where lies the oxygen atom from which the hydrogen atom is the closest. The ice-rule is thus recast as follows: at each vertex there must be exactly two in-going and two out-going arrows. There are six admissible configurations of arrows at a vertex, hence the name *six-vertex model*. The allowed configurations are depicted in the Figure 3.1.

The model admits a generalisation to the case where the ice-rule is relaxed: the permitted number of in-going and out-going arrows at each vertex is even. The model thus allows for the configurations with four arrows pointing inward or outward, as represented in the last two configurations of Figure 3.1, and is called the *eight-vertex model*.

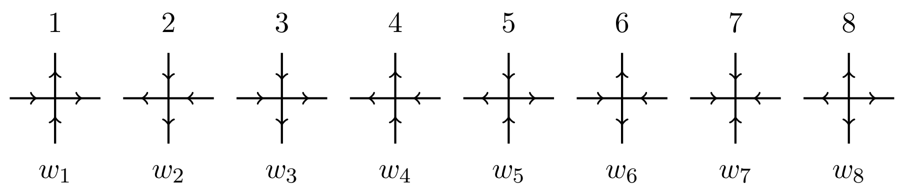



Figure 3.1: The admissible configurations of the vertices in the eight-vertex model, labeled by $1, \ldots, 8$. The configurations 7 and 8 violate the ice-rule and are forbidden in the six-vertex model. To the configuration $i$ is associated the weight $w_i$.

**Boundary conditions.** The boundary conditions, which are the permitted arrows configurations on the extremal edges of the domain have to be specified. In the following, we consider the eight-vertex model on a cylinder, hence with periodic boundary conditions along the horizontal direction. In later chapters, we revisit the model with so-called open boundary conditions.

If each edge carries an arrow such that all the vertices are in one of the eight vertex configurations, we say that the system is in a admissible configuration. An example of such a configuration of the eight-vertex model with periodic boundary conditions is given in Figure 3.2.

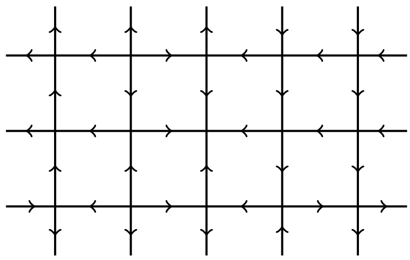


Figure 3.2: An example of an admissible configuration of the eight-vertex model with periodic boundary conditions along the horizontal direction with $L = 5$, $N = 3$.

**Weights and partition function.** To the $i$-th configuration, as ordered on Figure 3.1 is assigned a (Boltzmann) weight $w_i$. We call partition function the weigthed sum of all admissible configurations of the lattice. It is denoted by $\mathcal{Z}$ and is given by

$$\mathcal{Z} = \sum_{\sigma} \prod_{i=1}^{8} w_i^{n_i(\sigma)}, \tag{3.1}$$

where $n_i(\sigma)$ counts the number of vertex of type $i$ in the configuration $\sigma$. The partition function is a fundamental object in statistical mechanics as it allows for the computation of thermodynamic state functions [89]. As an example, the logarithm of the partition function yields the free energy per site for large systems:

$$f = -\lim_{L,N\to\infty} \frac{1}{LN} \ln(\mathcal{Z}). \tag{3.2}$$

In the following, we assume that the vertex weights are invariant under the simultaneous reversal of all the arrows. This condition is referred to as the zero-field assumption. It allows one to denote the weights by

$$w_1 = w_2 = a, \quad w_3 = w_4 = b, \quad w_5 = w_6 = c, \quad w_7 = w_8 = d. \tag{3.3}$$

**Transfer matrix.** A proven method to obtain the partition function is the *transfer matrix* method [89]. It consists in rewriting the partition function as specific matrix elements of a product of operators, the so-called transfer matrices. In essence, the transfer matrix contains the weights of the vertices of a row of the lattice. Its construction is based on the $R$-matrix, which encodes the possible weights of a vertex in a spin basis. Let us now translate the arrows configurations formalism into the spin language.

We label the edge at the left, below, at the right and above a vertex with $1, 2, 3$ and $4$, respectively. Furthermore, we assign a spin up ($\uparrow$) to an edge carrying an arrow pointing towards the north or the east and similarly a spin down ($\downarrow$) to each edge on which sits an arrow that points to the south or the west. The $R$-matrix of the eight-vertex model is an operator $R : V \otimes V \to V \otimes V$ such that a configuration with spin $s_i$ on the $i$-th edge has the weight given by the matrix element $\langle s_3 s_4 | R | s_1 s_2 \rangle$, as depicted on the Figure 3.3. In the standard basis $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ of $V \otimes V$ it is given by

$$R = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}. \tag{3.4}$$

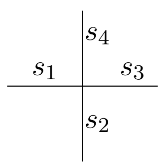
Expressed in terms of the Pauli matrices, the $R$-matrix reads

$$R = \frac{1}{2} \sum_{\alpha=0}^{3} r_\alpha \sigma^\alpha \otimes \sigma^\alpha \tag{3.5a}$$

where we used the notation $\sigma^0 = \mathbf{1}$. The coefficients $r_\alpha$ are given by

$$r_0 = a + b, \quad r_1 = c + d, \quad r_2 = c - d, \quad r_3 = a - b. \tag{3.5b}$$

The transfer matrix of the eight-vertex model on the square lattice with $L$ vertical lines with periodic boundary conditions along the horizontal




$$= \quad \langle s_3 s_4 | R | s_1 s_2 \rangle$$

Figure 3.3: The vertex configurations are encoded in the $R$ matrix. For a veretx configuration $s_i \in \{\uparrow, \downarrow\}$, $i = 1, \ldots, 4$, the corresponding weight is given by $\langle s_3 s_4 | R | s_1 s_2 \rangle$.

direction is an operator $\mathcal{T} : V^L \to V^L$, defined as

$$\mathcal{T} = \mathrm{tr}_0 \left( R_{0L} \cdots R_{01} \right) . \tag{3.6}$$

Here $R_{ij}$ is the $R$-matrix acting non-trivially only the factors $V_i$ and $V_j$ in the product space $V_0 \otimes V^L = V_0 \otimes V_1 \otimes \cdots \otimes V_L$, where $V_0 = V$ is referred to as the auxiliary space. The trace in (3.6) is taken over the space $V_0$.

The partition function for a $L \times N$ lattice is obtained by taking the $N$-th power of $\mathcal{T}$. Finding the spectrum of $\mathcal{T}$ is of physical importance, in particular the highest eigenvalue, as it allows for the computation of the free energy. As an example, let us apply periodic boundary conditions in the vertical direction. The partition function is

$$\mathcal{Z} = \mathrm{tr}(\mathcal{T}^N) = d(\lambda_0)\lambda_0^N + d(\lambda_1)\lambda_1^N + \cdots , \tag{3.7}$$

where the $\lambda_i$ are the distinct eigenvalues of $\mathcal{T}$ with respective degeneracy $d(\lambda_i)$, in decreasing order: $\lambda_i > \lambda_{i+1}$. We have

$$f = - \lim_{L,N\to\infty} \frac{1}{LN} \ln(\mathcal{Z}) = - \lim_{L\to\infty} \frac{1}{L} \ln(\lambda_0) \tag{3.8}$$

and the free energy per site is extracted from the largest eigenvalue of the transfer matrix.

## 3.2 The transfer matrix and XYZ spin chain

In this section, we rederive known results about the transfer matrix of the eight-vertex model and its relation with the XYZ spin-chain Hamiltonian. First we reproduce the result and proof of Sutherland [90] on the commutation of the $\mathcal{T}$ with $H_{\mathrm{XYZ}}$. Second,we use a parameterisation of the weights to investigate in more details the properties of $\mathcal{T}$.

The transfer matrix is invariant under translations, spin reversal and preserves the spin parity [89]. This is expressed by the following commutation relations

$$[\mathcal{T}, \mathcal{S}] = [\mathcal{T}, \mathcal{R}] = [\mathcal{T}, \mathcal{P}] = 0. \tag{3.9}$$

Furthermore, one can show [89] that $\mathcal{T}$ commutes with its transpose $[\mathcal{T}, \mathcal{T}^t] = 0$. Hence it is a real normal matrix and therefore diagonalisable by means of a unitary transformation.

Sutherland [90] showed that the transfer matrix with generic weights $a, b, c, d$ commutes with the Hamiltonian $H_{\mathrm{XYZ}}$ for a certain choice of anisotropy parameters. The proof of this proposition is based on the existence of operator $A$. This operator allows one to find a local commutation relation of the $R$-matrices with the operators $h_{jj+1}$.

**Proposition 3.2.1.** *The transfer matrix $\mathcal{T}$ commutes with the Hamiltonian of the XYZ spin chain,*

$$[\mathcal{T}, H_{\mathrm{XYZ}}] = 0, \tag{3.10}$$

*if the anisotropy parameters are given by*

$$J_1 = 1 + \frac{cd}{ab}, \quad J_2 = 1 - \frac{cd}{ab}, \quad J_3 = \frac{a^2 + b^2 - c^2 - d^2}{2ab}. \tag{3.11}$$

*Proof.* To compute the commutator of $\mathcal{T}$ with the Hamiltonian, we first establish a commutation relation between the transfer matrix and $h_{12}$. To this end, we introduce the operator $A : V \otimes V \to V \otimes V$, given by

$$A = \kappa \sum_{\alpha=0}^{3} \frac{1}{r_\alpha} \sigma^\alpha \otimes \sigma^\alpha \tag{3.12}$$

where the $r_\alpha$, $\alpha = 0, 1, 2, 3$ are the coefficients in the $R$-matrix (3.5b) and $\kappa = (c^2 - d^2)(a^2 - b^2)/(4ab)$. We further define $A_{01}, A_{02} : V_0 \otimes V^L \to V_0 \otimes V^L$ by

$$A_{01} = A \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{L-1}, \quad A_{02} = \mathcal{S} A_{01} \mathcal{S}^{-1}. \tag{3.13}$$

Here, the translation operator leaves $V_0$ unchanged and only acts on $V^L$. The operator $A_{0j}$ only acts non-trivially on $V_0$ and $V_j$ in the product space $V_0 \otimes V^L$. If the anisotropy parameters obey (3.11), then we have the following commutation relation between the $R$-matrix and operator $h_{12}$, which is shown by a straightforward computation:

$$[R_{02}R_{01}, h_{12}] = R_{02}A_{01} - A_{02}R_{01}. \tag{3.14}$$

Multiplying this equation on the left by $R_{0L} \cdots R_{03}$, and taking the trace with respect to the space $V_0$, we obtain

$$[\mathcal{T}, h_{12}] = \mathcal{A} - \mathcal{S}\mathcal{A}\mathcal{S}^{-1} \tag{3.15}$$

with $\mathcal{A} : V^L \to V^L$ defined by $\mathcal{A} = \mathrm{tr}_0(R_{0L} \cdots R_{02} A_{01})$. We conjugate this relation by $\mathcal{S}^{j-1}$ and find, using the translation property of the local hamiltonian (2.11),

$$[\mathcal{T}, h_{jj+1}] = \mathcal{S}^{j-1} \mathcal{A} \mathcal{S}^{-(j-1)} - \mathcal{S}^j \mathcal{A} \mathcal{S}^{-j}. \tag{3.16}$$

The sum of these equalities yields the telescopic cancellation

$$[\mathcal{T}, \sum_{j=1}^{L} h_{jj+1}] = \mathcal{A} - \mathcal{S}^L \mathcal{A} \mathcal{S}^{-L}. \tag{3.17}$$

The right-hand side of this equality vanishes on $V^L$ as $\mathcal{S}^L = \mathbf{1}$, hence the transfer matrix and the Hamiltonian with anisotropy parameters (3.11) commute. □

It follows from $[\mathcal{T}, H_{\mathrm{XYZ}}] = 0$ that the transfer matrix and the Hamiltonian with (2.14) can be simultaneously diagonalised.

**Parameterisation.** The commutation relation found by Sutherland is in fact a consequence of a stronger relation between the Hamiltonian and the transfer matrix. In order to establish this relation and other properties of the transfer matrix, it is convenient to use an explicit parameterisation of the vertex weights [24, 26]. We write the vertex weights of the model in terms of Jacobi theta functions [91]:

$$a(u) = \rho\vartheta_4(2\eta, p^2)\vartheta_1(u + 2\eta, p^2)\vartheta_4(u, p^2), \tag{3.18a}$$
$$b(u) = \rho\vartheta_4(2\eta, p^2)\vartheta_4(u + 2\eta, p^2)\vartheta_1(u, p^2), \tag{3.18b}$$
$$c(u) = \rho\vartheta_1(2\eta, p^2)\vartheta_4(u + 2\eta, p^2)\vartheta_4(u, p^2), \tag{3.18c}$$
$$d(u) = \rho\vartheta_1(2\eta, p^2)\vartheta_1(u + 2\eta, p^2)\vartheta_1(u, p^2). \tag{3.18d}$$

Here, $\rho$ is a normalisation constant, $\eta$ the so-called crossing parameter, $p$ the elliptic nome and $u$ the spectral parameter. With this parameterisation, the $R$-matrix $R = R(u)$ satisfies the Yang-Baxter equation:

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v) \tag{3.19}$$

for all $u, v$. It follows from the Yang-Baxter equation that transfer matrices with different spectral parameters commute [89, 25]:

**Proposition 3.2.2.** *Writing $\mathcal{T} = \mathcal{T}(u)$, we have*

$$[\mathcal{T}(u), \mathcal{T}(v)] = 0, \quad \textit{for all } u, v. \tag{3.20}$$

*Proof.* The proof follows from the unitary relation

$$R(u)R(-u) = a(u)a(-u) + d(u)d(-u) \tag{3.21}$$

and the Yang-Baxter relation: we have, for each $j$,

$$R_{0\bar{0}}(w)R_{0j}(u)R_{\bar{0}j}(v) = R_{\bar{0}j}(v)R_{0j}(u)R_{0\bar{0}}(w), \tag{3.22}$$

where we abbreviated $w = u - v$. Denoting by $C$ the term $a(w)a(-w) + d(w)d(-w)$, we have

$$\mathcal{T}(u)\mathcal{T}(v) = \frac{1}{C}\mathrm{tr}_{0\bar{0}}\left(R_{0\bar{0}}(-w)R_{0\bar{0}}(w)R_{0L}(u)R_{\bar{0}L}(v)\cdots R_{01}(u)R_{\bar{0}1}(v)\right).$$

In the left-hand side, $\mathrm{tr}_{0\bar{0}}$ indicates that the trace is taken over the auxilliary spaces $V_0$ and $V_{\bar{0}}$. We apply the Yang-Baxter equation (3.22) for each $j = L, \dots, 1$ and get

$$\begin{aligned}\mathcal{T}(u)\mathcal{T}(v) &= \frac{1}{C}\mathrm{tr}_{0\bar{0}}\left(R_{\bar{0}L}(v)R_{0L}(u)\cdots R_{\bar{0}1}(v)R_{01}(u)R_{0\bar{0}}(w)R_{0\bar{0}}(-w)\right)\\ &= \mathcal{T}(v)\mathcal{T}(u),\end{aligned}$$

where we used the cyclicity of the trace to obtain the last equality. This proves the commutation relation (3.20). □

As a consequence $\mathcal{T}(u)$ possesses an eigenbasis that is independent of the spectral parameter $u$. This commutation relation also provides a second proof of the commutation relation (3.10) between the Hamiltonian of the XYZ spin chain with anisotropy parameters (3.11) and the transfer matrix. This is a consequence of an observation by Baxter that the XYZ spin-chain Hamiltonian is proportional to the logarithmic derivative of the transfer matrix [24, 25].

Indeed, a standard calculation of the transfer matrix and its logarithmic derivative with respect to the parameter $u$, evaluated at $u = 0$, leads to

$$\mathcal{T}(0) = a(0)^L\mathcal{S}, \quad \mathcal{T}(0)^{-1}\mathcal{T}'(0) = \frac{L(a'(0) + c'(0))}{2a(0)} - \frac{b'(0)}{a(0)}H_{\mathrm{XYZ}}. \tag{3.23}$$

Here, the anisotropy parameters of the spin-chain Hamiltonian are

$$J_1 = 1 + \frac{d'(0)}{b'(0)}, \quad J_2 = 1 - \frac{d'(0)}{b'(0)}, \quad J_3 = \frac{a'(0) - c'(0)}{b'(0)}. \tag{3.24}$$

Using the theta function parameterisation (3.18) (and a few identities between the Jacobi theta functions [91]) one finds $J_1 = 1 + \zeta$ and $J_2 = 1 - \zeta$ with

$$\zeta = \left(\frac{\vartheta_1(2\eta, p^2)}{\vartheta_4(2\eta, p^2)}\right)^2 = \frac{c(u)d(u)}{a(u)b(u)}, \tag{3.25}$$

and

$$J_3 = \frac{\vartheta_2(2\eta, p^2)\vartheta_3(2\eta, p^2)\vartheta_4(0, p^2)^2}{\vartheta_2(0, p^2)\vartheta_3(0, p^2)\vartheta_4(2\eta, p^2)^2} = \frac{a(u)^2 + b(u)^2 - c(u)^2 - d(u)^2}{2a(u)b(u)}. \tag{3.26}$$

As expected, it follows from (3.23) that $[\mathcal{T}(u), H_{\text{XYZ}}] = 0$. We note that for real $0 < p < 1$, the parameter $\zeta$ given in (3.25) is real and verifies $0 < \zeta < 1$.

## 3.3 The transfer matrix and supersymmetry

From now on, we consider the case where the anisotropy parameters of the XYZ chain are parameterised according to (2.14) and yield an Hamiltonian that possesses supersymmetry. It follows from (3.11) that this is equivalent to

$$\zeta = \frac{cd}{ab}, \tag{3.27}$$

and to the relation

$$(a^2 + ab)(b^2 + ab) = (c^2 + ab)(d^2 + ab) \tag{3.28}$$

with non-zero vertex weights. (Furthermore, it corresponds to the value $\eta = \pi/3$ of the crossing parameter.) The eight-vertex model with the weights satisfying (3.28) is the supersymmetric eight-vertex model.

It was conjectured in [76] that the transfer matrix of the supersymmetric eight-vertex model and the supercharges of the XYZ Hamiltonian with $\zeta = cd/ab$ have a simple commutation relation. In this section, we prove this conjecture. The proof relies on a relation between the $R$-matrix of the supersymmetric eight-vertex model and the local supercharge, akin to the relation that appears in Sutherland's proof that we reproduced in Proposition 3.2.1. This sheds some light on the connection between integrability and supersymmetry.

We introduce an operator $\mathfrak{a} : V \to V \otimes V$. Its action on the basis states $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$ is

$$\mathfrak{a}|{\uparrow}\rangle = d\left(-\frac{c}{a}|{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle\right), \quad \mathfrak{a}|{\downarrow}\rangle = c\left(|{\uparrow\uparrow}\rangle - \frac{d}{b}|{\downarrow\downarrow}\rangle\right). \tag{3.29}$$

We define $\mathfrak{a}_0^1$, $\mathfrak{a}_0^2 : V_0 \otimes V^L \to V_0 \otimes V^{L+1}$ by

$$\mathfrak{a}_0^1 = \mathfrak{a} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{L}, \quad \mathfrak{a}_0^2 = \mathcal{S}\mathfrak{a}_0^1\mathcal{S}^{-1}. \tag{3.30}$$

Here, as in (3.13), the translation operator leaves $V_0$ unchanged and only acts on $V^L$. The operators $\mathfrak{a}_0^1$ and $\mathfrak{a}_0^2$ allow us to establish a relation between the $R$-matrix of the eight-vertex model and the local supercharge whose proof is a straightforward computation.

**Lemma 3.3.1.** *We have the equality*

$$R_{02}R_{01}(\mathbf{1} \otimes \mathfrak{q}_1) + (a+b)(\mathbf{1} \otimes \mathfrak{q}_1)R_{01} = \mathfrak{a}_0^2 R_{01} + R_{02}\mathfrak{a}_0^1 \tag{3.31}$$

*if and only if the relations* (3.28) *hold.*

By means of this lemma, we prove the following commutation relation between the supercharge and the transfer matrix of the eight-vertex model:

**Proposition 3.3.2.** *The transfer matrix of the supersymmetric eight-vertex model obeys the commutation relation*

$$\mathcal{T}\mathfrak{Q} + (a+b)\mathfrak{Q}\mathcal{T} = 0 \tag{3.32}$$

*on $V^L$ for each $L \geqslant 1$.*

*Proof.* We notice that this relation trivially holds on any eigenspace of the translation operator that is not equal to the space of alternate-cyclic states.

It is therefore sufficient to prove the relation on $W^L$. To this end, we multiply the relation (3.31) on the left by the product of $R$-matrices $R_{0L+1}\cdots R_{03}$ and take the trace over the space $V_0$. Using the identity $R_{0j}\mathfrak{q}_1 = \mathfrak{q}_1 R_{0j-1}$ for each $j = 3,\dots,L+1$, we obtain

$$\mathcal{T}\mathfrak{q}_1 + (a+b)\mathfrak{q}_1\mathcal{T} = \mathrm{tr}_0\left(R_{0L+1}\cdots R_{03}\mathfrak{a}_0^2 R_{01}\right) + \mathrm{tr}_0\left(R_{0L+1}\cdots R_{03}R_{02}\mathfrak{a}_0^1\right). \tag{3.33}$$

We define the operator $\mathfrak{A}: V^L \to V^{L+1}$ as

$$\mathfrak{A} = \mathrm{tr}_0\left(R_{0L+1}\cdots R_{03}R_{02}\mathfrak{a}_0^1\right), \tag{3.34}$$

and rewrite (3.33) in terms of $\mathfrak{A}$ and the translation operator $\mathcal{S}$. Using the definition $\mathcal{S}\mathfrak{a}_0^1\mathcal{S}^{-1} = \mathfrak{a}_0^2$ and the cyclic property of the trace operation, we find

$$\mathcal{T}\mathfrak{q}_1 + (a+b)\mathfrak{q}_1\mathcal{T} = \mathcal{S}\mathfrak{A}\mathcal{S}^{-1} + \mathfrak{A}. \tag{3.35}$$

By conjugation with $\mathcal{S}^{j-1}$ this equality generalises to

$$\mathcal{T}\mathfrak{q}_j + (a+b)\mathfrak{q}_j\mathcal{T} = \mathcal{S}^j\mathfrak{A}\mathcal{S}^{-j} + \mathcal{S}^{j-1}\mathfrak{A}\mathcal{S}^{-(j-1)}, \quad j = 0,\dots,L. \tag{3.36}$$

Here, we used the definition of $\mathfrak{q}$ (2.16) and the commutation relation (3.9) between the transfer matrix and the translation operator. We take an alternating sum of these equalities and obtain

$$\mathcal{T}\left(\sum_{j=0}^{L}(-1)^j\mathfrak{q}_j\right) + (a+b)\left(\sum_{j=0}^{L}(-1)^j\mathfrak{q}_j\right)\mathcal{T} = (-1)^L\mathcal{S}^L\mathfrak{A}\mathcal{S}^{-L} + \mathcal{S}^{-1}\mathfrak{A}\mathcal{S}. \tag{3.37}$$

The expression on the right-hand side can be simplified. Indeed, we have $\mathcal{S}^L = \mathbf{1}$ on $V^L$ and $\mathcal{S}^{L+1} = \mathbf{1}$ on $V^{L+1}$. Hence, we obtain $\mathcal{S}^L\mathfrak{A}\mathcal{S}^{-L} = \mathcal{S}^{-1}\mathcal{S}^{L+1}\mathfrak{A}\mathcal{S}^{-L} = \mathcal{S}^{-1}\mathfrak{A}$. We thus find

$$\mathcal{T}\left(\sum_{j=0}^{L}(-1)^j\mathfrak{q}_j\right) + (a+b)\left(\sum_{j=0}^{L}(-1)^j\mathfrak{q}_j\right)\mathcal{T} = \mathcal{S}^{-1}\mathfrak{A}\left((-1)^L + \mathcal{S}\right). \tag{3.38}$$

On $W^L$ the left-hand side is, up to a factor, equal to $\mathcal{T}\mathfrak{Q} + (a+b)\mathfrak{Q}\mathcal{T}$. The right-hand side vanishes on $W^L$. This proves that the commutation relation (3.32) holds on the subspace of alternate-cyclic states. □

## 3.4 The transfer-matrix eigenvalue

As stated in the introduction, the supersymmetric eight-vertex model has been studied for its connection with solutions to Painlevé equations and combinatorics. Many of these investigations rely on the existence of a remarkably simple eigenvalue of its transfer matrix. Stroganov conjectured in 2001 that for odd $L = 2n+1$, $n \geqslant 0$, the spectrum of the transfer matrix contains the doubly degenerate eigenvalue $\Theta_n = (a+b)^{2n+1}$ [54, 56]. In this section, we prove Stroganov's conjecture for $n \geqslant 1$. (The case $n = 0$ is trivial.) We summarise our main results in the following theorem:

**Theorem 3.4.1.** *For each $L = 2n+1$, $n \geqslant 1$, and non-zero vertex weights, the transfer matrix of the supersymmetric eight-vertex model possesses the doubly degenerate eigenvalue $\Theta_n = (a+b)^{2n+1}$. Its eigenspace is spanned by the ground states of the XYZ Hamiltonian (2.7) with $L = 2n+1$ sites and the anisotropy parameters (2.14) where $\zeta = cd/ab$.*

Moreover, we show that if the vertex weights are positive then $\Theta_n$ is the largest eigenvalue of the transfer matrix.

In the next proposition, we show that if the transfer matrix possesses the eigenvalue $\Theta_n$, then the corresponding eigenvectors are necessary alternate-cyclic zero-energy states of the Hamiltonian $H$. This is a direct consequence of (3.23).

**Proposition 3.4.2.** *Let $n \geqslant 1$ and suppose that $|\Psi\rangle \in V^{2n+1}$ is a non-zero solution of the eigenvalue equation*

$$\mathcal{T}|\Psi\rangle = (a+b)^{2n+1}|\Psi\rangle, \tag{3.39}$$

*then we have*

$$\mathcal{S}|\Psi\rangle = |\Psi\rangle, \quad H_{\mathrm{XYZ}}|\Psi\rangle = E_0|\Psi\rangle \tag{3.40}$$

*with $E_0 = -(2n+1)(3+\zeta^2)/4$ where $\zeta = cd/ab$.*

*Proof.* We use the parameterisation (3.18) and thus consider the eigenvalue problem

$$\mathcal{T}(u)|\Psi\rangle = (a(u)+b(u))^{2n+1}|\Psi\rangle \tag{3.41}$$

for fixed $n \geqslant 1$. Let us suppose that this equation has a non-zero solution $|\Psi\rangle \in V^{2n+1}$. Because of (3.20) we may suppose without loss of generality that it is independent of the spectral parameter $u$. For $u = 0$, we have $\mathcal{T}(0)|\Psi\rangle = a(0)^{2n+1}|\Psi\rangle$ with $a(0) \neq 0$. It follows from (3.23) that $\mathcal{S}|\Psi\rangle = |\Psi\rangle$. Next, we differentiate both sides of (3.41) with respect to $u$ and set $u = 0$. Using (3.23) we obtain

$$H_{\text{XYZ}}|\Psi\rangle = -(2n+1)\left(1 + \frac{a'(0) - c'(0)}{2b'(0)}\right)|\Psi\rangle. \tag{3.42}$$

On the right-hand side, we recognise the expression of $J_3$ given in (3.24). Since $J_3 = (\zeta^2 - 1)/2$ we find $H_{\text{XYZ}}|\Psi\rangle = E_0|\Psi\rangle$. □

In order to demonstrate Theorem 3.4.1, we need to prove the reciprocal of the previous proposition: an alternate-cyclic zero-energy state is an eigenvector of the transfer matrix $\mathcal{T}(u)$ with eigenvalue $\Theta_n$.

Since $[\mathcal{T}, H_{\text{XYZ}}] = [\mathcal{T}, H] = 0$, the space of alternate-cyclic zero-energy states is stable under the action of $\mathcal{T}$. Hence, we may deduce its action from the evaluation of the matrix elements of $\mathcal{T}$ between the states $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$. From Proposition 2.4.2 and (3.9) we infer that

$$\langle\Psi_n|\mathcal{T}|\bar{\Psi}_n\rangle = \langle\bar{\Psi}_n|\mathcal{T}|\Psi_n\rangle = 0. \tag{3.43}$$

It immediately follows that both $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$ are eigenstates of the transfer matrix. To find the corresponding eigenvalues, we consider the diagonal matrix elements

$$\Theta_n = \frac{\langle\bar{\Psi}_n|\mathcal{T}|\bar{\Psi}_n\rangle}{\langle\bar{\Psi}_n|\bar{\Psi}_n\rangle} = \frac{\langle\Psi_n|\mathcal{T}|\Psi_n\rangle}{\langle\Psi_n|\Psi_n\rangle}. \tag{3.44}$$

Here, the equality of the matrix elements for $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$ is again a consequence of Proposition 2.4.2 and (3.9).

We now utilise the Proposition 1.4.9 that allows us to calculate expectation values of operators with respect to supersymmetry singlets to evaluate $\Theta_n$. We use its expression in terms of $|\bar{\Psi}_n\rangle$. For $a + b \neq 0$, we compute the numerator $\langle\bar{\Psi}_n|\mathcal{T}|\bar{\Psi}_n\rangle$ by means of Proposition 1.4.9 with $\mathcal{B} = \mathcal{T}$, $\lambda = -(a+b)$ and the decompositions given in Theorem 2.4.3 and Proposition 2.4.4. Similarly, we obtain the denominator $\langle\bar{\Psi}_n|\bar{\Psi}_n\rangle$ with $\mathcal{B} = \mathbf{1}$, $\lambda = 1$ and the same decompositions. This leads to

$$\Theta_n = \frac{\langle\bar{\Phi}_n(\zeta^{-1})|\mathcal{T}|\uparrow\cdots\uparrow\rangle}{\langle\bar{\Phi}_n(\zeta^{-1})|\uparrow\cdots\uparrow\rangle} = \langle\bar{\Phi}_n(\zeta^{-1})|\mathcal{T}|\uparrow\cdots\uparrow\rangle, \tag{3.45}$$

where $|\bar{\Phi}_n(\zeta)\rangle$ is the state defined in (2.42). Here, we used (2.33) to compute $\langle\bar{\Phi}_n(\zeta^{-1})|\uparrow\cdots\uparrow\rangle = 1$. The case where $a+b=0$ can be treated as a suitable limit of this result. The resulting matrix element can be explicitly computed:

**Proposition 3.4.3.** *For each $n \geqslant 1$, we have $\Theta_n = (a+b)^{2n+1}$.*

*Proof.* We write $|\bar{\Phi}_n(\zeta^{-1})\rangle$ as a linear combination of the canonical basis states (2.4). To this end, we introduce the notation

$$||x_1,\dots,x_k\rangle\rangle = |\uparrow\cdots\uparrow\underset{x_1}{\downarrow}\uparrow\cdots\uparrow\underset{x_2}{\downarrow}\uparrow\underset{\cdots}{\cdots}\uparrow\underset{x_k}{\downarrow}\uparrow\cdots\uparrow\rangle, \tag{3.46}$$

where $k=1,\dots,L$. Using (2.42), we find

$$|\bar{\Phi}_n(\zeta^{-1})\rangle = |\uparrow\cdots\uparrow\rangle + \sum_{m=1}^{n}\zeta^{-m}\sum_{1\leqslant x_1<\cdots<x_{2m}\leqslant 2n+1}||x_1,\dots,x_{2m}\rangle\rangle. \tag{3.47}$$

Hence, we obtain

$$\begin{aligned}\Theta_n &= \langle\uparrow\cdots\uparrow|\mathcal{T}|\uparrow\cdots\uparrow\rangle\\ &\quad+\sum_{m=1}^{n}\zeta^{-m}\sum_{1\leqslant x_1<\cdots<x_{2m}\leqslant 2n+1}\langle\langle x_1,\dots,x_{2m}||\mathcal{T}|\uparrow\cdots\uparrow\rangle.\end{aligned} \tag{3.48}$$

The matrix elements on the right-hand side of this equality are readily evaluated. We have

$$\langle\uparrow\cdots\uparrow|\mathcal{T}|\uparrow\cdots\uparrow\rangle = a^{2n+1}+b^{2n+1}. \tag{3.49a}$$

Furthermore, for $m=1,\dots,n$ we obtain

$$\langle\langle x_1,\dots,x_{2m}||\mathcal{T}|\uparrow\cdots\uparrow\rangle = \zeta^m\left(\alpha(x_1,\dots,x_{2m})+\delta(x_1,\dots,x_{2m})\right), \tag{3.49b}$$

with $\zeta = cd/ab$ and

$$\alpha(x_1,\dots,x_{2m}) = a^{x_2-x_1}b^{x_3-x_2}\cdots a^{x_{2m}-x_{2m-1}}b^{2n+1-(x_{2m}-x_1)}, \tag{3.49c}$$

$$\delta(x_1,\dots,x_{2m}) = b^{x_2-x_1}a^{x_3-x_2}\cdots b^{x_{2m}-x_{2m-1}}a^{2n+1-(x_{2m}-x_1)}. \tag{3.49d}$$

We substitute these expressions into (3.48) and find

$$\begin{aligned}\Theta_n &= a^{2n+1}+b^{2n+1}\\ &\quad+\sum_{m=1}^{n}\sum_{1\leqslant x_1<\cdots<x_{2m}\leqslant 2n+1}(\alpha(x_1,\dots,x_{2m})+\delta(x_1,\dots,x_{2m})).\end{aligned} \tag{3.50}$$

The evaluation of this sum reduces to a combinatorial problem. To see this, we consider the set of all words $\gamma = (\gamma_1, \dots, \gamma_{2n+1})$ of length $2n+1$ with letters $\gamma_j \in \{a, b\}$. We assign a weight $\omega(\gamma) = \gamma_1\gamma_2 \cdots \gamma_{2n+1}$ to each word $\gamma$. Two simple examples are $\gamma = (a, a, \dots, a)$ and $\gamma = (b, b, \dots, b)$ whose weights are $\omega(\gamma) = a^{2n+1}$ and $\omega(\gamma) = b^{2n+1}$, respectively. Every other word contains both letters $a$ and $b$. For each such word there is an integer $m = 1, \dots, n$ and a sequence of integers $1 \leqslant x_1 < x_2 < \cdots < x_{2m} \leqslant 2n+1$ such that either

$$\gamma = (b, \dots, b, \underset{x_1}{a}, \dots, a, \underset{x_2}{b}, \dots, b, \dots, \underset{x_{2m-1}}{a}, \dots, a, \underset{x_{2m}}{b}, \dots, b),$$

and $\omega(\gamma) = \alpha(x_1, \dots, x_{2m})$ or

$$\gamma = (a, \dots, a, \underset{x_1}{b}, \dots, b, \underset{x_2}{a}, \dots, a, \dots, \underset{x_{2m-1}}{b}, \dots, b, \underset{x_{2m}}{a}, \dots, a),$$

in which case $\omega(\gamma) = \delta(x_1, \dots, x_{2m})$. We conclude that the sum (3.50) can be written as a sum over all words $\gamma$. The terms to sum up are their corresponding weights $\omega(\gamma)$. Hence we find

$$\Theta_n = \sum_{\gamma_1 = a,b} \cdots \sum_{\gamma_{2n+1} = a,b} \gamma_1 \cdots \gamma_{2n+1} = (a+b)^{2n+1}. \tag{3.51}$$

This concludes the proof. □

*Proof of Theorem 3.4.1.* According to Proposition 3.4.2 every eigenstate of the transfer matrix of the supersymmetric eight-vertex model with the eigenvalue $\Theta_n = (a+b)^{2n+1}$ is an alternate-cyclic zero-energy state of the Hamiltonian $H$. It follows from Theorem 2.4.1 that the space of these states is spanned by $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$. According to Proposition 3.4.3, we have

$$\mathcal{T}|\Psi_n\rangle = (a+b)^{2n+1}|\Psi_n\rangle, \quad \mathcal{T}|\bar{\Psi}_n\rangle = (a+b)^{2n+1}|\bar{\Psi}_n\rangle. \tag{3.52}$$

Hence, the eigenspace of $\Theta_n$ is two-dimensional and spanned by $|\Psi_n\rangle$ and $|\bar{\Psi}_n\rangle$. Furthermore, according to Theorem 2.4.5 it is the space of ground states of the XYZ Hamiltonian with the anisotropy parameters (2.14). The ground-state eigenvalue is $E_0 = -(2n+1)(3+\zeta^2)/4$. □

**Largest eigenvalue** We now prove that $\Theta_n$ is the largest eigenvalue of $\mathcal{T}$ if the vertex weights are positive:

**Theorem 3.4.4.** *If the vertex weights are positive, $a, b, c, d > 0$, then for each $L = 2n + 1$, $n \geqslant 1$, $\Theta_n = (a + b)^{2n+1}$ is the largest eigenvalue of the transfer matrix of the supersymmetric eight-vertex model.*

*Proof.* We denote by $\mathcal{T}_\pm$ the restriction of the transfer matrix to the eigenspace of the spin-parity operator with eigenvalue $\pm 1$. The matrix elements of $\mathcal{T}_\pm$ with respect to the canonical basis of this eigenspace can be explicitly computed [89]. If $a, b, c, d > 0$ then these matrix elements are positive and thus $\mathcal{T}_\pm$ is a positive matrix. It follows from the Perron-Frobenius theorem [81] that $\mathcal{T}_\pm$ has a largest positive eigenvalue $\Theta_\pm$ which is non-degenerate. There is a unique vector $|\Phi_\pm\rangle$ with positive components and norm 1 such that

$$\mathcal{T}_\pm|\Phi_\pm\rangle = \Theta_\pm|\Phi_\pm\rangle. \tag{3.53}$$

Furthermore, except for positive multiples of $|\Phi_\pm\rangle$, the matrix $\mathcal{T}_\pm$ has no other eigenstate with positive components.

Let us consider the state $|\Psi_\pm\rangle$ defined in the proof of Theorem 2.4.5. It has positive components and norm one. Furthermore, according to Theorem 3.4.1 it is an eigenstate of $\mathcal{T}_\pm$:

$$\mathcal{T}_\pm|\Psi_\pm\rangle = \Theta_n|\Psi_\pm\rangle. \tag{3.54}$$

By the uniqueness of $|\Phi_\pm\rangle$, we conclude that $|\Phi_\pm\rangle = |\Psi_\pm\rangle$ and consequently, $\Theta_\pm = \Theta_n$. $\square$

We notice that this result immediately implies that for positive vertex weights the free energy per site of the supersymmetric eight-vertex model is given by

$$f = -\lim_{n\to\infty} \frac{1}{2n+1} \ln \Theta_n = -\ln(a + b), \tag{3.55}$$

as follows from (3.8). As expected, this agrees with Baxter's result for the free energy per site of the eight-vertex model [89], if specialised to the supersymmetric case.

Throughout this chapter, we have considered non-zero vertex weights. However, for $L = 2n + 1$ the transfer matrix still possesses the eigenvalue

$\Theta_n$ if some of the vertex weights are zero. The reason is that the eigenvalues are continuous functions of the entries of $\mathcal{T}$. Hence, they are continuous with respect to $a, b, c, d$. Furthermore, $\Theta_n$ is still the largest transfer-matrix eigenvalue for $a, b, c > 0$ and $d \to 0$. In this limit, the supersymmetric eight-vertex model reduces to the six-vertex model (corresponding to the spin chain with anisotropy parameter $\Delta = -1/2$). In that case, the existence of the eigenvalue can be proven by other techniques [61].

# Chapter 4

# XYZ spin chain: open boundary conditions

In the preceding chapters, we have dealt with spin and vertex models with periodic boundary conditions. We considered the sites at the extremities of the system as neighbours and thus in interaction.

From now on, we consider systems in which the first and last sites do not interact directly with each other. We refer to this type of boundary conditions as *open boundary conditions*.

For a spin chain with $L \geqslant 1$ sites and open boundary conditions, the XYZ Hamiltonian is given by

$$H_{\mathrm{XYZ}} = \sum_{j=1}^{L-1} h^{\mathrm{XYZ}}_{jj+1} + (h_{\mathrm{B}}^{-})_1 + (h_{\mathrm{B}}^{+})_L, \tag{4.1a}$$

where $h^{\mathrm{XYZ}}$ is the Hamiltonian density (2.7b), which involves the spin chain's anisotropy parameters $J_1, J_2, J_3$. The terms $h_{\mathrm{B}}^{\pm}$ describe the interactions of the first and last spins with boundary magnetic fields. By convention, for $L = 1$, the bulk interaction term is absent and the Hamiltonian is given by the sum $H_{\mathrm{XYZ}} = h_{\mathrm{B}}^{+} + h_{\mathrm{B}}^{-}$.

**Open spin chains and Bethe ansatz.** One of the first appearances of open spin chains is the XY Hamiltonian, given by (4.1a) with $J_3 = 0$

and $h_{\mathrm{B}}^{\pm} = 0$. This Hamiltonian was solved using a Jordan-Wigner transformation [92]. Gaudin was the first to apply the coordinate Bethe ansatz to solve a XXZ model with open boundary conditions. He considered the Hamiltonian (4.1a) with $J_1 = J_2$ and $h_{\mathrm{B}}^{\pm} = 0$ [93, 13]. This result was generalised to the case of diagonal boundary fields in [17].

The study of open spin chains in the framework of the algebraic Bethe ansatz relies on the *boundary Yang-Baxter equation* (sometimes referred to as *reflection equation*), introduced by Cherednik and formalised by Sklyanin [94, 35]. This method allows for the computation of correlation functions of the XXZ chain with diagonal boundary fields [95, 96].

However, the application of the algebraic Bethe ansatz requires the existence of a reference state, which is a trivial eigenvector of the Hamiltonian.

Such a state may not exist in the case of the XXZ chain with non-diagonal boundary fields. Nevertheless, one can circumvent this absence and solve the model by finding functional equations (see [97] for a review), generalising the coordinate Bethe ansatz [98] or using the method of separation of variables [99].

The same limitation occurs for the XYZ spin chain with open boundary conditions. Hence, there are only a few existing results that analyse the spectrum of the corresponding transfer matrix. Two examples of these methods are the off-diagonal Bethe ansatz [40] and the quantum separation of variables method [47].

In this chapter, we do not use the integrability of the model. We rather exploit the supersymmetric structure of the Hamiltonian.

**Supersymmetry.** As in Chapter 2, we focus on the case where the anisotropy parameters are given by

$$J_1 = 1 + \zeta, \quad J_2 = 1 - \zeta, \quad J_3 = \frac{1}{2}(\zeta^2 - 1), \tag{4.1b}$$

with a real parameter $\zeta$. We also consider the specific boundary terms

$$h_{\mathrm{B}}^{+} = h_{\mathrm{B}}^{-} = \sum_{\alpha=1}^{3} \lambda_\alpha \sigma^\alpha, \tag{4.1c}$$

where

$$\lambda_1 = -\frac{(1+\zeta)\,\mathrm{Re}\,y}{1+|y|^2}, \quad \lambda_2 = -\frac{(1-\zeta)\,\mathrm{Im}\,y}{1+|y|^2}, \quad \lambda_3 = \frac{\zeta^2-1}{4}\left(\frac{1-|y|^2}{1+|y|^2}\right), \tag{4.1d}$$

and $y$ is a complex number.

We show that the Hamiltonian consisting of (4.1a) with anisotropy parameters (4.1b) and boundary terms (4.1c),(4.1d) is supersymmetric by constructing explicitly a corresponding supercharge and its adjoint. The supersymmetry implies that the Hamiltonian may have supersymmetry singlets. If they exist, then they are the Hamiltonian's ground states. We compute the (co)homology of the supercharge and its adjoint to analyse the (non-)existence of supersymmetry singlets.

The layout of this chapter is as follows: in Section 4.1, we investigate the supersymmetry for open spin chains. We construct the supercharge that yields the Hamiltonian (4.1) in Section 4.2 and we restrict the range of the parameters $\zeta$ and $y$. The theta function parameterisation of Section 4.3 allows us to define a new basis of the Hilbert space in which the action of the supercharge is simple. In Section 4.4, we characterise the space of the ground states of the Hamiltonian using the relation between supersymmetry and (co)homology.

## 4.1 Open spin chains and supersymmetry

In this section, we investigate the supersymmetry of systems with open boundary conditions. We define a local supercharge that yields the XYZ Hamiltonian (4.1).

**Local supercharges and supercharges.** The construction of the supersymmetry for the XYZ spin chain is based on local supercharges $\mathfrak{q}' : V \to V \otimes V$.[1] We consider local supercharges with the property (2.24): for all $|\psi\rangle \in V$,

$$(\mathfrak{q}' \otimes \mathbf{1} - \mathbf{1} \otimes \mathfrak{q}')\mathfrak{q}'|\psi\rangle = |\chi\rangle \otimes |\psi\rangle - |\psi\rangle \otimes |\chi\rangle. \tag{4.2}$$

[1] The prime notation for open system supersymmetry generator is conventional in the (co)homology literature [73, 74, 100], to differentiate it from the supercharge of the periodic spin chain. In particular, it is not a derivative.

Here $|\chi\rangle \in V \otimes V$ is a fixed state. If $|\chi\rangle = 0$ then (4.2) reduces to

$$(\mathfrak{q}' \otimes \mathbf{1} - \mathbf{1} \otimes \mathfrak{q}')\mathfrak{q}' = 0. \tag{4.3}$$

We call a local supercharge with this property *coassociative*. Coassociative local supercharges allow us to construct supercharges for open spin chains. To see this, we consider the local operators $\mathfrak{q}'_j$, $j = 1, \ldots, L$, on $V^L$ that are given by

$$\mathfrak{q}'_j = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{j-1} \otimes \mathfrak{q}' \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{L-j}. \tag{4.4}$$

They map $V^L$ to $V^{L+1}$. Using these operators, we define for each $L \geqslant 1$ the supercharge $\mathfrak{Q}' : V^L \to V^{L+1}$ as the linear combination

$$\mathfrak{Q}' = \sum_{j=1}^{L} (-1)^j \mathfrak{q}'_j. \tag{4.5}$$

For each $L \geqslant 2$, the adjoint supercharge $\mathfrak{Q}'^\dagger : V^L \to V^{L-1}$ is defined by means of the scalar product of the spin-chain Hilbert space.

The following proposition ensures the nilpotency of the supercharge and its adjoint. Its proof follows from the computation made in Proposition 2.2.1.

**Proposition 4.1.1.** *The operators $\mathfrak{Q}'$ and $\mathfrak{Q}'^\dagger$ are nilpotent,*

$$(\mathfrak{Q}')^2 = 0, \quad (\mathfrak{Q}'^\dagger)^2 = 0, \tag{4.6}$$

*if and only if the local supercharge $\mathfrak{q}'$ is coassociative.*

*Proof.* First, we suppose that the supercharge is nilpotent. In particular, for $L = 1$, we have

$$0 = \mathfrak{Q}'^2 = (\mathfrak{q}'_1 - \mathfrak{q}'_2)\mathfrak{q}'_1. \tag{4.7}$$

This implies that $\mathfrak{q}'$ is coassociative.

Second, we consider a coassociative local supercharge and show the nilpotency of $\mathfrak{Q}'$. (The case of its adjoint is similar.) We have, for each $L \geqslant 1$

$$(\mathfrak{Q}')^2 = \sum_{j=1}^{L} (\mathfrak{q}'_j - \mathfrak{q}'_{j+1})\mathfrak{q}'_j, \tag{4.8}$$

which identically vanishes. □

**Hamiltonian.** We use $\mathfrak{Q}'$ and $\mathfrak{Q}'^\dagger$ to define a Hamiltonian $H$ for open spin chains. For $L=1$, it is given by $H=\mathfrak{Q}'^\dagger\mathfrak{Q}'$. For $L \geqslant 2$, it is the anticommutator

$$H = \mathfrak{Q}'\mathfrak{Q}'^\dagger + \mathfrak{Q}'^\dagger\mathfrak{Q}'. \tag{4.9}$$

We compute the Hamiltonian (4.9) using the specific supercharge (4.5). It is the sum of a bulk part and boundary terms:

$$H = \sum_{j=1}^{L-1} h_{jj+1} + (h_{\mathrm{B}})_1 + (h_{\mathrm{B}})_L. \tag{4.10}$$

Here, $h_{jj+1}$ is the Hamiltonian density which can be expressed in terms of the local supercharge (2.27). Furthermore, the operator $h_{\mathrm{B}} : V \to V$ encodes the boundary interaction at the first and last site of the chain. In terms of the local supercharge, we find

$$h_{\mathrm{B}} = \frac{1}{2}\mathfrak{q}'^\dagger\mathfrak{q}'. \tag{4.11}$$

The Hamiltonian (4.10) has identical boundary interactions at both extremities of the spin chain. We relax this constraint and allow different boundary terms in Chapter 6. This necessitates generalising the construction (4.5) of $\mathfrak{Q}'$ from local supercharges.

## 4.2 Supercharge for the XYZ spin chain

We now construct a local supercharge that allows us to investigate the Hamiltonian (4.1). To this end, we define three local supercharges that satisfy (4.2).

First, we introduce the operator $\mathfrak{q}_\phi$ that acts on $|\psi\rangle \in V$ according to

$$\mathfrak{q}_\phi|\psi\rangle = |\phi\rangle \otimes |\psi\rangle + |\psi\rangle \otimes |\phi\rangle. \tag{4.12}$$

Here $|\phi\rangle \in V$ is a fixed state. Indeed, $\mathfrak{q}_\phi$ obeys (4.2) with $|\chi\rangle = |\phi\rangle \otimes |\phi\rangle$. Hence, if $|\phi\rangle$ is non-zero, then the local supercharge $\mathfrak{q}_\phi$ is not coassociative. We follow [77] and refer to $\mathfrak{q}_\phi$ as a local gauge supercharge.

Second, we define $\mathfrak{q}^{\uparrow}$ and $\mathfrak{q}^{\downarrow}$ through the following action on the basis vectors of $V$ [77]:

$$\mathfrak{q}^{\uparrow}|\uparrow\rangle = 0, \quad \mathfrak{q}^{\uparrow}|\downarrow\rangle = |\uparrow\uparrow\rangle - \zeta|\downarrow\downarrow\rangle, \tag{4.13a}$$

$$\mathfrak{q}^{\downarrow}|\downarrow\rangle = 0, \quad \mathfrak{q}^{\downarrow}|\uparrow\rangle = |\downarrow\downarrow\rangle - \zeta|\uparrow\uparrow\rangle. \tag{4.13b}$$

One checks that both $\mathfrak{q}^{\uparrow}$ and $\mathfrak{q}^{\downarrow}$ obey (4.2) with the vectors $|\chi\rangle = -\zeta|\uparrow\uparrow\rangle$ and $|\chi\rangle = -\zeta|\downarrow\downarrow\rangle$, respectively. Hence, these operators are not coassociative for non-zero $\zeta$.

We define the local supercharge $\mathfrak{q}'$ as a linear combination of $\mathfrak{q}^{\uparrow}$, $\mathfrak{q}^{\downarrow}$ and a local gauge supercharge:

$$\mathfrak{q}' = (1 - y^2\zeta)\mathfrak{q}^{\uparrow} + y(y^2 - \zeta)\mathfrak{q}^{\downarrow} + \mathfrak{q}_{\phi}. \tag{4.14a}$$

Here $|\phi\rangle$ is given by

$$|\phi\rangle = y(y^2\zeta - 1)|\uparrow\rangle + (\zeta - y^2)|\downarrow\rangle, \tag{4.14b}$$

and $y$ is a complex number. In this expression, we adjusted the multiplicative factors and the gauge term in such a way that $\mathfrak{q}'$ is coassociative. A straightforward calculation shows that this is indeed the case for all $\zeta$ and $y$.

**Hamiltonian.** We prove that the Hamiltonian (4.1) of the XYZ spin chain is supersymmetric, up to a rescaling and to adding a multiple of the identity matrix.

**Proposition 4.2.1.** *For each $L \geqslant 1$, the Hamiltonian (4.9) constructed from the local supercharge (4.14) is*

$$H = x\left(H_{\text{XYZ}} + \frac{(L-1)(3+\zeta^2)}{4} + 2\lambda_0\right), \tag{4.15}$$

*where $H_{\text{XYZ}}$ is defined in (4.1). We have*

$$\lambda_0 = \frac{1+3\zeta^2}{4} - \frac{(\zeta^2-1)((3+\zeta^2)|y|^2 - 4\zeta\,\text{Re}(y^2))}{2(1+|y|^4 + (\zeta^2-1)|y|^2 - 2\zeta\,\text{Re}(y^2))}, \tag{4.16}$$

*and*

$$x = (1+|y|^2)(1+|y|^4 + (\zeta^2-1)|y|^2 - 2\zeta\,\text{Re}(y^2)). \tag{4.17}$$

*Proof.* The proof consists in the calculation of the bulk and boundary interactions of (4.10). The Hamiltonian density (2.27) yields

$$h = x\left(h^{\mathrm{XYZ}} + \frac{3+\zeta^2}{4}\right), \tag{4.18}$$

where the anisotropy parameters of $h^{\mathrm{XYZ}}$ are defined by (4.1b) and $x$ is given by (4.17). This expression is similar to (2.30) found for the XYZ Hamiltonian with periodic boundary conditions. Furthermore, it is straightforward to verify that the boundary term (4.11) is

$$\frac{1}{2}\mathfrak{q}'^{\dagger}\mathfrak{q}' = x\left(\sum_{\alpha=1}^{3}\lambda_\alpha\sigma^\alpha + \lambda_0\right). \tag{4.19}$$

Here, $\lambda_0$ and the three parameters $\lambda_1, \lambda_2, \lambda_3$ are given by (4.16) and (4.1d), respectively. □

**Transformation of the parameters.** We analyse the transformation behaviour of the Hamiltonian (4.1) under spin rotations. To this end, we introduce the operators

$$\mathcal{R}^\alpha(\theta) = \exp\left(\frac{\mathrm{i}\theta}{2}(\sigma_1^\alpha + \cdots + \sigma_L^\alpha)\right), \quad \alpha = 1,2,3. \tag{4.20}$$

We write $H_{\mathrm{XYZ}} = H_{\mathrm{XYZ}}(\zeta, y)$ to stress the dependence of the Hamiltonian on $\zeta$ and $y$. For each $L \geqslant 1$, it transforms under rotations by the angle $\theta = \pi/2$ according to

$$\begin{aligned}
\mathcal{R}^1(\pi/2)H_{\mathrm{XYZ}}(\zeta,y)\mathcal{R}^1(-\pi/2) &= \left(\frac{1+\zeta}{2}\right)^2 H_{\mathrm{XYZ}}\left(\frac{3-\zeta}{1+\zeta}, \frac{y-\mathrm{i}}{1-\mathrm{i}y}\right),\\
\mathcal{R}^2(\pi/2)H_{\mathrm{XYZ}}(\zeta,y)\mathcal{R}^2(-\pi/2) &= \left(\frac{1-\zeta}{2}\right)^2 H_{\mathrm{XYZ}}\left(\frac{\zeta+3}{\zeta-1}, \frac{1+y}{1-y}\right),\\
\mathcal{R}^3(\pi/2)H_{\mathrm{XYZ}}(\zeta,y)\mathcal{R}^3(-\pi/2) &= H_{\mathrm{XYZ}}\left(-\zeta, -\mathrm{i}y\right).
\end{aligned} \tag{4.21}$$

Two successive applications of (4.21) lead to the following transformations under rotations by the angle $\theta = \pi$:

$$\begin{aligned}
\mathcal{R}^1(\pi)H_{\mathrm{XYZ}}(\zeta,y)\mathcal{R}^1(-\pi) &= H_{\mathrm{XYZ}}\left(\zeta, y^{-1}\right),\\
\mathcal{R}^2(\pi)H_{\mathrm{XYZ}}(\zeta,y)\mathcal{R}^2(-\pi) &= H_{\mathrm{XYZ}}\left(\zeta, -y^{-1}\right),\\
\mathcal{R}^3(\pi)H_{\mathrm{XYZ}}(\zeta,y)\mathcal{R}^3(-\pi) &= H_{\mathrm{XYZ}}(\zeta, -y).
\end{aligned} \tag{4.22}$$

The transformations (4.21) and (4.22) are unitary. Therefore, they do not change the spectrum of the Hamiltonian. Moreover, they allow us to transform a Hamiltonian with arbitrary parameters $\zeta$ and $y$ to a Hamiltonian whose parameters are restricted to a domain defined by the inequalities

$$0 \leqslant \zeta \leqslant 1,\ 0 \leqslant |y| \leqslant 1,\ \mathrm{Re}\, y \geqslant 0. \tag{4.23}$$

**Limit cases.** The case $\zeta = 1$ is trivial. Indeed, in this case, the Hamiltonian is

$$H_{\mathrm{XYZ}}(1, y) = -\sum_{j=1}^{L-1} \sigma_j^1 \sigma_{j+1}^1 - \frac{2\mathrm{Re}\, y}{1+|y|^2}\left(\sigma_1^1 + \sigma_L^1\right) \tag{4.24}$$

and its ground states are easily found: for each $L$, the space of the ground states is one-dimensional and is spanned by

$$|\Psi_L\rangle = \sum_{s_1=\uparrow,\downarrow} \cdots \sum_{s_{2n+1}=\uparrow,\downarrow} |s_1 \cdots s_{2n+1}\rangle. \tag{4.25}$$

The corresponding eigenvalue is $-(L-1) - \frac{4\mathrm{Re}\, y}{1+|y|^2}$.

If $\zeta = 0$, then $J_1 = J_2$ and $H_{\mathrm{XYZ}}$ becomes an XXZ Hamiltonian. Its properties substantially differ from the case $\zeta > 0$. As an example, this Hamiltonian commutes with $\mathcal{R}^3(\theta)$ for each $\theta$. This is a consequence of the commutation relation

$$\left[H_{\mathrm{XYZ}}(0, y), \sum_{j=1}^{L} \sigma_j^3\right] = 0. \tag{4.26}$$

This relation is akin to a conservation of magnetisation. It implies that the Hamiltonian decomposes into a direct sum of Hamiltonians acting on subspaces of the Hilbert space spanned by vectors with a definite number of up (or down) spins. In this case, the following treatment of the Hamiltonian's ground states does not apply. Hence, we postpone the investigation of the Hamiltonian (4.1a) with $\zeta = 0$, and some generalisations, to subsequent chapters.

Therefore, we focus in this chapter on $0 < \zeta < 1$, specifically on the case where $(\zeta, y)$ belongs to the domain

$$\mathbb{D} = \{(\zeta, y) : 0 < \zeta < 1,\ 0 \leqslant |y| \leqslant 1,\ \mathrm{Re}\, y \geqslant 0\}. \tag{4.27}$$

## 4.3 Theta function parameterisation

In this section, we introduce a parameterisation of the points $(\zeta, y) \in \mathbb{D}$ in terms of Jacobi theta functions. We employ this theta-function parameterisation to define a new basis of the spin Hilbert space. The action of the local supercharge (4.14) on the basis states yields simple results.

**Parameterisation.** We use the classical notation $\vartheta_j(u,p)$, $1 \leqslant j \leqslant 4$ and definitions for the Jacobi theta functions [91]. We only consider a real elliptic nome $p$ with

$$0 < p < 1. \tag{4.28}$$

Let us write $p = e^{-s}$, $s > 0$. We define the rectangle $\mathbb{R}_p = \{z \in \mathbb{C} : 0 \leqslant \operatorname{Re} z \leqslant \pi/2,\ -s/2 \leqslant \operatorname{Im} z \leqslant s/2\}$ and the domain

$$\bar{\mathbb{D}} = \{(p,t) : 0 < p < 1,\ t \in \mathbb{R}_p\}. \tag{4.29}$$

The parameterisation of $(\zeta, y) \in \mathbb{D}$ in terms of $(p,t) \in \bar{\mathbb{D}}$ is given by

$$\zeta = \left(\frac{\vartheta_1(2\pi/3, p^2)}{\vartheta_4(2\pi/3, p^2)}\right)^2, \quad y = \frac{\vartheta_1(t,p^2)}{\vartheta_4(t,p^2)}. \tag{4.30}$$

It has the following property:

**Proposition 4.3.1.** *The parameterisation (4.30) defines a bijection between $\bar{\mathbb{D}}$ and $\mathbb{D}$.*

*Proof.* We only sketch the proof. First, we note that $\zeta$ is a monotone function of $p$. Second, as a function of $t$, $y$ is the Jacobi elliptic function sn, up to a rescaling of its argument and a constant factor. The bijectivity can be established with the help of the monotonicity and the conformal mapping properties of sn [101]. $\square$

In the remainder of this section, we implicitly assume the parameterisation (4.30).

**Basis states.** In addition to the parameterisation, we introduce the states and dual states

$$|v_\epsilon\rangle = \vartheta_4(t+\epsilon\pi/3,p^2)|{\uparrow}\rangle + \vartheta_1(t+\epsilon\pi/3,p^2)|{\downarrow}\rangle, \tag{4.31}$$

$$\langle w_\epsilon| = \epsilon\left(-\vartheta_1(t-\epsilon\pi/3,p^2)\langle{\uparrow}| + \vartheta_4(t-\epsilon\pi/3,p^2)\langle{\downarrow}|\right), \tag{4.32}$$

where $\epsilon = \pm$. One checks that

$$\langle w_\epsilon|v_{\epsilon'}\rangle = \vartheta_1(\pi/3,p)\vartheta_2(t,p)\delta_{\epsilon\epsilon'}, \tag{4.33}$$

for each $\epsilon,\epsilon' = \pm$. In the next five lemmas, we establish several properties of these states.

**Lemma 4.3.2.** *For all $(p,t) \in \bar{\mathbb{D}}$ with $t \neq \pi/2$, the states $|v_+\rangle$ and $|v_-\rangle$ form a basis of $V$.*

*Proof.* The matrix

$$M = \begin{pmatrix} \vartheta_4(t+\pi/3,p^2) & \vartheta_4(t-\pi/3,p^2) \\ \vartheta_1(t+\pi/3,p^2) & \vartheta_1(t-\pi/3,p^2) \end{pmatrix}, \tag{4.34}$$

whose columns are given by $|v_+\rangle$ and $|v_-\rangle$, has the determinant

$$\det M = -\vartheta_1(\pi/3,p)\vartheta_2(t,p). \tag{4.35}$$

For $t \neq \pi/2$, this determinant is non-vanishing. Hence, the vectors are linearly independent. Therefore, they form a basis of $V$. □

If $t = \pi/2$, then $|v_+\rangle$ and $|v_-\rangle$ are not linearly independent: we have $|v_-\rangle = |v_+\rangle$. To find a suitable basis of $V$, we define

$$|\dot{v}_+\rangle = \frac{\mathrm{d}}{\mathrm{d}t}|v_+\rangle\bigg|_{t=\pi/2}. \tag{4.36}$$

**Lemma 4.3.3.** *For all $(p,t) \in \bar{\mathbb{D}}$ with $t = \pi/2$, the states $|v_+\rangle$ and $|\dot{v}_+\rangle$ form a basis of $V$.*

*Proof.* The matrix whose columns are given by the states $|v_+\rangle, |\dot{v}_+\rangle$ is

$$\dot{M} = \begin{pmatrix} \vartheta_3(\pi/3,p^2) & \vartheta_3'(\pi/3,p^2) \\ \vartheta_2(\pi/3,p^2) & \vartheta_2'(\pi/3,p^2) \end{pmatrix}. \tag{4.37}$$

Its determinant is given by $\det \dot{M} = -\frac{1}{2}\vartheta_1'(0,p)\vartheta_1(\pi/3,p)$, which is non-zero. Hence, the vectors are linearly independent. Therefore, they form a basis of $V$. □

**Lemma 4.3.4.** *For each $\epsilon = \pm$, we have*

$$\mathfrak{q}'|v_\epsilon\rangle = \Lambda_\epsilon |v_\epsilon\rangle \otimes |v_\epsilon\rangle, \tag{4.38}$$

*where*

$$\Lambda_\epsilon = \frac{2\epsilon\vartheta_1(\pi/3,p^2)\vartheta_4(0,p^2)^2}{\vartheta_4(\pi/3,p^2)\vartheta_2(0,p)}\frac{\vartheta_2(t+\epsilon\pi/3,p)}{\vartheta_4(t,p^2)^3}. \tag{4.39}$$

*Proof.* The proof follows from a number of identities for Jacobi theta functions. □

**Lemma 4.3.5.** *Let $t = \pi/2$, then*

$$\mathfrak{q}'|\dot{v}_+\rangle = \dot{\Lambda}_+|v_+\rangle \otimes |v_+\rangle + \Lambda_+(|\dot{v}_+\rangle \otimes |v_+\rangle + |v_+\rangle \otimes |\dot{v}_+\rangle), \tag{4.40}$$

*where*

$$\dot{\Lambda}_+ = \frac{\mathrm{d}}{\mathrm{d}t}\Lambda_+\Big|_{t=\pi/2}. \tag{4.41}$$

*Proof.* We differentiate (4.38) at $t = \pi/2$. We eliminate the terms that involve the derivative of $\mathfrak{q}$ with respect to $t$ by observing that for $t = \pi/2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_-\Big|_{t=\pi/2} = -\dot{\Lambda}_+ \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}|v_-\rangle\Big|_{t=\pi/2} = -|\dot{v}_+\rangle. \tag{4.42}$$

This leads to the action of $\mathfrak{q}$ on $|\dot{v}_+\rangle$. □

**Lemma 4.3.6.** *For each $\epsilon, \epsilon' = \pm$, we have*

$$(\langle w_\epsilon| \otimes \langle w_{\epsilon'}|)\,\mathfrak{q}' = \vartheta_1(\pi/3,p)\vartheta_2(t,p)\Lambda_\epsilon\delta_{\epsilon\epsilon'}\langle w_\epsilon|. \tag{4.43}$$

*Proof.* The proof is a straightforward calculation using standard identities for the Jacobi theta functions. □

## 4.4 Supersymmetry singlets and cohomology

The aim of this and the following section is to investigate the absence or existence of supersymmetry singlets for the Hamiltonian $H$ of Proposition 4.2.1 as a function of $(p,t) \in \bar{\mathbb{D}}$. To this end, we exploit the relation between supersymmetry and (co)homology presented in the Chapter 1.

Our goal is to compute the cohomology $\mathcal{H}^\bullet$ of the supercharge $\mathfrak{Q}'$ (1.45). We recall that $\mathcal{H}^\bullet$ is the direct sum of spaces $\mathcal{H}^L$, $L \geqslant 1$. We defined $\mathcal{H}^1 = \ker\{\mathfrak{Q}' : V \to V^2\}$ and

$$\mathcal{H}^L = \frac{\ker\{\mathfrak{Q}' : V^L \to V^{L+1}\}}{\operatorname{im}\{\mathfrak{Q}' : V^{L-1} \to V^L\}}. \tag{4.44}$$

Similarly, the homology of the adjoint supercharge is $\mathcal{H}_\bullet = \bigoplus_L \mathcal{H}_L$. Here, $\mathcal{H}_1 = V/\operatorname{im}\{\mathfrak{Q}'^\dagger : V^2 \to V\}$ and

$$\mathcal{H}_L = \frac{\ker\{\mathfrak{Q}'^\dagger : V^L \to V^{L-1}\}}{\operatorname{im}\{\mathfrak{Q}'^\dagger : V^{L+1} \to V^L\}}. \tag{4.45}$$

We use the same notation as in Chapter 1 and denote by $[|\psi\rangle]$ the equivalence class of $|\psi\rangle$.

**Auxiliary results.** To compute the (co)homology for the supercharge of the XYZ Hamiltonian, we establish three auxiliary results.

**Lemma 4.4.1.** *Let $|u_+\rangle$, $|u_-\rangle$ be a basis of $V$ and $\mathfrak{q}'$ a local supercharge defined by*

$$\mathfrak{q}'|u_+\rangle = |u_+\rangle \otimes |u_+\rangle, \quad \mathfrak{q}'|u_-\rangle = |u_-\rangle \otimes |u_-\rangle, \tag{4.46}$$

*then $\mathcal{H}^L = 0$ for each $L \geqslant 1$.*

*Proof.* For $L = 1$, the statement $\mathcal{H}^1 = 0$ is immediate since $|u_+\rangle$ and $|u_-\rangle$ form a basis of $V$.

For $L \geqslant 2$, let $|\psi\rangle \in \ker\{\mathfrak{Q}' : V^L \to V^{L+1}\}$. We write $|\psi\rangle = |u_+\rangle \otimes |\psi_+\rangle + |u_-\rangle \otimes |\psi_-\rangle$ with unique states $|\psi_+\rangle, |\psi_-\rangle \in V^{L-1}$. It follows from $\mathfrak{Q}'|\psi\rangle = 0$ that

$$|u_+\rangle\otimes(|u_+\rangle\otimes|\psi_+\rangle+\mathfrak{Q}'|\psi_+\rangle)+|u_-\rangle\otimes(|u_-\rangle\otimes|\psi_-\rangle+\mathfrak{Q}'|\psi_-\rangle) = 0. \tag{4.47}$$

Since $|u_+\rangle$ and $|u_-\rangle$ form a basis of $V$, we find $|u_\pm\rangle \otimes |\psi_\pm\rangle = -\mathfrak{Q}'|\psi_\pm\rangle$. Therefore, we have

$$|\psi\rangle = -\mathfrak{Q}'(|\psi_+\rangle + |\psi_-\rangle) \in \operatorname{im}\{\mathfrak{Q}' : V^{L-1} \to V^L\}. \tag{4.48}$$

This implies that $\mathcal{H}^L = 0$. $\square$

**Lemma 4.4.2.** *Let $|u_+\rangle$, $|u_-\rangle$ be a basis of $V$ and $\mathfrak{q}'$ a local supercharge defined by*

$$\mathfrak{q}'|u_+\rangle = |u_+\rangle \otimes |u_+\rangle, \quad \mathfrak{q}'|u_-\rangle = |u_+\rangle \otimes |u_-\rangle + |u_-\rangle \otimes |u_+\rangle, \tag{4.49}$$

*then $\mathcal{H}^L = 0$ for each $L \geqslant 1$.*

*Proof.* For $L = 1$, $\mathcal{H}^1 = 0$ follows immediately from the fact that $|u_+\rangle$, $|u_-\rangle$ is a basis of $V$.

For $L \geqslant 2$, let $|\psi\rangle \in \ker\{\mathfrak{Q}' : V^L \to V^{L+1}\}$. Again, we write $|\psi\rangle = |u_+\rangle \otimes |\psi_+\rangle + |u_-\rangle \otimes |\psi_-\rangle$ with unique states $|\psi_+\rangle, |\psi_-\rangle \in V^{L-1}$. The condition $\mathfrak{Q}'|\psi\rangle = 0$ yields

$$|u_+\rangle \otimes (|\psi\rangle + \mathfrak{Q}'|\psi_+\rangle) + |u_-\rangle \otimes (|u_+\rangle \otimes |\psi_-\rangle + \mathfrak{Q}'|\psi_-\rangle) = 0. \tag{4.50}$$

Since $|u_+\rangle$ and $|u_-\rangle$ span $V$, we obtain

$$|\psi\rangle = -\mathfrak{Q}'|\psi_+\rangle \in \mathrm{im}\{\mathfrak{Q}' : V^{L-1} \to V^L\}. \tag{4.51}$$

Hence, $\mathcal{H}^L = 0$. $\square$

**Lemma 4.4.3.** *Let $|u_+\rangle$, $|u_-\rangle$ be a basis of $V$ and $\mathfrak{q}'$ a local supercharge defined by*

$$\mathfrak{q}'|u_+\rangle = 0, \quad \mathfrak{q}'|u_-\rangle = |u_-\rangle \otimes |u_-\rangle, \tag{4.52}$$

*then $\mathcal{H}^L = \mathbb{C}[|u_+\rangle^{\otimes L}]$ for each $L \geqslant 1$.*

Here, and in the following, we use the notation

$$|u\rangle^{\otimes L} = |u\rangle \otimes \cdots \otimes |u\rangle \tag{4.53}$$

for the $L$-fold tensor product of $|u\rangle \in V$.

*Proof.* For each $L \geqslant 1$, we define a mapping $\mathfrak{S} : V^L \to V^{L+1}$ by

$$\mathfrak{S}|\psi\rangle = |u_+\rangle \otimes |\psi\rangle. \tag{4.54}$$

It satisfies the commutation relation $\mathfrak{S}\mathfrak{Q}' = -\mathfrak{Q}'\mathfrak{S}$ on $V^L$. Hence, the mapping $\mathfrak{S}^\sharp : \mathcal{H}^L \to \mathcal{H}^{L+1}$, given by

$$\mathfrak{S}^\sharp[|\psi\rangle] = [|u_+\rangle \otimes |\psi\rangle], \tag{4.55}$$

is well-defined [73]. We prove that $\mathfrak{S}^\sharp$ is a bijection.

First, we show that $\mathfrak{S}^\sharp$ is injective. This is straightforward for $L = 1$. For $L \geqslant 2$, we show that the kernel of $\mathfrak{S}^\sharp$ is zero in the cohomology. This is equivalent to the statement that any state $|\psi\rangle \in \ker\{\mathfrak{Q}' : V^L \to V^{L+1}\}$ with

$$\mathfrak{S}|\psi\rangle = \mathfrak{Q}'|\phi\rangle, \tag{4.56}$$

for some $|\phi\rangle \in V^L$, belongs to $\operatorname{im}\{\mathfrak{Q}' : V^{L-1} \to V^L\}$. To see this, we write $|\phi\rangle = |u_+\rangle\otimes|\phi_+\rangle+|u_-\rangle\otimes|\phi_-\rangle$ with unique states $|\phi_+\rangle$, $|\phi_-\rangle \in V^{L-1}$. It follows that

$$|u_+\rangle \otimes |\psi\rangle = -|u_+\rangle \otimes \mathfrak{Q}'|\phi_+\rangle - |u_-\rangle \otimes \big(|u_-\rangle \otimes |\phi_-\rangle + \mathfrak{Q}'|\phi_-\rangle\big)\,. \tag{4.57}$$

Since $|u_+\rangle$, $|u_-\rangle$ form a basis of $V$, we infer $|\psi\rangle = -\mathfrak{Q}'|\phi_+\rangle$, which proves the injectivity.

Second, we show that $\mathfrak{S}^\sharp$ is surjective. To this end, we fix $L \geqslant 2$ and consider a representative $|\psi\rangle \in V^L$ of an element of $\mathcal{H}^L$. As before, we write $|\psi\rangle = |u_+\rangle \otimes |\psi_+\rangle + |u_-\rangle \otimes |\psi_-\rangle$ with unique states $|\psi_+\rangle, |\psi_-\rangle \in V^{L-1}$. The equation $\mathfrak{Q}'|\psi\rangle = 0$ implies

$$\mathfrak{Q}'|\psi_+\rangle = 0, \quad \mathfrak{Q}'|\psi_-\rangle = -|u_-\rangle \otimes |\psi_-\rangle, \tag{4.58}$$

and therefore

$$|\psi\rangle = |u_+\rangle \otimes |\psi_+\rangle - \mathfrak{Q}'|\psi_-\rangle. \tag{4.59}$$

Hence, $[|\psi\rangle] = [|u_+\rangle \otimes |\psi_+\rangle] = \mathfrak{S}^\sharp[|\psi_+\rangle]$ with $|\psi_+\rangle \in \ker\{\mathfrak{Q}' : V^{L-1} \to V^L\}$. This proves the surjectivity.

Since $\mathfrak{S}^\sharp$ is a bijection, it follows that $\mathcal{H}^L = (\mathfrak{S}^\sharp)^{L-1}(\mathcal{H}^1)$ for each $L \geqslant 2$. One checks that $\mathcal{H}^1 = \mathbb{C}[|u_+\rangle]$. Hence, $\mathcal{H}^L = \mathbb{C}[|u_+\rangle^{\otimes L}]$. □

**Results for the XYZ supercharge.** In the remainder of this section, $\mathfrak{Q}'$ denotes the supercharge constructed from the local supercharge (4.14) for the XYZ Hamiltonian. We apply the auxiliary results to this case.

**Proposition 4.4.4.** *Let $L \geqslant 1$, and $(p,t) \in \bar{\mathbb{D}}$. We have*

$$\mathcal{H}^L = \begin{cases} 0, & \text{if } t \neq \pi/6, \\ \mathbb{C}[|v_+\rangle^{\otimes L}], & \text{if } t = \pi/6. \end{cases} \tag{4.60}$$

*Proof.* We distinguish three cases.

First, we consider $t \neq \pi/2, \pi/6$. In this case, it follows from Lemma 4.3.2 that $|v_+\rangle$ and $|v_-\rangle$ form a basis of $V$. Furthermore, the constants $\Lambda_\pm$, defined in (4.39), are non-vanishing. Hence, the states

$$|u_+\rangle = \Lambda_+|v_+\rangle, \quad |u_-\rangle = \Lambda_-|v_-\rangle \tag{4.61}$$

form a basis of $V$. We find from Lemma 4.3.4 that $\mathfrak{q}'|u_+\rangle = |u_+\rangle \otimes |u_+\rangle$, $\mathfrak{q}'|u_-\rangle = |u_-\rangle \otimes |u_-\rangle$. Hence, we apply Lemma 4.4.1 and conclude that $\mathcal{H}^L = 0$ for each $L \geqslant 1$.

Second, we suppose that $t = \pi/2$. It follows from Lemma 4.3.3 that the states $|v_+\rangle$ and $|\dot{v}_+\rangle$, defined in (4.36), form a basis of $V$. We define the states

$$|u_+\rangle = \Lambda_+|v_+\rangle, \quad |u_-\rangle = \dot{\Lambda}_+|v_+\rangle + \Lambda_+|\dot{v}_+\rangle. \tag{4.62}$$

These states form a basis of $V$ because $\Lambda_+, \dot{\Lambda}_+ \neq 0$ for $t = \pi/2$. Moreover, we have $\mathfrak{q}'|u_+\rangle = |u_+\rangle \otimes |u_+\rangle$, $\mathfrak{q}'|u_-\rangle = |u_+\rangle \otimes |u_-\rangle + |u_-\rangle \otimes |u_+\rangle$, thanks to Lemma 4.3.5. Therefore, it follows from Lemma 4.4.2 that $\mathcal{H}^L = 0$ for each $L \geqslant 1$.

Third, we analyse the case where $t = \pi/6$. In this case, we have $\Lambda_+ = 0$ and $\Lambda_- \neq 0$. The states

$$|u_+\rangle = |v_+\rangle, \quad |u_-\rangle = \Lambda_-|v_-\rangle \tag{4.63}$$

constitute a basis of $V$. They obey the relations $\mathfrak{q}'|u_+\rangle = 0$ and $\mathfrak{q}'|u_-\rangle = |u_-\rangle \otimes |u_-\rangle$. According to Lemma 4.4.3, we have

$$\mathcal{H}^L = \mathbb{C}[|u_+\rangle^{\otimes L}] = \mathbb{C}[|v_+\rangle^{\otimes L}], \tag{4.64}$$

for each $L \geqslant 1$. □

**Proposition 4.4.5.** *Let $L \geqslant 1$ and $(p,t) \in \bar{\mathbb{D}}$. We have*

$$\mathcal{H}_L = \begin{cases} 0, & \text{if } t \neq \pi/6, \\ \mathbb{C}[|w_+\rangle^{\otimes L}], & \text{if } t = \pi/6. \end{cases} \tag{4.65}$$

*Proof.* First, we consider $t \neq \pi/6$. In this case, $\mathcal{H}_L = 0$ for each $L \geqslant 1$ follows immediately from Proposition 4.4.4 and the fact that $\mathcal{H}^L$ and $\mathcal{H}_L$ are isomorphic.

Second, we consider $t = \pi/6$ and compute $\mathcal{H}_L$. To this end, we note that Lemma 4.3.6 implies

$$\mathfrak{Q}'^{\dagger}\left(|w_\epsilon\rangle \otimes |w_{\epsilon'}\rangle\right) = -\vartheta_1(\pi/3,p)^2\Lambda_\epsilon\delta_{\epsilon\epsilon'}|w_\epsilon\rangle, \tag{4.66}$$

for each $\epsilon, \epsilon' = \pm$. Furthermore, we have $\Lambda_+ = 0$ and $\Lambda_- \neq 0$. For $L = 1$, we find

$$\mathcal{H}_1 = V/\mathrm{im}\{\mathfrak{Q}'^{\dagger} : V^2 \to V\} = V/\mathbb{C}|w_-\rangle = \mathbb{C}[|w_+\rangle]. \tag{4.67}$$

For $L \geqslant 2$, Proposition 4.4.4 implies that $\mathcal{H}_L$ is one-dimensional. Hence, $\mathcal{H}_L = \mathbb{C}[|\omega\rangle]$ for some $|\omega\rangle \in V^L$ that is in the kernel of $\mathfrak{Q}'^{\dagger}$, but not in its image. We claim that $|\omega\rangle = |w_+\rangle^{\otimes L}$ is a valid choice. Indeed, on the one hand, (4.66) implies $\mathfrak{Q}'^{\dagger}|\omega\rangle = 0$. On the other hand, we use (4.33) to compute the scalar product

$$\langle\omega|\left(|v_+\rangle^{\otimes L}\right) = \langle w_+|v_+\rangle^L = \vartheta_1(\pi/3,p)^{2L}, \tag{4.68}$$

which is non-zero. If $|\omega\rangle = \mathfrak{Q}'^{\dagger}|\phi\rangle$ for some state $|\phi\rangle \in V^{L+1}$, then $\langle\omega|\left(|v_+\rangle^{\otimes L}\right) = \langle\phi|\mathfrak{Q}'\left(|v_+\rangle^{\otimes L}\right) = 0$. This is a contradiction and therefore proves the claim. $\square$

## 4.5 Spin-chain ground states

In this section, we examine if the Hamiltonian $H$ possesses supersymmetry singlets. To this end, we use the isomorphism provided by Proposition 1.4.2. If they exist, we establish multiple cohomology decompositions (1.49) and homology decompositions (1.61) of those supersymmetry singlets. Finally, we characterise the space of the ground states of the XYZ Hamiltonian (4.1).

**Theorem 4.5.1.** *Let $L \geqslant 1$ and $(p,t) \in \bar{\mathbb{D}}$. If $t \neq \pi/6$, then the Hamiltonian $H$ does not possess supersymmetry singlets. Conversely, if $t = \pi/6$ then the space of supersymmetry singlets of $H$ is one-dimensional, and spanned by*

$$|\Psi_L\rangle = \begin{cases} |v_+\rangle, & L = 1,\\ |v_+\rangle^{\otimes L} + \mathfrak{Q}'|\gamma_L\rangle, & L \geqslant 2, \end{cases} \tag{4.69}$$

*where $|\gamma_L\rangle \in V^{L-1}$.*

*Proof.* First, we consider $t \neq \pi/6$. In this case, it follows from Proposition 4.4.4 that $\mathcal{H}^L = 0$. Hence, $H$ does not possess supersymmetry singlets.

Second, for $t = \pi/6$, the Proposition 4.4.4 states that $\mathcal{H}^L = \mathbb{C}[|v_+\rangle^{\otimes L}]$. Hence, the space of the supersymmetry singlets of $H$ is one-dimensional. In fact, the decomposition for $L \geqslant 2$ follows from (1.49). □

**Proposition 4.5.2.** *For $t = \pi/6$ and each $L \geqslant 1$, the state (4.69) can be written as*

$$|\Psi_L\rangle = \mu_L |w_+\rangle^{\otimes L} + \mathfrak{Q}'^\dagger |\gamma'_L\rangle, \tag{4.70}$$

*with $|\gamma'_L\rangle \in V^{L+1}$. The constant $\mu_L$ is non-zero and given by*

$$\mu_L = \frac{\left(\langle v_+|^{\otimes L}\right)|\Psi_L\rangle}{\vartheta_1(\pi/3, p)^{2L}}. \tag{4.71}$$

*Proof.* The decomposition (4.70) follows from $\mathcal{H}_L = \mathbb{C}[|w_+\rangle^{\otimes L}]$ for $t = \pi/6$, found in Proposition 4.4.5. To find the coefficient $\mu_L$, it is sufficient to compute the scalar product of both sides of (4.70) with $|v_+\rangle^{\otimes L}$. It has to be non-zero because otherwise, $|\Psi_L\rangle$ would be in the image of $\mathfrak{Q}'^\dagger$. This would imply $|\Psi_L\rangle = 0$ [1] and thus contradicts Proposition 4.4.5. □

**Alternative decompositions.** We know from Chapter 1 that the (co)homology decompositions are not unique. We exploit the non-uniqueness to compute two alternative decompositions for the supersymmetry singlet $|\Psi_L\rangle$. To this end, we define

$$\begin{aligned} |\chi\rangle &= |v_+\rangle \otimes |v_+\rangle - \kappa^2 |v_-\rangle \otimes |v_-\rangle, \\ |\alpha\rangle &= |w_+\rangle \otimes |w_+\rangle + \kappa^{-1} |w_-\rangle \otimes |w_+\rangle, \end{aligned} \tag{4.72}$$

where $\kappa = \vartheta_3(\pi/3, p)/\vartheta_3(0, p)$.

**Proposition 4.5.3.** *For $t = \pi/6$ and each $L \geqslant 2$, the supersymmetry singlet $|\Psi_L\rangle$ can be written as*

$$|\Psi_L\rangle = |\chi\rangle \otimes |v_+\rangle^{\otimes(L-2)} + \mathfrak{Q}' |\delta_L\rangle, \tag{4.73}$$

*for some state $|\delta_L\rangle \in V^{L-1}$, and as*

$$|\Psi_L\rangle = \mu_L |\alpha\rangle \otimes |w_+\rangle^{\otimes(L-2)} + \mathfrak{Q}'^\dagger |\delta'_L\rangle, \tag{4.74}$$

*for some state $|\delta'_L\rangle \in V^{L+1}$. Here, $\mu_L$ is the constant defined in (4.71).*

*Proof.* The proof consists of two simple calculations. We focus on (4.73). Using $\mathfrak{q}'|v_+\rangle = 0$, $\mathfrak{q}'|v_-\rangle = \Lambda_-|v_-\rangle \otimes |v_-\rangle$ with $\Lambda_- \neq 0$ for $t = \pi/6$, we obtain

$$|v_+\rangle^{\otimes L} = |\chi\rangle \otimes |v_+\rangle^{\otimes(L-2)} - \mathfrak{Q}'\left(\kappa^2\Lambda_-^{-1}|v_-\rangle \otimes |v_+\rangle^{\otimes(L-2)}\right). \tag{4.75}$$

We use this in (4.69) and obtain (4.73) with

$$|\delta_L\rangle = |\gamma_L\rangle - \kappa^2\Lambda_-^{-1}|v_-\rangle \otimes |v_+\rangle^{\otimes(L-2)}. \tag{4.76}$$

The proof of (4.74) is similar. □

Finally, we point out that for $t = \pi/6$, the basis states $|v_\pm\rangle$ and their duals $|w_\pm\rangle$, as well as $|\chi\rangle$ and $|\alpha\rangle$, can up to factor be written in terms of polynomials in $\zeta$ and $y$. This property can be shown with the help of identities between Jacobi theta functions.

**Lemma 4.5.4.** *We have $|v_\pm\rangle = C_\pm|\bar v_\pm\rangle$ and $|w_\pm\rangle = C_\mp|\bar w_\pm\rangle$, where*

$$|\bar v_+\rangle = y(1-\zeta y^2)|{\uparrow}\rangle + (\zeta - y^2)|{\downarrow}\rangle, \qquad |\bar v_-\rangle = |{\uparrow}\rangle - y|{\downarrow}\rangle,$$
$$|\bar w_-\rangle = (\zeta - y^2)|{\uparrow}\rangle - y(1-\zeta y^2)|{\downarrow}\rangle, \qquad |\bar w_+\rangle = y|{\uparrow}\rangle + |{\downarrow}\rangle,$$

*and $C_+ = (1-\zeta^2)^{-2/3}y^{-1}\vartheta_3(\pi/3, p^2)$, $C_- = \vartheta_3(\pi/3, p^2)$.*

**Lemma 4.5.5.** *We have $|\chi\rangle = D_+|\bar\chi\rangle$ and $|\alpha\rangle = D_-|\bar\alpha\rangle$ with*

$$|\bar\chi\rangle = y^2(\zeta - 2 + \zeta y^2)|{\uparrow\uparrow}\rangle + y(y^2-1)(|{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle) - (\zeta + (\zeta-2)y^2)|{\downarrow\downarrow}\rangle,$$
$$|\bar\alpha\rangle = y\left(|{\uparrow\uparrow}\rangle - |{\downarrow\downarrow}\rangle\right) + |{\uparrow\downarrow}\rangle - y^2|{\downarrow\uparrow}\rangle,$$

*where $D_+ = \zeta(y^2-1)C_+^2$ and $D_- = \zeta(y^2-1)y^{-1}(\zeta-1)^{-1}C_-^2$.*

**The XYZ ground states.** We now return to the XYZ Hamiltonian defined in (4.1). We introduce the polynomial

$$P(\zeta, y) = \zeta(1+y^4) - (3-\zeta^2)y^2. \tag{4.77}$$

It is straightforward to see that, given $0 < \zeta < 1$, the biquadratic equation $P(\zeta, y) = 0$ for $y$ possesses four real solutions. They have particularly simple expressions in the parameterisation by Jacobi theta functions.

**Lemma 4.5.6.** *Let $0 < \zeta < 1$ and $y$ be parametrised according to (4.30) with $0 < p < 1$, then the solutions of $P(\zeta, y) = 0$ are given by*

$$y_0 = \frac{\vartheta_1(\pi/6, p^2)}{\vartheta_4(\pi/6, p^2)}, \quad y_1 = \frac{1}{y_0}, \quad y_2 = -\frac{1}{y_0}, \quad y_3 = -y_0. \tag{4.78}$$

*In particular, $P(\zeta, y) = 0$ for $(\zeta, y) \in \mathbb{D}$ if and only if $y = y_0$.*

*Proof.* First, we substitute the parameterisation (4.30) into the polynomial $P(\zeta, y)$ and find

$$P(\zeta, y) = \frac{C\vartheta_1(\pi/6 - t, p)\vartheta_1(\pi/6 + t, p)}{\vartheta_4(t, p^2)^4}, \tag{4.79}$$

where $C = (\vartheta_1(\pi/3, p)\vartheta_4(0, p^2)/\vartheta_4(\pi/3, p^2))^2$. The right-hand side vanishes if and only if

$$t = \pm\pi/6,\ t = \pm\pi/6 + \mathrm{i}s \mod \pi, 2\mathrm{i}s, \tag{4.80}$$

where $s > 0$ is defined through $p = e^{-s}$. The evaluations of $y$ at these values of $t$ lead to the four roots given in (4.78).

Second, we conclude from (4.78) that $(\zeta, y_0) \in \mathbb{D}$ but $(\zeta, y_\alpha) \notin \mathbb{D}$ for $\alpha = 1, 2, 3$. $\square$

In terms of the parameterisation (4.30), this lemma implies that $P(\zeta, y)$ vanishes for $(p, t) \in \bar{\mathbb{D}}$ if and only if $t = \pi/6$. We exploit this property in the following proof.

**Theorem 4.5.7.** *For each $L \geqslant 1$ and $0 < \zeta < 1$, the space of the ground states of the Hamiltonian (4.1) is equal to the space of supersymmetry singlets if and only if $y$ is a solution of the polynomial equation*

$$\zeta(1 + y^4) - (3 - \zeta^2)y^2 = 0. \tag{4.81}$$

*This space is one-dimensional, and the corresponding ground-state eigenvalue is given by*

$$E_0 = -\frac{(L-1)(3+\zeta^2)}{4} - \frac{(1+\zeta)^2}{2}. \tag{4.82}$$

*Proof of Theorem 4.5.7.* First, we prove the theorem for $(\zeta, y) \in \mathbb{D}$. To this end, we recall the relation (4.15) that expresses the Hamiltonian $H$ in terms of $H_{\text{XYZ}}$ for $L \geqslant 1$ sites:

$$H = x\left(H_{\text{XYZ}} + \frac{(L-1)(3+\zeta^2)}{4} + 2\lambda_0\right). \tag{4.83}$$

The factor $x$ in this relation is positive for all $(\zeta, y) \in \mathbb{D}$. Hence, the spaces of the ground states of $H$ and $H_{\text{XYZ}}$ are equal. We use the parameterisation of $(\zeta, y) \in \mathbb{D}$ by $(p, t) \in \bar{\mathbb{D}}$. According to Theorem 4.5.1, the space of the ground states of $H$ is spanned by the supersymmetry singlet $|\Psi_L\rangle$ if and only if $t = \pi/6$. We use Lemma 4.5.6 to conclude that the space of the ground states of $H_{\text{XYZ}}$ consists of supersymmetry singlets if and only if $y = y_0$. According to (4.15) the corresponding ground-state eigenvalue of this Hamiltonian is

$$E_0 = -\frac{(L-1)(\zeta^2+3)}{4} - 2\lambda_0\Big|_{y=y_0} = -\frac{(L-1)(\zeta^2+3)}{4} - \frac{(1+\zeta)^2}{2}. \tag{4.84}$$

Second, we consider $0 < \zeta < 1$ and $(\zeta, y) \notin \mathbb{D}$. In this case, it follows from (4.22) that there is an integer $1 \leqslant \alpha \leqslant 3$ such that

$$H_{\text{XYZ}}(\zeta, y) = \mathcal{R}^\alpha(-\pi) H_{\text{XYZ}}(\zeta, \bar{y}) \mathcal{R}^\alpha(\pi) \tag{4.85}$$

with $(\zeta, \bar{y}) \in \mathbb{D}$. Since $\mathcal{R}^\alpha(\pi)$ is a unitary operator, the two Hamiltonians in this equality have the same spectrum. Furthermore, writing $\mathfrak{Q}' = \mathfrak{Q}'(\zeta, y)$ to indicate the dependence of the supercharge on $\zeta$ and $y$, we have

$$\mathfrak{Q}'(\zeta, y_\alpha) = \mathcal{R}^\alpha(-\pi)\mathfrak{Q}'(\zeta, y_0)\mathcal{R}^\alpha(\pi). \tag{4.86}$$

The state $|\Psi_L^\alpha\rangle = \mathcal{R}^\alpha(-\pi)|\Psi_L\rangle$ is a supersymmetry singlet with respect to the supercharge $\mathfrak{Q}'(\zeta, y_\alpha)$. We conclude from these two observations that the space of the ground states of $H_{\text{XYZ}}(\zeta, y)$ is a space of supersymmetry singlets if and only if $\bar{y} = y_0$, and hence $y = y_\alpha$. This space is one-dimensional and spanned by the supersymmetry singlet $|\Psi_L^\alpha\rangle$. $\square$

# Chapter 5

# The supersymmetric eight-vertex model with open boundary conditions

In this chapter, we revisit the eight-vertex model introduced in the Chapter 3. We consider the model with open boundary conditions: the geometry of the domain is a strip. The transfer matrix of the eight-vertex model with open boundary conditions is built from $R$-matrices, containing the weights of a bulk vertex, and so-called $K$-matrices, which encode the boundary conditions. As in Chapter 3, we study the model by exploring the relation between its transfer matrix and an XYZ Hamiltonian.

We proceed with the investigation of the supersymmetric eight-vertex model, for which the weights $a, b, c, d$ satisfy

$$(a^2 + ab)(b^2 + ab) = (c^2 + ab)(d^2 + ab). \tag{5.1}$$

This choice is related to the Hamiltonian density of the supersymmetric XYZ spin-chain Hamiltonian. We focus on the case where $0 < \frac{cd}{ab} < 1$. Furthermore, we choose the $K$-matrices in accordance with the boundary terms $h_{\mathrm{B}}^{\pm}$ of the Hamiltonian (4.1). We specify those matrices later.

The main result of this chapter is to prove that the transfer matrix possesses a remarkable eigenvalue. To be precise, if the space of the ground states of the Hamiltonian (4.1) equals the space of supersymmetry singlets of the supercharge $\mathfrak{Q}'$; then the transfer matrix possesses the eigenvalue $\Lambda_L$ given by

$$\Lambda_L = (a+b)^{2L}\,\mathrm{tr}(K^+K^-), \tag{5.2}$$

where $K^\pm$ are specifically chosen $K$-matrices. The eigenvalue is non-degenerate, and its eigenspace is spanned by the ground states of the Hamiltonian (4.1) with $\zeta = \frac{cd}{ab}$.

The layout of this chapter is similar to the one of Chapter 3. In Section 5.1, we construct the transfer matrix of the eight-vertex model with open boundary conditions and its relation to the Hamiltonian of the XYZ spin chain. We focus on the supersymmetric case in Section 5.2: we establish a commutation relation between the transfer matrix and the supercharge of the spin chain. This relation allows us to prove the existence of the eigenvalue $\Lambda_L$ in Section 5.3. Finally, we analyse the positivity of the transfer matrix and use the Perron-Frobenius theorem to prove that $\Lambda_L$ is the largest eigenvalue of the transfer matrix if the vertex weights are positive.

## 5.1 Lattice formulation and transfer matrix

In this section, we consider the eight-vertex model with open boundary conditions, introduce the $K$-matrices, and construct the corresponding transfer matrix. We recall a few of its elementary properties and its relation to the XYZ Hamiltonian with open boundary conditions.

Let us consider the eight-vertex model introduced in Chapter 3 on a strip geometry. The domain of the model consists of a square lattice of $L$ vertical lines and $N$ pairs of horizontal lines. We call each intersection a *bulk vertex*. The boundaries are formed as follows: each of the $N$ pairs of consecutive rows merges at the extremities of the domain. Each of these merging points creates a *boundary vertex*. As previously, we refer to a segment between two adjacent vertices as an edge.

On each edge is present an arrow. A bulk vertex can be in one configuration amongst the eight admissible. Each boundary vertex connects two edges that can be in two possible states. Hence, a boundary vertex

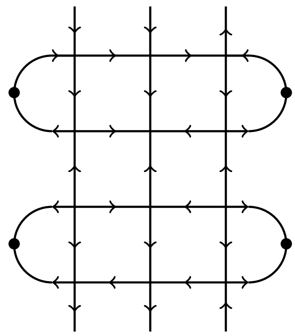


Figure 5.1: An admissible configuration of the eight-vertex model on a strip with $L = 3$, $N = 4$. The intersections of vertical and horizontal lines are bulk vertices while the boundary vertices are indicated by black dots.

has four admissible configurations. The Figure 5.1 gives an example of a configuration of the eight-vertex model on a strip.

***R* and *K* matrices.** The transfer matrix of the eight-vertex model on the strip is constructed from the $R$-matrix. We use the same notation as in Chapter 3: the $R$-matrix is given, in the canonical basis of $V \otimes V$ by

$$R = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}, \tag{5.3}$$

where $a, b, c, d$ are the vertex weights. We denote by $R_{ij}$, $0 \leqslant i < j \leqslant L$, the $R$-matrix acting non-trivially only on the factors $V_i$ and $V_j$ of the tensor product $V_0 \otimes V^L = V_0 \otimes V_1 \otimes \cdots \otimes V_L$. For convenience, we introduce the abbreviations

$$U_{0,[i,j]} = R_{0j} R_{0j-1} \cdots R_{0i}, \quad \bar{U}_{0,[i,j]} = R_{0i} R_{0i+1} \cdots R_{0j}, \tag{5.4}$$

for $1 \leqslant i \leqslant j \leqslant L$. We also define $U_{0,[j+1,j]} = \bar{U}_{0,[j+1,j]} = \mathbf{1}$ for $j = 0, \ldots, L$.

We encode the weights of the boundary vertex in the $K$-matrices. There are two boundaries that we treat separately. We denote by $K^+$ and $K^-$ the matrix containing the weight of the vertex on the right and the left

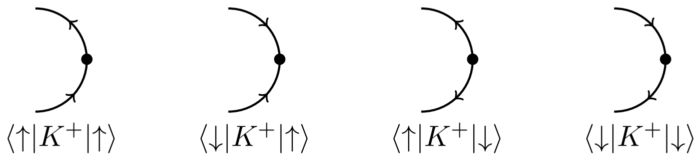



Figure 5.2: The four vertex configurations on the right boundary and the corresponding weights given in the $K^+$ matrix.

of the strip, respectively. Let us consider the $K^+$ matrix. It connects two edges on which are an arrow. As in Chapter 3, let us translate arrows configurations into the spin language. To this end, we label the edge on the top and the bottom of the boundary vertex with 1 and 2. We assign a spin up to an edge if it carries an arrow pointing toward the north, and a spin down if the arrow points to the south. The entry $\langle s_1|K^+|s_2\rangle$ of the $K^+$ matrix contains the weight of a configuration with spin $s_i$ on the $i$-th edge. The Figure 5.2 depicts the arrows configurations at the right boundary vertex and the corresponding weights.

The definition of the $K^-$ matrix is similar. We consider the two edges merging at a boundary vertex on the left of the domain and label them from the bottom to the top with 1 and 2. We assign a spin up to an arrow pointing south and, conversely, a spin down to an arrow that points to the north. We write the weight of a configuration with spin $s_i$ on the $i$-th edge in the entry $\langle s_1|K^+|s_2\rangle$ of the $K^-$ matrix. The Figure 5.3 represents generic spin configuration at the boundary vertex and their corresponding weight in the $K$-matrices.


Figure 5.3: Generic spin configuration $s_1, s_2$ at a boundary vertex and the corresponding weight $\langle s_1|K^\pm|s_2\rangle$.

**Transfer matrix.** The transfer matrix of the eight-vertex model for a strip with $L$ vertical lines and open boundary conditions is an operator

$\mathcal{T} : V^L \to V^L$ defined as

$$\mathcal{T} = \mathrm{tr}_0 \left( K_0^+ U_{0,[1,L]} K_0^- \bar{U}_{0,[1,L]} \right) . \tag{5.5}$$

The trace is taken with respect to the auxiliary space $V_0$. Moreover, $K_0^{\pm}$ are operators $K^{\pm} : V \to V$ acting on the auxiliary space.

To investigate the properties of the transfer matrix, we use the parameterisation (3.18) of the vertex weights in terms of Jacobi theta functions. We recall that with this parameterisation, the $R$-matrix of the eight-vertex model obeys the Yang-Baxter equation (3.19). Furthermore, we choose

$$K^- = K(u), \quad K^+ = K(u + 2\eta), \tag{5.6}$$

where the operator $K = K(u)$ is a solution of the *reflection equation*: for all $u$ and $v$ it obeys

$$R_{12}(u-v)K_1(u)R_{12}(u+v)K_2(v) = K_2(v)R_{12}(u+v)K_1(u)R_{12}(u-v), \tag{5.7}$$

where $R = R(u)$, and $K_i(u)$ denotes the operator $K(u)$ acting on $V_i$.

Let us write $\mathcal{T} = \mathcal{T}(u)$ to stress the dependence of the transfer matrix on the spectral parameter. The choice (5.6) implies that transfer matrices with different spectral parameters commute: we have

$$\mathcal{T}(u)\mathcal{T}(v) = \mathcal{T}(v)\mathcal{T}(u), \tag{5.8}$$

for all $u$ and $v$ [35]. The proof of this commutation relation is based on the Yang-Baxter equation (3.19) and the reflection equation (5.7).

**Transfer matrix and Hamiltonian.** We now recall the relation between the transfer matrix and the Hamiltonian of the XYZ spin chain [35]. To this end, we use the $K$-matrix

$$K(u) = \mathbf{1} + \sum_{\alpha=1}^{3} \frac{\vartheta_1(u,p)}{\vartheta_{5-\alpha}(u,p)} \mu_\alpha \sigma^\alpha. \tag{5.9}$$

Here $\mu_1, \mu_2, \mu_3$ are arbitrary complex numbers. Up to an overall factor, this $K$-matrix is the most general solution to the reflection equation (5.7) of the eight-vertex model [102, 103, 104].

**Proposition 5.1.1.** *We have the logarithmic derivative*

$$\mathcal{T}(0)^{-1}(\mathcal{T})'(0) = L\left(\frac{a'(0)+c'(0)}{a(0)}\right) - \frac{2b'(0)}{a(0)}H_{\text{XYZ}}. \tag{5.10}$$

*Here, $H_{\text{XYZ}}$ is the Hamiltonian (4.1a) of the open XYZ spin chain with the anisotropy parameters*

$$J_1 = 1 + \frac{d'(0)}{b'(0)}, \quad J_2 = 1 - \frac{d'(0)}{b'(0)}, \quad J_3 = \frac{a'(0)-c'(0)}{b'(0)}, \tag{5.11}$$

*and the boundary terms*

$$h_{\text{B}}^{\pm} = -\frac{\vartheta_1(2\eta,p)}{2}\sum_{\alpha=1}^{3}\frac{J_\alpha\mu_\alpha}{\vartheta_{5-\alpha}(2\eta,p)}\sigma^\alpha. \tag{5.12}$$

*Proof.* We have $R(0) = a(0)P$, where $P$ is the permutation operator on $V\otimes V$, $\operatorname{tr} K(u) = 2$, $\operatorname{tr} K'(u) = 0$ and $K(0) = \mathbf{1}$. After a standard calculation, we obtain the logarithmic derivative

$$\mathcal{T}(0)^{-1}(\mathcal{T})'(0) = \frac{2}{a(0)}\sum_{j=1}^{L-1}\check{R}'_{jj+1}(0) + K'_1(0) + \frac{1}{a(0)}\text{tr}_0\left(K_0(2\eta)\check{R}'_{0L}(0)\right), \tag{5.13}$$

where $\check{R}(u) = PR(u)$. The $\check{R}$-matrix has the property

$$\check{R}'(0) = \frac{a'(0)+c'(0)}{2} + \frac{b'(0)}{2}\sum_{\alpha=1}^{3}J_\alpha\sigma^\alpha\otimes\sigma^\alpha, \tag{5.14}$$

where the anisotropy parameters $J_1, J_2, J_3$ are given by (5.11). The insertion of this expression into (5.13) leads to (5.10) with the boundary terms

$$(h_{\text{B}}^-)_1 = -\frac{a(0)}{2b'(0)}K'_1(0), \quad (h_{\text{B}}^+)_L = -\frac{1}{4}\text{tr}_0\left(K_0(2\eta)\sum_{\alpha=1}^{3}J_\alpha\sigma_0^\alpha\otimes\sigma_L^\alpha\right). \tag{5.15}$$

The evaluation of the partial trace for $h_{\text{B}}^+$ is straightforward and leads to the expression given in (5.12). To see that $h_{\text{B}}^-$ is given by the same expression, we first note that the parameterisation (3.18) and (5.11) lead to

$$J_\alpha = J\frac{\vartheta_{5-\alpha}(2\eta,p)}{\vartheta_{5-\alpha}(0,p)} \quad \text{for} \quad \alpha = 1,2,3, \tag{5.16}$$

where $J = (\vartheta_4(0,p^2)/\vartheta_4(2\eta,p^2))^2$. Hence, we obtain

$$h_{\text{B}}^{-} = -\frac{a(0)\vartheta_1'(0,p)}{2b'(0)}\sum_{\alpha=1}^{3}\frac{\mu_\alpha}{\vartheta_{5-\alpha}(0,p)}\sigma^\alpha \tag{5.17}$$

$$= -\frac{a(0)\vartheta_1'(0,p)}{2Jb'(0)}\sum_{\alpha=1}^{3}\frac{J_\alpha\mu_\alpha}{\vartheta_{5-\alpha}(2\eta,p)}\sigma^\alpha. \tag{5.18}$$

It remains to be shown that $a(0)\vartheta_1'(0,p)/(Jb'(0)) = \vartheta_1(2\eta,p)$, which can be accomplished with the help of identities for Jacobi theta functions [91]. □

An immediate consequence of (5.8), (5.10) and $\mathcal{T}(0) = 2a(0)^{2L}$ is:

**Corollary 5.1.2.** *We have* $[H_{\text{XYZ}}, \mathcal{T}(u)] = 0$ *where the XYZ Hamiltonian has the anisotropy parameters* (5.11) *and boundary terms* (5.12).

## 5.2 The transfer matrix and supersymmetry

In this section, we focus on the supersymmetric eight-vertex model, for which the weights satisfy (5.1) and determine the corresponding $K$-matrices. We show that an eigenvector of the transfer matrix with the eigenvalue $\Lambda_L$ (5.2) is a ground state of the supersymmetric XYZ Hamiltonian that we investigate in the previous chapter. We establish a commutation relation between the transfer matrix of the supersymmetric eight-vertex model with open boundary conditions and the supercharge of the supersymmetric open XYZ spin chain.

**Supersymmetric eight-vertex model.** The parameterisation (3.18) of the vertex weights in terms of Jacobi theta functions depends on the parameters $u, \rho, \eta$ and $p$. From now, we consider the crossing parameter

$$\eta = \frac{\pi}{3}, \tag{5.19}$$

real $\rho, u$ and $0 < p < 1$. For this choice, the weights $a, b, c, d$ are real and obey the relation (5.1) that defines the supersymmetric eight-vertex model. The spin chain's anisotropy parameters (5.11) coincide with the expressions given in (4.1), where $0 < \zeta < 1$ is defined by

$$\zeta = \frac{cd}{ab}. \tag{5.20}$$

It follows from Corollary 5.1.2 that the transfer matrix of the eight-vertex model commutes with the Hamiltonian (4.1) provided that the parameters of the $K$-matrix are given by

$$\mu_1 = \frac{\vartheta_4(\eta,p)}{\vartheta_1(\eta,p)}\frac{2\operatorname{Re} y}{1+|y|^2}, \quad \mu_2 = \frac{\vartheta_3(\eta,p)}{\vartheta_1(\eta,p)}\frac{2\operatorname{Im} y}{1+|y|^2}, \quad \mu_3 = \frac{\vartheta_2(\eta,p)}{\vartheta_1(\eta,p)}\frac{1-|y|^2}{1+|y|^2}. \tag{5.21}$$

We express the corresponding $K$-matrices $K^{\pm}$ in terms of the vertex weights and the parameter $y$ in the following proposition. Its proof relies on a few identities for Jacobi theta functions.

**Proposition 5.2.1.** *For the choice (5.21), the $K$-matrices $K^{\pm}$ are given by*

$$K^- = \mathbf{1} + \frac{2\operatorname{Re} y}{1+|y|^2}\frac{ab+cd}{ac+bd}\sigma^1 + \frac{2\operatorname{Im} y}{1+|y|^2}\frac{ab-cd}{ac-bd}\sigma^2 + \frac{1-|y|^2}{1+|y|^2}\frac{b^2-d^2}{2ab+b^2+d^2}\sigma^3, \tag{5.22}$$

$$K^+ = \mathbf{1} + \frac{2\operatorname{Re} y}{1+|y|^2}\frac{ab+cd}{ad+bc}\sigma^1 + \frac{2\operatorname{Im} y}{1+|y|^2}\frac{ab-cd}{bc-ad}\sigma^2 + \frac{1-|y|^2}{1+|y|^2}\frac{b^2-c^2}{2ab+b^2+c^2}\sigma^3. \tag{5.23}$$

In the next proposition, we consider the transfer matrix of the supersymmetric eight-vertex model with these $K$-matrices and with $y$ being a solution of (4.81). This polynomial equation was given by:

$$\zeta(1+y^4) - (3-\zeta^2)y^2 = 0. \tag{5.24}$$

For this case, we show that if $\Lambda_L$, defined in (5.2), is a transfer-matrix eigenvalue, then its eigenspace is contained in the space of the supersymmetry singlet of the XYZ Hamiltonian.

**Proposition 5.2.2.** *Let $L \geqslant 1$, $0 < \zeta < 1$ and $y$ be a solution of (4.81). If $|\psi\rangle \in V^L$ obeys*

$$\mathcal{T}|\psi\rangle = \Lambda_L|\psi\rangle, \tag{5.25}$$

*where $\Lambda_L$ is given in (5.2), then $|\psi\rangle$ is a supersymmetry singlet of the $XYZ$ Hamiltonian (4.1) with $\zeta$ given by (5.20).*

*Proof.* We use the theta-function parameterisation of the eight-vertex model. It follows from (5.10) that $|\psi\rangle$ is an eigenstate of the XYZ Hamiltonian (4.1) for the eigenvalue

$$E = -L\left(\frac{a'(0)-c'(0)}{2b'(0)}+1\right) - \frac{a(0)}{4b'(0)}\mathrm{tr}\left(K'(0)K(2\eta)\right). \tag{5.26}$$

In the first term on the right-hand side of this equality, we recognise the expression (5.11) for the anisotropy parameter $J_3 = \frac{1}{2}(\zeta^2-1)$. To compute the second term, we use the parameterisation (5.9) of the $K$-matrix in terms of the parameters $\mu_1, \mu_2, \mu_3$ given by (5.21), as well as the expression (5.16) for the anisotropy parameters. We have

$$\frac{a(0)}{4b'(0)}\mathrm{tr}\left(K'(0)K(2\eta)\right) = \frac{1}{2}\sum_{\alpha=1}^{3} J_\alpha \frac{\vartheta_1^2(2\eta,p)}{\vartheta_{5-\alpha}^2(2\eta,p)}\mu_\alpha^2 = 2\sum_{\alpha=1}^{3}\frac{\lambda_\alpha^2}{J_\alpha}. \tag{5.27}$$

The constants $\lambda_1, \lambda_2, \lambda_3$ are given in (4.1). We use their explicit expression and the relation (4.81) between $\zeta$ and $y$ to compute $\sum_{\alpha=1}^{3}\lambda_\alpha^2/J_\alpha = (\zeta^2+4\zeta-1)/8$. This yields the eigenvalue

$$E = -\frac{L(3+\zeta^2)}{4} - \frac{\zeta^2+4\zeta-1}{4}. \tag{5.28}$$

We conclude that $E$ is the ground-state eigenvalue $E_0$, defined in (4.82). It follows from Theorem 4.5.7 that $|\psi\rangle$ is a supersymmetry singlet. □

**Transformations of the transfer matrix.** The transfer matrix of the supersymmetric eight-vertex model with the $K$-matrices (5.22) has a simple transformation behaviour under certain spin rotations. Let us write $\mathcal{T} = \mathcal{T}(a,b,c,d;y)$, to stress the dependence of the transfer matrix on the vertex weights $a,b,c,d$ and the parameter $y$. We have

$$\begin{aligned}
\mathcal{R}^1(\pi)\mathcal{T}(a,b,c,d;y)\mathcal{R}^1(-\pi) &= \mathcal{T}(a,b,c,d;y^{-1}),\\
\mathcal{R}^2(\pi)\mathcal{T}(a,b,c,d;y)\mathcal{R}^2(-\pi) &= \mathcal{T}(a,b,c,d;-y^{-1}),\\
\mathcal{R}^3(\pi)\mathcal{T}(a,b,c,d;y)\mathcal{R}^3(-\pi) &= \mathcal{T}(a,b,c,d;-y),
\end{aligned} \tag{5.29}$$

where $\mathcal{R}^\alpha(\theta)$ is the spin-rotation operator (4.20). These relations are similar to (4.22). (It is possible to work out the transformation behaviour under rotations by the angle $\theta = \pi/2$, but we will not use it.) We note that since these transformations are unitary, the transfer matrices on the right-hand side of these equalities have the same spectrum as $\mathcal{T}(a,b,c,d;y)$.

We now establish a commutation relation between $\mathcal{T}$ and the supercharge of the supersymmetric open XYZ spin chain. To this end, we first establish local relations between the $R$-matrix of the eight-vertex model, the $K$-matrices, the local supercharge of the XYZ Hamiltonian, and certain auxiliary operators. Second, we combine these local relations with the definition of the transfer matrix to obtain the commutation relation.

**Local relations.** We follow the strategy of the Chapter 3 and define two operators $\mathfrak{a}^\uparrow, \mathfrak{a}^\downarrow : V \to V \otimes V$. Their action on the basis states $|\uparrow\rangle$ and $|\downarrow\rangle$ is given by

$$\mathfrak{a}^\uparrow|\uparrow\rangle = d\left(-\frac{c}{a}|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right), \quad \mathfrak{a}^\uparrow|\downarrow\rangle = c\left(|\uparrow\uparrow\rangle - \frac{d}{b}|\downarrow\downarrow\rangle\right), \tag{5.30}$$

$$\mathfrak{a}^\downarrow|\uparrow\rangle = c\left(|\downarrow\downarrow\rangle - \frac{d}{b}|\uparrow\uparrow\rangle\right), \quad \mathfrak{a}^\downarrow|\downarrow\rangle = d\left(-\frac{c}{a}|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right). \tag{5.31}$$

We also define an operator $\mathfrak{a}_\phi : V \to V \otimes V$ through the following action on the basis states:

$$\begin{aligned}\mathfrak{a}_\phi|\uparrow\rangle &= (2a+b)\phi_\uparrow|\uparrow\uparrow\rangle + (a+2b)\phi_\downarrow|\uparrow\downarrow\rangle + c\phi_\downarrow|\downarrow\uparrow\rangle + d\phi_\downarrow|\downarrow\downarrow\rangle,\\ \mathfrak{a}_\phi|\downarrow\rangle &= (2a+b)\phi_\downarrow|\downarrow\downarrow\rangle + (a+2b)\phi_\uparrow|\downarrow\uparrow\rangle + c\phi_\uparrow|\uparrow\downarrow\rangle + d\phi_\uparrow|\uparrow\uparrow\rangle.\end{aligned} \tag{5.32}$$

Here, $\phi_\uparrow = y(y^2\zeta - 1)$ and $\phi_\downarrow = \zeta - y^2$ are the components of the state $|\phi\rangle$ defined in the local supercharge $\mathfrak{q}'$(4.14). We use the operators $\mathfrak{a}^\uparrow, \mathfrak{a}^\downarrow$, and $\mathfrak{a}_\phi$ to define the linear combination

$$\mathfrak{a} = (1 - y^2\zeta)\mathfrak{a}^\uparrow + y(y^2 - \zeta)\mathfrak{a}^\downarrow + \mathfrak{a}_\phi. \tag{5.33}$$

We also need an action of $\mathfrak{a}, \mathfrak{a}^\uparrow, \mathfrak{a}^\downarrow$ and $\mathfrak{a}_\phi$ on the space $V_0 \otimes V^L$. To this end, we use a notation similar to the one we introduced in Chapter 3: for each operator $\mathfrak{b} : V \to V \otimes V$ we define $\mathfrak{b}_0^j : V_0 \otimes V^L \to V_0 \otimes V^{L+1}$, $j = 1, \ldots, L+1$ by

$$\mathfrak{b}_0^1 = \mathfrak{b} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{L} \tag{5.34}$$

and, recursively,

$$\mathfrak{b}_0^{j+1} = P_{jj+1}\mathfrak{b}_0^j, \tag{5.35}$$

for each $j = 1, \ldots, L$. Here, $P_{jj+1}$, $j = 1, \ldots, L$ denotes the permutation operator acting on the factors $V_j$ and $V_{j+1}$ of the tensor product $V_0 \otimes V^{L+1}$.

In the next two lemmas, we establish several relations between the $R$-matrix of the supersymmetric eight-vertex model, the $K$-matrices $K^{\pm}$ defined in (5.22), the local supercharge $\mathfrak{q}'$ and the operator $\mathfrak{a}$.

**Lemma 5.2.3.** *For each $j = 1, \ldots, L$ we have*

$$R_{0j}R_{0j+1}(\mathbf{1}\otimes\mathfrak{q}_j') + (a+b)(\mathbf{1}\otimes\mathfrak{q}_j')R_{0j} = R_{0j}\mathfrak{a}_0^{j+1} + \mathfrak{a}_0^j R_{0j}, \tag{5.36a}$$

$$R_{0j+1}R_{0j}(\mathbf{1}\otimes\mathfrak{q}_j') + (a+b)(\mathbf{1}\otimes\mathfrak{q}_j')R_{0j} = R_{0j+1}\mathfrak{a}_0^{j} + \mathfrak{a}_0^{j+1} R_{0j}, \tag{5.36b}$$

*if and only if* (5.1) *holds.*

*Proof.* The multiplication of (5.36a) from the left by $P_{jj+1}$ yields (5.36b) by virtue of $P_{jj+1}\mathfrak{q}_j = \mathfrak{q}_j$. Hence, it is sufficient to prove (5.36a).

The key observation is that each of the relations

$$\begin{aligned}
&R_{01}R_{02}(\mathbf{1}\otimes(\mathfrak{q}^{\uparrow})_1) + (a+b)(\mathbf{1}\otimes(\mathfrak{q}^{\uparrow})_1)R_{01} = R_{01}(\mathfrak{a}^{\uparrow})_0^2 + (\mathfrak{a}^{\uparrow})_0^1 R_{01},\\
&R_{01}R_{02}(\mathbf{1}\otimes(\mathfrak{q}^{\downarrow})_1) + (a+b)(\mathbf{1}\otimes(\mathfrak{q}^{\downarrow})_1)R_{01} = R_{01}(\mathfrak{a}^{\downarrow})_0^2 + (\mathfrak{a}^{\downarrow})_0^1 R_{01},\\
&R_{01}R_{02}(\mathbf{1}\otimes(\mathfrak{q}_{\phi})_1) + (a+b)(\mathbf{1}\otimes(\mathfrak{q}_{\phi})_1)R_{01} = R_{01}(\mathfrak{a}_{\phi})_0^2 + (\mathfrak{a}_{\phi})_0^1 R_{01},
\end{aligned} \tag{5.37}$$

holds if and only if the vertex weights obey (5.1), as follows from a straightforward calculation. We obtain (5.36a), for $j = 1$, using the definition (5.33). Its generalisation to $j = 2, \ldots, L$ is readily obtained through the conjugation with appropriate products of permutation operators. □

**Lemma 5.2.4.** *The $K$-matrices* (5.22) *obey*

$$(a+b)\mathfrak{a}_0^1 K_0^- = R_{01}K_0^-\mathfrak{a}_0^1, \tag{5.38}$$

$$(a+b)(\mathfrak{a}_0^1)^{t_0}(K_0^+)^{t_0} = (R_{01})^{t_0}(K_0^+)^{t_0}(\mathfrak{a}_0^1)^{t_0}, \tag{5.39}$$

*if and only* (5.1) *holds. Here, the superscript $t_0$ denotes the transposition with respect to the auxiliary space.*

*Proof.* The proof is a straightforward calculation. □

**The commutation relation.** We now use the Lemmas 5.2.3 and 5.2.4 to compute a commutation relation between the transfer matrix and the supercharge. This generalises a relation established by Weston and Yang [105] for the six-vertex model, corresponding to $d = 0$ (hence $\zeta = 0$) and $y = 0$.

**Proposition 5.2.5.** *If (5.1) holds and the $K$-matrices $K^{\pm}$ are given by (5.22) then*

$$\mathcal{T}\mathfrak{Q}' = (a+b)^2\mathfrak{Q}'\mathcal{T}. \tag{5.40}$$

*Proof.* First, we evaluate a commutator between the transfer matrix and the local supercharge $\mathfrak{q}_j$. To this end, we use

$$R_{0k}(\mathbf{1}\otimes\mathfrak{q}_j') = (\mathbf{1}\otimes\mathfrak{q}_j')R_{0k}, \qquad \text{if } 1 \leqslant k < j \leqslant L, \tag{5.41}$$

$$R_{0k}(\mathbf{1}\otimes\mathfrak{q}_j') = (\mathbf{1}\otimes\mathfrak{q}_j')R_{0k-1}, \qquad \text{if } 1 \leqslant j < k-1 \leqslant L-1. \tag{5.42}$$

We apply them together with Lemma 5.2.3 to obtain

$$\begin{aligned}
&\mathcal{T}\mathfrak{q}_j' - (a+b)^2\mathfrak{q}_j'\mathcal{T}\\
&= \operatorname{tr}_0\Big(K_0^+U_{0,[1,L+1]}K_0^-\bar{U}_{0,[1,j-1]}\left(R_{0j}\mathfrak{a}_0^{j+1} + \mathfrak{a}_0^{j}R_{0j}\right)\bar{U}_{0,[j+1,L]}\Big)\\
&-(a+b)\operatorname{tr}_0\Big(K_0^+U_{0,[j+2,L+1]}\left(R_{0j+1}\mathfrak{a}_0^{j} + \mathfrak{a}_0^{j+1}R_{0j}\right)U_{0,[1,j-1]}K_0^-\bar{U}_{0,[1,L]}\Big),
\end{aligned}$$

for $j = 1, \dots, L$.

Second, we take an alternating sum of these equalities and find

$$\begin{aligned}
&\mathcal{T}\mathfrak{Q}' - (a+b)^2\mathfrak{Q}'\mathcal{T}\\
&\qquad = \operatorname{tr}_0\left(K_0^+U_{0,[2,L+1]}\left((a+b)\mathfrak{a}_0^1K_0^- - R_{01}K_0^-\mathfrak{a}_0^1\right)\bar{U}_{0,[1,L]}\right)\\
&\qquad\quad + (-1)^L\left(\operatorname{tr}_0\left(K_0^+R_{0L+1}\mathcal{U}\mathfrak{a}_0^{L+1}\right) - (a+b)\operatorname{tr}_0\left(K_0^+\mathfrak{a}_0^{L+1}\mathcal{U}\right)\right),
\end{aligned} \tag{5.43}$$

where we used the shorthand notation $\mathcal{U} = U_{0,[1,L]}K_0^-\bar{U}_{0,[1,L]}$. The relation (5.38) implies that the first term on the right-hand side of (5.43) vanishes. To evaluate the second term, we compute

$$\begin{aligned}
&\operatorname{tr}_0\left(K_0^+R_{0L+1}\mathcal{U}\mathfrak{a}_0^{L+1}\right) = \operatorname{tr}_0\left(\mathcal{U}^{t_0}(R_{0L+1})^{t_0}(K_0^+)^{t_0}(\mathfrak{a}_0^{L+1})^{t_0}\right)\\
&= (a+b)\operatorname{tr}_0\left(\mathcal{U}^{t_0}(\mathfrak{a}_0^{L+1})^{t_0}(K_0^+)^{t_0}\right) = (a+b)\operatorname{tr}_0\left(K_0^+\mathfrak{a}_0^{L+1}\mathcal{U}\right).
\end{aligned} \tag{5.44}$$

To establish this equality, we used the invariance of the trace under matrix transposition and applied the identity $(R_{0L+1})^{t_0}(K_0^+)^{t_0}(\mathfrak{a}_0^{L+1})^{t_0} =$

$(a+b)(\mathfrak{a}_0^{L+1})^{t_0}(K_0^+)^{t_0}$, which follows from (5.39) after an appropriate multiplication with permutation operators. Hence, we conclude that the second term on the right-hand side of (5.43) vanishes, too. $\square$

## 5.3 The transfer-matrix eigenvalue

In this final section, we prove the result claimed at the beginning of the chapter on the existence of the eigenvalue $\Lambda_L$. We prepare its proof by establishing a few auxiliary results. Below, we denote by $\mathcal{T}$ the transfer matrix of the supersymmetric eight-vertex model on a strip with $L \geqslant 1$ vertical lines, the $K$-matrices $K^\pm$ defined in (5.22) and $t = \pi/6$.

We compute the action of this transfer matrix on the supersymmetry singlet $|\Psi_L\rangle$ defined in (4.69). This singlet is an eigenstate of $H$, and thus of $H_{\text{XYZ}}$. Therefore, it is an eigenstate of $\mathcal{T}$. The eigenvalue $\Lambda_L$ can be obtained as

$$\Lambda_L = \frac{\langle \Psi_L|\mathcal{T}|\Psi_L\rangle}{\langle \Psi_L|\Psi_L\rangle}. \tag{5.45}$$

We evaluate this quotient by using the Proposition 1.4.9 of Chapter 1 which allows for the evaluation of matrix element with respect to supersymmetry singlets of operators that commute with the supercharge.

It follows from Proposition 5.2.5 that if $a+b \neq 0$, then we may apply Proposition 1.4.9 with $\mathcal{A} = \mathcal{T}$ and $\lambda = (a+b)^2$ to evaluate the matrix element $\langle \Psi_L|\mathcal{T}|\Psi_L\rangle$. Furthermore, we compute the square norm $\langle \Psi_L|\Psi_L\rangle$ with the help of this proposition for $\mathcal{A} = \mathbf{1}$ and $\lambda = 1$. The resulting expressions depend on the choice of the decompositions of $|\Psi_L\rangle$. First, using (4.69) and (4.70), we have

$$\Lambda_L = \frac{\left(\langle w_+|^{\otimes L}\right)\mathcal{T}\left(|v_+\rangle^{\otimes L}\right)}{\langle w_+|v_+\rangle^L}, \tag{5.46}$$

for each $L \geqslant 1$. Second, using the alternative representations (4.73) and (4.74), we find

$$\Lambda_L = \frac{\left(\langle \alpha| \otimes \langle w_+|^{\otimes (L-2)}\right)\mathcal{T}\left(|\chi\rangle \otimes |v_+\rangle^{\otimes (L-2)}\right)}{\langle \alpha|\chi\rangle\langle w_+|v_+\rangle^{L-2}}, \tag{5.47}$$

for each $L \geqslant 2$. These two relations still hold if $a+b=0$. Indeed, the eigenvalues of a matrix are continuous functions of its entries [81]. Hence, $\Lambda_L$ is a continuous function of $a, b, c, d$.

We exploit (5.46) and (5.47) to establish a recurrence relation for the eigenvalue $\Lambda_L$. To this end, we need the following two lemmas:

**Lemma 5.3.1.** *For $t = \pi/6$, the $K$-matrices (5.22) obey*

$$\frac{\langle w_+|\mathrm{tr}_0\left(K_0^+ R_{01} K_0^- R_{01}\right)|v_+\rangle}{\langle w_+|v_+\rangle} = (a+b)^2 \mathrm{tr}(K^+K^-). \tag{5.48}$$

*Proof.* By Lemma 4.5.4, it is sufficient to show that

$$I = \langle \bar{w}_+|\mathrm{tr}_0\left(K_0^+ R_{01} K_0^- R_{01}\right)|\bar{v}_+\rangle - \langle \bar{w}_+|\bar{v}_+\rangle (a+b)^2 \mathrm{tr}(K^+K^-) \tag{5.49}$$

vanishes. This difference is a rational expression of the vertex weights $a, b, c, d$, $\zeta$ and the parameter $y$. Using the polynomial equation (4.81), as well as the relations (5.1) and (5.20), we find after some algebra, that is indeed zero. □

**Lemma 5.3.2.** *For $t = \pi/6$, the matrix $K^-$, defined in (5.22), obeys*

$$\frac{(\mathbf{1}\otimes\langle\alpha|)R_{02}R_{01}K_0^- R_{01}R_{02}(\mathbf{1}\otimes|\chi\rangle)}{\langle\alpha|\chi\rangle} = (a+b)^4 K_0^-. \tag{5.50}$$

*Proof.* By virtue of Lemma 4.5.5, the equality holds if the $2\times 2$ matrix

$$\bar{I} = (\mathbf{1}\otimes\langle\bar{\alpha}|)R_{02}R_{01}K_0^- R_{01}R_{02}(\mathbf{1}\otimes|\bar{\chi}\rangle) - (a+b)^4\langle\bar{\alpha}|\bar{\chi}\rangle K_0^- \tag{5.51}$$

vanishes. Its entries are rational expressions of the vertex weights $a, b, c$, $d$, $\zeta$ and the parameter $y$. As above, we use (4.81), (5.1) and (5.20) to show that its entries are indeed zero. □

We combine the previous lemmas to prove that the eigenvalue of the transfer matrix with respect to the supersymmetry singlet $|\Psi_L\rangle$ is given by (5.2).

**Theorem 5.3.3.** *Let $L \geqslant 1$, $0 < \zeta < 1$ and $y$ be a solution of (4.81), then the transfer matrix of the supersymmetric eight-vertex model on a strip with $L$ vertical lines and the $K$-matrices (5.22) possesses the non-degenerate eigenvalue $\Lambda_L$.*

*The corresponding eigenspace is the space of the supersymmetry singlets of the XYZ Hamiltonian (4.1) with $\zeta$ given by $\frac{cd}{ab}$.*

*Proof.* According to Proposition 5.2.2, if $L \geqslant 1$, $0 < \zeta < 1$, and if $y$ is a solution of (4.81), then any solution $|\psi\rangle$ of $\mathcal{T}|\psi\rangle = \Lambda_L|\psi\rangle$ is a supersymmetry singlet. This observation does, however, not guarantee that $\Lambda_L$ is an eigenvalue of the transfer matrix because a solution of the eigenvalue problem might not exist. To see that it is an eigenvalue, we thus evaluate the action transfer matrix on $|\Psi_L\rangle$. To this end, we use (5.45).

First, we consider $t = \pi/6$ and hence $y = y_0$, where $y_0$ is the unique real solution of (4.81) with $0 < y < 1$. We suppose $L \geqslant 3$, and use the definition of the transfer matrix to rewrite (5.47) as

$$\Lambda_L = \frac{1}{\langle\alpha|\chi\rangle\langle w_+|v_+\rangle^{L-2}}\left(\langle\alpha|\otimes\langle w_+|^{\otimes(L-2)}\right)\cdot \\ \operatorname{tr}_0\left(K_0^+U_{0,[3,L]}R_{02}R_{01}K_0^-R_{01}R_{02}\bar{U}_{0,[3,L]}\right)\left(|\chi\rangle\otimes|v_+\rangle^{\otimes(L-2)}\right).$$

We apply Lemma 5.3.2 on the right-hand side of this equality and obtain, after a redefinition of labels, the expression

$$\Lambda_L = (a+b)^4\frac{\langle w_+|^{\otimes(L-2)}\operatorname{tr}_0\left(K_0^+U_{0,[1,L-2]}K_0^-\bar{U}_{0,[1,L-2]}\right)|v_+\rangle^{\otimes(L-2)}}{\langle w_+|v_+\rangle^{L-2}}. \tag{5.52}$$

Now, we use (5.46) to recognise on the right-hand side of this equality $\Lambda_{L-2}$. Therefore, we have the recurrence relation

$$\Lambda_L = (a+b)^4\Lambda_{L-2}. \tag{5.53}$$

To solve this recurrence, we compute the eigenvalues $\Lambda_L$ for $L = 1, 2$. They immediately follow from Lemmas 5.3.1 and 5.3.2. We find

$$\begin{aligned}\Lambda_1 &= \frac{\langle w_+|\operatorname{tr}_0\left(K_0^+R_{01}K_0^-R_{01}\right)|v_+\rangle}{\langle w_+|v_+\rangle} = (a+b)^2\operatorname{tr}(K^+K^-),\\ \Lambda_2 &= \frac{\langle\alpha|\operatorname{tr}_0\left(K_0^+R_{02}R_{01}K_0^-R_{01}R_{02}\right)|\chi\rangle}{\langle\alpha|\chi\rangle} = (a+b)^4\operatorname{tr}(K^+K^-).\end{aligned} \tag{5.54}$$

The solution of the recurrence relation with these initial conditions leads to the eigenvalue $\Lambda_L = (a+b)^{2L}\operatorname{tr}(K^+K^-)$, for each $L \geqslant 1$. The eigenspace of $\Lambda_L$ is by construction the space spanned by the supersymmetry singlet $|\Psi_L\rangle$. It is one-dimensional. Therefore, $\Lambda_L$ is non-degenerate.

Second, we consider the other real solutions $y = y_\alpha$, $\alpha = 1, 2, 3$, of (4.81). It follows from (5.29) that the corresponding transfer matrix has the property

$$\mathcal{T}(a,b,c,d;y_\alpha) = \mathcal{R}^\alpha(-\pi)\mathcal{T}(a,b,c,d;y_0)\mathcal{R}^\alpha(\pi). \tag{5.55}$$

The two transfer matrices in this equality are related by a unitary transformation. Therefore, they have the same eigenvalues with the same degeneracies. Hence, the transfer matrix possesses the eigenvalue $\Lambda_L$ in this case, too. Its eigenspace is the span of the supersymmetry singlet $|\Psi_L^\alpha\rangle$, defined in the proof of Theorem 4.5.7. □

**Largest eigenvalue** The relation (5.1) admits positive solutions. Indeed, using the parameterisation (3.18), we have $a,b,c,d > 0$ if $\rho > 0$, $\eta = \pi/3$, $0 < u < \pi/3$, and $0 < p < 1$. We now prove that in this case, $\Lambda_L$ is the largest eigenvalue of the transfer matrix $\mathcal{T}$ of the supersymmetric eight-vertex model with the $K$-matrices (5.22) and $y$ a solution of (4.81).

The proof is based on the Perron-Frobenius theorem for positive matrices and its variant for non-negative matrices. We use certain concepts from Perron theory and refer to the book [81] for details. We only recall that $|\psi\rangle \in V^L$ is called a *Perron vector* if all its components are positive and its norm is one.

**Proposition 5.3.4.** *For each $L \geqslant 1$, there is a constant $C_L$ such that $|\Psi'_L\rangle = C_L|\Psi_L\rangle$ is a Perron vector.*

*Proof.* First, we note that for all $(p,t) \in \bar{\mathbb{D}}$ with $t = \pi/6$, the off-diagonal matrix elements of the Hamiltonian $H_{\text{XYZ}}$ are zero or negative. Hence, there is a real number $\lambda$ such that the matrix $\lambda - H_{\text{XYZ}}$ has a positive diagonal and non-negative off-diagonal entries.

Second, we note that the action of $\lambda - H_{\text{XYZ}}$ on any basis state $|s_1 \cdots s_L\rangle$ of $V^L$ leads to a linear combination of basis states that are obtained from $|s_1 \cdots s_L\rangle$ by *(i)* flipping pairs adjacent aligned spins, *(ii)* exchanging pairs of adjacent anti-aligned spins, *(iii)* flipping the spin on the first or last site or *(iv)* leaving the basis state unchanged. The coefficients of this linear combination are positive. The repeated application of the operations *(i)*-*(iv)* allows one to generate any basis state from $|s_1 \cdots s_L\rangle$.

We conclude that there is an integer $m > 0$ such that $(\lambda - H_{\text{XYZ}})^m$ has positive entries. Hence, $\lambda - H_{\text{XYZ}}$ is a non-negative irreducible matrix.

Third, we apply the Perron-Frobenius theorem to the matrix $\lambda - H_{\text{XYZ}}$. It implies that its largest eigenvalue is non-degenerate and that the corresponding eigenspace is spanned by a Perron vector $|\Psi_L'\rangle$. By Theorem 4.5.7, this largest eigenvalue is $\lambda - E_0$, and the eigenspace spanned by $|\Psi_L\rangle$. Hence, there must be a constant $C_L$ such that $|\Psi_L'\rangle = C_L|\Psi_L\rangle$. $\square$

**Proposition 5.3.5.** *For each $L \geqslant 1$, positive vertex weights $a, b, c, d$, $0 < \zeta < 1$ and real $0 < y < 1$, the transfer matrix of the supersymmetric eight-vertex model on a strip of length $L$ with the $K$-matrices $K^\pm$ defined in (5.22) is a positive matrix.*

*Proof.* Let $V_0, V_{\bar{0}} = V$ be two copies of the single-spin Hilbert space. For each $s, \bar{s} \in \{\uparrow, \downarrow\}$, we define an operator $C^{s\bar{s}} : V_0 \otimes V_{\bar{0}} \to V_0 \otimes V_{\bar{0}}$ by

$$C^{s\bar{s}} = (\mathbf{1} \otimes \mathbf{1} \otimes \langle \bar{s}|)\, R_{01}(R_{\bar{0}1})^{t_{\bar{0}}}\, (\mathbf{1} \otimes \mathbf{1} \otimes |s\rangle)\,. \tag{5.56}$$

Its entries are non-negative. A direct calculation shows that for all $s, \bar{s} \in \{\uparrow, \downarrow\}$ and each $|p\rangle \in \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle\}$ there is a unique $|\bar{p}\rangle \in \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle\}$, depending on $s, \bar{s}$, such that $\langle \bar{p}|C^{s\bar{s}}|p\rangle > 0$. Moreover, we define two states $|k^\pm\rangle \in V_0 \otimes V_{\bar{0}}$ through their components, given by

$$\langle s\bar{s}|k^\pm\rangle = \langle \bar{s}|K^\pm|s\rangle, \tag{5.57}$$

for all $s, \bar{s} \in \{\uparrow, \downarrow\}$. These components are positive.

For each pair of basis states $|s_1 \cdots s_L\rangle, |\bar{s}_1 \cdots \bar{s}_L\rangle$, we write the matrix elements of the transfer matrix in terms of these operators and states:

$$\langle \bar{s}_1 \cdots \bar{s}_L|\mathcal{T}|s_1 \cdots s_L\rangle = \langle k^+|C^{s_L\bar{s}_L} \cdots C^{s_1\bar{s}_1}|k^-\rangle. \tag{5.58}$$

To investigate this matrix element, we use the identity

$$\sum_{|w\rangle \in \Omega} |w\rangle\langle w| = \mathbf{1}, \tag{5.59}$$

where $\Omega = \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ denotes the canonical basis of $V^2$. It allows us to write

$$\langle \bar{s}_1 \ldots \bar{s}_L|\mathcal{T}|s_1 \ldots s_L\rangle = \sum_{\substack{|w_i\rangle \in \Omega \\ i=0,\ldots,L}} \langle k^+|w_L\rangle \left( \prod_{j=1}^{L} \langle w_j|C^{s_j\bar{s}_j}|w_{j-1}\rangle \right) \langle w_0|k^-\rangle. \tag{5.60}$$

Each term inside the sum of the right-hand side is a product of non-negative factors. To show that the sum is positive, it is therefore sufficient to find a single choice for $|w_0\rangle, \dots, |w_L\rangle$ that yields a positive term. We determine such a choice by iteration. First, we set $|w_0\rangle = |p_0\rangle = |\uparrow\uparrow\rangle$. Second, we choose the unique state $|w_1\rangle = |p_1\rangle \in \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle\}$ such that $\langle w_1|C^{s_1\bar{s}_1}|w_0\rangle = \langle p_1|C^{s_1\bar{s}_1}|p_0\rangle > 0$. Next, we iterate this step and determine for each $i = 2, \dots, L$ the unique $|w_i\rangle = |p_i\rangle \in \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle\}$ such that $\langle w_i|C^{s_i\bar{s}_i}|w_{i-1}\rangle = \langle p_i|C^{s_i\bar{s}_i}|p_{i-1}\rangle > 0$. The term corresponding to this choice is a lower boundary for the sum:

$$\langle \bar{s}_1 \dots \bar{s}_L|\mathcal{T}|s_1 \dots s_L\rangle \geqslant \langle k^+|p_L\rangle \left(\prod_{j=1}^{L} \langle p_j|C^{s_j\bar{s}_j}|p_{j-1}\rangle\right) \langle p_0|k^-\rangle. \quad (5.61)$$

Each factor of the product on the right-hand side of this equality is positive. Hence, the matrix element is positive. □

We combine the results of the previous lemmas to prove the following:

**Theorem 5.3.6.** *Let $L \geqslant 1$, $0 < \zeta < 1$ and $y$ be a solution of (4.81). If $a, b, c, d > 0$, then $\Lambda_L$ is the largest eigenvalue of the transfer matrix of the supersymmetric eight-vertex model on a strip with $L$ vertical lines and $K$-matrices (5.22).*

*Proof.* First, let $y = y_0$ be the unique solution of the equation (4.81) with $0 < y < 1$. We denote by $\Lambda'_L$ the largest eigenvalue of the transfer matrix $\mathcal{T} = \mathcal{T}(a, b, c, d; y_0)$ of the supersymmetric eight-vertex model with the $K$-matrices (5.22) and positive vertex weights $a, b, c, d > 0$. By Proposition 5.3.5, $\mathcal{T}$ is a positive matrix. The Perron-Frobenius theorem states that the eigenspace of $\Lambda'_L$ is one-dimensional and spanned by a Perron vector, and that no other eigenspace contains a Perron vector. We have $\mathcal{T}|\Psi'_L\rangle = \Lambda_L|\Psi'_L\rangle$, where $|\Psi'_L\rangle$ is the Perron vector of Proposition 5.3.4. Hence, $\Lambda'_L = \Lambda_L$.

Second, let $y = y_\alpha$, $\alpha = 1, 2, 3$, be another solution of (4.81). We follow the reasoning of the proof of Theorem 5.3.3. The transfer matrix has the property

$$\mathcal{T}(a, b, c, d; y_\alpha) = \mathcal{R}^\alpha(-\pi)\mathcal{T}(a, b, c, d; y_0)\mathcal{R}^\alpha(\pi). \quad (5.62)$$

The two transfer matrices in this equality are related by a unitary transformation. Therefore, they have the same spectrum and, hence, the same largest eigenvalue $\Lambda_L$. $\square$

**The free energy.** Up to an irrelevant factor, the free energy per pair of horizontal lines of the eight-vertex model on a strip is given by the logarithm of the largest eigenvalue of its transfer matrix. For large $L$, it is expected to take the form

$$-\ln \Lambda_L = 2Lf + f_{\mathrm{B}} + O(L^{-1}), \tag{5.63}$$

where $f$ is the bulk free energy per site, and $f_{\mathrm{B}}$ the boundary free energy. The bulk free energy per site is known from Baxter's work [89]. As for $f_{\mathrm{B}}$, however, we are not aware of an explicit formula for general vertex weights and boundary conditions in the literature.

In the case studied in this chapter, it is trivial to compute the expansion (5.63), because we explicitly know $\Lambda_L$ for each $L \geqslant 1$. We obtain

$$f = -\ln(a+b), \quad f_{\mathrm{B}} = -\ln \operatorname{tr}(K^+K^-). \tag{5.64}$$

The finite-size corrections $O(L^{-1})$ are absent. We note that $f = -\ln(a+b)$ matches Baxter's results [89].

# Chapter 6

# The supersymmetric XXZ spin chain

The XYZ Hamiltonians studied in Chapters 2 and 4 depend on a parameter that we called $\zeta$. In our analysis, we did not address the case $\zeta = 0$. The goal of this chapter is to treat this specific case.

When we set the parameter $\zeta$ to zero, the anisotropy parameters of the XYZ spin-chain become

$$J_1 = 1, \quad J_2 = 1, \quad J_3 = -\frac{1}{2}. \tag{6.1}$$

As $J_1 = J_2$, the model is referred to as an XXZ spin-chain, in which case the parameter $J_3$ is usually denoted by $\Delta$. With $\Delta = -1/2$, the XXZ spin-chain Hamiltonian density reads

$$h^{\mathrm{XXZ}}_{jj+1} = -\frac{1}{2}\left(\sigma^1_j\sigma^1_{j+1} + \sigma^2_j\sigma^2_{j+1} - \frac{1}{2}\sigma^3_j\sigma^3_{j+1}\right). \tag{6.2}$$

We consider the spin-chain Hamiltonian with open boundary conditions

$$H_{\mathrm{XXZ}} = \sum_{j=1}^{L-1} h^{\mathrm{XXZ}}_{jj+1} + (h_{\mathrm{B}})_1 + (h_{\mathrm{B}})_L\,. \tag{6.3a}$$

Here, $h_{\mathrm{B}}$ is a boundary interaction given by

$$h_{\mathrm{B}} = \sum_{\alpha=1}^{3} \lambda_\alpha \sigma^\alpha, \tag{6.3b}$$

where

$$\lambda_1 = -\frac{\operatorname{Re} y}{1+|y|^2}, \quad \lambda_2 = -\frac{\operatorname{Im} y}{1+|y|^2}, \quad \lambda_3 = -\frac{1}{4}\left(\frac{1-|y|^2}{1+|y|^2}\right), \tag{6.3c}$$

and $y$ is a complex number. We further generalise this Hamiltonian by allowing for different boundary terms on each end of the chain.

**Periodic XXZ Hamiltonian and $\Delta = -1/2$.** As for the XXZ spin chain with periodic boundary conditions, its Hamiltonian is the sum

$$H_{\text{XXZ}}^{(\text{per})} = \sum_{j=1}^{L} h_{jj+1}^{\text{XXZ}}, \tag{6.4}$$

where we wrote the superscript to differentiate this Hamiltonian from (6.3).

The Hamiltonian $H_{\text{XXZ}}^{(\text{per})}$ acting on a chain with an odd finite number of sites has been studied intensively since the seminal work of [54] on the related transfer matrix.

Razumov and Stroganov observed that the components of the ground state are, in the proper normalisation, integers and are moreover related to the enumeration of combinatorial objects. As an example, if the least component of the ground state of a chain of size $L = 2n+1$ is normalised to one, then the largest one equals the number of $n \times n$ alternating sign matrices [48]. (We postpone the precise definition of those objects to Chapter 9 which is partially devoted to enumerative combinatorics.)

Following this discovery of a relation between spin chains and combinatorics, the beginning of the century saw several conjectures on the $\Delta = -1/2$ case and related models being made [48, 58, 55, 106, 107]. Some of these conjectures have already been proven or generalised [61, 59, 60].

The XXZ Hamiltonian with $\Delta = -1/2$ and periodic boundary conditions also possesses a supersymmetric structure [80, 108]. The supersymmetry allowed for proving the existence of the ground state energy $E_0 = -\frac{3L}{4}$ for each odd $L$ [80].

In this chapter, we revisit this periodic spin-chain Hamiltonian, and we compute (co)homology decomposition of the ground states, in the cases where it possesses supersymmetry singlets.

The layout of this chapter is as follows. In the first part of this chapter, we investigate the supersymmetry of the open spin-chain's Hamiltonian $H_{\text{XXZ}}$. In Section 6.1, we recall the definition of the local supercharge that depends on a parameter $y$. We generalise the action of the supercharge to obtain Hamiltonians that have unequal boundary terms at both ends of the chain. We examine the existence of supersymmetry singlets of the supercharges by computing their (co)homology. As we shall see, this depends on whether the parameter $y$ is non-zero or zero. We compute the cohomology for $y \neq 0$ in Section 6.2. For $y = 0$, we show in Section 6.3 that the supercharge possesses supersymmetry singlets. In Section 6.4, we use the representatives of the (co)homology classes to provide (co)homology decompositions of the zero-energy states.

In the second part of this chapter, we focus on the Hamiltonian with periodic boundary conditions (6.4). In Section 6.5, we recall known results related to the ground-state eigenvector and its properties. We explain the relation between the supercharges corresponding to periodic and open systems. The (co)homology decompositions of the ground states of the periodic spin-chain Hamiltonian, given in Section 6.6, use this relation.

## 6.1 Supersymmetry

We consider the local supercharge (4.14) with $\zeta = 0$. We recall that it reads

$$\mathfrak{q}' = \mathfrak{q}^{\uparrow} + y^3\mathfrak{q}^{\downarrow} + \mathfrak{q}_{\phi}, \quad \text{with } |\phi\rangle = -y|\uparrow\rangle - y^2|\downarrow\rangle. \tag{6.5}$$

The action of $\mathfrak{q}'$ on the canonical basis of $V$ is

$$\mathfrak{q}'|\uparrow\rangle = y^3|\downarrow\downarrow\rangle - 2y|\uparrow\uparrow\rangle - y^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \tag{6.6}$$

$$\mathfrak{q}'|\downarrow\rangle = |\uparrow\uparrow\rangle - 2y^2|\downarrow\downarrow\rangle - y(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle). \tag{6.7}$$

We construct the supercharge $\mathfrak{Q}' : V^L \to V^{L+1}$ as the alternating sum of the local supercharge:

$$\mathfrak{Q}' = \sum_{j=1}^{L}(-1)^j\mathfrak{q}'_j. \tag{6.8}$$

As previously, the subscript $j$ indicates that the local supercharge acts on the $j$-th factor of $V^L$. The supercharge is nilpotent by virtue of Proposition 4.1.1.

The supersymmetric Hamiltonian $H = \mathfrak{Q}'\mathfrak{Q}'^\dagger + \mathfrak{Q}'^\dagger\mathfrak{Q}'$ follows from Proposition 4.2.1:

$$H = (1+|y|^6)\left(H_{\text{XXZ}} + \frac{3L-1}{4} + 3\frac{|y|^2}{1-|y|^2+|y|^4}\right). \tag{6.9}$$

For non-zero $y$, the off-diagonal terms of $h_{\text{B}}$ are non-zero and therefore generalise the diagonal boundary interactions found by Yang and Fendley [80].

**Spatial parity.** Before generalising $H_{\text{XXZ}}$ by allowing different boundary terms on both ends of the chain, we examine the symmetries of the Hamiltonian. The parity operator $\mathcal{C}$ on $V^L$, $L \geqslant 1$, is the linear operator defined by the following action on the canonical basis states:

$$\mathcal{C}|s_1 s_2 \cdots s_L\rangle = |s_L s_{L-1} \cdots s_1\rangle. \tag{6.10}$$

The Hamiltonian (6.3) is invariant under the action of $\mathcal{C}$:

$$[H_{\text{XXZ}}, \mathcal{C}] = 0. \tag{6.11}$$

This parity-invariance is evident from the Hamiltonian density (6.3b) and since the boundary interactions are identical at both extremities of the chain.

**Magnetisation.** We define the magnetisation operator as

$$\mathcal{M} = \frac{1}{2}\sum_{j=1}^{L} \sigma_j^3. \tag{6.12}$$

Its action on a basis state $|s_1 s_2 \cdots s_L\rangle$ counts half of the number of up spins ($\uparrow$) amongst $s_1, \dots, s_L$ minus half of the number of down spins ($\downarrow$). We call magnetisation of an eigenstate of $\mathcal{M}$ the corresponding eigenvalue and refer to the corresponding eigenspace as a sector of magnetisation.

Using (6.2), we find that the Hamiltonian density satisfies $[h^{\text{XXZ}}_{jj+1}, \mathcal{M}] = 0$ for each $j = 1, \ldots, L-1$. Hence, the bulk part of the Hamiltonian $H_{\text{XXZ}}$ conserves the magnetisation.

For $y = 0$, the boundary field $h_{\text{B}}$ given by (6.3b) has the property $[h_{\text{B}}, \mathcal{M}] = 0$. This commutation relation implies that for $y = 0$, the full Hamiltonian preserves the magnetisation: $[H_{\text{XXZ}}, \mathcal{M}] = 0$. Conversely, for generic values of $y$ this conservation law is broken by the boundary terms. Indicating the dependence on the parameter $y$ of $h_{\text{B}}$, we have:

$$e^{\mathrm{i}\theta\mathcal{M}} h_{\text{B}}(y) e^{-\mathrm{i}\theta\mathcal{M}} = h_{\text{B}}(e^{-\mathrm{i}\theta} y). \tag{6.13}$$

**Boundary terms.** In this subsection, we modify the action of $\mathfrak{Q}'$ defined in (6.8) on the first and last site of the spin chain. This generalisation allows us to show that the lattice supersymmetry can be present for unequal boundary terms at both ends of the spin chain. Each boundary term depends on the parameter $y$ and is individually characterised by an integer label $j = -1, 0, 1$.

The main ingredients of our construction are the vectors

$$|\xi_k\rangle = |\phi(y)\rangle - |\phi(q^k y)\rangle, \quad k = -1, 0, 1, \tag{6.14}$$

where $q = e^{2\mathrm{i}\pi/3}$ is a third root of unity, and $|\phi(y)\rangle$ is defined in (6.5). We trivially have $|\xi_0\rangle = 0$. The action of $\mathfrak{q}'$ on these vectors is very simple. Indeed, using (6.5), it is not difficult to show that

$$\mathfrak{q}'|\xi_k\rangle = |\xi_k\rangle \otimes |\xi_k\rangle, \quad k = -1, 0, 1. \tag{6.15}$$

For any pair $-1 \leqslant \ell, k \leqslant 1$, we consider an operator $\mathfrak{Q}'_{\ell,k}$ that acts on any $|\psi\rangle \in V^L$ according to

$$\mathfrak{Q}'_{\ell,k}|\psi\rangle = |\xi_\ell\rangle \otimes |\psi\rangle + (-1)^{L-1}|\psi\rangle \otimes |\xi_k\rangle + \mathfrak{Q}'|\psi\rangle. \tag{6.16}$$

We note that the case $\ell = k = 0$ corresponds to unmodified boundary interactions: $\mathfrak{Q}'_{0,0} = \mathfrak{Q}'$. Using $\mathfrak{Q}'^2 = 0$ and (6.15), one checks that

$$\left(\mathfrak{Q}'_{\ell,k}\right)^2 = 0. \tag{6.17}$$

The corresponding Hamiltonian $H_{\ell,k} = \{\mathfrak{Q}'_{\ell,k}, (\mathfrak{Q}'_{\ell,k})^\dagger\}$ is readily evaluated. It is given by a sum of nearest-neighbour interactions and boundary terms that depend on $\ell$ and $k$:

$$H_{\ell,k} = x\left(\sum_{j=1}^{L-1} h^{\text{XXZ}}_{jj+1} + (h^{(\ell)}_{\text{B}})_1 + (h^{(k)}_{\text{B}})_L + \frac{3L-1}{4} + 3\frac{|y|^2}{1-|y|^2+|y|^4}\right), \tag{6.18}$$

where we used the shorthand notation $x = (1+|y|^6)$. Here, $h^{\text{XXZ}}_{jj+1}$ is the Hamiltonian density (6.2) and the boundary terms are given by

$$h^{(k)}_{\text{B}} = h_{\text{B}}(q^k y), \tag{6.19}$$

where we explicitly wrote the dependence of $h_{\text{B}}$ on $y$.

We conclude that all boundary conditions that result from a modification of the action of the supercharge on the first and last site of the spin chain are parameterised by two integers $-1 \leqslant \ell, k \leqslant 1$. However, not all choices of these parameters lead to unequal spectra.

To see this, let us write $H_{\ell,k} = H_{\ell,k}(y)$ to stress the dependence of this Hamiltonian on the parameter $y$. We note that the spectrum of $H_{\ell,k}(y)$ is the same as the spectrum of $e^{\mathrm{i}\theta\mathcal{M}} H_{\ell,k}(y) e^{-\mathrm{i}\theta\mathcal{M}} = H_{\ell,k}(e^{-\mathrm{i}\theta} y)$ (for any real value of $\theta$) and $(H_{\ell,k}(y))^* = H_{-\ell,-k}(y^*)$. An appropriate choice for $\theta$ allows us to conclude that it is sufficient to restrict the parameters to real values for $y$ and the two distinct cases $\ell = 1, k = 0$ and $\ell = k = 0$.

We have constructed a family of Hamiltonians $H_{\ell,k}$ that are supersymmetric. In the following two sections, we analyse whether they possess supersymmetry singlets. To this end, we exploit the relation between zero-energy states and the cohomology of the supercharge, explained in Chapter 1. The structure of the cohomology depends on whether the parameter $y$ is non-zero or zero. Hence, we treat each case independently.

## 6.2 Zero-energy states: the case $y \neq 0$

In this and the following section, we explicitly compute $\mathcal{H}^L$ for the supercharge $\mathfrak{Q}'_{\ell,k}$. The computation allows us to characterise the space of zero-energy states of the Hamiltonian $H_{\ell,k}$ as a function of the parameter $y$, the integer labels $\ell, k$ and the system size $L$.

Here, we consider the case $y \neq 0$. We prove the following theorem:

**Theorem 6.2.1.** *For $y \neq 0$ and each $\ell, k = -1, 0, 1$, the cohomology $\mathcal{H}^\bullet$ of the supercharge $\mathfrak{Q}'_{\ell,k}$ is trivial.*

This theorem implies that for $y \neq 0$ and any length of the chain $L$, the Hamiltonian $H_{\ell,k}$ does not possess zero-energy states. Thus, its spectrum is strictly positive.

The proof is based on two lemmas. The first lemma deals with a mapping $\mathfrak{s}$ that is akin to a so-called *contracting homotopy* [74].

**Lemma 6.2.2.** *Let $\mathfrak{Q}'$ be an arbitrary supercharge. Suppose that for each $L \geqslant 2$ there is a mapping $\mathfrak{s} : V^L \to V^{L-1}$ such that*

$$\mathfrak{s}\mathfrak{Q}' + \mathfrak{Q}'\mathfrak{s} = \mathbf{1}. \tag{6.20}$$

*Then for each $L \geqslant 2$, we have $\mathcal{H}^L = 0$.*

*Proof.* We show that any cocycle $|\psi\rangle \in V^L$ is a coboundary. Indeed, applying (6.20) to $|\psi\rangle$, we obtain

$$|\psi\rangle = (\mathfrak{s}\mathfrak{Q}' + \mathfrak{Q}'\mathfrak{s})|\psi\rangle = \mathfrak{Q}'(\mathfrak{s}|\psi\rangle). \tag{6.21}$$

Hence, $\mathcal{H}^L = 0$. □

We aim to construct such a mapping $\mathfrak{s}$ for the supercharge $\mathfrak{Q}'_{\ell,k}(y)$. To this end, we use the vectors $|\xi_{\pm 1}\rangle$ defined in (6.14). The second lemma needed for our proof of Theorem 6.2.1 establishes that for non-vanishing $y$ these vectors span the Hilbert space $V$ of a single spin:

**Lemma 6.2.3.** *For $y \neq 0$, the vectors $|\xi_{-1}\rangle$ and $|\xi_1\rangle$ constitute a basis of $V$.*

*Proof.* The matrix whose columns are given by $|\xi_1\rangle$ and $|\xi_{-1}\rangle$ in the canonical basis of $V$ is

$$\Xi = \begin{pmatrix} -y(1-q) & -y(1-q^{-1}) \\ -y^2(1-q^2) & -y^2(1-q^{-2}) \end{pmatrix}. \tag{6.22}$$

To prove the lemma, it is sufficient to show that the matrix $\Xi$ has a non-zero determinant. This is indeed the case as $\det \Xi = 3y^3(q^2 - q^1)$ is non-vanishing for $y \neq 0$. □

*Proof of Theorem 6.2.1.* We now prove that if $y \neq 0$ then the quotient space $\mathcal{H}^L$ of the supercharge $\mathfrak{Q}'_{\ell,k}$ is zero for each $L \geqslant 1$ and each $\ell, k = -1, 0, 1$. For $L = 1$, the proof is trivial: one readily checks that $\ker\{\mathfrak{Q}'_{\ell,k} : V \to V^2\} = 0$, using Lemma 6.2.3. Hence, we focus on $L \geqslant 2$. The proof is based on the construction of a mapping $\mathfrak{s}_\ell$ that obeys (6.20) for each $\ell = -1, 0, 1$. We separately consider the cases $\ell = \pm 1$ and $\ell = 0$.

Let us first consider $\ell = \pm 1$. It follows from Lemma 6.2.3 that for $y \neq 0$, every vector $|\psi\rangle \in V^L$ can be written as

$$|\psi\rangle = |\xi_{-1}\rangle \otimes |\psi_{-1}\rangle + |\xi_1\rangle \otimes |\psi_1\rangle, \tag{6.23}$$

with unique vectors $|\psi_{-1}\rangle, |\psi_1\rangle \in V^{L-1}$. We define the mapping $\mathfrak{s}_\ell$ by

$$\mathfrak{s}_\ell |\psi\rangle = |\psi_\ell\rangle. \tag{6.24}$$

Using the action (6.15) of the local supercharge on the basis vectors $|\xi_1\rangle$ and $|\xi_{-1}\rangle$, it is easy to see that

$$(\mathfrak{s}_\ell \mathfrak{Q}' + \mathfrak{Q}' \mathfrak{s}_\ell)|\psi\rangle = -|\xi_\ell\rangle \otimes |\psi_\ell\rangle = -|\xi_\ell\rangle \otimes \mathfrak{s}_\ell|\psi\rangle. \tag{6.25}$$

We combine this identity with the definition of the supercharge (6.16) and find that

$$\mathfrak{s}_\ell \mathfrak{Q}'_{\ell,k} + \mathfrak{Q}'_{\ell,k} \mathfrak{s}_\ell = \mathbf{1} \tag{6.26}$$

for each $\ell = \pm 1$ and $k = -1, 0, 1$.

Second, for $\ell = 0$, we define

$$\mathfrak{s}_0 = -\mathfrak{s}_{-1} - \mathfrak{s}_1. \tag{6.27}$$

Using the definition of $\mathfrak{s}_{\pm 1}$, it is easy to see that (6.26) holds for $\ell = 0$ and $k = -1, 0, 1$, too.

In both cases, it follows from Lemma 6.2.2 that the quotient space $\mathcal{H}^L$ corresponding to the supercharge $\mathfrak{Q}'_{\ell,k}$ equals zero for any $L \geqslant 2$ and each $\ell, k = -1, 0, 1$. This ends the proof of the theorem. $\square$

We notice that the proof only relies on the existence of a basis $|\xi_{-1}\rangle, |\xi_1\rangle$ of $V$ with the property $\mathfrak{q}'|\xi_k\rangle = |\xi_k\rangle \otimes |\xi_k\rangle$ for each $k = \pm 1$. This property, observed in a variety of other physically-relevant spin chains [77], is similar to the one presented and used in Chapter 4.

## 6.3 Zero-energy states: the case $y = 0$

The proof of Theorem 6.2.1 is not generalisable to the case $y = 0$. For this value of $y$, the local supercharge reduces to $\mathfrak{q}^{\uparrow}$, and its action on the basis of $V$ is

$$\mathfrak{q}'|\uparrow\rangle = 0, \qquad \mathfrak{q}'|\downarrow\rangle = |\uparrow\uparrow\rangle. \tag{6.28}$$

It follows that the adjoint supercharge $\mathfrak{q}'^{\dagger}$ acts on the basis vectors of $V^2$ according to

$$\mathfrak{q}'^{\dagger}|\uparrow\uparrow\rangle = |\downarrow\rangle, \qquad \mathfrak{q}'^{\dagger}|\uparrow\downarrow\rangle = \mathfrak{q}'^{\dagger}|\downarrow\uparrow\rangle = \mathfrak{q}'^{\dagger}|\downarrow\downarrow\rangle = 0. \tag{6.29}$$

We mention that the supercharge (6.28) is locally equivalent to the one of the so-called $M_1$ model. This model describes supersymmetric fermions on a one-dimensional lattice with an exclusion constraint that forbid the fermions to have neighbours. Hence, each fermion forms a cluster of size one. The supercharge splits a cluster of size one into a pair of adjacent clusters of size zero [109].

The $M_1$ model belongs to a family of $M_k$ models of Fendley, Nienhuis and Schoutens [110], which allows $k$ fermions to be adjacent in a given cluster. The $M_k$ model is locally equivalent to a spin chain with spin $k/2$ [77]. The corresponding ground-state eigenspaces have been investigated through the characterisation of the (co)homology [1].

As for the local supercharge (6.28), it was found in the spin language by Fendley and Yang [80]. The corresponding supercharge $\mathfrak{Q}'_{\ell,k}$ is independent of the indices $\ell, k$ if $y = 0$, and we simply denote it by $\mathfrak{Q}'$. Its cohomology is non-trivial.

The main result of this section is the following theorem:

**Theorem 6.3.1.** *The space $\mathcal{H}^L$ is spanned by the cohomology class of the state*

$$|\underbrace{\chi \cdots \chi}_{n\ \textit{times}}\rangle \quad \textit{if} \quad L = 2n, \tag{6.30a}$$

*and*

$$|\uparrow\rangle \otimes |\underbrace{\chi \cdots \chi}_{n-1\ \textit{times}}\rangle \quad \textit{if} \quad L = 2n - 1, \tag{6.30b}$$

*where $|\chi\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \in V^2$, and $n$ is a positive integer.*

Here, we abbreviate the tensor product $|\chi\rangle \otimes \cdots \otimes |\chi\rangle$ by $|\chi \cdots \chi\rangle$ in order to simplify the notation.

This theorem implies that for $y = 0$ the spin-chain Hamiltonian $H$ possesses a zero-energy state for each length $L$. The state is unique up to normalisation.

The proof of Theorem 6.3.1 is based on several lemmas. They establish the existence of an explicit bijection between $\mathcal{H}^L$ and $\mathcal{H}^{L+2}$ for each $L \geqslant 1$. Hence, we may construct $\mathcal{H}^L$ from $\mathcal{H}^1$ and $\mathcal{H}^2$. We explicitly compute them in the following lemma:

**Lemma 6.3.2.** *$\mathcal{H}^1$ and $\mathcal{H}^2$ are spanned by the cohomology classes of the states $|\uparrow\rangle$ and $|\chi\rangle$, respectively.*

*Proof.* For $L = 1$, recall that $\mathcal{H}^1 = \ker\{\mathfrak{Q}' : V^1 \to V^2\}$. According to (6.28), the only solution to $\mathfrak{q}'|\psi\rangle = 0$ is $|\psi\rangle = |\uparrow\rangle$, up to a factor.

For $L = 2$, we consider a cocycle $|\psi\rangle \in V^2$. We write $\psi_{s_1 s_2} = \langle s_1 s_2|\psi\rangle$ for its components with respect to the canonical basis of the Hilbert space $V$. From $\mathfrak{Q}'|\psi\rangle = 0$, it follows that

$$\psi_{\downarrow\downarrow} = 0, \quad \psi_{\uparrow\downarrow} = \psi_{\downarrow\uparrow}. \tag{6.31}$$

The state $|\psi\rangle$ thus reads

$$|\psi\rangle = \psi_{\uparrow\downarrow}|\chi\rangle + \mathfrak{Q}'\left(-\psi_{\uparrow\uparrow}|\downarrow\rangle\right). \tag{6.32}$$

Hence $[|\psi\rangle] = \psi_{\uparrow\downarrow}[|\chi\rangle]$ with an arbitrary coefficient $\psi_{\uparrow\downarrow}$. The cohomology class of $|\chi\rangle$ cannot be zero since this state is a linear combination of basis states of $V^2$ that are clearly not in the image of $\mathfrak{q}'$. $\square$

The preceding lemma gives us simple representatives of $\mathcal{H}^1$ and $\mathcal{H}^2$. In particular, we note that they satisfy

$$\mathfrak{Q}'|\uparrow\rangle = 0, \qquad \mathfrak{Q}'|\chi\rangle = 0. \tag{6.33}$$

Our next aim is to study $\mathcal{H}^L$ for $L \geqslant 3$. In the following technical lemma, we determine a convenient choice of their representatives.

**Lemma 6.3.3.** *For each $L \geqslant 3$ any element in $\mathcal{H}^L$ can be represented by a cocycle $|\psi\rangle \in V^L$ with*

$$|\psi\rangle = |\psi_\uparrow\rangle \otimes |\uparrow\rangle + |\psi_\downarrow\rangle \otimes |\downarrow\rangle \tag{6.34}$$

*such that $|\psi_\uparrow\rangle = |\psi_{\downarrow,\uparrow}\rangle \otimes |\downarrow\rangle$ for some vector $|\psi_{\downarrow,\uparrow}\rangle \in V^{L-2}$.*

*Proof.* Let us consider the cocycle $|\psi'\rangle$ representing an element of $\mathcal{H}^L$. Then for any $|\phi\rangle \in V^{L-1}$ the vector $|\psi\rangle = |\psi'\rangle + \mathfrak{Q}'|\phi\rangle$ is also a cocycle, representing the same element of $\mathcal{H}^L$. We write the vector $|\psi\rangle$ (and likewise $|\psi'\rangle$, $|\phi\rangle$) as a superposition

$$|\psi\rangle = |\psi_\uparrow\rangle \otimes |\uparrow\rangle + |\psi_\downarrow\rangle \otimes |\downarrow\rangle, \tag{6.35}$$

where $|\psi_\uparrow\rangle, |\psi_\downarrow\rangle \in V^{L-1}$. The equality $|\psi\rangle = |\psi'\rangle + \mathfrak{Q}'|\phi\rangle$ leads to

$$|\psi_\uparrow\rangle = |\psi'_\uparrow\rangle + \mathfrak{Q}'|\phi_\uparrow\rangle + (-1)^L|\phi_\downarrow\rangle \otimes |\uparrow\rangle, \tag{6.36}$$

$$|\psi_\downarrow\rangle = |\psi'_\downarrow\rangle + \mathfrak{Q}'|\phi_\downarrow\rangle. \tag{6.37}$$

We decompose the vector $|\psi'_\uparrow\rangle$ as follows: $|\psi'_\uparrow\rangle = |\psi'_{\uparrow,\uparrow}\rangle \otimes |\uparrow\rangle + |\psi'_{\downarrow,\uparrow}\rangle \otimes |\downarrow\rangle$ and substitute this decomposition in the first of these equations. We obtain

$$|\psi_\uparrow\rangle = |\psi'_{\uparrow,\uparrow}\rangle \otimes |\uparrow\rangle + |\psi'_{\downarrow,\uparrow}\rangle \otimes |\downarrow\rangle + \mathfrak{Q}'|\phi_\uparrow\rangle + (-1)^L|\phi_\downarrow\rangle \otimes |\uparrow\rangle. \tag{6.38}$$

In order to prove the lemma, we choose

$$|\phi_\uparrow\rangle = 0, \quad |\phi_\downarrow\rangle = (-1)^{L+1}|\psi'_{\uparrow,\uparrow}\rangle. \tag{6.39}$$

This choice leads to $|\psi_\uparrow\rangle = |\psi'_{\downarrow,\uparrow}\rangle \otimes |\downarrow\rangle$ and ends the proof. □

For the next two lemmas, we introduce the operator $\mathfrak{S}$ which acts on any vector $|\psi\rangle \in V^L$, $L \geqslant 1$, according to

$$\mathfrak{S}|\psi\rangle = |\psi\rangle \otimes |\chi\rangle. \tag{6.40}$$

One checks that it commutes with the supercharge $\mathfrak{Q}'$,

$$\mathfrak{S}\mathfrak{Q}' = \mathfrak{Q}'\mathfrak{S}, \tag{6.41}$$

because $|\chi\rangle$ is annihilated by the supercharge. It follows that we can extend $\mathfrak{S}$ to a mapping $\mathfrak{S}^\sharp$ defined on the cohomology [74, 73]: $\mathfrak{S}^\sharp : \mathcal{H}^L \to \mathcal{H}^{L+2}$. Its action on the cohomology classes is given by $\mathfrak{S}^\sharp[|\psi\rangle] = [\mathfrak{S}|\psi\rangle]$.

**Lemma 6.3.4.** *For each $L \geqslant 1$ the mapping $\mathfrak{S}^\sharp : \mathcal{H}^L \to \mathcal{H}^{L+2}$ is surjective.*

*Proof.* Let $|\psi\rangle$ be a cocycle representing an element of $\mathcal{H}^{L+2}$. We decompose it as in (6.35) with respect to the last site. The equation $\mathfrak{Q}'|\psi\rangle = 0$ leads to

$$\mathfrak{Q}'|\psi_\uparrow\rangle = (-1)^{L+1}|\psi_\downarrow\rangle \otimes |\uparrow\rangle, \tag{6.42}$$
$$\mathfrak{Q}'|\psi_\downarrow\rangle = 0. \tag{6.43}$$

Let us consider the first equation. From Lemma 6.3.3 it follows that we may choose $|\psi_\uparrow\rangle = |\psi_{\downarrow,\uparrow}\rangle \otimes |\downarrow\rangle$, for some state $|\psi_{\downarrow,\uparrow}\rangle \in V^L$, without loss of generality. This choice leads to

$$\mathfrak{Q}'|\psi_{\downarrow,\uparrow}\rangle \otimes |\downarrow\rangle + (-1)^{L+1}|\psi_{\downarrow,\uparrow}\rangle \otimes |\uparrow\uparrow\rangle = (-1)^{L+1}|\psi_\downarrow\rangle \otimes |\uparrow\rangle. \tag{6.44}$$

Comparing both sides, we obtain

$$\mathfrak{Q}'|\psi_{\downarrow,\uparrow}\rangle = 0, \quad \text{and} \quad |\psi_\downarrow\rangle = |\psi_{\downarrow,\uparrow}\rangle \otimes |\uparrow\rangle. \tag{6.45}$$

According to (6.28) and the definition of $|\chi\rangle$, we find

$$|\psi\rangle = |\psi_{\downarrow,\uparrow}\rangle \otimes |\downarrow\uparrow\rangle + |\psi_\downarrow\rangle \otimes |\downarrow\rangle = |\psi_{\downarrow,\uparrow}\rangle \otimes |\chi\rangle = \mathfrak{S}|\psi_{\downarrow,\uparrow}\rangle \tag{6.46}$$

with a cocycle $|\psi_{\downarrow,\uparrow}\rangle \in V^L$. For the corresponding cohomology classes, we find thus $[|\psi\rangle] = \mathfrak{S}^\sharp[|\psi_{\downarrow,\uparrow}\rangle]$. □

**Lemma 6.3.5.** *For each $L \geqslant 1$ the mapping $\mathfrak{S}^\sharp : \mathcal{H}^L \to \mathcal{H}^{L+2}$ is injective.*

*Proof.* Consider an element of $\ker\{\mathfrak{S}^\sharp : \mathcal{H}^L \to \mathcal{H}^{L+2}\}$. It can be represented by a cocycle $|\psi\rangle \in V^L$ such that $\mathfrak{S}|\psi\rangle = \mathfrak{Q}'|\phi\rangle$ for some vector $|\phi\rangle \in V^{L+1}$. As before, it is useful to decompose the state with respect to the last site: $|\phi\rangle = |\phi_\uparrow\rangle \otimes |\uparrow\rangle + |\phi_\downarrow\rangle \otimes |\downarrow\rangle$. We find

$$\mathfrak{S}|\psi\rangle = |\psi\rangle \otimes |\chi\rangle = \mathfrak{Q}'|\phi_\uparrow\rangle \otimes |\uparrow\rangle + \mathfrak{Q}'|\phi_\downarrow\rangle \otimes |\downarrow\rangle + (-1)^{L+1}|\phi_\downarrow\rangle \otimes |\uparrow\uparrow\rangle. \tag{6.47}$$

We select on both sides the terms corresponding to $|\uparrow\rangle$ on the last site and find

$$|\psi\rangle \otimes |\downarrow\rangle = \mathfrak{Q}'|\phi_\uparrow\rangle + (-1)^{L+1}|\phi_\downarrow\rangle \otimes |\uparrow\rangle. \tag{6.48}$$

We decompose once again $|\phi_\uparrow\rangle = |\phi_{\downarrow,\uparrow}\rangle \otimes |{\downarrow}\rangle + |\phi_{\uparrow,\uparrow}\rangle \otimes |{\uparrow}\rangle$. The action of the supercharge to this decomposition leads to

$$\begin{aligned}|\psi\rangle \otimes |{\downarrow}\rangle &= \mathfrak{Q}'|\phi_{\downarrow,\uparrow}\rangle \otimes |{\downarrow}\rangle \\ &\quad + \big((-1)^L|\phi_{\downarrow,\uparrow}\rangle \otimes |{\uparrow}\rangle + \mathfrak{Q}'|\phi_{\uparrow,\uparrow}\rangle + (-1)^{L+1}|\phi_\downarrow\rangle\big) \otimes |{\uparrow}\rangle. \end{aligned} \tag{6.49}$$

This equality implies $|\psi\rangle = \mathfrak{Q}'|\phi_{\downarrow,\uparrow}\rangle$. Hence, $|\psi\rangle$ is a coboundary. We conclude that $\ker\{\mathfrak{S}^\sharp : \mathcal{H}^L \to \mathcal{H}^{L+2}\} = 0$, which proves the claim. □

We are now ready to prove the main result of this section.

*Proof of Theorem 6.3.1.* From Lemmas 6.3.4 and 6.3.5 we conclude that for each $L \geqslant 1$ the mapping $\mathfrak{S}^\sharp : \mathcal{H}^L \to \mathcal{H}^{L+2}$ is both surjective and injective. Hence $\mathcal{H}^{L+2}$ is isomorphic to $\mathcal{H}^L$. By transitivity, we obtain

$$\mathcal{H}^{2n-1} = (\mathfrak{S}^\sharp)^{n-1}\mathcal{H}^1, \quad \mathcal{H}^{2n} = (\mathfrak{S}^\sharp)^{n-1}\mathcal{H}^2 \tag{6.50}$$

for each $n \geqslant 1$. We computed $\mathcal{H}^1$ and $\mathcal{H}^2$ in Lemma 6.3.2. This allows us to compute representatives of the elements of $\mathcal{H}^L$ for odd and even $L$ from the repeated action of $\mathfrak{S}$ on $|{\uparrow}\rangle$ and $|\chi\rangle$, respectively, which leads to (6.30). □

We note that the state $|\chi \cdots \chi\rangle$ can be seen as the limit case of the alternative decomposition of the supersymmetry singlet of the XYZ chain given in (4.73) when $\zeta$ and $y$ tend to zero. This decomposition involves the state $|\chi(y,\zeta)\rangle$, where we write the dependence in the parameters $\zeta, y$. We have, using Lemma 4.5.5,

$$\lim_{y,\zeta\to 0} \frac{-y}{\zeta}|\chi(y,\zeta)\rangle = |{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle. \tag{6.51}$$

Hence, in a properly normalised limit, one retrieves the state $|\chi\rangle$ of Theorem 6.3.1.

## 6.4 Decomposition of zero-energy states

In this section, we analyse the zero-energy states of the spin-chain Hamiltonians $H$ with $y = 0$, which correspond to the ground states of

$H_{\text{XXZ}}$. Our main goal is to unveil some of their properties with the help of Theorem 6.3.1. We discuss two decompositions of the zero-energy states arising from the representatives of the (co)homology. We deduce from these decompositions their magnetisation, parity and relations between specific components.

**Cohomology decomposition.** It follows from Proposition 1.4.3 that a spin-chain ground state can be written as the sum of a representative of $\mathcal{H}^L$ and an element of the image of the supercharge. Specifically, we have:

**Proposition 6.4.1.** *For each $L \geqslant 1$, the space of the ground states of the Hamiltonian (6.3) is one dimensional. For $L = 2n$, it is spanned by*

$$|\Psi'_{2n}\rangle = |\chi \cdots \chi\rangle + \mathfrak{Q}'|\phi'_{2n}\rangle \tag{6.52}$$

*with $|\phi'_{2n}\rangle \in V^{2n-1}$. For $L = 2n-1$, it is spanned by*

$$|\Psi'_{2n-1}\rangle = |\uparrow\rangle \otimes |\chi \cdots \chi\rangle + \mathfrak{Q}'|\phi'_{2n-1}\rangle \tag{6.53}$$

*with $|\phi'_{2n-1}\rangle \in V^{2(n-1)}$.*

*Proof.* The proof is a direct consequence of Theorem 6.3.1 and the cohomology decomposition of a supersymmetry singlet (1.49). □

From now, we write $|\Psi'_L\rangle$ for the state defined in (6.52) and (6.53) for even and odd $L$, respectively. Furthermore, we write

$$(\Psi'_L)_{s_1 s_2 \ldots s_L} = \langle s_1 s_2 \cdots s_L | \Psi'_L\rangle, \tag{6.54}$$

with $s_i \in \{\uparrow, \downarrow\}$ for its components with respect to the canonical basis of $V^L$.

The decompositions of the zero-energy states given in (6.52) and (6.53) allow us to derive several simple properties of $|\Psi'_L\rangle$. Two immediate consequences are:

**Corollary 6.4.2.** *For each $n \geqslant 1$, we have*

$$(\Psi'_{2n})_{\uparrow\downarrow\cdots\uparrow\downarrow} = 1 = (\Psi'_{2n})_{\downarrow\uparrow\cdots\downarrow\uparrow} \tag{6.55}$$

*and*

$$(\Psi'_{2n-1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow} = 1. \tag{6.56}$$

*Proof.* We consider $L = 2n$ and project both sides of the equality (6.52) onto the state $|\omega\rangle = |{\uparrow\downarrow} \cdots {\uparrow\downarrow}\rangle$. We compute $\langle\omega|\chi\cdots\chi\rangle = 1$ from the definition of $|\chi\rangle$. Furthermore, $|\omega\rangle$ is annihilated by the adjoint supercharge, as follows from the action (6.29) of the adjoint local supercharge $\mathfrak{q}'^{\dagger}$.

The proof of $(\Psi'_{2n})_{\downarrow\uparrow\cdots\downarrow\uparrow} = 1$ is similar and uses the vector $|\omega\rangle = |{\downarrow\uparrow}\cdots{\downarrow\uparrow}\rangle$. Finally, the case $L = 2n-1$ is shown similarly with $|\omega\rangle = |{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}\rangle$. □

**Corollary 6.4.3.** *We have $\mathcal{M}|\Psi'_{2n}\rangle = 0$ and $\mathcal{M}|\Psi'_{2n-1}\rangle = 1/2|\Psi'_{2n-1}\rangle$.*

*Proof.* The $E = 0$ eigenspace of $H$ is one-dimensional. Furthermore, the Hamiltonian conserves the magnetisation $[H, \mathcal{M}] = 0$. Hence, we must have $\mathcal{M}|\Psi'_L\rangle = m_L|\Psi'_L\rangle$ for any $L \geqslant 1$. To find $m_L$, it is sufficient to project this equality onto simple basis vectors. For $L = 2n$, we find

$$0 = \langle{\uparrow\downarrow}\cdots{\uparrow\downarrow}|\mathcal{M}|\Psi'_{2n}\rangle = m_{2n}\langle{\uparrow\downarrow}\cdots{\uparrow\downarrow}|\Psi'_{2n}\rangle. \tag{6.57}$$

According to Corollary 6.4.2, $(\Psi'_{2n})_{\uparrow\downarrow\cdots\uparrow\downarrow} = 1$ and therefore $m_{2n} = 0$. For $L = 2n-1$, the proof is similar. □

The Proposition 6.4.1 implies that $|\Psi'_L\rangle$ is even under the action of the parity operator.

**Corollary 6.4.4.** *For any $L \geqslant 1$ we have $\mathcal{C}|\Psi'_L\rangle = |\Psi'_L\rangle$.*

*Proof.* The proof follows the lines of Corollary 6.4.3: since the $E = 0$ eigenspace of $H$ has dimension one and $[H, \mathcal{C}] = 0$, we must have $\mathcal{C}|\Psi'_L\rangle = c_L|\Psi'_L\rangle$ for any $L \geqslant 1$. To fix $c_L$ we project this equation onto simple basis states.

For $L = 2n-1$, we consider the projection onto the parity-invariant state $|{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}\rangle$:

$$\langle{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}|\Psi'_{2n-1}\rangle = \langle{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}|\mathcal{C}|\Psi'_{2n-1}\rangle = c_{2n-1}\langle{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}|\Psi'_{2n-1}\rangle. \tag{6.58}$$

According to Corollary 6.4.2, $\langle{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}|\Psi'_{2n-1}\rangle = (\Psi'_{2n-1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow} = 1$ and therefore $c_{2n-1} = 1$.

For $L = 2n$, the projection onto the state $|{\uparrow\downarrow}\cdots{\uparrow\downarrow}\rangle$ leads to

$$\langle{\downarrow\uparrow}\cdots{\downarrow\uparrow}|\Psi'_{2n}\rangle = \langle{\uparrow\downarrow}\cdots{\uparrow\downarrow}|\mathcal{C}|\Psi'_{2n}\rangle = c_{2n}\langle{\uparrow\downarrow}\cdots{\uparrow\downarrow}|\Psi'_{2n}\rangle. \tag{6.59}$$

The Corollary 6.4.2 directly implies $c_{2n} = 1$. □

We notice in this proof that a zero-energy state can be parity-invariant even though its representative is not.

**Homology decomposition.** Up to now, we have focused on the cohomology of the supercharge $\mathfrak{Q}'$ and the resulting decomposition of the zero-energy states given in Proposition 6.4.1. We know from Chapter 1 that we could as well have considered the adjoint supercharge $\mathfrak{Q}'^\dagger$ and its homology, which is the direct sum of quotient spaces $\mathcal{H}_L$, $L \geqslant 1$, where $\mathcal{H}_1 = V/\mathrm{im}\{\mathfrak{Q}'^\dagger : V^2 \to V\}$ and

$$\mathcal{H}_L = \frac{\ker\{\mathfrak{Q}'^\dagger : V^L \to V^{L-1}\}}{\mathrm{im}\{\mathfrak{Q}'^\dagger : V^{L+1} \to V^L\}}, \quad \text{for} \quad L \geqslant 2. \tag{6.60}$$

The Corollary 1.4.7 states that the cohomology and homology are isomorphic:

$$\mathcal{H}_L \simeq \mathcal{H}^L \quad \text{for each} \quad L \geqslant 1. \tag{6.61}$$

We now determine an alternative decomposition of the zero-energy states $|\Psi'_L\rangle$ using this property.

**Proposition 6.4.5.** *For each $L \geqslant 1$, we have*

$$|\Psi'_L\rangle = \mu_L|{\uparrow\downarrow\uparrow\downarrow}\cdots\rangle + \mathfrak{Q}'^\dagger|\varphi'_L\rangle \tag{6.62}$$

*with $|\varphi'_L\rangle \in V^{L+1}$. The constants $\mu_L$ are given by*

$$\mu_{2n} = \langle\chi\cdots\chi|\Psi'_{2n}\rangle, \quad \mu_{2n-1} = (\langle\uparrow| \otimes \langle\chi\cdots\chi|)\,|\Psi'_{2n-1}\rangle \tag{6.63}$$

*for each $n \geqslant 1$.*

*Proof.* We focus on the case $L = 2n$. Since $\mathcal{H}_{2n}$ is one-dimensional, each of its non-zero elements can be represented by a non-zero multiple of a fixed vector in $V^{2n}$. This vector must be in the kernel (but not in the image) of $\mathfrak{Q}'^\dagger$. We claim that such a vector is given by

$$|\omega\rangle = |{\uparrow\downarrow}\cdots{\uparrow\downarrow}\rangle. \tag{6.64}$$

Using (6.29), one readily checks that the adjoint supercharge $\mathfrak{Q}'^\dagger$ annihilates $|\omega\rangle$. Furthermore, it cannot be in the image of $\mathfrak{Q}'^\dagger$. Otherwise, if $|\omega\rangle = \mathfrak{Q}'^\dagger|\tilde{\omega}\rangle$ for some $|\tilde{\omega}\rangle \in V^{2n+1}$ then it follows that $(\Psi'_{2n})_{\uparrow\downarrow\cdots\uparrow\downarrow} = \langle\omega|\Psi'_{2n}\rangle = \langle\tilde{\omega}|\mathfrak{Q}'|\Psi'_{2n}\rangle = 0$. This contradicts Corollary 6.4.2.

It follows that the zero-energy states for $L = 2n$ have the decomposition

$$|\Psi'_{2n}\rangle = \mu_{2n}|\uparrow\downarrow\cdots\uparrow\downarrow\rangle + \mathfrak{Q}'^\dagger|\varphi'_{2n}\rangle \tag{6.65}$$

for some non-zero scalar $\mu_{2n}$ and $|\varphi'_{2n}\rangle \in V^{2n+1}$. The vector $|\varphi'_{2n}\rangle$ cannot be determined from homological arguments. The factor $\mu_{2n}$, however, can be found by computing the scalar product of the zero-energy state with suitable states $|\gamma\rangle$ that are annihilated by the supercharge $\mathfrak{Q}'$ and have a non-zero scalar product with the representative. In the present case, the choice $|\gamma\rangle = |\chi\cdots\chi\rangle$ leads to

$$\mu_{2n} = \langle\chi\cdots\chi|\Psi'_{2n}\rangle. \tag{6.66}$$

The argument for $L = 2n-1$ is similar, with the choice $|\gamma\rangle = |\uparrow\rangle \otimes |\chi\cdots\chi\rangle$. □

A consequence of the (co)homology decompositions of the zero-energy states given in Proposition 6.4.1 and Proposition 6.4.5 is:

**Proposition 6.4.6.** *For each $n \geqslant 1$, we have*

$$\|\Psi'_{2n}\|^2 = (\Psi'_{2n})_{\uparrow\downarrow\cdots\uparrow\downarrow}\langle\chi\cdots\chi|\Psi'_{2n}\rangle \tag{6.67a}$$

$$\|\Psi'_{2n-1}\|^2 = (\Psi'_{2n-1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}\left(\langle\uparrow|\otimes\langle\chi\cdots\chi|\right)|\Psi'_{2n-1}\rangle. \tag{6.67b}$$

*Proof.* These results directly follow from the projection of both sides of the equalities (6.62) onto the state $|\Psi'_L\rangle$. We have

$$\|\Psi'_L\|^2 = \mu_L\langle\Psi'_L|\uparrow\downarrow\uparrow\downarrow\cdots\rangle + \langle\Psi'_L|\mathfrak{Q}'^\dagger|\varphi'_L\rangle. \tag{6.68}$$

The second term in the right-hand side vanishes and the definition of $\mu_L$ allows us to conclude. □

We use this factorisation property of the norm in the next chapter when we discuss scalar products between ground states.

## 6.5 Supersymmetry of the periodic XXZ spin chain

**Supercharge of the periodic spin chain.** We recall from Chapter 2 that $W^L$ is the space of alternate-cyclic states, which are eigenvectors of the translation operator $\mathcal{S}$ with eigenvalue $(-1)^{L+1}$. For each $L$, we define

$$\mathcal{N} = \frac{1}{L}\sum_{j=0}^{L-1}\left((-1)^{L+1}\mathcal{S}\right)^j . \tag{6.69}$$

This operator satisfies $\mathcal{N} = \mathcal{N}^\dagger$ and $\mathcal{N}^2 = \mathcal{N}$. It follows that $\mathcal{N}$ is an orthogonal projection. It maps states of $V^L$ onto the subspace $W^L$.

The definition of the supercharge $\mathfrak{Q}$ for spin chains with periodic boundary conditions uses the projector $\mathcal{N}$:

$$\mathfrak{Q} = \left(\alpha_L \sum_{j=0}^{L}(-1)^j \mathfrak{q}_j\right)\mathcal{N}, \tag{6.70}$$

where we used $\alpha_L = \sqrt{\frac{L}{L+1}}$ as a shorthand notation. This definition is identical to the one made in Chapter 2, with $\zeta$ set to 0: on $W^L$, the supercharge acts as the alternating sum of local supercharges, multiplied by $\alpha_L$. On every other eigenspace of $\mathcal{S}$, it is identically zero.

The corresponding Hamiltonian follows from (2.28): we have

$$\{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = H_{\mathrm{XXZ}}^{(\mathrm{per})} + \frac{3L}{4} \quad \text{on } W^L. \tag{6.71}$$

The Hamiltonian $H_{\mathrm{XXZ}}^{(\mathrm{per})}$ has been investigated by Yang and Fendley [80], using a mapping to the aforementioned $M_1$ model with periodic boundary conditions. We summarise their results for the $XXZ$ spin-chain with an odd number of sites in the following theorem.

**Theorem 6.5.1** ([80])**.** *For each odd $L$, the Hamiltonian $H_{\mathrm{XXZ}}^{(\mathrm{per})}$ has the doubly-degenerate ground-state energy $E = -\frac{3L}{4}$. The corresponding eigenvectors are alternate-cyclic states and belong to the direct sum of the eigenspaces of $\mathcal{M}$ with eigenvalue $+\frac{1}{2}$ and $-\frac{1}{2}$.*

Two remarks about this theorem are in order. First, the degeneracy is a consequence of the Hamiltonian being invariant under spin reversal $[H_{\mathrm{XXZ}}^{(\mathrm{per})}, \mathcal{R}] = 0$: if $|\psi\rangle$ is a ground state with magnetisation $+\frac{1}{2}$, then $\mathcal{R}|\psi\rangle$ is also a ground state, with magnetisation $-\frac{1}{2}$, and these states are orthogonal to each other. This symmetry implies that every ground state $|\psi\rangle$ can be written uniquely as the sum $|\psi_+\rangle + |\psi_-\rangle$ such that

$$\mathcal{M}|\psi_+\rangle = \frac{1}{2}|\psi_+\rangle, \quad \mathcal{M}|\psi_-\rangle = -\frac{1}{2}|\psi_-\rangle. \tag{6.72}$$

Hence, we can separately examine the ground states in each magnetisation subspace.

Second, the existence of the eigenvalue can be seen as a consequence of Theorem 2.4.5 for $\zeta = 0$, as eigenvalues are continuous functions of $\zeta$. However, the eigenvectors are not necessarily so. In particular, the ground states introduced in Chapter 2 are not defined for $\zeta = 0$.

The goal of this and the following section is to characterise the ground states using (co)homological arguments. We start by rephrasing the statement of Theorem 6.5.1: the cohomology of the supercharge $\mathfrak{Q}$ is the direct sum

$$\mathcal{H}^L = \mathcal{H}_+^L \oplus \mathcal{H}_-^L. \tag{6.73}$$

Here $\mathcal{H}_+^L$ and $\mathcal{H}_-^L$ are one-dimensional quotient spaces (hence they consist of a single equivalence class) and are such that each representative of $\mathcal{H}_+^L$ and $\mathcal{H}_-^L$ has magnetisation $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively.

To construct simple representatives of these equivalences classes, we investigate the relation between the supercharges $\mathfrak{Q}$ and $\mathfrak{Q}'$, and the projector $\mathcal{N}$.

First, we note that the supercharge $\mathfrak{Q}$ satisfies $\mathfrak{Q}\mathcal{N} = \mathfrak{Q}$. This relation follows from the definition of $\mathfrak{Q}$. Second, the following proposition establishes a connection between $\mathfrak{Q}$ and $\mathfrak{Q}'$.

**Proposition 6.5.2.** *For each $L$, the supercharges $\mathfrak{Q}$ and $\mathfrak{Q}'$ satisfy*

$$\alpha_L \mathfrak{Q} = \mathcal{N}\mathfrak{Q}'. \tag{6.74}$$

*Proof.* The action of the projector on the local supercharge $\mathfrak{q}_i$, for $i = 1, \dots, L$ yields

$$\mathcal{N}\mathfrak{q}_i = \frac{1}{L+1}\sum_{j=0}^{L}(-1)^{(L+2)j}\mathcal{S}^j\mathfrak{q}_i = \frac{1}{L+1}\left(\sum_{j=0}^{L}\mathfrak{q}_j\mathcal{S}^j(-1)^{Lj}\right)(-1)^{Li}\mathcal{S}^{-i}. \tag{6.75}$$

We take the alternating sum of these equalities:

$$\begin{aligned}\mathcal{N}\mathfrak{Q}' &= \frac{1}{L+1}\sum_{i=1}^{L}(-1)^i\left(\sum_{j=0}^{L}\mathfrak{q}_j\mathcal{S}^j(-1)^{Lj}\right)(-1)^{Li}\mathcal{S}^{-i}\\ &= \frac{1}{L+1}\sum_{j=0}^{L}(-1)^j\mathfrak{q}_j\sum_{i=1}^{L}\left((-1)^{L+1}\mathcal{S}\right)^{j-i} = \frac{L}{L+1}\sum_{j=0}^{L}(-1)^j\mathfrak{q}_j\mathcal{N}.\end{aligned}$$

We used the definition of the projector $\mathcal{N}$ to obtain the last equality, the right-hand side of which equals $\alpha_L\mathfrak{Q}\mathcal{N} = \alpha_L\mathfrak{Q}$. This ends the proof. □

The previous result implies a relation between adjoint supercharges. We have:

$$\mathfrak{Q}'^\dagger\mathcal{N} = \alpha_L\mathfrak{Q}^\dagger. \tag{6.76}$$

The relation between the supercharges for open and periodic systems does not come as a surprise. This result is expected from the connection between the supercharges and the structures that are used in cyclic cohomology [74].

## 6.6 Decomposition of zero-energy states

In this section, we construct representatives of $\mathcal{H}_+^L$ and $\mathcal{H}_-^L$ for each odd $L = 2n+1$. We use those representatives to characterise the ground states of $H_{\mathrm{XXZ}}^{(\mathrm{per})}$. The proofs are based on the representatives of the (co)homology of $\mathfrak{Q}'$ as well as on Proposition 6.5.2.

**Proposition 6.6.1.** *The quotient space $\mathcal{H}_+^L$ is spanned by the equivalence class of the state*

$$\mathcal{N}\left(|\uparrow\rangle \otimes |\chi\cdots\chi\rangle\right). \tag{6.77}$$

*Proof.* It is straightforward to check that the state $\mathcal{N}\left(|{\uparrow}\rangle \otimes |\chi\cdots\chi\rangle\right)$ has magnetisation $+\frac{1}{2}$. To prove that it is an adequate representative, we show that this state belongs to the kernel of $\mathfrak{Q}$ but is not in its image.

On the one hand, using Proposition 6.5.2, one checks that the state $\mathcal{N}\left(|{\uparrow}\rangle \otimes |\chi\cdots\chi\rangle\right)$ vanishes under the action of the supercharge:

$$\mathfrak{Q}\mathcal{N}\left(|{\uparrow}\rangle \otimes |\chi\cdots\chi\rangle\right) = \alpha_L^{-1}\mathcal{N}\mathfrak{Q}'\left(|{\uparrow}\rangle \otimes |\chi\cdots\chi\rangle\right) = 0. \tag{6.78}$$

On the other hand, we show that the state $\mathcal{N}\left(|{\uparrow}\rangle \otimes |\chi\cdots\chi\rangle\right)$ is not in the image of $\mathfrak{Q}$. To this end, we define the state $|\omega_L\rangle$ as follows:

$$|\omega_L\rangle = \sum_{j=0}^{n} |{\uparrow}\rangle \otimes |{\uparrow\downarrow}\rangle^j \otimes |{\downarrow\uparrow}\rangle^{n-j}. \tag{6.79}$$

Here, $|{\uparrow\downarrow}\rangle^j$ is the shorthand notation for the $j$-fold tensor product $|{\uparrow\downarrow}\rangle \otimes \cdots \otimes |{\uparrow\downarrow}\rangle$, and similarly for $|{\downarrow\uparrow}\rangle$.

Let us suppose that there exists a state $|\gamma\rangle$ such that the following relation holds

$$\mathcal{N}\left(|{\uparrow}\rangle \otimes |\chi\cdots\chi\rangle\right) = \mathfrak{Q}|\gamma\rangle. \tag{6.80}$$

We project both sides of this equality onto the state $\mathcal{N}|\omega_L\rangle$. On the left-hand side of this relation, the scalar product $\langle\omega_L|\mathcal{N}\left(|{\uparrow}\rangle \otimes |\chi\cdots\chi\rangle\right)$ is easily computed and equals $\frac{3n+1}{2n+1}$, which is non-zero. Furthermore, we show that $\mathfrak{Q}^\dagger\mathcal{N}|\omega_L\rangle = \mathfrak{Q}^\dagger|\omega_L\rangle$ vanishes. Indeed, it is clear from (6.29) that the action of $\mathfrak{q}_j^\dagger$ on $|\omega_L\rangle$ is non-trivial only for $j = 0, 1$. We have

$$\begin{aligned}\sum_{j=0}^{L}(-1)^j\mathfrak{q}_j|\omega_L\rangle &= \mathfrak{q}_0^\dagger|\omega_L\rangle - \mathfrak{q}_1^\dagger|\omega_L\rangle \\ &= \sum_{j=0}^{n-1}|{\uparrow\downarrow}\rangle^j \otimes |{\downarrow\uparrow}\rangle^{n-j-1} \otimes |{\downarrow\downarrow}\rangle - \sum_{j=1}^{n}|{\downarrow\downarrow}\rangle \otimes |{\uparrow\downarrow}\rangle^{j-1} \otimes |{\downarrow\uparrow}\rangle^{n-j}.\end{aligned} \tag{6.81}$$

To obtain the action of $\mathfrak{Q}^\dagger$ on the state $|\omega_L\rangle$, we apply the projector $\mathcal{N}$ on both sides of this equality. The right-hand side yields zero, which proves that $\mathfrak{Q}^\dagger|\omega_L\rangle = 0$ and leads to the desired contradiction. □

**Proposition 6.6.2.** *The quotient space $\mathcal{H}_-^L$ is spanned by the equivalence class of the state $\mathcal{N}|\bar{\omega}_L\rangle \in V^L$, where $|\bar{\omega}_L\rangle$ is defined as*

$$|\bar{\omega}_L\rangle = 2|{\downarrow}\rangle \otimes |\chi\cdots\chi\rangle + \sum_{j=1}^{n}|{\uparrow}\rangle \otimes |\underbrace{\chi\cdots\chi}_{j-1\ \textit{times}}\rangle \otimes |{\downarrow\downarrow}\rangle \otimes |\underbrace{\chi\cdots\chi}_{n-j\ \textit{times}}\rangle. \tag{6.82}$$

*Proof.* One readily checks that the vector $\mathcal{N}|\bar{\omega}_L\rangle$ has magnetisation $-1/2$. We may verify that it is annihilated by $\mathfrak{Q}$ and does not belong to its image.

First, we use Proposition 6.5.2 and compute the action of $\mathfrak{Q}'$ on $|\bar{\omega}_L\rangle$:

$$\begin{aligned}\mathfrak{Q}'|\bar{\omega}_L\rangle &= -2|{\uparrow\uparrow}\rangle \otimes |\chi\cdots\chi\rangle \\ &+ \sum_{j=1}^{n} |{\uparrow}\rangle \otimes |\underbrace{\chi\cdots\chi}_{j-1\,\text{times}}\rangle \otimes \big(|{\uparrow}\rangle\otimes|\chi\rangle - |\chi\rangle\otimes|{\uparrow}\rangle\big) \otimes |\underbrace{\chi\cdots\chi}_{n-j\,\text{times}}\rangle. \end{aligned} \tag{6.83}$$

The sum is telescopic and reduces to

$$\mathfrak{Q}'|\bar{\omega}_L\rangle = -|{\uparrow\uparrow}\rangle \otimes |\chi\cdots\chi\rangle - |{\uparrow}\rangle \otimes |\chi\cdots\chi\rangle \otimes |{\uparrow}\rangle. \tag{6.84}$$

We apply the projector $\mathcal{N}$ on both sides of the equality, its action on the right-hand side yields zero. Thus, we find $\mathfrak{Q}\mathcal{N}|\bar{\omega}_L\rangle = 0$.

Second, we show that $\mathcal{N}|\bar{\omega}_L\rangle$ is not in the image of $\mathfrak{Q}$, in the same vein as in Proposition 6.6.1. We suppose that there exists a state $|\bar{\gamma}\rangle$ such that

$$\mathcal{N}|\bar{\omega}_L\rangle = \mathfrak{Q}|\bar{\gamma}\rangle. \tag{6.85}$$

To obtain a contradiction, we project both sides of this identity onto a state which is in the kernel of $\mathfrak{Q}^\dagger$ and which has a non-zero scalar product with $\mathcal{N}|\bar{\omega}_L\rangle$. Such a state is given by $\mathcal{N}|{\downarrow\uparrow}\cdots{\downarrow\uparrow\downarrow}\rangle$. Indeed, it is clear from the action of the local supercharge that $\mathfrak{Q}^\dagger\mathcal{N}|{\downarrow\uparrow}\cdots{\downarrow\uparrow\downarrow}\rangle = 0$. Furthermore, one checks that $\langle{\downarrow\uparrow}\cdots{\downarrow\uparrow\downarrow}|\mathcal{N}|\bar{\omega}_L\rangle = \frac{3n+2}{2n+1} \neq 0$. $\square$

Using the two preceding propositions, we characterise the ground states of $H_{\text{XXZ}}^{(\text{per})}$.

**Theorem 6.6.3.** *For each odd $L$, the space of the ground states of the Hamiltonian $H_{\text{XXZ}}^{(\text{per})}$ is spanned by the states*

$$|\Psi_L\rangle = \mathcal{N}\left(|{\uparrow}\rangle \otimes |\chi\cdots\chi\rangle\right) + \mathfrak{Q}|\phi\rangle \tag{6.86}$$

*and*

$$|\bar{\Psi}_L\rangle = \mathcal{N}|\bar{\omega}_L\rangle + \mathfrak{Q}|\bar{\phi}\rangle, \tag{6.87}$$

*where $|\phi\rangle, |\bar{\phi}\rangle \in V^{L-1}$.*

*Proof.* The proof directly follows from (6.73) and the cohomology decomposition of a supersymmetry singlet (1.49). □

It is straightforward to verify that the states $|\Psi_L\rangle$ and $|\bar{\Psi}_L\rangle$ are alternate-cyclic, have magnetisation $\frac{1}{2}$ and $-\frac{1}{2}$, respectively, and are thus orthogonal to each other.

**Homology decompositions.** We determine an alternative homology decomposition of the states $|\Psi_L\rangle$ and $|\bar{\Psi}_L\rangle$ defined in (6.86) and (6.87). This is based on the equivalence between the homology and the cohomology as stated in Corollary 1.4.7.

**Proposition 6.6.4.** *For each $L = 2n+1$, the states $|\Psi_L\rangle$ and $|\bar{\Psi}_L\rangle$ possess the following homology decomposition:*

$$|\Psi_L\rangle = \lambda_L \mathcal{N}|\omega_L\rangle + \mathfrak{Q}^\dagger|\varphi\rangle \tag{6.88}$$

*and*

$$|\bar{\Psi}_L\rangle = \bar{\lambda}_L\mathcal{N}|{\downarrow\uparrow}\cdots{\downarrow\uparrow\downarrow}\rangle + \mathfrak{Q}^\dagger|\bar{\varphi}\rangle. \tag{6.89}$$

*Here, $|\varphi\rangle$, $|\bar{\varphi}\rangle$ are vectors belonging to $V^{L+1}$. The state $|\omega_L\rangle \in V^L$ is defined in (6.79). The constant $\lambda_L$ and $\bar{\lambda}_L$ are non-zero and given by*

$$\lambda_L = \frac{2n+1}{3n+1}\left(\langle\uparrow|\otimes\langle\chi\cdots\chi|\right)|\Psi_L\rangle, \qquad \bar{\lambda}_L = \frac{2n+1}{3n+2}\langle\bar{\omega}_L|\bar{\Psi}_L\rangle. \tag{6.90}$$

*Proof.* First, we prove that $|\Psi_L\rangle$ can be written according to (6.88). The state $\mathcal{N}|\omega_L\rangle$ is annihilated by the adjoint supercharge, as seen in the proof of Proposition 6.6.1 and is not in its image. Indeed, if it were the case, there should be a state $|\gamma'\rangle$ such that $\mathcal{N}|\omega_L\rangle = \mathfrak{Q}^\dagger|\gamma'\rangle$. By taking the scalar product of both sides of this identity with $\mathcal{N}\left(|\uparrow\rangle\otimes|\chi\cdots\chi\rangle\right)$, we obtain a contradiction.

Hence, the decomposition (6.88) follows from (1.61). We compute the constant $\lambda_L$ by projecting both sides of the equality (6.88) onto the state $\mathcal{N}\left(|\uparrow\rangle\otimes|\chi\cdots\chi\rangle\right)$.

Second, we show that (6.89) is a valid homology decomposition. The state $\mathcal{N}|{\downarrow\uparrow}\cdots{\downarrow\uparrow\downarrow}\rangle$ is in the kernel of $\mathfrak{Q}^\dagger$ and cannot be in its image.

Indeed, if it were the case, one would have $\mathcal{N}|{\downarrow\uparrow}\cdots{\downarrow\uparrow\downarrow}\rangle = \mathfrak{Q}^\dagger|\bar{\gamma}'\rangle$ for a particular state $|\bar{\gamma}'\rangle$. The projection of both sides of this equality onto the state $\mathcal{N}|\bar{\omega}_L\rangle$ (6.79) would give a contradiction.

We find the constant $\bar{\lambda}_L$ by projecting both sides of (6.89) onto the state $\mathcal{N}|\bar{\omega}_L\rangle$. This concludes the proof. □

The states $|\Psi_L\rangle$ and $|\bar{\Psi}_L\rangle$ can be mapped into each other via a spin-reversal operation: there exists a non-zero constant $\rho_L$ for each $L$ such that

$$\mathcal{R}|\Psi_L\rangle = \rho_L|\bar{\Psi}_L\rangle, \quad \mathcal{R}|\bar{\Psi}_L\rangle = \rho_L^{-1}|\Psi_L\rangle. \tag{6.91}$$

We use this symmetry as well as the (co)homology decompositions of the ground states $|\Psi_L\rangle$, $|\bar{\Psi}_L\rangle$ to unveil some of their properties.

**Corollary 6.6.5.** *For each $L \geqslant 1$, the components $(\Psi_L)_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}$ and $(\bar{\Psi}_L)_{\downarrow\uparrow\cdots\downarrow\uparrow\downarrow}$ are non zero.*

*Proof.* First we prove the statement for $|\bar{\Psi}_L\rangle$. The projection of both sides of the cohomology decomposition (6.87) onto the state $|{\downarrow\uparrow}\cdots{\downarrow\uparrow\downarrow}\rangle$ yields

$$\langle{\downarrow\uparrow}\cdots{\downarrow\uparrow\downarrow}|\bar{\Psi}_L\rangle = \langle{\downarrow\uparrow}\cdots{\downarrow\uparrow\downarrow}|\mathcal{N}|\bar{\omega}_L\rangle + \langle{\downarrow\uparrow}\cdots{\downarrow\uparrow\downarrow}|\mathfrak{Q}|\bar{\phi}\rangle. \tag{6.92}$$

The action of the supercharge is zero and one checks, as in the proof of Proposition 6.6.2, that $\langle{\downarrow\uparrow}\cdots{\downarrow\uparrow\downarrow}|\mathcal{N}|\bar{\omega}_L\rangle = \frac{3n+2}{2n+1} \neq 0$.

Second, the proof for $|\Psi_L\rangle$ follows from the spin-reversal symmetry. We have

$$(\Psi_L)_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow} = \langle{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}|\mathcal{R}\mathcal{R}|\Psi_L\rangle = \rho_L(\bar{\Psi}_L)_{\downarrow\uparrow\cdots\downarrow\uparrow\downarrow}, \tag{6.93}$$

which is non-zero. This ends the proof. □

We note that we can also prove this property by using the Perron-Frobenius theorem applied to the restriction of the Hamiltonian to the adequate subspace.

**Corollary 6.6.6.** *For any $L \geqslant 1$, we have*

$$\mathcal{C}|\Psi_L\rangle = |\Psi_L\rangle, \quad \mathcal{C}|\bar{\Psi}_L\rangle = |\bar{\Psi}_L\rangle. \tag{6.94}$$

*Proof.* The Hamiltonian $H_{\text{XXZ}}^{(\text{per})}$ is invariant under the parity operator, $[H_{\text{XXZ}}^{(\text{per})}, \mathcal{C}] = 0$. Moreover, we have the commutation relation $[\mathcal{C}, \mathcal{M}] = 0$. Hence we must have

$$\mathcal{C}|\Psi_L\rangle = c_L|\Psi_L\rangle, \quad \mathcal{C}|\bar{\Psi}_L\rangle = \bar{c}_L|\bar{\Psi}_L\rangle. \tag{6.95}$$

We consider the relation for $|\Psi_L\rangle$. We project the left-hand equation onto the basis state $|{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}\rangle$ and find

$$\langle{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}|\Psi_L\rangle = \langle{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}|\mathcal{C}|\Psi_L\rangle = c_L\langle{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}|\Psi_L\rangle. \tag{6.96}$$

Corollary 6.6.5 ensures that $\langle{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}|\Psi_L\rangle$ is non-zero. Therefore $c_L = 1$. The proof for $|\bar{\Psi}_L\rangle$ is similar. □

Finally, we observe a factorisation of the norm of $|\Psi_L\rangle$ and $|\bar{\Psi}_L\rangle$, in terms of (sums of) components. The proof of this proposition is straightforward and is similar to the one of Proposition 6.4.6.

**Proposition 6.6.7.** *For each $L$, we have*

$$\langle\Psi_L|\Psi_L\rangle = \frac{2n+1}{3n+1}\left(\langle\uparrow| \otimes \langle\chi\cdots\chi|\right)|\Psi_L\rangle\langle\omega_L|\Psi_L\rangle \tag{6.97a}$$

*and*

$$\langle\bar{\Psi}_L|\bar{\Psi}_L\rangle = \frac{2n+1}{3n+2}\langle{\downarrow\uparrow}\cdots{\downarrow\uparrow\downarrow}|\bar{\Psi}_L\rangle\langle\bar{\omega}_L|\bar{\Psi}_L\rangle. \tag{6.97b}$$

We note that these relations are normalisation independent. We use them in the next chapter to investigate specific scalar products between ground states.

# Chapter 7

# Multipartite fidelities

A purely quantum property of a system is the entanglement. A system is said to be entangled if a local measurement affects parts of the system that are far away. Quantifying the entanglement is an essential task in the understanding of this property. This question was first raised in the framework of information theory. Nowadays, there is a vast interest in entanglement in various domains of physics, such as quantum computing [111], condensed matter [112] and high energy physics.

There are multiple ways to measure the entanglement of a system [113]. Amongst these, the *entanglement entropy*, a specific type of the so-called *von Neumann entropy* [114], is arguably the most studied one. We consider a quantity that behaves similarly to the entanglement entropy: the *fidelity* [115, 116, 117]. It is defined as the scalar product (also called overlap) between ground states of Hamiltonians that differ by a perturbation parametrised by $\lambda$. Let $|\lambda\rangle$ be the ground state of the Hamiltonian $H(\lambda)$. The fidelity is

$$f(\lambda, \lambda') = \left| \frac{\langle \lambda | \lambda' \rangle}{\sqrt{\langle \lambda | \lambda \rangle \langle \lambda' | \lambda' \rangle}} \right|^2 . \tag{7.1}$$

We consider the specific example where the system is partitioned in $m$ complementary subsystems and $\lambda$ parametrises the interaction between the subsystems:

$$H(\lambda) = H^1 + \cdots + H^m + \lambda H^{\text{int}}. \tag{7.2}$$

Here $H^1, \dots, H^m$ are the Hamiltonians of the subsystems, and $H^{\text{int}}$ contains the interaction terms between different subsystems. Let us denote by $|\psi_1\rangle, \dots, |\psi_m\rangle$ and $|\psi\rangle$ the ground states of $H^1, \dots, H^m$ and $H(1)$, respectively. We define the *multipartite fidelity* as the ratio

$$f(1,0) = \left| \frac{\langle\psi| \left(|\psi_1\rangle \otimes \cdots \otimes |\psi_m\rangle\right)}{\|\psi\| \, \|\psi_1\| \cdots \|\psi_m\|} \right|^2 . \tag{7.3}$$

In particular, the case $m = 2$ is called the *bipartite fidelity* and was introduced and investigated by Dubail and Stéphan [118]. The bipartite fidelity has been obtained by lattice derivations in a few cases [119, 120, 121, 122, 5]. For one-dimensional quantum critical systems, this quantity has a large-$L$ asymptotic expansion whose first few terms have been predicted by conformal field theory (CFT) techniques [118, 119]. These CFT predictions match with the exact lattice computations at the leading and subleading orders [119, 122, 5].

In this chapter, we consider two types of multipartite fidelities, based on different scalar products. Specifically, let $L$, $L_1, \dots, L_m$ be such that $L = \sum_{i=1}^m L_i$. We investigate the two overlaps

$$\langle\Psi'_L| \left(|\Psi'_{L_1}\rangle \otimes \cdots \otimes |\Psi'_{L_m}\rangle\right), \quad \langle\Psi_L| \left(|\Psi'_{L_1}\rangle \otimes \cdots \otimes |\Psi'_{L_m}\rangle\right), \tag{7.4}$$

where we recall that $|\Psi'_L\rangle$ and $|\Psi_L\rangle$ are ground states of the Hamiltonians (6.3) and (6.4), with open and periodic boundary conditions, respectively. Moreover, we introduce a new class of fidelities. They are built upon tensor products of ground states and down spins such that two ground states are separated by down spins. As an example, we consider the scalar product

$$\langle\bar{\Psi}_L| \left(|\downarrow\rangle \otimes |\Psi'_{L_1}\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\Psi'_{L_m}\rangle \otimes |\downarrow\rangle\right) \tag{7.5}$$

with $L = \sum_{i=1}^m L_i + m + 1$. We refer to the fidelity corresponding to this kind of scalar product as a *dressed multipartite fidelity*.

We use the (co)homology decompositions of the ground states of the XXZ spin-chain Hamiltonians with open and periodic boundary conditions to compute multipartite fidelities as a product of simple (sum of) components of each involved state. Using exact finite-size expressions of these (sum of) components, we provide an explicit formula for the scalar products as a function of $L_1, \dots, L_m$.

The layout of this chapter is as follows. We define of the multipartite fidelity in Section 7.1 and prove that they can be written in terms of simple (sum of) components of the ground states. We introduce the *dressed multipartite fidelity* in Section 7.2. It allows for the computation of fidelities that involve ground states of periodic spin-chain Hamiltonians on both sides of the scalar product. In Section 7.3, we evaluate exactly the multipartite fidelities and provide their scaling limits.

## 7.1 Multipartite fidelities

In Proposition 6.4.6, we have obtained simple relations for the square norm of the supersymmetry singlets. This simplicity is a consequence of the supersymmetry. To be specific, let us consider $L = 2n$. We have

$$\frac{\langle \Psi'_{2n}|\chi\cdots\chi\rangle}{\|\Psi'_{2n}\|} = \frac{1}{(\Psi'_{2n})_{\uparrow\downarrow\cdots\uparrow\downarrow}/\|\Psi'_{2n}\|}. \tag{7.6}$$

On the left-hand side of this equality, we have the projection of the normalised zero-energy state onto an $n$-fold tensor product of zero-energy states of the two-site chain. On the right-hand side, we find the inverse of a special component of the normalised zero-energy state on $2n$ sites.

It is natural to investigate if the result remains equally simple when we replace the $|\chi\rangle$'s by the zero-energy energy states of spins chains of generic lengths. This construction leads to the definition of multipartite fidelity: the square of the projection of the ground state of the complete chain onto the tensor product of the ground states of $m$ subchains of lengths $L_1, \ldots, L_m > 0$.

For each $L = L_1 + \cdots + L_m$, we define

$$Z'(L_1, \ldots, L_m) = \frac{\langle \Psi'_L| \left(\bigotimes_{j=1}^m |\Psi'_{L_j}\rangle\right)}{\|\Psi'_L\| \prod_{j=1}^m \|\Psi'_{L_j}\|}. \tag{7.7}$$

If $L$ is odd, we define

$$Z(L_1, \ldots, L_m) = \frac{\langle \Psi_L| \left(\bigotimes_{j=1}^m |\Psi'_{L_j}\rangle\right)}{\|\Psi_L\| \prod_{j=1}^m \|\Psi'_{L_j}\|}. \tag{7.8}$$

These quantities are the square roots of multipartite fidelities. The division by the norms of the states makes them normalisation-independent. For $L = 2n$ and $L_j = 2$ for $j = 1, \dots, n$, the expression (7.7) allows us to recover (up to factor) the left-hand side of (7.6).

**Theorem 7.1.1.** *If $L_1, \dots, L_m$ are even, then*

$$Z'(L_1, \dots, L_m) = \frac{\|\Psi'_L\|}{(\Psi'_L)_{\uparrow\downarrow\cdots\uparrow\downarrow}} \prod_{j=1}^{m} \frac{(\Psi'_{L_j})_{\uparrow\downarrow\cdots\uparrow\downarrow}}{\|\Psi'_{L_j}\|}. \tag{7.9}$$

*If $L_k$ is odd (for some $1 \leqslant k \leqslant m$) and $L_1, \dots, L_{k-1}, L_{k+1}, \dots, L_m$ are even, then*

$$Z'(L_1, \dots, L_m) = \frac{\|\Psi'_L\|}{(\Psi'_L)_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}} \frac{(\Psi'_{L_k})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}}{\|\Psi'_{L_k}\|} \prod_{\substack{j=1\\ j\neq k}}^{m} \frac{(\Psi'_{L_j})_{\uparrow\downarrow\cdots\uparrow\downarrow}}{\|\Psi'_{L_j}\|}, \tag{7.10}$$

$$Z(L_1, \dots, L_m) = g(L) \frac{\|\Psi_L\|}{\langle \omega_L | \Psi_L \rangle} \frac{(\Psi'_{L_k})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}}{\|\Psi'_{L_k}\|} \prod_{\substack{j=1\\ j\neq k}}^{m} \frac{(\Psi'_{L_j})_{\uparrow\downarrow\cdots\uparrow\downarrow}}{\|\Psi'_{L_j}\|}, \tag{7.11}$$

*where $g(L) = (3L-1)/(2L)$ and $|\omega_L\rangle$ is defined in (6.79). In all other cases, the scalar product vanishes.*

We note that these results are remarkably simple. The zero-energy states typically are very complicated states with many non-zero components. For a state in $V^L$ of magnetisation 0, there are, *a priori*, $2^{\binom{L}{L/2}}$ such non-trivial components. Nonetheless, we can infer the scalar product from the sole knowledge of less than $n$ components of the complete chain's ground state, as well as a single component of each state of the subchains. Furthermore, the exchange of $L_i$ and $L_j$ for $i \neq j$ leaves the result invariant even though it can completely change the subdivision of the chain of length $L$ into $m$ smaller subchains.

The proof of Theorem 7.1.1 is based on the following lemma:

**Lemma 7.1.2.** *If $L_1, \dots, L_m$ are even, then*

$$\bigotimes_{i=1}^{m} |\Psi'_{L_i}\rangle = |\chi \cdots \chi\rangle + \mathfrak{Q}' |\phi'_{L_1, \cdots, L_m}\rangle \tag{7.12}$$

*for a certain state* $|\phi'_{L_1,\cdots,L_m}\rangle$*. If* $L_k$ *is odd (for some* $1 \leqslant k \leqslant m$*) and* $L_1, \dots, L_{k-1}, L_{k+1}, \dots, L_m$ *are even, then*

$$\bigotimes_{i=1}^{m} |\Psi'_{L_i}\rangle = |\uparrow\rangle \otimes |\chi \cdots \chi\rangle + \mathfrak{Q}'|\phi'_{L_1,\cdots,L_m}\rangle \tag{7.13}$$

*for a certain state* $|\phi'_{L_1,\cdots,L_m}\rangle$*.*

*Proof.* First, we show (7.12). Its proof is based on recurrence. The case $m = 1$ follows from the cohomology decomposition (6.52). We now assume that the relation is valid for $n = 1, \dots, m-1$ and show that it holds for $n = m$: we have

$$\bigotimes_{i=1}^{m} |\Psi'_{L_i}\rangle = \Big(|\chi \cdots \chi\rangle + \mathfrak{Q}'|\phi'_{L_1,\cdots,L_{m-1}}\rangle\Big) \otimes \Big(|\underbrace{\chi \cdots \chi}_{\frac{L_m}{2}\text{ times}}\rangle + \mathfrak{Q}'|\phi'_{L_m}\rangle\Big).$$

The tensor product expands into four terms. We rearrange three of those terms as a vector in the image of $\mathfrak{Q}'$. We obtain (7.12) by choosing

$$|\phi'_{L_1,\cdots,L_m}\rangle = |\phi'_{L_1,\cdots,L_{m-1}}\rangle \otimes |\chi \cdots \chi\rangle + |\chi \cdots \chi\rangle \otimes |\phi'_{L_m}\rangle \\ + |\phi'_{L_1,\cdots,L_{m-1}}\rangle \otimes \mathfrak{Q}'|\phi'_{L_m}\rangle. \tag{7.14}$$

Second, we prove (7.13). Similarly to the previous case, one shows that

$$\bigotimes_{i=1}^{m} |\Psi'_{L_i}\rangle = |\chi \cdots \chi\rangle \otimes |\uparrow\rangle \otimes |\chi \cdots \chi\rangle + \mathfrak{Q}'|\phi'_{L_1,\cdots,L_m}\rangle. \tag{7.15}$$

for a certain state $|\phi'_{L_1,\cdots,L_m}\rangle$. Here, the up ($\uparrow$) spin in the right-hand side is at the position $L_1 + \cdots L_{k-1} + 1$. We observe that

$$|\chi \cdots \chi\rangle \otimes |\uparrow\rangle \otimes |\chi \cdots \chi\rangle = |\uparrow\rangle \otimes |\chi \cdots \chi\rangle \\ + \mathfrak{Q}'\left(\sum_{j=1}^{(L_1+\cdots+L_{k-1})/2} |\underbrace{\chi \cdots \chi}_{j-1\,\text{times}}\rangle \otimes |\downarrow\downarrow\rangle \otimes |\chi \cdots \chi\rangle\right). \tag{7.16}$$

This identity follows from $\mathfrak{Q}'|\downarrow\downarrow\rangle = |\chi\rangle \otimes |\uparrow\rangle - |\uparrow\rangle \otimes |\chi\rangle$. We combine the identities (7.15) and (7.16) to conclude the proof. $\square$

*Proof of Theorem 7.1.1.* First, we focus on the proof of (7.9). Using Lemma 7.1.2, we have

$$\langle \Psi'_L | \left( \bigotimes_{j=1}^{m} |\Psi'_{L_j}\rangle \right) = \langle \Psi'_L | \chi \cdots \chi \rangle + \langle \Psi'_L | \mathfrak{Q}' | \phi'_{L_1,\cdots,L_m} \rangle \tag{7.17}$$

The second term on the right-hand side of this equation vanishes. We divide by the norms of the zero-energy states and use Corollary 6.4.2 as well as Proposition 6.4.6 to find (7.9). The proof of (7.10) is similar.

Second, we prove (7.11). According to Lemma 7.1.2, we have

$$\langle \Psi_L | \left( \bigotimes_{j=1}^{m} |\Psi'_{L_j}\rangle \right) = \langle \Psi_L | \left( |\uparrow\rangle \otimes |\chi \cdots \chi\rangle \right) + \langle \Psi_L | \mathfrak{Q}' | \phi'_{L_1,\cdots,L_m} \rangle. \tag{7.18}$$

The action of $(\mathfrak{Q}')^\dagger$ on the state $|\Psi_L\rangle$ yields zero, as a consequence of (6.76):

$$(\mathfrak{Q}')^\dagger |\Psi_L\rangle = (\mathfrak{Q}')^\dagger \mathcal{N} |\Psi_L\rangle = \alpha_L \mathfrak{Q}^\dagger |\Psi_L\rangle = 0. \tag{7.19}$$

Upon division by the norms of the states and use of (6.97), we obtain (7.11).

Finally, the scalar products vanish in all other cases because of the definite magnetisation of the zero-energy states. □

## 7.2 Dressed multipartite fidelities

The proof of Theorem 7.1.1 is based on the supersymmetry, and specifically on the fact that the action on $\mathfrak{Q}'$ in the scalar product yields zero, which allows us to express multipartite fidelities in terms simple (sum of) normalised components. In this section, we use a similar property of the adjoint supercharge to define new multipartite fidelities. In particular, we construct dressed multipartite fidelities that involve the ground states $|\Psi_L\rangle$ and $|\bar{\Psi}_L\rangle$ of the periodic spin chain on both sides of the scalar product.

**Dressed multipartite fidelity and open spin chains.** Let $L_1, \dots, L_m$ and $L$ be such that $L+1 = \sum_{i=1}^{m}(L_i+1)$. We split the set $\{1, \dots, m\}$ into the disjoint sets $I$ and $I'$, such that $I \cup I' = \{1, \dots, m\}$ and $k \in I'$

if $L_k$ is even. These subsets allow us to label the ground states of the periodic and open spin chains. We introduce the notation

$$|\psi_{L_i}\rangle = \begin{cases} |\Psi_{L_i}\rangle & \text{for } i \in I, \\ |\Psi'_{L_i}\rangle & \text{for } i \in I'. \end{cases} \tag{7.20}$$

For each $L_1, \ldots, L_m$ and subsets $I, I'$ of $\{1, \ldots, m\}$, we introduce the quantity

$$Z'_d(I, I'; L_1, \ldots, L_m) = \frac{\langle \Psi'_L | \left( |\psi_{L_1}\rangle \otimes \bigotimes_{j=2}^m (|{\downarrow}\rangle \otimes |\psi_{L_j}\rangle) \right)}{\|\Psi'_L\| \prod_{j=1}^m \|\psi_{L_j}\|}. \tag{7.21}$$

This defines the square root of the corresponding *dressed multipartite fidelity*.

**Theorem 7.2.1.** *If $L_1, \ldots, L_m$ are odd then*

$$Z'_d(I, I'; L_1, \ldots, L_m) \\ = \frac{(\Psi'_L)_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}}{\|\Psi'_L\|} \prod_{i\in I} g(L_i) \frac{\|\Psi_{L_i}\|}{\langle \omega_{L_i} | \Psi_{L_i}\rangle} \prod_{i \in I'} \frac{\|\Psi'_{L_i}\|}{(\Psi'_{L_i})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}}, \tag{7.22}$$

*where $g(L) = (3L-1)/(2L)$. If $L_k$ is even (for some $1 \leqslant k \leqslant m$, $k \in I'$) and $L_1, \ldots, L_{k-1}$, $L_{k+1}, \ldots, L_m$ are odd, then*

$$Z'_d(I, I'; L_1, \ldots, L_m) \\ = \frac{(\Psi'_L)_{\uparrow\downarrow\cdots\uparrow\downarrow}}{\|\Psi'_L\|} \prod_{i\in I} g(L_i) \frac{\|\Psi_{L_i}\|}{\langle \omega_{L_i} | \Psi_{L_i}\rangle} \prod_{\substack{i \in I' \\ i \neq k}} \frac{\|\Psi'_{L_i}\|}{(\Psi'_{L_i})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}} \frac{\|\Psi'_{L_k}\|}{(\Psi'_{L_k})_{\uparrow\downarrow\cdots\uparrow\downarrow}}. \tag{7.23}$$

*In all other cases, the scalar product vanishes.*

*Proof.* Let us prove (7.22). The reasoning is similar to the proof of Theorem 7.1.1. We use the cohomology decomposition of $|\Psi'_L\rangle$ (6.53) in order to write

$$\langle \Psi'_L | \left( |\psi_{L_1}\rangle \otimes \bigotimes_{j=2}^m (|{\downarrow}\rangle \otimes |\psi_{L_j}\rangle) \right) \\ = \left( \langle {\uparrow} | \otimes \langle \chi \cdots \chi | + \langle \phi'_L | (\mathfrak{Q}')^\dagger \right) \left( |\psi_{L_1}\rangle \otimes \bigotimes_{j=2}^m (|{\downarrow}\rangle \otimes |\psi_{L_j}\rangle) \right). \tag{7.24}$$

The term involving $(\mathfrak{Q}')^\dagger$ vanishes. Indeed, the action of $(\mathfrak{Q}')^\dagger$ on $|\psi_{L_i}\rangle$ yields zero for each $i = 1, \dots, m$. This is direct for $i \in I'$ and follows from (7.19) for $i \in I$. Moreover, one checks that if both $|\psi\rangle$ and $|\psi'\rangle$ are in the kernel of $(\mathfrak{Q}')^\dagger$ then $(\mathfrak{Q}')^\dagger(|\psi\rangle \otimes |{\downarrow}\rangle \otimes |\psi'\rangle) = 0$ as a consequence of (6.29). Applying this property repeatedly, we reduce (7.24) to

$$\langle\Psi'_L|\left(|\psi_{L_1}\rangle \otimes \bigotimes_{j=2}^{m}(|{\downarrow}\rangle \otimes |\psi_{L_j}\rangle)\right) = \prod_{j=1}^{m}\left(\langle{\uparrow}| \otimes \langle\chi\cdots\chi|\right)|\psi_{L_j}\rangle. \quad (7.25)$$

Dividing by the norms of the states, and using (6.67) and (6.97), we find (7.22).

The proof of (7.23) is similar, using in addition the parity invariance of the zero-energy states $|\Psi'_L\rangle$ and $|\Psi_L\rangle$. In all other cases, the scalar products vanish because of the definite magnetisation of the zero-energy states. □

It is possible to extend the definition of $Z'_d$ in order to account for the case where $L_k$ is 0 for some $1 \leqslant k \leqslant m$ ($k \in I'$), and the $L_1, \dots, L_{k-1}$, $L_{k+1}, \dots, L_m$ are odd. (This can be done by formally defining $|\Psi'_0\rangle = 1$.) In this case, we have

$$\begin{aligned}
&Z'_d(I, I'; L_1, \dots, L_k = 0, \dots, L_m) \\
&\qquad = \frac{\langle\Psi'_L|\left(\bigotimes_{j=1}^{k-1}(|\psi_{L_j}\rangle \otimes |{\downarrow}\rangle) \otimes \bigotimes_{j=k+1}^{m}(|{\downarrow}\rangle \otimes |\psi_{L_j}\rangle)\right)}{\|\Psi'_L\| \prod_{j=1, j\neq k}^{m} \|\psi_{L_j}\|} \\
&\qquad = \frac{(\Psi'_L)_{\uparrow\downarrow\cdots\uparrow\downarrow}}{\|\Psi'_L\|} \prod_{i\in I} g(L_i) \frac{\|\Psi_{L_i}\|}{\langle\omega_{L_i}|\Psi_{L_i}\rangle} \prod_{\substack{i\in I' \\ i\neq k}} \frac{\|\Psi'_{L_i}\|}{(\Psi'_{L_i})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}}.
\end{aligned}$$

**Dressed multipartite fidelity and periodic spin chains.** One could ask if we can use the supersymmetry to account for multipartite fidelities that involve periodic spin-chain ground states on both sides of the scalar product. This is indeed the case.

Let $L_1, \dots, L_m$ be odd. As previously, we divide the set $\{1, \dots, m\}$ into the disjoint sets $I$ and $I'$, such that $I \cup I' = \{1, \dots, m\}$. The subsets $I, I'$ allow us to label the ground states of the periodic and open spin chains as in (7.20). We define the quantity

$$Z_d(I,I';L_1,\dots,L_m)=\frac{\langle\bar{\Psi}_L|\left(|{\downarrow}\rangle\otimes\bigotimes_{j=1}^m\left(|\psi_{L_j}\rangle\otimes|{\downarrow}\rangle\right)\right)}{\|\bar{\Psi}_L\|\prod_{j=1}^m\|\psi_{L_j}\|},\tag{7.26}$$

where $L=L_1+\cdots+L_m+m+1$.

**Theorem 7.2.2.** *If $L_1,\dots,L_m$ are odd, then*

$$\begin{aligned}&Z_d(I,I';L_1,\dots,L_m)\\&\quad=\frac{(\bar{\Psi}_L)_{\downarrow\uparrow\cdots\downarrow\uparrow\downarrow}}{\|\bar{\Psi}_L\|}\prod_{i\in I}g(L_i)\frac{\|\Psi_{L_i}\|}{\langle\omega_{L_i}|\Psi_{L_i}\rangle}\prod_{i\in I'}\frac{\|\Psi'_{L_i}\|}{(\Psi'_{L_i})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}},\end{aligned}\tag{7.27}$$

*with g(L)=(3L-1)/(2L). In all other cases, the scalar product vanishes.*

Although the result of Theorem 7.2.2 is similar to the preceding theorems, its proof is more complex and requires the computation of the scalar product between the involved representatives. To this end, we define for $I',I$, the homology representative of $|\psi_{L_i}\rangle$:

$$|\varpi_{L_i}\rangle=\begin{cases}|{\uparrow\downarrow}\cdots{\uparrow\downarrow\uparrow}\rangle & \text{for } i\in I',\\ \mathcal{N}|\omega_{L_i}\rangle & \text{for } i\in I.\end{cases}\tag{7.28}$$

We further introduce the notation

$$g_i(L_i)=\langle\chi\cdots\chi|\big(|\varpi_{L_i}\rangle\otimes|{\downarrow}\rangle\big)=\begin{cases}1 & \text{if } i\in I',\\ \frac{3L_i-1}{2L_i} & \text{if } i\in I.\end{cases}\tag{7.29}$$

We now prove the following (technical) lemma:

**Lemma 7.2.3.** *Let $L_1,\dots,L_m$ be odd and $I,I'$ as in Theorem 7.2.2. Then we have*

$$\langle\bar{\omega}_L|\mathcal{N}\left(|{\downarrow}\rangle\otimes\bigotimes_{j=1}^m\left(|\varpi_{L_j}\rangle\otimes|{\downarrow}\rangle\right)\right)=\frac{3L+1}{2L}\prod_{j=1}^m g_j(L_j).\tag{7.30}$$

We recall that the state $|\bar{\omega}_L\rangle$ is defined for odd $L=2n+1$ and is given by $|\bar{\omega}_L\rangle=|\bar{\omega}_L^{(1)}\rangle+|\bar{\omega}_L^{(2)}\rangle$, where

$$|\bar{\omega}_L^{(1)}\rangle=2|{\downarrow}\rangle\otimes|\chi\cdots\chi\rangle,\tag{7.31}$$

$$|\bar{\omega}_L^{(2)}\rangle=\sum_{j=1}^n|{\uparrow}\rangle\otimes|\underbrace{\chi\cdots\chi}_{j-1\,\text{times}}\rangle\otimes|{\downarrow\downarrow}\rangle\otimes|\underbrace{\chi\cdots\chi}_{n-j\,\text{times}}\rangle.\tag{7.32}$$

For the sake of conciseness, we omit to write the tensor products in the proof of this lemma. Furthermore, we write $\hat{\mathcal{N}}$ for $L\mathcal{N}$ acting on $V^L$.

*Proof of Lemma 7.2.3.* The proof is based on recurrence. To prove the basis case, we set $m=1$, $L=2n+1$, $L_1=2n_1+1$ and $n=n_1+1$. We use the shorthand notation $|\Omega_1\rangle = |{\downarrow}\rangle|\varpi_{L_1}\rangle|{\downarrow}\rangle$. First, we consider the projection of $|\Omega_1\rangle$ onto $\hat{\mathcal{N}}|\bar{\omega}_L^{(1)}\rangle$. It reads

$$\langle\bar{\omega}_L^{(1)}|\hat{\mathcal{N}}|\Omega_1\rangle = 2\sum_{k=0}^{2n}\left(\langle{\downarrow}|\langle\chi\cdots\chi|\right)\mathcal{S}^{-k}|\Omega_1\rangle. \tag{7.33}$$

The terms in the sum that correspond to an odd value of $k$ give a zero contribution, which is clear from the inspection of the spins on the first and last sites of $|\Omega_1\rangle = |{\downarrow}\rangle|\varpi_{L_1}\rangle|{\downarrow}\rangle$. The values $k=0$ and $k=n$ both yield $2g_1(L_1)$, as follows from (7.29). Furthermore, for each $k=2\kappa$, $\kappa=1,\ldots,n-1$, we use the definition of $\varpi_{L_1}$ to compute

$$\Big(\langle\underbrace{\chi\cdots\chi}_{\kappa}|\langle{\downarrow}|\langle\underbrace{\chi\cdots\chi}_{n-\kappa}|\Big)|\Omega_1\rangle = \begin{cases}1 & \text{if } 1\in I',\\ \frac{n_1+1}{L_1} & \text{if } 1\in I.\end{cases} \tag{7.34}$$

The different contributions add up to

$$\langle\bar{\omega}_L^{(1)}|\hat{\mathcal{N}}|\Omega_1\rangle = \begin{cases}2(n+1) & \text{if } 1\in I',\\ 4g_1(L_1)+\frac{2nn_1}{L_1} & \text{if } 1\in I.\end{cases} \tag{7.35}$$

Second, we focus on the following scalar product that involves $|\bar{\omega}_L^{(2)}\rangle$:

$$\langle\bar{\omega}_L^{(2)}|\hat{\mathcal{N}}|\Omega_1\rangle = \sum_{j=1}^{n}\sum_{k=0}^{2n}\Big(\langle{\uparrow}|\langle\underbrace{\chi\cdots\chi}_{j-1}|\langle{\downarrow\downarrow}|\langle\underbrace{\chi\cdots\chi}_{n-j}|\Big)\mathcal{S}^{-k}|\Omega_1\rangle. \tag{7.36}$$

We analyse the terms of the sums depending on the value of $j$ and the parity of $k$. On the one hand, for each $j=1,\ldots,n$, the summand vanishes if $k$ is odd, apart from the case where $k=2(n-j)+1$, which gives a contribution $g_1(L_1)$. On the other hand, we consider $k=2\kappa$ with $\kappa=0,\ldots,n$. The sum over $j$ of the terms corresponding to $\kappa=n-j$ and $\kappa=n-j+1$ is

$$\begin{aligned}\sum_{j=1}^{n}\Big(\langle{\downarrow\downarrow}|\langle\chi\cdots\chi|\langle{\uparrow}|\langle\underbrace{\chi\cdots\chi}_{j-1}| + \langle\chi\cdots\chi|\langle{\uparrow}|\langle\underbrace{\chi\cdots\chi}_{j-1}|\langle{\downarrow\downarrow}|\Big)|\Omega_1\rangle&\\ = \begin{cases}0 & \text{if } 1\in I',\\ (n-1)g_1(L_1) & \text{if } 1\in I.\end{cases}&\end{aligned} \tag{7.37}$$

Moreover, for a given $j = 1, \ldots, n$, and $\kappa \neq n-j, n-j+1$, we evaluate the summand:

$$\Big(\langle\uparrow|\langle\chi\ldots\chi|\langle\downarrow\downarrow|\langle\chi\ldots\chi|\langle\uparrow|\langle\chi\ldots\chi|\langle\uparrow|\Big)|\varpi_{L_1}\rangle = \begin{cases} 0 & \text{if } 1 \in I', \\ \frac{1}{L_1} & \text{if } 1 \in I. \end{cases} \tag{7.38}$$

There are $(n_1-1)n_1$ such contributions. By summing the different terms resulting from (7.36), we have

$$\langle\bar{\omega}_L^{(2)}|\hat{\mathcal{N}}|\Omega_1\rangle = \begin{cases} n & \text{if } 1 \in I', \\ (2n-1)g_1(L_1) + \frac{(n_1-1)n_1}{L_1} & \text{if } 1 \in I. \end{cases} \tag{7.39}$$

Finally, we gather the contributions from $|\bar{\omega}_L^{(1)}\rangle$ and $|\bar{\omega}_L^{(2)}\rangle$. For $1 \in I'$, we find

$$\langle\bar{\omega}_L|\hat{\mathcal{N}}|\Omega_1\rangle = (3n+2)g_1(L_1) = \frac{3L+1}{2}g_1(L_1). \tag{7.40}$$

Similarly, for $1 \in I$, we have

$$\langle\bar{\omega}_L|\hat{\mathcal{N}}|\Omega_1\rangle = (2n+3)g_1(L_1) + n_1\frac{2n+n_1-1}{L_1} = \frac{3L+1}{2}g_1(L_1),$$

which concludes the proof of recurrence's basis case.

Let us suppose that (7.30) is satisfied for each $L_1, \ldots, L_{m-1}$. This means that we have

$$\langle\bar{\omega}_{L_1+\cdots+L_{m-1}+m}|\hat{\mathcal{N}}|\Omega\rangle = \frac{3(L_1+\cdots+L_{m-1}+m)+1}{2}\prod_{j=1}^{m-1} g_j(L_j), \tag{7.41}$$

where we introduced the shorthand notation

$$|\Omega\rangle = |\downarrow\rangle|\varpi_{L_1}\rangle|\downarrow\rangle|\varpi_{L_2}\rangle\cdots|\downarrow\rangle|\varpi_{L_{m-1}}\rangle|\downarrow\rangle. \tag{7.42}$$

Here, $|\Omega\rangle$ depends on the lengths $L_1, \ldots, L_{m-1}$ and the sets $I, I'$. We now show that (7.30) holds for $n = m$ and odd $L_1, \ldots, L_m$. As previously, we separately treat the projection of $|\bar{\omega}_L^{(1)}\rangle$ and $|\bar{\omega}_L^{(2)}\rangle$ onto $|\Omega\rangle|\varpi_{L_m}\rangle|\downarrow\rangle$.

First, we focus on the following scalar product:

$$\langle\bar{\omega}_L^{(1)}|\hat{\mathcal{N}}\big(|\Omega\rangle|\varpi_{L_m}\rangle|\downarrow\rangle\big) = \sum_{k=0}^{2n}\langle\bar{\omega}_L^{(i)}|\mathcal{S}^{-k}\Big(|\Omega\rangle|\varpi_{L_m}\rangle|\downarrow\rangle\Big). \tag{7.43}$$

The odd values of $k$ contribute with zero to the sum. We have

$$\begin{aligned}\langle\bar{\omega}_L^{(1)}|\hat{\mathcal{N}}\big(|\Omega\rangle|\varpi_{L_m}\rangle|\downarrow\rangle\big) &= 2\sum_{k\text{ even}}\big(\langle\downarrow|\langle\chi\cdots\chi|\big)\mathcal{S}^{-k}|\Omega\rangle\langle\chi\cdots\chi|\big(|\varpi_{L_m}\rangle|\downarrow\rangle\big)\\ &+2\big(\langle\chi\cdots\chi|\langle\downarrow|\big)|\Omega\rangle\times\sum_{k=0}^{n_m}\Big(\langle\uparrow|\langle\underbrace{\chi\cdots\chi}_{k}|\langle\downarrow|\langle\underbrace{\chi\cdots\chi}_{n_m-k}|\Big)\big(|\varpi_{L_m}\rangle|\downarrow\rangle\big).\end{aligned}$$

We can simplify this expression by completing the first sum in the right-hand side to create a projector, $\mathcal{N}$. Furthermore, we express the second sum as follows:

$$\begin{aligned}&2\sum_{k=0}^{n_m}\Big(\langle\uparrow|\langle\underbrace{\chi\cdots\chi}_{k}|\langle\downarrow|\langle\underbrace{\chi\cdots\chi}_{n_m-k}|\Big)\big(|\varpi_{L_m}\rangle|\downarrow\rangle\big)\\ &\qquad=\langle\bar{\omega}_{L_m+2}^{(1)}|\hat{\mathcal{N}}\big(|\downarrow\rangle|\varpi_{L_m}\rangle|\downarrow\rangle\big)-2\langle\chi\cdots\chi|\big(|\varpi_{L_m}\rangle|\downarrow\rangle\big).\end{aligned}\tag{7.44}$$

Hence we have

$$\begin{aligned}\langle\bar{\omega}_L^{(1)}|\hat{\mathcal{N}}\big(|\Omega\rangle|\varpi_{L_m}\rangle|\downarrow\rangle\big) &= \langle\bar{\omega}_{L_1+\cdots+L_{m-1}+m}^{(1)}|\hat{\mathcal{N}}|\Omega\rangle\times\langle\chi\cdots\chi|\big(|\varpi_{L_m}\rangle|\downarrow\rangle\big)\\ &+\big(\langle\chi\cdots\chi|\langle\downarrow|\big)|\Omega\rangle\times\langle\bar{\omega}_{L_m+2}^{(1)}|\hat{\mathcal{N}}\big(|\downarrow\rangle|\varpi_{L_m}\rangle|\downarrow\rangle\big)\\ &-2\big(\langle\chi\cdots\chi|\langle\downarrow|\big)|\Omega\rangle\times\langle\chi\cdots\chi|\big(|\varpi_{L_m}\rangle|\downarrow\rangle\big).\end{aligned}$$

Second, we consider the scalar product $\langle\bar{\omega}_L^{(2)}|\hat{\mathcal{N}}\big(|\Omega\rangle|\varpi_{L_m}\rangle|\downarrow\rangle\big)$. We use similar arguments as before to write

$$\begin{aligned}\langle\bar{\omega}_L^{(2)}|\hat{\mathcal{N}}\big(|\Omega\rangle|\varpi_{L_m}\rangle|\downarrow\rangle\big) &= \langle\bar{\omega}_{L_1+\cdots+L_{m-1}+m}^{(2)}|\hat{\mathcal{N}}|\Omega\rangle\times\langle\chi\cdots\chi|\big(|\varpi_{L_m}\rangle|\downarrow\rangle\big)\\ &+\big(\langle\chi\cdots\chi|\langle\downarrow|\big)|\Omega\rangle\times\langle\bar{\omega}_{L_m+2}^{(2)}|\hat{\mathcal{N}}\big(|\downarrow\rangle|\varpi_{L_m}\rangle|\downarrow\rangle\big).\end{aligned}$$

Eventually, we take the sum of both contributions and the recurrence to obtain

$$\langle\bar{\omega}_L|\hat{\mathcal{N}}\big(|\Omega\rangle|\varpi_{L_m}\rangle|\downarrow\rangle\big)=\frac{3L+1}{2}\prod_{j=1}^{m}g_j(L_j),\tag{7.45}$$

which is the desired result. □

*Proof of Theorem 7.2.2.* We use the cohomology decomposition of $|\bar{\Psi}_L\rangle$ to write

$$\langle\bar{\Psi}_L|\left(|{\downarrow}\rangle\otimes\bigotimes_{j=1}^{m}\big(|\psi_{L_j}\rangle\otimes|{\downarrow}\rangle\big)\right) = \Big(\langle\bar{\omega}_L|\mathcal{N}+\langle\bar{\phi}_L|\mathfrak{Q}^\dagger\Big)\left(|{\downarrow}\rangle\otimes\bigotimes_{j=1}^{m}\big(|\psi_{L_j}\rangle\otimes|{\downarrow}\rangle\big)\right).$$

Similarly to the proof of Theorem 7.2.1, the term that involves $\mathfrak{Q}^\dagger$ vanishes. Furthermore, we express the tensor product as

$$|{\downarrow}\rangle\otimes\bigotimes_{j=1}^{m}\big(|\psi_{L_j}\rangle\otimes|{\downarrow}\rangle\big) = \prod_{i\in I'}\mu_{L_i}\prod_{i\in I}\lambda_{L_i}|{\downarrow}\rangle\otimes\bigotimes_{j=1}^{m}\big(|\varpi_{L_j}\rangle\otimes|{\downarrow}\rangle\big)+\mathfrak{Q}^\dagger|\Phi\rangle$$

for a certain $|\Phi\rangle$. (We could, in principle, compute the state $|\Phi\rangle$ but we do not need its explicit expression.) We recall that $\mu_L$ and $\lambda_L$ are defined in Proposition 6.4.5 and Proposition 6.6.4, respectively. The action of $\mathfrak{Q}^\dagger$ vanishes and the multipartite fidelity reduces to

$$Z_d(I,I';L_1,\dots,L_m) = \prod_{i\in I'}\mu_{L_i}\prod_{i\in I}\lambda_{L_i}\frac{\langle\bar{\omega}_L|\mathcal{N}\big(|{\downarrow}\rangle\otimes\bigotimes_{j=1}^{m}\big(|\varpi_{L_j}\rangle\otimes|{\downarrow}\rangle\big)\big)}{\|\bar{\Psi}_L\|\prod_{j=1}^{m}\|\psi_{L_j}\|}.$$

We finally use Lemma 7.2.3, the definition of $\mu_L, \lambda_L$, and the factorisation of the norms (6.67), (6.97) to conclude the proof. □

We can further generalise the quantity $Z_d$ by allowing for a single $L_k, k = 1,\dots,m$ to be even. In this case, we have

$$\begin{aligned}&Z_d(I,I';L_1,\dots,L_m)\\&\quad=\frac{(\bar{\Psi}_L)_{\downarrow\uparrow\cdots\downarrow\uparrow\downarrow}}{\|\bar{\Psi}_L\|}\prod_{i\in I}g(L_i)\frac{\|\Psi_{L_i}\|}{\langle\omega_{L_i}|\Psi_{L_i}\rangle}\prod_{\substack{i\in I'\\ i\neq k}}\frac{\|\Psi'_{L_i}\|}{(\Psi'_{L_i})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}}\frac{\|\Psi'_{L_k}\|}{(\Psi'_{L_k})_{\uparrow\downarrow\cdots\uparrow\downarrow}}.\end{aligned}$$

The proof of this statement is similar to the one of Theorem 7.2.2, which can formally be seen as the case $k = 0$.

## 7.3 Scaling limits

In this section, we discuss the multipartite fidelities found in Theorems 7.1.1, 7.2.1 and 7.2.2. We provide exact finite-size expressions for special (sum of) components of the normalised zero-energy state. We use them to exactly evaluate the fidelities $Z, Z', Z_d$ and $Z'_d$, as well as their scaling limits.

**Special components of $|\Psi'_L\rangle$.** The ratio of the two components $(\Psi'_{2n})_{\uparrow\downarrow\cdots\uparrow\downarrow}$ and $(\Psi'_{2n+1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}$ by the norm of the corresponding states can be expressed in terms of two integer sequences $A_{\rm V}(2n+1)$ and $N_8(2n)$. These two sequences enumerate $(2n+1)\times(2n+1)$ vertically-symmetric alternating sign matrices and cyclically-symmetric transpose complement plane partitions in a $2n\times 2n\times 2n$ cube, respectively [53, 123]. (We give the precise definitions of these objects in Chapter 9.) Explicitly, they are given by

$$A_{\rm V}(2n+1)=\frac{1}{2^n}\prod_{k=1}^{n}\frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!},\quad N_8(2n)=\prod_{k=0}^{n-1}\frac{(3k+1)(6k)!(2k)!}{(4k)!(4k+1)!}.$$

**Theorem 7.3.1.** *For each $n$, we have*

$$\frac{(\Psi'_{2n+1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}}{\|\Psi'_{2n+1}\|}=\sqrt{\frac{N_8(2n+2)}{A_{\rm V}(2n+3)}},\quad \frac{(\Psi'_{2n})_{\uparrow\downarrow\cdots\uparrow\downarrow}}{\|\Psi'_{2n}\|}=\sqrt{\frac{A_{\rm V}(2n+1)}{N_8(2n+2)}}. \tag{7.46}$$

We provide a proof of Theorem 7.3.1 along with many other combinatorial properties of the ground state $|\Psi'_L\rangle$ in subsequent chapters. This proof does not use the supersymmetry of the model. It exploits the quantum integrability of the XXZ spin chain.

The sequences $A_{\rm V}(2n+1)$ and $N_8(2n)$ are given by ratios of products of factorials. We can evaluate the components (7.46) for large system sizes:

$$\frac{(\Psi'_{2n+1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}}{\|\Psi'_{2n+1}\|}=C\ (2n)^{1/12}\ \alpha^{-2n-2}\left(1+O(n^{-1})\right), \tag{7.47a}$$

$$\frac{(\Psi'_{2n})_{\uparrow\downarrow\cdots\uparrow\downarrow}}{\|\Psi'_{2n}\|}=\frac{\sqrt{2}}{C}(2n)^{-1/12}\alpha^{-2n-1}\left(1+O(n^{-1})\right). \tag{7.47b}$$

Here $C$, $\alpha$ are the constants

$$C = \frac{\sqrt{\Gamma(1/3)}}{\pi^{1/4}}, \quad \alpha = \frac{3^{3/4}}{2}. \tag{7.47c}$$

**Special (sum of) components of $|\Psi_L\rangle$ and $|\bar{\Psi}_L\rangle$.** The normalised components and sums of components of $|\Psi_L\rangle$ and $|\bar{\Psi}_L\rangle$ appearing in the fidelities are given in terms of the sequences $A(n)$ and $A_{\mathrm{HT}}(2n+1)$. These sequences enumerate $n\times n$ alternating sign matrices and $(2n+1)\times(2n+1)$ half-turn symmetric alternating sign matrices, respectively, and are given by [53, 123, 124]

$$A(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}, \quad A_{\mathrm{HT}}(2n+1) = \prod_{k=1}^{n} \frac{4}{3}\left(\frac{(3k)!k!}{(2k)!^2}\right)^2. \tag{7.48}$$

The components of the normalised vector $|\Psi_L\rangle$ have been exactly computed, and the following combinatorial results hold [60, 61]:

$$\frac{(\Psi_{2n+1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}}{\|\Psi_{2n+1}\|} = \frac{A(n)}{\sqrt{A_{\mathrm{HT}}(2n+1)}}, \tag{7.49}$$

$$\frac{(\langle\uparrow| \otimes \langle\chi\cdots\chi|)\,|\Psi_{2n+1}\rangle}{\|\Psi_{2n+1}\|} = \frac{A(n+1)}{\sqrt{A_{\mathrm{HT}}(2n+1)}}. \tag{7.50}$$

Similar relations hold for $|\bar{\Psi}_L\rangle$ too, as a consequence of the spin reversal symmetry. Furthermore, we can use (6.97) to find exact expressions for $\langle\omega_L|\Psi_L\rangle$ and $\langle\bar{\omega}_L|\bar{\Psi}_L\rangle$.

Similarly to the case of the open spin chains, we evaluate the (sum of) components (7.49), (7.50) in the limit of large system size:

$$\frac{(\Psi_{2n+1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow}}{\|\Psi_{2n+1}\|} = \frac{\sqrt{2}}{C^2}(2n)^{-1/6}\alpha^{-2n-2/3}\left(1+O(n^{-1})\right), \tag{7.51a}$$

$$\frac{(\langle\uparrow| \otimes \langle\chi\cdots\chi|)\,|\Psi_{2n+1}\rangle}{\|\Psi_{2n+1}\|} = \frac{\sqrt{2}}{C^2}(2n)^{-1/6}\alpha^{2n+4/3}\left(1+O(n^{-1})\right). \tag{7.51b}$$

**Scaling behaviour of multipartite fidelities.** We use the asymptotic expansions (7.47), (7.51) to extract the scaling behaviour of the multipartite fidelities. It is obtained when the lengths of the subintervals $L_1, \dots, L_m$ become large in such a way that the ratio $L_i/L$ approaches the scaling variable $0 < x_i < 1$, for each $i = 1, \dots, m$. The fidelity is then given by an asymptotic series with respect to the system size $L$. The series coefficients are functions of $x_1, \dots, x_m$. Notice that the multipartite fidelities (and thus the corresponding asymptotic series) are only well-defined if the parity of the integers $L_1, \dots, L_m$ is fixed.

First, we compute the scaling behaviour of $Z$ and $Z'$. There are two interesting cases treated in Theorem 7.1.1:

If $L_i$ is even for each $i = 1, \dots, m$, then $L$ is even and

$$Z'(L_1, \dots, L_m) = \left(\frac{\sqrt{2}}{\alpha C}\right)^{m-1} L^{-(m-1)/12} \prod_{i=1}^{m} x_i^{-1/12} \left(1 + O(L^{-1})\right).$$

If $L_k$ is odd for a certain $k$ and $L_i$ is even for each $i = 1, \dots, k-1, k+1, \dots, m$. In this case, $L$ is odd, and

$$Z'(L_1, \dots, L_m) = \left(\frac{\sqrt{2}}{\alpha C}\right)^{m-1} L^{-(m-1)/12} x_k^{1/12} \prod_{\substack{i=1\\ i\neq k}}^{m} x_i^{-1/12} \left(1 + O(L^{-1})\right),$$

$$Z(L_1, \dots, L_m) = \left(\frac{\sqrt{2}}{\alpha C}\right)^{m} \alpha^{1/3} L^{-m/12} \quad x_k^{1/12} \prod_{\substack{i=1\\ i\neq k}}^{m} x_i^{-1/12} \left(1 + O(L^{-1})\right).$$

Second, we focus on the dressed multipartite fidelities. We compute the scaling limits of the quantities $Z'_d$ and $Z_d$ using the results of Theorems 7.2.1 and 7.2.2. If $L_i$ is odd for each $i = 1, \dots, m$, then $L$ is odd and

$$Z'_d(L_1, \dots, L_m) = \left(\frac{\sqrt{2}}{\alpha^{2/3}}\right)^{|I|} C^{1-2|I|-|I'|} \times L^{(1-2|I|-|I'|)/12} \prod_{i\in|I|} x_i^{-1/6} \prod_{i\in|I'|} x_i^{-1/12} \left(1 + O(L^{-1})\right), \quad (7.52)$$

and

$$Z_d(L_1, \dots, L_m) = \left(\frac{\sqrt{2}}{\alpha^{2/3}}\right)^{|I|+1} C^{-2-2|I|-|I'|} \times L^{(-2-2|I|-|I'|)/12} \prod_{i\in|I|} x_i^{-1/6} \prod_{i\in|I'|} x_i^{-1/12} \left(1 + O(L^{-1})\right). \quad (7.53)$$

In the case where $L_k$ is even for a single $k$, we obtain similar results.

**Scaling behaviour and conformal field theory.** The power-law decay of the fidelities' scaling limits as well as their algebraic dependence on the coordinates $x_1, \ldots, x_m$ suggest that they are related to correlation functions of conformal field theory (CFT).

The case $m = 2$ is related to the the *logarithmic bipartite fidelity* mentioned above and introduced by Dubail and Stéphan [118, 119]. In particular, they predicted the leading-order terms of the asymptotic expansion of the logarithmic bipartite fidelity with respect to the system size $L$ for one-dimensional quantum critical systems from CFT arguments. In [1], we compared our exact results for $Z'(L_1, L_2)$ with the CFT predictions of Dubail and Stéphan and observed a perfect match.

Hence, a natural question to ask is whether one can reproduce such a matching for the other types of multipartite fidelities that we introduced. The first calculations that we made confirm that this is the case [5]. The CFT calculations and the comparison with our exact results are, however, beyond the scope of this dissertation. We discuss this open problem and some outlook in the conclusion chapter.

# Chapter 8

# Boundary quantum Knizhnik-Zamolodchikov equations

In this and the following chapter, we study a generalisation of the XXZ spin chain considered in Chapter 6, with diagonal boundary magnetic fields that differ on both extremities of the chain. Its Hamiltonian is given by

$$H = -\frac{1}{2}\sum_{i=1}^{L-1}\left(\sigma_i^1\sigma_{i+1}^1 + \sigma_i^2\sigma_{i+1}^2 - \frac{1}{2}\sigma_i^3\sigma_{i+1}^3\right) + p\sigma_1^3 + \bar{p}\sigma_L^3, \tag{8.1}$$

where the boundary magnetic fields are given by

$$p = \frac{1}{2}\left(\frac{1}{2} - x\right), \quad \bar{p} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{x}\right). \tag{8.2}$$

Here, $x$ is an arbitrary complex parameter. A connection of the Hamiltonian with these parameters to the $O(1)$ model on a strip and the one-boundary Temperley-Lieb algebra has been discussed in [125, 126].

This Hamiltonian generalises the supersymmetric Hamiltonian considered in Chapters 6 and 7. In that case, we had $x = 1$, and we observed some combinatorial properties of the ground state related to the enumeration of alternating sign matrices and plane partitions. We stated these properties in Theorem 7.3.1.

The goal of this and the following chapter is to construct the ground-state vector explicitly and explore its properties, for generic $x$. To this end, we adapt a strategy of Razumov, Stroganov and Zinn-Justin [61] to the case of open boundary conditions: we find a solution to the so-called *boundary quantum Knizhnik-Zamolodchikov* equations. This construction allows us to prove Theorem 7.3.1.

**Boundary quantum Knizhnik-Zamolodchikov equations.** The Knizhnik-Zamolodchikov equations appeared in the context of conformal field theories that possess a Lie algebra symmetry [127]. They are a system of partial differential equations. They define correlation functions of operators that intertwine representations of the algebra [128]. The analogue of these equations for quantum algebras is a system of difference equations, known as *quantum Knizhnik-Zamolodchikov* (qKZ) equations. They are given in terms of the corresponding $R$-matrix and depend on the deformation parameter $q$ [129]. We refer to [130] for an overview of classical and quantum Knizhnik-Zamolodchikov equation and their relation with representation theory.

Di Francesco and Zinn-Justin established a connection between finite spin chains and qKZ equations. In order to prove some conjectures of Razumov and Stroganov for the related $O(1)$ model, they constructed a polynomial vector depending on inhomogeneity parameters [131]. Their vector satisfied relations akin to the qKZ equations [59]. This connection was used to find an explicit expression for the ground state components of the periodic XXZ chain, for $\Delta = -1/2$, and allowed for the proof of some of its combinatorial properties [61]

For systems with open boundary conditions, the corresponding equations are the *boundary quantum Knizhnik-Zamolodchikov* (bqKZ) equations [132]. Solutions to the bqKZ equations have been constructed to investigate specific models: semi-infinite XXZ and XYZ spin chains (with a single boundary) [133, 134, 120] or loop model and their relation with enumerative combinatorics [135, 62]. Furthermore, solutions to the bqKZ equation have been found in terms of (sums of off-shell) Bethe vectors [136], orthogonal polynomials [137], or multiple contour integrals [138].

The purpose of this chapter is to construct and analyse a (Laurent polynomial) solution to the bqKZ equations for the $R$-matrix of the six-vertex model and a diagonal $K$-matrix. We give this solution in terms of multiple contour integrals. We show that it is an eigenvector of the transfer matrix of the inhomogeneous six-vertex model on a strip if $q = \mathrm{e}^{2\pi \mathrm{i}/3}$. We study the homogeneous limit of this vector, its relation with the Hamiltonian (8.1), and its combinatorial properties in the next chapter.

The layout of this chapter is as follows. In Section 8.1, we define through multiple contour integrals the components of a vector $|\Psi_L\rangle \in V^L$ that depends on $L$ complex variables $z_1, \dots, z_L$ and a complex parameter $q$. We use the contour integrals in Section 8.2 and Section 8.3 to show that the vector obeys the *exchange* and *reflection relations*. These relations imply that the vector is a solution to the bqKZ equations. We discuss the polynomiality of the solution in Section 8.4 and obtain the vector's behaviour under a parity transformation in Section 8.5. In Section 8.6, we consider the transfer matrix of the inhomogeneous six-vertex model. Finally, we establish a relation between this transfer matrix and the bqKZ equations in Section 8.7. In particular, we show that if $q = \mathrm{e}^{2\pi \mathrm{i}/3}$, $|\Psi_L\rangle$ is an eigenvector of the transfer matrix and we compute the corresponding eigenvalue.

## 8.1 Integral solution to the boundary quantum Knizhnik-Zamolodchikov equations

Throughout this and the following chapter, we systematically use the notation $[z] = z - z^{-1}$. Furthermore, we denote by $n$ and $\bar{n}$ the integers

$$n = \lfloor L/2 \rfloor, \quad \bar{n} = \lceil L/2 \rceil. \tag{8.3}$$

Moreover, $a_1, \dots, a_n$ are integers that satisfy $1 \leqslant a_1 < \cdots < a_n \leqslant L$.

Inspired by [62, 139, 140], we define the multiple contour integral

$$\begin{aligned}(\Psi_L)_{a_1,\dots,a_n}(z_1, \dots, z_L) = (-[q])^n \prod_{1 \leqslant i < j \leqslant L} [q z_j / z_i]\, [q^2 z_i z_j] \\ \times \oint \cdots \oint \prod_{\ell=1}^{n} \frac{\mathrm{d}w_\ell}{\pi \mathrm{i} w_\ell}\, \Xi_{a_1,\dots,a_n}(w_1, \dots, w_n | z_1, \dots, z_L),\end{aligned} \tag{8.4}$$

where $z_1, \ldots, z_L$ and $w_1, \ldots, w_n$ are complex numbers. Moreover, $q \in \mathbb{C}\backslash\{0, 1, -1\}$ is a complex parameter. The integrand contains the meromorphic function

$$\Xi_{a_1,\ldots,a_n}(w_1, \ldots, w_n|z_1, \ldots, z_L) \\ = \frac{\prod_{1\leqslant i<j\leqslant n}[qw_j/w_i][w_i/w_j][qw_iw_j]\prod_{1\leqslant i\leqslant j\leqslant n}[q^2w_iw_j]\prod_{i=1}^{n}[\beta w_i]}{\prod_{i=1}^{n}\left(\prod_{j=1}^{a_i}[z_j/w_i]\prod_{j=a_i}^{L}[qz_j/w_i]\prod_{j=1}^{L}[q^2w_iz_j]\right)}, \tag{8.5}$$

where $\beta \in \mathbb{C}\backslash\{0\}$. The integration contour of $w_i$ in (8.4) is a collection of positively-oriented curves surrounding the poles $w_i = z_j$. This contour does not surround any other singularities situated at $w_i = 0, -z_j, \pm qz_j, \pm q^{-2}z_j^{-1}$.

Similarly, let $b_1, \ldots, b_{\bar{n}}$ be integers with $1 \leqslant b_1 < \cdots < b_{\bar{n}} \leqslant L$. We define the multiple contour integral

$$(\bar{\Psi}_L)_{b_1,\ldots,b_{\bar{n}}}(z_1, \ldots, z_L) = [q]^{\bar{n}}\prod_{1\leqslant i<j\leqslant L}[qz_j/z_i]\,[qz_iz_j]\prod_{i=1}^{L}[\beta z_i] \\ \times\oint\cdots\oint\prod_{\ell=1}^{\bar{n}}\frac{\mathrm{d}w_\ell}{\pi\mathrm{i}w_\ell}\,\bar{\Xi}_{b_1,\ldots,b_{\bar{n}}}(w_1, \ldots, w_{\bar{n}}|z_1, \ldots, z_L), \tag{8.6}$$

whose integrand contains the meromorphic function

$$\bar{\Xi}_{b_1,\ldots,b_{\bar{n}}}(w_1, \ldots, w_{\bar{n}}|z_1, \ldots, z_L) \\ = \frac{\prod_{1\leqslant i<j\leqslant \bar{n}}[qw_j/w_i][w_i/w_j][q^2w_iw_j]\prod_{1\leqslant i\leqslant j\leqslant \bar{n}}[qw_iw_j]}{\prod_{i=1}^{\bar{n}}\left(\prod_{j=1}^{b_i}[qw_i/z_j]\prod_{j=b_i}^{L}[w_i/z_j]\prod_{j=1}^{L}[qw_iz_j]\right)\prod_{i=1}^{\bar{n}}[\beta w_i]}. \tag{8.7}$$

The integration contour of $w_i$ in (8.6) is a collection of positively-oriented curves surrounding the poles $w_i = z_j$, but not the singularities located at $w_i = 0, -z_j, \pm q^{-1}z_j, \pm q^{-1}z_j^{-1}, \pm\beta^{-1}$.

It is possible to apply the residue theorem and compute an explicit (combinatorial) formula for the multiple contour integrals defined in (8.4) and (8.6). This combinatorial formula is, however, not quite useful for explicit computations. It only simplifies in the two cases $a_i = i$ and $b_i = \bar{n} + i$. In these cases, we obtain:

**Proposition 8.1.1.** *For each $L \geqslant 2$, we have*

$$(\Psi_L)_{1,\dots,n}(z_1,\dots,z_L) = (\bar{\Psi}_L)_{\bar{n}+1,\dots,L}(z_1,\dots,z_L) \\ = \prod_{i=1}^{n}[\beta z_i] \prod_{1\leqslant i<j\leqslant n}[qz_iz_j][qz_j/z_i] \prod_{n+1\leqslant i<j\leqslant L}[qz_j/z_i][q^2z_iz_j]. \quad (8.8)$$

*Proof.* We only sketch the evaluation of $(\Psi_L)_{1,\dots,n}(z_1,\dots,z_L)$. To this end, we iteratively compute the contour integrals (8.4) with respect to $w_1,\dots,w_n$ for $a_i = i$. We observe that the only pole that contributes to the contour integral with respect to $w_\ell$ is $z_\ell$. The evaluation of its residue leads to (8.8). The computation of $(\bar{\Psi}_L)_{\bar{n}+1,\dots,L}(z_1,\dots,z_L)$ is similar. □

We now introduce[1] two vectors $|\Psi_L\rangle = |\Psi_L(z_1,\dots,z_L)\rangle$ and $|\bar{\Psi}_L\rangle = |\bar{\Psi}_L(z_1,\dots,z_L)\rangle$. For $L=1$, they are given by $|\Psi_1\rangle = |\bar{\Psi}_1\rangle = |{\uparrow}\rangle$. For $L \geqslant 2$, we use (8.4) and (8.6) to define them as

$$|\Psi_L(z_1,\dots,z_L)\rangle = \\ \sum_{1\leqslant a_1<\cdots<a_n\leqslant L} (\Psi_L)_{a_1,\dots,a_n}(z_1,\dots,z_L)\,|{\uparrow}\cdots{\uparrow}\underset{a_1}{\downarrow}{\uparrow} \quad \cdots \quad {\uparrow}\underset{a_n}{\downarrow}{\uparrow}\cdots{\uparrow}\rangle, \quad (8.9)$$

$$|\bar{\Psi}_L(z_1,\dots,z_L)\rangle = \\ \sum_{1\leqslant b_1<\cdots<b_{\bar{n}}\leqslant L} (\bar{\Psi}_L)_{b_1,\dots,b_{\bar{n}}}(z_1,\dots,z_L)\,|{\downarrow}\cdots{\downarrow}\underset{b_1}{\uparrow}{\downarrow} \quad \cdots \quad {\downarrow}\underset{b_{\bar{n}}}{\uparrow}{\downarrow}\cdots{\downarrow}\rangle. \quad (8.10)$$

Here and in the following, we label the components of a vector in terms of the positions of the up or down spins of the associated spin configuration. We only write out the dependence on $z_1,\dots,z_L$ if necessary.

It follows from Proposition 8.1.1 that these vectors do not identically vanish. The purpose of the following section is to investigate their properties.

[1] The vectors $|\Psi_L\rangle$ and $|\bar{\Psi}_L\rangle$ depend on the variables $z_1,\dots,z_L$ and must not be confused with the ground states of the periodic XXZ spin chain previously introduced.

## 8.2 The exchange relations

To formulate the exchange relations, we recall from Chapter 3 the definition of the $R$-matrix of the six-vertex model. It is an operator on $V^2$ that acts on the canonical basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ as the matrix (3.4) where the weight $d$ is absent:

$$R(z) = \begin{pmatrix} a(z) & 0 & 0 & 0 \\ 0 & b(z) & c(z) & 0 \\ 0 & c(z) & b(z) & 0 \\ 0 & 0 & 0 & a(z) \end{pmatrix}. \tag{8.11}$$

We choose the following parameterisation for the entries of $R(z)$:

$$a(z) = [qz]/[q/z], \quad b(z) = [z]/[q/z], \quad c(z) = [q]/[q/z]. \tag{8.12}$$

The $R$-matrix obeys the Yang-Baxter equation, similar to (3.19). On $V^L$ and with the parameterisation (8.12), it is given by

$$R_{i\,i+1}(z/w)R_{i\,i+2}(z)R_{i+1\,i+2}(w) = R_{i+1\,i+2}(w)R_{i\,i+2}(z)R_{i\,i+1}(z/w). \tag{8.13}$$

Moreover, we have $R(1) = P$, where $P$ is the permutation operator acting according to $P(|v\rangle \otimes |w\rangle) = |w\rangle \otimes |v\rangle$ for any $|v\rangle, |w\rangle \in V$. Using this operator, we define the $\check{R}$-matrix by

$$\check{R}(z) = PR(z). \tag{8.14}$$

It follows from (8.13) that it obeys the braid version of the Yang-Baxter equation

$$\check{R}_{i\,i+1}(z/w)\check{R}_{i+1\,i+2}(z)\check{R}_{i\,i+1}(w) = \check{R}_{i+1\,i+2}(w)\check{R}_{i\,i+1}(z)\check{R}_{i+1\,i+2}(z/w). \tag{8.15}$$

**Exchange relations.** We say that a vector $|\Phi\rangle = |\Phi(z_1, \dots, z_L)\rangle \in V^L$, $L \geqslant 2$, that depends on the complex numbers $z_1, \dots, z_L$, obeys the *exchange relations* if

$$\check{R}_{i\,i+1}(z_i/z_{i+1})|\Phi(\dots, z_i, z_{i+1}, \dots)\rangle = |\Phi(\dots, z_{i+1}, z_i, \dots)\rangle, \tag{8.16}$$

for each $i = 1, \dots, L-1$. The compatibility of these equations follows from (8.15).

**Proposition 8.2.1.** *For each $L \geqslant 2$, the vectors $|\Psi_L\rangle$ and $|\bar{\Psi}_L\rangle$ obey the exchange relations* (8.16).

*Proof.* The proofs of the exchange relations for $|\Psi_L\rangle$ and $|\bar{\Psi}_L\rangle$ are similar and follow the lines of [61]. Hence, we focus on $|\Psi_L\rangle$. To prove that it obeys the exchange relations, we consider integers $a_1, \dots, a_n$ with $1 \leqslant a_1 < a_2 < \cdots < a_n \leqslant L$ and an integer $1 \leqslant i \leqslant L-1$. We examine four cases, depending on whether $i$ and $i+1$ belong to $\{a_1, \dots, a_n\}$. In this proof, we use $\Xi_{a_1,\dots,a_n}(z_1, \dots, z_L)$ as a shorthand notation for $\Xi_{a_1,\dots,a_n}(w_1, \dots, w_n|z_1, \dots, z_L)$

*Case 1: $i, i+1 \notin \{a_1, \dots, a_n\}$.* In this case, we note that the polynomial $\Xi_{a_1,\dots,a_n}(\dots, z_i, z_{i+1}, \dots)$ is symmetric under the exchange of $z_i$ and $z_{i+1}$. We combine this observation with (8.4) and conclude that the quotient of $(\Psi_L)_{a_1,\dots,a_n}(\dots, z_i, z_{i+1}, \dots)$ and $[qz_{i+1}/z_i]$ is symmetric under the exchange of $z_i$ and $z_{i+1}$. Hence, we find the relation

$$\frac{[qz_i/z_{i+1}]}{[qz_{i+1}/z_i]}(\Psi_L)_{a_1,\dots,a_n}(\dots, z_i, z_{i+1}, \dots) = (\Psi_L)_{a_1,\dots,a_n}(\dots, z_{i+1}, z_i, \dots). \tag{8.17}$$

*Case 2: $i, i+1 \in \{a_1, \dots, a_n\}$.* Let $1 \leqslant \ell \leqslant n-1$ be the integer such that $a_\ell = i$. We have

$$\begin{aligned} &\Xi_{a_1,\dots,a_n}(\dots, z_{i+1}, z_i, \dots) - \Xi_{a_1,\dots,a_n}(\dots, z_i, z_{i+1}, \dots) \\ &\qquad = \frac{[qw_\ell/w_{\ell+1}][z_i/z_{i+1}]}{[qz_i/w_{\ell+1}][z_{i+1}/w_\ell]}\Xi_{a_1,\dots,a_n}(\dots, z_i, z_{i+1}, \dots) \end{aligned} \tag{8.18}$$

The inspection of (8.5) allows us to conclude that this difference is antisymmetric under the exchange of $w_\ell$ and $w_{\ell+1}$. Hence, the multiple contour integral over the difference vanishes. It follows that the quotient of $(\Psi_L)_{a_1,\dots,a_n}(\dots, z_i, z_{i+1}, \dots)$ and $[qz_{i+1}/z_i]$ is symmetric under the exchange of $z_i$ and $z_{i+1}$. Therefore, the relation (8.17) holds in this case, too.

*Case 3: $i \in \{a_1,\dots,a_n\}$ and $i+1 \notin \{a_1,\dots,a_n\}$.* Let $1 \leqslant \ell \leqslant n-1$ be the integer such that $a_\ell = i$. We find

$$\Xi_{a_1,\dots,i+1,\dots,a_n}(\dots,z_i,z_{i+1},\dots) = \frac{[qz_i/w_\ell]}{[z_{i+1}/w_\ell]}\Xi_{a_1,\dots,i,\dots,a_n}(\dots,z_i,z_{i+1},\dots), \tag{8.19}$$

$$\Xi_{a_1,\dots,i,\dots,a_n}(\dots,z_{i+1},z_i,\dots) = \frac{[z_i/w_\ell]}{[z_{i+1}/w_\ell]}\Xi_{a_1,\dots,i,\dots,a_n}(\dots,z_i,z_{i+1},\dots). \tag{8.20}$$

We combine these relations into the equality

$$\begin{aligned}[q]\Xi_{a_1,\dots,i,\dots,a_n}(\dots,z_i,z_{i+1},\dots) \\ + [z_i/z_{i+1}]\Xi_{a_1,\dots,i+1,\dots,a_n}(\dots,z_i,z_{i+1},\dots) \\ = [qz_i/z_{i+1}]\Xi_{a_1,\dots,i,\dots,a_n}(\dots,z_{i+1},z_i,\dots).\end{aligned} \tag{8.21}$$

Using this equality, it is straightforward to show that

$$\begin{aligned}\frac{[q]}{[qz_{i+1}/z_i]}(\Psi_L)_{a_1,\dots,i,\dots,a_n}(\dots,z_i,z_{i+1},\dots) \\ + \frac{[z_i/z_{i+1}]}{[qz_{i+1}/z_i]}(\Psi_L)_{a_1,\dots,i+1,\dots,a_n}(\dots,z_i,z_{i+1},\dots) \\ = (\Psi_L)_{a_1,\dots,i,\dots,a_n}(\dots,z_{i+1},z_i,\dots).\end{aligned} \tag{8.22}$$

*Case 4: $i \notin \{a_1,\dots,a_n\}$ and $i+1 \in \{a_1,\dots,a_n\}$* The analysis of this case is very similar to the previous one. One obtains the relation

$$\begin{aligned}\frac{[q]}{[qz_{i+1}/z_i]}(\Psi_L)_{a_1,\dots,i+1,\dots,a_n}(\dots,z_i,z_{i+1},\dots) \\ + \frac{[z_i/z_{i+1}]}{[qz_{i+1}/z_i]}(\Psi_L)_{a_1,\dots,i,\dots,a_n}(\dots,z_i,z_{i+1},\dots) \\ = (\Psi_L)_{a_1,\dots,i+1,\dots,a_n}(\dots,z_{i+1},z_i,\dots).\end{aligned} \tag{8.23}$$

To conclude, we note that (8.17), (8.22) and (8.23) are equal to the exchange relations, written for the components of $|\Psi_L\rangle$, which ends the proof. □

**Properties of solutions to the exchange relations.** We now investigate a few simple properties of $|\Phi\rangle = |\Phi(z_1,\dots,z_L)\rangle \in V^L$, $L \geqslant 2$, a vector with magnetisation $(\bar{n}-n)/2$ that obeys the exchange relations (8.16). We define its components through the expansion

$$|\Phi\rangle = \sum_{1\leqslant a_1<\dots<a_n\leqslant L} \Phi_{a_1,\dots,a_n}(z_1,\dots,z_L)|\uparrow\cdots\uparrow\underset{a_1}{\downarrow}\uparrow \quad \cdots \quad \uparrow\underset{a_n}{\downarrow}\uparrow\cdots\uparrow\rangle. \tag{8.24}$$

Following the proof of Proposition 8.2.1, we rewrite the exchange relations for the components of the vector. There are four different cases. The next lemma addresses two of them.

**Lemma 8.2.2.** *Let $i$ be an integer with $1 \leqslant i \leqslant L-1$ such that either $i, i+1 \notin \{a_1,\dots,a_n\}$ or $i, i+1 \in \{a_1,\dots,a_n\}$ then*

$$\Phi_{a_1,\dots,a_n}(\dots,z_i,z_{i+1},\dots) = [qz_{i+1}/z_i]\bar{\Phi}_{a_1,\dots,a_n}(\dots,z_i,z_{i+1},\dots), \tag{8.25}$$

*where $\bar{\Phi}_{a_1,\dots,a_n}(\dots,z_i,z_{i+1},\dots)$ is symmetric under the exchange of $z_i$ and $z_{i+1}$.*

For the two other cases, we introduce the divided difference operator $\delta$. It acts on a function $f$ of two complex variables $z, w$ according to

$$\delta f(z,w) = \frac{[qw/z]f(w,z) - [q]f(z,w)}{[z/w]}. \tag{8.26}$$

More generally, for a function $f$ depending on $z_1,\dots,z_L$, we write $\delta_i f$ for the action of the divided difference operator $\delta$ on $f$ with $z = z_i$ and $w = z_{i+1}$.

**Lemma 8.2.3.** *Let $i$ be an integer with $1 \leqslant i \leqslant L-1$ such that $i \in \{a_1,\dots,a_n\}$ and $i+1 \notin \{a_1,\dots,a_n\}$, then*

$$\Phi_{a_1,\dots,i+1,\dots,a_n}(\dots,z_i,z_{i+1},\dots) = \delta_i\Phi_{a_1,\dots,i,\dots,a_n}(\dots,z_i,z_{i+1},\dots), \tag{8.27}$$

$$\Phi_{a_1,\dots,i,\dots,a_n}(\dots,z_i,z_{i+1},\dots) = \delta_i\Phi_{a_1,\dots,i+1,\dots,a_n}(\dots,z_i,z_{i+1},\dots). \tag{8.28}$$

This lemma allows us to prove the following useful property of the solutions to the exchange relations:

**Proposition 8.2.4.** *Suppose that there are integers $\bar{a}_1, \ldots, \bar{a}_n$ with $1 \leqslant \bar{a}_1 < \cdots < \bar{a}_n \leqslant L$ such that $\Phi_{\bar{a}_1,\ldots,\bar{a}_n}(z_1, \ldots, z_L)$ vanishes identically then the vector $|\Phi\rangle$ vanishes identically.*

*Proof.* Lemma 8.2.3 allows us to write for all integers $a_1, \ldots, a_n$ with $1 \leqslant a_1 < \cdots < a_n \leqslant L$ the relation

$$\Phi_{a_1,\ldots,a_n}(z_1, \ldots, z_L) = \left( \prod_{i=1,\ldots,n} \overset{\curvearrowleft}{\prod_{j=i,\ldots,a_i-1}} \delta_j \right) \Phi_{1,\ldots,n}(z_1, \ldots, z_L). \quad (8.29)$$

Here, $\curvearrowleft$ indicates that we take the products of operators in reverse order. Since the component $\Phi_{\bar{a}_1,\ldots,\bar{a}_n}(z_1, \ldots, z_L)$ vanishes identically, we find

$$0 = \left( \overset{\curvearrowleft}{\prod_{i=1,\ldots,n}} \prod_{j=i,\ldots,\bar{a}_i-1} \delta_j \right) \Phi_{\bar{a}_1,\ldots,\bar{a}_n}(z_1, \ldots, z_L) \quad (8.30)$$

$$= \left( \overset{\curvearrowleft}{\prod_{i=1,\ldots,n}} \prod_{j=i,\ldots,\bar{a}_i-1} \delta_j \right) \left( \prod_{i=1,\ldots,n} \overset{\curvearrowleft}{\prod_{j=i,\ldots,\bar{a}_i-1}} \delta_j \right) \Phi_{1,\ldots,n}(z_1, \ldots, z_L). \quad (8.31)$$

One checks that for each $j = 1, \ldots, L-1$, the divided difference operator $\delta_j$ has the property $\delta_j^2 = \mathrm{id}$. Hence, we conclude from (8.31) that $\Phi_{1,\ldots,n}(z_1, \ldots, z_L)$ vanishes identically, and from (8.29) that all components vanish identically. □

We now show that the two vectors defined in Section 8.1 are equal. This equality allows us to limit our investigation to $|\Psi_L\rangle$. For each of its (non-trivial) components, we have two different multiple contour integral formulas.

**Proposition 8.2.5.** *We have $|\bar{\Psi}_L\rangle = |\Psi_L\rangle$.*

*Proof.* For $L = 1$, the proposition holds by the definition of the vectors. Hence, we consider the difference $|\Phi\rangle = |\bar{\Psi}_L\rangle - |\Psi_L\rangle$ for $L \geqslant 2$. It follows from (8.9) and (8.10) that this vector is of the form (8.24). Moreover, Proposition 8.2.1 implies that it obeys the exchange relations. It has the component

$$\Phi_{1,\ldots,n}(z_1, \ldots, z_L)$$
$$= (\bar{\Psi}_L)_{\bar{n}+1,\ldots,L}(z_1, \ldots, z_L) - (\Psi_L)_{1,\ldots,n}(z_1, \ldots, z_L) = 0, \quad (8.32)$$

as follows from Proposition 8.1.1. By virtue of Proposition 8.2.4, we conclude that $|\Phi\rangle$ vanishes identically. □

## 8.3 The reflection relations

The reflection relations for the vector $|\Psi_L\rangle$ are written in terms of a $K$-matrix. It is an operator $K(z)$ on $V$ that solves the boundary Yang-Baxter equation for the six-vertex model [35],

$$R_{12}(z/w)K_1(z)R_{12}(zw)K_2(w) = K_2(w)R_{12}(zw)K_1(z)R_{12}(z/w). \tag{8.33}$$

The most general solution of this equation can be found in [104]. Here, we consider Cherednik's diagonal solution $K(z) = K(z;\beta)$ [132]. It acts on the canonical basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ as the matrix

$$K(z;\beta) = \begin{pmatrix} 1 & 0 \\ 0 & [\beta z]/[\beta/z] \end{pmatrix}, \tag{8.34}$$

where $\beta$ is a non-zero complex number.

**Reflection relations.** A vector $|\Phi\rangle = |\Phi(z_1,\dots,z_L)\rangle \in V^L$ that depends on the complex numbers $z_1,\dots,z_L$ obeys the reflection relations if

$$K_1(z_1^{-1};\beta)|\Phi(z_1,\dots,z_L)\rangle = |\Phi(z_1^{-1},\dots,z_L)\rangle, \tag{8.35}$$
$$K_L(sz_L;s\bar\beta)|\Phi(z_1,\dots,z_L)\rangle = |\Phi(z_1,\dots,s^{-2}z_L^{-1})\rangle. \tag{8.36}$$

Here, $s$ and $\bar\beta$ are two complex parameters. Throughout this section, we assume that they obey the relations

$$s^4 = q^6, \tag{8.37}$$

and

$$\bar\beta^2\beta^2q^2 = 1. \tag{8.38}$$

**Proposition 8.3.1.** *For each $L \geqslant 1$, the vector $|\Psi_L\rangle$ obeys the reflection relations.*

*Proof.* The case $L = 1$ is trivial. Hence, we consider $L \geqslant 2$. We present the proof of the second reflection relation (8.36) for $|\Phi\rangle = |\Psi_L\rangle$ in detail. To this end, we establish the two equations

$$(\Psi_L)_{a_1,\dots,a_n}(z_1,\dots,z_L) = (\Psi_L)_{a_1,\dots,a_n}(z_1,\dots,s^{-2}z_L^{-1}), \quad (8.39)$$

if $a_n < L$, and

$$\frac{[s^2\bar{\beta}z_L]}{[\bar{\beta}z_L^{-1}]}(\Psi_L)_{a_1,\dots,a_n}(z_1,\dots,z_L) = (\Psi_L)_{a_1,\dots,a_n}(z_1,\dots,s^{-2}z_L^{-1}), \quad (8.40)$$

if $a_n = L$. These two equations are equivalent to the reflection relations.

*Case 1: $a_n < L$.* Using (8.8) and (8.37), it is straightforward to show that

$$(\Psi_L)_{1,\dots,n}(z_1,\dots,z_L) = (\Psi_L)_{1,\dots,n}(z_1,\dots,s^{-2}z_L^{-1}). \quad (8.41)$$

For $n = 1$, there is nothing left to prove. For $n \geqslant 2$, we apply Lemma 8.2.3 to obtain (8.39).

*Case 2: $a_L = L$.* We consider the difference

$$\begin{aligned}\Delta(z_1,\dots,z_L) = [s^2\bar{\beta}z_L](\Psi_L)_{1,\dots,n-1,L}(z_1,\dots,z_L)\\ - [\bar{\beta}z_L^{-1}](\Psi_L)_{1,\dots,n-1,L}(z_1,\dots,s^{-2}z_L^{-1}).\end{aligned} \quad (8.42)$$

We compute it using the contour integral formula (8.4). The integrations with respect to $w_1,\dots,w_{n-1}$ are straightforward. Using (8.37), we find

$$\Delta(z_1,\dots,z_L) = p(z_1,\dots,z_L)\int_C \mathrm{d}w_n\, f(w_n), \quad (8.43)$$

where

$$\begin{aligned}p(z_1,\dots,z_L) = -[q][s^2z_L^2]\prod_{i=1}^{n-1}[\beta z_i]\\ \times \prod_{1\leqslant i<j\leqslant n-1}[qz_j/z_i][qz_iz_j]\prod_{n\leqslant i<j\leqslant L}[qz_j/z_i][q^2z_iz_j],\end{aligned} \quad (8.44)$$

and

$$f(w) = \frac{[\bar{\beta}q^3w][\beta w][q^2w^2]\prod_{i=1}^{n-1}[qw/z_i][qwz_i]}{\mathrm{i}\pi w\prod_{i=n}^{L+1}[z_i/w][q^2wz_i]}. \quad (8.45)$$

Here, we abbreviated $z_{L+1} = s^{-2}z_L^{-1}$. The integration contour $C$ is a collection of positively-oriented curves around the simple poles $z_n, \ldots, z_{L+1}$, but not around any other pole of $f$.

We now analyse the contour integral in (8.43). To this end, we make three simple observations. First, $f$ has a removable singularity at $w = 0$. All other singularities of $f$ are simple poles, located at $w = z_n, \ldots, z_{L+1}$ and $w = \varphi_i(z_n), \ldots, \varphi_i(z_{L+1})$, $i = 1, 2, 3$, where

$$\varphi_1(w) = -w, \quad \varphi_2(w) = q^{-2}w^{-1}, \quad \varphi_3(w) = -q^{-2}w^{-1}. \tag{8.46}$$

Second, we note that $f$ obeys $\varphi_i'(w)f(\varphi_i(w)) = f(w)$. This is trivial for $i = 1$. For $i = 2, 3$, it follows from (8.38). These properties of $f$ allow us to write

$$\int_{\varphi_i(C)} \mathrm{d}w_n\, f(w_n) = \int_C \mathrm{d}w_n\, f(w_n), \quad i = 1, 2, 3. \tag{8.47}$$

Third, we note that $f$ tends to zero at infinity and has no residue at infinity. This observation allows us to push the integration contour $C$ in (8.43) to infinity, which results in

$$\int_C \mathrm{d}w_n\, f(w_n) = -\sum_{i=1}^{3} \int_{\varphi_i(C)} \mathrm{d}w_n\, f(w_n) = -3\int_C \mathrm{d}w_n\, f(w_n). \tag{8.48}$$

Here, we used (8.47). We conclude from this equation that $\Delta(z_1, \ldots, z_L)$ vanishes, and therefore

$$\frac{[s^2\bar{\beta}z_L]}{[\bar{\beta}z_L^{-1}]}(\Psi_L)_{1,\ldots,n-1,L}(z_1, \ldots, z_L) = (\Psi_L)_{1,\ldots,n-1,L}(z_1, \ldots, s^{-2}z_L^{-1}) \tag{8.49}$$

For $n = 1$, there is nothing left to prove. For $n \geqslant 2$, we use Lemma 8.2.3 to obtain (8.40). This ends the proof of (8.36).

Finally, we comment on the proof of (8.35). It amounts to establishing the relations

$$\frac{[\beta/z_1]}{[\beta z_1]}(\Psi_L)_{a_1,\ldots,a_n}(z_1, \ldots, z_L) = (\Psi_L)_{a_1,\ldots,a_n}(z_1^{-1}, \ldots, z_L), \tag{8.50}$$

if $a_1 = 1$, and

$$(\Psi_L)_{a_1,\ldots,a_L}(z_1, \ldots, z_L) = (\Psi_L)_{a_1,\ldots,a_n}(z_1^{-1}, \ldots, z_L), \tag{8.51}$$

if $a_1 > 1$. The first relation is easily proven for the specific choice $a_i = i$, using the component (8.8). The general relation (8.50) then follows from Lemma 8.2.3. The proof of the second relation is based on showing that the difference

$$\bar{\Delta}(z_1,\ldots,z_L) = (\Psi_L)_{2,\ldots,n+1}(z_1,\ldots,z_L) - (\Psi_L)_{2,\ldots,n+1}(z_1^{-1},\ldots,z_L). \tag{8.52}$$

vanishes. Proposition 8.2.5 allows us to rewrite this difference as

$$\bar{\Delta}(z_1,\ldots,z_L) \\ = (\bar{\Psi}_L)_{1,n+2,\ldots,L}(z_1,\ldots,z_L) - (\bar{\Psi}_L)_{1,n+2\ldots,L}(z_1^{-1},\ldots,z_L). \tag{8.53}$$

Using the alternative integral formula (8.6), we write this difference in terms of a single contour integral similar to (8.43). Following the same lines as above, we show that this contour integral vanishes. This proves (8.51) for the choice $a_i = i+1$. The general relation (8.51) follows from Lemma 8.2.3. □

**The bqKZ equations.** Let us introduce for each $i = 1,\ldots,L$ an operator

$$S^{(i)}(z_1,\ldots,z_L) = \\ \overset{\curvearrowleft}{\prod_{j=1,\ldots,i-1}} \check{R}_{jj+1}(s^2 z_i/z_j) K_1(s^2 z_i;\beta) \overset{\curvearrowright}{\prod_{j=1,\ldots,i-1}} \check{R}_{jj+1}(s^2 z_i z_j) \\ \times \overset{\curvearrowright}{\prod_{j=i,\ldots,L-1}} \check{R}_{jj+1}(s^2 z_i z_{j+1}) K_L(s z_i; s\bar{\beta}) \overset{\curvearrowleft}{\prod_{j=i,\ldots,L-1}} \check{R}_{jj+1}(z_i/z_{j+1}). \tag{8.54}$$

Proposition 8.2.1 and Proposition 8.3.1 imply that the vector $|\Psi_L\rangle$ obeys the bqKZ equations [132, 137, 136, 138]

$$S^{(i)}(z_1,\ldots,z_L)|\Psi_L(\ldots,z_i,\ldots)\rangle = |\Psi_L(\ldots,s^2 z_i,\ldots)\rangle, \quad i = 1,\ldots,L. \tag{8.55}$$

This system of difference equations is compatible thanks to the commutation relations

$$S^{(i)}(z_1,\ldots,s^2 z_j,\ldots,z_L) S^{(j)}(z_1,\ldots,z_L) \\ = S^{(j)}(z_1,\ldots,s^2 z_i,\ldots,z_L) S^{(i)}(z_1,\ldots,z_L), \tag{8.56}$$

for each $1 \leqslant i, j \leqslant L$. These commutation relations follow from the Yang-Baxter equation (8.13) and the boundary Yang-Baxter equation (8.33).

## 8.4 Polynomiality

In this section, we show that the components of $|\Psi_L\rangle$ are Laurent polynomials in the variables $z_1, \ldots, z_L$, and determine their degrees. To this end, we examine the action of the divided difference operator on Laurent polynomials. We then apply the results of this investigation to the components.

**The divided difference operator.** We consider the divided difference operator $\delta$ defined in (8.26) acting on a Laurent polynomial $f$ in $z, w$. For a special class of Laurent polynomials, the action again yields a Laurent polynomial.

**Lemma 8.4.1.** *Let $f$ be a Laurent polynomial with*

$$f(-z,w) = \epsilon f(z,w), \qquad f(z,-w) = -\epsilon f(z,w), \tag{8.57}$$

*where $\epsilon^2 = 1$, then $\delta f$ is a Laurent polynomial with the property*

$$\delta f(-z,w) = -\epsilon f(z,w), \quad \delta f(z,-w) = \epsilon f(z,w). \tag{8.58}$$

*Proof.* The definition of the divided difference operator implies that $\delta f$ is a rational function that obeys (8.58). We compute the limits

$$\lim_{z\to w} \delta f(z,w) = \frac{w}{2}\left(-\frac{q^2+1}{qw} f(w,w) + [q]\left(\frac{\partial f(w,w)}{\partial w} - \frac{\partial f(w,w)}{\partial z}\right)\right), \tag{8.59}$$

and, using (8.58),

$$\lim_{z\to -w} \delta f(z,w) = -\epsilon \lim_{z\to w} \delta f(z,w). \tag{8.60}$$

They imply that $\delta f(z,w)$ has no poles at $z = \pm w$. Hence, it is a Laurent polynomial with respect to $z$. A similar calculation shows that it is a Laurent polynomial with respect to $w$, too. $\square$

For each Laurent polynomial $f$ in $z, w$, there are integers $d^\pm$ and $\bar{d}^\pm$, with $d^- \leqslant d^+$ and $\bar{d}^- \leqslant \bar{d}^+$, such that

$$f(z,w) = \sum_{k=d^-}^{d^+} c_k(w) z^k = \sum_{k=\bar{d}^-}^{\bar{d}^+} \bar{c}_k(z) w^k, \tag{8.61}$$

with non-zero $c_{d^\pm}(w)$, $\bar{c}_{\bar{d}^\pm}(z)$. We refer to $d^-$ as the lower degree and $d^+$ as the upper degree of $f$ with respect to $z$, and use the same terminology for $\bar{d}^-$ and $\bar{d}^+$ for the degrees of $f$ with respect to $w$. We also use the following notation:

$$\deg_z^\pm f = d^\pm, \quad \deg_w^\pm f = \bar{d}^\pm. \tag{8.62}$$

The action of the divided difference operator does not preserve the degrees. They can, however, not change arbitrarily as show the next two lemmas.

**Lemma 8.4.2.** *Let $f$ be as in Lemma 8.4.1.*

*(i) Let $m = \deg_z^+ f - 1$ and suppose that $\deg_w^+ f \leqslant m$, then*

$$\deg_z^+ \delta f \leqslant m \quad \textit{and} \quad \deg_w^+ \delta f = m+1. \tag{8.63}$$

*(ii) Let $m = \deg_w^+ f - 1$ and suppose that $\deg_z^+ f \leqslant m$, then*

$$\deg_z^+ \delta f = m+1 \quad \textit{and} \quad \deg_w^+ \delta f \leqslant m. \tag{8.64}$$

*Proof.* The proofs of *(i)* and *(ii)* are very similar. We only present the proof of *(i)*.

Let $m' = \deg_w^+ f$. First, we analyse $\delta f(z,w)$ for $z \to \infty$. We find

$$\delta f(z,w) = \big( -q^{-1}\bar{c}_{m'}(w) z^{m'} + O(z^{m'-1})\big) \\ + \big([q] w c_{m+1}(w) z^m + O(z^{m-1})\big). \tag{8.65}$$

This expression allows us to conclude that $\deg_z^+ \delta f \leqslant \max(m, m') \leqslant m$. Second, we consider $w \to \infty$ and obtain

$$\delta f(z,w) = \big( -q c_{m+1}(z) w^{m+1} + O(w^m)\big) \\ + \big( -[q] z \bar{c}_{m'}(z) w^{m'-1} + O(w^{m'-2})\big). \tag{8.66}$$

This expression implies that $\deg_w^+ \delta f = m+1$, since $q c_{m+1}(z)$ does not vanish identically and $m+1 > m'-1$. $\square$

**Lemma 8.4.3.** *Let $f$ be as in Lemma 8.4.1.*

*(i) Let $m = \deg_z^- f$ and suppose that $\deg_w^- f \geqslant m+1$, then*

$$\deg_z^- \delta f \geqslant m+1 \quad \text{and} \quad \deg_w^- \delta f = m. \tag{8.67}$$

*(ii) Let $m = \deg_w^- f$ and suppose that $\deg_z^- f \geqslant m+1$, then*

$$\deg_z^- \delta f = m \quad \text{and} \quad \deg_w^- \delta f \geqslant m+1. \tag{8.68}$$

*Proof.* The proof follows from the analysis of $\delta f(z,w)$ as $z \to 0$ and $w \to 0$. It is similar to the proof of Lemma 8.4.2. □

**Polynomiality of the vector.** We now show that the components of the vector $|\Psi_L\rangle$ are Laurent polynomials in the variables $z_1, \dots, z_L$ and find (bounds for) their degrees with respect to each $z_i$. We treat the cases of even and odd $L$ separately in the two following propositions.

**Proposition 8.4.4.** *Let $L = 2n$, $n \geqslant 1$ and $i$ be integers with $1 \leqslant i \leqslant L$, then we have:*

*(i) The component $(\Psi_{2n})_{a_1,\dots,a_n}$ is a Laurent polynomial with respect to $z_i$.*

*(ii) If $i \in \{a_1, \dots, a_n\}$ then $(\Psi_{2n})_{a_1,\dots,a_n}$ is an odd function of $z_i$ with*

$$\deg_{z_i}^{\pm} (\Psi_{2n})_{a_1,\dots,a_n} = \pm(2n-1). \tag{8.69}$$

*(iii) If $i \notin \{a_1, \dots, a_n\}$ then $(\Psi_{2n})_{a_1,\dots,a_n}$ is an even function of $z_i$ with*

$$\deg_{z_i}^- (\Psi_{2n})_{a_1,\dots,a_n} \geqslant -2(n-1), \quad \deg_{z_i}^+ (\Psi_{2n})_{a_1,\dots,a_n} \leqslant 2(n-1). \tag{8.70}$$

*Proof.* The proof is based on recurrence. First, we note that *(i)*, *(ii)* and *(iii)* hold for the special component $(\Psi_{2n})_{1,\dots,n}$, as readily follows from (8.8).

Second, we show that *(i)*, *(ii)* and *(iii)* are preserved under the action of a divided difference operator on any component. To this end, let us consider integers $\bar{a}_1, \dots, \bar{a}_n$ with $1 \leqslant \bar{a}_1 < \dots < \bar{a}_n \leqslant L$ such that there is $j = 1, \dots, 2n-1$ with $j \in \{\bar{a}_1, \dots, \bar{a}_n\}$ but $j+1 \notin \{\bar{a}_1, \dots, \bar{a}_n\}$. According to Lemma 8.2.3, we have

$$\begin{aligned}(\Psi_{2n})_{\bar{a}_1,\dots,j+1,\dots,\bar{a}_n}(\dots, z_j, z_{j+1}, \dots)\\ = \delta_j(\Psi_{2n})_{\bar{a}_1,\dots,j,\dots,\bar{a}_n}(\dots, z_j, z_{j+1}, \dots).\end{aligned} \tag{8.71}$$

Let us now suppose that *(i)*, *(ii)* and *(iii)* hold for the component $\Psi_{\bar{a}_1,\dots,j,\dots,\bar{a}_n}(\dots, z_j, z_{j+1}, \dots)$. We apply Lemma 8.4.1 to (8.71). It implies that $\Psi_{\bar{a}_1,\dots,j+1,\dots,\bar{a}_n}(\dots, z_j, z_{j+1}, \dots)$ is an even Laurent polynomial in $z_j$, and an odd Laurent polynomial in $z_{j+1}$. Moreover, it follows from Lemma 8.4.2*(i)* that

$$\deg^+_{z_j}(\Psi_{2n})_{\bar{a}_1,\dots,j+1,\dots,\bar{a}_n} \leqslant 2(n-1), \quad \deg^+_{z_{j+1}}(\Psi_{2n})_{\bar{a}_1,\dots,j+1,\dots,\bar{a}_n} = 2n-1. \tag{8.72}$$

Likewise, we may apply Lemma 8.4.3*(ii)* to conclude that

$$\deg^-_{z_j}(\Psi_{2n})_{\bar{a}_1,\dots,j+1,\dots,\bar{a}_n} \geqslant 2(1-n), \quad \deg^-_{z_{j+1}}(\Psi_{2n})_{\bar{a}_1,\dots,j+1,\dots,\bar{a}_n} = 1-2n. \tag{8.73}$$

Since all other variables remain unaffected, we conclude that *(i)*, *(ii)* and *(iii)* hold for the component $(\Psi_{2n})_{\bar{a}_1,\dots,j+1,\dots,\bar{a}_n}(\dots, z_j, z_{j+1}, \dots)$.

Third, the statements *(i)*, *(ii)* and *(iii)* follow for each component $(\Psi_{2n})_{a_1,\dots,a_n}$ as we can obtain it through the action of a (finite) product of divided difference operators on the component $(\Psi_{2n})_{1,\dots,n}$. □

**Proposition 8.4.5.** *Let $L = 2n+1$, $n \geqslant 1$ and $i$ be integers with $1 \leqslant i \leqslant L$, then we have:*

*(i) The component $(\Psi_{2n+1})_{a_1,\dots,a_n}$ is a Laurent polynomial with respect to $z_i$.*

*(ii) If $i \in \{a_1, \dots, a_n\}$ then $(\Psi_{2n+1})_{a_1,\dots,a_n}$ is an odd function of $z_i$ with*

$$\deg^-_{z_i}(\Psi_{2n+1})_{a_1,\dots,a_n} \geqslant -(2n-1), \quad \deg^+_{z_i}(\Psi_{2n+1})_{a_1,\dots,a_n} \leqslant 2n-1. \tag{8.74}$$

*(iii) If $i \notin \{a_1, \dots, a_n\}$ then $(\Psi_{2n+1})_{a_1,\dots,a_n}$ is an even function of $z_i$ with*

$$\deg^{\pm}_{z_i}(\Psi_{2n+1})_{a_1,\dots,a_n} = \pm 2n. \tag{8.75}$$

*Proof.* The proof is very similar to the proof of Proposition 8.4.4. We only mention a few minor differences. First, we note that *(i)*, *(ii)* and *(iii)* hold for the component $(\Psi_{2n+1})_{1,\ldots,n}$. Second, the argument that the action of the divided difference operators preserves *(i)*, *(ii)* and *(iii)* follows through but uses Lemma 8.4.2*(ii)* and Lemma 8.4.3*(i)*. Third, the statements hold for each component $(\Psi_{2n+1})_{a_1,\ldots,a_n}$ as we can obtain it through the action of a (finite) product of divided difference operators on $(\Psi_{2n+1})_{1,\ldots,n}$. $\square$

**Relations between even and odd size.** For each $s \in \{\uparrow, \downarrow\}$ and $i = 1, \ldots, L+1$, let $\Theta_i^s : V^L \to V^{L+1}$ be the linear operator whose action on the canonical basis vectors of $V^L$ is given by

$$\Theta_i^s |s_1 \cdots s_{i-1} s_i \cdots s_L\rangle = |s_1 \cdots s_{i-1}\rangle \otimes |s\rangle \otimes |s_i \cdots s_L\rangle. \tag{8.76}$$

In the next proposition, we use this operator to establish a relation between the vectors $|\Psi_L\rangle$ and $|\Psi_{L-1}\rangle$.

**Proposition 8.4.6.** *Let $n \geqslant 1$. For each $i = 1, \ldots, 2n$, we have*

$$\lim_{z_i \to 0} z_i^{2n-1} |\Psi_{2n}\rangle = (-1)^{n+i+1} \beta^{-1} q^{-\frac{3(i-1)}{2} + \frac{1}{2}\sum_{j=1}^{i-1} \sigma_j^3} \Theta_i^{\downarrow} |\Psi_{2n-1}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{2n})\rangle, \tag{8.77}$$

$$\lim_{z_i \to \infty} z_i^{-(2n-1)} |\Psi_{2n}\rangle = (-1)^{n+i} \beta q^{\frac{3(i-1)}{2} - \frac{1}{2}\sum_{j=1}^{i-1} \sigma_j^3} \Theta_i^{\downarrow} |\Psi_{2n-1}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{2n})\rangle. \tag{8.78}$$

*Likewise, for each $i = 1, \ldots, 2n+1$, we have*

$$\lim_{z_i \to 0} z_i^{2n} |\Psi_{2n+1}\rangle = (-1)^{i-1} q^{-\frac{3(i-1)}{2} - \frac{1}{2}\sum_{j=1}^{i-1} \sigma_j^3} \Theta_i^{\uparrow} |\Psi_{2n}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{2n+1})\rangle, \tag{8.79}$$

$$\lim_{z_i \to \infty} z_i^{-2n} |\Psi_{2n+1}\rangle = (-1)^{i-1} q^{\frac{3(i-1)}{2} + \frac{1}{2}\sum_{j=1}^{i-1} \sigma_j^3} \Theta_i^{\uparrow} |\Psi_{2n}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{2n+1})\rangle. \tag{8.80}$$

*Proof.* The proofs of the first two relations are similar. Hence, we focus on the proof of (8.77). First, we consider the case where $i = 1$. It follows from Proposition 8.4.4 that there is a vector $|\Phi\rangle = |\Phi(z_2, \dots, z_{2n})\rangle \in V^{2n-1}$ such that

$$\lim_{z_1 \to 0} z_1^{2n-1} |\Psi_{2n}(z_1, z_2 \dots, z_{2n})\rangle = |\downarrow\rangle \otimes |\Phi(z_2, \dots, z_{2n})\rangle. \tag{8.81}$$

This vector is of the form (8.24) (with $n$ replaced by $n-1$) and obeys the exchange relations. Moreover, it has the component

$$\begin{aligned} \Phi_{1,\dots,n-1}(z_2, \dots, z_{2n}) &= \lim_{z_1 \to 0} z_1^{2n-1} (\Psi_{2n})_{1,\dots,n}(z_1, z_2 \dots, z_{2n}) \\ &= (-1)^n \beta^{-1} (\Psi_{2n-1})_{1,\dots,n-1}(z_2, \dots, z_{2n}). \end{aligned} \tag{8.82}$$

We apply Lemma 8.2.3 and conclude $|\Phi\rangle = (-1)^n \beta^{-1} |\Psi_{2n-1}(z_2, \dots, z_{2n})\rangle$. This ends the proof for $i = 1$.

Second, for $i = 2, \dots, 2n$, we write

$$\begin{aligned} &|\Psi_{2n}(z_1, \dots, z_i, \dots, z_{2n})\rangle \\ &\quad = \overset{\curvearrowleft}{\prod_{j=1,\dots,i-1}} \check{R}_{jj+1}(z_i/z_j) |\Psi_{2n}(z_i, z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{2n})\rangle. \end{aligned} \tag{8.83}$$

Using the relation $\lim_{z\to 0} \check{R}_{jj+1}(z) = -q^{-\frac{3}{2} - \frac{1}{2}\sigma_j^3 \sigma_{j+1}^3} P_{jj+1}$ and the result for $i = 1$ leads to the relation (8.77).

Moreover, the proofs of (8.79) and (8.80) are also similar. Therefore, we only prove (8.79). First, we consider the case $i = 2n+1$. By Proposition 8.4.4, there exists a vector $|\Phi\rangle = |\Phi(z_1, \dots, z_{2n})\rangle \in V^{2n}$ such that

$$\lim_{z_{2n+1} \to 0} z_{2n+1}^{2n} |\Psi_{2n+1}(z_1, \dots, z_{2n+1})\rangle = |\Phi(z_1, \dots, z_{2n})\rangle \otimes |\uparrow\rangle \tag{8.84}$$

This vector is of the form (8.24), satisfies the exchange relations and has the component

$$\begin{aligned} \Phi_{1,\dots,n}(z_1, \dots, z_{2n}) &= \lim_{z_{2n+1} \to 0} z_{2n+1}^{2n} (\Psi_{2n+1})_{1,\dots,n}(z_1, z_2 \dots, z_{2n+1}) \\ &= q^{-3n} (\Psi_{2n})_{1,\dots,n}(z_1, \dots, z_{2n}). \end{aligned} \tag{8.85}$$

Lemma 8.2.3 implies that $|\Phi\rangle = q^{-3n} |\Psi_{2n}(z_1, \dots, z_{2n})\rangle$. A direct inspection shows that this is the result (8.79) for $i = 2n+1$.

Second, for $i = 1, \dots, 2n$, we have

$$|\Psi_{2n+1}(z_1, \dots, z_{2n+1})\rangle = \prod_{j=i,\dots,2n} \check{R}_{jj+1}(z_{j+1}/z_i) \times |\Psi_{2n+1}(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{2n+1}, z_i)\rangle. \quad (8.86)$$

We use the relation $\lim_{z\to\infty} \check{R}_{jj+1}(z) = -q^{\frac{3}{2}+\frac{1}{2}\sigma_j^3\sigma_{j+1}^3} P_{jj+1}$ and the result for the case $i = 2n+1$ to obtain (8.79). □

The preceding proposition allows us to show that the bounds in the inequalities (8.70) and (8.74) of Propositions 8.4.4 and 8.4.5 are actually equal to the degrees.

**Proposition 8.4.7.** *The property (ii) of Proposition 8.4.4 holds with* (8.70) *replaced by*

$$\deg^{\pm}_{z_i}(\Psi_{2n})_{a_1,\dots,a_n} = \pm 2(n-1). \quad (8.87)$$

*Likewise, the property (iii) of Proposition 8.4.5 holds with* (8.74) *replaced by*

$$\deg^{\pm}_{z_i}(\Psi_{2n+1})_{a_1,\dots,a_n} = \pm(2n-1). \quad (8.88)$$

*Proof.* The proofs of the two properties are similar. Hence, let us prove that Proposition 8.4.4 holds with (8.87). To this end, let $j$ be an integer with $1 \leqslant j \leqslant 2n$ and $a_\ell = j$. From Proposition 8.4.6, it follows that

$$\lim_{z_j\to 0} z_j^{2n-1}(\Psi_{2n})_{a_1,\dots,a_{\ell-1},j,a_{\ell+1},\dots,a_n} = (-1)^{n+j+1}\beta^{-1}q^{-(\ell+j-2)} \times (\Psi_{2n-1})_{a_1,\dots,a_{\ell-1},a_{\ell+1}-1,\dots,a_n-1}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_{2n}). \quad (8.89)$$

Now, choose an integer $1 \leqslant i \leqslant 2n$ with $i \notin \{a_1, \dots, a_{\ell-1}, a_{\ell+1}, \dots, a_n\}$. By Proposition 8.4.5, the right-hand side of (8.89) is a Laurent polynomial in $z_i$ with lower degree $-2(n-1)$ and upper degree $+2(n-1)$. According to Proposition 8.4.4, these are equal to the bounds on the degrees of $(\Psi_{2n})_{a_1,\dots,a_{\ell-1},j,a_{\ell+1},\dots,a_n}$ with respect to $z_i$. Since the lower degree can only increase and the upper degree can only decrease when taking the limit on the left-hand side of (8.89), we have

$$\deg^{\pm}_{z_i}(\Psi_{2n})_{a_1,\dots,a_{\ell-1},j,a_{\ell+1},\dots,a_n} = \pm 2(n-1). \quad (8.90)$$

Since this statement holds for all $j \in \{a_1, \dots, a_n\}$, we obtain (8.87). □

## 8.5 Parity

In this section, we establish the behaviour of $|\Psi_L\rangle$ under a parity transformation. To this end, we use an explicit formula for the component $(\Psi_L)_{\bar{n}+1,\dots,L}$. Obtaining it from the contour-integral formulas appears to be complicated. Hence, we compute it via factor exhaustion.

**Proposition 8.5.1.** *We have the component*

$$(\Psi_L)_{\bar{n}+1,\dots,L} = (-1)^{(L+1)n} \prod_{i=\bar{n}+1}^{L} [q\beta z_i] \\ \times \prod_{1\leqslant i<j\leqslant \bar{n}} [qz_j/z_i][qz_iz_j] \prod_{\bar{n}+1\leqslant i<j\leqslant L} [qz_j/z_i][q^2z_iz_j]. \quad (8.91)$$

*Proof.* Lemma 8.2.2 implies the factorisation

$$(\Psi_L)_{\bar{n}+1,\dots,L} = f_L(z_1,\dots,z_{\bar{n}};z_{\bar{n}+1},\dots,z_L) \\ \times \prod_{1\leqslant i<j\leqslant \bar{n}} [qz_j/z_i] \prod_{\bar{n}+1\leqslant i<j\leqslant L} [qz_j/z_i], \quad (8.92)$$

where $f_L(z_1,\dots,z_{\bar{n}};z_{\bar{n}+1},\dots,z_L)$ is a Laurent polynomial with respect to each of its arguments. It is separately symmetric in the variables $z_1,\dots,z_{\bar{n}}$, and $z_{\bar{n}+1},\dots,z_L$. Furthermore, the reflection relations (8.40) and (8.51) lead to

$$[\bar{\beta}s^2z_L] \prod_{i=\bar{n}+1}^{L-1} [qz_L/z_i] f_L(\dots;\dots,z_L) = \\ [\bar{\beta}z_L^{-1}] \prod_{i=\bar{n}+1}^{L-1} [q/(s^2z_Lz_i)] f_L(\dots;\dots,s^{-2}z_L^{-1}),$$

and

$$\prod_{i=2}^{\bar{n}}[qz_j/z_1] f_L(z_1,\dots;\dots) = \prod_{i=2}^{\bar{n}}[qz_jz_1] f_L(z_1^{-1},\dots;\dots).$$

Taking into account the symmetry of $f_L$, we obtain

$$f_L(z_1,\dots,z_{\bar{n}};z_{\bar{n}+1},\dots,z_L) = C_L \prod_{i=\bar{n}+1}^{L} [\bar{\beta}z_i^{-1}] \\ \times \prod_{1\leqslant i<j\leqslant \bar{n}} [qz_iz_j] \prod_{\bar{n}+1\leqslant i<j\leqslant L} [q/(s^2z_iz_j)]. \quad (8.93)$$

It follows from Propositions 8.4.4 and 8.4.5 that $C_L$ is a Laurent polynomial in $z_i$ with degrees $\deg_{z_i}^{\pm} C_L = 0$ for each $i = 1, \dots, L$. Therefore, $C_L$ is a constant. To find it, we consider the limits where $z_1 \to \infty$ and $z_L \to \infty$. Using Proposition 8.4.6, we obtain the recurrence relations

$$C_{2n+1} = (-1)^n C_{2n}, \quad C_{2n} = (q^3/s^2)^{n-1} \bar{\beta}\beta q \, C_{2n-1}, \tag{8.94}$$

for each $n \geqslant 1$. The initial condition $C_1 = 1$ implies

$$C_{2n+1} = (-1)^n C_{2n} = (-1)^{n(n+1)/2} (q^3/s^2)^{n(n-1)/2} (\bar{\beta}\beta q)^n. \tag{8.95}$$

We substitute (8.93) and (8.95) into (8.92) and simplify the resulting expression with the help of (8.37) and (8.38). This yields (8.91). $\square$

In the next proposition, we compute the action of the parity operator $\mathcal{C}$, defined in Chapter 6, onto $|\Psi_L\rangle$. To this end, we write $|\Psi_L\rangle = |\Psi_L(z_1, \dots, z_L; \beta)\rangle$ to stress the vector's dependence on the parameter $\beta$.

**Proposition 8.5.2.** *We have*

$$\mathcal{C}|\Psi_L(z_1, \dots, z_L; \beta)\rangle = \epsilon_L |\Psi_L(s^{-1} z_L^{-1}, \dots, s^{-1} z_1^{-1}; q^2 s^{-1} \beta^{-1})\rangle, \tag{8.96}$$

*where* $\epsilon_L = (q^3 s^{-2})^{(L+1)n}$.

*Proof.* We consider the vector

$$|\Phi\rangle = \epsilon_L \mathcal{C} |\Psi_L(s^{-1} z_L^{-1}, \dots, s^{-1} z_1^{-1}; q^2 s^{-1} \beta^{-1})\rangle. \tag{8.97}$$

Using the fact that $\check{R}(z)$ is a symmetric matrix, it is straightforward to show that this vector obeys the exchange relations. Moreover, it has the component

$$\Phi_{1,\dots,n} = \epsilon_L (\Psi_L)_{\bar{n}+1,\dots,L} (s^{-1} z_L^{-1}, \dots, s^{-1} z_1^{-1}; q^2 s^{-1} \beta^{-1}). \tag{8.98}$$

We compute the right-hand side by using the explicit formula for the component $(\Psi_L)_{\bar{n}+1,\dots,L}$ given in Proposition 8.5.1. This leads to $\Phi_{1,\dots,n} = (\Psi_L)_{1,\dots,n}$. Hence, by Proposition 8.2.4 we have $|\Phi\rangle = |\Psi_L\rangle$. $\square$

## 8.6 The transfer matrix of the six-vertex model

In this section, we consider the transfer matrix of the inhomogeneous six-vertex model on a strip and discuss some of its properties.

Following [35], we define the double-row transfer matrix of the inhomogeneous six-vertex model on a strip with $L$ horizontal lines. It is an operator on the space $V^L$, given by the partial trace

$$T(z|z_1,\ldots,z_L) = \mathrm{tr}_0\Big(K_0(qz;\bar{\beta})\overset{\curvearrowleft}{\prod_{i=1,\ldots,L}} R_{0i}(z/z_i)K_0(z;\beta)\overset{\curvearrowright}{\prod_{i=1,\ldots,L}} R_{0i}(zz_i)\Big). \tag{8.99}$$

The operators inside the trace act on $V_0 \otimes V^L$, where $V_0$ is the auxiliary space. The trace is taken with respect to the space $V_0$. This transfer matrix is the inhomogeneous version, for the six-vertex model, of the one defined in Chapter 5.

The Yang-Baxter equation (8.13) and the boundary Yang-Baxter equation (8.33) imply the commutation relation

$$[T(z|z_1,\ldots,z_L), T(w|z_1,\ldots,z_L)] = 0 \tag{8.100}$$

for all $z, w$ and $z_1,\ldots,z_L$ [35]. The common eigenvectors of the family of commuting transfer matrices $T(z|z_1,\ldots,z_L)$ parameterised by $z$ are therefore independent of $z$. In the following, we construct a common eigenvector of the family of commuting transfer matrices and compute its eigenvalue.

To this end, we use a few properties of the transfer matrices that follow from the properties of the $R$-matrix (8.11) and the $K$-matrix (8.34). As for the $R$-matrix, let us introduce for $i < j$ the operator $R_{ji}(z) = P_{ij}R_{ij}(z)P_{ij}$. By virtue of the symmetry of the $R$-matrix, we find

$$R_{ji}(z) = R_{ij}(z). \tag{8.101}$$

Furthermore, the $R$-matrix satisfies the unitarity relation

$$R_{ij}(z)R_{ij}(1/z) = \mathbf{1}, \tag{8.102}$$

and obeys the following relations:

$$[q/z]\sigma_i^1 \left(R_{ij}(z)\right)^{t_i} \sigma_i^1 = -[q^2 z] R_{ij}(-1/(qz)), \tag{8.103}$$
$$[q/z]\sigma_i^2 \left(R_{ij}(z)\right)^{t_i} \sigma_i^2 = -[q^2 z] R_{ij}(1/(qz)), \tag{8.104}$$
$$\sigma_i^3 R_{ij}(z) \sigma_i^3 = R_{ij}(-z). \tag{8.105}$$

Here, the superscript $t_i$ indicates the transposition with respect to the space $V_i$.

**Lemma 8.6.1.** *We have the relations*

$$T(-z|z_1,\dots,z_L) = T(z|z_1,\dots,z_L), \tag{8.106}$$

*and*

$$T(1/(qz)|z_1,\dots,z_L) = \frac{[\beta/z][\bar\beta/(qz)]}{[\bar\beta z][q\beta z]} \prod_{i=1}^{L} \left( \frac{[qz_i/z][q/(zz_i)]}{[q^2 z/z_i][q^2 z z_i]} \right) T(z|z_1,\dots,z_L). \tag{8.107}$$

*Proof.* First, we use (8.105) to write

$$T(-z|z_1,\dots,z_L) = \mathrm{tr}_0\Big( K_0(-qz;\bar\beta)\sigma_0^3 \overset{\curvearrowleft}{\prod_{i=1,\dots,L}} R_{0i}(z/z_i)\sigma_0^3 K_0(-z;\beta)\sigma_0^3 \overset{\curvearrowright}{\prod_{i=1,\dots,L}} R_{0i}(zz_i)\sigma_0^3 \Big).$$

The relations $K_0(-z;\beta) = \sigma_0^3 K_0(z;\beta)\sigma_0^3 = K_0(z;\beta)$ and the cyclicity of the trace allow us to conclude that $T(-z|z_1,\dots,z_L) = T(z|z_1,\dots,z_L)$.

Second, we use (8.104), take the transpose on the space $V_0$ and use the cyclicity of the trace to obtain

$$T(1/(qz)|z_1,\dots,z_L) = \prod_{i=1}^{L} \frac{[qz_i/z][q/(zz_i)]}{[q^2 z/z_i][q^2 z z_i]} \times$$
$$\mathrm{tr}_0\Big( \sigma_0^2 \overset{\curvearrowright}{\prod_{i=1,\dots,L}} R_{0i}(zz_i)\sigma_0^2 K_0(1/z;\bar\beta)\sigma_0^2 \overset{\curvearrowleft}{\prod_{i=1,\dots,L}} R_{0i}(z/z_i)\sigma_0^2 K_0(1/(qz);\beta) \Big).$$

The $K$-matrix satisfies $\sigma_0^2 K_0(1/z;\beta)\sigma_0^2 = \frac{[\beta/z]}{[z\beta]} K_0(z;\beta)$ as well as the following relation

$$\mathrm{tr}_{\bar 0}\left( K_{\bar 0}(qz;\bar\beta) R_{\bar 0 0}(z^2) P_{\bar 0 0} \right) = \frac{[q^2 z^2][\bar\beta/z]}{[z^2/q][qz/\bar\beta]} K_0(z;\bar\beta), \tag{8.108}$$

where we introduced a new auxiliary space $V_{\bar{0}} = V$. The operators inside trace act on $V_{\bar{0}} \otimes V_0 \otimes V^L$. We use these relations to write

$$T(1/(qz)|z_1,\dots,z_L) = \prod_{i=1}^{L} \frac{[qz_i/z][q/(zz_i)]}{[q^2z/z_i][q^2zz_i]} \frac{[qz/\bar{\beta}][\beta/(qz)]}{[z\bar{\beta}][\beta qz]} \frac{[z^2/q]}{[q^2z^2]} \times$$
$$\mathrm{tr}_{\bar{0}}\Big(\mathrm{tr}_0\Big(K_{\bar{0}}(qz;\bar{\beta}) \overset{\curvearrowright}{\prod_{i=1,\dots,L}} R_{0i}(zz_i) R_{\bar{0}0}(z^2) \overset{\curvearrowleft}{\prod_{i=1,\dots,L}} R_{\bar{0}i}(z/z_i) P_{\bar{0}0} K_0(qz;\beta)\Big)\Big).$$

We rearrange the products of $R$-matrix inside the trace by using the Yang-Baxter equation in the form

$$R_{0i}(zz_i)R_{\bar{0}0}(z^2)R_{\bar{0}i}(z/z_i) = R_{\bar{0}i}(z/z_i)R_{\bar{0}0}(z^2)R_{0i}(zz_i) \tag{8.109}$$

After rearrangement, we note that the permutation operator $P_{\bar{0}0}$ allows us to take the products out of the trace over the space $V_0$. We obtain

$$T(1/(qz)|z_1,\dots,z_L) = \prod_{i=1}^{L} \frac{[qz_i/z][q/(zz_i)]}{[q^2z/z_i][q^2zz_i]} \frac{[qz/\bar{\beta}][\beta/(qz)]}{[z\bar{\beta}][\beta qz]} \frac{[z^2/q]}{[q^2z^2]} \times$$
$$\mathrm{tr}_{\bar{0}}\Big(K_{\bar{0}}(qz;\bar{\beta}) \overset{\curvearrowleft}{\prod_{i=1,\dots,L}} R_{\bar{0}i}(z/z_i) \mathrm{tr}_0\left(R_{\bar{0}0}(z^2)P_{\bar{0}0}K_0(qz;\beta)\right) \overset{\curvearrowright}{\prod_{i=1,\dots,L}} R_{\bar{0}i}(zz_i)\Big).$$

We again use the relation (8.108) and recognise the transfer matrix with auxiliary space index $\bar{0}$. This concludes the proof. □

**Lemma 8.6.2.** *Let $z$ be a solution of $z^4 = 1$ then*

$$T(z|z_1,\dots,z_L) = \frac{[q^2][\bar{\beta}/z]}{[q][\bar{\beta}/(qz)]}\mathbf{1}. \tag{8.110}$$

*Proof.* We separately consider the cases where $z = \pm 1$ and $z = \pm \mathrm{i}$. First, if $z = \pm 1$, then we have $K_0(z;\beta) = \mathbf{1}$ and $R_{0i}(z/z_i) = R_{0i}(1/(zz_i))$. These relations allow us to write

$$T(z|z_1,\dots,z_L) = \mathrm{tr}_0\Big(K_0(qz;\bar{\beta}) \overset{\curvearrowleft}{\prod_{i=1,\dots,L}} R_{0i}(1/(zz_i)) \overset{\curvearrowright}{\prod_{i=1,\dots,L}} R_{0i}(zz_i)\Big).$$

Moreover, using the unitarity relation (8.102), we find

$$T(z|z_1,\dots,z_L) = \mathrm{tr}_0\left(K_0(qz;\bar{\beta})\right) = \frac{[q^2][\bar{\beta}]}{[q][\bar{\beta}/q]}\mathbf{1}. \tag{8.111}$$

This proves (8.110) for $z = \pm 1$.

Second, if $z = \pm\mathrm{i}$, then we have $K_0(z;\beta) = \sigma_0^3$, and $R_{0i}(z/z_i)\sigma_0^3 = \sigma_0^3 R_{0i}(1/(zz_i))$. These relations allow us to write

$$T(z|z_1,\ldots,z_L) = \mathrm{tr}_0\Big(K_0(qz;\bar\beta)\sigma_0^3 \overset{\curvearrowleft}{\prod_{i=1,\ldots,L}} R_{0i}(1/(zz_i)) \overset{\curvearrowright}{\prod_{i=1,\ldots,L}} R_{0i}(zz_i)\Big).$$

Using the unitarity relation (8.102), we find

$$T(z|z_1,\ldots,z_L) = \mathrm{tr}_0\left(K_0(qz;\bar\beta)\sigma_0^3\right) = \frac{[q^2][\bar\beta/z]}{[q][\bar\beta/(qz)]}\mathbf{1}. \tag{8.112}$$

This proves (8.110) for $z = \pm\mathrm{i}$. $\square$

## 8.7 The eigenvector

In this section, we establish a relation between the double-row transfer matrix of the inhomogeneous six-vertex model on the strip and the operators $S^{(i)}$ (8.54). (We refer to [141] for a general discussion on the relation between transfer matrices and bqKZ equations.) We use this relation to show that if $q = \mathrm{e}^{2\pi\mathrm{i}/3}$ then the vector $|\Psi_L\rangle$ is an eigenvector of the transfer matrix. Moreover, we explicitly compute the corresponding eigenvalue.

**Proposition 8.7.1.** *If $q = \mathrm{e}^{2\pi\mathrm{i}/3}$ and (8.38) holds then*

$$T(z_i|z_1,\ldots,z_L) = -\frac{[q\beta z_i]}{[q^2\beta z_i]} S^{(i)}(z_1,\ldots,z_L), \tag{8.113}$$

*for each $i = 1,\ldots,L$. Here, $S^{(i)}(z_1,\ldots,z_L)$ is the operator defined in (8.54) with $s = 1$.*

*Proof.* First, we assume that $q, \beta, \bar\beta$ are generic. We use $R_{0i}(1) = P_{0i}$, the symmetry of the $R$-matrix (8.101), and the properties of partial traces to write

$$\begin{aligned} T(z_i|z_1,\ldots,z_L) = &\overset{\curvearrowleft}{\prod_{j=1,\ldots,i-1}} R_{ij}(z_i/z_j) K_i(z_i;\beta) \overset{\curvearrowright}{\prod_{j=1,\ldots,i-1}} R_{ij}(z_iz_j) \\ \times\, \mathrm{tr}_0\Big(K_0(qz_i;\bar\beta) &\overset{\curvearrowleft}{\prod_{j=i+1,\ldots,L}} R_{0j}(z_i/z_j) R_{0i}(z_i^2) \overset{\curvearrowright}{\prod_{j=i+1,\ldots,L}} R_{ij}(z_iz_j)P_{0i}\Big). \end{aligned}$$

We rearrange the products of $R$-matrix using the Yang-Baxter equation as in the proof of Lemma 8.6.1 and obtain

$$\begin{aligned} T(z_i|z_1,\dots,z_L) = &\overset{\curvearrowleft}{\prod_{j=1,\dots,i-1}} R_{ij}(z_i/z_j)K_i(z_i;\beta) \overset{\curvearrowright}{\prod_{j=1,\dots,i-1}} R_{ij}(z_iz_j) \\ \times &\overset{\curvearrowright}{\prod_{j=i+1,\dots,L}} R_{ij}(z_iz_j)\,\mathrm{tr}_0\left(K_0(qz_i;\bar\beta)R_{0i}(z_i^2)P_{0i}\right) \overset{\curvearrowleft}{\prod_{j=i+1,\dots,L}} R_{ij}(z_i/z_j). \end{aligned} \tag{8.114}$$

The remaining trace is given by

$$\mathrm{tr}_0\left(K_0(qz_i;\bar\beta)R_{0i}(z_i^2)P_{0i}\right) = \frac{[q^2z_i^2][\bar\beta/z_i]}{[z_i^2/q][qz_i/\bar\beta]}K_i(z_i;\bar\beta). \tag{8.115}$$

We rewrite the products of $R$-matrices as products of $\check{R}$-matrices and obtain

$$\begin{aligned} T(z_i|z_1,\dots,z_L) = &\frac{[q^2z_i^2][\bar\beta/z_i]}{[z_i^2/q][qz_i/\bar\beta]} \overset{\curvearrowleft}{\prod_{j=1,\dots,i-1}} \check{R}_{jj+1}(z_i/z_j)\,K_1(z_i;\beta) \\ \times &\overset{\curvearrowright}{\prod_{j=1,\dots,i-1}} \check{R}_{jj+1}(z_iz_j) \overset{\curvearrowright}{\prod_{j=i,\dots,L-1}} \check{R}_{jj+1}(z_iz_{j+1})\,K_L(z_i;\bar\beta) \overset{\curvearrowleft}{\prod_{j=i,\dots,L-1}} \check{R}_{jj+1}(z_i/z_{j+1}). \end{aligned} \tag{8.116}$$

Second, we assume $q=\mathrm{e}^{2\pi\mathrm{i}/3}$ and that (8.38) holds. In (8.116), these specialisations lead to the pre-factor

$$\frac{[q^2z_i^2][\bar\beta/z_i]}{[z_i^2/q][qz_i/\bar\beta]} = -\frac{[q\beta z_i]}{[q^2\beta z_i]}. \tag{8.117}$$

The products of $\check{R}$- and $K$-matrices that remain yield the operator $S^{(i)}(z_1,\dots,z_L)$ defined in (8.54) with $s=1$. □

**Proposition 8.7.2.** *Let $q=\mathrm{e}^{2\pi\mathrm{i}/3}$ and suppose that (8.38) holds then we have*

$$T(z|z_1,\dots,z_L)|\Psi_L\rangle = \Lambda_L(z)|\Psi_L\rangle, \tag{8.118}$$

*where the eigenvalue is*

$$\Lambda_L(z) = -\frac{[q\beta z]}{[q^2\beta z]}. \tag{8.119}$$

*Proof.* We define the operator

$$\bar{T}(z|z_1,\dots,z_L) = -[\beta/z][q^2\beta z]\Big(\prod_{j=1}^{L}[qz_j/z][q/(z_jz)]\Big)\,T(z|z_1,\dots,z_L). \tag{8.120}$$

We are going to prove that if $q = \mathrm{e}^{2\pi\mathrm{i}/3}$ and (8.38) holds then

$$\bar{T}(z|z_1,\dots,z_L)|\Psi_L\rangle = [\beta/z][q\beta z]\Big(\prod_{j=1}^{L}[qz_j/z][q/(z_jz)]\Big)\,|\Psi_L\rangle, \tag{8.121}$$

which is equivalent to the statement of the theorem. To this end, we note that the pre-factor on the right-hand side of (8.121) is a Laurent polynomial in $z$ with lower degree $-2(L+1)$ and upper degree $2(L+1)$. Likewise, the matrix elements of $\bar{T}(z|z_1,\dots,z_L)$ are Laurent polynomials in $z$ with lower degree at least $-2(L+1)$ and upper degree at most $2(L+1)$. Therefore, it is sufficient to show that (8.121) holds for at least $4L+5$ distinct values of $z$.

First, it follows from Proposition 8.7.1 that

$$\begin{aligned}\bar{T}(z_i|z_1,\dots,z_L)&\\ &= [\beta/z_i][q\beta z_i]\Big(\prod_{j=1}^{L}[qz_j/z_i][q/(z_jz_i)]\Big)S^{(i)}(z_1,\dots,z_L),\end{aligned} \tag{8.122}$$

where $S^{(i)}(z_1,\dots,z_L)$ is the operator defined in (8.54) with $s=1$. It follows from this equality and from the bqKZ equations (8.55) that (8.121) holds if $z=z_i$ for each $i=1,\dots,L$. Moreover, Lemma 8.6.1 allows us to conclude that it holds if $z=-z_i, 1/(qz_i), -1/(qz_i)$ for each $i=1,\dots,L$, too.

Second, according to Lemma 8.6.1 and Lemma 8.6.2, for any solution $z$ of $z^4=1$ we have

$$\bar{T}(z|z_1,\dots,z_L) = \Big(\prod_{j=1}^{L}[qz_j/z][q/(z_jz)]\Big)[\beta/z][q\beta z]\mathbf{1}. \tag{8.123}$$

This implies that (8.121) trivially holds for the values $z=\pm1,\pm\mathrm{i}$, $\pm q^{-1},\pm\mathrm{i}q^{-1}$.

In summary, the relation (8.121) holds for $4L+8>4L+5$ distinct values of $z$ and, hence, for all $z$. □

Two remarks about the eigenvalue $\Lambda_L(z)$ are in order. First, we note that we can write it as the trace

$$\Lambda_L(z) = \mathrm{tr}_0 \left( K_0(qz;\bar{\beta}) K_0(z;\beta) \right) , \tag{8.124}$$

where $\bar{\beta}$ is a solution of (8.38). Formally, the right-hand side of this equality is the transfer matrix of the six-vertex model on a strip with $L = 0$ vertical lines. Second, for $q = \beta = \mathrm{e}^{2\pi \mathrm{i}/3}$ and $z_1 = \cdots = z_L = 1$, the eigenvalue is equivalent to the trigonometric limit of the eigenvalue of the supersymmetric eight-vertex model on a strip studied in Chapter 5 [3].

# Chapter 9

# Spin chain and combinatorics

In this chapter, we study the homogeneous limit $z_1 = \cdots = z_L = 1$ of the vector $|\Psi_L\rangle$ defined in Chapter 8. It is convenient to define the rescaled version

$$|\psi_L\rangle = (-1)^{\bar{n}(\bar{n}-1)/2}[\beta]^{-n}[q]^{-n(n-1)-\bar{n}(\bar{n}-1)}|\Psi_L(1,\ldots,1)\rangle. \tag{9.1}$$

This vector is non-vanishing, as follows from Proposition 8.1.1, and depends on the parameters $q$ and $\beta$.

For generic $q, \beta$, the state $|\psi_L\rangle$ is unrelated to the XXZ spin chain. If $q = \mathrm{e}^{2\mathrm{i}\pi/3}$, then we show that $|\psi_L\rangle$ becomes an eigenvector of the XXZ Hamiltonian (8.1) with the parameters (8.2), where

$$x = -[q\beta]/[\beta]. \tag{9.2}$$

Moreover, for $x > 0$, it spans the ground-state eigenspace. Hence, if $q = \mathrm{e}^{2\pi i/3}$, we refer to $|\psi_L\rangle$ as the ground-state vector (even though, strictly speaking, this is only valid for $x > 0$).

As already mentioned in Chapters 6 and 8, there is a close connection between XXZ spin-chain ground-states at $\Delta = -1/2$ (and, to a larger extent, the solutions of qKZ equations) and enumerative combinatorics. As for $|\psi_L\rangle$, we show that scalar products are related to the enumeration of symmetry classes of plane partitions and alternating sign matrices. In particular, we conjecture a relation between the sum of components of $|\psi_L\rangle$, for generic $q$, and a weighted enumeration of the so-called totally-symmetric alternating sign matrices.

The layout of this chapter is as follows. In Section 9.1, we show that if $q = \mathrm{e}^{2\pi \mathrm{i}/3}$ then $|\psi_L\rangle$ is an eigenvector of the XXZ spin-chain Hamiltonian (8.1) with parameters (8.2) and (9.2). Furthermore, we compute the corresponding eigenvalue. The purpose of Section 9.2 is to express the components of the homogeneous vector in terms of multiple contour integrals and to investigate some of its properties. We discuss scalar products involving the vector $|\psi_L\rangle$ in Section 9.3. In Section 9.4, we recall the definition and enumeration of (special kinds of) plane partitions and alternating sign matrices and discuss the relation of these combinatorial objects with the components of $|\psi_L\rangle$.

## 9.1 Transfer matrix and the XXZ Hamiltonian

In this section, we show the relation between the transfer matrix and the Hamiltonian (8.1) and prove that the latter possesses a simple eigenvalue in the sector of magnetisation $\mu = (\bar{n} - n)/2$. To this end, we define the transfer matrix of the homogeneous six-vertex model on the strip by

$$t(z) = T(z|1, \dots, 1). \tag{9.3}$$

It follows from the calculation made in Chapter 5 that its logarithmic derivative at $z = 1$ yields the XXZ Hamiltonian of the open XXZ spin chain [35]. Using our parameterisation of the $R$- and $K$-matrix, we find

$$t(1)^{-1}t'(1) = -\frac{4}{[q]}\left(H - C\mathbf{1}\right), \tag{9.4}$$

where $H$ is the Hamiltonian (8.1) with the parameters

$$\Delta = \frac{[q^2]}{2[q]}, \quad p = \frac{[q][\beta^2]}{4[\beta]^2}, \quad \bar{p} = \frac{[q][\bar{\beta}^2]}{4[\bar{\beta}]^2}. \tag{9.5}$$

Moreover, the constant $C$ is given by

$$C = \frac{3L[q^2]}{4[q]} + \frac{[q][\beta^2]}{4[\beta]^2} - \frac{[q]^2[\bar{\beta}^2]}{2[q^2][\bar{\beta}][q/\bar{\beta}]}. \tag{9.6}$$

**Theorem 9.1.1.** *The Hamiltonian (8.1) with the parameters (8.2) possesses the eigenvalue*

$$E_0 = -\frac{3L-1}{4} - \frac{(1-x)^2}{2x}. \tag{9.7}$$

*For real $x > 0$, it is the non-degenerate ground-state eigenvalue in the sector of magnetisation $\mu = (\bar{n} - n)/2$.*

*Proof.* Let $q = \mathrm{e}^{2\pi\mathrm{i}/3}$ and suppose that (8.38) holds. In this case, it follows from (9.5) that $\Delta = -1/2$ and that the parameters $p, \bar{p}$ satisfy (8.2), where $x$ is given in terms of $\beta$ by (9.2).

First, we prove the existence of the eigenvalue $E_0$. To this end, we use (9.4) to write

$$H|\psi_L\rangle = C|\psi_L\rangle - \frac{[q]}{4} t(1)^{-1} t'(1)|\psi_L\rangle. \tag{9.8}$$

By Proposition 8.7.2, $|\psi_L\rangle$ is an eigenvector of $t(z)$ with the eigenvalue $\Lambda_L(z)$. Hence, we find $H|\psi_L\rangle = E_0|\psi_L\rangle$ with

$$E_0 = C - \frac{[q]}{4}\frac{\Lambda_L'(1)}{\Lambda_L(1)}. \tag{9.9}$$

Using (9.6) and the explicit expression of $\Lambda_L(z)$, we find that the right-hand side yields (9.7).

Second, let us denote by $E^\mu(x)$ the ground-state eigenvalue of the restriction $H_\mu$ of $H$ to the sector of magnetisation $\mu = (\bar{n} - n)/2$. We show that if $x > 0$, then $E_0(x) = E^\mu(x)$ and $E_0(x)$ is non-degenerate. Here, we write $E_0 = E_0(x)$ to stress its dependence on $x$.

We show that $E^\mu(x)$ is non-degenerate following an argument of Yang and Yang [14]. For each real $x > 0$, we use a similar argument as in [3] to show that there exists a real number $\lambda$ such that the $\lambda - H_\mu$ is a non-negative irreducible matrix. Hence, we may apply the Perron-Frobenius theorem. It implies that the largest eigenvalue of $\lambda - H_\mu$ is non-degenerate. Therefore, the ground-state eigenvalue $E^\mu$ of $H_\mu$ is non-degenerate.

For $x = 1$, $E_0(1)$ is the non-degenerate ground-state eigenvalue of the Hamiltonian. The corresponding eigenvector has magnetisation $\mu = (\bar{n} - n)/2$ [1]. This implies that $E_0(1) = E^\mu(1)$.

One finally checks that for $x > 0$, both $E_0(x)$ and $E^\mu(x)$ are continuous functions of $x$. We show that $E_0(x) = E^\mu(x)$ for each $x > 0$. Indeed, if $E_0(x) \neq E^\mu(x)$ for $x \neq 1$ (without loss of generality, we can assume that $0 < x < 1$), this would imply that there exists $x'$ with $x < x' \leq 1$ such that $E_0(x') = E^\mu(x')$ is (at least) doubly degenerate. This is a contradiction. Hence, $E_0(x) = E^\mu(x)$ and $E_0(x)$ is non-degenerate $\square$

Theorem 9.1.1 gives an explicit expression for the ground-state eigenvalue of the Hamiltonian in the sector of magnetisation $\mu = (\bar{n} - n)/2$, for $x > 0$. Its existence was conjectured in [126]. The state $|\psi_L\rangle$ spans the corresponding eigenspace.

## 9.2 Components

In this section, we investigate the properties of the vector $|\psi_L\rangle$. For $L \geqslant 2$, we may write

$$|\psi_L\rangle = \sum_{1\leqslant a_1<\cdots<a_n\leqslant L} (\psi_L)_{a_1,\ldots,a_n} |\uparrow\cdots\uparrow\underset{a_1}{\downarrow}\uparrow \quad \cdots \quad \uparrow\underset{a_n}{\downarrow}\uparrow\cdots\uparrow\rangle. \tag{9.10}$$

Likewise, we have

$$|\psi_L\rangle = \sum_{1\leqslant b_1<\cdots<b_{\bar{n}}\leqslant L} (\bar{\psi}_L)_{b_1,\ldots,b_{\bar{n}}} |\downarrow\cdots\downarrow\underset{b_1}{\uparrow}\downarrow \quad \cdots \quad \downarrow\underset{b_{\bar{n}}}{\uparrow}\downarrow\cdots\downarrow\rangle. \tag{9.11}$$

We note that (9.1) fixes the following component:

$$(\psi_L)_{1,\ldots,n} = (\bar{\psi}_L)_{n+1,\ldots,L} = \tau^{\bar{n}(\bar{n}-1)/2}. \tag{9.12}$$

Here, and in the following, $\tau = -q - q^{-1}$.

The value $q = \mathrm{e}^{2\pi\mathrm{i}/3}$ corresponds to $\tau = 1$. In this case, $|\psi_L\rangle$ is an eigenvector of the Hamiltonian and the normalisation (9.12) becomes

$$(\psi_L)_{\downarrow\cdots\downarrow\uparrow\cdots\uparrow} = 1. \tag{9.13}$$

**Contour integral formulas.** There are several contour-integral formulas for the components of $|\psi_L\rangle$. The first type of formulas follows from the evaluation of (8.4) with $z_1 = \cdots = z_L = 1$. Changing the integration variables to $u_i = [w_i]/[w_i/q]$ leads to the following expression:

$$(\psi_L)_{a_1,\ldots,a_n} = \tau^{L(L-1)/2} \times$$
$$\oint \cdots \oint \prod_{k=1}^{n} \frac{\mathrm{d}u_k}{2\pi \mathrm{i}} \frac{(1+xu_k)(1+\tau u_k)(\tau+(\tau^2-2)u_k)(1+\tau u_k+u_k^2)^{L-2n}}{u_k^{a_k}(\tau+(\tau^2-1)u_k)^L}$$
$$\times \prod_{1\leqslant i<j\leqslant n}(u_j-u_i)(1+\tau(u_i+u_j)+(\tau^2-1)u_iu_j)(1+\tau u_j+u_iu_j)$$
$$\times (\tau+(\tau^2-1)(u_i+u_j)+\tau(\tau^2-2)u_iu_j). \quad (9.14)$$

The integration contour of $u_i$ goes around 0, but not around $-\tau/(\tau^2-1)$. Likewise, the evaluation of (8.6) with $z_1 = \cdots = z_L = 1$ and a change of the integration variables to $u_i = [w_{\bar{n}+1-i}]/[qw_{\bar{n}+1-i}]$ leads to a second contour integral formula

$$(\bar{\psi}_L)_{b_1,\ldots,b_{\bar{n}}} = \oint \cdots \oint \prod_{k=1}^{\bar{n}} \frac{\mathrm{d}u_k}{2\pi \mathrm{i}} \frac{(1+\tau u_k+u_k^2)^{L+1-2\bar{n}}}{u_k^{L+1-b_{\bar{n}+1-k}}(1+(x-\tau)u_k)}$$
$$\times \prod_{1\leqslant i\leqslant j\leqslant \bar{n}}(1-u_iu_j) \prod_{1\leqslant i<j\leqslant \bar{n}}(u_j-u_i)(1+\tau u_j+u_iu_j)(\tau+u_i+u_j)$$
$$(9.15)$$

Here, the integration contour of $u_i$ goes around 0. We also use the following third contour-integral representation of the components:

**Proposition 9.2.1.** *For each increasing sequence $1 \leqslant a_1 < \cdots < a_n \leqslant L$, we have*

$$(\psi_L)_{a_1,\ldots,a_n} = \oint \cdots \oint \prod_{k=1}^{n} \frac{\mathrm{d}u_k}{2\pi \mathrm{i}} \frac{(u_k+x)(1+\tau u_k+u_k^2)^{L-2n}}{u_k^{L+1-a_{n+1-k}}}$$
$$\times \prod_{1\leqslant i\leqslant j\leqslant n}(1-u_iu_j) \prod_{1\leqslant i<j\leqslant n}(u_j-u_i)(1+\tau u_j+u_iu_j)(\tau+u_i+u_j).$$
$$(9.16)$$

*The integration contour of $u_i$ goes around* 0.

*Proof.* It follows from Proposition 8.5.2 that

$$(\Psi_L)_{a_1,\ldots,a_n} = \epsilon_L(\Psi_L)_{L+1-a_n,\ldots,L+1-a_1}(s^{-1}z_L^{-1},\ldots,s^{-1}z_1^{-1};q^2s^{-1}\beta^{-1}),$$
$$(9.17)$$

where $s$ obeys (8.37). Using the integral formula (8.4) on the right-hand side and performing the change of variables $w_i \to s^{-1}w_{n+1-i}^{-1}$ lead, after some algebra, to

$$(\Psi_L)_{a_1,\dots,a_n} = [q]^n \prod_{1\leqslant i<j\leqslant L}[qz_j/z_i][qz_iz_j] \oint\cdots\oint\prod_{k=1}^{n}\frac{\mathrm{d}w_k}{\mathrm{i}\pi w_k}[q\beta w_k] \\ \times \frac{\prod_{1\leqslant i<j\leqslant n}[qw_j/w_i][w_i/w_j][q^2w_iw_j]\prod_{1\leqslant i\leqslant j\leqslant n}[qw_iw_j]}{\prod_{i=1}^{n}\left(\prod_{j=1}^{a_i}[qw_i/z_j]\prod_{j=a_i}^{L}[w_i/z_j]\prod_{j=1}^{L}[qw_iz_j]\right)}. \quad (9.18)$$

The integration contour of $w_i$ is collections of positively-oriented curves around $z_j$, but not $0, -z_j, \pm q^{-1}z_j$, $\pm q^{-1}z_j^{-1}$.

We now set $z_1 = \cdots = z_L = 1$ and change the integration variables to $u_i = [w_{n+1-i}]/[qw_{n+1-i}]$. Using the definitions (9.1) and (9.2), we obtain (9.16). □

For $x = 0$, this proposition shows that $|\psi_L\rangle$ is the vector studied in [139]. We now show that we can compute this vector from $x = \tau$. To this end, we recall that the spin-reversal operator $\mathcal{R}$ on $V^L$ is defined as

$$\mathcal{R} = \prod_{i=1}^{L}\sigma_i^1. \quad (9.19)$$

**Proposition 9.2.2.** *We have* $|\psi_L(0)\rangle = \mathcal{R}|\psi_{L-1}(\tau)\rangle \otimes |{\uparrow}\rangle$.

*Proof.* For $x = 0$, the contour-integral expression (9.16) becomes

$$(\psi_L)_{a_1,\dots,a_n} = \oint\cdots\oint\prod_{k=1}^{n}\frac{\mathrm{d}u_k}{2\pi\mathrm{i}}\frac{(1+\tau u_k+u_k^2)^{L-2n}}{u_k^{L-a_{n+1-k}}} \\ \times \prod_{1\leqslant i\leqslant j\leqslant n}(1-u_iu_j)\prod_{1\leqslant i<j\leqslant n}(u_j-u_i)(1+\tau u_j+u_iu_j)(\tau+u_i+u_j). \quad (9.20)$$

It implies that the component vanishes for $a_n = L$ because the integrand has no pole at $u_1 = 0$. For $a_n < L$, we find by comparison with (9.15) the relation

$$(\psi_L(0))_{a_1,\dots,a_n} = (\bar{\psi}_{L-1}(\tau))_{a_1,\dots,a_n} \quad (9.21)$$

Combining this relation with (9.10) and (9.11) proves the statement of the proposition. □

**Polynomiality.** Using the expression of the components in terms of multiple contour integrals, we prove the following property:

**Theorem 9.2.3.** *The components $(\psi_L)_{s_1\cdots s_L}$ are polynomials in $x$ and $\tau$ with integer coefficients. Furthermore, for $\tau = 1$ and each $1 \leqslant m \leqslant n$, the degree of the polynomial $(\psi_L)_{\downarrow\cdots\downarrow s_{m+1}\cdots s_L}$ in $x$ is at most $n - m$.*

*Proof.* First, it follows from the residue theorem that the contour integral (9.16) yields a polynomial in $x$ and $\tau$ with integer coefficients.

Second, for $\tau = 1$, the contour-integral formula (9.14) simplifies to

$$\begin{aligned}(\psi_L)_{a_1,\dots,a_n} = \oint\cdots\oint \prod_{k=1}^{n}\frac{\mathrm{d}u_k}{2\pi\mathrm{i}}\prod_{k=1}^{n}\frac{(1+xu_k)(1+u_k+u_k^2)^{L-2n}}{u_k^{a_k}}\\ \times \prod_{1\leqslant i\leqslant j\leqslant n}(1-u_iu_j)\prod_{1\leqslant i<j\leqslant n}(u_j-u_i)(1+u_j+u_iu_j)(1+u_i+u_j).\end{aligned} \tag{9.22}$$

If $a_i = i$ for $i = 1,\dots,m$, then the integrations with respect to $u_1,\dots,u_m$ are trivial. We find

$$\begin{aligned}&(\psi_L)_{1,\dots,m,a_{m+1},\dots,a_n}\\ &= \oint\cdots\oint \prod_{k=m+1}^{n}\frac{\mathrm{d}u_k}{2\pi\mathrm{i}}\frac{(1+xu_k)(1+u_k)^{2m}(1+u_k+u_k^2)^{L-2n}}{u_k^{a_k-m}}\\ &\times \prod_{m+1\leqslant i\leqslant j\leqslant n}(1-u_iu_j)\prod_{m+1\leqslant i<j\leqslant n}(u_j-u_i)(1+u_j+u_iu_j)(1+u_i+u_j).\end{aligned} \tag{9.23}$$

The right-hand side is a polynomial in $x$ of degree is at most $n - m$. □

More generally, we observe that the coefficients of the polynomials in $x$ and $\tau$ are non-negative but do not have a proof of this statement.

**Parity.** Next, we consider the vector's behaviour under a parity transformation $\mathcal{C}$, for $\tau = 1$. To express the action of the parity operator on the ground-state vector, we stress its dependence of $x$ by writing $|\psi_L\rangle = |\psi_L(x)\rangle$.

**Theorem 9.2.4.** *For $\tau = 1$, we have $\mathcal{C}|\psi_L(x)\rangle = x^n|\psi_L(x^{-1})\rangle$.*

*Proof.* For $\tau = 1$, we find by comparison of (9.14) and (9.16) the relation

$$(\psi_L)_{L+1-a_n,\ldots,L+1-a_1}(x) = x^n(\psi_L)_{a_1,\ldots,a_n}(x^{-1}). \tag{9.24}$$

It is equivalent to $\mathcal{C}|\psi_L(x)\rangle = x^n|\psi_L(x^{-1})\rangle$. □

## 9.3 Scalar products

In this section, we consider scalar products involving the vector $|\psi_L\rangle$. We investigate the overlap of $|\psi_L\rangle$ with a tensor product of states belonging to $V^2$, as well as the sum of its components.

Let us introduce the co-vector $\langle\xi(\alpha)| = \langle\uparrow\downarrow| + \alpha\langle\downarrow\uparrow|$, where $\alpha$ is a complex number. We define the scalar products

$$\begin{aligned} F_{2n} &= \big(\langle\xi(\alpha)| \otimes \cdots \otimes \langle\xi(\alpha)|\big)|\psi_{2n}\rangle, \\ F_{2n+1} &= \big(\langle\uparrow| \otimes \langle\xi(\alpha)| \otimes \cdots \otimes \langle\xi(\alpha)|\big)|\psi_{2n+1}\rangle. \end{aligned} \tag{9.25}$$

The scalar products $F_L$ are polynomials in $\alpha$ and $x$ of degree at most $n$. The following theorem gives closed-form expressions for $F_L$:

**Theorem 9.3.1.** *For each $n \geqslant 0$, we have*

$$F_{2n} = \det_{i,j=1}^{n}\left(\alpha x f_{i,j}^{-2} + (\alpha + x) f_{i,j}^{-1} + f_{i,j}^{0}\right). \tag{9.26}$$

*and*

$$F_{2n+1} = \det_{i,j=1}^{n}\left(\alpha x f_{i,j+1}^{0} + (\alpha + x) f_{i,j+1}^{1} + f_{i,j+1}^{2}\right), \tag{9.27}$$

*where*

$$f_{i,j}^{k} = \tau^{2(i-j)+k+1}\sum_{m=0}^{\infty}\binom{i-1}{2(i-j)+m+k+1}\binom{j-1}{m}\tau^{2m}. \tag{9.28}$$

This theorem reveals that the scalar products are symmetric under the exchange of $x$ and $\alpha$. We have not been able to find a simple symmetry of the ground state that could explain this curious property.

The proof follows the lines of [62, 139], and uses an antisymmetriser identity. We recall that the antisymmetriser $\mathcal{A}f$ of a function $f$ of the variables $u_1, \ldots, u_n$ is

$$(\mathcal{A}f)(u_1, \ldots, u_n) = \sum_{\sigma} \operatorname{sgn}\sigma\, f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}). \tag{9.29}$$

Here, the sum runs over all permutations $\sigma$ of $\{1,\dots,n\}$. We use two elementary properties of the antisymmetriser. First, the Vandermonde determinant can be written as an antisymmetriser:

$$\Delta(u_1,\dots,u_n) = \prod_{1\leqslant i<j\leqslant n}(u_j-u_i) = \mathcal{A}\left(\prod_{i=1}^n u_i^{i-1}\right). \tag{9.30}$$

Second, if $f$ and $g$ are functions of $u_1,\dots,u_n$ then we have

$$\begin{aligned}\oint\cdots\oint\prod_{i=1}^n\frac{\mathrm{d}u_i}{2\pi\mathrm{i}}(\mathcal{A}f)(u_1,\dots,u_n)\,g(u_1,\dots,u_n)\\ = \oint\cdots\oint\prod_{i=1}^n\frac{\mathrm{d}u_i}{2\pi\mathrm{i}}f(u_1,\dots,u_n)(\mathcal{A}g)(u_1,\dots,u_n),\end{aligned} \tag{9.31}$$

where the integration contour of each $u_i$ on both sides is a collection of positively-oriented curves around 0, but no other singularity of the integrand.

*Proof of Theorem 9.3.1.* We compute the overlap (9.25) for the vector (9.1) with arbitrary $\tau$. In terms of the components, we obtain

$$F_L = \sum_{\epsilon_1,\dots,\epsilon_n=0,1}\alpha^{\sum_{i=1}^n\epsilon_i}(\psi_L)_{L-2(n-1)-\epsilon_1,L-2(n-2)-\epsilon_2,\dots,L-\epsilon_n}. \tag{9.32}$$

We use the integral formulas (9.16) to rewrite this sum in terms of a contour integral:

$$\begin{aligned}F_L = \oint\cdots\oint\prod_{k=1}^n\frac{\mathrm{d}u_k}{2\pi\mathrm{i}}\frac{(u_k+x)(u_k+\alpha)(1+\tau u_k+u_k^2)^{L-2n}}{u_k^{2k}}\\ \times\prod_{1\leqslant i\leqslant j\leqslant n}(1-u_iu_j)\prod_{1\leqslant i<j\leqslant n}(u_j-u_i)(1+\tau u_j+u_iu_j)(\tau+u_i+u_j).\end{aligned} \tag{9.33}$$

The integrand contains a Vandermonde determinant. We use (9.30) and rewrite it as an antisymmetriser. Using (9.31), we obtain

$$F_L = \oint\cdots\oint\prod_{k=1}^n\frac{\mathrm{d}u_k}{2\pi\mathrm{i}\,u_k}h(u_k)u_k^{k-1}(\tau+u_k)^{k-1}g(u_1,\dots,u_n), \tag{9.34}$$

where

$$h(u) = (u+x)(u+\alpha)(1+\tau u+u^2)^{L-2n}, \tag{9.35}$$

and

$$g(u_1,\ldots,u_n)=\prod_{1\leqslant i\leqslant j\leqslant n}(1-u_iu_j)\mathcal{A}\Big(\prod_{k=1}^{n}u_i^{-2k+1}\prod_{1\leqslant i<j\leqslant n}(1+\tau u_j+u_iu_j)\Big). \quad (9.36)$$

Let us denote by $g(u_1,\ldots,u_n)_{\leqslant 0}$ the polynomial in $u_1^{-1},\ldots,u_n^{-1}$ obtained from $g(u_1,\ldots,u_n)$ by removing all monomials that contain at least one positive power in $u_1,\ldots,u_n$. We have [139]

$$g(u_1,\ldots,u_n)_{\leqslant 0}=\mathcal{A}\left(\prod_{i=1}^{n}u_i^{-i}(\tau+u_i^{-1})^{i-1}\right). \quad (9.37)$$

This identity allows us to apply (9.31) a second time. We obtain

$$F_L=\oint\cdots\oint\prod_{k=1}^{n}\frac{\mathrm{d}u_k}{2\pi\mathrm{i}}h(u_k)u_k^{-k-1}(\tau+u_k^{-1})^{k-1} \times\Delta(u_1(\tau+u_1),\ldots,u_n(\tau+u_n)). \quad (9.38)$$

The Vandermonde determinant in the integrand allows us to rewrite $F_L$ as the determinant of a single contour integral:

$$F_L=\det_{i,j=1}^{n}\left(\oint\frac{\mathrm{d}u}{2\pi\mathrm{i}}h(u)(\tau+u)^{i-1}(\tau+u^{-1})^{j-1}u^{i-j-2}\right). \quad (9.39)$$

We evaluate the contour integral inside the determinant in terms of

$$\begin{aligned}f_{i,j}^{k}&=\oint\frac{\mathrm{d}u}{2\pi\mathrm{i}}(\tau+u)^{i-1}(\tau+u^{-1})^{j-1}u^{i-j+k}\\&=\tau^{2(i-j)+k+1}\sum_{m=0}^{\infty}\binom{i-1}{2(i-j)+m+k+1}\binom{j-1}{m}\tau^{2m}.\end{aligned} \quad (9.40)$$

The sum on the right-hand side is finite. Therefore, $f_{i,j}^{k}$ is a polynomial in $\tau$. In terms of these polynomials, we find for even $L=2n$ the determinant (9.26). If $L=2n+1$ is odd then the evaluation of the contour integral, combined with the identity $f_{i,j}^{k}=f_{i,j+1}^{k+2}-\tau f_{i,j}^{k+1}$ and elementary column operations, yields (9.27). □

The theorem allows us to obtain determinant formulas for the components of $|\psi_L\rangle$ that are labelled by alternating spin configurations. Indeed, we obtain through the specialisations $\alpha = 0$ and $\alpha \to \infty$ the expressions

$$(\psi_{2n})_{\uparrow\downarrow\cdots\uparrow\downarrow} = \det_{i,j=1}^{n} \left(f^0_{i,j} + x f^{-1}_{i,j}\right), \tag{9.41}$$

$$(\psi_{2n+1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow} = \det_{i,j=1}^{n} \left(x f^0_{i,j+1} + f^1_{i,j+1}\right). \tag{9.42}$$

Furthermore, we consider the sum of components

$$S_L(x,\tau) = \sum_{1\leqslant a_1<\cdots<a_n\leqslant L} (\psi_L)_{a_1,\ldots,a_n}. \tag{9.43}$$

Using (9.16), we can evaluate it in terms of the following multiple contour integrals

$$S_L(x,\tau) = \oint \cdots \oint \prod_{k=1}^{n} \frac{\mathrm{d}u_k}{2\pi\mathrm{i}} \frac{(u_k+x)(1+\tau u_k+u_k^2)^{L-2n}}{u_k^{L-n+k}\left(1-\prod_{j=1}^{k} u_j\right)}$$
$$\times \prod_{1\leqslant i\leqslant j\leqslant L}(1-u_iu_j) \prod_{1\leqslant i<j\leqslant n}(u_j-u_i)(\tau+u_i+u_j)(1+\tau u_j+u_iu_j). \tag{9.44}$$

We have not been able to simplify further this formula to express it in terms of a single determinant (as in Theorem 9.3.1) or a pfaffian. In the next section, we conjecture a relation between this sum of components and the enumeration of combinatorial objects.

## 9.4 The XXZ spin chain and combinatorics

In this section, we give the definition of two types of combinatorial objects, namely the plane partitions and the alternating sign matrices. We discuss certain symmetry classes and their enumeration. We show that they are related to certain scalar products that we discussed in the preceding section. In particular, we prove Theorem 7.3.1.

**Plane partitions.** A *plane partition* is a two-dimensional array of integers, $\pi_{i,j}$, $i,j \geqslant 1$, such that each row and column is a non-increasing sequence of integers

$$\pi_{i,j} \geqslant \pi_{i,j+1} \quad \pi_{i,j} \geqslant \pi_{i+1,j}. \tag{9.45}$$

| | | | | | |
|---|---|---|---|---|---|
| 6 | 6 | 6 | 5 | 3 | 3 |
| 6 | 6 | 5 | 3 | 3 | 3 |
| 6 | 6 | 5 | 3 | 3 | 1 |
| 4 | 3 | 3 | 1 | 1 | |
| 4 | 3 | 3 | | | |
| 3 | 2 | 2 | | | |

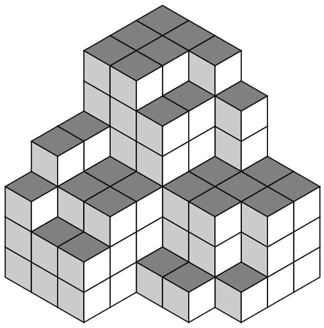


Figure 9.1: A cyclically-symmetric transpose-complement plane partitions in a $6 \times 6 \times 6-$box as an array and as the graphical representation of its diagram.

For each plane partition $\pi$, we can define its diagram (also called Ferrer graph) as the set of integers $D(\pi)$ given by

$$D(\pi) = \{(i, j, k) | \pi_{i,j} \geqslant k\}. \tag{9.46}$$

We say that the plane partition is contained in a $r \times s \times t-$box if $i \leqslant r, j \leqslant s, k \leqslant t$ for each $(i, j, k) \in D(\pi)$.

The diagram allows one to visualise the plane partition as a pile of cubes in the corner of a box. The Figure 9.1 gives an example of a plane partition and its graphical representation. In the following, we identify a diagram with the corresponding plane partition and denote both by $\pi$.

**A class of symmetry.** Various symmetry classes have been considered for plane partitions [142, 53]. Here, we discuss the cyclically-symmetric transpose-complement plane partitions. A plane partition is *cyclically symmetric* if $(i, j, k) \in \pi$ whenever $(j, k, i) \in \pi$.

Furthermore, let $\pi$ be a plane partition contained in a $r \times s \times t-$box. Its complement, $\pi^c$ is the set of points in the box that do not belong to $\pi$:

$$\pi^c = \{(i, j, k) | i \leqslant r, j \leqslant s, k \leqslant t, (r-i-1, s-j-1, t-k-1) \notin \pi\}. \tag{9.47}$$

A plane partition is *transpose-complement* if $\pi^c = \pi^t$, where $\pi^t$ is the plane partition obtained by transposition of $\pi$: $(\pi^t)_{i,j} = \pi_{j,i}$.

We consider *cyclically-symmetric transpose-complement plane partitions* (CSTCPP) that are plane partitions that are both cyclically symmetric

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},$$
$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

Figure 9.2: The seven ASMs of size $3 \times 3$.

and transpose-complement. Due to the symmetry constraints, CSTCPPs only exist in cubic boxes of size $2n$. The plane partition depicted in Figure 9.1 is a CSTCPP. The number of CSTCPPs in a $2n \times 2n \times 2n-$box is denoted by $N_8(2n)$ and is given by [143, 53]

$$N_8(2n) = \prod_{k=0}^{n-1} \frac{(3k+1)(6k)!(2k)!}{(4k)!(4k+1)!}. \tag{9.48}$$

**Alternating sign matrices.** An *alternating sign matrix* (ASM) is a square matrix with entries $-1, 0$ or $+1$, such that each row and column sum equals one and the non-zero entries along each row and column alternate in sign [49, 50, 53]. The Figure 9.2 gives the ASMs of size $3 \times 3$.

We consider ASMs that are invariant under a certain symmetry of the square. There are eight classes of ASMs that are of interest [144, 123]. Here, we focus on two of them.

**Vertically symmetric alternating sign matrices.** The *vertically symmetric alternating sign matrices* are the ASMs that are invariant under a reflection with respect to the vertical symmetry axis. Let $A = (a_{ij})_{i,j=1}^N$ be a vertically symmetric ASM. In terms of its entries, the symmetry reads

$$a_{ij} = a_{i(N+1-j)}. \tag{9.49}$$

We denote by $A_{\mathrm{V}}(N)$ the number of vertically symmetric ASMs of size $N \times N$. (Due to the symmetry, there are no such matrices of even size.)

These numbers are given by [123]

$$A_{\mathrm{V}}(2n+1) = \frac{1}{2^n}\prod_{k=1}^{n}\frac{(6k-2)!(2k-1)!}{(4k-2)!(4k-1)!}. \tag{9.50}$$

Before introducing the second class of ASMs, we revisit the supersymmetric model that we investigate in Chapters 6 and 7 and prove Theorem 7.3.1.

**The supersymmetric point.** We consider the vector $|\psi_L\rangle$ with $x = \tau = 1$. It is a ground state of the supersymmetric Hamiltonian $H$ of Chapter 6. The determinant expressions of Theorem 9.3.1 simplify for $\tau = 1$, we have

$$f_{i,j}^{k}\Big|_{\tau=1} = \binom{i+j-2}{2i-j+k}. \tag{9.51}$$

Our first result concerns the components (9.41) and (9.42). For $x = 1$, we can explicitly evaluate their determinant expressions with the help of Krattenthaler's formula [53]. We obtain

$$(\psi_{2n})_{\uparrow\downarrow\cdots\uparrow\downarrow} = A_{\mathrm{V}}(2n+1), \quad (\psi_{2n+1})_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow} = N_8(2n+2). \tag{9.52}$$

Similarly, the scalar product $F_L$ with $x = 1$ and $\alpha = 1$ is given by

$$(\langle\chi|\otimes\cdots\otimes\langle\chi|)\,|\psi_{2n}\rangle = N_8(2n+2), \tag{9.53}$$
$$(\langle\uparrow|\otimes\langle\chi|\otimes\cdots\otimes\langle\chi|)\,|\psi_{2n+1}\rangle = A_{\mathrm{V}}(2n+3), \tag{9.54}$$

where we used the definition of $|\chi\rangle = |\xi(1)\rangle$.

These results allow us to prove Theorem 7.3.1:

*Proof of Theorem 7.3.1.* The proof is a direct consequence of (9.52) and (9.53), together with Proposition 6.4.6. □

**Totally-symmetric alternating sign matrices.** The other symmetry class of ASM that we consider are the *totally-symmetric alternating sign matrices* (TSASMs), which are ASMs of odd size that are invariant

$$\begin{pmatrix} 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & - & 0 & + & 0 & 0 \\ 0 & + & - & 0 & + & 0 & - & + & 0 \\ 0 & 0 & 0 & + & - & + & 0 & 0 & 0 \\ + & - & + & - & + & - & + & - & + \\ 0 & 0 & 0 & + & - & + & 0 & 0 & 0 \\ 0 & + & - & 0 & + & 0 & - & + & 0 \\ 0 & 0 & + & 0 & - & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 9.3: A totally-symmetric alternating sign matrix of size $9 \times 9$, where $\pm$ represents the non-zero entry $\pm 1$. The horizontal and vertical median are fixed to alternating sequences of $+1$ and $-1$.

under all symmetries of the square [145, 144]. (There are no totally-symmetric alternating sign matrices of even size.) Figure 9.3 shows an example of a TSASM.

Let $A = (a_{ij})_{i,j=1}^{2m+1}$ be a TSASM. The invariance under the symmetries of the square reads

$$a_{ij} = a_{ji} = a_{i(2m+2-j)}. \tag{9.55}$$

In particular, the symmetries imply that the horizontal and vertical medians are alternating sequences of $+1$ and $-1$:

$$a_{mj} = (-1)^{j+1}, \quad a_{im} = (-1)^{i+1}. \tag{9.56}$$

The medians divide the matrix into four equivalent sub-matrices of size $m \times m$.

Due to the symmetries, TSASM's entries are determined by the upper-triangular part of the upper-left sub-matrix:

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1m} \\ & a_{22} & & a_{2m} \\ & & \ddots & \vdots \\ & & & a_{mm}. \end{matrix} \tag{9.57}$$

Let $\mu(A)$ and $\nu(A)$ be the number of non-zero entries along and above its diagonal, respectively. In terms of the sub-matrix' entries, they are

$$
\begin{array}{cccc} 0 & 0 & 0 & 0 \\ & 0 & 0 & + \\ & & 0 & 0 \\ & & & 0 \end{array}
\qquad
\begin{array}{cccc} 0 & 0 & 0 & 0 \\ & + & 0 & 0 \\ & & 0 & 0 \\ & & & + \end{array}
\qquad
\begin{array}{cccc} 0 & 0 & 0 & 0 \\ & 0 & + & 0 \\ & & - & 0 \\ & & & + \end{array}
\qquad
\begin{array}{cccc} 0 & 0 & 0 & 0 \\ & 0 & 0 & + \\ & & + & - \\ & & & + \end{array}
$$

$$
\tau \qquad\qquad\qquad t^2 \qquad\qquad\qquad t^2\tau \qquad\qquad\qquad t^2\tau^2
$$

Figure 9.4: The upper-triangular parts of the upper-left submatrix for each of the four totally-symmetric alternating sign matrices of size $9 \times 9$ ($m = 4$) and their corresponding weights. The corresponding generating function is $A_{\mathrm{TS}}(9; t, \tau) = \tau + t^2(1 + \tau + \tau^2)$.

given by

$$
\mu(A) = \sum_{i=1}^{m} |a_{ii}|, \quad \nu(A) = \sum_{1 \leqslant i < j \leqslant m} |a_{ij}|. \tag{9.58}
$$

We use them to assign the weight $t^{\mu(A)}\tau^{\nu(A)}$ to the matrix $A$. Figure 9.4 displays the triangular parts of all TSASMs of size $9 \times 9$, as well as the corresponding weights. Using the weights, we introduce the following generating function:

$$
A_{\mathrm{TS}}(2m + 1; t, \tau) = \sum_{A \in \mathcal{A}_{\mathrm{TS}}(2m+1)} t^{\mu(A)}\tau^{\nu(A)}. \tag{9.59}
$$

Here, the sum runs over the set $\mathcal{A}_{\mathrm{TS}}(2m + 1)$ of all TSASMs of size $(2m + 1) \times (2m + 1)$. For $t = \tau = 1$, we simply write $A_{\mathrm{TS}}(2m + 1) = A_{\mathrm{TS}}(2m + 1; 1, 1)$. These are the TSASM numbers, given by

$$
A_{\mathrm{TS}}(2m + 1) = 1, 1, 1, 2, 4, 13, 46, 248, 1516, 13654, \dots \tag{9.60}
$$

for $m = 0, \dots, 9$. To our best knowledge, no (product) formula for these numbers is currently known or conjectured.

We note that the specification $t = 0, \tau = 1$ yields the enumeration of TSASMs with no entries on the diagonal, except for the central entry. Namely, these matrices are vertically and off-diagonally symmetric alternating sign matrices, introduced by Okada [146]. Their enumeration and some weighted enumerations are known and are given in terms of products of pfaffians [146, 147].

We have numerically generated the TSASMs for small $m$ with MATHEMATICA and computed the corresponding generating functions $A_{\mathrm{TS}}(2m+$

$1; t, \tau)$. We refer the reader to Appendix A, where we discuss the implementation and computation of TSASMs.

We observe that the generating functions are related to the sums of the components $S_L = S_L(x,\tau)$ discussed above and formulate this relation as the following conjecture:

**Conjecture 9.4.1.** *We have*

$$S_L(x,\tau) = (1 + x(x-\tau))^{\bar{n}/2} A_{\mathrm{TS}}(2L+3; t, \tau), \tag{9.61}$$

*where* $t = (1+x)/(1+x(x-\tau))^{1/2}$.

This conjecture has several interesting consequences. First, it implies that the generating functions for the weighted enumeration of TSASMs introduced in this section can be obtained from the contour-integral formula (9.44). Even though not very practical, it seems to be the first formula related to a TSASM enumeration that appears in the literature. In particular, the TSASM numbers are

$$A_{\mathrm{TS}}(2m+1) = S_{m-1}(0,1), \tag{9.62}$$

and can thus be computed from the contour-integral formula. Second, using Proposition 9.2.2 one easily finds that the sum of the components obeys the relation $S_L(0,\tau) = S_{L-1}(\tau,\tau)$. Hence, we obtain the curious relation

$$A_{\mathrm{TS}}(2L+3; 1, \tau) = A_{\mathrm{TS}}(2L+1; 1+\tau, \tau). \tag{9.63}$$

Finally, specialising the conjecture to $t = \tau = 1$, we obtain the relation $S_L(1,1) = A_{\mathrm{TS}}(2L+3)$. Hence, the sums of the ground-state components of the spin chain at its supersymmetric point $x = 1$ yield the sequence of TSASM numbers. We have numerically verified this corollary for $m = 1, \ldots, 16$.

# Conclusion and outlook

In this final chapter, we briefly recall the results obtained throughout this dissertation. We present an outlook and discuss open problems afterwards.

## Conclusion

In this thesis, we have investigated different finite-size integrable models at particular values of their parameters where they exhibit an additional symmetry beyond their integrability: supersymmetry. We have discussed the supersymmetric XYZ spin chains and related eight vertex models in Chapters $2-5$. The rest of the text focuses on XXZ spin chains. We have explored some of their properties using supersymmetry in Chapters 6 and 7. In the last two chapters, we have studied a solution to the bqKZ equations and we have proved some properties of this solution related to combinatorics.

In the first chapter, we have introduced some concepts pertaining to supersymmetric quantum mechanics. A central definition of Chapter 1 is the supercharge. The anticommutator of the supercharge and its adjoint yields a supersymmetric Hamiltonian. We have shown a few properties of the spectrum and eigenvectors of a supersymmetric Hamiltonian. In particular, we have discussed the correspondence between the space of supersymmetry singlets and the (co)homology of the supercharge and its adjoint.

In Chapter 2, we have studied the XYZ Hamiltonian with periodic boundary conditions and anisotropy parameters obeying (2.1):

$$J_1 J_2 + J_2 J_3 + J_1 J_3 = 0. \tag{9.64}$$

We have shown that the space of the ground-states is two-degenerate and spanned by supersymmetry singlets.

We have proven Stroganov's conjecture [54, 56] about the transfer matrix of the supersymmetric eight-vertex model on the square lattice in Chapter 3. It states that this transfer matrix with $L = 2n + 1$ vertical lines and periodic boundary conditions possesses the doubly degenerate eigenvalue $\Theta_n = (a + b)^{2n+1}$. The corresponding eigenvectors are the supersymmetry singlets characterised in the preceding chapter.

The Chapter 4 has focused on the supersymmetric Hamiltonian of an XYZ spin chain with open boundary conditions. We showed that if the parameters of the Hamiltonian are carefully adjusted, then its ground states are supersymmetry singlets.

We have investigated the corresponding supersymmetric eight-vertex model on a strip with $L$ vertical lines in Chapter 5. We have shown that the space of supersymmetry singlets is an eigenspace of the transfer matrix. We have computed the corresponding non-degenerate eigenvalue: $\Lambda_L = (a + b)^{2L}\mathrm{tr}(K^+K^-)$.

In Chapter 6, we have revisited the spin chains in the XXZ case. First, for the open spin chain, we have identified a family of supercharges that yield non-diagonal boundary interactions. We have computed their cohomology and characterised the space of zero-energy states. Second, we have discussed the supersymmetry singlets of the periodic spin-chain Hamiltonian using a relation with the supercharge of the open spin chain.

The Chapter 7 has revealed that we can compute a large family of scalar products involving an arbitrary number of normalised supersymmetry singlets in terms of certain distinguished (sum of) components. We have evaluated these scalar products in the large-system-size limit.

The evaluation of fidelities are based on combinatorial expressions for certain components of the supersymmetry singlets. We have proved these combinatorial properties in the last two chapters by investigating the integrability of the XXZ chain.

We have studied the boundary quantum Knizhnik-Zamolodchikov equations for the $R$-matrix of the six-vertex model and a diagonal $K$-matrix in Chapter 8. We have found a solution in terms of multiple contour integrals, proved some of its properties and shown its relation with the transfer matrix of the corresponding inhomogeneous six-vertex model.

Finally, in Chapter 9, we have investigated the properties of our solution to bqKZ equations in the homogeneous limit. In particular, we have explicitly computed linear sum rules and special components in terms of determinants and discussed relations with the enumeration of combinatorial objects.

## Outlook

The topics treated in this thesis have natural generalisations. Furthermore, there are various questions and conjectures that our results lead us to formulate. Here, we present a few open problems that are of interest to better understand the supersymmetric models that we investigated, as well as their relation with integrability and combinatorics. We also suggest strategies and ideas to address these questions.

**Inhomogeneous transfer-matrix eigenvalue.** The Theorem 3.4.1 proves a twenty-years-old conjecture made by Stroganov on the existence of the special eigenvalue $\Theta_n$ of the transfer matrix. One may consider the transfer matrix of the inhomogeneous eight-vertex model with vertex weights that locally fulfil (3.28), corresponding to $\eta = \pi/3$. In the case of periodic boundary conditions with $L = 2n+1$, the transfer matrix is

$$\mathcal{T}(u|u_1,\dots,u_L) = \operatorname{tr}_0\left(R_{0L}(u-u_L)\cdots R_{01}(u-u_1)\right), \tag{9.65}$$

where $u_1,\dots,u_L$ are inhomogeneity parameters. Razumov and Stroganov conjectured that $\mathcal{T}(u|u_1,\dots,u_L)$ possesses the following doubly degenerate eigenvalue [57]:

$$\Theta_n(u_1,\dots,u_L) = \prod_{j=1}^{2n+1}\left(a(u-u_j)+b(u-u_j)\right). \tag{9.66}$$

This formula reduces to $\Theta_n = (a+b)^{2n+1}$ in the homogeneous limit.

As for the eight-vertex model on a strip with $L \geqslant 1$ vertical lines, we give a conjecture that generalises the transfer-matrix eigenvalue to the inhomogeneous model. Its transfer matrix is

$$\mathcal{T}(u|u_1,\dots,u_L) = \mathrm{tr}_0\left(K_0^+(u)U_{0,[1,L]}(u|u_1,\dots,u_L)\cdot \right. \\ \left. K_0^-(u)\bar{U}_{0,[1,L]}(u|u_1,\dots,u_L)\right), \quad (9.67)$$

where $K^-(u) = K(u)$ and $K^+(u) = K(u+2\eta)$, and

$$\begin{aligned} U_{0,[1,L]}(u|u_1,\dots,u_L) &= R_{0L}(u+u_L)\cdots R_{01}(u+u_1), \\ \bar{U}_{0,[1,L]}(u|u_1,\dots,u_L) &= R_{01}(u-u_1)\cdots R_{0L}(u-u_L). \end{aligned} \quad (9.68)$$

**Conjecture 9.4.2.** *Let $\eta = \pi/3$, $K(u)$ be the $K$-matrix (5.9) with the coefficients (5.21), evaluated at $t = \pi/6$, then the transfer matrix (9.67) possesses the eigenvalue*

$$\Lambda_L(u|u_1,\dots u_L) = \mathrm{tr}(K^+(u)K^-(u))\cdot \\ \prod_{j=1}^{L}\left(a(u+u_j)+b(u+u_j)\right)\left(a(u-u_j)+b(u-u_j)\right). \quad (9.69)$$

This conjecture holds in the trigonometric limit $p \to 0$, as a consequence of Proposition 8.7.2. Furthermore, we checked that it is compatible with functional equations obeyed by the transfer matrix [40], and simplifications that occur for certain specialisations of the spectral parameter $u$.

Both these conjectures remain to be proven. Their proof is of interest since the inhomogeneous models allow one to determine interesting properties of the corresponding eigenvectors. For periodic boundary conditions, Zinn-Justin initiated this rigorous investigation in [88].

**Supersymmetry and quantum groups.** The proofs of the Theorems 3.4.1 and 5.3.3, stating the existence of the eigenvalues $\Theta_n$ and $\Lambda_L$, both rely on a commutation relation between the corresponding transfer matrix and supercharge. These relations are based on the relation (3.31) between the $R$-matrix and the local supercharge. (And, additionally for the open boundary case, on the relations (5.38), (5.39) between $\mathfrak{q}$ and the $R$ and $K$-matrices.)

As the Yang-Baxter equation (and the boundary Yang-Baxter equation) are the cornerstone of the construction of commuting transfer matrices, it seems that the relations (3.31) (and (5.38), (5.39)) may be the key for a better understanding of the connection between the supersymmetry and quantum integrability.

In the trigonometric case, the $R$-matrix can be understood in the framework of quasi-triangular Hopf algebras, also known as quantum groups, in which the Yang-Baxter equation arises naturally. As an example, in our case, the Hopf algebra is built upon $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$, a deformation of the universal enveloping of the affine algebra $\widehat{\mathfrak{sl}}_2$; and a representation yields the trigonometric $R$-matrix (8.11).

An open question is the algebraic origin of the supercharge in the context of quantum groups. We have dedicated our first investigations to the behaviour of the supercharge under the action of $\mathcal{U}_q(\mathfrak{sl}_2)$. This work, which yielded promising preliminary results, is in progress.

**Fidelity and CFT.** In Chapter 7, we have considered multipartite fidelities and have expressed them in terms of simple components or sum of components using the supersymmetry. We have computed the scaling behaviour of these quantities. Here, we briefly discuss the connection with conformal field theory (CFT) that we mentioned.

For the sake of simplicity, we focus here on the *logarithmic bipartite fidelity* (LBF) where we divide the system into two parts [118, 119]. The LBF for spin chains is given in terms of scalar products:

$$\mathcal{F} = -\ln\left|\frac{\langle\psi_L|\,(|\psi_{L_1}\rangle\otimes|\psi_{L_2}\rangle)}{\|\psi_L\|\,\|\psi_{L_1}\|\,\|\psi_{L_2}\|}\right|^2, \tag{9.70}$$

with $L = L_1 + L_2$. In this dissertation, we explored different variations of $\mathcal{F}$ involving the ground states of periodic and open spin chains.

In these cases, one can equivalently express the LBF in terms of partition functions $\mathcal{Z}$ of the corresponding two-dimensional six-vertex models. It reads

$$\mathcal{F} = \lim_{N\to\infty} -\ln\left|\frac{(\mathcal{Z}_{1,2})^2}{\mathcal{Z}_1\mathcal{Z}_2\mathcal{Z}_{1\cup 2}}\right|. \tag{9.71}$$

Here, the partition functions are defined on a square lattice with $N$ rows. The number of columns ($L_1$, $L_2$ or $L$) and the boundary conditions of each partition function depend on the corresponding scalar product. As examples, $\|\Psi'_L\|$ and $\|\Psi_L\|$ correspond to partition functions on a strip of length $L$ and a cylinder of perimeter $L$, respectively.

In the large-$L$ limit, and with the ration $x = L_1/L$ kept fixed, we express $\mathcal{F}$ as a linear combination of free energies. We have

$$\mathcal{F} = 2f_{1,2} - f_1 - f_2 - f_{1\cup 2}. \tag{9.72}$$

Dubail and Stéphan argued that the leading terms of the asymptotic expansion of this quantity have a universal behaviour. They computed the LBF in the case where all the subsystems have open boundary conditions [118, 119]. Morin-Duchesne *et al.* adapted their strategy to compute fidelities for models with periodic boundary conditions [5].

These results are universal in the sense that they are obtained by making little assumptions on the CFT. The formulas for $\mathcal{F}$ depend on the considered geometry, as well as the central charge and the dimension of the involved fields.

Hence, we can particularise those conformal predictions to the XXZ chain. Indeed, it is well-known that the XXZ chain with anisotropy parameter $-1 \leqslant \Delta \leqslant 1$ is described by a CFT with central charge $c = 1$ [148].

In [1], we have compared the CFT result of Dubail and Stéphan with our lattice derivation of $Z'(L_1, L_2)$ in the scaling limit. We observed a perfect matching between its scaling behaviour and the predictions from CFT at both leading and sub-leading orders. The results obtained in Chapter 7 also coincide with the CFT calculation of [5].

We have obtained the CFT result that corresponds to the fidelity $Z'(L_1, \dots, L_m)$ [149]. However, CFT predictions for generic multipartite fidelities are still lacking and would be of great interest.

**Deviation from supersymmetry.** The Hamiltonian (8.1), which depends on $x$, is not supersymmetric for $x \neq 1$. Nevertheless, it exhibits features that are akin to a supersymmetric structure. We discuss here observations related to the spectrum and multipartite fidelities. Afterwards, we propose an algebraic framework to prove the various conjectures.

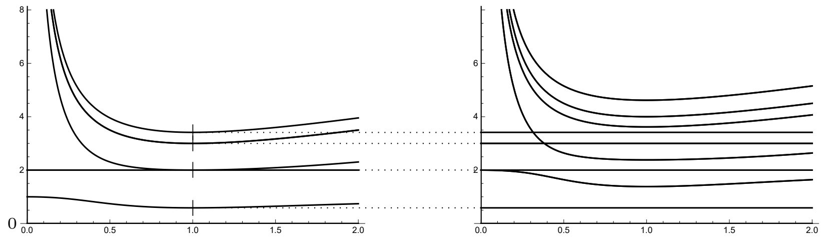


Figure 9.5: Spectrum of the Hamiltonian $H - E_0$, where $H$ is the Hamiltonian (8.1) with (8.2), for $L = 3$ (left panel) and $L = 4$ (right panel), as a function of $x$. The eigenvalues depending on $x$ for $L = 3$ are present in the spectrum for $L = 4$, evaluated at $x = 1$.

The first observations concern the spectrum of the Hamiltonian (8.1). Our numerical analysis suggests that Theorem 9.1.1 holds without the restriction on the sector of magnetisation:

**Conjecture 9.4.3.** *For real $x > 0$, the eigenvalue* (9.7) *is the non-degenerate ground-state eigenvalue of the Hamiltonian* (8.1)*.*

Moreover, the spectrum reveals that (9.7) is not the ground-state eigenvalue for $x < 0$.

The rest of the spectrum has another interesting property. For each $L$ and $x > 0$, the spectrum decomposes into a set of constants with respect to $x$ and a set of functions of $x$. We have observed that, for each $L$, the constant part of the spectrum consists of the ground-state eigenvalue and the set of functions present in the spectrum for $L - 1$, evaluated at $x = 1$. The Figure 9.5 illustrates this phenomenon for $L = 3$ and 4. We numerically checked that this property holds up to $L = 9$.

This observation suggests that, for generic $x$, there exists a doublets structure: $\{|\psi\rangle \in V^L, |\psi'\rangle \in V^{L+1}\}$ such that

$$H|\psi\rangle = f(x)|\psi\rangle, \quad H|\psi'\rangle = f(1)|\psi'\rangle. \tag{9.73}$$

Restricted to $x = 1$, this property is a direct consequence of supersymmetry, as explained in Chapter 1: We have $|\psi'\rangle = \mathfrak{Q}|\psi\rangle$.

The second observation relates to multipartite fidelities. To formulate it, we introduce the following scalar products between ground states of the

Hamiltonian (8.1):

$$O_{L_1,\ldots,L_m} = \langle \psi_L^* | \left( |\psi_{L_1}\rangle \otimes \cdots \otimes |\psi_{L_m}\rangle \right), \tag{9.74}$$

where $L = L_1 + \cdots + L_m$. Here, $\langle \psi_L^* | = |\psi_L\rangle^t$ is defined by transposition (without complex conjugation).

We have found the scalar products for up to $L = 12$ sites through the exact computation of the ground-state vector with MATHEMATICA. Our results suggest the following conjecture:

**Conjecture 9.4.4.** *Let $L_1, \ldots, L_m$ be integers and $L = L_1 + \cdots + L_m$. If $L_1, \ldots, L_m$ are even or if $L_k$ is odd for $1 \leqslant k \leqslant m$ and $L_1, \ldots, L_{k-1}, L_{k+1} \ldots, L_m$ are even, then we have the scalar product*

$$O_{L_1,\ldots,L_m} = x^n F_L(x, x^{-1}) \prod_{j=1}^{m} F_{L_j}(1,0), \tag{9.75}$$

*where we wrote $F_L(x,\alpha)$ for $F_L$ defined in (9.25) to stress its dependence on $x$ and $\alpha$. In all other cases, the scalar product vanishes due to the magnetisation of the vectors.*

This factorisation of $O_{L_1,\ldots,L_m}$ into a product of determinants is extraordinary simple. It generalises the result for open spin chains obtained in Chapter 7, for $x = 1$. For arbitrary $x$, the proof of the conjecture and the evaluation of the behaviour of the fidelity in the limit $L \to \infty$ remain interesting open problems.

The two preceding observations and conjectures have been shown to hold for $x = 1$. In that case, the proofs rely almost entirely on the supersymmetry present at this point. Therefore, a possible strategy to understand these properties for generic $x$ is to use a *deformed* version of the supersymmetry. This approach is based on the following observation: there exist operators $\mathfrak{Q}(x)$ such that the Hamiltonian (8.1), which depends on $x$, reads

$$H = \mathfrak{Q}(x)\mathfrak{Q}(x)^t + \mathfrak{Q}(x)^t\mathfrak{Q}(x) - \Gamma\mathbf{1}. \tag{9.76}$$

Here, $\Gamma$ is a function of $x$, the superscript $t$ denotes the transposition (without complex conjugation), and the operator $\mathfrak{Q}(x)$ is a deformation of the supercharge. An example of an operator $\mathfrak{Q}(x)$ is given by $\mathfrak{Q}(x) = \mathfrak{Q}' + \mathfrak{Q}^g$ where

$$\mathfrak{Q}^g|\psi\rangle = \sqrt{x-1}|{\downarrow}\rangle \otimes |\psi\rangle + (-1)^L\sqrt{x^{-1}-1}|\psi\rangle \otimes |{\downarrow}\rangle. \tag{9.77}$$

This choice leads to $\Gamma = (3L-1)/4 + 3(x-1)^2/(2x)$. The relation (9.76) is akin to the definition of a supersymmetric Hamiltonian. We note that $\mathfrak{Q}(x)$ is not a supercharge as it is not nilpotent (except for $x = 1$ in which case we recover the supercharge $\mathfrak{Q}'$ of Chapter 6). Nevertheless, this observation is promising to investigate the structure of the spectrum and eigenvectors, away from the supersymmetric point.

**Totally-symmetric alternating sign matrices.** The enumeration of TSASMs is arguably the most striking missing part of the puzzle of the alternating sign matrices. Therefore, a proof of Conjecture 9.4.1, relating the vector $|\psi_L\rangle$ and a weighted enumeration of TSASMs, is of great interest. We commented on the importance and some consequences of this conjecture, as well as its relations with other combinatorial problems in Chapter 9. Here, we discuss a strategy for proving it.

A possible approach consists in constructing multivariable Laurent polynomials $P_1$, $P_2$ such that the homogeneous limits yield the sum of the components of $|\psi_L\rangle$ and the enumeration of TSASMs, respectively. In the case of $|\psi_L\rangle$, a candidate for $L = 2n$ is

$$P_1 = (\langle\chi(x_1)| \otimes \cdots \otimes \langle\chi(x_n)|)|\Psi_L(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\rangle. \tag{9.78}$$

(A similar candidate exists for odd $L$.) In this equation, $|\chi(x)\rangle$ is a solution to the boundary Yang-Baxter equation in its vector form [150]. As for $P_2$, one can construct it using the connection between TSASMs and the six-vertex model with domain-wall boundary conditions, as explained in Appendix A. The proof of Conjecture 9.4.1 amounts to showing that these polynomials are equal. We have started working on this project, which has not been finalised yet.

The next step would be to express the formula for the sum of the components in a simpler form. Examples from the literature suggest that it could be expressed in terms of determinants or pfaffians [146, 147]. Whether such an expression exists is itself an interesting question. So far, no formula for the enumeration of the TSASMs has been conjectured. We hope that our results and observations will allow for the unravelling of this mystery.

# Appendix A

# TSASM Enumeration

We based Conjecture 9.4.1, related to the sum of components of $|\psi_L\rangle$, on the construction and observation of totally-symmetric alternating sign matrices (TSASMs) of small sizes. The goal of this appendix is to provide details about the implementation of this problem.

The layout of this appendix is as follows. In Section A.1, we discuss the equivalence between the alternating sign matrices and configurations of the six-vertex model on a square lattice with so-called domain-wall boundary conditions. We investigate this equivalence in the case of TSASMs in Section A.2. It allows us to express the weighted enumeration $A_{\text{TS}}(2m+1, t, \tau)$ of Chapter 9 as a matrix element of a specific operator. We discuss the implementation of this computation on Mathematica in Section A.3.

## A.1 Equivalence with the six-vertex model

The construction of the TSASMs is based on the equivalence between alternating sign matrices and configurations of the six-vertex model with specific boundary conditions, as first exploited by Kuperberg [52]. We recall that the vertices are of six types, as labelled on Figure A.1.

The equivalence is as follows. We consider the six-vertex model on a $N \times N$ square lattice with *domain-wall boundary conditions*. These boundary conditions consist of arrows on the left and right boundaries of

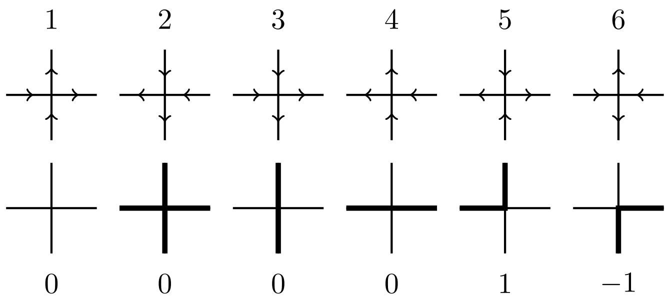



Figure A.1: The configurations of the six-vertex model in terms of arrows and in terms of lines, labelled from 1 to 6, and the corresponding entry in the ASM.

the square domain pointing outward and arrows on the top and bottom pointing inward. The Figure A.2 depicts a configuration of the six-vertex model on a $3 \times 3$ square lattice with domain-wall boundary conditions.

We create a matrix $A = (a_{ij})_{i,j=1}^{N}$ such that $a_{ij} = 1, -1$ or $0$ if the vertex in the $i$-th row and $j$-th column of the square lattice is a vertex of type 5, 6 or different from those two values, respectively. The domain-wall boundary conditions ensure that the matrix obtained is an ASM and that the mapping is a bijection. (Hence, one can construct a unique configuration of the vertex model, given an ASM.)

The partition function of the six-vertex model with domain-wall boundary conditions is a matrix element of a product of $R$-matrices acting on a tensor product of $V$. We refer the reader to Chapter 3 for an explanation of the correspondence between arrows and spins configurations. The $R$-matrix is an operator acting on $V \otimes V$. It encodes the weights $w_1, \ldots, w_6$ of the configurations. In the canonical basis $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ of $V^2$, it reads

$$R = \begin{pmatrix} w_1 & 0 & 0 & 0 \\ 0 & w_3 & w_5 & 0 \\ 0 & w_6 & w_4 & 0 \\ 0 & 0 & 0 & w_2 \end{pmatrix}. \tag{A.1}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

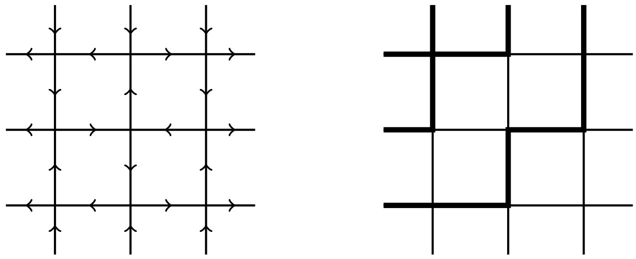


Figure A.2: From left to right, a $3 \times 3$ (totally symmetric) ASM, the corresponding six-vertex model configuration in terms of arrows, the same configuration in terms of lines.

## A.2 Symmetry classes of ASM and six-vertex model

In the following, we consider the partition function of the six-vertex model corresponding to ASM in a particular symmetry class.

**Symmetry class and vertex restrictions.** If an ASM belongs to a symmetry class, the corresponding six-vertex model configuration does not necessarily have the same symmetry. As an example, the matrix depicted in Figure A.2 possesses the quarter-turn symmetry while the vertex configuration does not. However, the symmetry of the ASM may give constraints on the configuration of the vertices. The vertical symmetry gives a trivial example: it imposes entries $\pm 1$ on the vertical median of the ASM, which implies that the corresponding vertices are of type 5 or 6.

The diagonal symmetry imposes a more subtle constraint: it forbids vertices in the $i$-th row and $i$-th column, $i = 1, \ldots, N$, to be of type 3 or 4 (we refer to those as vertices *on the diagonal*). Furthermore, vertices on the anti-diagonal cannot be of type 1 or 2. This restriction is difficult to understand from the arrow picture. We rather draw the vertex configurations in terms of *lines*, which we obtain by drawing a thick line on each arrow pointing down or to the left. The Figure A.1 depicts this mapping, and the Figure A.2 gives an example of a vertex configuration in terms of lines.

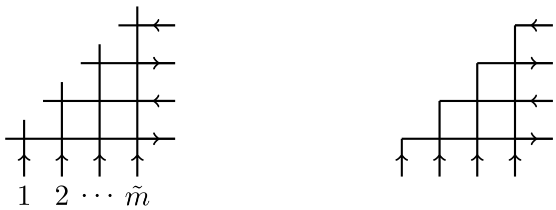



Figure A.3: Left panel: a fundamental domain of the six-vertex model corresponding to an TSASM of size $(2\tilde{m}+3)\times(2\tilde{m}+3)$. In the picture, $\tilde{m}=4$ . Right panel: the same fundamental domain with vertices on the anti-diagonal replaced by boundary $K$ matrices.

In this line picture, a configuration of the six-vertex model corresponding to a TSASM is invariant under the diagonal symmetry. Hence, the only vertices allowed on the diagonal are those invariant under the diagonal symmetry, namely vertices of type $1, 2, 5$ and $6$. The restriction of the vertices on the anti-diagonal to the vertices of type $3, 4, 5$ and $6$ follows a similar argument.

**Fundamental domain.** To study the TSASMs, it is sufficient to consider one octant of the matrix, which has the shape of a triangle, as explained in Chapter 9. The matrix entries in an octant completely determine the TSASM.

Accordingly, we restrict the square lattice of the six-vertex model to a triangular domain. We refer to this domain as the *fundamental domain*. For convenience, we choose the sixth octant which contains the vertices in the $i$-th row and $j$-th column, such that

$$1 \leqslant j \leqslant m, \quad N+1-j \leqslant i \leqslant N. \tag{A.2}$$

The spins in the fundamental domain completely fix the square lattice configuration. We can further restrict the fundamental domain to the vertices satisfying

$$1 < j \leqslant m, \quad N+1-j \leqslant i < N. \tag{A.3}$$

Here, we removed from the square lattice the $N$-th row, as the configuration of its vertices are fixed by the TSASM's symmetries. We note $\tilde{m}=m-1$ the number of columns of the fundamental domain.

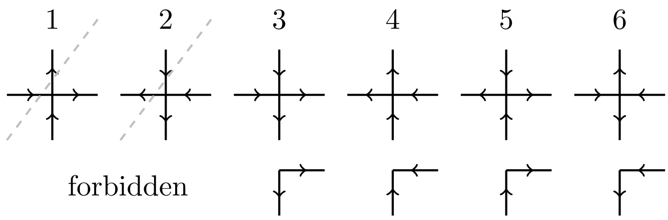



Figure A.4: Configurations of a vertex on the anti-diagonal and the corresponding corner vertex, with weights encoded in the matrix $K$. The vertices of type 1 and 2 are forbidden by the symmetry of the corresponding ASM.

Figure A.3 illustrates the restriction of the $11 \times 11$ square lattice to (one of) its fundamental domain.

The boundary conditions of the fundamental domain are arrows pointing north on the bottom boundary, and an alternating sequence of arrows pointing right and left (starting from the bottom) on the vertical boundary. This condition is a consequence of the vertices being of type 5 and 6 on the vertical median.

We note that a configuration of the six-vertex model in the fundamental domain with $\tilde{m}$ columns and with these boundary conditions corresponds to a TSASM of size $2\tilde{m}+3 \times 2\tilde{m}+3$.

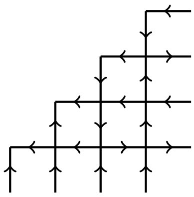
We now focus on the vertices on the anti-diagonal. They can be in four possible configurations. Each of these configurations is entirely determined by the two arrows below and at the right of the vertex (those arrows lie below the anti-diagonal), as can be seen on Figure A.4. Hence, it is sufficient to consider these arrows. We encode the weight of the corresponding vertex in a $2 \times 2$ matrix that we denote by $K$:

$$K = (\langle v| \otimes \mathbf{1}) R (|v\rangle \otimes \mathbf{1}) \tag{A.4}$$

where $|v\rangle = |\uparrow\rangle + |\downarrow\rangle$. In terms of the weights, the $K$ matrix reads

$$K = \begin{pmatrix} w_5 & w_3 \\ w_4 & w_6 \end{pmatrix}. \tag{A.5}$$

It is simpler to see $K$ as an operator acting on a single Hilbert space as a boundary matrix. The right panel of Figure A.3 shows a fundamental



$$
\begin{array}{cccccc}
 & & & & & -1 \\
 & & & & -1 & 1 \\
 & & & -1 & 1 & -1 \\
 & & 0 & 0 & 0 & 1 \\
 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Figure A.5: Left panel: a configuration of the six-vertex model on the fundamental domain with $\tilde{m} = 4$. Right panel: the sixth octant of the corresponding TSASM of size $11 \times 11$. The grey entries are fixed by the symmetries of the TSASM and the corresponding vertex are not present in the fundamental domain.

domain with these boundary $K$ matrices. Furthermore, the Figure A.5 gives an example of a configuration for $\tilde{m} = 4$ and (an octant of) the corresponding $11 \times 11$ TSASM.

**Partition function.** We compute the partition function of the six-vertex model on the triangular fundamental domain by evaluating the matrix element

$$\langle \uparrow\downarrow\uparrow\downarrow \cdots | \overset{\curvearrowleft}{\prod_{1 \leqslant j \leqslant \tilde{m}}} \big(K_j \overset{\curvearrowleft}{\prod_{1 \leqslant i < j}} R_{ij}\big) | \uparrow\uparrow \cdots \uparrow \rangle. \tag{A.6}$$

We obtain the enumeration of TSASM by setting $w_1 = \cdots = w_6 = 1$. Our weighted enumeration assigns a different weight to the vertices of type 5, 6 on the diagonal. We can obtain this by modifying the $K$ matrix to

$$\tilde{K} = \begin{pmatrix} \tilde{w}_5 & w_3 \\ w_4 & \tilde{w}_6 \end{pmatrix}. \tag{A.7}$$

In order to obtain the weighted enumeration of TSASMs, as discussed in Chapter 9, we take the weights as follows:

$$w_1 = w_2 = w_3 = w_4 = 1, \quad \tilde{w}_5 = \tilde{w}_6 = t, \quad w_5 = w_6 = \tau. \tag{A.8}$$

## A.3 Implementation

In this section, we implement the computation of the matrix element

$$\langle\uparrow\downarrow\uparrow\downarrow\cdots|\overset{\curvearrowleft}{\prod_{1\leqslant j\leqslant\tilde{m}}}(\tilde{K}_j\overset{\curvearrowleft}{\prod_{1\leqslant i<j}}R_{ij})|\uparrow\uparrow\cdots\uparrow\rangle, \tag{A.9}$$

with the parameterisation (A.8).

**Naive approach.** A (naive) way to compute (A.9) is to define the operators $R$ and $K$ as sparse matrices, and directly evaluate their product. This is done by the following code:

Definitions of spare matrices and initialisation

```
Clear["Global`*"]
x__⊗y___ := KroneckerProduct[x, y]
e1 = SparseArray[{{1, 2} -> 1}, {2, 2}];
e2 = SparseArray[{{2, 1} -> 1}, {2, 2}];
id[L_] := IdentityMatrix[2^L, SparseArray]
```

Definition of the operators

```
R[L_,i_,j_,tau_:1] := KroneckerProduct @@
    {id[i-1],id[j-i+1] + tau e1⊗id[j-i-1]⊗e2 +
    tau e2⊗id[j-i-1]⊗e1,id[L-j]};
K[L_,i_,t_:1] := KroneckerProduct @@
      {id[i - 1], {{t, 1}, {1, t}}, id[L - i]};
w[1] = {1,0}; w[2] = {0,1,0,0};
w[L_] := Flatten[w[2]⊗w[L - 2]];
```

Actual computation

```
TSASM[L_,t_:1,tau_:1] := w[L].K[L,L,t]. Dot@@Table[
      Dot@@Table[R[L,i,j+1,tau], {i,j,1,-1}].
      K[L,j,t],{j,L-1,1,-1}].UnitVector[2^L,1]
```

**Optimisation.** The above code is explicit but not optimal. It allows[1] us to compute $A_{\mathrm{TS}}(2m+1,t,\tau)$ up to $m=8$ and $A_{\mathrm{TS}}(2m+1,1,1)$ for

[1] All the computations have been done under 60 seconds on a intel(R) Core (TM) i7-67000HQ CPU @ 2.6 GHz

$m = 1, \ldots, 12$. We optimise it as follows: (i) we recursively construct the two states of the scalar product (A.9), (ii) we use memoisation (the storage of intermediate results of the computation).

```
w[1] = {1,0}; w[2] = {0,1,0,0};
w[L_] := w[L] = Flatten[w[2]⊗w[L-2]]
v[1] = {1,1}; v[L_] := v[L] =
    K[L,L,t].Dot@@Table[R[L,i,L,tau],{i,L-1,1,-1}].
    Flatten[v[L-1]⊗{1,0}];
TSASM[L_] := w[L].v[L]
```

This code is particularly efficient for the computation of the non-weighted enumeration. It allows us to evaluate $A_{\mathrm{TS}}(2m+1, 1, 1)$ up to $m = 19$ (this corresponds to TSASM of size $L = 39$) in reasonable time. Furthermore, this computation provides more terms to the sequence of enumerating TSASMs available on the On-Line Encyclopedia of Integer Sequences (limited to $m \leqslant 13$) [151]. We list the number of TSASMs in the Table A.1.

| L | $A_{\mathrm{TS}}(2m+1,1,1)$ |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 2 | 1 |
| 3 | 2 |
| 4 | 4 |
| 5 | 13 |
| 6 | 46 |
| 7 | 248 |
| 8 | 1516 |
| 9 | 13654 |
| 10 | 142873 |
| 11 | 2156888 |
| 12 | 38456356 |
| 13 | 974936056 |
| 14 | 29540545024 |
| 15 | 1259111024288 |
| 16 | 64726478396896 |
| 17 | 4641989615977216 |
| 18 | 404396533544588344 |
| 19 | 48825344233129714772 |

Table A.1: Number of TSASM of size $2m+1 \times 2m+1$.

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