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The option of joint purchase in vertically differentiated markets*

Jean J. Gabszewicz[1] **and Xavier Y. Wauthy**[2]

[1] CORE, Université Catholique de Louvain, 34 voie du Roman Pays, 1348 Louvain-la-Neuve, BELGIUM
[2] CEREC, Facultés Universitaires Saint-Louis, 43 boulevard du jardin botanique, 1000 Bruxelles, BELGIUM (e-mail:xwauthy@fusl.ac.be)

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Summary. Within the framework proposed by Mussa and Rosen (1978) for modelling quality differentiation, consumers are assumed to make mutually exclusive purchases. A unique pure strategy equilibrium exists in this case. In this note, we allow consumers to buy simultaneously different variants of the differentiated good. We call this the "joint purchase option". The paper proposes a detailed analysis of price competition when this option is opened: first, we show that either uniqueness, or multiplicity, or absence of price equilibrium arise, depending on the utility derived from joint purchase relative to exclusive purchase. Second, we characterize these equilibria, whenever they exist.

Keywords and Phrases: Joint purchase, Price, Vertical differentiation.

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1 Introduction

Two variants of a product are said vertically differentiated when, sold at the same price, all consumers buy one of the variants at the exclusion of the other when assumed to be making mutually exclusive purchases. Even though this definition is generally used to qualify consumption differentiated goods, it may also apply in other contexts. Think, for instance, of the relative merits of press medias as advertising supports. When faced with equal advertising fees opposed by two different newspapers'editors, advertisers, when forced to advertise in a single newspaper

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Correspondence to: X.Y. Wauthy

only, will certainly advertise in the one with the broader coverage : the newspapers are vertically differentiated, and the source of differentiation is the size of the readership. Now if the exclusive purchase constraint is dropped, advertisers may now be interested in advertising in both newspapers : even if the readerships of these newspapers overlap, buying a further ad-space should increase the number of readers exposed when the overlap is not perfect. Yet the larger the overlap, the smaller the gain to be expected from advertising in both newspapers (this example is drawn from Gabszewicz et al. (2001)).

A second example could be made of operating systems: different OS perform the same basic tasks, but may differ by the fact that some applications either are running only with specific OS, or are optimized for a particular one. In such a case, one could argue that OS are vertically differentiated, their "quality" essentially depending on the number of applications they can run. What is the value added of buying two different OS? Intuitively, the second purchase enlarges the set of applications that a user can run on its desktop, so that the specific value of buying a second OS comes from the number of applications which are specific to each OS. Notice that in these two examples the motivation of multiple purchases is definitely related to the fact that multiple purchases concern different variants of the same basic good. Notice also that buying a single unit of one variant only still remains a relevant option for the consumer.

For the goods considered in the two above examples, the markets embody two types of agents, some of them buying both variants and others who prefer buying only one of them. How should we model price competition in such cases? Does the co-existence of the two consumption profiles -single versus joint purchase- affect firms' strategic behaviour?

Standard models of product differentiation are not designed to address these questions. They assume indeed that those consumers who decide to buy a product, single out a particular variant of it among the various substitutes provided by the industry, and buy a single unit of that variant. This way of defining the decision set open to consumers reduces the *quantity* decision of households to a binary choice : one or zero unit. There is no question that reality is quite different. The mere improvement of living standards through the population allows many households to be equipped with several variants of the same indivisible product. It is indeed far from seldom to observe households equipped with two or three different cars, or several TV-sets or P.C.'s. Similarly, it is not difficult to identify consumers owning two or three different houses for their personal use only. Buying two vertically differentiated variants of an indivisible good is thus quite common. This note precisely addresses the nature of price competition in this case.

To this end, we consider an industry where two firms sell goods that are vertically differentiated. We keep the property that each variant is consumed in indivisible units. Still, unlike the traditional assumption, we now suppose that consumers who are interested in consuming both variants at the existing price constellation are allowed to do so. The extent to which this possibility of joint purchase influences price competition between duopolists is the problem considered in this note. Note that virtually any paper dealing with address-models of product differentiation makes this assumption (see Anderson, De Palma and Thisse, 1992; Tirole, 1988).

A noticeable exception is the recent contribution by Caillaud, Grilo and Thisse (2000). Another related paper is De Palma, Leruth and Regibeau (1999) who allows for joint purchase within a Cournot framework. Their focus however is on product complementarity with network effects. Let us mention also that, even though we develop a model of vertical differentiation, most of our results carry over to markets where products are horizontally differentiated.

Our results are best summarized in comparison with those obtained in the standard vertical differentiation model. To ease comparison, it is sufficient to remind that not allowing for joint purchase formally amounts to assume that buying two units does not add utility compared with buying the high quality product. This is the standard case and it is well-known (see Gabszewicz and Thisse, 1979) that in such case there always exists a unique price equilibrium in pure strategies. The implication of the joint purchase option on price competition will obviously depend on the additional utility gained by a consumer when she buys a second unit. First, when buying the two variants does not add much in utility compared with the utility obtained when buying the best variant only, price competition is not influenced by opening to consumers the option of buying both variants : equilibrium prices are the same as those obtained in the "traditional" model, and no consumer in the population takes advantage of the new option available to them (in the following, we refer to this equilibrium as the *exclusive purchase equilibrium).* Second, when buying the two variants starts to add a more substantial amount in utility compared with the utility of the best variant, multiple price equilibria arise, among which the equilibrium of the traditional model is still present. Yet another equilibrium appears along at which some consumers buy both variants of the product. At the new price equilibrium (referred below as the *joint purchase equilibrium)*, both prices are lower than at the standard one, in which no joint purchase is allowed.This reflects the fact that the low-quality seller has to lower his price in order to attract some consumers, who already buy the high-quality product, to purchase as well the low-quality one. When the utility of joint purchase is still higher, only the joint purchase equilibrium exists. Finally, when the utility of buying both variants is close to the sum of the utilities corresponding to consuming each variant separately, no price equilibrium in pure strategies still exists. However, mixed strategy equilibria exist in this case and we exhibit one class of them.

The note is organised as follows. In the next section, we introduce briefly the "pure" model of vertical product differentiation, and set how this model has to be adapted in order to take also into account the option of consuming both variants. This essentially consists in deriving demand functions In section 3, we examine how price competition develops when this new option is opened to consumers. Section 4 concludes

2 The joint purchase option

Consider a model "à la Mussa-Rosen", with two variants of a product, indexed by their quality u_i, $i = 1, 2$ (Mussa and Rosen, 1978). We assume, without loss of generality, that $u_2 > u_1$. Firms produce at zero cost. They choose prices non-cooperatively in order to maximise revenue. Consumers' types are indexed by a parameter θ which expresses the intensity of their preferences for buying a unit of the good. Types are uniformly distributed in the $[0, 1]$-interval, with one consumer per type. If consumer θ buys one unit of variant i at price p_i, his utility is given by

$$u_i\theta - p_i. \tag{1}$$

We denote by θ_i, $i = 1, 2$, the consumer who is indifferent between the option of buying one unit of variant i at price p_i and the option of not buying. If we assign zero utility to the latter option, we obtain

$$\theta_i = \frac{p_i}{u_i}. \tag{2}$$

Similarly we denote by θ_{12} the consumer who is indifferent between buying one unit of variant 1 at price p_1 and one unit of variant 2 at price p_2, i.e.

$$\theta_{12} = \frac{p_2 - p_1}{u_2 - u_1}. \tag{3}$$

The standard model of vertical product differentiation assumes that consumers, when they buy, select which variant they wish to buy, *at the exclusion of the other.* Using (2) and (3), demands adressed to the sellers are then easily derived as

$$D_1(p_1, p_2) = \theta_{12} - \theta_1 = \frac{p_2 u_1 - p_1 u_2}{u_1(u_2 - u_1)}; \tag{4a}$$

$$D_2(p_1, p_2) = 1 - \theta_{12} = 1 - \frac{p_2 - p_1}{(u_2 - u_1)}. \tag{4b}$$

The corresponding price game has a unique price equilibrium (*exclusive purchase equilibrium*), namely

$$p_1^* = \frac{u_1(u_2 - u_1)}{4u_2 - u_1}, \tag{5a}$$

$$p_2^* = \frac{2u_2(u_2 - u_1)}{4u_2 - u_1}; \tag{5b}$$

in the sequel we simply refer to this equilibrium as p^*.

Now let us assume that, unlike the "traditional" assumption, the quantity decision set of each household is extended to also include the possibility of buying both variants, and denote by u_3 the utility index derived from such a joint consumption. In order to preserve the fact that variants 1 and 2 are *substitutes* of the same product, we shall assume that[1]

$$u_2 < u_3 < u_1 + u_2.$$

In the case of joint purchase , the utility of consumer θ is assumed to be given by $u_3\theta - p_1 - p_2$. As above, we denote by θ_{i3} the consumer who is indifferent between buying one unit of variant i at price p_i and one unit of *both* variants at prices p_1 and p_2, namely

$$\theta_{13} = \frac{p_2}{u_3 - u_1}; \tag{6a}$$

[1] The case $u_3 \geq u_1 + u_2$ is considered in Gabszewicz, Sonnac and Wauthy (2000)

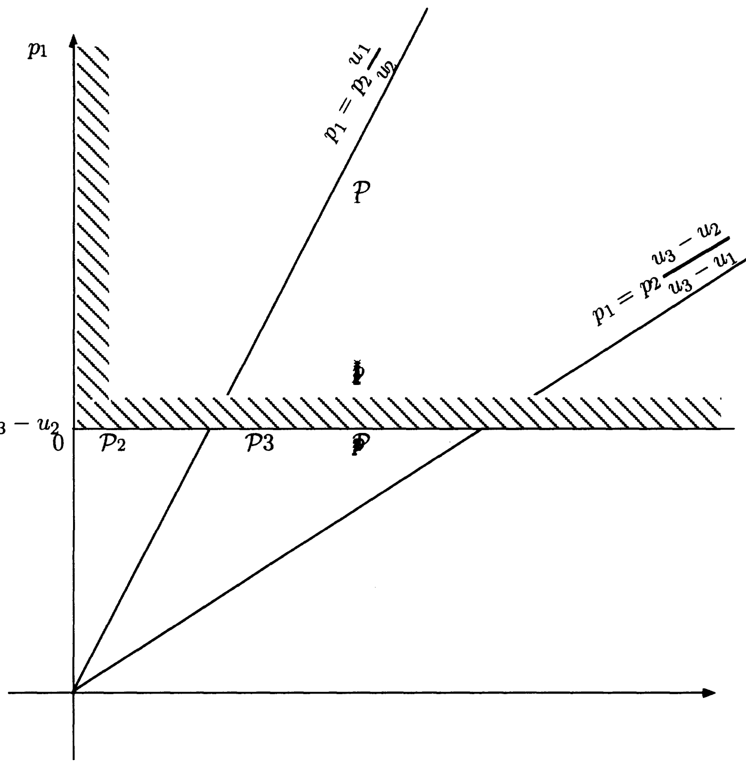

Figure 1

$$\theta_{23} = \frac{p_1}{u_3 - u_2}. \tag{6b}$$

With these definitions, it is a matter of patience to derive the demand functions of the duopolists, which can be best understood using the following diagram providing a partition of the domain of (p_1, p_2)-prices into four sub-domains $P_i, i = 1, ...4$. The frontiers between the different regions are defined hereafter.

The price -subdomain P_1, which is delimited from below by the line $p_1 = u_3 - u_2$, is defined as

$$P_1 = \{(p_1, p_2) : p_1 \geq u_3 - u_2\}.$$

In this domain, the demand functions D_1 and D_2 are given by (4) : the price p_1 is so high that even consumer $\theta = 1$, who has the highest willingness to pay for consuming both variants, is not willing to buy them at that price. Accordingly, in the domain P_1, demand functions are as in the "pure" vertical differentiation model since nobody in the market is considering to buy both variants. Yet, this changes as soon as $p_1 < u_3 - u_2$: then some consumers - those with the highest θ's - start to buy both variants. Consider then the sub-domain P_2 defined by

$$P_2 = \left\{(p_1, p_2) : p_1 < u_3 - u_2; p_1 \geq p_2 \frac{u_1}{u_2}\right\}.$$

In P_2, demands are given by

$$D_1(p_1, p_2) = 1 - \theta_{23} = 1 - \frac{p_1}{u_3 - u_2}; \tag{7a}$$

$$D_2(p_1, p_2) = 1 - \theta_2 = 1 - \frac{p_2}{u_2}. \tag{7b}$$

In the sub-domain P_2, all consumers who buy variant 1 also buy variant 2, so that the market of firm 2 extends up to θ_2. This changes as soon as the inequality $p_1 \geq p_2 \frac{u_1}{u_2}$ is reversed. Then a new class of consumers appears at prevailing prices : those who start to buy only variant 1. Then we enter into the sub-domain P_3 defined by

$$P_3 = \left\{ (p_1, p_2) : p_1 < u_3 - u_2; p_1 \leq p_2 \frac{u_1}{u_2}; p_1 \geq p_2 \frac{u_3 - u_2}{u_3 - u_1} \right\}.$$

In this sub-domain, the demand adressed to firm 2 now coincides with the demand adressed to this firm in the "pure" vertical differentiation model.Yet the demand adressed to firm 1 is made of those consumers who buy both variants (the interval $[\theta_{23}, 1]$), as well as of those who buy variant 1 only (the interval $[\theta_1, \theta_{12}]$), that is

$$D_1(p_1, p_2) = (1 - \theta_{23}) + (\theta_{12} - \theta_1) = 1 + \frac{p_2}{u_2 - u_1} - p_1 K \tag{8a}$$

$$D_2(p_1, p_2) = 1 - \theta_{12} = 1 - \frac{p_2 - p_1}{u_2 - u_1}, \tag{8b}$$

with K defined by

$$K = \frac{(u_3 - u_2)(u_2 - u_1) + u_1(u_3 - u_1)}{u_1(u_2 - u_1)(u_3 - u_2)}. \tag{9}$$

Finally, we can define the sub-domain P_4, which is the symmetric of P_2 where all consumers who buy variant 2 also buy variant 1. It is defined as

$$P_4 = \left\{ (p_1, p_2) : p_1 < u_3 - u_2; p_1 \leq p_2 \frac{u_3 - u_2}{u_3 - u_1} \right\},$$

we get

$$D_1(p_1, p_2) = 1 - \theta_1 = 1 - \frac{p_1}{u_1}; \tag{10a}$$

$$D_2(p_1, p_2) = 1 - \theta_{13} = 1 - \frac{p_2}{u_3 - u_1} : \tag{10b}$$

now the boundary between markets of firms 1 and 2 corresponds to the consumer who is indifferent between the options of buying only variant 1 or buying both variants, and no longer to the consumer who is indifferent between buying variant 1 only and buying variant 2 only, as it was the case in the sub-domains P_i, $i = 1, 2, 3$, defined above.

At this point, three remarks are in order. Notice first that, compared with the standard vertical differentiation model, allowing for joint purchase essentially amounts to alter the definition of demands when the price of firm 1 is below the value $u_3 - u_2$.

The dividing line between region P_1 (where the standard model applies) and regions $P_{2,}$, P_3 and P_4 does not depend on p_2. When choosing p_1 firm 1 "decides" whether the demands corresponding to those of the standard analysis apply or not. Unilateral deviations of p_2 cannot achieve the same result. Second, it should be noticed that demand addressed to firm 2 in region P_2 (respectively firm 1 in region P_4) is the standard monopoly demand, while firm 1's in this region is independent of p_2. This demand can be viewed as a residual demand defined on the "second purchase" market, i.e. the set of consumers considering to buy a second unit. By fixing one price, say p_1 and moving from regions $P_{2,}$, P_3 and P_4 by increasing p_2 we may summarizes the nature of price competition. When firm 2 names low prices (within P_2) it demand is the monopoly one while firm 1 sells only the those who buy two units: firms are not truly competing with each other. On the other hand, in Region P_3, firm 2 and firm 1 are competing for those consumers who buy one unit only, even though firm 1 also sells a second unit on the right side of the market. Last, in region P_4 firm 2 focuses on double purchasers and charges to them a high price. Therefore, because of the joint purchae option, a firm faces two different pricing strategies: either it charges relatively low prices and fights for market shares or it "retreats" with high price on the "rich" side of the market where "joint-purchasers" are located. Third, the payoffs in this game, obtained as the revenue functions resulting from the demands addressed to each firm in the various sub-domains of prices, are continuous functions throughout the whole space of prices.

Equipped with the above material, we can now tackle the equilibrium analysis assuming that consumers are also allowed to make joint purchases (*joint purchase price game*). This is done in the next section.

3 Equilibrium analysis

In the situation in which joint purchase does not add much in utility, compared with the utility index corresponding to the top quality variant, a first question which seems natural to raise is whether the equilibrium p^* of the original price game is still an equilibrium in the joint purchase price game. Since the increase in utility obtained by consumers from joint purchase is assumed to be small, it may be conjectured that firms in the latter may have no interest to set prices at equilibrium taking advantage of this new opportunity. Proposition 1 below provides a positive answer to this conjecture.

Proposition 1. *There exists an interval* $[u_2, u_2 + \varepsilon^*]$, *with* $\epsilon^* > 0$, *such that, whenever* $u_3 \in [u_2, u_2 + \varepsilon^*]$, *the exclusive purchase equilibrium* p^* *is still an equilibrium in the joint purchase price game.*

Proof. First, it is clear that, for the standard equilibrium $p^* = (p_1^*, p_2^*)$ to be an equilibrium in the joint purchase price game, we must have $p_1^* > u_3 - u_2$ and $p_2^* u_1 > p_1^* u_2$: the first inequality follows from the fact that, at p^*, even consumer $\theta = 1$ should not be willing to buy both variants, while the second inequality follows from $D_1(p_1^*, p_2^*) > 0$. The equilibrium $p^* = (p_1^*, p_2^*)$ is thus located in the (p_1, p_2) - plane as depicted on figure 1.

Notice that no unilateral deviation from the equilibrium p^* which would let the resulting pair of prices in P_1, can be advantageous to any of the two firms : recall that p^* is an equilibrium in the original game which is, in particular, defined in the sub-domain P_1, so that unilateral deviations leaving the pair of prices in this sub-domain cannot be profitable. Since any unilateral deviation of firm 2 from p_2^* maintains the pair of prices in the sub-domain P_1, it cannot be advantageous to firm 2 : in this sub-domain, we know that p_2^* is a best response against p_1^*. To destroy the equilibrium p^* as an equilibrium in the joint purchase price game, we can thus rely only on unilateral deviations p_1 of firm 1 which drive the resulting pair of prices in P_3 orP_4. For deviations in P_3, it follows from (8) that the revenue of firm 1 obtains as

$$R_1(p_1, p_2^*) = p_1 \left(1 + \frac{p_2^*}{u_2 - u_1} - p_1 K\right),$$

which is maximal in P_3 for p_1 given by

$$p_1^{'} = \frac{u_2 - u_1 + p_2^*}{2(u_2 - u_1)K}. \tag{11}$$

On the other hand, revenue at the equilibrium p^* is given by

$$R_1(p_1^*, p_2^*) = p_1^* \left(\frac{p_2^* u_1 - p_1^* u_2}{u_1(u_2 - u_1)}\right).$$

Comparing $R_1(p_1^*, p_2^*)$ and $R_1(p_1^{'}, p_2^*)$ reveals that the former exceeds the latter as long as $u_3 \leq u_2 + \varepsilon^*$, with

$$\varepsilon^* = \frac{4u_1u_2(u_2 - u_1)}{32u_2^2 - 12u_1u_2 + u_1^2} > 0,$$

where the last strict inequality follows from the fact that $u_2 > u_1$. Consequently, when $u_3 \in [u_2, u_2 + \varepsilon^*]$, there exists no unilateral advantageous deviation for firm 1 in P_3. Similarly, it can be checked that no unilateral advantageous deviation for firm 1 which would bring the pair of prices in P_4, exists either. Indeed, in region P_4, $D_1(.)$ is the monopoly demand so that the best reply candidate is the monopoly price, which lies outside this region. Stated otherwise, the best reply candidate for firm 1 in region P_4 is defined by the strategy lying at the frontier with region P_2, which, by continuity, is itself dominated by the best reply in the interior of P_2.

Consequently, when $u_3 \in [u_2, u_2 + \varepsilon^*]$, the pair of prices (p_1^*, p_2^*) remains a price equilibrium in the joint purchase price game. Q.E.D.

The above proposition shows that the equilibrium p^* remains robust to the introduction of the joint purchase option, at least when u_3 is in a sufficiently small neighborhood of u_2. But this does not preclude the possibility that, for some u_3-values, another price equilibrium would co-exist with p^* when the joint purchase option becomes available. That this is indeed the case follows from the following

Proposition 2. *There exists a non-degenerate interval of values for u_3 in which both the exclusive and the joint purchase equilibria coexist.*

Proof. Consider the pair of prices which are best responses to each other in the sub-domain P_3, with payoffs (revenues) obtained from the demand functions in P_3 (see 8), namely

$$R_1(p_1, p_2) = p_1 \left(1 + \frac{p_2}{u_2 - u_1} - p_1 K\right)$$
$$R_2(p_1, p_2) = p_2 \left(1 - \frac{p_2 - p_1}{u_2 - u_1}\right),$$

with K as defined by (9). These best responses in P_3 are easily identified from the first-order conditions, i.e.

$$\phi_1(p_2) = \frac{u_2 - u_1 + p_2}{2(u_2 - u_1)K} \tag{12}$$

for firm 1, and

$$\phi_2(p_1) = \frac{u_2 - u_1 + p_1}{2} \tag{13}$$

for firm 2. Combining these best responses yields a candidate price equilibrium $(\overset{\circ}{p}_1, \overset{\circ}{p}_2)$ which is given by

$$\overset{\circ}{p}_1 = \frac{3(u_2 - u_1)}{4(u_2 - u_1)K - 1};$$
$$\overset{\circ}{p}_2 = \frac{(2K(u_2 - u_1) + 1)(u_2 - u_1)}{4(u_2 - u_1)K - 1}.$$

Now we study the necessary and sufficient conditions under which this candidate is, indeed, a price equilibrium. First, it is easy to check that $\overset{\circ}{p}_1 < u_3 - u_2$, so that $\overset{\circ}{p}_1 \in P_3$ (the candidate equilibrium is indeed defined in Region P_3) and, by definition, no unilateral deviation can be advantageous if it leaves the pair of prices in this sub-domain. Let us then consider deviations that lead us outside region P_3.

To remain robust against unilateral deviations of firm 2 driving the pair of prices in P_4, it is necessary and sufficient that

$$R_2(\overset{\circ}{p}_1, \overset{\circ}{p}_2) \geq R_2(\overset{\circ}{p}_1, \psi_2(\overset{\circ}{p}_1)),$$

with $R_2(\overset{\circ}{p}_1, \psi_2(\overset{\circ}{p}_1))$ denoting the revenue of firm 2 at its best response against $\overset{\circ}{p}_1$ in P_4. Using (10), a direct comparison between these two numbers reveals that the desired inequality holds if, and only if, $u_3 - u_2 < \eta$, with $\eta > 0$.[2] Ruling out an advantageous deviation for firm 1 driving the pair of prices in P_2 follows from applying the same method as the one used above for deviations of firm 2 driving the pair of prices in P_4. The resulting comparison reveals the existence of a particular value η', $\eta < \eta'$, such that no deviation for firm 1 in P_2 can be profitable if, and only if, $u_3 - u_2 < \eta'$.[3] Last, we must check that there exists no profitable

[2] Computations show that
$\eta = \frac{u_1(u_2-u_1)(11u_1-8u_2+\sqrt{17u_1^2-48u_1u_2+64u_2^2})}{2(4u_2-u_1)^2}$

[3] Computations show that $\eta' = \frac{2u_1(u_2-u_1)(2u_1+u_2+3\sqrt{u_1^2-4u_1u_2+9u_2^2})}{(4u_2-u_1)^2}$

deviation for firm 1 in P_4, nor for firm 2 in P_2. To show this we may refer to the fact that in these regions, respective demands are the monopoly demands (see 7b, 10a). Accordingly, the corresponding best reply candidates are $\frac{u_i}{2}$, which lie outside the corresponding regions. The best reply candidate in each such regions is thus defined by the strategy lying at the frontier with P_3, which is itself dominated by $\phi_i(p_j)$. Thus, whenever $u_3 - u_2 < \text{Min}\{\eta, \eta'\}$, no profitable deviations from $(\overset{\circ}{p}_1, \overset{\circ}{p}_2)$ exist towards regions P_2 and P_4. It then remains to exclude the possibility of profitable deviations from $\overset{\circ}{p}_1$ for firm 1 which would lead the pair of prices in P_1. Such advantageous deviations are excluded if, and only if, the inequality

$$R_1(\overset{\circ}{p}_1, \overset{\circ}{p}_2) \geqq R_1(\psi_1(\overset{\circ}{p}_2), \overset{\circ}{p}_2)$$

holds, with $\psi_1(\overset{\circ}{p}_2)$ denoting the best response of firm 1 to $\overset{\circ}{p}_2$ in the standard model of vertical differentiation (remind that the demand function of firm 1 is defined in P_1 as in this model). An additional computation shows that the above inequality holds if, and only if, $u_3 > u_2 + \delta^*$, with δ^* as defined in footnote 5; furthermore, it is easily checked that $0 < \delta^* < \varepsilon^*$.[4] A direct comparison between the numbers δ^* and $\eta = \min\{\eta, \eta'\}$ shows that $\delta^* < \eta$. Consequently, when the difference $u_3 - u_2$ starts to be larger than the number δ^*, the pair of prices $(\overset{\circ}{p}_1, \overset{\circ}{p}_2)$ is indeed a price equilibrium, namely the joint purchase equilibrium, in the interval of u_3-values $[u_2 + \delta^*, u_2 + \eta]$. This interval includes the non-degenerate interval $[u_2 + \delta^*, u_2 + \varepsilon^*]$ in which the pair of prices (p_1^*, p_2^*) defined by (5) has been already shown to be a price equilibrium (see Proposition 1). This completes the proof of Proposition 2. Q.E.D

Proposition 1 indicates that, when joint purchase only adds little utility to the utility of the high quality variant, the traditional model and the joint purchase price game give the same outcome to price competition. Yet, Proposition 2 shows that this is no longer true when the increase in utility corresponding to joint purchase becomes more significant. Even if the exclusive purchase equilibrium (p_1^*, p_2^*) still belongs to the set of equilibria, another price equilibrium, - the joint purchase equilibrium $(\overset{\circ}{p}_1, \overset{\circ}{p}_2)$ -, starts to coexist. However, this pair of prices does not remain an equilibrium for *all* values of u_3 in the admissible range $]u_2, u_1 + u_2[$: we know from the above proof that, when $u_3 - u_2 > \eta$, firm 2 has an advantageous deviation from $\overset{\circ}{p}_2$ by letting the pair of prices to enter into P_4. Accordingly, for values of u_3 exceeding $u_2 + \eta$, neither (p_1^*, p_2^*), nor $(\overset{\circ}{p}_1, \overset{\circ}{p}_2)$ are still price equilibria. That no other price equilibrium exists in that case follows from the following reasoning. First, due to the concavity of the revenue functions of both firms when restricted to the sub-domains P_1 and P_3, it is clear that no pair of prices differing from (p_1^*, p_2^*) or $(\overset{\circ}{p}_1, \overset{\circ}{p}_2)$ could be a price equilibrium in these sub-domains. Furthermore, given the definition (10) of demands in the sub-domain P_4, any candidate equilibrium in this sub-domain is excluded by the fact that the best response $\frac{u_1}{2}$ of firm 1 lies outside

[4] The explicit value of δ^* obtains as

$$\delta^* = \frac{-2(u_1^3 - 8u_1^2 u_2 + 7u_1 u_2^2 - 3\sqrt{u_1^2 u_2 (9u_2 - u_1)(u_2 - u_1)})}{32u_2^2 - 4u_1 u_2 + u_1^2}$$

the projection of the sub-domain P_4 on the p_1-axis. A similar argument reveals that no candidate equilibrium could exist either in P_2. Since a direct comparison between $u_2 + \eta$ and $u_1 + u_2$ shows that $u_2 + \eta < u_1 + u_2$, we obtain the following

Proposition 3. *In the non-degenerate interval* $]u_2 + \eta, u_1 + u_2[$ *of* u_3*-values, there exists no price equilibrium in pure strategies.*

It is important to notice that Proposition 3 only precludes the existence of *pure-strategy* price equilibria in the relevant domain, but does not do it for *mixed* strategies. In fact, since the revenue functions of both firms are continuous, we know that non-degenerate mixed-strategy price equilibria must exist whenever $u_3 \in \,]u_2 + \eta, u_1 + u_2[$. Furthermore, since these revenue functions are piecewise concave, one should expect that equilibrium mixed strategies must have a finite support in prices. Exploiting this property, we have been able to identify mixed-strategy equilibria in the relevant range. Without entering into details,[5] let us simply notice that these mixed-strategy equilibria consist, for one firm, in playing a pure strategy and, for the other, in playing with some probability a "low" price and, with the complementary probability, a "high" price. Furthermore, the closer u_3 to $u_1 + u_2$, the higher the probability assigned to the "high" price option, and the closer this option to the pure monopoly price. This is interesting because, at the only price equilibrium corresponding to the limiting case $u_3 = u_1 + u_2$, both firms set their monopoly price $\frac{u_i}{2}$ (for a formal proof, see Gabszewicz, Sonnac and Wauthy, 2000). In other words, the sequence of mixed-strategy equilibria which we have identified, converges to the pair $(\frac{u_1}{2}, \frac{u_2}{2})$ of monopoly prices when $u_3 \to u_1 + u_2$.

4 Final remarks

Considering the above analysis, it seems fair to say that introducing the joint purchase option considerably enriches the nature of price competition between firms, compared with the standard model of vertical product differentiation. The joint purchase option alters price competition in two different ways. First, it may induce the low quality firm to sell one unit to the "poors" (low θs) plus one unit to the "richs", who then end up buying two units at the market equilibrium. As compared to the standard model, this implies lower prices in equilibrium and may lead to multiple equilibria. Straightforward computations indicate that the high quality firm's profits are lower in the joint purchase equilibrium than in the exclusive purchase one. In this sense, the joint purchase option enhances price competition. However, if the utility gained when buying a second unit is large enough, joint purchase may be viewed as relaxing price competition. Indeed, it introduces a new strategic option to the firms: that of focusing on those "rich" consumers who are likely to buy two units, instead of fighthing for market shares. When this is the case, direct competition tends to disappear because one firm focuses on the "first purchase" market while the second focuses on the "second purchase" one. This allows firms to sustain higher price in equilibrium.

[5] The derivation of such a mixed strategy equilibrium is given in the Appendix.

The analysis has been conducted under the assumption that consumers do not buy two units of the same good. Our approach can be interpreted as a Lancasterian multi-characteristics representation of goods, in which each good is endowed with a limited set of characterisitics from the overall characteristics space. The quality of a good can then be interpreted in terms of the numbers of different characteristics it embodies. Accordingly, buying several goods is valuable only to the extent that it enlarges the number of characteristics embodied in the bundle, i.e. to the extent that it endows the consumer with more characteristics. Therefore, buying several identical goods is not an interesting option because it cannot increase the number of different characteristics present in the bundle. A natural extension of the present model would thus consist of introducing the quantity of each characteristic, in addition to the number of different characteristics as an argument in the utility function. It is difficult to speculate about the consequences on the equilibrium pattern of introducing this further option for the consumers. Clearly, this would make equilibrium analysis much more intricate by multiplying in a combinatorial way the number of regions in which different segments of demands have to be defined.

Two additional lines of research deserve to be mentioned as well. First, the implications of joint purchase on collusion possibilities should be studied. In the present note we have restricted the analysis to purely non-cooperative strategies, however the joint purchase option allows firms to implement strategies which are in fact based on the joint value of their respective products. There is thus more scope for cooperation between them. This is in particular the case in the region where mixed strategy equilibria prevail. Accordingly, we may expect that firms will be able to sustain more collusive outcomes than under standard collusive outcomes.

Second, a natural next step to follow would consist in studying the implications of joint purchase on *quality selection* by firms. Preliminary work for the case of a monopoly (Gabszewicz and Wauthy, 2002) suggests that the implications of multiple purchases on quality choices are likely to be non-trivial.

Appendix: A mixed strategy equilibrium

In this appendix, we characterize an equilibrium in mixed strategies in which firm 2 randomizes while firm 1 plays a pure strategy. The alternative equilibrium where firm 1 randomizes can be derived using the same methodology.

Recall first that each firm's payoff is concave, region by region. Therefore, against a pure strategy of firm 1, we may identify a unique best reply in each region of the price space. Let us then consider firm 2's best reply candidate against a "low" p_1 (i.e. region P_1 irrelevant).

As already argued in Section 3, firm 2's best reply candidate in region P_2 is defined as the frontier between between region P_2 and P_3. By continuity of firm 2's payoffs, this best reply candidate must be dominated by the best reply candidate in region P_3, which is given by $\phi_2(p_1)$ as defined by (13). Using (10b), it is immediate to derive firm's candidate best reply in region P_4 as $p_2 = \frac{u_3-u_1}{2}$.

In order to identify which of $\phi_2(.)$ or $\frac{u_3-u_1}{2}$ is the "true" best reply against p_1 we only need to compare firm 2's payoffs in the two cases and identify the critical level of p_1 which makes firm 2 indifferent between the two strategies. Solving $\frac{u_3-u_1}{4} = \phi_2(p_1)(1 - \frac{\phi_2(p_1)-p_1}{u_2-u_1})$, for p_1, we obtain the critical value $\hat{p_1}$.

A candidate equilibrium may therefore be identified as follows: firm 2 randomizes over $\phi_2(\hat{p_1})$ and $\frac{u_3-u_1}{2}$ with probability $(\mu, 1-\mu)$ while firm 1 plays the pure strategy $\hat{p_1}$. In order to prove that this is indeed an equilibrium, we only need to show that there exists a μ such that $\hat{p_1}$ is a best reply for firm 1 against firm 2's mixed strategy $(\mu, 1-\mu)$.

The profit function of firm 1 against firm 2's mixed strategy defines as

$$\pi_1(p_1, p_2, \mu) = p_1[\mu(1 - \frac{p_1}{u_1}) + (1-\mu)(1 + \frac{p_2}{u_2 - u_1} - p_1 K)].$$

In order for $\hat{p_1}$ to be part of an equilibrium, it must be true that the first order condition for the above function is satisfied at $(\hat{p_1}, \phi_2(p_1))$, i.e.

$$\mu(1 - \frac{2\hat{p_1}}{u_1}) + (1-\mu)(1 + \frac{\phi_2(\hat{p_1})}{u_2 - u_1} - \hat{p_1} K) = 0.$$

The first term is positive while the second is negative, so that there must exist some μ^* which satisfies the previous equation. Straightforward computations yield

$$\mu^* = \frac{1 + \frac{\phi_2(\hat{p_1})}{u_2-u_1} - 2\hat{p_1}K}{1 + \frac{\phi_2(\hat{p_1})}{u_2-u_1} - 2\hat{p_1}K - (1 - \frac{2\hat{p_1}}{u_1})}$$

Additional computations show that $\mu \geq 0$ whenever the pure strategy equilibrium candidate defined in region P_3 ceases to exist whereas it is less than 1 whenever $\hat{p_1} < p_1^m$.

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