Université catholique de Louvain

Selected Topics in Goursat Categories

Idriss Tchoffo Nguefeu

Dissertation submitted in partial fulfillment of the requirements
for the degree of Docteur en Sciences

Examination committee:

Prof. Marino Gran (UCLouvain, supervisor)
Prof. Diana Rodelo (University of Algarve, supervisor)
Dr. Pierre-Alain Jacqmin (UCLouvain, secretary)
Prof. Andrea Montoli (University of Milan)
Prof. Manuela Sobral (University of Coimbra)
Prof. Michel Willem (UCLouvain, chairperson)

August 2020

Dedications

To my mother, Antoinette Magni

and

to the loving memory of my father, Daniel Nguefeu.

"Listen, my son, to your father's instruction
and do not forsake your mother's teaching.
They are a garland to grace your head
and a chain to adorn your neck."
Proverbs 1: 8-9

Remerciements

Tout d'abord, je tiens à exprimer mes profonds et sincères remerciements à mes promoteurs de thèse, Marino Gran et Diana Rodelo, pour la qualité de leur supervision, pour leur grande disponibilité et pour leur soutien depuis le début jusqu'à la fin de cette thèse.

Sans me connaître personnellement, Marino Gran m'a fait confiance en me donnant l'opportunité de quitter Yaoundé pour Louvain-la-Neuve afin d'effectuer cette thèse sous sa direction. Je lui suis très reconnaissant pour ses qualités pédagogiques et scientifiques, sa franchise, sa sympathie et sa disponibilité malgré son emploi de temps très souvent chargé. J'ai beaucoup appris à ses côtés tant sur le plan scientifique que professionnel et humain. Marino, infiniment je te dis merci... Grazie mille signore!

D'une gentillesse remarquable, Diana Rodelo m'a également très tôt fait confiance. Je lui exprime toute ma gratitude pour son attention à tout instant sur mes travaux, pour ses conseils avisés et son écoute qui ont été prépondérants pour la bonne réussite de cette thèse. Son énergie et sa disponibilité ont été des éléments moteurs pour moi. Je n'oublierai jamais mes séjours scientifiques toujours fructueux à l'Université d'Algarve. Diana, du fond du coeur je te dis merci... Muito Obrigado senhora!

Je remercie sincèrement tous les membres de mon jury de thèse: Michel Willem, Manuela Sobral, Andrea Montoli et Pierre-Alain Jacqmin qui ont gentiment accepté de lire cette thèse et dont les remarques et suggestions intéressantes m'ont permis d'améliorer ce travail.

Je remercie également les membres de mon comité d'accompagnement de thèse: Enrico Vitale, Tim Van der Linden et Celestin Nkuimi pour leurs suggestions et commentaires lors de mon épreuve de confirmation de thèse qui m'ont permis d'améliorer mes premiers résultats.

Mes remerciements vont également à l'endroit de Maurice Kianpi qui a guidé mes premiers pas dans la recherche à l'Université de Yaoundé I et qui m'a lancé dans le champ de la théorie des catégories. Un grand merci à mes enseignants de Mathématiques du secondaire André Mbakop et Richard Mouafo qui m'ont donné le goût des Mathematiques.

Je remercie également le F.R.S.-FNRS pour le soutien financier lors de mes séjours scientifiques au Portugal, ainsi que tous mes collègues du CMUC pour l'invitation à donner un exposé à leur séminaire.

Ma gratitude va aussi à l'endroit de Paul-Arnaud qui m'a beaucoup aidé lors de ma venue en Belgique et pour ses précieux conseils et encouragements tout au long de ce travail. Merci à mes co-bureaux Miradain et Cyrille pour leur bonne humeur, nos débats intenses sur differents sujets scientifiques et non scientifiques. Je remercie également les anciens doctorants que j'ai croisés au Cyclotron. Merci à Valerian, Olivette, Justin, Kader, Gabriel... pour leur sympathie et leur gentillesse.

Un grand merci à tous mes collègues du Cyclotron: en particulier à Cathy pour sa gentillesse et son aide au quotidien. Sans oublier Carine et Martine pour leur aide dans toutes les démarches administratives. Ces années de thèse ne sauraient être dissociées des enseignements que j'ai assurés à l'UCLouvain en tant qu'assistant. Je voudrais ainsi remercier tous les différents enseignants avec qui j'ai travaillé.

J'adresse toute mon affection à ma famille, et en particulier à ma maman, à qui je dois tout: merci pour ton amour et ton soutien inconditionnel. Un grand merci à tous mes frères et soeurs pour leur encouragement constant et leur soutien indefectible depuis le début de mes études.

J'exprime toute ma reconnaissance à mes amis qui, avec cette question recurrente "quand est-ce que tu soutiens ta thèse ?", bien qu'angoissante, m'ont permis de rester concentrer sur mon objectif final. Merci à Achille, Israël, Florent, Arnaud, Léticia, Aurel, Carrele, Laureine, Alex, Marlène, Igor, Loïc, Rudja, Charles, Leslie, Pauline, Yves, Arnold, Auberlin..., pour leur encouragement et soutien multiforme.

Une pensée pour terminer ces remerciements pour toi qui n'a pas vu l'aboutissement de mon travail mais je sais que tu en aurais été très fier de ton fils, que ton âme repose en paix papa!!!

Abstract

Over the last thirty years, many interesting results have been discovered in the categorical extension of 2-permutable varieties, called Mal'tsev categories. Many of these results still hold in regular categories satisfying the strictly weaker property of 3-permutability, called Goursat categories. A nice feature of regular Mal'tsev and Goursat categories is that Gumm's Shifting Lemma holds in these categories, a property which allowed, for example, to develop commutator theory in universal algebras. The aim of this thesis is twofold: on the one hand, we extend to Goursat categories the main results obtained for the theory of projective covers and internal structures in Mal'tsev categories. In particular, we give some characterizations of the categories which are the projective covers of Goursat categories. Then, we show that the structure of internal connector is stable under quotients in any Goursat category. As a consequence, the category of internal connectors in a Goursat category is again a Goursat category. This implies that Goursat categories can be characterized in terms of a simple property of internal groupoids and internal categories. On the other hand, we study the Shifting Lemma in regular Mal'tsev and Goursat categories. We prove that regular Mal'tsev and Goursat categories can be characterized through suitable variations of the Shifting Lemma. We also investigate two properties related to the Shifting Lemma and called the Triangular Lemma and the Trapezoid Lemma in universal algebras. We establish some characterizations of regular Mal'tsev and Goursat categories with distributive lattice of equivalence relations through variations of the Triangular Lemma and Trapezoid Lemma.

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Introduction

A. I. Mal'tsev (1909-1967)

E. Goursat (1858-1936)

A bit of history

In his 1954 paper [75], A. I. Mal'tsev proved that, for a variety $\mathbb{V}$ of universal algebras, the following conditions are equivalent:

1. for any pair of congruences R and S on the same algebra X in $\mathbb{V}$, the equality $RS = SR$ holds;

2. the algebraic theory of $\mathbb{V}$ contains a ternary operation p satisfying the equations
$$\begin{cases} p(x,y,y) &= x \\ p(x,x,y) &= y \end{cases}$$

where

$$RS=\{(x,z)\in X\times X \mid \exists y\in X:\ (x,y)\in S\ \wedge\ (y,z)\in R\}$$

and

$$SR=\{(x,z)\in X\times X \mid \exists y\in X:\ (x,y)\in R\ \wedge\ (y,z)\in S\}$$

are the usual composites of congruences. A variety of universal algebras satisfying these conditions is now called a **Mal'tsev variety** (**or** 2-**permutable variety**), and such an operation p a **Mal'tsev operation**. Many classical varieties of algebras are Mal'tsev varietes: for instance, the theory of the varieties of groups contains a Mal'tsev operation given by $p(x,y,z)=xy^{-1}z$.

In his 1976 book [88], J. Smith introduced and developed the notions of centrality and commutator of congruences in Mal'tsev varieties. In the case of groups, for instance, the commutator $[A,B]$ of two normal subgroups A and B of a group G is the usual (normal) subgroup generated by all elements $[a,b]=a^{-1}b^{-1}ab$, such that $a\in A$, $b\in B$. Similarly for rings, the commutator $[I,J]$ of two ideals I and J of a ring R is the ideal generated by all elements $[i,j]=ij+ji$, such that $i\in I$, $j\in J$. This theory was extended to congruence modular varieties by J. Hagemman and C. Hermann in [52] by using a "lattice-theoretic approach". But in doing so, the "geometrical intuition" that was present in J. Smith's lecture notes became less visible. A variety of universal algebras is **(congruence) modular** when for any congruences R, S and T on the same object $X\in\mathbb{V}$ such that $R\leqslant T$, one has: $R\vee(S\wedge T)=(R\vee S)\wedge T$.

The notion of Mal'tsev variety was extended to Mal'tsev categories by A. Carboni, J. Lambek, M. C. Pedicchio in [23] by replacing congruences with internal equivalence relations, allowing one to explore some interesting new (non-varietal) examples, and to establish some new versions of the classical homological lemmas in a non-abelian categorical context [13]. Among the fundamental examples of Mal'tsev categories, one has the above mentioned categories **Grp** of groups and **Rng** of rings, but also **Heyt** of Heyting algebras. As examples of regular Mal'tsev categories that are not (finitary) varieties of algebras we list, for instance, any abelian category and the non-abelian categories of C*-algebras, $\mathbf{Hopf}_{K,coc}$ of cocomutative Hopf algebras over a fixed field K [49], and the dual category of any elementary topos [21].

One of the results that A. Carboni, J. Lambek, M. C. Pedicchio had in mind in [23] was the extension to Mal'tsev categories of the Goursat Lemma [33], due to E. Goursat and stated in the category of groups as follows: *every subgroup of the direct product of two groups determines an isomorphism between factor groups of subgroups of the given groups*; this means that, given a homomorphic relation R from a group G to a group H, the following quotient groups are isomorphic:

$$GR/R^{\circ}R \;\cong\; RH/RR^{\circ}$$

where $GR = \{h \in H \mid \exists g \in G, gRh\}$, $\quad RH = \{g \in G \mid \exists h \in H, gRh\}$, and R° is the opposite relation to R.

This result was used to obtain general forms of the Zassenhaus lemma and the Jordan-Hölder-Schreier theorem for normal series in [91] and was generalized to Mal'tsev varieties by J. Lambek in [73]. After an in-depth analysis, A. Carboni, J. Lambek and M. C. Pedicchio proved that this result of E. Goursat holds not only in the Mal'tsev categories but also in those regular categories wherein each relation P from A to B satisfies the equality

$$PP^{\circ}PP^{\circ} = PP^{\circ}. \tag{1}$$

That is how **Goursat categories** were born: these are the regular categories in which each relation P satisfies the condition (1).

In 1993, A. Carboni, M. Kelly and M. C. Pedicchio observed in [21] that the condition (1) is in fact strictly weaker than the difunctionality $PP^{\circ}P = P$ equivalent to the Mal'tsev condition $RS = SR$ in regular categories. They also observed that condition (1) is equivalent in a regular category to the 3-permutability of equivalence relations

$$RSR = SRS, \tag{2}$$

for any pair of equivalence relations R and S on the same object. This justifies the fact that the 3-permutable categories are still called Goursat categories. Thus, the Mal'tsev and Goursat conditions are "similar", this suggests that some properties which were known to hold in the Mal'tsev context also hold in the weaker Goursat context. This was one of our motivations to study Goursat categories. As examples of Goursat categories that are not regular Mal'tsev categories we have the category **ImplAlg** of implication algebras [78] and the category **RCSGrp** of right complemented semigroups [53].

An important property common to regular Mal'tsev and Goursat categories is that the lattice of equivalence relations on the same object is modular [21], a property playing a crucial role in commutator theory [32, 51], and that distinguishes them from general n-permutable categories. This property was also useful to simplify and improve some results in categorical Galois theory. A general notion of central extensions was investigated (see [61]).

Over the last thirty years, Mal'tsev categories were widely studied in connection with the theory of commutators, centrality, and central extensions (see [5, 7, 15, 31, 34, 35, 61, 64, 79, 80], for instance, and references therein). In his habilitation's thesis [51], H. P. Gumm introduced a geometrical approach to congruence modularity, which allowed him to prove all the main properties of commutators in modular varieties: the purely algebraic manipulations previously used by J. Hagemann and C. Herrman to prove the properties of the modular commutator became more intuitive, thanks to this geometrical approach. He also introduced a new property, called the Shifting Lemma, and proved that a variety of universal algebras satisfies the Shifting Lemma precisely when it is congruence modular. For a variety $\mathbb{V}$ of universal algebras, the **Shifting Lemma** is stated as follows: given congruences R, S and T on the same algebra X in $\mathbb{V}$ such that $R \wedge S \leqslant T$, whenever x, y, u, v are elements in X with $(x, y) \in R \wedge T$, $(x, u) \in S$, $(y, v) \in S$ and $(u, v) \in R$, it then follows that $(u, v) \in T$. We display this condition as

$$\begin{tikzcd}
x \arrow[r, dash, "S"] \arrow[d, dash, "R"'] \arrow[d, dash, bend right=60, "T"'] & u \arrow[d, dash, "R"] \arrow[d, dash, dashed, bend left=60, "T"] \\
y \arrow[r, dash, "S"'] & v.
\end{tikzcd} \tag{3}$$

In any regular category it is easy to see that congruence modularity implies that the Shifting Lemma holds. Since regular Mal'tsev and Goursat categories are such that their lattices of equivalence relations on any object are modular, then the Shifting Lemma holds in both contexts.

In [79] M. C. Pedicchio developed a categorical approach to commutator theory in exact Mal'tsev categories with coequalizers thanks to the notions of pregroupoid in the sense of Kock [70, 71]. Motivated by their work on internal categories and internal groupoids in [63], this categorical approach to commutator theory was generalized in general categories by G. Janelidze and M. C. Pedicchio in [64] by introducing the notion of internal pseudogroupoid. However, the main results were actually obtained only in the varietal context.

In [14], D. Bourn and M. Gran introduced a categorical version of the Shifting Lemma, called the **Shifting Property**, in any finitely complete category, and this leads to the notion of a **Gumm category**. They also proved that, whenever a finitely complete category satisfies the categorical formulation of Gumm's Shifting Lemma, a pseudogroupoid structure is unique, when it exists. So, for two equivalence relations R and S having a pseudogroupoid structure becomes a property, and the path to an entirely categorical approach to commutator theory (in full generality) was open. Indeed, the existence of pseudogroupoid structure is equivalent to the triviality of the modular commutator. The pseudgroupoid structure can be simplified in Gumm categories and it has some good properties reflecting those of commutators in the modular context.

In [15] D. Bourn and M. Gran developed a new categorical approach to centrality, by emphasizing the role of the notion of internal connector between two equivalence relations introduced in [16]. If (R, r_1, r_2) and (S, s_1, s_2) are two equivalence relations on an object X and $R \times_X S$ the pullback of r_2 along s_1, a **connector** between R and S is an arrow $p\colon R \times_X S \longrightarrow X$ in $\mathbb{C}$

$$\begin{tikzcd}
R \times_X S \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, dashed, "p"] & S \arrow[d, "s_1"] \\
R \arrow[r, "r_2"'] & X,
\end{tikzcd}$$

such that

1. $xSp(x,y,z)Rz$;
2. $p(x,x,y) = y$;
3. $p(x,y,y) = x$;
4. $p(x,y,p(z,u,v)) = p(p(x,y,z),u,v)$,

when each term is defined.

The notion of connector is deeply related to the notion of pregroupoid. In fact, given two regular epimorphisms $d\colon X \to Y$ and $c\colon X \to Z$, a connector on the effective equivalence relations $\mathrm{Eq}(d)$ and $\mathrm{Eq}(c)$ is the same thing as an internal pregroupoid (see also the introduction of [15], for instance, for a comparison between these two related notions and some additional references). So, the notion of connector allows to study the centrality of all equivalence relations, even of those that are not effective. In [15], the authors used this notion of connector to prove the important basic centrality properties corresponding to the classical properties of the commutator in the context of regular Mal'tsev categories. They also gave some new characterizations of regular Mal'tsev categories in terms of properties of internal categories and internal groupoids. In this thesis we generalize these results to the context of Goursat categories.

All the previous results in the regular Mal'tsev and Goursat categories in terms of properties of connectors were obtained thanks to the validity of the Shifting Lemma in both contexts. For this reason we also thoroughly investigate the Shifting Lemma in this thesis: it allows us to show that regular Mal'tsev and Goursat categories can be characterized through suitable variations of the Shifting Lemma. These results are new, even in the context of universal algebras.

In 2000, motivated by the Shifting Lemma characterizing congruence modular varieties, J. Duda introduced in [29, 30] other properties related to the Shifting Lemma and that characterize congruence distributive varieties. These properties are called the Triangular Lemma and the Trapezoid Lemma in the varietal context. A variety $\mathbb{V}$ of universal algebras satisfies the **Triangular Lemma** [27] if, given congruences R, S

and T on the same algebra X in $\mathbb{V}$ such that $R \wedge S \leqslant T$, whenever y, u, v are elements in X with $(u, y) \in T$, $(y, v) \in S$ and $(u, v) \in R$, it then follows that $(u, v) \in T$. We display this condition as

$$\begin{tikzcd} & u \arrow[d, dash, "R"] \arrow[d, dash, dashed, bend left=60, "T"] \\ y \arrow[ur, dash, "T"] \arrow[r, dash, "S"'] & v. \end{tikzcd} \tag{4}$$

A variety $\mathbb{V}$ of universal algebras satisfies the **Trapezoid Lemma** [27] if, given congruences R, S and T on the same algebra X in $\mathbb{V}$ such that $R \wedge S \leqslant T$, whenever x, y, u, v are elements in X with $(x, y) \in T$, $(x, u) \in S$, $(y, v) \in S$ and $(u, v) \in R$, it then follows that $(u, v) \in T$. We display this condition as

$$\begin{tikzcd} & x \arrow[r, dash, "S"] & u \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=60, "T"] \\ y \arrow[ur, dash, "T"] \arrow[rr, dash, "S"'] & & v. \end{tikzcd} \tag{5}$$

In their 2003 paper [27], I. Chajda, G. Czédli, E. Horváth studied these properties and proved that, for a variety $\mathbb{V}$ of universal algebras, the fact that both the Shifting Lemma and the Triangular Lemma hold in a variety $\mathbb{V}$ is equivalent to $\mathbb{V}$ being a congruence distributive variety, and this is also equivalent to the fact that the Trapezoid Lemma holds in $\mathbb{V}$. Consequently, by considering stronger versions of the Triangular Lemma we were hoping to get at once 2-permutability (or 3-permutability) and congruence distributivity in a varietal context, and to extend these observations to a categorical context.

Explaining how this is indeed possible is one of the results of this thesis, where suitable variations of the Triangular Lemma and of the Trapezoid Lemma are shown to be the right properties to characterize **equivalence distributive** categories (the natural generalization of congruence distributive varieties). This also led us to give some new characterizations of equivalence distributive Mal'tsev and Goursat categories through variations of the Triangular Lemma and of the Trapezoid Lemma. As examples of equivalence distributive categories we list, for instance, the categories **Heyt**, **BoolAlg** of boolean algebras. The dual category of any elementary topos and more generally any arithmetical category [80] are equivalence distributive categories.

Over the past, Mal'tsev categories were also studied in the theory of regular and exact completions [26]. In particular, J. Rosický and E. Vitale studied Mal'tsev categories in relationship with the construction of the free exact and regular completion of a category with weak finite limits (*weakly lex*) in [86].

An important aspect in the study of these completions concerns the possibility of characterizing projective covers of certain algebraic categories through simpler properties involving projectives, and to relate those properties to the known varietal characterizations in terms of the existence of operations and identities of their varietal theories. Such kind of studies have been done for the projective covers of categories which are: Mal'tsev [86], protomodular and semi-abelian [37], (strongly) unital and subtractive [42]. In this thesis, we give a characterization of those categories with weak finite limits which are projective covers of Goursat categories. This result also applies to 3-permutable (quasi)varieties, yielding a Mal'tsev condition characterization.

Overview of the contents

In Chapter 1, we introduce the basic categorical notions and results we shall need in the following chapters. In particular we recall the relationship between some special epimorphisms such as strong epimorphisms, regular epimorphisms and split epimorphisms. We then recall some basics facts about regular categories and relations in regular categories. We also recall the notion of projective cover, and Barr's Methatheorem concerning the internal logic of regular categories (Theorem 1.23, 1.24).

In Chapter 2, we begin by recalling the definition and some important characterizations of Goursat categories such as the characterization through the (denormalized) 3-by-3 Lemma, the characterization through a special kind of pushouts, called **Goursat pushouts**, and the characterization in terms of fibrations of points. We then introduce the notion of weak Goursat category and give the characterization of the categories with weak finite limits whose regular completion is a Goursat category (Propositions 2.25 and 2.27). As an application, we relate them to the existence of the quaternary operations which characterize the varieties of universal algebras which are 3-permutable (Proposition 2.28).

In Chapter 3, we study some internal structures such as internal connectors, internal categories and internal groupoids in Goursat categories. We show that, for any Goursat category $\mathbb{C}$, the category $\mathrm{Equiv}(\mathbb{C})$ of equivalence relations in $\mathbb{C}$ is also a Goursat category (Proposition 3.3). We use this result to prove some properties of Goursat categories in terms of connectors. More precisely, we show that, when $\mathbb{C}$ is a Goursat category, then connectors are stable under quotients in $\mathbb{C}$ (Proposition 3.9), and this implies that the category $\mathrm{Conn}(\mathbb{C})$ of connectors in $\mathbb{C}$ is again a Goursat category (Theorem 3.12). As a consequence, we show that Goursat categories can be characterized in terms of properties of internal groupoids and internal categories (Theorem 3.19). It turns out that a regular category $\mathbb{C}$ is a Goursat category if and only if the category $\mathrm{Grpd}(\mathbb{C})$ of internal groupoids (equivalently, the category $\mathrm{Cat}(\mathbb{C})$ of internal categories) in $\mathbb{C}$ is closed under quotients in the category $\mathrm{RG}(\mathbb{C})$ of reflexive graphs in $\mathbb{C}$. The main results of this chapter are summarized in Table 3.1.

In Chapter 4, we study the Shifting Lemma in regular Mal'tsev and Goursat categories. We begin by giving some new characterizations of regular Mal'tsev and Goursat categories in terms of positive relations (Theorems 4.6 and 4.7). We then use these characterizations to show that Mal'tsev and Goursat categories can be characterized through variations of the Shifting Lemma. More precisely, we prove that a regular category $\mathbb{C}$ is a Mal'tsev category if and only if the Shifting Lemma holds for reflexive relations on the same object in $\mathbb{C}$ (Theorems 4.11 and 4.12). Moreover, we prove that a regular category $\mathbb{C}$ is a Goursat category if and only if the Shifting Lemma holds for a reflexive relation S and reflexive and positive relations R and T in $\mathbb{C}$ (Theorem 4.17).

In Chapter 5, we study the Triangular and Trapezoid Lemma in regular Mal'tsev and Goursat categories. We introduce the notion of equivalence distributive category and show that when $\mathbb{C}$ is a regular Mal'tsev category, or even a Goursat category, the Triangular Lemma is equivalent to the Trapezoid Lemma, and both of these properties are equivalent to $\mathbb{C}$ being equivalence distributive (Propositions 5.7 and 5.10). We then give some new characterizations of equivalence distributive Mal'tsev and Goursat categories through variations of the Triangular and Trapezoid Lemmas involving reflexive and positive relations (Theorems 5.13 and 5.19).

We conclude the thesis in Chapter 6 with some directions for future research.

Chapter 1

Preliminaries

In this chapter, we recall some elementary categorical notions and properties needed in the subsequent chapters.

1.1 Special epimorphisms

In this section, we examine various types of epimorphisms in order to understand the notions of regular and exact categories.

Definition 1.1. A morphism $f\colon X \longrightarrow Y$ in a category $\mathbb{C}$ is an *epimorphism* if, for any pair of parallel arrows $u, v\colon Y \longrightarrow Z$ such that $uf = vf$, one has $u = v$.

In the category **Set** of sets, the epimorphisms are precisely the surjective maps. In the category **Grp** and **Ab** of abelian groups, the epimorphisms are the surjective homomorphisms.

The notion of monomorphism is defined dually:

Definition 1.2. A morphism $f\colon X \longrightarrow Y$ in a category $\mathbb{C}$ is a *monomorphism* if, for any pair of parallel arrows $u, v\colon W \longrightarrow X$ such that $fu = fv$, one has $u = v$.

Monomorphisms in **Set** are the injective maps, in **Grp** and in **Ab** the injective homomorphisms, in **Top** (the category of topological spaces) the continuous injective maps.

We shall generally use the arrow $f\colon X \rightarrowtail Y$ to express the fact that f is a monomorphism.

For an object X in a category $\mathbb{C}$ and two monomorphisms $m\colon M \rightarrowtail X$ and $n\colon N \rightarrowtail X$ we say that m *factors through* n (and we will write $m \leqslant n$) when there exists $s : M \to N$ such that $m = ns$. A *subobject* of an object X in $\mathbb{C}$ is an equivalence class of monomorphisms with codomain X, where two monomorphisms $m\colon M \rightarrowtail X$ and $n\colon N \rightarrowtail X$ are equivalent if and only if $m \leqslant n$ and $n \leqslant m$.

Definition 1.3. An epimorphism $f\colon X \longrightarrow Y$ in a category $\mathbb{C}$ is a *strong epimorphism* if, given any commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ W \arrow[r, tail, "m"'] & Z, \end{tikzcd}$$

where m is a monomorphism, then there exists a unique arrow
$t\colon\ Y \longrightarrow\ W$ such that $mt = h$ and $tf = g$.

Strong epimorphisms have the following properties:

1. if $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are two strong epimorphisms, then $gf\colon X \longrightarrow Z$ is a strong epimorphism.

2. if $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are such that $gf\colon X \longrightarrow Z$ is a strong epimorphism, then $g\colon Y \longrightarrow Z$ is a strong epimorphism.

Definition 1.4. A morphism $f\colon X \longrightarrow Y$ in a category $\mathbb{C}$ is a *regular epimorphism* if there exist two arrows $u, v\colon C \to X$ in $\mathbb{C}$

$$\begin{tikzcd} C \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X \arrow[r, "f"] & Y, \end{tikzcd}$$

such that f is the coequalizer of u and v.

We shall generally use the arrow $f\colon X \twoheadrightarrow Y$ to express the fact that f is a regular epimorphism.

For an object X in a category $\mathbb{C}$ and two regular epimorphisms
$p\colon X \twoheadrightarrow Y$ and $q\colon X \twoheadrightarrow Z$ we will write $p \leqslant q$ when there exists

$e : Z \to Y$ such that $p = eq$. A *quotient object* of an object X in $\mathbb{C}$ is an equivalence class of regular epimorphisms with domain X, where two regular epimorphisms $p\colon X \twoheadrightarrow Y$ and $q\colon X \twoheadrightarrow Z$ are equivalent if and only if $p \leqslant q$ and $q \leqslant p$.

Definition 1.5. A morphism $f\colon X \longrightarrow Y$ in a category $\mathbb{C}$ is a *split epimorphism* if there is an arrow $i\colon Y \longrightarrow X$ such that $fi = 1_Y$.

In a category $\mathbb{C}$, regular, strong and split epimorphisms are linked as follows:

Proposition 1.6. In any category $\mathbb{C}$, the following properties hold.

- every split epimorphism is regular;
- every regular epimorphism is strong.

1.2 Regular categories

Definition 1.7. A finitely complete category $\mathbb{C}$ is called *regular* [4] if

- every kernel pair has a coequalizer;
- regular epimorphisms are pullback stable, that is, in any pullback

$$\begin{tikzcd} X \times_Z Y \arrow[r, two heads, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "f"] \\ X \arrow[r, two heads, "g"'] & Z, \end{tikzcd}$$

  the morphism $p_2 : X \times_Z Y \longrightarrow Y$ is a regular epimorphism whenever $g\colon X \longrightarrow Z$ is a regular epimorphism.

Example 1.8. The categories **Set** of sets, **Grp** of groups, **Ab** of abelian groups, **Mon** of Monoids, **Rng** of rings, **Mod-R** of modules over some fixed ring R are regular categories. More generally, any **variety of universal algebras** is a regular category, where regular epimorphisms are surjective homomorphisms, and finite limits (in particular, pullbacks) are computed as in the category of sets. The same is true for any **quasi-variety** of algebras (see [81], for example). As examples of regular categories that are not (finitary) varieties of universal algebras we have: the categories **CGrp** of compact groups, **Grp(Top)** of topological groups.

The categories **Top** of topological spaces, **Cat** of small categories, for instance, are not regular categories, since regular epimorphisms are not always pullback stable in these categories.

In any regular category $\mathbb{C}$, regular epimorphisms have the following properties:

Proposition 1.9. Let $\mathbb{C}$ be a regular category, then the following properties hold:

1. given regular epimorphisms $f\colon X \twoheadrightarrow Y$ and $g\colon X' \twoheadrightarrow Y'$, their product $f \times g\colon X \times X' \longrightarrow Y \times Y'$ is a regular epimorphism as well;

2. the notions of regular epimorphism and strong epimorphism are equivalent;

3. a morphism which is both a monomorphism and a regular epimorphism is an isomorphism.

The notion of regular category can be also reformulated as follows:

Theorem 1.10. Let $\mathbb{C}$ be a finitely complete category. Then $\mathbb{C}$ is a regular category if and only if

1. any arrow $f\colon X \to Y$ has a (unique up to isomorphism) factorization as a regular epimorphism $p\colon X \to I$ followed by a monomorphism $m\colon I \to Y$,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & Y \\ & I \arrow[ur, tail, "m"'] & \end{tikzcd}$$

2. these factorizations are pullback stable.

The subobject determined by the monomorphism $m\colon I \rightarrowtail Y$ is unique, and it is called the *regular image* of the arrow f.

In **Grp**, for instance, the factorization is obtained as follows: given a group homomorphism $f\colon A \to B$, consider the kernel pair of f which is the equivalence relation $\mathrm{Eq}(f) = \{(x, y) \in A \times A \mid f(x) = f(y)\}$ also obtained by building the pullback of f along f

$$\begin{tikzcd}
\mathrm{Eq}(f) \arrow[r, "f_2"] \arrow[d, "f_1"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B.
\end{tikzcd}$$

The canonical quotient $p\colon A \to A/\mathrm{Eq}(f)$ is a group homomorphism and allows one to get the following commutative diagram in **Grp**

$$\begin{tikzcd}
\mathrm{Eq}(f) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & A \arrow[rr, "f"] \arrow[dr, "p"'] & & B. \\
& & A/\mathrm{Eq}(f) \arrow[ur, dashed, "m"'] &
\end{tikzcd}$$

The canonical quotient $p\colon A \to A/\mathrm{Eq}(f)$ is actually the coequalizer of f_1 and f_2 and thus a regular epimorphism in **Grp** and the induced morphism $m\colon A/\mathrm{Eq}(f) \to B$ is a monomorphism since it is injective.

The construction of the factorization is performed similarly in any regular category.

1.3 Relations in regular categories

Regular categories provide a good context for the calculus of relations. As usual, a *relation* R from X to Y is a subobject $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times Y$. The *opposite relation* of R, denoted R^o, is the relation from Y to X given by the subobject $\langle r_2, r_1\rangle\colon R \rightarrowtail Y \times X$. A relation R from X to X is called a relation on X. We shall identify a morphism $f\colon X \longrightarrow Y$ with the relation $\langle 1_X, f\rangle\colon X \rightarrowtail X \times Y$ and write f^o for its opposite relation. In particular, the identity arrow $1_X\colon X \longrightarrow X$ yields the identity relation, denoted by 1_X, given by $\langle 1_X, 1_X\rangle \colon X \rightarrowtail X \times X$. An important aspect of regular categories is that in these categories one can define a composition of relations. In **Set**, if $R \rightarrowtail X \times Y$ is a relation from X to Y, and $S \rightarrowtail Y \times Z$ a relation from Y to Z, one usually defines the relation $SR \rightarrowtail X \times Z$ by setting

$$SR = \{(x, z) \in X \times Z \mid \exists y \in Y : (x, y) \in R \wedge (y, z) \in S\}.$$

Thanks to the existence of regular images (see Theorem 1.10), this construction is actually possible in any regular category $\mathbb{C}$. In fact, given two relations $\langle r_1, r_2\rangle \colon R \rightarrowtail X \times Y$ and $\langle s_1, s_2\rangle \colon S \rightarrowtail Y \times Z$ in a regular category $\mathbb{C}$, to construct the relational composite $SR \rightarrowtail X \times Z$, we first build the pullback of r_2 and s_1,

$$\begin{tikzcd}
 & & R \times_Y S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\
X & & Y & & Z
\end{tikzcd}$$

and the composite relation SR of S and R is given by the regular image of the arrow $\langle r_1 p_1, s_2 p_2\rangle \colon R \times_Y S \longrightarrow X \times Z$,

$$\begin{tikzcd}
R \times_Y S \arrow[rr, "\langle r_1 p_1, s_2 p_2\rangle"] \arrow[dr, two heads, "p"'] & & X \times Z. \\
 & SR \arrow[ur, tail] &
\end{tikzcd}$$

This composition is then associative, thanks to the fact that regular epimorphisms are assumed to be pullback stable. With the above notation, any relation $\langle r_1, r_2\rangle \colon R \rightarrowtail X \times Y$ can be seen as the relational composite $r_2 r_1^o$.

Relations are partially ordered by the usual inclusion of subobjects. The following properties are well known and easy to prove (see [21] for instance); we collect them in the following lemma:

Lemma 1.11. Let $f \colon X \longrightarrow Y$ be an arrow in a regular category $\mathbb{C}$, and let $f = ir$ be its (regular epimorphism, monomorphism) factorization. Then:

1. $f^o f$ is the kernel pair of f, thus $1_X \leqslant f^o f$; moreover, $1_X = f^o f$ if and only if f is a monomorphism;

2. $f f^o$ is $\langle i, i \rangle$, thus $f f^o \leqslant 1_Y$; moreover, $f f^o = 1_Y$ if and only if f is a regular epimorphism;

3. $f f^o f = f$ and $f^o f f^o = f^o$.

Definition 1.12. A relation (R, r_1, r_2) on an object X is said to be:

- *reflexive* when $1_X \leqslant R$, i.e. there is an arrow $r\colon X \longrightarrow R$ such that $r_1 r = 1_X = r_2 r$;

- *symmetric* when $R^\circ \leqslant R$, i.e. there is an arrow $\sigma\colon R \longrightarrow R$ such that $r_2 = r_1\sigma$ and $r_1 = r_2\sigma$;

- *transitive* when $RR \leqslant R$, i.e. by considering the pullback $(R \times_X R, p_1, p_2)$ of r_2 and r_1, there is an arrow $t\colon R \times_X R \longrightarrow R$ such that the diagram

$$\begin{tikzcd}
 & & R \arrow[lldd, bend right, "r_1"'] \arrow[rrdd, bend left, "r_2"] & & \\
 & & R \times_X R \arrow[u, "t"'] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
X & & X & & X
\end{tikzcd}$$

  commutes;

- an *equivalence relation* if R is reflexive, symmetric and transitive.

In particular, the kernel pair $\langle f_1, f_2 \rangle\colon \mathrm{Eq}(f) \rightarrowtail X \times X$ of a morphism $f\colon X \longrightarrow Y$ (obtained by building the pullback of f along f) is an equivalence relation. The equivalence relations that occur as kernel pairs of some morphism in a category $\mathbb{C}$ are called *effective*.

We denoted by Equiv($\mathbb{C}$) the category whose objects are equivalence relations in $\mathbb{C}$ and arrows from $\langle r_1, r_2\rangle\colon R \rightarrowtail X\times X$ to $\langle s_1, s_2\rangle\colon S \rightarrowtail Y\times Y$ are pairs (f, g) of arrows in $\mathbb{C}$ making the following diagram commute

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

i.e. $s_1 g = f r_1$ and $s_2 g = f r_2$.

Definition 1.13. [2] A category $\mathbb{C}$ is said to be *exact* if it is regular and every equivalence relation in $\mathbb{C}$ is effective.

Example 1.14. The category **Set** is exact: each equivalence relation R on a set A is the kernel pair of the canonical quotient $\pi_R\colon A \longrightarrow A/R$. Similarly, the categories **Grp**, **Rng**, **Mon** and, more generally, any variety of universal algebras are exact categories. The category **CMon** of cancellative monoids is regualr, but not exact. The same is true more generaly for any quasi-variety [81].

Definition 1.15. Let $\mathbb{C}$ be a regular category, (R, r_1, r_2) a relation on X and $f\colon X \twoheadrightarrow Y$ a regular epimorphism. We define the *regular image of R along $f\colon X \longrightarrow Y$* to be the relation $f(R)$ on Y induced by the (regular epimorphism, monomorphism) factorization $\langle s_1, s_2\rangle\psi$ of the composite $(f\times f)\langle r_1, r_2\rangle$:

$$\begin{tikzcd} R \arrow[r, dotted, two heads, "\psi"] \arrow[d, tail, "\langle r_1, r_2\rangle"'] & f(R) \arrow[d, dotted, tail, "\langle s_1, s_2\rangle"] \\ X\times X \arrow[r, two heads, "f\times f"'] & Y\times Y. \end{tikzcd}$$

Note that the regular image $f(R)$ can be obtained as the relational composite $f(R) = fRf^o = fr_2 r_1^o f^o$. When R is an equivalence relation, $f(R)$ is also reflexive and symmetric. In a general regular category $f(R)$ is not necessarily an equivalence relation. This is the case in a Goursat category (Theorem 2.4).

By using the previous definition and Lemma 1.11 we prove the following:

Proposition 1.16. Let $\mathbb{C}$ be a regular category. Given the commutative diagram

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, two heads, "f"'] & Y. \end{tikzcd}$$

where R and S are relations and f a regular epimorphism, the morphism g is a regular epimorphism if and only if $S = f(R)$.

Proof. Suppose g to be a regular epimorphism; one has:

$$\begin{aligned} f(R) &= fr_2r_1^\circ f^\circ \\ &= s_2gg^\circ s_1^\circ \quad \text{(since } fr_2 = s_2g \text{ and } fr_1 = s_1g) \\ &= s_2s_1^\circ \quad \text{(since } g \text{ is a regular epimorphism)} \\ &= S \end{aligned}$$

For the converse, if $S = f(R)$, then g is a regular epimorphism by the definition of the regular image of R. $\square$

1.4 Projective cover

Here, we recall the notion of projective cover and the construction of the free regular completion of a weakly lex category.

The construction of the free exact category over a category with finite limits was introduced in [20]. It was later improved to the construction of the free exact category over a category with finite weak limits (*weakly lex*) in [26]. This was possible because the uniqueness requirement in the definition of finite limits of the original category was never used in the construction, but only the existence requirement. In [26], the authors also considered the free regular category over a weakly lex one.

An important property of the free exact (or regular) construction is that such categories always have enough (regular) projectives. In fact, an exact category $\mathbb{A}$ may be seen as the exact completion of a weakly lex category if and only if it has enough projectives. If so, then $\mathbb{A}$ is the exact completion of any of its *projective covers*. Such a phenomenon is captured by varieties of universal algebras: they are the exact completions of their full subcategory of free algebras.

Definition 1.17. An object P in a category is (regular) *projective* if, for any arrow $f\colon P \longrightarrow X$ and for any regular epimorphism $g\colon Y \twoheadrightarrow X$, there exists an arrow $h\colon P \longrightarrow Y$

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "f"] \\ Y \arrow[r, two heads, "g"'] & X \end{tikzcd}$$

such that $gh = f$.

We say that a full subcategory $\mathbb{C}$ of $\mathbb{A}$ is a *projective cover* of $\mathbb{A}$ if two conditions are satisfied:

- any object of $\mathbb{C}$ is regular projective in $\mathbb{A}$;
- for any object X in $\mathbb{A}$, there exists a ($\mathbb{C}$-)cover of X, that is an object C in $\mathbb{C}$ and a regular epimorphism $C \twoheadrightarrow X$.

When $\mathbb{A}$ admits a projective cover, one says that $\mathbb{A}$ has *enough projectives.*

The following property holds in any category:

Lemma 1.18. Any regular epimorphism with a projective codomain is a split epimorphism.

Proof. Let $f\colon X \twoheadrightarrow Y$ be a regular epimorphism in a category such that Y is a projective object. Since Y is projective, there exists an arrow $h\colon Y \to X$

$$\begin{tikzcd} & Y \arrow[dl, dashed, "h"'] \arrow[d, "1_Y"] \\ X \arrow[r, two heads, "f"'] & Y \end{tikzcd}$$

such that $fh = 1_Y$, and then f is a split epimorphism. $\square$

By dropping the assumption of uniqueness of the factorization in the definition of a limit, one obtains the definition of a weak limit. So, weak limits are not unique in general (not even up to isomorphisms). We call *weakly lex* a category with weak finite limits.

We shall generally use the arrow $X \underset{\leftarrow}{\overset{f}{\twoheadrightarrow}} Y$ to express the fact that f is a split epimorphism in a weakly lex category.

Remark 1.19. If $\mathbb{C}$ is a projective cover of a weakly lex category $\mathbb{A}$, then $\mathbb{C}$ is also weakly lex [26]. For example, let X and Y be objects in $\mathbb{C}$ and $X \longleftarrow W \longrightarrow Y$ a *weak product* of X and Y in $\mathbb{A}$. Then, for any cover $\bar{W} \twoheadrightarrow W$ of W, $X \longleftarrow \bar{W} \longrightarrow Y$ is a weak product of X and Y in $\mathbb{C}$. Furthermore, if $\mathbb{A}$ is a finitely complete category and $X \times Y$ is the usual product of X and Y, then the induced morphism $W \twoheadrightarrow X \times Y$ is a regular epimorphism. Similar remarks apply to all weak finite limits.

Definition 1.20. [26] Let $\mathbb{C}$ be a weakly lex category:

1. a *pseudo-relation* on an object X of $\mathbb{C}$ is a pair of parallel arrows $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$; a pseudo-relation is a relation if r_1 and r_2 are jointly monomorphic;

2. a pseudo-relation $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ on X is said to be:

   - *reflexive* when there is an arrow $r\colon X \longrightarrow R$ such that $r_1 r = 1_X = r_2 r$;
   - *symmetric* when there is an arrow $\sigma\colon R \longrightarrow R$ such that $r_2 = r_1\sigma$ and $r_1 = r_2\sigma$;
   - *transitive* if by considering a weak pullback

$$\begin{tikzcd} W \arrow[r, "p_2"] \arrow[d, "p_1"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_2"'] & X, \end{tikzcd}$$

     there is an arrow $t\colon W \longrightarrow R$ such that $r_1 t = r_1 p_1$ and $r_2 t = r_2 p_2$.
   - a *pseudo-equivalence relation* if it is reflexive, symmetric and transitive.

Remark that the transitivity of a pseudo-relation $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ does not depend on the choice of the weak pullback of r_1 and r_2; in fact, if

$$\begin{tikzcd} \bar{W} \arrow[r, "\bar{p_2}"] \arrow[d, "\bar{p_1}"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_2"'] & X, \end{tikzcd}$$

is another weak pullback, the factorization $\bar{W} \longrightarrow W$ composed with the transitivity $t\colon W \longrightarrow R$ ensures that the pseudo-relation is transitive also with respect to the second weak pullback.

Example 1.21. Any weak kernel pair in a weakly lex category is a pseudo-equivalence relation.

It was proved in [26] that regular categories with enough projectives are the regular completions of their full subcategories of projective objects. Note that the free regular completion $\mathbb{C}_{reg}$ of a weakly lex category $\mathbb{C}$ is built as follows:

- objects: an object in $\mathbb{C}_{reg}$ is a finite family of arrows $(f_i\colon X \to X_i)_{i\in I}$ in $\mathbb{C}$ (all the arrows f_i of the family have the same domain)

- arrows: an arrow between two objects $(f_i\colon X \to X_i)_{i\in I}$ and $(g_j\colon Y \to Y_j)_{j\in J}$ is an equivalence class of arrow $h\colon X \to Y$ in $\mathbb{C}$ such that $g_j h t_1 = g_j h t_2$

$$\begin{tikzcd} \bar{X} \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] & \\ X \arrow[r, "h"] \arrow[d, "f_i"'] & Y \arrow[d, "g_j"] \\ X_i & Y_j \end{tikzcd}$$

where $\bar{X} \overset{t_1}{\underset{t_2}{\rightrightarrows}} X$ is a weak joint kernel pair of the family $(f_i\colon X \to X_i)$, i.e. it is weakly universal with respect to the property $f_i t_1 = f_i t_2$, $\forall i \in I$. Two arrows of this kind $h\colon X \to Y$ and $k\colon X \to Y$ are said to be equivalent if $\forall j \in J$, $g_j k = g_j h$.

Theorem 1.22. [26] When $\mathbb{C}$ has weak finite limits, then the category $\mathbb{C}_{reg}$ is regular.

1.5 The Yoneda embedding

In this section, we recall two important results due to Yoneda and Barr which will allow us to partially use set-theoretic terms to develop proofs in a regular category.

Let $\mathbb{C}$ be a small category. The *Yoneda embedding* of $\mathbb{C}$ is the functor

$$Y_{\mathbb{C}}\colon \mathbb{C} \longrightarrow [\mathbb{C}^{op}, \mathbf{Set}], \quad C \mapsto \mathbb{C}(-, C).$$

This functor $Y_{\mathbb{C}}$ is a full and faithful embedding of $\mathbb{C}$ in the category $[\mathbb{C}^{op}, \mathbf{Set}]$ of contravariant functors from $\mathbb{C}$ to Set and natural transformations between them. Each functor $\mathbb{C}(-, C)$ preserves limits. This implies that the Yoneda embebbing preserves all limits which exits in $\mathbb{C}$. As a consequence, these properties of the Yoneda embedding allow us to reduce the proofs of statements about finite limits in any finitely complete category to a proof in the category of sets. However, statements about specific arrows being regular epimorphisms cannot be proved in the same way, since the Yoneda embedding does not preserve regular epimorphisms. Also, the Yoneda embedding does not allow one to prove the existence of some arrow in a category directly in the category of sets. Thanks to the Yoneda embedding, Barr proved an important theorem (also called "**Barr's embedding thereom**") which allowed to develop part of the internal logic of a topos in any regular category. It also allows to reduce the proofs about finite limits and regular epimorphisms in a regular category by just proving them in the category of sets.

Theorem 1.23. [3] For every small regular category $\mathbb{C}$, there is a small category $\mathbb{D}$ and a full and faithful embedding

$$Z\colon \mathbb{C} \longrightarrow [\mathbb{D}^{op}, \mathbf{Set}],$$

which preserves and reflects finite limits and regular epimorphisms.

This theorem has been adapted to the context of a regular category by Borceux and Bourn as follows:

Metatheorem 1.24. [7] Consider a statement of the form $A \Longrightarrow B$, where A and B are conjunctions of properties which can be expressed as:

- some finite diagram is commutative;
- some morphism is a monomorphism;
- some morphism is a regular epimorphism;
- some morphism is an isomorphism;
- some finite diagram is a limit diagram;
- some arrow factors through some specified monomorphism.

If the statement $A \Longrightarrow B$ is true in the category of sets, then it is also true in every regular category.

As an application, we will partially use this result in the proof of Theorems 5.16 and 5.20. In a regular context, thanks to Barr's Metatheorem (Theorem 1.24), one can use set-theoretic terms, so that the properties given in the diagrams (3), (4) and (5) may still be expressed by using generalized elements.

Chapter 2

Goursat categories

In this chapter, we recall the definition of Goursat categories introduced by A. Carboni, M. Kelly, J. Lambek and M. C. Pedicchio in [21, 23] and investigate some of their properties. We then introduce the notion of weak Goursat categories and give some characterizations of categories with weak finite limits whose regular completion is a Goursat category.

2.1 Definition and examples

As mentioned in the Introduction, Goursat categories were discovered by A. Carboni, J. Lambek and M. C. Pedicchio in [23] but were defined explicitely and first studied by A. Carboni, M. Kelly, and M. C. Pedicchio in [21].

Definition 2.1. [23, 21] A regular category $\mathbb{C}$ is called a *Goursat category* when the equivalence relations in $\mathbb{C}$ are 3-permutable, i.e. $RSR = SRS$ for any pair of equivalence relations R and S on the same object.

Goursat categories are also characterized by other properties on (equivalence) relations, as follows:

Theorem 2.2. [21] Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:

(i) $\mathbb{C}$ is a Goursat category;

(ii) $\forall R, S \in \mathrm{Equiv}(X)$, $RSR \in \mathrm{Equiv}(X)$, for any object X in $\mathbb{C}$;

(iii) $\forall R, S \in \mathrm{Equiv}(X)$, $R \vee S = RSR(= SRS)$, for any object X in $\mathbb{C}$;

(iv) any relation P is such that $PP^{\circ}PP^{\circ} = PP^{\circ}$;

(v) for any reflexive relation E, the relation EE° is an equivalence relation;

(vi) for any reflexive relation E, $EE^{\circ} = E^{\circ}E$.

From this theorem it follows that, for any object X of a Goursat category $\mathbb{C}$, the lattice of equivalence relation on X is modular (Proposition 3.3 [21]).

Example 2.3. There are many important algebraic examples of Goursat categories. Indeed, by a classical theorem in [53], a variety of universal algebras is a Goursat category precisely when it is a 3-permutable variety: this property is known to be equivalent to the existence of two ternary operations r and s satisfying the following the identities:

$$\begin{cases} r(x,y,y) & = & x \\ r(x,x,y) & = & s(x,y,y) \\ s(x,x,y) & = & y. \end{cases}$$

We shall see in Proposition 2.28 another algebraic characterization of 3-permutable varieties via quaternary operations. Accordingly, the variety of groups with the operations $r(x,y,z) = xy^{-1}z$ and $s(x,y,z) = z$, the category **Ab**, **Mod-R**, **Rng**, **Hey**, **X-Mod** of crossed modules, **QGrp** of quasi-groups, **AssAlg** of associative algebras, and **ImplAlg** of implication algebras are all Goursat categories. As examples of Goursat categories that are not (finitary) varieties of universal algebras we have: the categories **CGrp** of compact groups, **Grp(Top)** of topological groups [68], $\mathbf{Hopf}_{K,coc}$ [49], the dual category of sets $\mathbf{Set}^{op}$ and more generaly the dual category of any topos [21].

Another characterization of Goursat categories in terms of (equivalence) relations is given by the preservation of equivalence relations through the regular image by a regular epimorphism as follows:

Theorem 2.4. [21] A regular category $\mathbb{C}$ is a Goursat category if and only if for any regular epimorphism $f\colon X \twoheadrightarrow Y$ and any equivalence relation R on X, the regular image $f(R) = fRf^o$ of R along f is an equivalence relation.

Proof. The relation $f(R)$ is always reflexive and symmetric in a regular category $\mathbb{C}$ since
$$1_Y = ff^\circ \leqslant fRf^\circ = f(R)$$
and
$$(f(R))^\circ = (fRf^\circ)^\circ = fR^\circ f^\circ = fRf^\circ = f(R).$$

For the transitivity, one has:

$$\begin{array}{rcll} f(R)f(R) & = & fRf^\circ fRf^\circ & \\ & = & ff^\circ fRf^\circ ff^\circ & \text{(by assumption)} \\ & = & fRf^\circ & \text{(by Lemma 1.11 3.)} \\ & = & f(R) & \end{array}$$

For the converse, let $E = e_2e_1^\circ$ be a reflexive relation, we are going to prove that EE° is an equivalence relation (Theorem 2.2 (v)).

One has
$$EE^\circ = e_2e_1^\circ e_1 e_2^\circ = e_2(\mathrm{Eq}(e_1))$$
so, EE° is the regular image of the equivalence relation $\mathrm{Eq}(e_1)$ along the regular epimorphism e_2 and is then, by assumption, an equivalence relation. ☐

2.2 Properties of Goursat categories

There are many interesting known properties of Goursat categories. In this section, we present some of these properties in terms of a special kind of pushout, through a specific stability property of regular epimorphisms with respect to pullbacks and in terms of fibrations of points.

2.2.1 The 3-by-3 Lemma

Here, we observe that the (denormalized) 3-by-3 Lemma holds in any regular Goursat category.

In any abelian category, the classical 3-by-3 Lemma is stated as follows:

Theorem 2.5. [7] (**The 3-by-3 Lemma**)
Let $\mathbb{C}$ be an abelian category. Then "*The 3-by-3 Lemma*" holds in $\mathbb{C}$: given a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[r, "f"] \arrow[d] & Z' \arrow[d] \\
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

where $ff' = 0$, then if any two rows are short exact sequences, then so is the third.

We refer the reader to [6, 7, 19] for more details about the 3-by-3 Lemma and abelian categories.

In a category which is not pointed, the 3-by-3 Lemma has a "denormalized version" where short exact sequences are replaced by "*exact forks*": these are diagrams of the form

$$\begin{tikzcd}
\mathrm{Eq}(r) \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, two heads, "r"] & Y
\end{tikzcd}$$

in which $(\mathrm{Eq}(r), r_1, r_2)$ is the kernel pair of its coequalizer r. The corresponding 3-by-3 Lemma, called "*The denormalized 3-by-3 Lemma*" [13], would then concern a diagram

$$
\begin{tikzcd}
\mathrm{Eq}(\varphi) \arrow[r, shift left, "\bar{h_1}"] \arrow[r, shift right, "\bar{h_2}"'] \arrow[d, shift right, "\varphi_2"'] \arrow[d, shift left, "\varphi_1"] & \mathrm{Eq}(f) \arrow[r, "\bar{h}"] \arrow[d, shift right, "f_2"'] \arrow[d, shift left, "f_1"] & \mathrm{Eq}(g) \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "g_2"] \\
\mathrm{Eq}(h) \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] \arrow[d, two heads, "\varphi"] & A \arrow[r, two heads, "h"] \arrow[d, two heads, "f"'] & C \arrow[d, two heads, "g"] \\
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & B \arrow[r, two heads, "k"'] & D
\end{tikzcd}
\tag{6}
$$

satisfying the usual commutativity conditions in which the three columns and the middle row are exact forks. The lemma states that the top row is an exact fork if and only if the bottom row is an exact fork.

Note that the "middle" version of the denormalized 3-by-3 Lemma (which states that top and bottom rows are exact fork imply that middle row is an exact fork) always holds in a regular category.

From [13, 72], we have:

Proposition 2.6. Let $\mathbb{C}$ be a Goursat category. Then the denormalized 3-by-3 lemma holds in $\mathbb{C}$.

It was proved in [13] that this result holds when a regular category $\mathbb{C}$ satisfies the 2-permutability property, also known as the Mal'tsev property: in this case, $\mathbb{C}$ is called a *Mal'tsev category* [21], that is a regular category in which the composition of equivalence relations permute, i.e. $RS = SR$ for any pair of equivalence relations R and S on the same object. A rich theory of Mal'tsev categories has been developed over the past thirty years by various authors, whose mains features are collected in [7, 9, 18]. For instance, the 2-permutable version of Theorem 2.2 is:

Theorem 2.7. [23] Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:

(i) $\mathbb{C}$ is a Mal'tsev category;

(ii) $\forall R, S \in \mathrm{Equiv}(X)$, $RS \in \mathrm{Equiv}(X)$, for any object X in $\mathbb{C}$;

(iii) $\forall R, S \in \mathrm{Equiv}(X)$, $R \vee S = RS (= SR)$, for any object X in $\mathbb{C}$;

(iv) any relation P is difunctional: $PP^\circ P = P$;

(v) any reflexive relation E is symmetric: $E^\circ = E$;

(vi) any reflexive relation E is an equivalence relation.

Theorem 2.7 implies that any regular Mal'tsev category is a Goursat category. In fact, if $\mathbb{C}$ is a Mal'tsev category, one has

$$\begin{array}{rcll} RSR & = & RRS & \text{(by assumption } RS = SR\text{)} \\ & = & RS & \text{(by transitivity of } R\text{)} \\ & = & RSS & \text{(by transitivity of } S\text{)} \\ & = & SRS & \text{(by assumption } RS = SR\text{)} \end{array}$$

As examples of Goursat categories that are not regular Mal'tsev categories we have the category **ImplAlg** of implication algebras [78] and the category **RCSGrp** of right complemented semigroups [53].

2.2.2 Goursat pushout

In this subsection, we present a characterization of Goursat categories in terms of a special kind of pushouts, called Goursat pushouts and some of their useful consequences.

Theorem 2.8. [39] Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:

(i) $\mathbb{C}$ is a Goursat category;

(ii) any commutative diagram of type (I) in $\mathbb{C}$, where α and β are regular epimorphisms and f and g are split epimorphisms

$$\begin{tikzcd} X \arrow[r, two heads, "\alpha"] \arrow[d, shift left, "f"] \arrow[dr, phantom, "\mathrm{(I)}"] & U \arrow[d, shift left, "g"] \\ Y \arrow[r, two heads, "\beta"'] \arrow[u, shift left, "s"] & W, \arrow[u, shift left, "t"] \end{tikzcd} \qquad \begin{array}{l} g\alpha=\beta f \\ \alpha s=t\beta \end{array}$$

(which is necessarily a pushout) is a *Goursat pushout*: the morphism $\lambda\colon \mathrm{Eq}(f) \longrightarrow \mathrm{Eq}(g)$, induced by the universal property of kernel pair $\mathrm{Eq}(g)$ of g, is a regular epimorphism.

In [39], the authors give a proof of the implication $(i) \Rightarrow (ii)$ in terms of the calculus of relations and a diagrammatical proof for the implication $(ii) \Rightarrow (i)$. Here we give a simple proof of the implication $(ii) \Rightarrow (i)$ in terms of the calculus of relations.

Proof. $(i) \Rightarrow (ii)$ To prove that the induced morphism $\lambda\colon \mathrm{Eq}(f) \longrightarrow \mathrm{Eq}(g)$ is a regular epimorphism, by Proposition 1.16 it suffices to prove that $\alpha(\mathrm{Eq}(f)) = \mathrm{Eq}(g)$. Since f and g are split epimorphisms, the induced factorization $\gamma\colon \mathrm{Eq}(\alpha) \longrightarrow \mathrm{Eq}(\beta)$ is necessarily a split epimorphism and then $f(\mathrm{Eq}(\alpha)) = \mathrm{Eq}(\beta)$. One has

$$\begin{array}{rcll}
\alpha(\mathrm{Eq}(f)) & = & \alpha f^{\circ} f \alpha^{\circ} & (\text{since } \mathrm{Eq}(f) = f^{\circ} f) \\
& = & \alpha\alpha^{\circ}\alpha f^{\circ} f \alpha^{\circ}\alpha\alpha^{\circ} & (\text{by Lemma 1.11}) \\
& = & \alpha f^{\circ} f \alpha^{\circ}\alpha f^{\circ} f \alpha^{\circ} & (\text{by Goursat property}) \\
& = & \alpha f^{\circ}\beta^{\circ}\beta f \alpha^{\circ} & (\text{since } f(\mathrm{Eq}(\alpha)) = \mathrm{Eq}(\beta)) \\
& = & \alpha\alpha^{\circ} g^{\circ} g \alpha\alpha^{\circ} & (\beta f = g\alpha) \\
& = & g^{\circ} g & (\text{since } \alpha \text{ is a regular epimorphism}) \\
& = & \mathrm{Eq}(g) &
\end{array}$$

$(ii) \Rightarrow (i)$ Let R be an equivalence relation on X and $f\colon X \twoheadrightarrow Y$ a regular epimorphism. We are going to prove that the regular image

$$\begin{tikzcd}
R \arrow[r, two heads, "g"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & f(R) = S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

$f(R) = S$ is an equivalence relation.

The relation S is always reflexive and symmetric. It remains to prove that S is transitive. The following diagram is of the type (I)

$$\begin{tikzcd}
R \arrow[r, two heads, "g"] \arrow[d, shift left, "r_1"] & S \arrow[d, shift left, "s_1"] \\
X \arrow[r, two heads, "f"'] \arrow[u, shift left, "e_R"] & W, \arrow[u, shift left, "e_S"]
\end{tikzcd}$$

where e_R and e_S are given by the reflexivity of R and S, respectively. So, by assumption the factorization $\lambda\colon \mathrm{Eq}(r_1) \longrightarrow \mathrm{Eq}(s_1)$ is a regular epimorphism and then $g(\mathrm{Eq}(r_1)) = \mathrm{Eq}(s_1)$. One has

$$
\begin{array}{rclc}
SS & = & s_2\ s_1^\circ\ s_1\ s_2^\circ & \text{(since } S \text{ is symmetric)} \\
& = & s_2\ \mathrm{Eq}(s_1)\ s_2^\circ & \\
& = & s_2\ g\ r_1^\circ\ r_1\ g^\circ\ s_2^\circ & \text{(since } g(\mathrm{Eq}(r_1)) = \mathrm{Eq}(s_1)) \\
& = & f\ r_2\ r_1^\circ\ r_1\ r_2^\circ\ f^\circ & \text{(since } s_2\ g = f\ r_2) \\
& = & f\ R\ f^\circ & \text{(since } R \text{ is symmetric and transitive)} \\
& = & S &
\end{array}
$$

So, $S = f(R)$ is transitive and then an equivalence relation, as desired.

□

Remark 2.9. Diagram (I) is a Goursat pushout precisely when the regular image of $\mathrm{Eq}(f)$ along α is (isomorphic to) $\mathrm{Eq}(g)$. From Theorem 2.8, it then follows that a regular category $\mathbb{C}$ is a Goursat category if and only if for any commutative diagram of type (I) one has $\alpha(\mathrm{Eq}(f)) = \mathrm{Eq}(g)$. So, Theorem 2.4 characterizes Goursat categories through the property that regular images of equivalence relations are equivalence relations, while Theorem 2.8 characterizes them through the property that regular images of certain kernel pairs are kernel pairs.

Definition 2.10. In any regular category $\mathbb{C}$, a commutative diagram of regular epimorphisms

$$
\begin{tikzcd}
W \arrow[r, two heads, "p_1"] \arrow[d, two heads, "p_2"'] & Z \arrow[d, two heads, "g"] \\
X \arrow[r, two heads, "f"'] & Y,
\end{tikzcd}
$$

is said to be a **regular pushout** (or **double extension** as in [48]) when the factorization $t\colon W \to X \times_Y Z$ towards the pullback of f along g is a regular epimorphism.

Equivalently, this commutative diagram of regular epimorphisms is a regular pushout if and only if $p_1 p_2^o = g^o f$.

Remark 2.11. It was shown in [13] that regular Mal'tsev categories can also be characterized by a diagram of type (I) as follows: a regular category $\mathbb{C}$ is a Mal'tsev category if and only if any commutative diagram of type (I) in $\mathbb{C}$ is a *regular pushout*: this means that the canonical factorization $e\colon X \longrightarrow Y \times_W U$ induced by the universal property of the pullback $Y \times_W U$ of β and g

$$\begin{tikzcd}
X \arrow[dr, "e"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "f"'] & & \\
& Y \times_W U \arrow[r, two heads, "p_1"] \arrow[d, two heads, "p_2"'] & U \arrow[d, two heads, shift left, "g"] \\
& Y \arrow[r, two heads, "\beta"'] \arrow[uul, bend left=50, "s"] & W. \arrow[u, shift left, "t"]
\end{tikzcd}$$

is a regular epimorphism.

Goursat pushouts allowed to show that the validity of the (denormalized) 3-by-3 Lemma is actually equivalent to the Goursat property as follows:

Theorem 2.12. [39] Let $\mathbb{C}$ be a regular category. Then the following conditions are equivalent:

(i) $\mathbb{C}$ is a Goursat category;

(ii) the (denormalized) 3-by-3 Lemma holds in $\mathbb{C}$;

(iii) the lower 3-by-3 Lemma holds in $\mathbb{C}$: given any commutative diagram (6), the lower row is an exact fork whenever the upper row is an exact fork;

(iv) the upper 3-by-3 Lemma holds in $\mathbb{C}$: given any commutative diagram (6), the upper row is an exact fork whenever the lower row is an exact fork.

Remark 2.13. Regular Mal'tsev categories can also be characterized by a stronger version of the denormalized 3-by-3 Lemma, called the *Cuboid Lemma* [43], by replacing the kernels pairs in the three exact columns in (6) with pullbacks of regular epimorphisms along arbitrary morphisms.

Theorem 2.14. [43] Let $\mathbb{C}$ be a regular category and consider any commutative diagram in $\mathbb{C}$

$$
\begin{tikzcd}
S \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, two heads] \arrow[dd] & & U \arrow[rr, "f"] \arrow[dr, two heads] \arrow[dd, dashed] & & A \arrow[dr, two heads] \arrow[dd] & \\
& \mathrm{Eq}(g) \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd] & & V \arrow[rr, two heads, "g"] \arrow[dd] & & B \arrow[dd] \\
\mathrm{Eq}(h) \arrow[rr, dashed, shift left] \arrow[rr, dashed, shift right] \arrow[dr, two heads] & & Z \arrow[rr, dashed, two heads, "h"] \arrow[dr, two heads] & & C \arrow[dr, two heads] & \\
& Y \arrow[rr, shift left] \arrow[rr, shift right] & & W \arrow[rr, two heads, "k"] & & D
\end{tikzcd}
\tag{7}
$$

where the three diamonds are pullbacks of regular epimorphisms along arbitrary morphisms and the two middle rows are exacts forks. Then the following conditions are equivalent:

(i) $\mathbb{C}$ is a Mal'tsev category;

(ii) the Cuboid Lemma holds in $\mathbb{C}$: given any commutative diagram (7), the upper row is an exact fork whenever the lower row is an exact fork.

Note that, this characterization is still true when the three diamonds are pullbacks of split epimorphisms along arbitrary morphisms [43]; this variation of the Cuboid Lemma is called *split Cuboid Lemma*.

Another application of Goursat pushouts is that they allow one to obtain a characterization of Goursat categories through a stability property of regular epimorphisms with respect to pullbacks of split epimorphisms as follows:

Theorem 2.15. [41] Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:

(i) $\mathbb{C}$ is a Goursat category;

(ii) for any commutative cube

$$
\begin{tikzcd}
X\times_Y Z \arrow[rr, dotted, "\lambda"] \arrow[dr, shift right] \arrow[dd, shift right] & & U\times_W V \arrow[dr, shift right] \arrow[dd, dashed, shift right] & \\
& Z \arrow[ul, shift right] \arrow[rr, two heads, "\gamma"] \arrow[dd, shift right, "l"'] & & V \arrow[ul, shift right] \arrow[dd, shift right, "h"'] \\
X \arrow[uu, shift right] \arrow[rr, dashed, two heads, "\alpha"] \arrow[dr, shift right, "f"'] & & U \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift right, "g"'] & \\
& Y \arrow[uu, shift right, "k"'] \arrow[ul, shift right, "i"'] \arrow[rr, two heads, "\beta"'] & & W, \arrow[uu, shift right] \arrow[ul, dashed, shift right, "j"']
\end{tikzcd}
\tag{8}
$$

where the left and right faces are pullbacks of split epimorphisms and α, β and γ are regular epimorphisms (commuting also with the splittings), then the comparison morphism $\lambda \colon X \times_Y Z \to U \times_W V$ is also a regular epimorphism;

(iii) for any commutative cube

$$
\begin{tikzcd}
X\times_Y Z \arrow[rr, two heads, "\delta"] \arrow[dr, shift right, "\pi_Z"'] \arrow[dd, shift right, "\pi_X"'] & & A \arrow[dr, shift right] \arrow[dd, dashed, shift right] & \\
& Z \arrow[ul, shift right, "{\langle il, 1_Z \rangle}"'] \arrow[rr, two heads, "\gamma"] \arrow[dd, shift right, "l"'] & & V \arrow[ul, shift right] \arrow[dd, shift right, "h"'] \\
X \arrow[uu, shift right, "{\langle 1_X, kf \rangle}"'] \arrow[rr, dashed, two heads, "\alpha"] \arrow[dr, shift right, "f"'] & & U \arrow[uu, dashed, shift right] \arrow[dr, dashed, shift right, "g"'] & \\
& Y \arrow[uu, shift right, "k"'] \arrow[ul, shift right, "i"'] \arrow[rr, two heads, "\beta"'] & & W, \arrow[uu, shift right] \arrow[ul, dashed, shift right, "j"']
\end{tikzcd}
\tag{9}
$$

where the left face is a pullback of split epimorphisms, the right face is a commutative diagram of split epimorphisms and the horizontal arrows $\alpha, \beta, \gamma, \delta$ are regular epimorphisms (commuting also with the splittings), then the right face is a pullback.

The previous theorem can also be expressed as a restricted Beck-Chevalley condition, with respect to the fibration of points, for a special class of commutative squares.

Let $\mathrm{Pt}(\mathbb{C})$ be the category whose objects are split epimorphisms with a chosen splitting (also called points) and morphisms the commutative squares between these data. When $\mathbb{C}$ has pullbacks of split epimorphisms, the functor sending a point to its codomain

$$\begin{array}{ccc} \mathrm{Pt}(\mathbb{C}) & \to & \mathbb{C} \\ U \underset{g}{\overset{j}{\leftrightarrows}} W & \mapsto & W \end{array}$$

is a fibration, called the *fibration of points* [11]. Given a morphism $\beta\colon Y \to W$, the change-of-base functor with respect to this fibration is denoted by $\beta^*\colon \mathrm{Pt}_W(\mathbb{C}) \to \mathrm{Pt}_Y(\mathbb{C})$. If $\mathbb{C}$ has, moreover, pushouts along split monomorphisms, then any pullback functor β^* has a left adjoint

$$\begin{array}{cccc} \beta_! : & \mathrm{Pt}_Y(\mathbb{C}) & \to & \mathrm{Pt}_W(\mathbb{C}), \\ & X \underset{f}{\overset{i}{\leftrightarrows}} Y & \mapsto & \beta_!(X) \underset{\beta_!(f)}{\overset{\beta_!(i)}{\leftrightarrows}} W \end{array}$$

where $(\beta_!(X), \beta_!(f), \beta_!(i)) \in \mathrm{Pt}_W(\mathbb{C})$ is determined by the right hand part of the following pushout:

$$\begin{tikzcd} X \arrow[r, "\overline{\beta}"] & \beta_!(X) \\ Y \arrow[u, "i"] \arrow[r, "\beta"'] & W. \arrow[u, "\beta_!(i)"'] \end{tikzcd}$$

Theorem 2.16. [41] Let $\mathbb{C}$ be a regular category with pushouts along split monomorphisms. Then the following conditions are equivalent:

(i) $\mathbb{C}$ is a Goursat category;

(ii) for any regular epimorphism $\beta\colon Y \twoheadrightarrow W$ in $\mathbb{C}$ the functor $\beta_!\colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_W(\mathbb{C})$ preserves binary products;

(iii) for any commutative square

$$\begin{tikzcd} X \arrow[r, two heads, "\alpha"] \arrow[d, shift right, "f"'] & U \arrow[d, shift right, "g"'] \\ Y \arrow[r, two heads, "\beta"'] \arrow[u, shift right, "i"'] & W \arrow[u, shift right, "j"'] \end{tikzcd}$$

where f and g are split epimorphisms and α and β are regular epimorphisms (commuting also with the splittings), the *Beck-Chevalley* condition holds: there is a functor isomorphism $\alpha_! f^* \cong g^* \beta_!$.

So, we can add Goursat categories to the list of (many) algebraic categories characterized in terms of the fibration of points (see [10, 7] for the case of protomodular, semi-abelian and Mal'tsev categories).

Remark 2.17. The equivalence $(i) \Leftrightarrow (iv)$ in Theorem 2.7 allows to study Mal'tsev categories in a finitely complete context, without the assumption of regularity. Goursat categories can be also defined in a finitely complete context, without the assumption of regularity, and such that in a regular context it coincides with Definition 2.1, see [12]. But in this thesis, we shall always work in regular Goursat categories that we shall call Goursat categories for simplicity.

Remark 2.18. We observe a parallelism between certain properties of Mal'tsev and Goursat categories:

Goursat categories	**Mal'tsev categories**
$RSR = SRS$	$RS = SR$
Goursat pushout	Regular pushout
Denormalized 3-by-3 Lemma	Cuboid Lemma

Regular Mal'tsev (2-permutable) and Goursat (3-permutable) categories are the first two in an infinite sequence of categories. In general, if R and S are two equivalence relations on the same object X and $n \geqslant 0$, we define the relation $(R, S)_n$ by: $(R, S)_0 = 1_X$, $(R, S)_1 = R$, $(R, S)_2 = RS$, $(R, S)_3 = RSR$, $(R, S)_4 = RSRS$, ...

Definition 2.19. A regular category $\mathbb{C}$ is called *n-permutable* whenever $(R, S)_n = (S, R)_n$ for any pair of equivalence relations R and S on the same object.

Regular Mal'tsev and Goursat categories differ from the others n-permutable categories by the following property:

Proposition 2.20. [23, 21] In any regular Mal'tsev and Goursat categories, the lattice of equivalence relations on the same object is modular:

given equivalence relations R, S and T on the same object $X \in \mathbb{C}$ such that $R \leqslant T$, one has:

$$R \vee (S \wedge T) = (R \vee S) \wedge T.$$

In the 4-permutable case, for instance, this property does not hold. In fact, it was shown in [28] that the Polin variety is 4-permutable but the lattice of equivalence relations on the same object is not modular.

Note that the meet of equivalence relations (R, r_1, r_2) and (S, s_1, s_2) on the same object X always exists in any finitely complete category and it is defined by the following pullback:

$$\begin{tikzcd}
R \wedge S \arrow[r] \arrow[d] & S \arrow[d, tail, "\langle s_1, s_2 \rangle"] \\
R \arrow[r, tail, "\langle r_1, r_2 \rangle"'] & X \times X.
\end{tikzcd}$$

A simple calculation shows that every n-permutable category is a $(n+1)$-permutable category. An n-permutable version of Theorem 2.2 is:

Theorem 2.21. [21] For any regular category $\mathbb{C}$ and for any $n \geqslant 2$, the following conditions are equivalent:

(i) $\mathbb{C}$ is n-permutable;

(ii) $\forall R, S \in \mathrm{Equiv}(X)$, the relation $(R, S)_n \in \mathrm{Equiv}(X)$, for any object X in $\mathbb{C}$;

(iii) $\forall R, S \in \mathrm{Equiv}(X)$, $R \vee S = (R, S)_n (= (S, R)_n)$, for any object X in $\mathbb{C}$;

(iv) for any relation $P \rightarrowtail A \times B$ in $\mathbb{C}$ we have $(P, P^o)_{n+1} = (P, P^o)_{n-1}$;

(v) for any reflexive relation $E \rightarrowtail X \times X$ in $\mathbb{C}$, the relation $(E, E^o)_{n-1}$ is an equivalence relation;

(vi) for any reflexive relation $E \rightarrowtail X \times X$ in $\mathbb{C}$, we have $(E, E^o)_{n-1} = (E^o, E)_{n-1}$.

2.3 Weak Goursat categories

In this section, we introduce the notion of weak Goursat category and characterize categories with weak finite limits whose regular completions give rise to Goursat categories. This kind of characterizations have been obtained for the projective covers of categories which are: Mal'tsev [86], extensive [50], topos [77], (locally) cartesian closed [25, 85] , protomodular and semi-abelian [37], (strongly) unital and subtractive [42] and 2-star permutable [1]. As an application, we then relate them to the existence of the quaternary operations characterizing the varieties of universal algebras which are 3-permutable varieties. Most results of this section come from the joint paper with D. Rodelo [84].

The following property from [90] (Proposition 1.1.9) will be useful in the sequel:

Proposition 2.22. [90] Let $\mathbb{C}$ be a projective cover of a regular category $\mathbb{A}$. Let $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ be a pseudo-relation in $\mathbb{C}$ and consider its (regular epimorphism, monomorphism) factorization in $\mathbb{A}$

$$\begin{tikzcd} R \arrow[rr, "(r_1,r_2)"] \arrow[dr, two heads, "p"'] & & X \times X. \\ & E \arrow[ur, tail, "(e_1,e_2)"'] & \end{tikzcd}$$

Then, R is a pseudo-equivalence relation in $\mathbb{C}$ if and only if E is an equivalence relation in $\mathbb{A}$.

In order to characterize the projective covers $\mathbb{C}$ of Goursat categories, we should consider good properties characterizing Goursat categories which could be translated to the weakly lex context. A possible translation of the property in Theorem 2.4 should replace equivalence relations in $\mathbb{A}$ with pseudo-equivalence relations in $\mathbb{C}$ and regular epimorphisms in $\mathbb{A}$ with split epimorphisms in $\mathbb{C}$ since a regular epimorphism in $\mathbb{A}$ with a projective codomain is necessarily a split epimorphism (Lemma 1.18). Thus, we introduce:

Definition 2.23. Let $\mathbb{C}$ be a weakly lex category. We call $\mathbb{C}$ a *weak*

Goursat category if, for any pseudo-equivalence relation $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ and any split epimorphism $X \overset{f}{\rightleftarrows} Y$, the composite $R \overset{fr_1}{\underset{fr_2}{\rightrightarrows}} Y$ is also a pseudo-equivalence relation.

Lemma 2.24. If $\mathbb{C}$ is a regular weak Goursat category, then $\mathbb{C}$ is a Goursat category.

Proof. We shall prove that for any reflexive relation $\langle e_1, e_2 \rangle : E \rightarrowtail X \times X$, EE^o is an equivalence relation (Theorem 2.2).

Consider the (pseudo-)equivalence relation $\mathrm{Eq}(e_1) \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} E$ and the split epimorphism e_2 (which is split by the reflexivity arrow). By assumption $\mathrm{Eq}(e_1) \overset{e_2\pi_1}{\underset{e_2\pi_2}{\rightrightarrows}} X$ is a pseudo-equivalence relation. Its (regular epimorphism, monomorphism) factorization defines the regular image $e_2(\mathrm{Eq}(e_1)) = EE^o$

$$\begin{tikzcd}
\mathrm{Eq}(e_1) \arrow[r, two heads] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & e_2(\mathrm{Eq}(e_1)) = EE^o \arrow[d, shift right] \arrow[d, shift left] \\
E \arrow[r, "e_2"'] & X; \arrow[l, dashed, shift right]
\end{tikzcd}$$

thus EE^o is an equivalence relation. □

We use Remark 1.19 repeatedly in the next results.

Proposition 2.25. Let $\mathbb{C}$ be a projective cover of a regular category $\mathbb{A}$. Then $\mathbb{A}$ is a Goursat category if and only if $\mathbb{C}$ is a weak Goursat category.

Proof. Since $\mathbb{C}$ is a projective cover of a regular category $\mathbb{A}$, $\mathbb{C}$ is weakly lex. Suppose that $\mathbb{A}$ is a Goursat category. Let $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ be a pseudo-equivalence relation in $\mathbb{C}$ and let $X \overset{f}{\rightleftarrows} Y$ be a split epimorphism in $\mathbb{C}$. For the (regular epimorphism, monomorphism) factorizations of $\langle r_1, r_2 \rangle$

and $\langle fr_1, fr_2\rangle$ we get the following diagram

$$\begin{tikzcd}
R \arrow[rr, "\langle r_1,r_2\rangle"] \arrow[dddd, equal] \arrow[dr, two heads, "p"'] & & X \times X \arrow[dddd, "f\times f"] \\
& E \arrow[ur, tail, "\langle e_1,e_2\rangle"'] \arrow[dd, dotted, "w"'] & \\
& & \\
& S \arrow[dr, tail, "\langle s_1,s_2\rangle"] & \\
R \arrow[ur, two heads, "q"] \arrow[rr, "\langle fr_1,fr_2\rangle"'] & & Y \times Y,
\end{tikzcd} \tag{10}$$

where $w : E \longrightarrow S$ is induced by the strong epimorphism p

$$\begin{tikzcd}
R \arrow[r, two heads, "p"] \arrow[d, "q"'] & E \arrow[d, "(f\times f)\langle e_1,e_2\rangle"] \arrow[dl, dotted, "w"'] \\
S \arrow[r, tail, "\langle s_1,s_2\rangle"'] & Y \times Y.
\end{tikzcd}$$

Then w is a regular epimorphism and by the commutativity of the right side of (10), one has $S = f(E)$. By Proposition 2.22, we know that E is an equivalence relation in $\mathbb{A}$. Since $\mathbb{A}$ is a Goursat category and f is a regular epimorphism (being a split one), then $S = f(E)$ is also an equivalence relation in $\mathbb{A}$ (Theorem 2.4) and by Proposition 2.22, we can conclude that $R \overset{fr_1}{\underset{fr_2}{\rightrightarrows}} X$ is a pseudo-equivalence relation in $\mathbb{C}$.

Conversely, suppose that $\mathbb{C}$ is a weak Goursat category. Let $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ be an equivalence relation in $\mathbb{A}$ and $f \colon X \twoheadrightarrow Y$ a regular epimorphism. We are going to show that $f(R) = S$

$$\begin{tikzcd}
R \arrow[r, two heads, "h"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & f(R) = S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X \arrow[r, two heads, "f"'] & Y
\end{tikzcd}$$

is an equivalence relation; it is obviously reflexive and symmetric. In order to conclude that $\mathbb{A}$ is a Goursat category, we must prove that S is transitive.

We begin by covering the regular epimorphism f in $\mathbb{A}$ with a split epimorphism $\bar{f}$ in $\mathbb{C}$. For that we take a cover $y\colon \bar{Y} \twoheadrightarrow Y$ (with $\bar{Y} \in \mathbb{C}$), consider the pullback of y and f in $\mathbb{A}$ and take a cover $\alpha\colon \bar{X} \twoheadrightarrow X \times_Y \bar{Y}$

$$\begin{tikzcd}
\bar{X} \arrow[dr, two heads, "\alpha"] \arrow[drr, bend left, "\bar{f}=f'\alpha"] \arrow[ddr, bend right, two heads, "x=y'\alpha"'] & & \\
& X \times_Y \bar{Y} \arrow[r, two heads, "f'"] \arrow[d, two heads, "y'"'] & \bar{Y} \arrow[d, two heads, "y"] \arrow[ull, dashed, bend right] \\
& X \arrow[r, two heads, "f"'] & Y.
\end{tikzcd}$$

Since $\bar{f} = f'\alpha$ is a regular epimorphism in $\mathbb{A}$ with a projective codomain, it is a split epimorphism (Lemma 1.18). Note that the above outer diagram is a *regular pushout*, so that

$$f^o y = x\bar{f}^o \quad \text{and} \quad y^o f = \bar{f} x^o \tag{11}$$

(Definition 2.10).

Next, we take the inverse image $x^{-1}(R)$ in $\mathbb{A}$, defined by the following pullback

$$\begin{tikzcd}
x^{-1}(R) \arrow[r, two heads, "\pi_R"] \arrow[d, tail, "\langle \rho_1,\rho_2 \rangle"'] & R \arrow[d, tail, "\langle r_1,r_2 \rangle"] \\
\bar{X} \times \bar{X} \arrow[r, two heads, "x \times x"'] & X \times X,
\end{tikzcd}$$

which is an equivalence relation since R is, and cover it with an element $W \in \mathbb{C}$ to obtain a pseudo-equivalence $W \rightrightarrows \bar{X}$ in $\mathbb{C}$. By assumption $W \rightrightarrows \bar{X} \xrightarrow{\bar{f}} \bar{Y}$ is a pseudo-equivalence relation in $\mathbb{C}$ so it factors through an equivalence relation, say $V \underset{v_2}{\overset{v_1}{\rightrightarrows}} \bar{Y}$, in $\mathbb{A}$. We have the commutative diagram

$$
\begin{tikzcd}
W \arrow[rrrr, equal] \arrow[d, two heads, "w"'] & & & & W \arrow[dr, two heads, "v"] \arrow[ddd] & & \\
x^{-1}(R) \arrow[rrrrr, dotted, two heads, "\gamma"] \arrow[dr, two heads, "\pi_R"'] \arrow[dd, tail, "\langle \rho_1,\rho_2 \rangle"'] & & & & & V \arrow[dr, dotted, two heads, "\lambda"] \arrow[ddl, tail, "\langle v_1,v_2 \rangle"'] & \\
 & R \arrow[rrrrr, two heads, "h"] \arrow[dd, tail, "\langle r_1,r_2 \rangle"'] & & & & & S \arrow[dd, tail, "\langle s_1,s_2 \rangle"] \\
\bar{X} \times \bar{X} \arrow[rrrr, two heads, "\bar{f} \times \bar{f}"] \arrow[dr, two heads, "x \times x"'] & & & & \bar{Y} \times \bar{Y} \arrow[drr, two heads, "y \times y"] & & \\
 & X \times X \arrow[rrrrr, two heads, "f \times f"'] & & & & & Y \times Y,
\end{tikzcd}
$$

where γ and λ are induced by the strong epimorphisms w and v, respectively

$$
\begin{tikzcd}
W \arrow[r, two heads, "w"] \arrow[dd, two heads, "v"'] & x^{-1}(R) \arrow[d, tail, "\langle \rho_1,\rho_2 \rangle"] \arrow[ddl, dotted, "\gamma"'] \\
 & \bar{X} \times \bar{X} \arrow[d, two heads, "\bar{f} \times \bar{f}"] \\
V \arrow[r, tail, "\langle v_1,v_2 \rangle"'] & \bar{Y} \times \bar{Y}
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
W \arrow[r, two heads, "v"] \arrow[dd, two heads, "h\pi_R w"'] & V \arrow[d, tail, "\langle v_1,v_2 \rangle"] \arrow[ddl, dotted, "\lambda"'] \\
 & \bar{Y} \times \bar{Y} \arrow[d, two heads, "y \times y"] \\
S \arrow[r, tail, "\langle s_1,s_2 \rangle"'] & Y \times Y.
\end{tikzcd}
$$

Since γ is a regular epimorphism, we have $V = \bar{f}(x^{-1}(R))$. Since λ is a regular epimorphism, we have $S = y(V)$. One also has $V = y^{-1}(S)$

because

$$\begin{array}{rcl} y^{-1}(S) & = & y^o S y \\ & = & y^o f(R) y \\ & = & y^o f R f^o y \\ & = & \bar{f} x^o R x \bar{f}^o \quad \text{(by (11))} \\ & = & \bar{f}(x^{-1}(R)) \\ & = & V. \end{array}$$

Finally, S is transitive since

$$\begin{array}{rcll} SS & = & yy^o S y y^o S y y^o & \text{(Lemma 1.11(2))} \\ & = & y y^{-1}(S) y^{-1}(S) y^o & \\ & = & yVVy^o & \\ & = & yVy^o & \text{(since V is transitive)} \\ & = & y(V) & \\ & = & S. & \end{array}$$

□

We may also consider weak Goursat categories through a property which is more similar to the one mentioned in Theorem 2.4:

Lemma 2.26. Let $\mathbb{C}$ be a projective cover of a regular category $\mathbb{A}$. The following conditions are equivalent:

(i) $\mathbb{C}$ is a weak Goursat category;

(ii) For any commutative diagram in $\mathbb{C}$

$$\begin{tikzcd} R \arrow[r, shift left, "\varphi"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[l, dashed, shift left] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, shift right, "f"'] & Y \arrow[l, dashed, shift right] \end{tikzcd} \tag{12}$$

such that f and φ are split epimorphism and R is a pseudo-equivalence relation, then S is a pseudo-equivalence relation.

Proof. $(i) \Rightarrow (ii)$ Since $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ is a pseudo-equivalence relation, by assumption $R \underset{fr_2}{\overset{fr_1}{\rightrightarrows}} X$ is also a pseudo-equivalence relation and then its (regular epimorphism, monomorphism) factorization gives an equivalence relation $E \underset{e_2}{\overset{e_1}{\rightrightarrows}} Y$ in $\mathbb{A}$ (Proposition 2.22). We have the following commutative diagram

$$\begin{tikzcd}
R \arrow[r, equal] \arrow[dd, shift right, "r_1"'] \arrow[dd, shift left, "r_2"] & R \arrow[rr, shift left, "\varphi"] \arrow[dd, shift right, "fr_1"'] \arrow[dd, shift left, "fr_2"] \arrow[dr, "\rho"] & & S \arrow[ll, shift left, dashed] \arrow[dl, dotted, "\sigma"] \arrow[ddll, bend left, shift right, "s_1"'] \arrow[ddll, bend left, shift left, "s_2"] \\
& & E \arrow[dl, shift right, "e_1"'] \arrow[dl, shift left, "e_2"] & \\
X \arrow[r, shift right, "f"'] & Y \arrow[l, shift right, dashed] & &
\end{tikzcd}$$

where $\sigma \colon S \longrightarrow E$ is induced by the strong (split) epimorphism φ

$$\begin{tikzcd}
R \arrow[r, shift left, "\varphi"] \arrow[d, "\rho"'] & S \arrow[l, shift left, dashed] \arrow[d, "\langle s_1, s_2 \rangle"] \arrow[dl, dotted, "\sigma"'] \\
E \arrow[r, tail, "\langle e_1, e_2 \rangle"'] & Y \times Y.
\end{tikzcd}$$

Then σ is a regular epimorphism and $S \underset{s_2}{\overset{s_1}{\rightrightarrows}} Y$ is a pseudo-equivalence relation (Proposition 2.22).

$(ii) \Rightarrow (i)$ Let $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ be a pseudo-equivalence relation in $\mathbb{C}$ and $X \overset{f}{\rightleftarrows} Y$ a split epimorphism. The following diagram is of the type (12)

$$\begin{tikzcd}
R \arrow[r, equal] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & R \arrow[d, shift right, "fr_1"'] \arrow[d, shift left, "fr_2"] \\
X \arrow[r, shift right, "f"'] & Y. \arrow[l, shift right, dashed]
\end{tikzcd}$$

Since $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ is a pseudo-equivalence relation, then by assumption $R \underset{fr_2}{\overset{fr_1}{\rightrightarrows}} Y$ is also a pseudo-equivalence relation. $\square$

Alternatively, weak Goursat categories can be characterized through a property more similar to the one mentioned in Remark 2.9. A diagram of type (I) in a weakly lex context should have the regular epimorphisms

α and β replaced by split epimorphisms; we call it of type (II):

$$\begin{tikzcd}
X \arrow[r, shift left, "\alpha"] \arrow[d, shift left, "f"] \arrow[dr, phantom, "\text{(II)}"] & U \arrow[l, shift left, dashed] \arrow[d, shift left, "g"] \\
Y \arrow[u, shift left, "s"] \arrow[r, shift right, "\beta"'] & W. \arrow[l, shift right, dashed] \arrow[u, shift left, "t"]
\end{tikzcd}
\qquad
\begin{array}{l} g\alpha=\beta f \\ \alpha s=t\beta \end{array}$$

Note that such a diagram does not necessarily commute with the left ward splittings of α and β.

Proposition 2.27. Let $\mathbb{C}$ be a projective cover of a regular category $\mathbb{A}$. The following conditions are equivalent:

(i) $\mathbb{A}$ is a Goursat category;

(ii) $\mathbb{C}$ is a weak Goursat category;

(iii) For any commutative diagram of type (II) in $\mathbb{C}$

$$\begin{tikzcd}
F \arrow[r, shift left, "\lambda"] \arrow[d, shift right, "\beta_1"'] \arrow[d, shift left, "\beta_2"] & G \arrow[l, shift left, dashed] \arrow[d, shift right, "\rho_1"'] \arrow[d, shift left, "\rho_2"] \\
X \arrow[r, shift left, "\alpha"] \arrow[d, shift left, "f"] \arrow[dr, phantom, "\text{(II)}"] & U \arrow[l, shift left, dashed] \arrow[d, shift left, "g"] \\
Y \arrow[u, shift left, "s"] \arrow[r, shift right, "\beta"'] & W \arrow[l, shift right, dashed] \arrow[u, shift left, "t"]
\end{tikzcd}$$

where F is a weak kernel pair of f and λ is a split epimorphism, then G is a weak kernel pair of g.

Proof. $(i) \Leftrightarrow (ii)$ By Proposition 2.25.

$(i) \Rightarrow (iii)$ If we take the kernel pairs of f and g, then the induced morphism $\bar{\alpha}\colon \mathrm{Eq}(f) \longrightarrow \mathrm{Eq}(g)$ is a regular epimorphism by Theorem 2.8. Moreover, the induced morphism $\varphi\colon F \longrightarrow \mathrm{Eq}(f)$ is also a regular epimorphism (Remark 1.19). We get the commutative diagram

$$\begin{tikzcd}
F \arrow[r, shift left, "\lambda"] \arrow[d, "\varphi"] & G \arrow[l, dashed, shift left] \arrow[d, dotted, "\omega"] \arrow[dd, bend left=60, shift left, "\rho_2"] \arrow[dd, bend left=50, "\rho_1"'] \\
\mathrm{Eq}(f) \arrow[r, two heads, "\bar{\alpha}"] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] & \mathrm{Eq}(g) \arrow[d, shift right, "g_1"'] \arrow[d, shift left, "g_2"] \\
X \arrow[r, shift left, "\alpha"] \arrow[d, shift left, "f"] & U \arrow[l, dashed, shift left] \arrow[d, shift left, "g"] \\
Y \arrow[u, shift left, "s"] \arrow[r, shift right, "\beta"'] & W, \arrow[l, dashed, shift right] \arrow[u, shift left, "t"]
\end{tikzcd}$$

where $w\colon G \longrightarrow \mathrm{Eq}(g)$ is induced by the strong (split) epimorphism λ

$$\begin{tikzcd}
F \arrow[r, shift left, "\lambda"] \arrow[d, "\bar{\alpha}.\varphi"'] & G \arrow[l, dashed, shift left] \arrow[d, "\langle \rho_1, \rho_2 \rangle"] \arrow[dl, dotted, "w"'] \\
\mathrm{Eq}(g) \arrow[r, tail, "\langle g_1, g_2 \rangle"'] & U \times U.
\end{tikzcd}$$

This implies that ω is a regular epimorphism and then $G \overset{\rho_1}{\underset{\rho_2}{\rightrightarrows}} U$ is a weak kernel pair of g.

(iii) $\Rightarrow$ *(ii)* Consider diagram (12) in $\mathbb{C}$ where $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ is a pseudo-equivalence relation. We want to prove that $S \overset{s_1}{\underset{s_2}{\rightrightarrows}} Y$ is also a pseudo-equivalence. Take the (regular epimorphism, monomorphism) factorization of R and S in $\mathbb{A}$ and the induced morphism μ making the following diagram commute

$$\begin{tikzcd}
R \arrow[rr, shift left, "\varphi"] \arrow[dr, "\rho"] \arrow[dd, shift right, "r_1"'] \arrow[dd, shift left, "r_2"] & & S \arrow[ll, dashed, shift left] \arrow[dr, "\sigma"] \arrow[dd, shift right, "s_1"'] \arrow[dd, shift left, "s_2"] & \\
& U \arrow[rr, dotted, "\mu"] \arrow[dl, shift right, "u_1"'] \arrow[dl, shift left, "u_2"] & & V \arrow[dl, shift right, "v_1"'] \arrow[dl, shift left, "v_2"] \\
X \arrow[rr, shift right, "f"'] & & Y. \arrow[ll, dashed, shift right] &
\end{tikzcd}$$

Since μ is a regular epimorphism, $V = f(U)$ and consequently, V is reflexive and symmetric, as the regular image of the equivalence relation U.

To conclude that S is a pseudo-equivalence relation, we just need to prove that V is transitive. We apply our assumption to the diagram

$$
\begin{tikzcd}
F \arrow[rr, two heads, "\lambda=\chi'\beta"] \arrow[dr, two heads, "\beta"'] \arrow[dd, two heads, "\delta=\alpha'\beta"'] & & G \arrow[ll, dashed, bend left=10] \arrow[dd, two heads, "\alpha"] \\
& \mathrm{Eq}(r_1) \times_{\varphi(\mathrm{Eq}(r_1))} G \arrow[ur, two heads, "\chi'"'] \arrow[dl, two heads, "\alpha'"] & \\
\mathrm{Eq}(r_1) \arrow[rr, two heads, "\chi"'] \arrow[d, shift left] \arrow[d, shift right] & & \varphi(\mathrm{Eq}(r_1)) \arrow[d, shift left] \arrow[d, shift right] \\
R \arrow[rr, "\varphi"] \arrow[d, shift left, "r_1"] & & S \arrow[ll, dashed, bend left=10] \arrow[d, shift left, "s_1"] \\
X \arrow[u, shift left, "e_R"] \arrow[rr, "f"'] \arrow[urr, phantom, "\mathrm{(II)}"] & & Y \arrow[u, shift left, "e_S"] \arrow[ll, dashed, bend right=10]
\end{tikzcd}
$$

where G is a cover of the regular image $\varphi(\mathrm{Eq}(r_1))$ and F is a cover of the pullback $\mathrm{Eq}(r_1) \times_{\varphi(\mathrm{Eq}(r_1))} G$. Note that $\lambda = \chi'\beta$ is a regular epimorphism in $\mathbb{A}$ with a projective codomain, so it is a split epimorphism (Lemma 1.18). Since δ is a regular epimorphism, then $F \rightrightarrows R$ is a weak kernel pair of r_1. By assumption $G \rightrightarrows S$ is a weak kernel pair of s_1, thus $\varphi(\mathrm{Eq}(r_1)) = \mathrm{Eq}(s_1)$. We then have

$$
\begin{array}{rclc}
VV & = & v_2 v_1^o v_1 v_2^o & \text{(since } V \text{ is symmetric)} \\
& = & v_2 \sigma\sigma^o v_1^o v_1 \sigma\sigma^o v_2^o & \text{(Lemma 1.11 2)} \\
& = & s_2 s_1^o s_1 s_2^o & (v_i\sigma = s_i) \\
& = & s_2 \varphi r_1^o r_1 \varphi^o s_2^o & (\varphi(\mathrm{Eq}(r_1)) = \mathrm{Eq}(s_1)) \\
& = & f r_2 r_1^o r_1 r_2^o f^o & (s_i\varphi = f r_i\ ) \\
& = & f u_2 \rho\rho^o u_1^o u_1 \rho\rho^o u_2^o f^o & (u_i\rho = r_i) \\
& = & f u_2 u_1^o u_1 u_2^o f^o & \text{(Lemma 1.11 2)} \\
& = & f U U f^o & \text{(since } U \text{ is symmetric)} \\
& = & f U f^o & \text{(since } U \text{ is transitive)} \\
& = & V. & (\ f(U) = V)
\end{array}
$$

$\square$

As an application of these results, one obtains the existence of the quaternary operations which characterize the varieties of universal algebras which are 3-permutable.

Proposition 2.28. Let $\mathbb{A}$ be a variety of universal algebras and $\mathbb{C}$ its full subcategory of free algebras. Then the following conditions are equivalent:

(i) $\mathbb{A}$ is a Goursat category;

(ii) $\mathbb{C}$ is a weak Goursat category;

(iii) the algebraic theory of $\mathbb{A}$ contains two quaternary operations p and q satistying

$$\begin{cases} p(x,y,y,z) &= x \\ p(x,x,y,y) &= q(x,x,y,y) \\ q(x,y,y,z) &= z. \end{cases}$$

Proof. By assumption $\mathbb{C}$ is a projective cover of $\mathbb{A}$, thus conditions (i) and (ii) are equivalent by Proposition 2.25.

$(ii) \Rightarrow (iii)$ Let X denote the free algebra on one element. Diagram (II) below belongs to $\mathbb{C}$

$$\begin{tikzcd}
F \arrow[r, equal] \arrow[d, two heads, "\mu"] & F \arrow[d, two heads, "\lambda\mu"] \\
\mathrm{Eq}(\nabla_2+\nabla_2) \arrow[r, dotted, "\lambda"] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & \mathrm{Eq}(\nabla_3) \arrow[d, shift right] \arrow[d, shift left] \\
4X \arrow[r, shift left, "1_X+\nabla_2+1_X"] \arrow[d, shift left, two heads, "\nabla_2+\nabla_2"] \arrow[dr, phantom, "\mathrm{(II)}"] & 3X \arrow[l, shift left, dashed] \arrow[d, shift left, two heads, "\nabla_3"] \\
2X \arrow[u, shift left, "\iota_2+\iota_1"] \arrow[r, shift right, "\nabla_2"'] & X. \arrow[u, shift left, "\iota_2"] \arrow[l, shift right, dashed]
\end{tikzcd}$$

Here ∇_i denotes the codiagonal from the i-indexed copower of X to X and ι_k the k-*th* injection into a copower. If F is a cover of $\mathrm{Eq}(\nabla_2+\nabla_2)$, then $F \rightrightarrows 4X$ is a weak kernel pair of $\nabla_2+\nabla_2$. By assumption $F \rightrightarrows 3X$ is a weak kernel pair of ∇_3, so that $\lambda\mu$ is surjective. We then conclude that λ is surjective.

The terms $p_1(x,y,z)=x$ and $p_2(x,y,z)=z$ are such that $(p_1,p_2)\in \mathrm{Eq}(\nabla_3)$. By surjectivity of λ, $\exists(p,q)\in \mathrm{Eq}(\nabla_2+\nabla_2)$ such that $p(x,y,y,z)=p_1(x,y,z)=x$, $q(x,y,y,z)=p_2(x,y,z)=z$ and $p(x,x,y,y)=q(x,x,y,y)$ since $(p,q)\in \mathrm{Eq}(\nabla_2+\nabla_2)$.

$(iii)\Rightarrow(i)$ It suffices to prove that $\mathbb{A}$ is a 3-permutable variety (see [53] and the references therein). Let R and S be two congruences on the same algebra X in $\mathbb{A}$, we are going to prove that $RSR=SRS$. For that it suffices to prove that $RSR\leqslant SRS$. Suppose that $(x,v)\in RSR$, then there exist elements y and u such that $(x,y)\in R$, $(y,u)\in S$ and $(u,v)\in R$

One has

$$\begin{array}{c} xSx \\ ySy \\ ySu \\ vSv, \end{array}$$

and by applying the quaternary operations p and q, one obtains $xSp(x,y,u,v)$ and $vSq(x,y,u,v)$.

One also has

$$\begin{array}{c} xRy \\ yRy \\ uRv \\ vRv, \end{array}$$

and by applying the quaternary operations p and q, on obtains $p(x,y,u,v)Rp(y,y,v,v)$ and $q(x,y,u,v)Rq(y,y,v,v)$.
Since $q(y,y,v,v)=p(y,y,v,v)$, it the follows that

$$xSp(x,y,u,v)Rq(x,y,u,v)Sv,$$

and then $(x,v)\in SRS$. □

Remark 2.29. The Proposition 2.28 is also true for quasi-varieties.

Chapter 3

Internal structures in Goursat categories

In [16] D. Bourn and M. Gran introduced the notion of connector between two internal equivalence relations which is deeply related to the notion of pregroupoid [70, 71] and then to Commutator Theory. One of the main interests of the notion of connector is that it enables us to understand centrality even without defining the commutator of equivalence relations. Indeed, thanks to this notion one can prove the important basic centrality properties which correspond to the classical properties of the commutator. In [15] Bourn and Gran developed this notion of connector in the context of Mal'tsev categories and used it to characterize Mal'tsev categories. In this chapter, we establish some basic properties of Goursat categories in terms of connectors, as it was done in [15] for the case of Mal'tsev categories. These results have turned out to be useful to develop a monoidal approach to internal structures [38]. We then give a new characterization of Goursat categories in terms of properties of internal categories and internal groupoids, on the model of what was done in [36] in the case of Mal'tsev categories. The main results of this chapter come from the joint paper with M. Gran and D. Rodelo [45].

3.1 Equivalence relations in Goursat categories

In this section we investigate the category $\mathrm{Equiv}(\mathbb{C})$ of internal equivalence relations in a regular category $\mathbb{C}$. We show that $\mathrm{Equiv}(\mathbb{C})$ is a Goursat category whenever $\mathbb{C}$ is.

The category $\mathrm{Equiv}(\mathbb{C})$ is finitely complete whenever $\mathbb{C}$ is: any finite limit in $\mathrm{Equiv}(\mathbb{C})$ is computed "levelwise". In particular, the terminal object in $\mathrm{Equiv}(\mathbb{C})$ is the discrete equivalence relation

$$1 \rightrightarrows 1$$

on the terminal object 1 of $\mathbb{C}$. The kernel pair of a morphism (f, g) in $\mathrm{Equiv}(\mathbb{C})$ is given by the kernel pairs $\mathrm{Eq}(f)$ of f and $\mathrm{Eq}(g)$ of g in $\mathbb{C}$

$$\begin{tikzcd} \mathrm{Eq}(g) \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[d, shift right, "\bar{r_1}"'] \arrow[d, shift left, "\bar{r_2}"] & R \arrow[r, "g"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ \mathrm{Eq}(f) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & X \arrow[r, "f"'] & Y. \end{tikzcd} \tag{13}$$

Consequently, a morphism (f, g) is a monomorphism in $\mathrm{Equiv}(\mathbb{C})$ if and only if both f and g are monomorphisms in $\mathbb{C}$. When $\mathbb{C}$ is a Goursat category, a similar property holds with respect to regular epimorphisms:

Lemma 3.1. Let R and S be two equivalence relations in a Goursat category $\mathbb{C}$ and $(f, g)\colon R \to S$ a morphism

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{14}$$

in $\mathrm{Equiv}(\mathbb{C})$. Then (f, g) is a regular epimorphism in $\mathrm{Equiv}(\mathbb{C})$ if and only if both f and g are regular epimorphisms in $\mathbb{C}$.

Proof. When f and g are regular epimorphisms in $\mathbb{C}$, it is not difficult to check that (f, g) is necessarily the coequalizer of its kernel pair in $\mathrm{Equiv}(\mathbb{C})$ given in (13) (one uses the fact that $g = \mathit{coeq}(g_1, g_2)$ and $f = \mathit{coeq}(f_1, f_2)$ in $\mathbb{C}$).

Conversely, let (f, g) be a morphism in $\mathrm{Equiv}(\mathbb{C})$ as in (14) that is a regular epimorphism in $\mathrm{Equiv}(\mathbb{C})$. Consider the kernel pairs of f and g, the (regular epimorphism, monomorphism) factorization $f = iq$ of f, and the regular image $(q(R), t_1, t_2)$ of (R, r_1, r_2) along q. We obtain the following commutative diagram

$$\begin{tikzcd}
\mathrm{Eq}(g) \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[dd, shift right, "\bar{r_1}"'] \arrow[dd, shift left, "\bar{r_2}"] & R \arrow[rr, "g"] \arrow[dr, "\alpha"'] \arrow[dd, shift right, "r_1"'] \arrow[dd, shift left, "r_2"] & & S \arrow[dd, shift right, "s_1"'] \arrow[dd, shift left, "s_2"] \\
& & q(R) \arrow[ur, dotted, "j"'] \arrow[dd, shift right, "t_1"'] \arrow[dd, shift left, "t_2"] & \\
\mathrm{Eq}(f) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y, \\
& & Z \arrow[ur, tail, "i"'] &
\end{tikzcd} \tag{15}$$

where $(q(R), t_1, t_2) \in \mathrm{Equiv}(\mathbb{C})$ (by Theorem 2.4) and (i, j) is the morphism in $\mathrm{Equiv}(\mathbb{C})$ such that $(i, j)(q, \alpha) = (f, g)$. Note that j is induced by the fact that $(i \times i)\langle t_1, t_2\rangle\alpha$ is the (regular epimorphism, monomorphism) factorization of $\langle s_1, s_2\rangle g$, thus it is a monomorphism

$$\begin{tikzcd}
R \arrow[r, two heads, "\alpha"] \arrow[d, "g"'] & q(R) \arrow[dl, dotted, tail, "j"'] \arrow[d, tail, "(i\times i)\langle t_1,t_2\rangle"] \\
S \arrow[r, tail, "\langle s_1,s_2\rangle"'] & Y \times Y.
\end{tikzcd}$$

From the fact that (f, g) is the coequalizer of its kernel pair in $\mathrm{Equiv}(\mathbb{C})$ and that $(q,\ \alpha)$ is the coequalizer of $(f_1,\ g_1)$ and $(f_2,\ g_2)$ in $\mathrm{Equiv}(\mathbb{C})$ (since $(i,\ j)$ is a monomorphism), it easily follows that (i, j) is an isomorphism in $\mathrm{Equiv}(\mathbb{C})$. This implies that f and g are regular epimorphisms in $\mathbb{C}$. $\square$

Proposition 3.2. Let $\mathbb{C}$ be a Goursat category. Then the category $\mathrm{Equiv}(\mathbb{C})$ is a regular category.

Proof. As mentioned above, the category $\mathrm{Equiv}(\mathbb{C})$ is finitely complete because $\mathbb{C}$ is so. Lemma 3.1 implies that regular epimorphisms in $\mathrm{Equiv}(\mathbb{C})$ are stable under pullbacks since regular epimorphisms are stable in $\mathbb{C}$, and regular epimorphisms in Equiv($\mathbb{C}$) are "levelwise" regular epimorphisms. The existence of the (regular epimorphism, monomorphism) factorization of a morphism (f, g) as in (14) in the category Equiv($\mathbb{C}$) follows from the construction of diagram (15): the (regular epimorphism, monomorphism) factorization $f = iq$ of f in $\mathbb{C}$ gives rise to the (regular epimorphism, monomorphism) factorization $g = j\alpha$ of g in $\mathbb{C}$. Thus $(i, j)(q, \alpha)$ is the (regular epimorphism, monomorphism) factorization of (f, g) in Equiv($\mathbb{C}$). ☐

Proposition 3.3. Let $\mathbb{C}$ be a Goursat category. Then the category Equiv($\mathbb{C}$) is a also a Goursat category.

Proof. Let $(R, (p_1, q_1), (p_2, q_2))$ be an equivalence relation on (S, s_1, s_2) in the category Equiv($\mathbb{C}$) and $f = (f_1, f_2)$ a regular epimorphism in Equiv($\mathbb{C}$). We must prove that $f(R)$ is an equivalence relation in Equiv($\mathbb{C}$). The relation $f(R)$ is obtained through the following diagram

$$\begin{tikzcd}
R \arrow[rr, two heads, "t"] \arrow[dr, shift left, "r_2"] \arrow[dr, shift right, "r_1"'] \arrow[dd, shift right, "p_1"'] \arrow[dd, shift left, "p_2"] & & U \arrow[dr, dotted, shift left, "\beta_2"] \arrow[dr, dotted, shift right, "\beta_1"'] \arrow[dd, dashed, shift right, "\alpha_1"'] \arrow[dd, dashed, shift left, "\alpha_2"] & \\
& X \arrow[rr, two heads, "k"] \arrow[dd, shift right, "q_1"'] \arrow[dd, shift left, "q_2"] & & V \arrow[dd, shift right, "a"'] \arrow[dd, shift left, "b"] \\
S \arrow[rr, dashed, two heads, "f_2"] \arrow[dr, shift left, "s_2"] \arrow[dr, shift right, "s_1"'] & & W \arrow[dr, dashed, shift left, "d"] \arrow[dr, dashed, shift right, "c"'] & \\
& Y \arrow[rr, two heads, "f_1"'] & & Z,
\end{tikzcd} \tag{16}$$

where $U = f_2(R)$ and $V = f_1(X)$ are the regular images in $\mathbb{C}$.

One has:

$$\begin{aligned}
akr_1 &= f_1 q_1 r_1 \\
&= f_1 s_1 p_1 \\
&= c f_2 p_1 \\
&= c\alpha_1 t
\end{aligned}$$

and in the same way $bkr_1 = c\alpha_2 t$, thus the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, two heads, "t"] \arrow[d, "kr_1"'] & U \arrow[d, "\langle c\alpha_1, d\alpha_2 \rangle"] \\ V \arrow[r, tail, "\langle a,b \rangle"'] & Z \times Z. \end{tikzcd}$$

Since t is a strong epimorphism and $\langle a, b\rangle$ is a monomorphism, there exists $\beta_1 \colon U \to V$ such that $\langle a, b\rangle\beta_1 = \langle c\alpha_1, c\alpha_2\rangle$. The existence of β_2 is obtained similarly. Since $(\langle a, b\rangle \times \langle a, b\rangle)\langle \beta_1, \beta_2\rangle = \langle c \times c, d \times d\rangle\langle \alpha_1, \alpha_2\rangle$, it follows that $\langle \beta_1, \beta_2\rangle \colon U \to V \times V$ is a monomorphism and thus (U, β_1, β_2) is a relation on V. All parallel morphisms of the left face represent equivalence relations and all horizontal morphisms are regular epimorphisms, so that all parallel morphisms of the right face also represent equivalence relations (Theorem 2.4), and then $f(R)$ is an equivalence relation in Equiv($\mathbb{C}$). $\square$

Remark 3.4. Lemma 3.1 and Proposition 3.2 were proved in [40, 45] and, independently, in [12].

3.2 Connectors in Goursat categories

In this section we prove that connectors are stable under quotients in any Goursat category $\mathbb{C}$. We then define the category Conn($\mathbb{C}$) of connectors in $\mathbb{C}$ whose objects are pairs of equivalence relations equipped with a connector, and prove that Conn($\mathbb{C}$) is a Goursat category whenever the base category $\mathbb{C}$ is.

Definition 3.5. Let (R, r_1, r_2) and (S, s_1, s_2) be two equivalence relations on an object X and $(R \times_X S,\ p_1,\ p_2)$ the pullback of r_2 along s_1

$$\begin{tikzcd} R \times_X S \arrow[r, "p_2"] \arrow[d, "p_1"'] & S \arrow[d, "s_1"] \\ R \arrow[r, "r_2"'] & X. \end{tikzcd}$$

A *connector* [15] between R and S is an arrow $p \colon R \times_X S \longrightarrow X$ in $\mathbb{C}$ satisfying:

1. $xSp(x,y,z)Rz$;

2. Mal'tsev identities: $p(x,x,y)=y$ and $p(x,y,y)=x$;

3. associativity: $p(x,y,p(z,u,v))=p(p(x,y,z),u,v)$,

when each term is defined.

Example 3.6. 1. If ∇_X is the largest equivalence relation on an object X, then an associative Mal'tsev operation

$$p\colon X\times X\times X\longrightarrow X$$

is precisely a connector between ∇_X and ∇_X.

2. Let X and Y be two objects in a finitely complete category $\mathbb{C}$ and $(X\times Y,p_X,p_Y)$ their product. Then the canonical arrow $p\colon X\times X\times Y\times Y\to X\times Y$ defined by

$$p(x,x',y,y')=(x',y)$$

is a connector between the kernel pairs $\mathrm{Eq}(p_X)$ and $\mathrm{Eq}(p_Y)$ of the product projections p_X and p_Y.

It is well known that Goursat categories satisfy the so-called *Shifting Property* [51, 14]. In this context connectors are unique when they exist (Theorem 2.13 and Proposition 5.1 in [14]): accordingly, for a given pair of equivalence relations on the same object in a Goursat category the fact of having a connector becomes a *property.*

The notion of connector is also related to double equivalence relations.

Definition 3.7. Let R and S be two equivalence relations on an object X. A *double equivalence relation* on R and S is an internal equivalence relation from R to S in the category of internal equivalence relations, i.e. an object $C\in\mathbb{C}$ equipped with two equivalence relations

$(\pi_1, \pi_2)\colon C \rightrightarrows S$ and $(p_1, p_2)\colon C \rightrightarrows R$ such that the following diagram

$$\begin{tikzcd} C \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \end{tikzcd}$$

commutes (in the "obvious" way).

A double equivalence relation C on R and S is called a *centralizing relation* [24] when the square

$$\begin{tikzcd} C \arrow[r, "\pi_1"] \arrow[d, "p_1"'] & S \arrow[d, "r_1"] \\ R \arrow[r, "s_1"'] & X \end{tikzcd}$$

is a pullback.

By the symmetry of the equivalence relations it follows that any of the four commutative squares in the definition of a centralizing relation is a pullback.

The definition of connectors in terms of centralizing relations is given by the following lemma.

Lemma 3.8. [15] If $\mathbb{C}$ is a category with finite limits and R and S are two equivalence relations on the same object X, then the following conditions are equivalent:

(i) there exists a connector between R and S;

(ii) there exists a centralizing relation on R and S.

Proof. (i) $\Rightarrow$ (ii) Let $p\colon R \times_X S \longrightarrow X$ be a connector between R and S. Then by defining the arrows $\beta_1\colon R \times_X S \longrightarrow S$ and $\beta_2\colon R \times_X S \longrightarrow R$ by $\beta_1(x, y, z) = (x, p(x, y, z))$ and $\beta_2(x, y, z) = (p(x, y, z), z)$, one obtains the following centralizing relation on R and S:

$$\begin{tikzcd} R \times_X S \arrow[r, shift left, "\beta_1"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "\beta_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \end{tikzcd}$$

(ii) ⇒ (i) Let

$$\begin{tikzcd}
C \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X
\end{tikzcd}$$

be a centralizing relation on R and S, then the morphism
$r_1 p_1 \colon C = R \times_X S \longrightarrow X$ defines a connector between R and S. $\square$

In [15] D. Bourn and M. Gran established some basic centrality properties in a Mal'tsev category. In particular, they proved that when $\mathbb{C}$ is a Mal'tsev category, R and S are equivalence relations on an object X with a connector and $i : I \rightarrowtail X$ is a monomorphism, then the inverse images $i^{-1}(R)$ and $i^{-1}(S)$ also have a connector. By Theorem 12 in [10] it follows that the converse of this result is also satisfied. We establish a similar property for Goursat categories, with respect to regular epimorphisms:

Proposition 3.9. Let $\mathbb{C}$ be a Goursat category, R and S two equivalence relations on an object X, and let $f \colon X \twoheadrightarrow Y$ be a regular epimorphism. If there exists a connector between R and S, then there exists a connector between the regular images $f(R)$ and $f(S)$.

Proof. Suppose that there exists a connector between R and S. This implies that there exists a centralizing relation $(C, (\pi_1, \pi_2), (p_1, p_2))$ on R and S. Consider the regular image $(f(R), a, b)$ and $(f(S), c, d)$ of R and S along f. We obtain the following diagram

$$\begin{tikzcd}
C \arrow[rr, two heads, "\alpha"] \arrow[dr, shift left, "\pi_2"] \arrow[dr, shift right, "\pi_1"'] \arrow[dd, shift right, "p_1"'] \arrow[dd, shift left, "p_2"] & & f_R(C) \arrow[dr, shift left, "\beta_2"] \arrow[dr, shift right, "\beta_1"'] \arrow[dd, dashed, shift right, "\alpha_1"'] \arrow[dd, dashed, shift left, "\alpha_2"] & \\
& R \arrow[rr, two heads, "f_R"] \arrow[dd, shift right, "r_1"'] \arrow[dd, shift left, "r_2"] & & f(R) \arrow[dd, shift right, "a"'] \arrow[dd, shift left, "b"] \\
S \arrow[rr, dashed, two heads, "f_S"] \arrow[dr, shift left, "s_2"] \arrow[dr, shift right, "s_1"'] & & f(S) \arrow[dr, dashed, shift left, "d"] \arrow[dr, dashed, shift right, "c"'] & \\
& X \arrow[rr, two heads, "f"'] & & Y,
\end{tikzcd} \tag{17}$$

where $(f_R(C), \beta_1, \beta_2)$ is the regular image of the equivalence relation (C, π_1, π_2) along the regular epimorphism f_R. The fact that the square

$$\begin{tikzcd} C \arrow[r, two heads, "\alpha"] \arrow[d, "f_S p_1"'] & f_R(C) \arrow[d, "\langle a\beta_1, a\beta_2\rangle"] \arrow[dl, dotted, "\alpha_1"'] \\ f(S) \arrow[r, tail, "\langle c,d\rangle"'] & Y \times Y \end{tikzcd}$$

commutes, α is a strong epimorphism and $\langle c, d\rangle$ is a monomorphism, implies the existence of an arrow $\alpha_1 \colon f_R(C) \longrightarrow f(S)$ making the above diagram commute. Similarly, from the commutativity

$$\begin{tikzcd} C \arrow[r, two heads, "\alpha"] \arrow[d, "f_S p_2"'] & f_R(C) \arrow[d, "\langle b\beta_1, b\beta_2\rangle"] \arrow[dl, dotted, "\alpha_2"'] \\ f(S) \arrow[r, tail, "\langle c,d\rangle"'] & Y \times Y \end{tikzcd}$$

we obtain an arrow $\alpha_2 \colon f_R(C) \longrightarrow f(S)$.

The relations $(f_R(C), \beta_1, \beta_2)$, $(f(R), a, b)$ and $(f(S), c, d)$ are all equivalence relations by Theorem 2.4. It is then easy to check that the relation $(f_R(C), \alpha_1, \alpha_2)$ is an equivalence relation on $f(S)$. In fact, the morphism $\langle \alpha_1, \alpha_2\rangle \colon f_R(C) \to f(S) \times f(S)$ is a monomorphism since $\langle c \times c, d \times d\rangle\langle \alpha_1, \alpha_2\rangle = \langle a, b\rangle \times \langle a, b\rangle\langle \beta_1, \beta_2\rangle$. So, $\langle \alpha_1, \alpha_2\rangle$ is the regular image of $\langle p_1, p_2\rangle$ along f_S, thus it is an equivalence relation on $f(S)$ by Theorem 2.4.

By assumption all the left squares of (17) are pullbacks and all the horizontal morphisms are regular epimorphisms, so it follows that all the right squares of (17) are pullbacks as well by Theorem 2.15 (iii). It then follows that $(f_R(C), (\alpha_1, \alpha_2), (\beta_1, \beta_2))$ is a centralizing relation on $f(R)$ and $f(S)$. By Lemma 3.8 there is a connector between $f(R)$ and $f(S)$. $\square$

We are now going to show that the category whose objects are pairs of equivalence relations equipped with a connector is a Goursat category whenever the base category is a Goursat category. For this, let us first fix some notation: if $\mathbb{C}$ is a finitely complete category, we write 2-Eq($\mathbb{C}$) for the category whose objects (R, S, X) are pairs of equivalence relations R

and S on the same object X, $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X \overset{s_1}{\underset{s_2}{\leftleftarrows}} S$, and arrows are triples (f_R, f_S, f) making the following diagram commute:

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, "f_R"'] & X \arrow[d, "f"'] & S \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "s_2"] \arrow[d, "f_S"] \\ \bar{R} \arrow[r, shift left, "\bar{r_1}"] \arrow[r, shift right, "\bar{r_2}"'] & \bar{X} & \bar{S}. \arrow[l, shift right, "\bar{s_1}"'] \arrow[l, shift left, "\bar{s_2}"] \end{tikzcd} \tag{18}$$

We write Conn($\mathbb{C}$) for the subcategory of 2-Eq($\mathbb{C}$) whose objects (R, S, X, p) are pairs of equivalence relations R and S on an object X with a given connector $p : R \times_X S \to X$; arrows in Conn($\mathbb{C}$) are arrows in 2-Eq($\mathbb{C}$) respecting the connectors. This means that, given a diagram (18) where both (R, S, X) and $(\bar{R}, \bar{S}, \bar{X})$ are in Conn($\mathbb{C}$), with $p\colon R \times_X S \to X$ and $\bar{p}\colon \bar{R} \times_{\bar{X}} \bar{S} \to \bar{X}$ the corresponding connectors, then the diagram

$$\begin{tikzcd} R \times_X S \arrow[r, "f_R \times_X f_S"] \arrow[d, "p"'] & \bar{R} \times_{\bar{X}} \bar{S} \arrow[d, "\bar{p}"] \\ X \arrow[r, "f"'] & \bar{X} \end{tikzcd}$$

commutes, where $f_R \times_X f_S$ is the natural map induced by the universal property of the pullback $\bar{R} \times_{\bar{X}} \bar{S}$.

We say that a subcategory $\mathbb{P}$ is *closed under (regular) quotients* in a category $\mathbb{Q}$ if, for any regular epimorphism $f\colon A \twoheadrightarrow B$ in $\mathbb{Q}$ such that $A \in \mathbb{P}$, then $B \in \mathbb{P}$.

Proposition 3.10. If $\mathbb{C}$ is a Goursat category, then Conn($\mathbb{C}$) is a full subcategory of 2-Eq($\mathbb{C}$), that is closed in 2-Eq($\mathbb{C}$) under quotients.

Proof. The fullness of the forgetful functor Conn($\mathbb{C}$) $\to$ 2-Eq($\mathbb{C}$) follows from Corollary 3.2 in [14], by taking into account the fact that any Goursat category satisfies the Shifting Property.

Let us then consider a regular epimorphism in 2-Eq($\mathbb{C}$)

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, two heads, "f_R"'] & X \arrow[d, two heads, "f"'] & S \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "s_2"] \arrow[d, two heads, "f_S"] \\ \bar{R} \arrow[r, shift left, "\bar{r_1}"] \arrow[r, shift right, "\bar{r_2}"'] & \bar{X} & \bar{S} \arrow[l, shift right, "\bar{s_1}"'] \arrow[l, shift left, "\bar{s_2}"] \end{tikzcd}$$

(this means that f, f_R and f_S are regular epimorphisms in $\mathbb{C}$) such that its domain (R, S, X) belongs to $\mathrm{Conn}(\mathbb{C})$. The equalities $f(R) = \bar{R}$ and $f(S) = \bar{S}$, together with Proposition 3.9, imply that there exists a connector between $\bar{R}$ and $\bar{S}$. □

Lemma 3.11. Let $\mathbb{D}$ be a finitely complete category, and $\mathbb{C}$ a full subcategory of $\mathbb{D}$ closed in $\mathbb{D}$ under finite limits and quotients. Then:

1. $\mathbb{C}$ is regular whenever $\mathbb{D}$ is regular.

2. $\mathbb{C}$ is a Goursat category whenever $\mathbb{D}$ is a Goursat category.

Proof. The (regular epimorphism, monomorphism) factorization in $\mathbb{D}$ of an arrow in $\mathbb{C}$ is also its factorization in $\mathbb{C}$, since $\mathbb{C}$ is closed in $\mathbb{D}$ under quotients. Since finite limits in $\mathbb{C}$ are calculated as in $\mathbb{D}$, it follows that regular epimorphisms are stable under pullbacks. Now the second statement easily follows from the fact that the composition of relations is computed in the same way in $\mathbb{C}$ and in $\mathbb{D}$. □

Theorem 3.12. If $\mathbb{C}$ is a Goursat category then $\mathrm{Conn}(\mathbb{C})$ is a Goursat category.

Proof. Using similar arguments as those given in the proof of Proposition 3.3 with respect to $\mathrm{Equiv}(\mathbb{C})$, one may deduce that $2\text{-}\mathrm{Eq}(\mathbb{C})$ is a Goursat category. The result then follows from Proposition 3.10 and Lemma 3.11. □

3.3 Internal groupoids in Goursat categories

In this section, we give a new characterization of Goursat categories in terms of properties of internal categories and internal groupoids.

Definition 3.13. An internal *reflexive graph* in a category $\mathbb{C}$ is a diagram of the form

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0 \arrow[l, "e" description] \end{tikzcd}$$

such that $d\ e = 1_{X_0} = c\ e$.

We write RG($\mathbb{C}$) for the category of internal reflexive graphs in $\mathbb{C}$ with obvious morphisms.

Definition 3.14. An *internal category* in a category $\mathbb{C}$ with pullbacks is a reflexive graph with a morphism $m\colon X_1 \times_{X_0} X_1 \to X_1$

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0, \arrow[l, "e" description] \end{tikzcd}$$

where $(X_1 \times_{X_0} X_1, p_1, p_2)$ is the pullback of d and c

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] & X_1 \arrow[d, "d"] \\ X_1 \arrow[r, "c"'] & X_0 \end{tikzcd}$$

and such that:

- $dm = dp_1, \quad cm = cp_2$;
- $m\langle ed, 1_{X_1}\rangle = 1_{X_1} = m\langle 1_{X_1}, ec\rangle$;
- $m(1_{X_1} \times_{X_0} m) = m(m \times_{X_1} 1_{X_1})$.

The object X_0 is called the "object of objects", X_1 the "object of arrows", $X_1 \times_{X_0} X_1$ the "object of composable pairs of arrows". The morphisms d and c are called "domain" and "codomain" respectively, e is the "identity", and m is the "composition".

Definition 3.15. An internal category in a category $\mathbb{C}$ with pullbacks

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, shift left=2, "p_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "p_2"'] & X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0, \arrow[l, "e" description] \end{tikzcd}$$

is an *internal groupoid* when there is an additional morphism
$i\colon X_1 \longrightarrow X_1$, called "inversion", satisfying the axioms:

- $di = c, \quad ci = d$;
- $m\langle i, 1_{X_1}\rangle = ec \quad$ and $\quad m\langle 1_{X_1}, i\rangle = ed$.

Example 3.16. An internal category in the category **Set** is a small category. An equivalence relation is a special kind of groupoid, where its domain and codomain morphisms are jointly monomorphic; also any reflexive and transitive relation, that is a preorder, is in particular an internal category.

We write Cat($\mathbb{C}$) and Grpd($\mathbb{C}$) for the categories of internal categories and internal groupoids in $\mathbb{C}$, respectively. The morphisms in these categories are called internal functors: they are pairs of arrows (f_0, f_1) in $\mathbb{C}$, as in the diagram

$$\begin{tikzcd}
X_1 \times_{X_0} X_1 \arrow[r, dashed, "f_2"] \arrow[d, "m_X"'] & X'_1 \times_{X_0} X'_1 \arrow[d, "m_{X'}"'] \\
X_1 \arrow[r, "f_1"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & X'_1 \arrow[d, shift right=2, "d'"'] \arrow[d, shift left=2, "c'"] \\
X_0 \arrow[r, "f_0"'] \arrow[u, "e"] & X'_0, \arrow[u, "e'"]
\end{tikzcd}$$

such that:

- $f_0\ d = d'\ f_1, \qquad f_0\ c = c'\ f_1$
- $f_1\ e = e'\ f_0, \qquad f_1\ m_X = m_{X'}\ f_2$

(where f_2 is the arrow induced by the universal property of the pullback).

Connectors provide a way to distinguish groupoids amongst reflexive graphs:

Proposition 3.17. [24] Given a reflexive graph

$$\begin{tikzcd}
X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0 \arrow[l, "e" description]
\end{tikzcd}$$

in a finitely complete category $\mathbb{C}$, the connectors between Eq(c) and Eq(d) are in bijection with the groupoid structures on this reflexive graph.

In fact, if p is a connector between Eq(d) and Eq(c) then the arrow $m\colon X_1 \times_{X_0} X_1 \to X_1$ (internally) defined by: $\forall (x, y) \in X_1 \times_{X_0} X_1$,

$m(x, y) = p(y, (ec)(x), x)$ gives the composition of a groupoid structure. Conversely, if X is equipped with a groupoid composition m and an inversion i, a connector p between $\mathrm{Eq}(c)$ and $\mathrm{Eq}(d)$ is obtained by setting $p(x, y, z) = m(m(z, i(y)), x)$ for any $(x, y, z) \in \mathrm{Eq}(c) \times_{X_1} \mathrm{Eq}(d)$.

To obtain our new characterization of Goursat categories, the following theorem will be useful:

Theorem 3.18. [76] Let $\mathbb{C}$ be a regular category, then the following conditions are equivalent:

(i) all reflexive and transitive relations in $\mathbb{C}$ are equivalence relations;

(ii) all internal categories in $\mathbb{C}$ are internals groupoids.

Also, *(i)*, *(ii)* hold if $\mathbb{C}$ is n-permutable, $n \geqslant 2$

It then follows that if $\mathbb{C}$ is a Goursat category, then any reflexive and transitive relation is an equivalence relation or, equivalently, any internal category is a groupoid. Then Theorem 2.4, which could equivalently be stated through the property that $\mathrm{Equiv}(\mathbb{C})$ (or the category of reflexive and transitive relations in $\mathbb{C}$) is closed in the category of reflexive relations in $\mathbb{C}$ under quotients, has an *extended* counterpart given below. This characterization leads to the observation that the structural aspects of Goursat categories mainly concern groupoids (rather than equivalence relations).

We are now ready to prove the main theorem of this section.

Theorem 3.19. Let $\mathbb{C}$ be a regular category. Then the following conditions are equivalent:

(i) $\mathbb{C}$ is a Goursat category;

(ii) $\mathrm{Grpd}(\mathbb{C})$ is closed in $\mathrm{RG}(\mathbb{C})$ under quotients;

(iii) $\mathrm{Cat}(\mathbb{C})$ is closed in $\mathrm{RG}(\mathbb{C})$ under quotients.

Proof. (i) $\Rightarrow$ (ii) Let

$$\begin{tikzcd} X_1 \arrow[r, two heads, "g"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & X_1' \arrow[d, shift right=2, "d'"'] \arrow[d, shift left=2, "c'"] \\ X_0 \arrow[u, "e" description] \arrow[r, two heads, "f"'] & X_0' \arrow[u, "e'" description] \end{tikzcd}$$

be a regular epimorphism (f, g) in $\mathrm{RG}(\mathbb{C})$ (which means that f and g are regular epimorphisms in $\mathbb{C}$), with

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0 \arrow[l, "e" description] \end{tikzcd}$$

a groupoid in $\mathbb{C}$. Let $\mathrm{Eq}(d)$, $\mathrm{Eq}(c)$, $\mathrm{Eq}(d')$ and $\mathrm{Eq}(c')$ be the kernel pairs of the arrows d, c, d' and c', respectively. By Proposition 3.17, there exists a connector between $\mathrm{Eq}(d)$ and $\mathrm{Eq}(c)$. Let $\lambda : \mathrm{Eq}(d) \to \mathrm{Eq}(d')$ and $\beta \colon \mathrm{Eq}(c) \to \mathrm{Eq}(c')$ be the arrows induced by the universal property of kernel pairs $\mathrm{Eq}(d')$ and $\mathrm{Eq}(c')$, respectively. By Theorem 2.8, λ and β are regular epimorphisms, so that $g(\mathrm{Eq}(d)) = \mathrm{Eq}(d')$ and $g(\mathrm{Eq}(c)) = \mathrm{Eq}(c')$. By Proposition 3.9 there is then a connector between $\mathrm{Eq}(d')$ and $\mathrm{Eq}(c')$, thus

$$\begin{tikzcd} X_1' \arrow[r, shift left=2, "d'"] \arrow[r, shift right=2, "c'"'] & X_0' \arrow[l, "e'" description] \end{tikzcd}$$

is a groupoid (Proposition 3.17).

(ii) $\Rightarrow$ (i) This implication follows from Theorem 2.4 and the fact that equivalence relations are in particular groupoids (whose domain and codomain morphisms are jointly monomorphic).

(i) $\Rightarrow$ (iii) This implication follows from (i) $\Rightarrow$ (ii) and the fact that $\mathrm{Grpd}(\mathbb{C}) \cong \mathrm{Cat}(\mathbb{C})$ in a Goursat context (Theorem 3.18).

(iii) $\Rightarrow$ (i) Let (R, r_1, r_2) be an equivalence relation on X, $f \colon X \twoheadrightarrow Y$ a regular epimorphism and $(f(R), t_1, t_2)$ the regular image of R along f

$$\begin{tikzcd} R \arrow[r, two heads, "g"] \arrow[d, shift right=1, "r_1"'] \arrow[d, shift left=1, "r_2"] & f(R) \arrow[d, shift right=1, "t_1"'] \arrow[d, shift left=1, "t_2"] \\ X \arrow[r, two heads, "f"] & Y. \end{tikzcd}$$

$(f(R), t_1, t_2)$ is reflexive and symmetric being the image of the equivalence relation R along a regular epimorphism f. By assumption, $(f(R), t_1, t_2)$ is an internal category, thus it is transitive and then an equivalence relation. It follows that $\mathbb{C}$ is a Goursat category (by Theorem 2.4). $\square$

Remark 3.20. Observe that Theorem 3.19 implies that $\mathrm{Grpd}(\mathbb{C})$ and

Cat($\mathbb{C}$) are Goursat categories whenever $\mathbb{C}$ is, again thanks to Lemma 3.11 (the category RG($\mathbb{C}$) obviously being a Goursat category since it is a functor category). This simplifies and slightly extends Proposition 4.3 in [40], where the existence of coequalizers in $\mathbb{C}$ was assumed. Also, Theorem 3.19 implies that the converse of Proposition 3.9 is also satisfied.

Remark 3.21. A result analogous to Theorem 3.19 holds in the Mal'tsev context: a category $\mathbb{C}$ is a Mal'tsev category if and only if Grpd($\mathbb{C}$) (or, equivalently, Cat($\mathbb{C}$)) is closed in RG($\mathbb{C}$) under subobjects [10]. Together with the comments made before Proposition 3.9 we observe the existence of a sort of "duality" between Mal'tsev categories and Goursat categories: similar results hold for Mal'tsev categories with respect to monomorphisms and for Goursat categories with respect to regular epimorphisms as shown in the following table (with R and S two equivalence relations on the same object in a regular category $\mathbb{C}$):

Goursat category $\mathbb{C}$	**Mal'tsev category** $\mathbb{C}$
$RSR = SRS$	$RS = SR$
If there exists a connector between R and S, then there exists a connector between the regular images $f(R)$ and $f(S)$, for a <u>regular epimorphism</u> f	If there exists a connector between R and S, then there exists a connector between the inverse images $i^{-1}(R)$ and $i^{-1}(S)$, for a <u>monomorphism</u> i
Conn($\mathbb{C}$) closed in 2-Eq($\mathbb{C}$) under <u>quotients</u>	Conn($\mathbb{C}$) closed in 2-Eq($\mathbb{C}$) under <u>subobjects</u>
Grpd($\mathbb{C}$) closed in RG($\mathbb{C}$) under <u>quotients</u>	Grpd($\mathbb{C}$) closed in RG($\mathbb{C}$) under <u>subobjects</u>
Cat($\mathbb{C}$) closed in RG($\mathbb{C}$) under <u>quotients</u>	Cat($\mathbb{C}$) closed in RG($\mathbb{C}$) under <u>subobjects</u>
Conn($\mathbb{C}$), RG($\mathbb{C}$), Grpd($\mathbb{C}$) and Cat($\mathbb{C}$) are Goursat whenever $\mathbb{C}$ is	Conn($\mathbb{C}$), RG($\mathbb{C}$), Grpd($\mathbb{C}$) and Cat($\mathbb{C}$) are Mal'tsev whenever $\mathbb{C}$ is

Table 3.1: "Duality" between Mal'tsev and Goursat categories.

Chapter 4

Shifting Lemma and Goursat categories

In Chapter 3, we have seen that the structure of internal connector is unique in Goursat categories because the categorical version of Gumm's Shifting Lemma holds in any Goursat category. Many other important results and properties hold in regular Mal'tsev and Goursat categories thanks to the validity of the Shifting Lemma in these categories; as examples: the admissibility in the sense of Galois theory of the subcategory of abelian objects in any Mal'tsev category [34], the uniqueness of the pseudogroupoid structure on two equivalence relations (when it exists) in any Mal'tsev or Goursat category [14], the characterization of Goursat categories given in Theorem 2.15, the validity of some properties in commutator theory [32, 51]. In this chapter, we focus our attention on the Shifting Lemma. More precisely, we study some variations of the Shifting Lemma in order to obtain new characterizations of regular Mal'tsev and Goursat categories. These results apply in particular to 2-permutable and 3-permutable quasi-varieties, since these latter categories are known to be regular. The main results of this chapter (in section 2 and 3) are joint work with M. Gran and D. Rodelo [46].

The Shifting Lemma

For a variety $\mathbb{V}$ of universal algebras, Gumm's *Shifting Lemma* [51] is stated as follows. Given congruences R, S and T on the same algebra X in $\mathbb{V}$ such that $R \wedge S \leqslant T$, whenever x, y, u, v are elements in X with $(x, y) \in R \wedge T$, $(x, u) \in S$, $(y, v) \in S$ and $(u, v) \in R$, it then follows that $(u, v) \in T$. We display this condition as

$$\begin{tikzcd} x \arrow[r, dash, "S"] \arrow[d, dash, "R"'] \arrow[d, dash, bend right=60, "T"'] & u \arrow[d, dash, "R"] \arrow[d, dash, dashed, bend left=60, "T"] \\ y \arrow[r, dash, "S"'] & v. \end{tikzcd} \tag{19}$$

A variety $\mathbb{V}$ of universal algebras satisfies the Shifting Lemma precisely when it is congruence modular [51], this means that the lattice of congruences on any algebra in $\mathbb{V}$ is modular: given congruences R, S and T on the same object $X \in \mathbb{V}$ such that $R \leqslant T$, one has:

$$R \vee (S \wedge T) = (R \vee S) \wedge T.$$

In a finitely complete category, the validity of the Shifting Lemma is equivalent to the following property, called the *Shifting property* [14]. In a finitely complete category $\mathbb{C}$, given two equivalence relations R and S on the same object, we write $R \square S$ for the largest double equivalence relation on R and S given by the following pullback

$$\begin{tikzcd} R \square S \arrow[r, tail, "\langle \pi_{12}, \pi_{34} \rangle"] \arrow[d, tail, "\langle \pi_{13}, \pi_{24} \rangle"'] & S \times S \arrow[d, tail, "\langle s_1, s_2 \rangle \times \langle s_1, s_2 \rangle"] \\ R \times R \arrow[r, tail, "\langle r_1 \times r_1, r_2 \times r_2 \rangle"'] & X \times X \times X \times X. \end{tikzcd}$$

We have that $(a, b, c, d) \in X^4$ belongs to the double equivalence relation $R \square S$ if and only if

$$\begin{tikzcd} a \arrow[r, dash, "S"] \arrow[d, dash, "R"'] & b \arrow[d, dash, "R"] \\ c \arrow[r, dash, "S"'] & d. \end{tikzcd}$$

Definition 4.1. [14] A finitely complete category $\mathbb{C}$ satisfies the *Shifting property* and is called *Gumm category* if for any equivalence relation R, S and T on the same object $X \in \mathbb{C}$ with $R \wedge S \leqslant T \leqslant R$, the canonical inclusion of equivalence relations $(i, j) \colon T \square S \to R \square S$

$$\begin{tikzcd} T\square S \arrow[r, tail, "j"] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & R\square S \arrow[d, shift right, "\lambda_1"'] \arrow[d, shift left, "\lambda_2"] \\ T \arrow[r, tail, "i"'] & R, \end{tikzcd} \tag{20}$$

is a discrete fibration: this means that any of the commutative squares in (20) is a pullback.

Lemma 4.2. [14] A finitely complete category $\mathbb{C}$ satisfies the Shifting Lemma if and only if it satisfies the Shifting property.

The Shifting Lemma can then be expressed in any category with finite limits.

In a regular context one can show that the lattice of equivalence relations in any Goursat category is modular [21], and that this latter property implies that the Shifting Lemma holds. However it is not true that the Shifting Lemma implies modularity in general. For instance, in the case of a variety of infinitary algebras there is a counterexample given by G. Janelidze in Example 12.5 in [60].

Remark 4.3. In some formulations of the Shifting Lemma, one may find the assumption $R \wedge S \leqslant T \leqslant R$ on the equivalence relations instead of the usual $R \wedge S \leqslant T$, as above. However, both statements are equivalent, as observed in [14]. In fact, if the properties hold when $R \wedge S \leqslant T$, then it obviously holds for $R \wedge S \leqslant T \leqslant R$. Conversely, consider $T' = R \wedge T$, which is such that $R \wedge S \leqslant T' \leqslant R$. By applying the assumption to R, S and T',

$$\begin{tikzcd} x \arrow[r, dash, "S"] \arrow[d, dash, "R"] \arrow[d, dash, bend right=60, "T'"'] & u \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=60, "T'"] \\ y \arrow[r, dash, "S"'] & v \end{tikzcd}$$

one concludes that $uT'v$, thus uTv.

In the varietal context, H.-P. Gumm has also considered a slight variation of the Shifting Lemma called the *Shifting Principle* [51]: given congruences R and T and a reflexive, symmetric and compatible relation S on the same algebra X such that $R \wedge S \leqslant T \leqslant R$, whenever x, y, u, v are elements in X with $(x, y) \in R \wedge T$, $(x, u) \in S$, $(y, v) \in S$ and $(u, v) \in R$ as in (19), it then follows that $(u, v) \in T$. The Shifting Principle, although apparently stronger, turns out to be equivalent to the Shifting Lemma in the varietal case.

With this observation in mind, it seems reasonable to expect that considering variations on the assumptions on the relations R, S or T appearing in the Shifting Lemma might provide characterizations of other types of categories. The variations we have in mind for R, S and T are to make those assumptions weaker, so that they give stronger versions of the Shifting Lemma. This idea comes from the well known characterization of Mal'tsev categories through the fact that reflexive relations are equivalence relations (Theorem 2.7), and from a more recent one of Goursat categories in terms of positive relations (Theorem 4.7).

We first investigate this notion of positive relation which allows us to obtain some new characterizations of regular Mal'tsev and Goursat Categories. We then use these results to show that stronger versions of the Shifting Lemma characterize regular Mal'tsev and Goursat categories.

In this chapter, we always assume that the base category in which we are working is a regular category, thus the proofs are partially given in set-theoretical terms (Theorem 1.23 and 1.24).

4.1 Positive relations and n-permutable categories

Here, we first simplify a result in [89] to give a new characterization of Mal'tsev categories in terms of positive relations. Then we also give a new characterization of Goursat categories and more generally of n-permutable categories in terms of positive relations.

Definition 4.4. A relation E on X is called *positive* [87] when it is of the form $E = R^o R$ for some relation $R \rightarrowtail X \times Y$.

Positive relations have the following properties:

Lemma 4.5. Let $\mathbb{C}$ be a regular category. Then:

(i) any positive relation is symmetric;

(ii) any equivalence relation is positive.

(iii) positive relations are stable under regular images.

Proof. (i) Let E be a positive relation and R a relation such that $E = R^oR$. One has $E^o = (R^oR)^o = R^oR = E$.

(ii) When R is an equivalence relation, one has $R = R^oR$.

(iii) Let E be a positive relation on X, and R a relation such that $E = R^oR$. Given a regular epimorphism $f : X \twoheadrightarrow Y$ a regular epimorphism, one has $f(E) = fEf^o = fR^oRf^o = (Rf^o)^oRf^o$. □

In [89] positive relations were used to identify those n-permutable categories which are actually Mal'tsev categories. Studying this article, we noticed that regular Mal'tsev categories can be characterized through the positivity of reflexive relations. As a consequence, condition (iv) of Theorem 4 in [89] could be replaced by condition (ii) as follows:

Theorem 4.6. For a regular category $\mathbb{C}$, the following conditions are equivalent:

(i) $\mathbb{C}$ is a Mal'tsev category;

(ii) every reflexive relation in $\mathbb{C}$ is positive.

Proof. $(i) \Rightarrow (ii)$ Let E be a reflexive relation. By Theorem 2.7(v) E is an equivalence relation and then a positive relation by Lemma 4.5 (ii).

$(ii) \Rightarrow (i)$ Let E be a reflexive relation. Then it is positive by assumption. By Lemma 4.5 (i) E is symmetric, thus by Theorem 2.7 (iv), $\mathbb{C}$ is a Mal'tsev category. □

Still using the positivity of relations we also obtained the following characterization for Goursat categories (which is a slight variation of Theorem 2.2 (v)).

Theorem 4.7. For a regular category $\mathbb{C}$, the following conditions are equivalent:

(*i*) $\mathbb{C}$ is a Goursat category;

(*ii*) any reflexive and positive relation in $\mathbb{C}$ is an equivalence relation.

Proof. (*i*) $\Rightarrow$ (*ii*) Let E be a reflexive and positive relation and R a relation such that $E = R^oR$. We are going to show that E is an equivalence relation.

By Lemma 4.5 (i) E is symmetric. One has

$$\begin{aligned} EE &= R^oRR^oR \\ &= R^oR \qquad \text{(Theorem 2.2 (iv))} \\ &= E \end{aligned}$$

Thus E is transitive and then an equivalence relation.

(*ii*) $\Rightarrow$ (*i*) Let R be an equivalence relation and f a regular epimorphism. We are going to prove that $f(R)$ is an equivalence relation. By assumption, it suffices to prove that $f(R)$ is positive ($f(R)$ is necessarily reflexive). Since R is an equivalence relation, by Lemma 4.5 (ii) and (iii), $f(R)$ is positive. □

The previous characterizations of regular Mal'tsev and Goursat categories can be extended to n-permutable categories.

Remark 4.8. For any relation $E \rightarrowtail X \times Y$ and $k \geqslant 1$, one has:

(i) $((E, E^o)_{2k})^o = (E, E^o)_{2k}$

(ii) $((E, E^o)_{2k+1})^o = (E^o, E)_{2k+1}$

Thanks to these observations, n-permutable categories can be characterized in terms of positive relations as follows:

Theorem 4.9. For a regular category $\mathbb{C}$ and $k \geqslant 1$, the following conditions are equivalent:

(*i*) $\mathbb{C}$ is a $(2k)$-permutable category;

(*ii*) for any reflexive relation E, the relation $(E, E^o)_{2k-1}$ is positive.

Proof. $(i) \Rightarrow (ii)$ Let E be a reflexive relation. By assumption the relation $(E, E^o)_{2k-1}$ is an equivalence relation (Theorem 2.21 (v)) and then positive (Lemma 4.5 (ii)).

$(ii) \Rightarrow (i)$ Let E be a reflexive relation and R a relation such that $(E, E^o)_{2k-1} = R^o R$. One has

$$\begin{array}{rcll} (E^o, E)_{2k-1} & = & ((E, E^o)_{2k-1})^o & \text{(Remark 4.8 (ii))} \\ & = & (R^o R)^o & \\ & = & R^o R & \\ & = & (E, E^o)_{2k-1} & \end{array}$$

and then by Theorem 2.21 (vi), $\mathbb{C}$ is $(2k)$-permutable. □

Theorem 4.10. For a regular category $\mathbb{C}$ and $k \geqslant 1$, the following conditions are equivalent:

(i) $\mathbb{C}$ is a $(2k+1)$-permutable category;

(ii) for any reflexive and positive relation E, the relation E^k is an equivalence relation.

Proof. $(i) \Rightarrow (ii)$ Since E is reflexive, E^k is also reflexive ($1 \leqslant E \leqslant E^k$). Let R be a relation such that $E = R^o R$. One has :

$$\begin{array}{rcll} (E^k)^o & = & ((R^o, R)_{2k})^o & \\ & = & (R^o, R)_{2k} & \text{(Remark 4.8 (i))} \\ & = & E^k, & \end{array}$$

thus E^k is symmetric. One also has

$$\begin{array}{rcll} E^k E^k & = & (R^o, R)_{2k} (R^o, R)_{2k} & \\ & = & (R^o, R)_{2k+2} (R^o, R)_{2k-2} & \\ & = & (R^o, R)_{2k} (R^o, R)_{2k-2} & \text{(Theorem 2.21 (iv))} \\ & = & (R^o, R)_{2k+2} (R^o, R)_{2k-4} & \\ & = & (R^o, R)_{2k} (R^o, R)_{2k-4} & \text{(Theorem 2.21 (iv))} \\ & = & . & \\ & = & . & \\ & = & . & \\ & = & (R^o, R)_{2k} & \\ & = & E^k & \end{array}$$

Thus E^k is transitive and then an equivalence relation.

$(ii) \Rightarrow (i)$ Let E be a reflexive relation, by Theorem 2.21 (v), it suffices to prove that the relation $(E, E^o)_{2k}$ is an equivalence relation. Since E is reflexive, EE^o is also reflexive. Moreover, one has $EE^o = (E^o)^o E^o$, thus the relation EE^o is positive and by assumption the relation $(EE^o)^k = (E, E^o)_{2k}$ is an equivalence relation. □

4.2 Mal'tsev categories and the Shifting Lemma

Motivated by the fact that in a Mal'tsev category reflexive relations coincide with equivalence relations, we are now going to show that regular Mal'tsev categories can be characterized through a stronger version of the Shifting Lemma where, in the assumption, the equivalence relations are replaced by reflexive relations. Note that, for a diagram such as (19) where R, S or T are not equivalence relations, the relations are always to be considered from left to right and from top to bottom. To avoid ambiguity with the interpretation of such diagram, from now on we will write $x \xrightarrow{U} y$ to mean that $(x, y) \in U$, whenever U is a non-symmetric relation.

Theorem 4.11. Let $\mathbb{C}$ be a finitely complete category. The following conditions are equivalent:

(i) $\mathbb{C}$ is a Mal'tsev category;

(ii) The Shifting Lemma holds in $\mathbb{C}$ when R, S and T are reflexive relations.

Proof. (i) $\Rightarrow$ (ii) This implication follows from the fact that reflexive relations are necessarily equivalence relations and the Shifting Lemma holds in any regular Mal'tsev category.
(ii) $\Rightarrow$ (i) We shall prove that every reflexive relation E is symmetric (which suffices by Theorem 2.7(iv)). Suppose that $(x, y) \in E$, and consider the reflexive relations T and R on E defined by the following

pullbacks

$$\begin{tikzcd}
T \arrow[r] \arrow[d, tail, "\langle t_1,t_2\rangle"'] & E \arrow[d, tail, "\langle e_1,e_2\rangle"] \\
E\times E \arrow[r, "e_1\times e_2"'] & X\times X
\end{tikzcd}$$

and

$$\begin{tikzcd}
R \arrow[rr] \arrow[d, tail, "\langle r_1,r_2\rangle"'] & & E \arrow[d, tail, "\langle e_1,e_2\rangle"] \\
E\times E \arrow[r, "e_2\times e_1"'] & X\times X \arrow[r, "\cong", "\langle \pi_2,\pi_1\rangle"'] & X\times X,
\end{tikzcd}$$

where $\pi_1\colon X\times X\to X$ and $\pi_2\colon X\times X\to X$ are the product projections. We have $(aEb, cEd)\in T$ if and only if $(a,d)\in E$, and $(aEb, cEd)\in R$ if and only if $(c,b)\in E$.

The third reflexive relation on E we consider is the kernel pair $\mathrm{Eq}(e_2)$ of e_2, defined as the following pullback

$$\begin{tikzcd}
\mathrm{Eq}(e_2) \arrow[r] \arrow[d] & E \arrow[d, "e_2"] \\
E \arrow[r, "e_2"'] & X.
\end{tikzcd}$$

$\mathrm{Eq}(e_2)$ is an equivalence relation, with the property that $\mathrm{Eq}(e_2)\leqslant R$ and $\mathrm{Eq}(e_2)\leqslant T$, so that $R\wedge \mathrm{Eq}(e_2)=\mathrm{Eq}(e_2)\leqslant T$. We can apply the assumption to the following relations given in solid lines

$$\begin{tikzcd}
xEy \arrow[r, "\mathrm{Eq}(e_2)"] \arrow[d, "R"] \arrow[d, bend right=60, "T"'] & yEy \arrow[d, "R"'] \arrow[d, dashed, bend left=60, "T"] \\
xEx \arrow[r, "\mathrm{Eq}(e_2)"'] & xEx
\end{tikzcd}$$

(xEx and yEy by the reflexivity of the relation E). We conclude that $(yEy, xEx)\in T$ and, consequently, that $(y,x)\in E$. $\square$

In the proof of the implication (ii) $\Rightarrow$ (i) we only used two "genuine" reflexive relations R and T. This observation gives:

Corollary 4.12. Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:

(i) $\mathbb{C}$ is a Mal'tsev category;

(ii) The Shifting Lemma holds in $\mathbb{C}$ when R and T are reflexive relations and S is an equivalence relation.

Example 4.13. Let $\mathbb{T}$ be the algebraic theory of a Mal'tsev variety, and $\mathbb{T}(\mathsf{Top})$ the category of *topological Mal'tsev algebras*. This category is regular, essentially because the regular epimorphisms are the *open* surjective homomorphisms [68], that are stable under pullbacks. The category $\mathbb{T}(\mathsf{Top})$ is also a Mal'tsev category, so that $\mathbb{T}(\mathsf{Top})$ satisfies the Shifting Lemma for reflexive relations (by Theorem 4.11). The same is true for the exact Mal'tsev category $\mathbb{T}(\mathsf{Comp})$ of *compact Hausdorff Mal'tsev algebras*.

4.3 Goursat categories and the Shifting Lemma

Here, thanks to Theorem 4.7, we prove that Goursat categories can be characterized through a stronger version of the Shifting Lemma.

Remark 4.14. As already observed in Theorem 2.2, for any pair of equivalence relations R and S on the same object X in a Goursat category, one has that RSR is an equivalence relation, that is then the supremum $R \vee S$ of R and S as equivalence relations on X

$$R \vee S = RSR.$$

When $\mathbb{C}$ is a Goursat category, the lattice of equivalence relations on the same object is modular [21] and, consequently, the Shifting Lemma holds. Moreover, the Shifting Lemma still holds when S is just a reflexive relation, as we show next. The following result is partly based on Lemma 2.2 in [69], and it gives a first step towards the characterization we aim to obtain for Goursat categories.

Proposition 4.15. In any regular Goursat category $\mathbb{C}$, the Shifting Lemma holds when S is a reflexive relation and R and T are equivalence relations.

Proof. Let R and T be equivalence relations and let S be a reflexive relation on an object X such that $R \wedge S \leqslant T$. Suppose that we have $(x, y) \in R \wedge T$, $(x, u) \in S$, $(y, v) \in S$ and $(u, v) \in R$ as in (19). We consider the two equivalence relations on S, $R \square S$ and W determined by the following pullback

$$\begin{tikzcd}
W \arrow[r, tail, "\langle \pi_{12}, \pi_{34} \rangle"] \arrow[d, tail, "\langle \pi_{13}, \pi_{24} \rangle"'] & S \times S \arrow[d, tail, "\langle s_1, s_2 \rangle \times \langle s_1, s_2 \rangle"] \\
T \times R \arrow[r, tail, "\langle t_1 \times r_1, t_2 \times r_2 \rangle"'] & X \times X \times X \times X.
\end{tikzcd}$$

We have $(aSb, cSd) \in W$ if and only if

$$\begin{tikzcd}
a \arrow[r, dash, "S"] \arrow[d, dash, "T"'] & b \arrow[d, dash, "R"] \\
c \arrow[r, dash, "S"'] & d.
\end{tikzcd}$$

Note that they are in fact equivalence relations on S since R and T are both equivalence relations.
Given the equivalence relations $R \square S$, $\mathrm{Eq}(s_2)$ and W on S, Remark 4.14 yields the following description of the supremum of
$R \square S \ \wedge\ \mathrm{Eq}(s_2)$ and W as equivalence relations on S:

$$\begin{aligned}
(\, R \square S \wedge \mathrm{Eq}(s_2)\,) \ \vee\ W \quad &= \quad (\, R \square S \wedge \mathrm{Eq}(s_2)\,)\, W \,(\, R \square S \wedge \mathrm{Eq}(s_2)\,) \\
&= \quad W \,(\, R \square S \wedge \mathrm{Eq}(s_2)\,)\, W.
\end{aligned}$$

Since

$$R \square S \wedge \mathrm{Eq}(s_2) \leqslant (\, R \square S \wedge \mathrm{Eq}(s_2)\,) \ \vee\ W$$

we can apply the Shifting Lemma to the following diagram

$$\begin{tikzcd}
xSu \arrow[r, dash, "\mathrm{Eq}(s_2)"] \arrow[d, dash, "R \square S"] \arrow[d, dash, bend right=60, "(\, R \square S \wedge \mathrm{Eq}(s_2)\,) \vee W"'] & uSu \arrow[d, dash, "R \square S"'] \arrow[d, dash, dashed, bend left=60, "(\, R \square S \wedge \mathrm{Eq}(s_2)\,) \vee W"] \\
ySv \arrow[r, dash, "\mathrm{Eq}(s_2)"'] & vSv.
\end{tikzcd}$$

Note that, uSu and vSv by the reflexivity of S. We then obtain

$$(uSu, vSv) \in (\, R\Box S \wedge \mathrm{Eq}(s_2)\,) \,\vee\, W.$$

Using

$$(\, R\Box S \wedge \mathrm{Eq}(s_2)\,) \,\vee\, W = (\, R\Box S \wedge \mathrm{Eq}(s_2)\,)\, W\, (\, R\Box S \wedge \mathrm{Eq}(s_2)\,)$$

this means that there exist a, b in X (Theorem 1.23 and 1.24) such that

$$(uSu)\,(R\Box S \wedge \mathrm{Eq}(s_2))\,(aSu) W (bSv)\,(R\Box S \wedge \mathrm{Eq}(s_2))\,(vSv),$$

i.e.

$$\begin{tikzcd}
u \arrow[r, dash, "S"] \arrow[d, dash, "R"'] & u \arrow[d, dash, "R"] \\
a \arrow[r, dash, "S"] \arrow[d, dash, "T"'] & u \arrow[d, dash, "R"] \\
b \arrow[r, dash, "S"] \arrow[d, dash, "R"'] & v \arrow[d, dash, "R"] \\
v \arrow[r, dash, "S"'] & v.
\end{tikzcd}$$

Since aRu (R is symmetric), aSu and $R \wedge S \leqslant T$, it follows that aTu; similarly bTv. From uTa (T is symmetric), aTb and bTv, we conclude that uTv (T is transitive), as desired. $\square$

Remark 4.16. The Shifting Lemma when S is a reflexive relation and R and T are equivalence relations, as stated in Proposition 4.15, is the categorical version of the Shifting Principle recalled at the beginning of this chapter. Firstly, assuming that $R \wedge S \leqslant T$ is equivalent to assuming that $R \wedge S \leqslant T \leqslant R$ for the property of diagram (19) to hold (Remark 4.3). Secondly, going carefully through the proofs in [51], one may check that the symmetry of S is not necessary. So the Shifting Principle could equivalently be stated by asking that S is just a reflexive and compatible relation.

We now use Theorem 4.7 and Proposition 4.15 to obtain the characterization of Goursat categories through a variation of the Shifting

Lemma:

Theorem 4.17. Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:

(i) $\mathbb{C}$ is a Goursat category;

(ii) The Shifting Lemma holds in $\mathbb{C}$ when S is a reflexive relation and R and T are reflexive and positive relations.

Proof. (i) $\Rightarrow$ (ii) This implication follows from the fact that reflexive and positive relations are necessarily equivalence relations in the Goursat context (Theorem 4.7) and from Proposition 4.15.
(ii) $\Rightarrow$ (i) We shall prove that for any reflexive relation E on X in $\mathbb{C}$, $EE^{\circ} = E^{\circ}E$ (see Theorem 2.2 (vi)). Suppose that $(x, y) \in EE^{\circ}$. Then, for some z in X, one has that $(z, x) \in E$ and $(z, y) \in E$. Consider the reflexive and positive relations $R = EE^{\circ}$ and $T = E^{\circ}E$, and the reflexive relation E on X. From the reflexivity of E, we get $E \leqslant EE^{\circ}$ and $E \leqslant E^{\circ}E$; thus $EE^{\circ} \wedge E = E \leqslant E^{\circ}E$. We can apply our assumption to the following relations given in solid lines:

$$\begin{tikzcd}
z \arrow[r, "E"] \arrow[d, "EE^{\circ}"'] \arrow[d, bend right=60, "E^{\circ}E"'] & x \arrow[d, "EE^{\circ}"] \arrow[d, dashed, bend left=60, "E^{\circ}E"] \\
z \arrow[r, "E"'] & y
\end{tikzcd} \tag{21}$$

to conclude that $(x, y) \in E^{\circ}E$. Having proved that $EE^{\circ} \leqslant E^{\circ}E$ for *every* reflexive relation E, the inequality $E^{\circ}E \leqslant EE^{\circ}$ follows immediately. $\square$

In the proof of the implication (ii) $\Rightarrow$ (i) we used the relations EE^{o} and $E^{o}E$ with E a reflexive relation. This observation gives:

Corollary 4.18. Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:

(i) $\mathbb{C}$ is a Goursat category;

(ii) The Shifting Lemma holds in $\mathbb{C}$ for a diagram as (21), with E any reflexive relation.

As an application, the fact that any quasi-variety is a regular category [81] implies the following:

Corollary 4.19. Let $\mathbb{V}$ be a quasi-variety. The following conditions are equivalent:

(i) $\mathbb{V}$ is 3-permutable;

(ii) The Shifting Lemma holds in $\mathbb{V}$ when S is a reflexive compatible relation, and R and T are reflexive and positive compatible relations.

Remark 4.20. Theorem 4.11 holds in any finitely complete category, and then gives a new characterisation of Mal'tsev categories without the assumption of regularity. It is an open problem to compare condition *(ii)* in Theorem 4.7 with the definition of Goursat category without the assumption of regularity [12]. As observed by P.-A Jacqmin, Theorem 4.17 could then lead to a new definition of Goursat category without the assumption of regularity.

Chapter 5

Equivalence distributivity in Goursat categories

In Chapter 4, we proved that, for a regular category, the property of being a Mal'tsev category, or of being a Goursat category, can be both characterized through suitable variations of the Shifting Lemma. These variations considered the Shifting Lemma for relations which were not necessarily equivalence relations, but only reflexive or positive ones, thus giving rise to stronger versions of the Shifting Lemma.

There are other properties similar to the Shifting Lemma and which allow one to characterize *congruence distributive* varieties. These properties are related to the Shifting Lemma, and are called the *Triangular Lemma* and the *Trapezoid Lemma* in the varietal context [27]. These properties were first introduced by J. Duda in [29, 30] where the Trapezoid Lemma was called the *Upright Principle*. This led us to further study the connections between these results and the property, for a regular Mal'tsev and Goursat category, of having distributive equivalence relation lattices on any of its objects.

From [27] we know that, for a variety $\mathbb{V}$ of universal algebras, the fact that both the Shifting Lemma and the Triangular Lemma hold in $\mathbb{V}$ is equivalent to $\mathbb{V}$ being a congruence distributive variety, and is also equivalent to the fact that the Trapezoid Lemma holds in $\mathbb{V}$.

$$\begin{aligned} \text{Shifting Lemma} + \text{Triangular Lemma} \;&\Leftrightarrow\; \text{Trapezoid Lemma} \\ &\Leftrightarrow\; \text{Congruence distributivity} \end{aligned}$$

Consequently, by considering stronger versions of the Triangular Lemma we were hoping to get at once 2-permutability (or 3-permutability) and congruence distributivity in a varietal context, and to extend these observations to a categorical context.

Explaining how this is indeed possible is the main goal of this chapter, where suitable variations of the Triangular Lemma and of the Trapezoid Lemma are shown to be the right properties to characterize *equivalence distributive* categories (the natural generalization of congruence distributive varieties). More precisely, when $\mathbb{C}$ is a regular Mal'tsev category, or even a Goursat category, the Triangular Lemma is equivalent to the Trapezoid Lemma, and both of them are equivalent to $\mathbb{C}$ being equivalence distributive (Propositions 5.7 and 5.10). We also give new characterizations of equivalence distributive Mal'tsev categories through variations of the Triangular Lemma and of the Trapezoid Lemma (Theorem 5.13), which then apply to arithmetical varieties [82] and arithmetical categories [80]. Inspired by the ternary Pixley term of arithmetical varieties [82], we consider a condition for relations, stronger than difunctionality [83], which captures the property for a regular category to be a Mal'tsev and equivalence distributive one (Theorem 5.16). In the last section we characterize equivalence distributive Goursat categories (Theorem 5.19) through variations on the Triangular and Trapezoid Lemmas involving reflexive and positive relations.

The main results of this chapter are joint work with M. Gran and D. Rodelo [44].

5.1 Triangular Lemma and Trapezoid Lemma

Here, we recall the Triangular Lemma and Trapezoid Lemma in a varietal context. We then give their interpretations in the context of regular categories.

The Triangular Lemma

A variety $\mathbb{V}$ of universal algebras satisfies the *Triangular Lemma* [27] if, given congruences R, S and T on the same algebra X in $\mathbb{V}$ such that $R \wedge S \leqslant T$, whenever y, u, v are elements in X with $(u, y) \in T$, $(y, v) \in S$ and $(u, v) \in R$, it then follows that $(u, v) \in T$. We display this condition as

$$\begin{tikzcd}
& u \arrow[d, dash, "R"] \arrow[d, dash, dashed, bend left=60, "T"] \\
y \arrow[ur, dash, "T"] \arrow[r, dash, "S"'] & v.
\end{tikzcd} \tag{22}$$

The Trapezoid Lemma

A variety $\mathbb{V}$ of universal algebras satisfies the *Trapezoid Lemma* [27] if, given congruences R, S and T on the same algebra X in $\mathbb{V}$ such that $R \wedge S \leqslant T$, whenever x, y, u, v are elements in X with $(x, y) \in T$, $(x, u) \in S$, $(y, v) \in S$ and $(u, v) \in R$, it then follows that $(u, v) \in T$. We display this condition as

$$\begin{tikzcd}
& x \arrow[r, dash, "S"] & u \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=60, "T"] \\
y \arrow[ur, dash, "T"] \arrow[rr, dash, "S"'] & & v.
\end{tikzcd} \tag{23}$$

If the Trapezoid Lemma holds in a variety, then also the Shifting Lemma and the Triangular Lemma hold. In fact, let R, S and T be congruences on the same algebra X in $\mathbb{V}$ such that $R \wedge S \leqslant T$. Suppose that x, y, u, v are elements in X related as in (22). We apply the Trapezoid Lemma to

$$\begin{tikzcd}
& u \arrow[r, dash, "S"] & u \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=60, "T"] \\
y \arrow[ur, dash, "T"] \arrow[rr, dash, "S"'] & & v.
\end{tikzcd}$$

to conclude that $(u, v) \in T$ and thus the Triangular Lemma holds. Similarly, if x, y, u, v are elements in X related as in (19), we apply the

Trapezoid Lemma to

$$\begin{tikzcd}
& x \arrow[r, dash, "S"] \arrow[dl, dash, "T"'] & u \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=60, "T"] \\
y \arrow[rr, dash, "S"'] & & v.
\end{tikzcd}$$

to conclude that $(u, v) \in T$ and thus the Shifting Lemma holds.

One can easily check that both the properties expressed by the Triangular Lemma and by the Trapezoid Lemma only involve finite limits. It is then possible to speak of the validity of these properties in any finitely complete category. So, in a regular category $\mathbb{C}$, given equivalence relations R, S and T on the same object X such that $R \wedge S \leqslant T$, the lemmas recalled above can be interpreted as follows:

Shifting Lemma:	$R \wedge S(R \wedge T)S \leqslant T$	**(SL)**
Triangular Lemma:	$R \wedge ST \leqslant T$	**(TL)**
Trapezoid Lemma:	$R \wedge STS \leqslant T$	**(TpL)**

We would like to point out that in some recent papers the notion of *majority category* has been introduced and investigated [55, 56]. This notion is closely related to the validity of the properties just recalled. For a regular category $\mathbb{C}$, the property of being a majority category can be equivalently defined as follows (see [55]): for any reflexive relations R, S and T on the same object X in $\mathbb{C}$, the inequality

$$R \wedge (ST) \leqslant (R \wedge S)(R \wedge T)$$

holds. We then observe that any regular majority category satisfies the Trapezoid Lemma (and, consequently, also the weaker Triangular Lemma and Shifting Lemma):

Lemma 5.1. [54] The Trapezoid Lemma holds true in any regular majority category $\mathbb{C}$.

Proof. Given equivalence relations R, S and T on the same object such that $R \wedge S \leqslant T$, then

$$\begin{aligned} R \wedge (STS) &\leqslant (R \wedge S)(R \wedge (TS)) \\ &\leqslant T(R \wedge T)(R \wedge S) \\ &\leqslant TTT \\ &= T. \end{aligned}$$

□

5.2 Equivalence distributivity

In this section, we introduce the notion of equivalence distributive category and we prove that in any regular Mal'tsev or Goursat category $\mathbb{C}$, the Triangular Lemma is equivalent to the Trapezoid Lemma and both of them are equivalent to $\mathbb{C}$ being equivalence distributive.

- A lattice L is called *distributive* when

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \forall a, b, c \in L.$$

  Equivalently, L is distributive if and only if it satisfies the Horn sentence

$$a \wedge b \leqslant c \Rightarrow a \wedge (b \vee c) \leqslant c. \tag{24}$$

- A variety $\mathbb{V}$ of universal algebras is called *(congruence) distributive* when the lattice $\mathrm{Cong}(X)$ of congruences on any algebra X in $\mathbb{V}$ is distributive.

Definition 5.2. A regular category $\mathbb{C}$ is *equivalence distributive* when the meet semilattice $\mathrm{Equiv}(X)$ of equivalence relations is a distributive lattice, for all objects X in $\mathbb{C}$.

Example 5.3. Any congruence distributive variety gives an example of an equivalence distributive category. The varieties of Boolean algebras, Heyting algebras and Von Neumann regular rings [47] are also examples. As categorical examples that are not varieties of universal algebras one has the dual category of any (pre)topos. These are actually *arith-*

metical categories [80], i.e. Barr-exact Mal'tsev equivalence distributive categories.

Remark 5.4. The notion of equivalence distributive category is different from the notion of distributive category defined by Carboni, Lack and Walters in [22]: a category with finite products and sums is *distributive* if the canonical arrow

$$\alpha\colon A \times B + A \times C \to A \times (B + C)$$

is an isomorphism.

The congruence distributive varieties can be characterized as follows:

Theorem 5.5. [27] Let $\mathbb{V}$ be a variety of universal algebras. The following conditions are equivalent:

(i) $\mathbb{V}$ is congruence distributive;

(ii) the Trapezoid Lemma holds in $\mathbb{V}$;

(iii) the Shifting Lemma and the Triangular Lemma hold in $\mathbb{V}$.

The equivalence between the Triangular Lemma and Trapezoid Lemma holds for any algebra X which is *congruence permutable*, meaning that 2-permutability holds "locally" in $\mathrm{Cong}(X)$:

Proposition 5.6. [27] Let $\mathbb{V}$ be a variety of universal algebras and X a congruence permutable algebra. The following conditions are equivalent:

(i) the Triangular Lemma holds for X;

(ii) the Trapezoid Lemma holds for X;

(iii) $\mathrm{Cong}(X)$ is distributive.

This result can be extended to the context of regular categories. To do so we apply Barr's Metatheorem (Theorem 1.23) which allows us to use part of the internal logic of a topos to develop proofs in a regular category. In particular, finite limits can be described elementwise as in the category of sets and regular epimorphisms via the usual formula describing surjections (Theorem 1.24).

Proposition 5.7. Let $\mathbb{C}$ be a regular Mal'tsev category. The following conditions are equivalent:

(i) the Triangular Lemma holds in $\mathbb{C}$;

(ii) the Trapezoid Lemma holds in $\mathbb{C}$;

(iii) $\mathbb{C}$ is equivalence distributive.

Proof. (i) $\Rightarrow$ (ii) Let R, S and T be equivalence relations on an object X such that $R \wedge S \leqslant T$ and suppose that x, y, u, v are related as in (23). Since $\mathbb{C}$ is a Mal'tsev category, then TS is an equivalence relation on X (Theorem 2.7(ii)). We can apply the Triangular Lemma to

$$\begin{tikzcd} & u \arrow[d, dash, "R"] \arrow[d, dash, dashed, bend left=60, "TS"] \\ y \arrow[ur, dash, "TS"] \arrow[r, dash, "S"'] & v \end{tikzcd}$$

$(R \wedge S \leqslant T \leqslant TS)$, to conclude that $(u, v) \in TS(= ST)$. So, there exists a in X such that

$$\begin{tikzcd} & u \arrow[d, dash, "R"] \arrow[d, dash, dashed, bend left=60, "T"] \\ a \arrow[ur, dash, "T"] \arrow[r, dash, "S"'] & v. \end{tikzcd}$$

Applying the Triangular Lemma again, we conclude that $(u, v) \in T$.

(ii) $\Rightarrow$ (iii) We prove that (24) holds with respect to the lattice $\mathrm{Equiv}(X)$ of equivalence relations on an object X. Let R, S and $T \in \mathrm{Equiv}(X)$ be such that $R \wedge S \leqslant T$. Then

$$\begin{array}{rcll} R \wedge (S \vee T) & = & R \wedge ST, & \text{by Theorem 2.7(iii)} \\ & \leqslant & R \wedge STS & \\ & \leqslant & T, & \text{by } \mathbf{(TpL)}. \end{array}$$

(iii) $\Rightarrow$ (ii) Let R, S and T be equivalence relations in $\mathrm{Equiv}(X)$ such that $R \wedge S \leqslant T$. Then

$$\begin{array}{rcll} R \wedge STS & \leqslant & R \wedge (S \vee T) & \\ & \leqslant & T, & \text{by (24)} \end{array}$$

thus **(TpL)** holds.

(ii) $\Rightarrow$ (i) Obvious. $\square$

Note that the implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) of Proposition 5.7 hold in any regular category.

Remark 5.8. It is known from Corollary 3.2 in [55] that a regular Mal'tsev category $\mathbb{C}$ is equivalence distributive if and only if $\mathbb{C}$ is a majority category. That every Mal'tsev equivalence distributive category is a majority category was already known from [56]. We remark that the converse implication also easily follows from Lemma 5.1 and Proposition 5.7.

Next we show that the same equivalent conditions hold in the weaker context of Goursat categories. The most difficult implication to prove is that a Goursat category which satisfies the Triangular Lemma also satisfies the Trapezoid Lemma. We start by giving a direct proof of this fact in the varietal context to then obtain a categorical translation of the proof via matrix conditions [67]. Note that, for varieties, this result actually follows from Theorem 1 in [27]; however, we give an alternative proof which is suitable to be extended to the categorical context of regular categories.

Lemma 5.9. If $\mathbb{V}$ is a 3-permutable variety which satisfies the Triangular Lemma, then the Trapezoid Lemma also holds in $\mathbb{V}$.

Proof. Let R, S and T be congruences on the same algebra X in $\mathbb{V}$ such that $R \wedge S \leqslant T$. Suppose that x, y, u, v are elements in X related as in (23). From the relations

$$\begin{aligned} &xTxSxRx\\ &xTxSuRu\\ &xTySvRu\\ &yTySyRy, \end{aligned} \tag{25}$$

we can deduce the following ones by applying the quaternary operations p and q (see Theorem 2.28 (iii)), respectively:

$$xTp(x,x,y,y)Sp(x,u,v,y)Rx$$

and

$$yTq(x,x,y,y)Sq(x,u,v,y)Ry.$$

We apply the Triangular Lemma to

$$\begin{tikzcd} & x \arrow[dl, dash, "T"'] \arrow[d, dash, "R"] \arrow[d, dash, dashed, bend left=60, "T"] \\ p(x,x,y,y) \arrow[r, dash, "S"'] & p(x,u,v,y) \end{tikzcd} \tag{26}$$

and

$$\begin{tikzcd} & y \arrow[dl, dash, "T"'] \arrow[d, dash, "R"] \arrow[d, dash, dashed, bend left=60, "T"] \\ q(x,x,y,y) \arrow[r, dash, "S"'] & q(x,u,v,y). \end{tikzcd} \tag{27}$$

Next, we apply the Shifting Lemma to

$$\begin{tikzcd} x \arrow[r, dash, "S"] \arrow[d, dash, "R"] \arrow[d, dash, bend right=60, "T"'] \arrow[d, phantom, shift right=4ex, "(26)"] & u=p(u,u,u,v) \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=60, "T"] \\ p(x,u,v,y) \arrow[r, dash, "S"'] & p(u,u,v,v) \end{tikzcd} \tag{28}$$

and

$$\begin{tikzcd} y \arrow[r, dash, "S"] \arrow[d, dash, "R"] \arrow[d, dash, bend right=60, "T"'] \arrow[d, phantom, shift right=4ex, "(27)"] & v=q(u,u,u,v) \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=60, "T"] \\ q(x,u,v,y) \arrow[r, dash, "S"'] & q(u,u,v,v). \end{tikzcd} \tag{29}$$

From (28) and (29), we obtain $uTp(u,u,v,v)=q(u,u,v,v)Tv$; it follows that $(u,v)\in T$. $\square$

We adapt this varietal proof into a categorical one using an appropriate matrix and the corresponding relations which can be deduced from it (see [67] or [58] for more details). The kind of matrix we use translates the quaternary identities (Theorem 2.28 (iii)) into the property on relations given in Theorem 2.2 (iv):

$$\left(\begin{array}{cccc|cc} x & y & y & z & x & z \\ u & u & v & v & \alpha & \alpha \end{array}\right) \tag{30}$$

The first and second columns after the vertical separation in the matrix are the result of applying p and q, respectively, to the elements in the lines before the vertical separation. Thus, the introduction of a new element α, to represent the identity $p(u,u,v,v) = q(u,u,v,v)(= \alpha)$. We then "interpret" the matrix as giving relations between top elements and bottom elements as follows. Whenever the relations before the vertical separation in the matrix are assumed to hold, then we may conclude that the relations after the vertical separation also hold. For this matrix, the interpretation gives: for any binary relation P, if xPu, yPu, yPv and zPv, then $xP\alpha$ and $zP\alpha$, for some α; this gives the property $PP^\circ PP^\circ \leqslant PP^\circ$. Since $PP^\circ \leqslant PP^\circ PP^\circ$ is always true, we get precisely $PP^\circ PP^\circ = PP^\circ$ from Theorem 2.2 (iv).

Proposition 5.10. Let $\mathbb{C}$ be a Goursat category. The following conditions are equivalent:

(i) the Triangular Lemma holds in $\mathbb{C}$;

(ii) the Trapezoid Lemma holds in $\mathbb{C}$;

(iii) $\mathbb{C}$ is equivalence distributive.

Proof. (i) $\Rightarrow$ (ii) We extend the proof of Lemma 5.9 to a categorical context by constructing an appropriate matrix of the type (30). In that proof we applied p and q to the 4-tuples (x,x,x,y), (x,x,y,y), (x,u,u,y), (u,u,u,v) and (u,u,v,v). We put them in the matrix so that (x,x,x,y), (x,u,u,y) and (u,u,u,v) go to the top lines and (x,x,y,y) and (u,u,v,v) go to the bottom lines as follows

$$\left(\begin{array}{cccc|cc} x & x & x & y & x & y \\ x & u & u & y & x & y \\ u & u & u & v & u & v \\ & & & & & \\ x & x & y & y & \alpha & \alpha \\ u & u & v & v & \varepsilon & \varepsilon \end{array}\right)$$

We also used the 4-tuple (x, u, v, y), but it does not "fit" into this type of matrix; it will be used in the definition of the binary relation P. From the matrix, we see that the relation P should be defined from X^3 to X^2. The relations between the 4-tuples in the matrix above and (x, u, v, y) given in (25), and the bottom and right hand relations in (28) and (29) tell us that P should be defined as:

$$(a, b, c)P(d, e) \Leftrightarrow \exists z \text{ such that } aTdSzRb, zSe \text{ and } eRc.$$

From the matrix we see that $(x, x, u)PP^\circ PP^\circ(y, y, v)$, from which we conclude that $(x, x, u)PP^\circ(y, y, v)$. It then follows that $(x, x, u)P(\alpha, \varepsilon)$ and $(y, y, v)P(\alpha, \varepsilon)$, for some (α, ε), i.e. there exist β and δ such that

$$\begin{gathered} xT\alpha S\beta Rx, \beta S\varepsilon \text{ and } \varepsilon Ru \\ yT\alpha S\delta Ry, \delta S\varepsilon \text{ and } \varepsilon Rv. \end{gathered}$$

Next we apply the Triangular Lemma to

$$\begin{tikzcd} & x \arrow[dl, dash, "T"'] \arrow[d, dash, "R"] \arrow[d, dash, dashed, bend left=70, "T"] \\ \alpha \arrow[r, dash, "S"'] & \beta \end{tikzcd} \tag{31}$$

and

$$\begin{tikzcd} & y \arrow[dl, dash, "T"'] \arrow[d, dash, "R"] \arrow[d, dash, dashed, bend left=70, "T"] \\ \alpha \arrow[r, dash, "S"'] & \delta. \end{tikzcd} \tag{32}$$

We now apply the Shifting Lemma to

$$\begin{tikzcd} x \arrow[r, dash, "S"] \arrow[d, dash, "R"] \arrow[d, dash, bend right=80, "T"'] \arrow[d, phantom, shift right=3ex, "(31)"] & u \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=80, "T"] \\ \beta \arrow[r, dash, "S"'] & \varepsilon \end{tikzcd} \tag{33}$$

and

$$\begin{tikzcd} y \arrow[r, dash, "S"] \arrow[d, dash, "R"] \arrow[d, dash, bend right=80, "T"'] \arrow[d, phantom, shift right=3ex, "(32)"] & v \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=80, "T"] \\ \delta \arrow[r, dash, "S"'] & \varepsilon. \end{tikzcd} \tag{34}$$

From (33) and (34) we obtain $uT\varepsilon Tv$, thus $(u,v) \in T$.

(ii) $\Rightarrow$ (iii) We prove that (24) holds with respect to the lattice $\mathrm{Equiv}(X)$ of equivalence relations on an object X.

Let $R, S, T \in \mathrm{Equiv}(X)$ be such that $R \wedge S \leqslant T$. Then

$$\begin{array}{rcll} R \wedge (S \vee T) & = & R \wedge STS, & \text{by Theorem 2.2(iii)} \\ & \leqslant & T, & \text{by } \textbf{(TpL)}. \end{array}$$

The converse implications always hold in a regular context, as observed after the proof of Proposition 5.7. □

Remark 5.11. In a varietal context, we know that the validity of the Shifting Lemma and the Triangular Lemma is equivalent to the validity of the Trapezoid Lemma (Theorem 5.5). We do not know if this result can be generalized to the context of a regular Gumm category [14, 17]. However, Propositions 5.7 and 5.10 show that this equivalence between the validity of the Triangular Lemma and the Trapezoid Lemma does hold under the stronger conditions that the base category is regular Mal'tsev and Goursat, respectively.

Remark 5.12. Note that another characterization of regular Goursat categories which are equivalence distributive is given in [8]. A regular Goursat category is equivalence distributive if and only if the regular image of equivalence relations preserves binary meets:

$$f(R \wedge S) = f(R) \wedge f(S),$$

for any regular epimorphism $f\colon X \twoheadrightarrow Y$ and $R, S \in \mathrm{Equiv}(X)$.

5.3 Equivalence distributive Mal'tsev categories

In Chapter 4, we proved that regular Mal'tsev categories can be characterized through variations of the Shifting Lemma. Thanks to the results in the previous section we can now give some new characterizations of equivalence distributive Mal'tsev categories and through similar variations of the Triangular and of the Trapezoid Lemmas.

The variations of the Triangular and of the Trapezoid Lemmas that we have in mind take R, S or T to be just reflexive relations. Note that, for diagrams such as (19), (22) or (23), where R, S or T are not symmetric, the relations are always to be considered from left to right and from top to bottom. To avoid ambiguity with the interpretation of such diagrams, from now on we will also write $x \xrightarrow{U} y$ to mean that $(x, y) \in U$, whenever U is a non-symmetric relation.

Theorem 5.13. Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:

(i) $\mathbb{C}$ is an equivalence distributive Mal'tsev category;

(ii) the Trapezoid Lemma holds in $\mathbb{C}$ when R, S and T are reflexive relations;

(iii) the Triangular Lemma holds in $\mathbb{C}$ when R, S and T are reflexive relations.

Proof. (i) $\Rightarrow$ (ii) Since $\mathbb{C}$ is a Mal'tsev category, reflexive relations are necessarily equivalence relations. Since $\mathbb{C}$ is also equivalence distributive, by Proposition 5.7, the Trapezoid Lemma holds for any reflexive relations in $\mathbb{C}$.
(ii) $\Rightarrow$ (iii) is obvious.
(iii) $\Rightarrow$ (i) We follow the proof of Theorem 4.11 with respect to the implication: if the Shifting Lemma holds in $\mathbb{C}$ for reflexive relations, then $\mathbb{C}$ is a Mal'tsev category. The main issue is to fit the rectangle to which we applied the Shifting Lemma in that result, into a suitable triangle to which we shall now apply the Triangular Lemma (to get the same conclusion that $\mathbb{C}$ is a Mal'tsev category).

To prove that $\mathbb{C}$ is a Mal'tsev category, we show that any reflexive relation $\langle e_1, e_2 \rangle \colon E \rightarrowtail X \times X$ in $\mathbb{C}$ is also symmetric (Theorem 2.7 (iv)). Suppose that $(x, y) \in E$, and consider the reflexive relations T and R on E defined as follows:

$$(aEb, cEd) \in R \text{ if and only if } (a, d) \in E, \text{ and}$$
$$(aEb, cEd) \in T \text{ if and only if } (c, b) \in E.$$

The third reflexive relation on E we consider is the *kernel pair* $\mathrm{Eq}(e_2)$ of the second projection e_2. $\mathrm{Eq}(e_2)$ is an equivalence relation, with the property that $\mathrm{Eq}(e_2) \leqslant R$ and $\mathrm{Eq}(e_2) \leqslant T$, so that $R \wedge \mathrm{Eq}(e_2) = \mathrm{Eq}(e_2) \leqslant T$. We can apply the assumption to the following relations given in solid lines

$$\begin{tikzcd}
 & xEx \arrow[dl, "T"'] \arrow[d, "R"'] \arrow[d, dashed, bend left=60, "T"] \\
xEy \arrow[r, "\mathrm{Eq}(e_2)"'] & yEy
\end{tikzcd}$$

(xEx and yEy by the reflexivity of the relation E). We conclude that $(xEx, yEy) \in T$ and, consequently, that $(y, x) \in E$, so that $\mathbb{C}$ is a Mal'tsev category.

Since the Triangular Lemma holds in $\mathbb{C}$, by Proposition 5.7 the category $\mathbb{C}$ is equivalence distributive. $\square$

In the proof of the implication (iii) $\Rightarrow$ (i) we only used two "genuine" reflexive relations R and T. This observation gives:

Corollary 5.14. Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:

(i) $\mathbb{C}$ is an equivalence distributive Mal'tsev category;

(ii) the Trapezoid Lemma holds in $\mathbb{C}$ when R and T are reflexive relations and S is an equivalence relation;

(iii) the Triangular Lemma holds in $\mathbb{C}$ when R and T are reflexive relations and S is an equivalence relation.

Remark 5.15. An arithmetical category $\mathbb{C}$ is an equivalence distributive and Mal'tsev category which is, moreover, Barr-exact. Note that in this thesis we do not assume the existence of coequalizers, differently from what was done in Pedicchio's original definition of arithmetical category [80]. So, given a Barr-exact category $\mathbb{C}$, the same equivalent conditions stated in Theorem 5.13(ii), Theorem 5.13(iii), Corollary 5.14(ii) and Corollary 5.14(iii) give characterizations of the fact that $\mathbb{C}$ is an arithmetical category.

We finish this section with a characterization of equivalence distributive Mal'tsev categories through a property on ternary relations which is stronger than difunctionality (Theorem 2.7(v)). The difunctionality of a binary relation $D \rightarrowtail X \times U$, $DD^{\circ}D = D$ can be pictured as

$$\begin{array}{c} xDu \\ yDu \\ yDv \\ \hline xDv. \end{array}$$

Whenever the first three relations hold, we can conclude that the bottom relation xDv holds.

Recall from [82] that an arithmetical variety is such that there exists a Pixley term $p(x, y, z)$ such that

$$\begin{cases} p(x,y,y) & = & x \\ p(x,y,x) & = & x \\ p(x,x,y) & = & y. \end{cases}$$

We translate these Mal'tsev conditions into a property on relations (following the technique in [65]) which is expressed for ternary relations $D \rightarrowtail (X \times A) \times U$, seen as binary relations from $X \times A$ to U. It may be pictured as

$$\begin{array}{c} (x,a)Du \\ (y,b)Du \\ (y,a)Dv \\ \hline (x,a)Dv. \end{array} \tag{35}$$

This condition on the relation D follows from applying the Pixley term to each column of elements, and writing the result in the bottom line. In a regular context, property (35) can be expressed as follows:

$$D(\mathrm{Eq}(\pi_A) \wedge D^{\circ}D\mathrm{Eq}(\pi_X)) \leqslant D.$$

Theorem 5.16. Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:

(i) $\mathbb{C}$ is an equivalence distributive Mal'tsev category;

(ii) any relation $D \rightarrowtail (X \times A) \times U$ has property (35).

Proof. (i) $\Rightarrow$ (ii) Suppose that the first three relations in (35) hold. Consider the equivalence relations $\mathrm{Eq}(d_1), \mathrm{Eq}(d_2)$ and $\mathrm{Eq}(d_3)$ on D given by the kernel pairs of the projections of D. We have

$(x, a, u)\, \mathrm{Eq}(d_2)\, (y, a, v) \Rightarrow (x, a, u)\, \mathrm{Eq}(d_1)\mathrm{Eq}(d_2)\, (y, a, v)$

$(x, a, u)\, \mathrm{Eq}(d_3)\, (y, b, u)\, \mathrm{Eq}(d_1)\, (y, a, v) \Rightarrow (x, a, u)\, \mathrm{Eq}(d_1)\mathrm{Eq}(d_3)\, (y, a, v).$

By assumption,

$$\mathrm{Eq}(d_1)(\mathrm{Eq}(d_2) \wedge \mathrm{Eq}(d_3)) = (\mathrm{Eq}(d_1)\mathrm{Eq}(d_2)) \wedge (\mathrm{Eq}(d_1)\mathrm{Eq}(d_3))$$

(by distributivity and by Theorem 2.7(iii)). Thus

$$(x, a, u)\, \mathrm{Eq}(d_1)(\mathrm{Eq}(d_2) \wedge \mathrm{Eq}(d_3))\, (y, a, v),$$

i.e.

$$(x, a, u)\, \mathrm{Eq}(d_2) \wedge \mathrm{Eq}(d_3)\, (y, a, u)\, \mathrm{Eq}(d_1)\, (y, a, v)$$

and, consequently, $(y, a, u) \in D$. Now we use the difunctionality of D (Theorem 2.7(v))

$$\begin{array}{c} (x, a)Du \\ (y, a)Du \\ (y, a)Dv \\ \hline (x, a)Dv, \end{array}$$

to conclude that $(x, a)Dv$.

(ii) $\Rightarrow$ (i) The assumption applied to the case when $A = 1$, is precisely difunctionality of any binary relation, so $\mathbb{C}$ is a Mal'tsev category (Theorem 2.7(v)).

Since $\mathbb{C}$ is a Mal'tsev category, we just need to prove the Triangular Lemma to conclude that $\mathbb{C}$ is equivalence distributive (Proposition 5.7). Consider equivalence relations R, S and T on an object X, such that $R \wedge S \leqslant T$ and that the relations in (22) hold.

We consider a relation $D \rightarrowtail (X \times X) \times X$ defined by

$$(a, b)Dc \Leftrightarrow \exists d \in X : dSa, dTb \text{ and } dRc.$$

We have the following first three relations for $d = u, d = v$ and $d = y$, respectively,

$$\frac{\begin{array}{c}(u, y)Dv \\ (y, v)Dv \\ (y, y)Dy\end{array}}{(u, y)Dy;}$$

by assumption, we conclude that $(u, y)Dy$. By the definition of D, there exists $w \in X$ such that wSu, wTy and wRy. We can then apply the Shifting Lemma to

$$\begin{tikzcd} w \arrow[r, dash, "S"] \arrow[d, dash, "R"] \arrow[d, dash, bend right=60, "T"'] & u \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=60, "T"] \\ y \arrow[r, dash, "S"'] & v, \end{tikzcd}$$

to conclude that uTv. □

5.4 Equivalence distributive Goursat categories

In Chapter 4, we showed that Goursat categories can be characterized through variations of the Shifting Lemma. Together with the results from Section 5.3, we are going to characterize equivalence distributive Goursat categories through similar variations of the Triangular and the Trapezoid Lemmas. Such variations use the notion of positive relation.

Let us begin with the following observation:

Proposition 5.17. In any equivalence distributive Goursat category $\mathbb{C}$, the Trapezoid Lemma holds when S is a reflexive relation and R and T are equivalence relations.

Proof. The proof of this result is based on that of Proposition 4.15 which claims that a Goursat category satisfies the Shifting Lemma when S is a reflexive relation and R and T are equivalence relations.

Let R and T be equivalence relations and let S be a reflexive relation on an object X such that $R \wedge S \leqslant T$. Suppose that we have $(x, y) \in T$,

$(x, u) \in S$, $(y, v) \in S$ and $(u, v) \in R$

$$\begin{tikzcd} & x \arrow[r, "S"] \arrow[dl, dash, "T"'] & u \arrow[d, dash, "R"] \\ y \arrow[rr, "S"'] & & v. \end{tikzcd}$$

We are going to show that $(u, v) \in T$.

Consider the two relations P and W on S defined as follows:
$(aSb, cSd) \in P$ if and only if aRc and bRd:

$$\begin{tikzcd} a \arrow[r, "S"] \arrow[d, dash, "R"'] & b \arrow[d, dash, "R"] \\ c \arrow[r, "S"'] & d \end{tikzcd}$$

while $(aSb, cSd) \in W$ if and only if aTc and bRd:

$$\begin{tikzcd} a \arrow[r, "S"] \arrow[d, dash, "T"'] & b \arrow[d, dash, "R"] \\ c \arrow[r, "S"'] & d \end{tikzcd}$$

The relations P and W are equivalence relations on S since R and T are both equivalence relations. Given the equivalence relations P, $\mathrm{Eq}(s_2)$ and W on S, since $\mathbb{C}$ is Goursat category, one has

$$\begin{aligned} (\, P \wedge \mathrm{Eq}(s_2)\,) \vee W &= (\, P \wedge \mathrm{Eq}(s_2)\,)\, W \,(\, P \wedge \mathrm{Eq}(s_2)\,) \\ &= W \,(\, P \wedge \mathrm{Eq}(s_2)\,)\, W, \end{aligned}$$

which is an equivalence relation (Theorem 2.7 (iii)).

Since

$$P \wedge \mathrm{Eq}(s_2) \leqslant (\, P \wedge \mathrm{Eq}(s_2)\,) \vee W$$

and $\mathbb{C}$ is a Goursat and equivalence distributive category, by Proposition 5.10, we can apply the Trapezoid Lemma to the following diagram

$$\begin{tikzcd}
 & xSu \arrow[r, dash, "\mathrm{Eq}(s_2)"] & uSu \arrow[d, dash, "P"'] \arrow[d, dash, dashed, bend left=60, "(\,P\wedge \mathrm{Eq}(s_2)\,)\vee W"] \\
ySv \arrow[ur, dash, "(\,P\wedge \mathrm{Eq}(s_2)\,)\vee W"] \arrow[rr, dash, "\mathrm{Eq}(s_2)"'] & & vSv.
\end{tikzcd}$$

Note that, uSu and vSv by the reflexivity of S. We then obtain

$$(uSu, vSv) \in (\, P \wedge \mathrm{Eq}(s_2)\,) \,\vee\, W = (\, P \wedge \mathrm{Eq}(s_2)\,)\, W\, (\, P \wedge \mathrm{Eq}(s_2)\,),$$

this means that there are a and b in X such that

$$(uSu)\,(P \wedge \mathrm{Eq}(s_2))\,(aSu)W(bSv)\,(P \wedge \mathrm{Eq}(s_2))\,(vSv),$$

i.e.

$$\begin{tikzcd}
u \arrow[r, "S"] \arrow[d, dash, "R"'] & u \arrow[d, dash, "R"] \\
a \arrow[r, "S"] \arrow[d, dash, "T"'] & u \arrow[d, dash, "R"] \\
b \arrow[r, "S"] \arrow[d, dash, "R"'] & v \arrow[d, dash, "R"] \\
v \arrow[r, "S"'] & v.
\end{tikzcd}$$

Since aRu (R is symmetric), aSu and $R \wedge S \leqslant T$, it follows that aTu; similarly one checks that bTv. From uTa (T is symmetric), aTb and bTv, we conclude that uTv (T is transitive), as desired. □

Since the Trapezoid Lemma implies the Triangular Lemma, we get the following:

Corollary 5.18. In any equivalence distributive Goursat category $\mathbb{C}$, the Triangular Lemma holds when S is a reflexive relation and R and T are equivalence relations.

We are now ready to prove the main result in this section:

Theorem 5.19. Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:

(i) $\mathbb{C}$ is an equivalence distributive Goursat category;

(ii) the Trapezoid Lemma holds in $\mathbb{C}$ when S is a reflexive relation and R and T are reflexive and positive relations;

(iii) the Triangular Lemma holds in $\mathbb{C}$ when S is a reflexive relation and R and T are reflexive and positive relations.

Proof. (i) $\Rightarrow$ (ii) Since $\mathbb{C}$ is a Goursat category, by Theorem 4.7, reflexive and positive relations are necessarily equivalence relations. Since $\mathbb{C}$ is also equivalence distributive, by Proposition 5.17, the Trapezoid Lemma holds when S is a reflexive relation and R and T are reflexive and positive relations.
(ii) $\Rightarrow$ (iii) is obvious.
(iii) $\Rightarrow$ (i) We follow the proof of Theorem 4.17 with respect to the implication: if the Shifting Lemma holds in $\mathbb{C}$ when S is a reflexive relation and R and T are reflexive and positive relations, then $\mathbb{C}$ is a Goursat category. The main issue is to fit the rectangle to which we applied the Shifting Lemma in that result, into a suitable triangle to which we shall now apply the Triangular Lemma (to get the same conclusion that $\mathbb{C}$ is a Goursat category).

To prove that $\mathbb{C}$ is a Goursat category, we show that for any reflexive relation E on X in $\mathbb{C}$, $EE^\circ = E^\circ E$ (Theorem 2.2(vi)). Suppose that $(x, y) \in EE^\circ$. Then, for some z in X, one has that $(z, x) \in E$ and $(z, y) \in E$. Consider the reflexive and positive relations EE° and $E^\circ E$, and the reflexive relation E on X. From the reflexivity of E, we get $E \leqslant EE^\circ$ and $E \leqslant E^\circ E$; thus $EE^\circ \wedge E = E \leqslant E^\circ E$. We can apply our assumption (for $R = EE^\circ, S = E, T = E^\circ E$) to the following relations given in solid lines:

$$\begin{tikzcd}
& x \arrow[dl, "E^\circ E"'] \arrow[d, "EE^\circ"'] \arrow[d, dashed, bend left=70, "E^\circ E"] \\
z \arrow[r, "E"'] & y
\end{tikzcd}$$

to conclude that $(x, y) \in E^\circ E$. Having proved that $EE^\circ \leqslant E^\circ E$ for *every* reflexive relation E, the equality $E^\circ E \leqslant EE^\circ$ follows immediately, and then $\mathbb{C}$ is a Goursat category.

Since the Triangular Lemma holds in $\mathbb{C}$, by Proposition 5.10 the category $\mathbb{C}$ is equivalence distributive. $\square$

We finish this section with a characterization of equivalence distributive Goursat categories through a property on ternary relations which is stronger than property (iv)

$$P^{\circ}PP^{\circ}P = P^{\circ}P \tag{36}$$

of Theorem 2.2. The process to obtain such a characterization is similar to what was done to obtain Theorem 5.16 for the Mal'tsev context. The property (36) of a binary relation $P \rightarrowtail X \times U$, can be pictured as

$$\begin{array}{c} xPu \\ yPu \\ yPv \\ zPv \\ \hline xPw \\ zPw. \end{array}$$

Whenever the first four relations hold, we can conclude that the bottom relations xPw and zPw hold for some w in U.

Recall from [74] that a 3-permutable congruence distributive variety is such that there exists ternary terms $r(x,y,z)$ and $s(x,y,z)$ such that

$$\begin{cases} r(x,y,y) & = & x \\ r(x,x,y) & = & s(x,y,y) \\ s(x,x,y) & = & y \\ r(x,y,x) & = x = & s(x,y,x). \end{cases}$$

It is easy to check that, equivalently, such varieties admit quaternary terms $p(x,y,z,w)$ and $q(x,y,z,w)$ such that

$$\begin{cases} p(x,y,y,z) & = & x \\ p(x,x,y,y) & = & q(x,x,y,y) \\ q(x,y,y,z) & = & z \\ p(x,y,z,x) & = x = & q(x,y,z,x). \end{cases}$$

These Mal'tsev conditions translate into a property on relations (following the technique in [66]) which is expressed for ternary relations $P \rightarrowtail (X \times A) \times U$, seen as binary relations from $X \times A$ to U, as

$$\frac{\begin{array}{c}(x,a)Pu\\(y,b)Pu\\(y,c)Pv\\(z,a)Pv\end{array}}{\begin{array}{c}(x,a)Pw\\(z,a)Pw,\end{array}} \tag{37}$$

for some w in U.

In a regular context, property (37) means that:

$$\mathrm{Eq}(\pi_A) \wedge P^{\circ}P\mathrm{Eq}(\pi_X)P^{\circ}P \leqslant P^{\circ}P,$$

and one can prove the following:

Theorem 5.20. Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:

(i) $\mathbb{C}$ is an equivalence distributive Goursat category;

(ii) any relation $P \rightarrowtail (X \times A) \times U$ has property (37).

Proof. (i) $\Rightarrow$ (ii) Suppose that the first four relations in (37) hold. Consider the equivalence relations $\mathrm{Eq}(d_1), \mathrm{Eq}(d_2)$ and $\mathrm{Eq}(d_3)$ on P given by the kernel pairs of the projections of P. We have

$$(x,a,u)\,\mathrm{Eq}(d_2)\,(z,a,v) \Rightarrow (x,a,u)\,\mathrm{Eq}(d_3)\mathrm{Eq}(d_2)\mathrm{Eq}(d_3)\,(z,a,v).$$

$$\begin{gathered}(x,a,u)\,\mathrm{Eq}(d_3)\,(y,b,u)\,\mathrm{Eq}(d_1)\,(y,c,v)\,\mathrm{Eq}(d_3)\,(z,a,v)\\ \Rightarrow (x,a,u)\,\mathrm{Eq}(d_3)\mathrm{Eq}(d_1)\mathrm{Eq}(d_3)\,(z,a,v).\end{gathered}$$

By assumption,

$$\begin{gathered}\mathrm{Eq}(d_3)(\mathrm{Eq}(d_1) \wedge \mathrm{Eq}(d_2))\mathrm{Eq}(d_3) =\\ (\mathrm{Eq}(d_3)\mathrm{Eq}(d_1)\mathrm{Eq}(d_3)) \wedge (\mathrm{Eq}(d_3)\mathrm{Eq}(d_2)\mathrm{Eq}(d_3))\end{gathered}$$

(by distributivity and by Theorem 2.2(iii)). Thus

$$(x,a,u)\,\mathrm{Eq}(d_3)(\mathrm{Eq}(d_1) \wedge \mathrm{Eq}(d_2))\mathrm{Eq}(d_3)\,(z,a,v),$$

i.e. $(x, a, u)\mathrm{Eq}(d_3)(\alpha, \beta, u)\,\mathrm{Eq}(d_1) \wedge \mathrm{Eq}(d_2)\,(\alpha, \beta, v)\,\mathrm{Eq}(d_3)\,(z, a, v)$ and, consequently, $(\alpha, \beta, u) \in P$ and $(\alpha, \beta, v) \in P$. Now we use the property (36) of P

$$\begin{array}{c} (x,a)Pu \\ (\alpha,\beta)Pu \\ (\alpha,\beta)Pv \\ (z,a)Pv \\ \hline (x,a)Pw, \\ (z,a)Pw \end{array}$$

to conclude that $(x, a)Pw$ and $(z, a)Pw$ for some $w \in U$.

(ii) $\Rightarrow$ (i) The assumption applied to the case when $A = 1$ implies property (36) of any binary relation, so $\mathbb{C}$ is a Goursat category (Theorem 2.2(iv)).

Since $\mathbb{C}$ is a Goursat category, we just need to prove the Triangular Lemma to conclude that $\mathbb{C}$ is equivalence distributive (Proposition 5.10). Consider equivalence relations R, S and T on an object X, such that $R \wedge S \leqslant T$ and that the relations in (22) hold.

We consider a relation $P \rightarrowtail (X \times X) \times X$ defined by

$$(a, b)Pc \Leftrightarrow \exists d \in X : dSa, dTb \;\text{ and }\; dRc.$$

We have the following first four relations for $d = u, d = v$, $d = y$ and $d = y$ again, respectively,

$$\begin{array}{c} (u,y)Pv \\ (y,v)Pv \\ (y,u)Py \\ (y,y)Py \\ \hline (u,y)Pw \\ (y,y)Pw; \end{array}$$

by assumption, we conclude that $\exists w \in X$ such that$(u, y)Pw$ and $(y, y)Pw$. By the definition of P, there exists $m, n \in X$ such that mSu, mTy, mRw; nSy, nTy, nRw.

One has:

$$\begin{aligned} mTyTn &\Rightarrow mTn \\ mRwRn &\Rightarrow mRn \\ nSySv &\Rightarrow nSv. \end{aligned}$$

We can then apply the Shifting Lemma to

$$\begin{tikzcd} m \arrow[r, dash, "S"] \arrow[d, dash, "R"] \arrow[d, dash, bend right=80, "T"'] & u \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=80, "T"] \\ n \arrow[r, dash, "S"'] & v, \end{tikzcd}$$

to conclude that uTv. □

Chapter 6

Perspectives

Here, we present some directions for future research.

6.1 Centrality properties in Goursat categories

One of the main interests of the notion of connector is that it allows us to understand centrality even without defining the commutator of equivalence relations. Indeed, we can prove the important basic centrality properties which correspond to the classical properties of the commutator. So, thanks to the properties of connectors in the Goursat categories that we obtained, we then expect to prove some basic centrality properties which correspond to the classical properties of the commutator as it was done in [15] for the case of Mal'tsev categories:

1. Symmetry: $[R, S] = [S, R]$;
2. Monotonicity: if $S_1 \leqslant S_2$, then $[R, S1] \leqslant [R, S2]$;
3. Inclusion of the commutator in the intersection: $[R, S] \leqslant R \cap S$;
4. Stability with respect to products: $[R_1 \times R_2, S_1 \times S_2] \leqslant [R_1, S_1] \times [R_2, S_2]$;
5. Stability with respect to restriction: if $i : Y \rightarrowtail X$ is a monomorphism, then $[R, S] = \Delta_X$ implies $[i^{-1}(R), i^{-1}(S)] = \Delta_Y$, where Δ_X is the smallest equivalence relation on X;

6. Stability with respect to joins: $[R, S_1 \vee S_2] = [R, S_1] \vee [R, S_2]$.

(where R, R_1, R_2, S, S_1 and S_2 are equivalence relations on a given object X).

The properties 1., 2. and 3. are always true in any regular category. The property 5. cannot be true in Goursat categories since it characterizes Mal'tsev categories. However, we think that the properties 4. and 6. could be true in Goursat categories.

These properties will then be useful for studying Smith-Pedicchio commutators in the context of Goursat categories.

6.2 Abelian objects in Goursat categories

The new characterization of Goursat categories in terms of properties of internal groupoids (Theorem 3.19) allows us to easily verify that the subcategory $\mathbb{C}_{Ab}$ of abelian objects of any Goursat category $\mathbb{C}$ is closed under quotients in $\mathbb{C}$.

It is an open problem whether $\mathbb{C}_{Ab}$ is also closed under subobjects in $\mathbb{C}$. The characterization of Goursat categories in terms of positive relations perhaps gives us a track to prove this property in a more general framework by proving that the subcategory $\mathbb{C}_{Ab}$ of a Goursat category $\mathbb{C}$ is a Birkhoff subcategory as in the case of exact Mal'tsev categories [35].

It is well-known that the subcategory $\mathbb{C}_{Ab}$ of abelian objects of any factor permutable category [34] $\mathbb{C}$ is closed under subobjects in $\mathbb{C}$ (Corollary 3.15 [34]). So, another way to solve the problem is to prove that any Goursat category is a factor permutable category. This property is true in a varietal context (Corollary 4.5 [51]). So, we can try to use some of the known techniques to extend varietal proofs into categorical ones such as the matrix technique [67], the embedding theoreoms [3, 57, 58].

We can then deduce that the subcategory of abelian objects $\mathbb{C}_{Ab}$ of a Goursat category $\mathbb{C}$ is admissible from the point of view of categorical Galois theory [61].

6.3 n-permutable completions

In Section 2.3, we characterized categories with weak finite limits whose regular completions give rise to Goursat categories and related them to the existence of the quaternary operations characterizing the varieties of universal algebras which are 3-permutable varieties. Such kind of studies have been done for Mal'tsev categories in [86]. So, the natural question that arises is: can we generalize these studies to n-permutable categories?

The characterizations of n-permutable categories in terms of positive relations (Theorem 4.9 and 4.10) and through certain stability properties of regular epimorphisms (Theorem 3.3 [59]) give us some interesting leads to approach this question.

In the same direction, one could also try to find the categorical properties characterizing the projective covers of congruence distributive and congruence modular varieties.

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