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# Homophily and Segregation in Social Networks when Individuals are Limitedly Forward-Looking

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*To Chaoran, Weizhen, and Qinhui*

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# Contents

# Chapter 1

# Introduction

Network formation models are enjoying growing favor in studying social and economic behaviors of human beings. This is because network structures are ubiquitous in our everyday life such as market sharing agreements in international trade, information diffusion in the labor market, and the emergence of power in the middle ages of Florence. It is also because it has its own merits compared with other competing models, such as cooperative games and non-cooperative games. For instance, how the Medici family rose to prominence in the presence of other equally powerful competing families? Cooperative game theory has little to say about this phenomenon. Indeed, important cooperative solutions such as the Shapley value, the nucleolus satisfy equal treatment of equals. On the other hand, if we model the bonding behavior across families such as marriage, trade, and making loans as a non-cooperative game, then a superfluous coordination problem arises: there exists a Nash equilibrium where no bonding behavior exists. It is the aim of this thesis to make a contribution to formulating reasonable solution concepts and understanding real-world issues in the framework of social and economic networks.

## 1.1 Motivations

Since ancient times, it is well-known that a deviating player may face a further deviation as a consequence of the deviation he or she initiated.[1] But the solution concepts incorporating this idea are lacking in the context of network formation. We explore this issue with two distinct approaches in Chapters 2 and 3. Then in Chapter 4, we apply our new solution concepts to a version of de Marti and Zenou's (2017) friendship network model augmented by a novel type of heterogeneity in players'

[1] For example, according to Woodman (2016), Caesar's famous last words "Et tu, Brute" should be interpreted as a reminder to Brutus that one day, Brutus will face his own betrayal just like Brutus's betrayal to Caesar.

rationality. We explicitly introduce the motivation of each chapter as follows:

The concept of stability in the sense of traditional economics refers to a stationary state in which any player who considers initiating a deviation would be deterred by an immediate loss of utility. However, this stability notion is controversial: as Harsanyi (1974) points out, restricting a player's counterfactual reasoning to the immediate consequence of a hypothetical deviation is presumptuous. Since a deviation may intrigue a chain of reactions, a farsightedly rational player should concern himself or herself with the ensuing effects before initiating a deviation. A technical issue may arise since this chain of reactions may form a closed cycle, thus there is no ending state for a player to decide whether this deviation brings a gain or a loss. In chapter 2, we circumvent this issue by formulating an analogue of von Neumann-Morgenstern stable set[2] by redefining a network " dominates" another network if there exists an improving path from the later to the former. The merit of this solution concept is that it naturally allows us to incorporate both types of rational players which are farsighted and myopic players in one single solution concept: the myopic-farsighted stable set.

It should be noted that there exists another alternative way to circumvent the issue that a deviation may intrigue a never-ending cycle. Bernheim et al. (1987) require that a further deviation from a deviation should only occur from within. The first merit of this requirement is that it imposes the same criterion of stability on the grand network as on the deviation which yields an appealing recursive structure in the definition. The second merit is that in contrast to the stable set, which is a set-valued solution concept, their solution concept is a point-value solution. Under this methodology, we introduce a new solution concept in chapter 3 by modeling a network structure: a coalition-proof stable network.

Except for the heterogeneous population, the second-dimensional heterogeneity in communities also intrigues our interest. We are seeking what will happen in a network where the players are not only identified with different rationality but also with a heterogeneously observable characteristic (e.g. ethnicity). In chapter 4, we adopt the two-community friendship model of de Marti and Zenou (2017) in which the cost of a link depends on the type of involved players. The intra-connection cost in one community is usually lower than the inter-connection cost, and the inter-connection cost is endogenous and diminishes with the rate of exposure of each of them to the other community. We are mainly addressing the following two questions in this paper: (i) Does farsightedness help to avoid ending up in segregation and

[2]A set of payoff vectors is a von Neumann-Morgenstern stable set if there doesn't exist a payoff vector dominates another in the set and each payoff vector outside the set is dominated by some payoff vector in the set.

alleviate the tension between efficiency and stability in a two-community friendship network? (ii) What are the stable networks for different levels of intra-community and inter-community cost?

## 1.2 Foundations

In chapter 2, we study the stability and efficiency of networks by considering the mixture with myopic and farsighted players. In Chapter 3, we assume group deviation is allowed and discuss the coalition-proof stability with credible group deviation. In Chapter 4, based on the main results of Chapter 2, we expand de Marti and Zenou's (2017) model to address segregation problems. We explicitly introduce the foundation of each chapter as follows:

### 1.2.1 Myopic and farsighted players

In chapter 2, we discuss a situation of the heterogeneous population consisting of myopic and farsighted players. A myopic player in a stable state chooses not to deviate because he or she reasons that if a hypothetical situation that he or she initiates a deviation, his or her utility will decrease immediately. By contrast, a farsighted player doesn't concern himself or herself with the immediate consequence of a deviation, instead, they can anticipate other players' reaction to their changes, so they may not add or cut a link that appears valuable to them if this can induce the formation and deletion of other links which ultimately lowering their payoffs.

### 1.2.2 Group deviation and credible group deviation

We not only discuss the pairwise deviation in the dissertation but also consider the group deviation to analyze the stability of networks. In the non-cooperative game, strong Nash Equilibrium allows players to communicate freely without a binding agreement, which contributes to selecting a Pareto-dominant NE equilibrium. In modeling network formation, a coalition $S$ is said to have a group deviation if all the members in $S$ are strictly better off by cutting or forming links. It requires that the players who add links between them should be all in $S$, and there must be at least one player belonging to $S$ for the deletion of any link. For environments in which players can communicate freely, it's natural to assume that any meaningful agreement to deviate must also be self-enforcing. So it's quite intuitive to consider credible group deviation which satisfies internal consistency since the stability requirement on a group deviation should be the same as the requirement on the whole network.

### 1.2.3 Diversity of communities

In the last chapter of my dissertation, we consider de Marti and Zenou's (2017) model of friendship networks where individuals belong to different communities. The communication cost for players from the same community and different communities are different. The inter-connection cost is endogenous which is based on the friendship composition of the two players. The intra-connection cost is always lower than the inter-connection cost, while the inter-connection cost diminishes with the rate of exposure of each of them to the other community. Knowing how social networks involving different communities are likely to be formed can help the policymaker to establish future policies in eliminating segregation and improving social welfare.

## 1.3 Contributions

In this section, we discuss our contributions to each chapter in the presence of related results in the literature.

### 1.3.1 Chapter 2

The chapter 2 "Network formation with myopic and farsighted players", joint with Ana Mauleon and Vincent Vannetelbosch, mainly studies the stability of networks in a heterogeneous population. We introduce the notion of myopic-farsighted stable set, which is an analogue of the von Neumann-Morgenstern stable set in the network formation model. In contrast to the von Neumann-Morgenstern stable set (Lucas, 1969), we show the existence of the myopic-farsighted stable set when all players are myopic. Furthermore, we provide conditions on the utility function that guarantee the existence of a myopic-farsighted stable set. In this paper, we discuss two cases: distance-based utility function and degree-based utility function, which respectively, exhibit positive externality and negative externality. In distance-based utility function, we show that mixture with myopic and farsighted players could help to eliminate the tension between stability and efficiency. So turning myopic players into farsighted is beneficial for social welfare to some extent. But there is a threshold of the number of farsighted players, once there are enough farsighted players in the population, the network becomes efficient and there is no need for turning more myopic players into farsighted ones. The degree-based utility function appears a segregated situation with myopic players being over-connected and farsighted players holding the social-optimal links. In this case, turning the myopic player into farsighted could help to improve social welfare.

### 1.3.2 Chapter 3

The chapter 3 "Coalition-proof stable network", which is also joint with Ana Mauleon and Vincent Vannetelbosch, studies the consequences of allowing group deviation instead of pairwise deviation only. Instead of excluding all possible group deviations, we consider the group deviation which satisfies internal consistency that doesn't have further sub-group deviation. We compare the strong stability/coalition-proof stability in network formation models with the networks resulting from strong Nash equilibrium/coalition-proof Nash equilibrium in Myerson's linking game, finding that even the strongly stable networks are equivalent to the networks induced by a strong Nash equilibrium of Myerson's linking game, there is no relationship between the coalition-proof stable networks and the networks resulting from coalition-proof Nash equilibrium in Myerson's linking game. In general, there's no guarantee for the existence of the coalition-proof stable network. But for component-wise egalitarian utility function, we show the existence of coalition-proof stable networks, and the coalition-proof stable networks are equivalent to the strongly stable networks, which are both efficient. Also, under the component-wise egalitarian utility function, the coalition-proof stability with farsighted players but restricted to pairwise deviation is equivalent to the coalition-proof stability with group deviations.

### 1.3.3 Chapter 4

The chapter 4 "segregation versus assimilation in friendship networks with farsighted and myopic agents" is a following-up of the chapter 2, by considering the myopic-farsighted stable set in the de Marti and Zenou's (2017) two-community friendship networks. It's also co-authored with Ana Mauleon and Vincent Vannetelbosch. Contrast with the results in de Marti and Zenou (2017), we find that the network in which the small community fully assimilated to the large community is always more efficient than the completely segregated network. Under intermediate intra-community cost with a heterogeneous population, the set of networks that consists of all star networks with a myopic player being the center is the unique myopic-farsighted stable set. Moreover, we show that in a two-community network with a heterogeneous population, the most inefficient networks have been destabilized. Farsightedness helps to alleviate the tension between efficiency and stability in friendship networks when players belong to different communities.

# Chapter 2

# Network Formation with Myopic and Farsighted Players [1]

Joint work with Ana Mauleon and Vincent Vannetelbosch

**Abstract**

We adopt the notion of myopic-farsighted stable set to study the stability of networks when myopic and farsighted individuals decide with whom they want to form a link, according to some utility function that weighs the costs and benefits of each connection. A myopic-farsighted stable set is the set of networks satisfying internal and external stability with respect to the notion of myopic-farsighted improving path. We first provide conditions on the utility function that guarantee the existence of a myopic-farsighted stable set and we show that, when the population becomes mixed, the myopic-farsighted stable set refines the set of pairwise stable networks by eliminating some Pareto-dominated networks. In the end, when all players are farsighted, the myopic-farsighted stable set only consists of all strongly efficient networks. We next show that, in the case of a distance-based utility function, a tension between stability and efficiency is likely to arise when the population is homogeneous (either all myopic or all farsighted). But, once the population is mixed, the tension vanishes if there are enough farsighted individuals. In the case of a degree-based utility function, myopic and farsighted individuals may end up segregated with myopic individuals being overconnected and farsighted ones getting the socially optimal payoff.

Keywords: networks; stable sets; myopic and farsighted players; egalitarian utility; positive convex externalities; distance-based utility; degree-based utility.

JEL Classification: A14, C70, D20.

[1] Luo, C., Mauleon, A. & Vannetelbosch, V. Network formation with myopic and farsighted players. Econ Theory (2020). *Economic Theory*, https://doi.org/10.1007/s00199-020-01288-8

## 2.1 Introduction

The organization of individuals into networks plays an important role in the determination of the outcome of many social and economic interactions. For instance, a communication or friendship network in which individuals have very few acquaintances with whom they share information will result in different employment patterns than one in which individuals have many such acquaintances. A central question is predicting the networks that individuals will form. Up to now, it has been assumed that all individuals are either myopic or farsighted when they decide with whom they want to link. Jackson and Wolinsky (1996) propose the notion of pairwise stability to predict the networks that one might expect to emerge in the long run. A network is pairwise stable if no individual benefits from deleting a link and no two individuals benefit from adding a link between them. Pairwise stability presumes that individuals are myopic: they do not anticipate that other individuals may react to their changes. Farsighted individuals may not add a link that appears valuable to them as this can induce the formation of other links, ultimately lowering their payoffs.[2]

However, recent experiments provide evidence in favor of a mixed population consisting of both myopic and farsighted individuals. Kirchsteiger, Mantovani, Mauleon, and Vannetelbosch (2016) test the myopic and the farsighted models of network formation, and compare the stability notions that are based on them. They find that most subjects are best classified as myopic but many others are limitedly farsighted.[3] So, the outcomes of real-life network formation problems are likely to be affected by the degree of farsightedness of the individuals. Consider the situation where the worth of link creation turns nonnegative after some threshold in the connectedness of the network is reached, both for the individuals and on aggregate, but the individual benefits are negative below this threshold. If network externalities take this form, myopic individuals can be stuck in insufficiently dense networks. Farsightedness may take care of this problem and achieve efficiency. In the presence of both myopic and farsighted individuals, their ability to pass the threshold will depend on the number of farsighted individuals. Only if there are enough farsighted individuals that, by linking among them, could pass the threshold, the myopic individuals would also start forming links achieving the efficient network.

Moreover, it is important to understand what happens when myopic players in-

[2] Mauleon and Vannetelbosch (2016) provide a comprehensive overview of the (myopic and farsighted) solution concepts for solving network formation games.

[3] Teteryatnikova and Tremewan (2020) compare the predictive power of myopic and farsighted stability concepts in a network formation experiment with a stream of payoffs. Their results show that there exist environments where farsighted stability concepts identify empirically stable networks that are not identified by myopic stability concepts.

teract with farsighted players since, in general, some networks that are nor stable when all players are myopic nor stable when all players are farsighted could now emerge in the long run. Is turning myopic players into farsighted players beneficial for the society? Could it be that a heterogeneous society does better than a homogeneous society in terms of efficiency? And if yes, when?

To address those questions we adopt the notion of myopic-farsighted stable set. This concept will help us to determine the networks that emerge when myopic and farsighted individuals decide with whom they want to form a link, according to some utility function that weighs the costs and benefits of each connection.[4] A myopic-farsighted stable set is the set of networks satisfying internal and external stability with respect to the notion of myopic-farsighted improving path. When all individuals are farsighted, the definition of a myopic-farsighted stable set boils down to the farsighted stable set.[5]

We first provide general results that are useful for characterizing the myopic-farsighted stable set in applications. If a network is optimal for the farsighted players and pairwise stable for the myopic players, then it belongs to any myopic-farsighted stable set. A set of networks is the unique myopic-farsighted stable set if there is no myopic-farsighted improving path from any network within the set, and from any network outside the set there is a myopic-farsighted improving path leading to some network within the set. We next provide conditions on the utility function that guarantee the existence and uniqueness of a myopic-farsighted stable set. We find that, under the egalitarian utility function or in the presence of positive convex externalities or in the case of no externality, the unique myopic-farsighted stable set consists of all pairwise stable networks when all players are myopic. When the population is composed of myopic and farsighted players, the myopic-farsighted stable set refines the set of pairwise stable networks by eliminating some Pareto-dominated networks. In the end, when all players are farsighted, the unique myopic-farsighted stable set only consists of all strongly efficient networks. Hence, under the egalitarian utility function or in the presence of positive convex externalities or in the case of no externality, turning myopic players into farsighted players alleviates the tension between stability and efficiency. In addition, myopic players can only be

[4]Herings, Mauleon, and Vannetelbosch (2017b) define first the myopic-farsighted stable set for two-sided matching problems, and Mauleon, Sempere-Monerris, and Vannetelbosch (2018) extend it to R&D network formation with pairwise deviations.

[5]See Chwe (1994), Herings, Mauleon, and Vannetelbosch (2009), Mauleon, Vannetelbosch, and Vergote (2011), Ray and Vohra (2015, 2019), Roketskiy (2018) for definitions of the farsighted stable set. Alternative notions of farsightedness are suggested by Diamantoudi and Xue (2003), Dutta, Ghosal, and Ray (2005), Dutta and Vohra (2017), Herings, Mauleon, and Vannetelbosch (2004, 2019), Page, Wooders, and Kamat (2005), Page and Wooders (2009), Xue (1998) among others.

better off by becoming farsighted since the least preferred pairwise stable networks are progressively discarded.

We then analyze two specific utility functions: distance-based utility function (where the formation of a link exerts positive externalities) and degree-based utility function (where the formation of a link exerts negative externalities).

First, we reconsider Bloch and Jackson (2007) model of network formation where individuals decide with whom they want to form a link, according to a distance-based utility function that weighs the costs and benefits of each connection. Benefits of a connection decrease with distance in the network, while the cost of a link represents the time an individual must spend with another individual for maintaining a direct link. Adding a link requires the consent of both individuals, while deleting a link can be done unilaterally. We now allow the population of individuals to include not only myopic individuals but also farsighted ones. Farsighted individuals are able to anticipate that once they add or delete some links, other individuals could add or delete links afterwards.

We focus on the range of costs and benefits such that a star network is the unique strongly efficient network.[6] When all individuals are myopic, Jackson (2008) shows that a conflict between stability and efficiency is likely to occur. In addition, starting from the empty network, a random process where pairs of players meet to add or to delete links becomes unlikely to reach a star network as the number of players increases (see Watts, 2001; Jackson, 2008). When the population consists of both myopic and farsighted individuals, we show that the conflict between stability and efficiency vanishes if there are enough farsighted individuals. Indeed, the set consisting of all star networks where the center of the star is a myopic individual is the unique myopic-farsighted stable set. However, once all individuals become farsighted, every set consisting of a star network encompassing all players is a myopic-farsighted stable set, but there may be other myopic-farsighted stable sets. For instance, the set of circles among four farsighted players can be a myopic-farsighted stable set.

One can then conclude that diversity guarantees the emergence in the long run of the efficient outcomes. When all individuals are myopic or all individuals are farsighted, a tension between stability and efficiency can occur. However, if the population is mixed, then this tension disappears. Farsighted individuals try to avoid ending up in the central position of the star, and so, if all of them are farsighted, this can lead to a worse inefficient outcome. But, if some individuals are myopic,

[6] In cases of intermediate link costs relative to benefits, individuals obtain their highest possible payoff when they are the peripherals in a star network. The center of the star is worse off compared to the peripherals.

farsighted individuals are able to place myopic individuals in positions where they have myopic incentives to move towards some star network where one of the myopic individuals ends up being the center of the star. However, if there are too many myopic individuals with respect to farsighted ones, farsighted individuals may fail to engage a path from some inefficient network towards a star network.[7]

Second, we reconsider Morrill (2011) model of network formation where the individual's utility from a link is a decreasing function of the number of links the other individuals maintain. Benefits of a link now decrease with the degree of the neighbors while costs of a link still represent the time an individual must spend with another individual for maintaining a link. Degree-based utility functions exhibit negative externalities. In general, there is a conflict between stability and efficiency. Morrill (2011) shows that when individuals are all myopic and are able to make transfers to their neighbors, then stable networks coincide with strongly efficient ones. When the population is mixed (and without transfers), we show that myopic and farsighted individuals may end up segregated with myopic individuals being overconnected and farsighted ones getting the socially optimal payoff. The more farsighted individuals in the population are, the less likely inefficient networks will emerge. In the limit, when all individuals are farsighted, the set of all strongly efficient networks is stable without the use of any transfers.

Finally, we study how networks evolve when myopic players may become farsighted over time. Players are initially unconnected to each other. Over time, pairs of players decide whether or not to form or cut links with each other. A link can be cut unilaterally but agreement by both players is needed to form a link. All players are initially myopic, and thus decide to form or cut links if doing so increases their current payoffs. The length of a period is sufficiently long so that the process can converge to some stable network. At the beginning of each period after the initial period, some myopic players become farsighted. The likelihood of becoming farsighted may be related to some endogenous factors (e.g. number of farsighted players in the neighborhood, average payoff of the neighbors, ...) or some exogenous factors (e.g. a policy for improving individuals' cognitive ability, ...). Depending on their positions in the network, the process either stays at the same network or evolves to another stable network. For instance, in the distance-based utility model, the dynamic pro-

[7] Another strand of the literature that was initiated by Bala and Goyal (2000) studies the formation of two-way flow networks where individuals unilaterally form costly links in order to access the benefits generated by other individuals. Benefits flow in both directions, irrespective of who pays the cost of the link. In Galeotti, Goyal, and Kamphorst (2006), individuals are heterogeneous with respect to benefits and costs of forming links. In Bloch and Dutta (2009), individuals choose how much to invest in each link. See also Hojman and Szeidl (2008) and Feri (2007) among others.

cess first converges to some pairwise stable network. Once the number of myopic players who have become farsighted is large enough, the dynamic process evolves to a star network with some myopic player in the center. Such star network will be dismantled once the myopic player in the center of the star becomes farsighted. In this case, the process evolves next to another star network with one of the remaining myopic player in the center.

The paper is organized as follows. In Section 2 we introduce networks, myopic-farsighted improving paths, myopic-farsighted stable sets, and we provide general results for characterizing a myopic-farsighted stable set of networks. In Section 3 we consider distance-based utility functions and we characterize the myopic-farsighted stable sets when the population consists of a mixture of myopic and farsighted individuals. In Section 4 we consider degree-based utility functions. In Section 5 we study the evolution and the dynamics of networks and we discuss the robustness of our results with respect to deviations by groups and limited farsightedness. In Section 6 we conclude.

## 2.2 Network formation

### 2.2.1 Modelling networks

We study networks where players form links with each other in order to exchange information. The population consists of both myopic and farsighted players. The set of players is denoted by $N = M \cup F$, where $M$ is the set of myopic players and $F$ is the set of farsighted players. Let $n$ be the total number of players and $m \geq 0$ ($n - m \geq 0$) be the number of myopic (farsighted) players. A network $g$ is a list of pairs of players who are linked to each other and $ij \in g$ indicates that $i$ and $j$ are linked under $g$. The complete network on the set of players $S \subseteq N$ is denoted by $g^S$ and is equal to the set of all subsets of $S$ of size 2.[8] It follows in particular that the empty network is denoted by $g^{\emptyset}$. The set of all possible networks on $N$ is denoted by $\mathcal{G}$ and consists of all subsets of $g^N$. The network obtained by adding link $ij$ to an existing network $g$ is denoted $g + ij$ and the network that results from deleting link $ij$ from an existing network $g$ is denoted $g - ij$. Let $N(g) = \{i \mid$there is $j$ such that $ij \in g\}$ be the set of players who have at least one link in the network $g$. Let $N_i(g) = \{j \in N \mid ij \in g\}$ be the set of neighbors of player $i$ in $g$. The degree of player $i$ in network $g$, denoted $d_i(g)$, is the cardinality of $i$'s set of neighbors, $d_i(g) = \#N_i(g)$. A star network is a network such that there exists some player

[8]Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. Finally, $\#$ will refer to the notion of cardinality.

$i$ (the center) who is linked to every other player $j \neq i$ (the peripherals) and that contains no other links (i.e. $g$ is such that $N_i(g) = N \setminus \{i\}$ and $N_j(g) = \{i\}$ for all $j \in N \setminus \{i\}$). A $d$-regular network is a network where all players have the same degree $d$. A path in a network $g$ between $i$ and $j$ is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = i$ and $i_K = j$. A network $g$ is connected if for all $i \in N$ and $j \in N \setminus \{i\}$, there exists a path in $g$ connecting $i$ and $j$. A nonempty subnetwork $h \subseteq g$ is a component of $g$, if for all $i \in N(h)$ and $j \in N(h) \setminus \{i\}$, there exists a path in $h$ connecting $i$ and $j$, and for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$. The set of components of $g$ is denoted by $H(g)$.

A network utility function (or payoff function) is a mapping $U_i : \mathcal{G} \to \mathbb{R}$ that assigns to each network $g$ a utility $U_i(g)$ for each player $i \in N$. A network $g \in \mathcal{G}$ is strongly efficient if $\sum_{i \in N} U_i(g) \geq \sum_{i \in N} U_i(g')$ for all $g' \in \mathcal{G}$. Let $E$ be the set of strongly efficient networks. Jackson and Wolinsky (1996) propose the notion of pairwise stability to analyze the networks that one might expect to emerge in the long run when all players are myopic. A network $g \in \mathcal{G}$ is pairwise stable if (i) for all $ij \in g$, $U_i(g) \geq U_i(g - ij)$ and $U_j(g) \geq U_j(g - ij)$, (ii) for all $ij \notin g$, if $U_i(g) < U_i(g + ij)$ then $U_j(g) > U_j(g + ij)$. Let $P$ be the set of pairwise stable networks.

### 2.2.2 Myopic-farsighted improving paths and stable sets

We adopt the notion of myopic-farsighted stable set to determine the networks that are stable when some players are myopic while others are farsighted.[9] A set of networks $G$ is said to be a myopic-farsighted stable set if it satisfies the following two types of stability. Internal stability: No network in $G$ is dominated by any other network in $G$. External stability: Every network not in $G$ is dominated by some network in $G$. A network $g'$ is said to be dominated by a network $g$ if there is a myopic-farsighted improving path from $g'$ to $g$. Hence, a set of networks is a myopic-farsighted stable set if (internal stability) there is no myopic-farsighted improving path between networks within the set and (external stability) there is a myopic-farsighted improving path from any network outside the set to some network within the set.

A myopic-farsighted improving path is a sequence of distinct networks that can emerge when farsighted players form or delete links based on the improvement the

[9]Stability concepts have their roots in cooperative game theory. They predict which network will emerge independently of the network formation process, but they are silent on how a network architecture is expected to emerge through the strategic decisions of the players. As such, myopia and farsightedness are not models of individual strategic behavior, because strategic behavior is cached in the stability approach.

end network offers relative to the current network while myopic players form or delete links based on the improvement the resulting network offers relative to the current network. Since we only allow for pairwise deviations, each network in the sequence differs from the previous one in that either a new link is formed between two players or an existing link is deleted. If a link is deleted, then it must be that either a myopic player prefers the resulting network to the current network or a farsighted player prefers the end network to the current network. If a link is added between some myopic player $i$ and some farsighted player $j$, then the myopic player $i$ must prefer the resulting network to the current network and the farsighted player $j$ must prefer the end network to the current network.

Along a myopic-farsighted improving path, myopic players do not care whether other players are myopic or farsighted. They behave as if all players are myopic and they compare their resulting network's payoff to their current network's payoff for taking a decision. However, farsighted players know exactly who is farsighted and who is myopic and they compare their end network's payoff to their current network's payoff for taking a decision.[10]

**Definition 2.1.** A myopic-farsighted improving path from a network $g$ to a network $g'$ is a finite sequence of distinct networks $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$ either

**(i)** $g_{k+1} = g_k - ij$ for some $ij$ such that $U_i(g_{k+1}) > U_i(g_k)$ and $i \in M$ or $U_j(g_K) > U_j(g_k)$ and $j \in F$; or

**(ii)** $g_{k+1} = g_k + ij$ for some $ij$ such that $U_i(g_{k+1}) > U_i(g_k)$ and $U_j(g_{k+1}) \geq U_j(g_k)$ if $i, j \in M$, or $U_i(g_K) > U_i(g_k)$ and $U_j(g_K) \geq U_j(g_k)$ if $i, j \in F$, or $U_i(g_{k+1}) \geq U_i(g_k)$ and $U_j(g_K) \geq U_j(g_k)$ (with one inequality holding strictly) if $i \in M, j \in F$.

If there exists a myopic-farsighted improving path from a network $g$ to a network $g'$, then we write $g \to g'$. The set of all networks that can be reached from a network $g \in \mathcal{G}$ by a myopic-farsighted improving path is denoted by $\phi(g)$, $\phi(g) = \{g' \in \mathcal{G} \mid g \to g'\}$. When all players are myopic, our notion of myopic-farsighted improving path reverts to Jackson and Watts (2002) notion of improving

[10] The distinction between myopic players and farsighted players can be linked to their cognitive ability or their patience. For instance, Dutta, Ghosal and Ray (2005) propose a dynamic approach with payoffs accruing in real time along with the network formation process. Players get a discounted stream of payoffs, and the discount factor becomes a natural proxy for the degree of farsightedness (discount factor close to 0 for myopic players and discount factor close to one for farsighted players).

path.[11] When all players are farsighted, our notion of myopic-farsighted improving path reverts to Herings, Mauleon and Vannetelbosch (2009) notion of farsighted improving path.

A set of networks $G$ is a myopic-farsighted stable set if the following two conditions hold. Internal stability: for any two networks $g$ and $g'$ in the myopic-farsighted stable set $G$ there is no myopic-farsighted improving path from $g$ to $g'$ (and vice versa). External stability: for every network $g$ outside the myopic-farsighted stable set $G$ there is a myopic-farsighted improving path leading to some network $g'$ in the myopic-farsighted stable set $G$ (i.e. there is $g' \in G$ such that $g \to g'$).

**Definition 2.2.** A set of networks $G \subseteq \mathcal{G}$ is a myopic-farsighted stable set if: **(IS)** for every $g, g' \in G$ $(g \neq g')$, it holds that $g' \notin \phi(g)$; and **(ES)** for every $g \in \mathcal{G} \setminus G$, it holds that $\phi(g) \cap G \neq \emptyset$.

When all players are farsighted, the myopic-farsighted stable set is simply the farsighted stable set as defined in Herings, Mauleon and Vannetelbosch (2009) or Ray and Vohra (2015). When all players are myopic, the myopic-farsighted stable set boils down to the pairwise CP vNM set as defined in Herings, Mauleon, and Vannetelbosch (2017a) for two-sided matching problems.[12]

**Example 2.1.** Consider a situation where four players can form links. The utilities they obtained from the different network configurations are as follows. For the empty network $g^{\emptyset}$, $U_i(g^{\emptyset}) = 8$ for all $i \in N$. For the complete network $g^N$, $U_i(g^N) = 9$ for all $i \in N$. For a line network $g^{L4}$ with four players, $U_i(g^{L4}) = 2 + 3d_i(g^{L4})$ for all $i \in N$. For a line network $g^{L3}$ with three players, $U_i(g^{L3}) = 14/d_i(g^{L3}) - (d_i(g^{L3}) + 1)^2$ for all $i \in N(g^{L3})$ and $U_j(g^{L3}) = 0$ for $j \in N \setminus N(g^{L3})$. For all other networks $g$, $U_i(g) = -d_i(g)$. Figure 3.4 gives some of the network configurations. Both the empty network and the complete network are pairwise stable networks. The complete network is also the Pareto-dominant network. When all players are farsighted, $\{g^N\}$ is the unique myopic farsighted stable set since $g^N \in \phi(g)$ for all $g \neq g^N$ and $\phi(g^N) = \emptyset$. When all players are myopic, $\{g^N, g^{\emptyset}\}$ is the unique myopic farsighted stable set since $g^{\emptyset} \in \phi(g)$ for all $g \neq g^N, g^{\emptyset}$, $\phi(g^{\emptyset}) = \emptyset$ and $\phi(g^N) = \emptyset$. Suppose now that players 1 and 3 are farsighted while players 2 and 4 are myopic. That is, $M = \{2, 4\}$ and $F = \{1, 3\}$. We still have $\phi(g^{\emptyset}) = \emptyset$ and $\phi(g^N) = \emptyset$. But now, there are no myopic-farsighted improving paths from the line networks $\{12, 13, 34\}$ and $\{14, 13, 23\}$ to the empty network since myopic players are worse off at the adjacent

[11] Mauleon, Roehl and Vannetelbosch (2018, 2019) extend Jackson and Watts notion of improving path to overlapping group structures.

[12] The pairwise CP vNM set follows the approach by Page and Wooders (2009) who define the stable set with respect to path dominance, i.e. the transitive closure of $\phi$.

networks to $\{12, 13, 34\}$ and $\{14, 13, 23\}$ and farsighted players prefer $\{12, 13, 34\}$ and $\{14, 13, 23\}$ to the empty network. Obviously, there are no myopic-farsighted improving paths from the line networks $\{12, 13, 34\}$ and $\{14, 13, 23\}$ to the complete network since along any path towards the complete network utilities are decreasing before reaching the complete network. The external stability condition implies that both $g^{\emptyset}$ and $g^N$ have to belong to any myopic-farsighted stable set. One can easily check that $\phi(g) \cap \{g^{\emptyset}, g^N\} \neq \emptyset$ for all $g \neq \{12, 13, 34\}, \{14, 13, 23\}, g^{\emptyset}, g^N$, while $\phi(\{12, 13, 34\}) = \{\{12, 13\}, \{13, 34\}\}$ and $\phi(\{14, 13, 23\}) = \{\{14, 13\}, \{13, 23\}\}$. So, if $\{12, 13, 34\}$ would not belong to a myopic-farsighted stable set, then either $\{12, 13\}$ or $\{13, 34\}$ has to be included in it. But, then the internal stability condition would be violated since $\phi(\{12, 13\}) \supseteq g^{\emptyset}$ and $\phi(\{13, 34\}) \supseteq g^{\emptyset}$. Hence, $\{g^{\emptyset}, g^N, \{12, 13, 34\}, \{14, 13, 23\}\}$ is the unique myopic farsighted stable set when $M = \{2, 4\}$ and $F = \{1, 3\}$. Thus, a mixed population can stabilize networks that are not stable when the population is homogeneous (i.e. where players are either all farsighted or all myopic).

In the external stability condition it is implicitly assumed some optimism on behalf of the players. A network $g'$ is said to be dominated by a network $g$ if there exists a myopic-farsighted improving path from $g'$ to $g$. But along the path from $g'$ to $g$ both farsighted and myopic players who are moving on the path may have a better alternative than the one prescribed by the path when they are called on to move. This is true not only for myopic players but also for farsighted players. However, Ray and Vohra (2019) show that every (farsighted) stable set satisfying some reasonable and easily verifiable properties is unaffected by the imposition of stringent maximality constraints. These constraints are satisfied by all (farsighted) stable sets consisting of networks with a single payoff.

In the definition of myopic-farsighted stable sets it is implicitly assumed that myopic players stay myopic and cannot become farsighted. However, one could argue that, a myopic player could learn and become less myopic overtime when interacting in an environment composed mainly of farsighted players. To address this issue we look in Section 2.5 at the evolution and dynamics of networks when at the beginning of each period, some myopic player become farsighted. The likelihood of becoming farsighted may be random or may depend on the network (number of farsighted players in the neighborhood, average payoff of neighbors, ...).

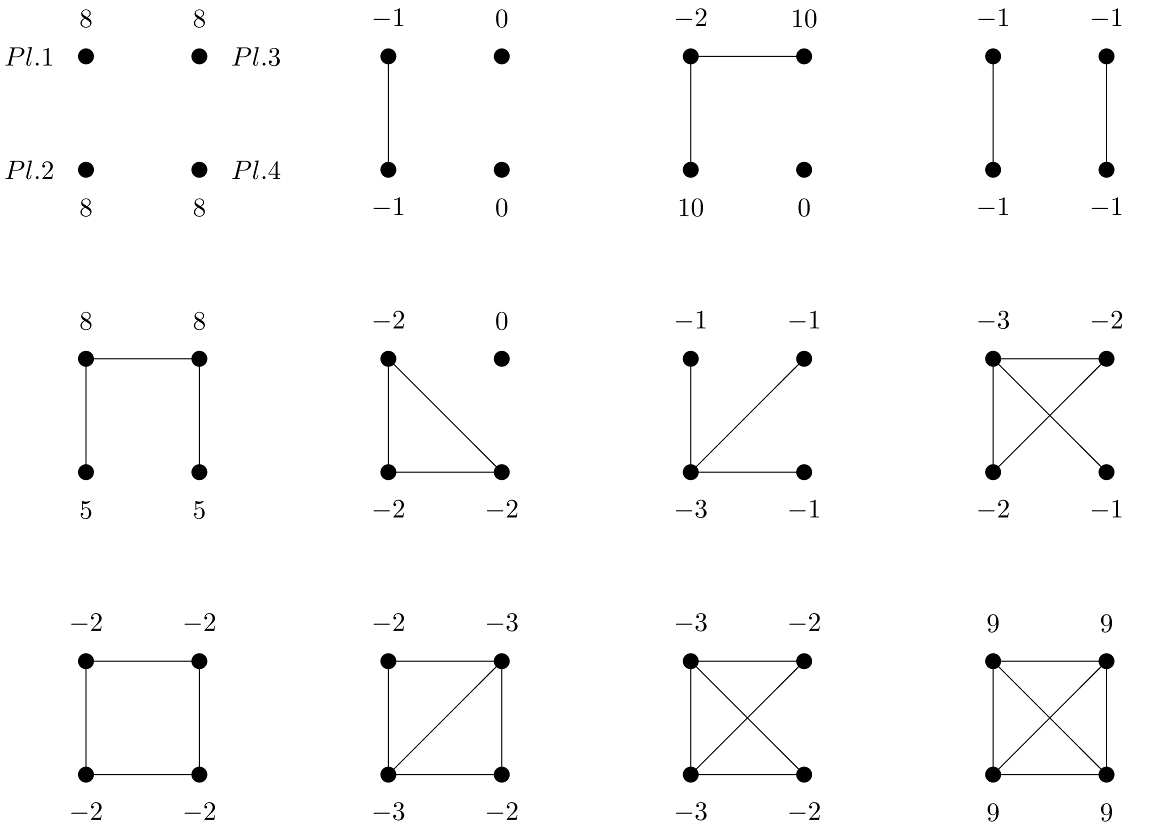


Figure 2.1: A line network with four players is stable when two players are farsighted while the other two are myopic, but is not stable when players are either all farsighted or all myopic.

### 2.2.3 General results

#### Existence and uniqueness

Let $\phi^2(g) = \phi(\phi(g)) = \{g'' \in \mathcal{G} \mid \exists g' \in \phi(g) \text{ such that } g'' \in \phi(g')\}$ be the set of networks that can be reached by a composition of two myopic-farsighted improving paths from $g$. We extend this definition and, for $r \in \mathbb{N}$, we define $\phi^r(g)$ as those networks that can be reached from $g$ by means of $r$ compositions of myopic-farsighted improving paths. The transitive closure of $\phi$ is denoted by $\phi^\infty$ and defined as $\phi^\infty(g) = \bigcup_{r \in \mathbb{N}} \phi^r(g)$. Since the set $\mathcal{G}$ is finite, it holds that, for some $r' \in \mathbb{N}$, for every $g \in \mathcal{G}$, $\phi^\infty(g) = \bigcup_{r=1}^{r'} \phi^r(g)$. We now extend Jackson and Watts (2002) notions of cycle and closed cycle to myopic and farsighted players. A set of networks $C$, forms a cycle if for any $g \in C$ and $g' \in C$ there exists a sequence of myopic-farsighted improving paths connecting $g$ to $g'$, i.e. $g' \in \phi^\infty(g)$. A cycle $C$ is a closed cycle if no network in $C$ lies on a myopic-farsighted improving path leading to a network that is not in $C$, i.e. $\phi^\infty(C) = C$.

**Proposition 2.1.** *Let $G \subseteq \mathcal{G}$ be a myopic-farsighted stable set. If $\phi(g) = \emptyset$ then $g \in G$.*

*Proof.* Take any $g$ such that $\phi(g) = \emptyset$. Then, $g$ should belong to the myopic-farsighted stable set $G$. Otherwise, $G$ would violate the external stability condition (**ES**). □

Suppose that $G \subseteq \mathcal{G}$ is a myopic-farsighted stable set. Proposition 2.1 tells us that, if there is a network such that there is no myopic-farsighted improving path leaving it, then this network belongs to any myopic-farsighted stable set. In addition, each myopic-farsighted stable set and each closed cycle have a non-empty intersection. That is, if $C^1, ..., C^R$ are the closed cycles, then $C^k \cap G \neq \emptyset$ for $k = 1, ..., R$. Indeed, if $C^k \cap G = \emptyset$, then $G$ would violate (**ES**) since for every $g \in C^k$ we have $\phi(g) \subseteq C^k$. The following result follows as a corollary of Proposition 2.1. If a network is optimal for the farsighted players and pairwise stable for the myopic players, then it belongs to any myopic-farsighted stable set.

**Corollary 2.1.** Let $G \subseteq \mathcal{G}$ be a myopic-farsighted stable set. If there is $g \in P$ such that for all $g' \in \mathcal{G} \setminus \{g\}$ it holds $U_i(g) > U_i(g')$ for all $i \in F$, then $g \in G$.

Proposition 2.2 tells us when a set of networks is the unique myopic-farsighted stable set. A set $G \subseteq \mathcal{G}$ is the unique myopic-farsighted stable set if (i) there is no myopic-farsighted improving path from any network within the set, and (ii) from any network outside the set there is a myopic-farsighted improving path leading to some network within the set.

**Proposition 2.2.** *If $G \subseteq \mathcal{G}$ is such that* (**i**) *for every $g \in \mathcal{G} \setminus G$, it holds that $\phi(g) \cap G \neq \emptyset$, and* (**ii**) *for every $g \in G$, it holds that $\phi(g) = \emptyset$, then $G$ is the unique myopic-farsighted stable set.*

*Proof.* From (**i**) the set $G$ satisfies (**ES**) and from (**ii**) the set $G$ satisfies (**IS**). Hence, $G$ is a myopic-farsighted stable set. We now show that it is the unique one. Suppose that $G' \neq G$ is a myopic-farsighted stable set. Since for every $g \in G$, it holds that $\phi(g) = \emptyset$, then $G \subseteq G'$. Otherwise, $G'$ violates (**ES**). But, if $G \subsetneq G'$ then $G'$ violates (**IS**). Hence, $G$ is the unique myopic-farsighted stable set. □

### Characterization when all players are myopic

Suppose now that all players are myopic, i.e. $F = \emptyset$. Lemma 1 in Jackson and Watts (2002) shows that there always exists at least one pairwise stable network or closed cycle of networks. Starting from any network, either it is pairwise stable (and no improving path leaves it) or it lies on an improving path to another network. Either the network reached is pairwise stable or the improving path can be continued forever and ends up running into a closed cycle. Using Lemma 1 of Jackson and Watts (2002) we provide a characterization of the (myopic-farsighted) stable set when all players are myopic. A set of networks is a (myopic-farsighted) stable set if and only if it consists of all pairwise stable networks and one network from each closed cycle.[13]

**Proposition 2.3.** *Suppose that all players are myopic, $F = \emptyset$. Let $C^1, ..., C^R$ be the set of closed cycles. A set of networks $G \subseteq \mathcal{G}$ is a myopic-farsighted stable set if and only if $G = P \cup \{g^1, ..., g^R\}$ with $g^k \in C^k$ for $k = 1, ..., R$.*

*Proof.* We first show that any $G = P \cup \{g^1, ..., g^R\}$ with $g^k \in C^k$ for $k = 1, ..., R$ satisfies (**IS**) and (**ES**). Since all players are myopic, the set $G$ satisfies (**IS**) by definition of a pairwise stable network and of a closed cycle; i.e. for every $g, g' \in G$ we have that $g \notin \phi(g')$. From Lemma 1 in Jackson and Watts (2002) we have that, for every $g \notin G$, $\phi(g) \cap G \neq \emptyset$, and so $G$ satisfies (**ES**).

Suppose now that $G$ is a (myopic-farsighted) stable set. First, $P \subseteq G$, otherwise, $G$ would violate (**ES**). Second, $C^k \cap G \neq \emptyset$ for $k = 1, ..., R$, otherwise, $G$ would violate (**ES**). Third, take any $G, G'$ such that $G \supsetneq G' = P \cup \{g^1, ..., g^R\}$ with $g^k \in C^k$ for $k = 1, ..., R$. Then, from Lemma 1 in Jackson and Watts (2002) we have that there is $g, g' \in G$ such that $g \in \phi(g')$ and $G$ violates (**IS**). □

Since there always exists at least one pairwise stable network or closed cycle of networks (Jackson and Watts, 2002), the existence of a myopic-farsighted stable set is guaranteed when all players are myopic.[14] When all players are myopic (i.e. $N = M$), if $(g, ..., g')$ and $(g', ..., g'')$ are myopic-farsighted improving paths, then $(g, .., g', .., g'')$ is also a myopic-farsighted improving path. However, when some players are farsighted ($F \neq \emptyset$), if $(g, ..., g')$ and $(g', ..., g'')$ are myopic-farsighted improving paths, then $(g, .., g', .., g'')$ may not be a myopic-farsighted improving path

[13] Demuynck, Herings, Saulle and Seel (2019) propose the myopically stable set for social environments, which generalizes the pairwise myopically stable set introduced by Herings, Mauleon, and Vannetelbosch (2009) for generic network problems. Theorem 1 of Herings, Mauleon, and Vannetelbosch (2009) shows that the pairwise myopically stable set is the union of all pairwise stable networks and closed cycles.

[14] van Deemen (1991) introduces the generalized stable set for abstract systems and shows its existence. Page and Wooders (2009) define it for abstract network formation game with the path dominance relation. It is not hard to see that the generalized stable set for abstract systems due to van Deemen (1991) coincides with the myopic-farsighted stable set for generic network problems when all players are myopic. For such network problems with myopic players, Theorem 2 in van Deemen (1991) and Theorem 3 in Page and Wooders (2009) are therefore equivalent to Proposition 2.3 that characterizes the myopic-farsighted stable sets as all pairwise stable networks and one network from each closed cycle.

since some farsighted players who move along the first myopic-farsighted improving path $(g, ..., g')$ may now decide not to move once they look forward towards the end network $g''$ of the second myopic-farsighted improving path. This is why it is much harder to analyze situations involving farsighted players.

**Existence and characterization under the egalitarian utility function**

Suppose that the utility function $U$ is such that, for any given network, all players get the same payoff: $U_i(g) = U_j(g)$ for all $i, j \in N$. With the egalitarian utility function $U$, each player's payoff depends on the network but not on the specific role she plays within the network. Proposition 2.4 shows that there is a unique myopic-farsighted stable set under the egalitarian utility function. Let $G^{\emptyset} = \{g \in \mathcal{G} \mid \phi(g) = \emptyset\}$ be the set of networks such that there are no myopic-farsighted improving paths emanating from them.

**Proposition 2.4.** *Take any $U$ such that $U_i(g) = U_j(g)$ for all $i, j \in N$. The set $G^{\emptyset} = \{g \in \mathcal{G} \mid \phi(g) = \emptyset\}$ is the unique myopic-farsighted stable set.*

*Proof.* First, we show that for every $g \in \mathcal{G} \setminus G^{\emptyset}$, it holds that $\phi(g) \cap G \neq \emptyset$. Given the egalitarian utility function $U$, we have that, for $0 \leq m \leq n$, there are no closed cycles: for all $g \in \mathcal{G}$ we have $g \notin \phi^{\infty}(g)$. Thus, all sequences of myopic-farsighted improving paths starting from any $g$ such that $\phi(g) \neq \emptyset$ will reach after a finite number of myopic-farsighted improving paths some network $g'$ such that $\phi(g') = \emptyset$. In addition, given the egalitarian utility function $U$, we have that $U_i(g_k) < U_i(g_K)$ for all $i \in N$, $k = 1, ..., K-1$, along any myopic-farsighted improving path $(g_1, ..., g_K)$. It follows that if $g' \in \phi(g)$ and $g'' \in \phi(g')$ then $g'' \in \phi(g)$. Hence, $\phi(g) = \phi^{\infty}(g)$. Thus, for any $g$ such that $\phi(g) \neq \emptyset$ we have $g \cap G^{\emptyset} \neq \emptyset$, and the set $G^{\emptyset}$ satisfies (**ES**). Second, since $\phi(g) = \emptyset$ for all $g \in G^{\emptyset}$, the set $G^{\emptyset}$ satisfies (**IS**). Third, uniqueness follows from Proposition 2.2. □

Let $G^{\emptyset}_{|M,F}$ and $G^{\emptyset}_{|M',F'}$ denote the set $G^{\emptyset}$ when $N = M \cup F$ and $N = M' \cup F'$, respectively. Proposition 2.5 characterizes the unique myopic-farsighted stable set under the egalitarian utility function. When all players are myopic, the unique myopic-farsighted stable set consists of all pairwise stable networks. When all players are farsighted, the unique myopic-farsighted stable set consists of all strongly efficient networks. When the population is mixed, the unique myopic-farsighted stable set is a subset of the set of pairwise stable networks. In fact, under the egalitarian utility function, turning myopic players into farsighted players improves efficiency by removing Pareto-dominated pairwise stable networks, and in the end fully eliminates the tension between stability and efficiency.

**Proposition 2.5.** *Take any $U$ such that $U_i(g) = U_j(g)$ for all $i, j \in N$. The unique myopic-farsighted stable set is such that*

**(i)** $G^{\emptyset} = P$ *for* $N = M$ $(F = \emptyset)$*;*

**(ii)** $G^{\emptyset}_{|M',F'} \subseteq G^{\emptyset}_{|M,F}$ *for* $M' \subsetneq M$*;*

**(iii)** $G^{\emptyset} = E$ *for* $N = F$ $(M = \emptyset)$*.*

*Proof.* **(i)** First, take $N = M$ $(F = \emptyset)$. Given the egalitarian utility function $U$, we have that there are no closed cycles. Hence, it follows from Proposition 2.3 that the set of pairwise stable networks $P$ is the unique myopic-farsighted stable set and is equal to $\{g \in \mathcal{G} \mid \phi(g) = \emptyset\}$.

**(ii)** Second, take the unique myopic-farsighted stable set $G^{\emptyset}_{|M,F} = \{g \in \mathcal{G} \mid \phi(g) = \emptyset\}$ for $N = M \cup F$. Now take $N = M' \cup F'$ with $M' \subsetneq M$. We show that the unique myopic-farsighted stable set $G^{\emptyset}_{|M',F'}$ for $N = M' \cup F'$ is included in $G^{\emptyset}_{|M,F}$, i.e. $G^{\emptyset}_{|M',F'} \subseteq G^{\emptyset}_{|M,F}$ for $M' \subsetneq M$. Given the egalitarian utility function $U$, we have that along any myopic-farsighted improving path $(g_1, ..., g_K)$, $U_i(g_k) < U_i(g_K)$ for all $i \in N$, $k = 1, ..., K-1$. In particular, if $(g_1, ..., g_K)$ is a myopic-farsighted improving path when all players are myopic $(F = \emptyset)$ then we have $U_i(g_1) < U_i(g_2) < ... < U_i(g_K)$ for all $i \in N$. Hence, if $(g_1, ..., g_K)$ is a myopic-farsighted improving path for $N = M \cup F$, then $(g_1, ..., g_K)$ remains a myopic-farsighted improving path for $N = M' \cup F', M' \subsetneq M$. Let $\phi(g)_{|M,F}$ be the set of all networks that can be reached from $g$ by a myopic-farsighted improving path given the set of players $N = M \cup F$. It follows that $\phi(g)_{|M,F} \subseteq \phi(g)_{|M' \subsetneq M,F'}$, and so $G^{\emptyset}_{|M',F'} \subseteq G^{\emptyset}_{|M,F}$ for $M' \subsetneq M$.

**(iii)** Third, given the egalitarian utility function $U$, any strongly efficient network $g \in E$ Pareto-dominates any network $g' \notin E$. That is, $U_i(g) > U_i(g')$ for all $g \in E, g' \notin E$, for all $i \in N$. Hence, once all players are farsighted $(N = F, M = \emptyset)$, we have that $\phi(g) = \emptyset$ for all $g \in E$ and $\phi(g') \cap E \neq \emptyset$ for all $g' \in \mathcal{G} \setminus E$. □

### 2.2.4 Externalities

#### No externality

Suppose that the utility function $U$ exhibits no externality and is given by $U_i(g) = a(d_i)$ for all $i \in N$. The function $a(d_i)$ is assumed to be single peaked: there exists an integer $d^*$ such that $a(d_i) - a(d_i - 1) > 0$ for every $d_i \leq d^*$ and $a(d_i) - a(d_i - 1) < 0$ for every $d_i > d^*$. For $d^* = n - 1$, each player wants to be linked to all players. For $d^* = 0$, each player does not want to form any link. Thus, $d^*$ is the maximum number of links a player would like to form. If she has more than $d^*$ links she will cut some of them. To guarantee the existence of $d$-regular networks $(1 \leq d \leq n-1)$ we restrict the analysis to an even number of players. Let $G^{d^*} = \{g \in \mathcal{G} \mid d_i = d^* \text{ for all } i \in N\}$ be the set of $d^*$-regular networks. Obviously, $E = G^{d^*}$ when $U$ exhibits no externality and is single peaked.

Proposition 2.6 tells us that, when all players are myopic, the unique myopic-farsighted stable set consists of all pairwise stable networks. Notice that $d^*$-regular networks are pairwise stable. In fact, in a pairwise stable network, each player has either $d^*$ links or less than $d^*$ links and those players who have less than $d^*$ links are fully linked among themselves. When all players are farsighted, the unique myopic-farsighted stable set consists of all strongly efficient networks $E = G^{d^*}$. When the population is mixed, a myopic-farsighted stable set consists not only of all $d^*$-regular networks, but also of all networks where farsighted players have $d^*$ links and are not linked to myopic players, each myopic player has either $d^*$ links or less than $d^*$ links and those myopic players who have less than $d^*$ links are fully linked among themselves. Hence, turning myopic players into farsighted players improves again efficiency by removing Pareto-dominated pairwise stable networks, and in the end fully eliminates the tension between stability and efficiency.

**Proposition 2.6.** *Take any $U$ such that $U_i(g) = a(d_i)$ for all $i \in N$ where $a(d_i)$ is single peaked. Suppose that the number of players $n$ is even.*

**(i)** *If all players are farsighted ($M = \emptyset$), then the set $G^{d^*}$ is the unique myopic-farsighted stable set.*

**(ii)** *If all players are myopic ($F = \emptyset$), then the set $P = \{g \in \mathcal{G} \mid d_i = d^* \text{ for all } i \in N \setminus S,\ d_j < d^* \text{ for all } j \in S,\ \#S \leq d^*,\ g^S \subsetneq g\}$ is the unique myopic-farsighted stable set.*

**(iii)** *If the population is mixed ($M \neq \emptyset, F \neq \emptyset$) with $n - m > d^*$ and $n - m$ even, then the set $G^{d^*} \cup \{g \in \mathcal{G} \mid \phi(g) \cap G^{d^*} = \emptyset,\ d_i = d^* \text{ for all } i \in F,\ N_j(g) \cap F = \emptyset \text{ if } j \in M,\ d_j = d^* \text{ for all } j \in M \setminus S,\ d_k < d^* \text{ for all } k \in S,\ \#S \leq d^*,\ g^S \subsetneq g\}$ is a myopic-farsighted stable set.*

*Proof.* Since $n$ is even we have that $G^{d^*} = \{g \in \mathcal{G} \mid d_i = d^* \text{ for all } i \in N\} \neq \emptyset$.

**(i)** Take $N = F$ ($M = \emptyset$). Take any $g \notin G^{d^*}$. We will show that $\phi(g) \cap G^{d^*} \neq \emptyset$. First, players who have more than $d^*$ links successively cut their links to reach a network $g'$ where all players have at most $d^*$ links. If $g' \in G^{d^*}$ we are done. Otherwise, players who have less than $d^*$ links successively cut all their links looking forward some $d^*$-regular network $g^* \in G^{d^*}$. We reach a network $g''$ where players have either no links (they are isolated) or $d^*$ links (they are part of a $d^*$-regular subnetwork). **(*)** From $g''$, looking forward to $g^*$, one isolated player, say $i$, build a link to some player $j$ who has $d^*$ links and belongs to some component $h \subseteq g''$. Player $j$ is indifferent between the current network $g''$ and the end network $g^*$ while player $i$ strictly prefers the end network. From $g'' + ij$, player $j$ cuts a link with another player $k$ who has $d^*$ links. At $g''+ij-jk$, player $k$ has $d^*-1$ links and has now incentives to cut all her links looking forward to $g^*$. Player $k$ is now isolated. Next, players who were linked to player $k$ have now less than $d^*$ links and so have incentives to cut all their links looking forward to $g^*$. They become isolated. Next, we repeat the process where all players who have less than $d^*$ links cut all their links until we reach a network $g''' \subsetneq g'$ where all players of the component $h$ have now become isolated (player $i$ is again isolated). Next, we repeat the process **(*)** with another component where all players have $d^*$ links, until we reach the empty network $g^{\emptyset}$ where all players are isolated. From the empty network $g^{\emptyset}$, players add successively links to form $g^* \in G^{d^*}$. Hence, $G^{d^*} = \{g \in \mathcal{G} \mid d_i = d^* \text{ for all } i \in N\}$ satisfies (**ES**). Obviously, $\phi(g') = \emptyset$ for all $g' \in G^{d^*}$, and thus, this set satisfies (**IS**). Uniqueness follows from Proposition 2.2.

**(ii)** Take $N = M$ ($F = \emptyset$). Take any $g \notin \{g \in \mathcal{G} \mid d_i = d^* \text{ for all } i \in N \setminus S,$ $d_j < d^* \text{ for all } j \in S,\ \#S \leq d^*,\ g^S \subsetneq g\}$. We will show that $\phi(g) \cap \{g \in \mathcal{G} \mid d_i = d^*$ for all $i \in N \setminus S, d_j < d^*$ for all $j \in S, \#S \leq d^*, g^S \subsetneq g\} \neq \emptyset$. First, players who have more than $d^*$ links successively cut their links to reach a network $g'$ where all players have at most $d^*$ links. If $g' \in \{g \in \mathcal{G} \mid d_i = d^* \text{ for all } i \in N \setminus S,\ d_j < d^* \text{ for all } j \in S,\ \#S \leq d^*,\ g^S \subsetneq g\}$ we are done. (*) Otherwise, two players $i$ and $j$ who have less than $d^*$ links build the link $ij$. We repeat this process (*) until we reach a network $g'' \in \{g \in \mathcal{G} \mid d_i = d^* \text{ for all } i \in N \setminus S,\ d_j < d^* \text{ for all } j \in S,\ \#S \leq d^*,$ $g^S \subsetneq g\}$ where players have either $d^*$ links or less than $d^*$ links with players who have less than $d^*$ links being all linked to each other. Hence, $\{g \in \mathcal{G} \mid d_i = d^* \text{ for all } i \in N \setminus S,\ d_j < d^* \text{ for all } j \in S,\ \#S \leq d^*,\ g^S \subsetneq g\}$ satisfies (**ES**). Obviously, $\phi(g') = \emptyset$ for all $g' \in \{g \in \mathcal{G} \mid d_i = d^* \text{ for all } i \in N \setminus S,\ d_j < d^* \text{ for all } j \in S,$ $\#S \leq d^*,\ g^S \subsetneq g\} = P$, and thus, this set satisfies (**IS**). Uniqueness follows from Proposition 2.2.

**(iii)** Take $M \neq \emptyset, F \neq \emptyset$ with $n - m > d^*$ and $n - m$ even. Take any $g \notin G^{d^*} \cup G^{M,F}$ where $G^{M,F} = \{g \in \mathcal{G} \mid \phi(g) \cap G^{d^*} = \emptyset,\ d_i = d^* \text{ for all } i \in F,$ $N_j(g) \cap F = \emptyset \text{ if } j \in M,\ d_j = d^* \text{ for all } j \in M \setminus S,\ d_k < d^* \text{ for all } k \in S,\ \#S \leq d^*,$ $g^S \subsetneq g\}$. First, players who have more than $d^*$ links successively cut their links to reach a network $g'$ where all players have at most $d^*$ links. If $g' \in G^{d^*} \cup G^{M,F}$ we are done. (*) Otherwise, two players $i$ and $j$ who have less than $d^*$ links build the link $ij$. We repeat this process (*) until we reach a network $g'' \in \{g \in \mathcal{G} \mid d_i = d^*$ for all $i \in N \setminus S$, $d_j < d^*$ for all $j \in S$, $\#S \leq d^*$, $g^S \subsetneq g\}$ where players have either $d^*$ links or less than $d^*$ links, with players who have less than $d^*$ links being all linked to each other. If $g'' \in G^{d^*} \cup G^{M,F}$ we are done. Otherwise, looking forward to some $g^* \in G^{d^*} \cup G^{M,F}$, one farsighted player, say $i$, forms a link with some player $j$ who has less than $d^*$ links (player $j$ may be farsighted or not but strictly improves, while player $i$ is indifferent). Now, player $i$ has more than $d^*$ links. Next, looking forward to some $g^* \in G^{d^*} \cup G^{M,F}$, player $i$ has now strict incentives to build successively the missing links with the other farsighted players who are at $g^*$ either strictly better off or equally off. (**) Next another farsighted player who has now $d^* + 1$ links after having linked to $j$ (if any) forms successively the missing links with the other farsighted players who are at $g^*$ either strictly better off or equally off. We repeat this process (**) until we reach a network where each farsighted player is linked to all other farsighted players. Next, farsighted players successively cut all their links with myopic players. We reach a network $g''$ where $N_j(g) \cap F = \emptyset$ if $j \in M$ and $g^F \subseteq g'''$. From $g'''$, farsighted players successively delete links to form a $d^*$-regular subnetwork. Next, myopic players form links between myopic players until they reach a pairwise stable subnetwork. We reach a network $\widehat{g}$ where $d_i = d^*$ for all $i \in F$, $N_j(\widehat{g}) \cap F = \emptyset$ if $j \in M$, $d_j = d^*$ for all $j \in M \setminus S$, $d_k < d^*$ for all $k \in S \subseteq M$, $\#S \leq d^*$, $g^S \subsetneq \widehat{g}$. If $\phi(\widehat{g}) \cap G^{d^*} = \emptyset$ we are done and $g^* = \widehat{g}$. If $\phi(\widehat{g}) \cap G^{d^*} \neq \emptyset$, from $\widehat{g}$ to some $g^* \in G^{d^*}$, there is a myopic-farsighted improving path that involves only farsighted players and myopic players who have less than $d^*$ in $\widehat{g}$. Notice that farsighted players are indifferent between $\widehat{g}$ and $g^* \in G^{d^*}$ while myopic players who have less than $d^*$ links strictly prefers $g^*$ to $\widehat{g}$. Thus, the myopic-farsighted improving path from $g$ to $\widehat{g}$ followed by the myopic-farsighted improving path from $\widehat{g}$ to some $g^* \in G^{d^*}$ constitutes a myopic-farsighted improving path from $g$ to $g^* \in G^{d^*}$. Hence, $G^{d^*} \cup G^{M,F}$ satisfies (**ES**). Obviously, $\phi(g) = \emptyset$ for all $g \in G^{d^*} \cup G^{M,F}$, and thus, $G^{d^*} \cup G^{M,F}$ satisfies (**IS**). □

**Positive convex externalities**

Suppose now that the utility function $U$ is given by $U_i(g) = \sum_{ij \in g} \alpha(d_j)$ for all $i \in N$ where the function $\alpha(d_j)$ exhibits positive convex externalities: (i) there exists an integer $d^*$ such that $\alpha(d_j) > 0$ for $d_j \geq d^*$ and $\alpha(d_j) \leq 0$ for $d_j < d^*$, (ii) $\alpha(d_j + 2) - \alpha(d_j + 1) > \alpha(d_j + 1) - \alpha(d_j) > 0$ for $d_j \geq d^*$. Once some player $j$ has at least $d^* - 1$ links then each player who is not yet linked to $j$ has incentives to form a link with $j$. Under positive convex externalities, we have that $E = \{g^N\}$. Proposition 2.7 tells us that, under positive convex externalities, there is a threshold with respect to the number of farsighted players such that if the number of farsighted players is above the threshold then the set $E$ is the unique myopic-farsighted stable set while if the number of farsighted players is below the threshold then the unique myopic-farsighted stable set consists of the empty network $g^{\emptyset}$ and the complete networks $g^S$ on the set of players $S$ for all $S$ large enough. Thus, under positive convex externalities, we do not need all the population to be farsighted to guarantee the emergence of the strongly efficient network.

**Proposition 2.7.** *Take any $U$ such that $U_i(g) = \sum_{ij \in g} \alpha(d_j)$ for all $i \in N$ where $\alpha(d_j)$ exhibits positive convex externalities.*

- **(i)** *Take $d^* = 1$. The set $\{g^N\}$ is the unique myopic-farsighted stable set.*
- **(ii)** *Take $n-1 \geq d^* > 1$. If $d^* \leq n-m$, then $\{g^N\}$ is the unique myopic-farsighted stable set. If $d^* > n-m$, then $\{g^S \mid \#S \geq d^*+1\} \cup \{g^{\emptyset}\}$ is the unique myopic-farsighted stable set.*

*Proof.* **(i)** Take $d^* = 1$. From any $g \neq g^N$ both myopic and farsighted players have incentives to add links to all other players. Thus, $\phi(g) \cap \{g^N\} \neq \emptyset$. Obviously, $\phi(g^N) = \emptyset$. Hence, the set $\{g^N\}$ satisfies (**ES**) and (**IS**) and is the unique myopic-farsighted stable set from Proposition 2.2.

**(ii)** Take $n-1 \geq d^* > 1$.
**(ii.a)** Suppose $d^* \leq n-m$. Take any $g \neq g^N$ we will show that $\phi(g) \cap \{g^N\} \neq \emptyset$. First, looking forward to $g^N$ farsighted players successively build links among themselves to reach a network $g' \supseteq g^F$. In $g'$ farsighted players have at least $d^*-1$ links. Hence, myopic players have now incentives to link to all farsighted players. At the end of this process we reach a network $g''$ where all myopic players have at least $d^*-1$ links, and so now each myopic player has incentives to link to all other myopic players to finally reach the complete network $g^N$. From $g^N$ we have $\phi(g^N) = \emptyset$ since $g^N$ Pareto dominates all other networks. Hence, the set $\{g^N\}$ satisfies (**ES**) and (**IS**) and is the unique myopic-farsighted stable set from Proposition 2.2.
**(ii.b)** Suppose $d^* > n-m$. Take any $g \notin G^{M,F} = \{g^S \mid \#S \geq d^*+1\} \cup \{g^{\emptyset}\}$ we will show that $\phi(g) \cap G^{M,F} \neq \emptyset$. **Step 1.** Myopic players successively cut the links they have to myopic and farsighted players who have less than $d^*$ links (If there is no myopic or farsighted player who has at least one link but less than $d^*$ links we go directly to Step 3). We repeat this process until we reach a network where myopic players have either at least $d^*$ links or no link. Now, farsighted players who have less than $d^*$ links can only be linked to farsighted players. Since the number of farsighted players is too small, $d^* > n-m$, myopic players with no link and farsighted players with less than $d^*$ links will never reach the threshold of $d^*-1$ links so that other players (myopic or farsighted) would have incentives to link to them. Hence, all

farsighted players will delete their links with those farsighted players who have less than $d^*$ links and they become isolated. **Step 2.** If some other farsighted players now have less than $d^*$ links, we repeat Step 1; otherwise we go to Step 3. **Step 3.** We have reached a network where $n - \#S$ players have no links and $\#S$ players belong to a component where each player has at least $d^*$ links. If $\#S = 0$, we end up at $g^\emptyset$. Otherwise, the players belonging to $S$ have now incentives to link to each other until they form the network $g^S$, and we end up at $g^S$. Obviously, $\phi(g^\emptyset) = \emptyset$ and $\phi(g^S) = \emptyset$. Hence, the set $\{g^S \mid \#S \geq d^* + 1\} \cup \{g^\emptyset\}$ satisfies (**ES**) and (**IS**) and is the unique myopic-farsighted stable set from Proposition 2.2. □

## 2.3 Distance-based utility

Distance-based utility functions exhibit positive externalities as any player weakly benefits from any new link between any two other players. As in Bloch and Jackson (2007) or Jackson (2008), if player $i$ is connected to player $j$ by a path of $t$ links, then player $i$ receives a benefit of $b(t)$ from her indirect connection with player $j$. It is assumed that $b(t) \geq b(t+1) > 0$ for any $t$.[15] Each direct link $ij \in g$ results in a benefit $b(1)$ and a cost $c$ to both $i$ and $j$. This cost can be interpreted as the time a player must spend with another player in order to maintain a direct link. Player $i$'s distance-based utility or payoff from a network $g$ is given by

$$U_i(g) = \sum_{j \neq i} b(t(ij)) - d_i(g) \cdot c,$$

where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between $i$ and $j$), $c \geq 0$ is a cost per link, and $b$ is a nonincreasing function. The symmetric connections model ($b(t) = \delta^t$) and the truncated connections model of Jackson and Wolinsky (1996) are special cases of distance-based payoffs.[16]

Proposition 4 in Bloch and Jackson (2007) tells us that the unique strongly efficient network is (i) the complete network $g^N$ if $c < b(1) - b(2)$, (ii) a star encompassing everyone if $b(1) - b(2) < c < b(1) + ((n-2)/2)b(2)$, and (iii) the empty network $g^\emptyset$ if $b(1) + ((n-2)/2)b(2) < c$. Are the strongly efficient networks likely

[15] In communication networks, players directly communicate with the players to whom they are linked. They benefit not only from direct communication but also from indirect communication from the players to whom their neighbors are linked. But, the benefit obtained from indirect communication decreases with the distance.

[16] Johnson and Gilles (2000) extend the connection model by introducing a cost of creating a link that is proportional to the geographical distance between two individuals. In Jackson and Rogers (2005) or de Marti and Zenou (2017), individuals belong to two different communities, and the cost for creating links depends whether it is an intracommunity link or an intercommunity link.

to arise when all players are myopic?

Jackson (2008) characterizes the pairwise stable networks. He shows that a conflict between pairwise stability and efficiency is likely to occur except if link costs are small. For $c < b(1) - b(2)$, the unique pairwise stable network is the complete network $g^N$. For $b(1) - b(2) < c < b(1)$, a star encompassing all players is pairwise stable, but not necessarily the unique pairwise stable network. For $b(1) < c$, any pairwise stable network which is nonempty is such that each player has at least two links and thus is inefficient. Only for $c < b(1) - b(2)$, there is no conflict between efficiency and pairwise stability. When $b(1) - b(2) < c < b(1)$, the efficient network is pairwise stable, but there are other pairwise stable networks that are not efficient. For $b(1) < c < b(1) + ((n-2)/2)b(2)$, the efficient network is never pairwise stable. And, finally, for $b(1) + ((n-2)/2)b(2) < c$, the efficient network is pairwise stable, but there could be other pairwise stable networks that are not efficient.

Hence, from Proposition 2.3, the concept of myopic-farsighted stable set confirms that, for a large range of parameter values, a conflict between stability and efficiency is likely to occur when all players are myopic.

We denote by $g^{*i}$ the star network where player $i$ is the center of the star. For intermediate linking costs, $b(1) - b(2) < c < b(1)$, we next show that, if there are enough farsighted players, the set consisting of all star networks where the center of the star is a myopic player is the unique myopic-farsighted stable set.

**Proposition 2.8.** *Consider the distance-based utility model in the case* $b(1) - b(2) < c < b(1)$. *If* $n > \#F \geq 1 + b(2)/(b(2) - b(3))$ *then the set* $G^* = \{g^{*i} \mid i \in M\}$ *is the unique myopic-farsighted stable set.*

*Proof.* We first show that $G^* = \{g^{*i} \mid i \in M\}$ satisfies both internal stability (i.e. condition (**IS**) in Definition 4.3) and external stability (i.e. condition (**ES**) in Definition 4.3).

**IS.** Farsighted players are peripherals in all networks in $G^*$ so that they always obtain the same payoff: $U_i(g) = b(1) + (n-2)b(2) - c$ for all $i \in F$, $g \in G^*$. Myopic players who are peripherals have no incentive to delete their single link ($b(1) + (n-2)b(2) - c > 0$) or to add a new link ($2b(1) + (n-3)b(2) - 2c < b(1) + (n-2)b(2) - c$ since $b(1) - b(2) < c$). The center who is myopic has no incentive to delete one link since $c < b(1)$. Hence, for every $g, g' \in G^*$, it holds that $g' \notin \phi(g)$.

**ES.** Take any network $g \notin G^*$. We build in steps a myopic-farsighted improving path from $g$ to some $g^{*i} \in G^*$.

**Step 1:** Starting in $g$, farsighted players delete all their links successively looking forward to some $g^{*i} \in G^*$, where they obtain their highest possible payoff given $b(1) - b(2) < c$. Notice that if $g$ is a star network where the center is a farsighted player, then the center starts by deleting all her links since only the center is better off in $g^{*i}$ compared to $g$ (and we go directly to Step 8). We reach a network $g^1$ where all farsighted players have no link and myopic players only keep the links to myopic players they had in $g$.

**Step 2:** From $g^1$, looking forward to $g^{*i} \in G^*$, farsighted players build a star network $g^{*jF}$ restricted to farsighted players with player $j$ being the center (i.e. $g^{*jF}$ is such that $j \in F$, $N_j(g^{*jF}) = F \setminus \{j\}$ and $N_k(g^{*jF}) = \{j\}$ for all $k \in F \setminus \{j\}$), and we obtain $g^2 = g^1 \cup g^{*jF}$ where all farsighted players are still disconnected from the myopic ones.

**Step 3:** From $g^2$, looking forward to $g^{*i} \in G^*$, the farsighted player $j$ who is the center of $g^{*jF}$ adds a link to some myopic player, say player 1. Player $j$ is better off in $g^{*i}$ compared to $g^2$, $b(1) + (n-2)b(2) - c > (n-m-1)(b(1) - c)$, while player 1 is better in $g^2 + j1$ since $b(1) > c$.

**Step 4:** From $g^2 + j1$, looking forward to $g^{*i} \in G^*$, the farsighted player $j$ adds a link successively to the myopic players who are neighbors of player 1 (if any), say player 2. Player 2 who is myopic and linked to player 1 has an incentive to add the link $j2$ if and only if $b(2) + (n-m-1)b(3) < b(1) - c + (n-m-1)b(2)$. Thus, the necessary and sufficient condition for adding the link is

$$c < b(1) - b(2) + (n-m-1)(b(2) - b(3)). \tag{2.3.1}$$

Since $c < b(1)$, a sufficient condition is

$$b(1) \leq b(1) - b(2) + (n-m-1)(b(2) - b(3)) \text{ or } 1 + \frac{b(2)}{b(2) - b(3)} \leq n - m \tag{2.3.2}$$

where $n-m$ is the number of farsighted players ($\#F$). In $g^2+j1+\{jl \mid l \in N_1(g^2 + j1) \cap M\}$, player $j$ is (directly) linked to all other farsighted players, player 1 and all neighbors of player 1.

**Step 5:** From $g^2+j1+\{jl \mid l \in N_1(g^2 + j1) \cap M\}$, the myopic players who are neighbors of player 1 and have just added a link to the farsighted player $j$ delete their link successively with player 1. They have incentives to do so since $b(1)-b(2) < c < b(1)$ and we reach $g^2 + j1 + \{jl \mid l \in N_1(g^2 + j1) \cap M\} - \{1l \mid l \in N_1(g^2 + j1) \cap M\}$.

**Step 6:** Next, looking forward to $g^{*i} \in G^*$, the farsighted player $j$ adds a link successively to the myopic players who are neighbors of some $l \in N_1(g^2 + j1) \cap M$ and we proceed as in Step 4 and Step 5. We repeat this process until we reach a network $g^3$ where there is no myopic player linked directly to the myopic neighbors of player $j$ (i.e. $N_k(g^3) \cap M = \emptyset$ for all $k \in N_j(g^3) \cap M$).

**Step 7:** From $g^3$, player $j$ adds a link to some myopic player belonging to another component (if any) as in Step 3 and we proceed as in Step 4 to Step 6. We repeat this process until we end up with a star network $g^{*j}$ with player $j$ (who is farsighted) in the center (i.e. $N_j(g^{*j}) = N \setminus \{j\}$ and $N_k(g^{*j}) = \{j\}$ for all $k \in N \setminus \{j\}$).

**Step 8:** From $g^{*j}$, looking forward to $g^{*i} \in G^*$, the farsighted player $j$ deletes all her links successively to reach the empty network $g^{\emptyset}$. From $g^{\emptyset}$, myopic and farsighted players have both incentives (since $b(1) > c$) to add links successively to build the star network $g^{*i} \in G^*$ where some myopic player $i \in M$ is the center.

**Uniqueness.** We now show that $G^*$ is the unique myopic-farsighted stable set. Farsighted players who are peripherals in all networks in $G^*$ obtain their highest possible payoff. Myopic players who are peripherals have no incentive to delete their single link or to add a new link. The center who is myopic has no incentive to delete one link. Hence, $\phi(g) = \emptyset$ for every $g \in G^*$. Suppose that $G \neq G^*$ is another myopic-farsighted stable set. (1) $G$ does not include $G^*$: $G \not\supseteq G^*$. External stability would be violated since $\phi(g) = \emptyset$ for every $g \in G^*$. (2) $G$ includes $G^*$: $G \supsetneq G^*$. Internal stability would be violated since for every $g \in \mathcal{G} \setminus G^*$, it holds that $\phi(g) \cap G^* \neq \emptyset$. □

In fact, the set $G^*$ satisfies a stronger external stability requirement: for every $g \in \mathcal{G} \setminus G^*$, it holds that $\phi(g) \supseteq G^*$. The internal stability condition is satisfied for $G^*$ even when $\#F < 1 + b(2)/(b(2) - b(3))$.[17]

If $b(1) - b(2) < c < b(1) - b(3)$ then the sufficient condition for having external stability becomes $b(1) - b(3) \leq b(1) - b(2) + (n - m - 1)(b(2) - b(3))$ or $2 \leq n - m$. Thus, once linking costs are intermediate but not so high, it suffices to have two farsighted players to guarantee that only efficient networks are going to emerge in the long run.

**Corollary 2.2.** Consider the distance-based utility model in the case $b(1) - b(2) < c < b(1) - b(3)$. If $n > \#F \geq 2$ then the set $G^* = \{g^{*i} \mid i \in M\}$ is the unique myopic-farsighted stable set.

What happens if $\#F < 1 + b(2)/(b(2) - b(3))$ and (4.5.1) is not satisfied? If a myopic-farsighted stable set exists then $G^*$ should be included in it. Otherwise, external stability would be violated since $\phi(g) = \emptyset$ for all $g \in G^*$.

[17] In the symmetric connections model where $b(t) = \delta^t$, the lower bound on the number of farsighted players, $1 + b(2)/(b(2) - b(3))$, becomes $1 + 1/(1 - \delta)$. Hence, the number of farsighted players needed for guaranteeing the emergence of the efficient networks increases with $\delta$.

**Corollary 2.3.** Consider the distance-based utility model in the case $b(1)-b(2) < c < b(1)$. If $1 \leq \#M \leq 3$ then the set $G^* = \{g^{*i} \mid i \in M\}$ is the unique myopic-farsighted stable set.

Notice that in a society with only three players, a star network is the unique pairwise stable network. Hence, if the population is mixed but the number of myopic players is less or equal than 3, then our main result holds without any condition on the number of farsighted players: the set consisting of all star networks where the center of the star is a myopic player is the unique myopic-farsighted stable set. But, what happens if the population consists of only farsighted players?

**Proposition 2.9.** *Consider the distance-based utility model in the case* $b(1)-b(2) < c < b(1)$. *Suppose that all players are farsighted,* $N = F$. *If* $g$ *is a star network then* $\{g\}$ *is a myopic-farsighted stable set.*

*Proof.* Since each set is a singleton set, internal stability (**IS**) is satisfied. (**ES**) Take any network $g \neq g^{*i}$, we need to show that $\phi(g) \ni g^{*i}$. (i) Suppose $g \neq g^{*j}$ $(j \neq i)$. From $g$, looking forward to $g^{*i}$ (where they obtain their highest possible payoff), farsighted players ($\neq i$) delete all their links successively to reach the empty network. From $g^{\emptyset}$, farsighted players have incentives (since $b(1) > c$) to add links successively to build the star network $g^{*i}$ with player $i$ in the center. (ii) Suppose $g = g^{*j}$ $(j \neq i)$. From $g$, looking forward to $g^{*i}$, the farsighted player $j$ deletes all her links successively to reach the empty network. From $g^{\emptyset}$, farsighted players have incentives (since $b(1) > c$) to add links successively to build the star network $g^{*i}$ with player $i$ in the center. □

Once all players become farsighted (i.e. $N = F$), for $b(1) - b(2) < c < b(1)$, every set consisting of a star network encompassing all players is a myopic-farsighted stable set, but they are not necessarily the unique myopic-farsighted stable sets. For instance, when $n = 4$, the set of circles among the four farsighted players can be a myopic-farsighted stable set.[18]

**Example 2.2.** Take $N = F = \{1, 2, 3, 4\}$ and $b(1)-b(2) < c < b(1)-b(3) < b(1)$ in the distance-based utility model. Let $G^{c,4} = \{\{12, 23, 34, 14\}, \{13, 12, 34, 24\}, \{13, 14, 23, 24\}\}$

[18] Dutta and Vohra (2017) propose two related solution concepts: the rational expectations farsighted stable set (REFS) and the strong rational expectations farsighted stable set (SREFS) where they restrict coalitions (or pairs in our case) to hold common, history independent expectations that incorporate maximality regarding the continuation path. REFS and SREFS coincide with a farsighted stable set when the latter consists of networks with a single payoff (Theorem 1 of Dutta and Vohra, 2017). Since every set consisting of a star network encompassing all players is a myopic-farsighted stable set, it is also a REFS and SREFS. When $n = 4$, the same holds for the set of circles among the four farsighted player.

be the set of circles among the four farsighted players. The set $G^{c,4}$ is a myopic-farsighted stable set. It satisfies (**IS**) since the four players obtain the same payoffs in all circle networks. We now show that (**ES**) is satisfied: for every $g \notin G^{c,4}$, it holds that $\phi(g) \cap G^{c,4} \neq \emptyset$. (i) Take any $g$ such that there is $g' \in G^{c,4}$ and $g \subsetneq g'$. In $g$, looking forward to $g'$, players have incentives to add links successively to form $g'$ since $c < b(1) - b(3)$, and so $g' \in \phi(g)$. (ii) Take any $g^S$ such that $\#S = 3$. Players belonging to $S$ have two links and are better off in any circle network $g' \in G^{c,4}$ than in $g^S$: $2b(1) - 2c < 2b(1) - 2c + b(2)$. Hence, from $g^S$, looking forward to some circle network $g'$, some player deletes one of her links and we reach a network belonging to case (i) from which players have incentives to add links successively to form some circle network $g'$, and so $g' \in \phi(g^S)$. (iii) Take any $g$ such that at least one player has three links. Any star network $g^{*i}$ is one of such network. Players who have three links are better off in any circle network $g'$ than in $g$: $3b(1) - 3c < 2b(1) - 2c + b(2)$ or $b(1) - b(2) < c$. Hence, from $g$, looking forward to some circle network $g'$, players who have three links successively delete one of their links and we reach either a circle network or a network belonging to case (i) or case (ii) from which players have incentives to add links successively to form some circle network $g'$, and so $g' \in \phi(g)$.

We have focused on the range of costs and benefits such that a star network is the unique strongly efficient network. In the case of small (very large) link costs relative to benefits, there is no conflict between stability and efficiency. The set consisting of the complete (empty) network is the unique myopic-farsighted stable set whatever the mixture of myopic and farsighted individuals.[19]

*Remark* 2.1. Consider the distance-based utility model in the case $c < b(1) - b(2)$. The set $\{g^N\}$ is the unique myopic-farsighted stable set.

Suppose now that player $i$'s distance-based utility from a network $g$ is given by $U_i(g) = \sum_{j \neq i} b_i(t(ij)) - d_i(g)c_i$ where $c_i \geq 0$ and $b_i$ is a nonincreasing function. Assume that $b_i(1) - b_i(2) < c_i < b_i(1)$ for all $i \in N$. If $n > \#F$ and $\#F \geq 1 + b_i(2)/(b_i(2) - b_i(3))$ for all $i \in M$, then the set $G^* = \{g^{*i} \mid i \in M\}$ is the unique myopic-farsighted stable set, and Proposition 2.8 still holds. However, such asymmetries in benefits and costs would imply that a conflict between stability and efficiency could again arise. For instance, the efficient network might even lie outside the set $G^*$ if it is a star network with some farsighted player in the center. Transfers might then be a solution for avoiding any conflict.[20]

[19]Let $\overline{c}(n) = \max\{c \in \mathbb{R} \mid \exists g \in \mathcal{G}$ such that $g \neq g^\emptyset$ and $U_i(g) \geq 0\ \forall i \in N\}$ be the highest cost such that the utility of all players is nonnegative in at least one network other than the empty network. For $n > 3$, it follows from the proof of Proposition 2 in Grandjean, Mauleon and Vannetelbosch (2011) that if $b(1) < \overline{c}(n) < c < b(1) + ((n-2)/2)b(2)$ then $\{g^\emptyset\}$ is the unique myopic-farsighted stable set when all players are farsighted.

[20]When all players are myopic, Bloch and Jackson (2007) show that peripheral players can sub-

## 2.4 Degree-based utility

Another common utility function in network formation is one where a player's payoff from a link is a decreasing function of the number of links the other players maintain.[21] Degree-based utility functions exhibit negative externalities. If player $i$ is linked to player $j$, then player $i$ receives a benefit of $\beta(d_j)$ from her link with player $j$. It is assumed that $\beta(d_j)$ is decreasing with $d_j$, i.e. $\beta(d_j) > \beta(d_j+1) > 0$ for any $d_j$. Each direct link $ij \in g$ results in a cost $c$ to both $i$ and $j$ for maintaining this direct link. As in Morrill (2011), player $i$'s degree-based utility or payoff from a network $g$ is given by

$$U_i(g) = \sum_{j \in N_i(g)} \beta(d_j(g)) - d_i(g) \cdot c,$$

where $d_i$ is player $i$'s degree, $d_j$ is player $j$'s degree, $c \geq 0$ is a cost per link, and $\beta$ is a decreasing function. A special case of degree-based utility function is Morrill's co-author model where $\beta(d_j) = \gamma^{d_j}$ with $0 < \gamma < 1$.[22] We assume as in Morrill (2011) that $c \neq \beta(d)$ for any $d \in \mathbb{N}$ and we let $\overline{d}$ be such that $\beta(\overline{d}+1) < c < \beta(\overline{d})$.

To simplify the analysis we focus on the case where the population consists of an even number of myopic players ($m \geq 0$ is even) and an even number of farsighted players ($n - m \geq 0$ is even). Morrill (2011) shows that a network $g$ is strongly efficient if and only if for every player $i$, $d_i \in \arg\max x(\beta(x) - c)$. It follows that, for any $d \in \arg\max x(\beta(x) - c)$, all $d$-regular networks are strongly efficient. When all players are myopic, there is often a conflict between efficiency and stability for degree-based utility functions. For instance, take $\beta(d_j) = (1/2)^{d_j}$, $n = 10$, $0 < c < (1/2)^9$. Then, $\overline{d} \geq 9$ and $\arg\max x(\beta(x) - c) = 1$. Hence, the strongly efficient networks are regular ones where every player has exactly one link. However, myopic players may have a tendency to form overconnected networks: the complete network is pairwise stable. Morrill (2011) shows that, although the strongly efficient and stable networks diverge in general, they coincide when players are able to make transfers to their partners.

What happens when myopic players coexist with farsighted ones? Could we stabilize the strongly efficient networks without transfers when the population is

sidize the center of the star to keep their links formed. Any (efficient) star network is supportable as a pairwise equilibrium of the direct transfer game when $b(1) - b(2) < c < b(1) + b(2)(n-2)/2$.

[21] Möhlmeier, Rusinowska and Tanimura (2016) consider a utility function that incorporates both the effects of distance and of neighbors' degree.

[22] It is an alternative functional form to Jackson and Wolinsky (1996) original coauthor model. In the coauthor model a researcher benefits from having a coauthor as it increases her research output. But, if her coauthor works on a new project with someone else, she has less time to devote to their project and the benefit of the collaboration decreases.

mixed?

**Proposition 2.10.** *Consider the degree-based utility model with an even number of myopic players and an even number of farsighted players. Suppose* $d^* = \arg\max x(\beta(x) - c)$,[23] $m-1 > d^*$, $n-m > d^*$, $\overline{d} \geq n-1$ *and* $m\beta(m)+(x-m)\beta(x)-xc < d^*(\beta(d^*)-c)$ *for* $x = m+1, ..., n-1$. *If* $g^s \in G^s = \{g \in \mathcal{G} \mid d_j(g) = m-1$ *if* $j \in M$, $d_i(g) = d^*$ *if* $i \in F$ *and* $N_j(g) \cap F = \emptyset$ *if* $j \in M\}$ *then* $\{g^s\}$ *is a myopic-farsighted stable set.*

*Proof.* We show that each $\{g^s\}$ such that $g^s \in G^s$ satisfies both internal stability (i.e. condition (**IS**) in Definition 4.3) and external stability (i.e. condition (**ES**) in Definition 4.3). Since $d^* = \arg\max x(\beta(x) - c)$, a network $g^e$ is strongly efficient if and only if it is a $d^*$-regular network. We have $U_i(g^e) = U_i(g^s) = d^*(\beta(d^*) - c)$ for all $i \in F$.

**IS.** Since each set $\{g^s\}$, $g^s \in G^s$, is a singleton set, internal stability (**IS**) is satisfied.

**ES.** Take any network $g^s \in G^s$. Take any $g \neq g^s$. We build in steps a myopic-farsighted improving path from $g$ to $g^s$. Let $I(g) = \{i \in N \mid d_i(g) = d^*$ and $d_j(g) = d^*$ for all $j \in N_i(g)\}$ be the set of players who have $d^*$ links in $g$ and their neighbors have $d^*$ links too.
**1.** Since $\overline{d} \geq n-1$, we have that $\beta(x) - c$ is positive for all $x \leq n-1$. Hence, myopic players have always incentives to form additional links. Starting from $g$, myopic players form successively the missing links between them to reach a network $g'$ where $g^M \subseteq g'$ (remember that $g^M$ is the complete network on the set of myopic players $M$). Notice that $d_j \geq m-1 > d^*$ for all $j \in M$.
**2. Step (2.1)** Take any farsighted player $i \in F$ such that $N_i(g') \subseteq M$. That is, player $i$ has only links with myopic players. Since $U_i(g') \leq m(\beta(m) - c) < d^*(\beta(d^*) - c) = U_i(g^e)$, player $i$ who looks forward towards $g^s$ has incentives to cut successively her links with the myopic players. Player $i$ becomes an isolated player. We proceed similarly with all other farsighted players who are only linked to myopic players. We end up with $g''$. **Step (2.2)** Take any $i \in F$ such that $d_i(g'') = 1$ and $i \notin I(g'')$. We do have $U_i(g'') < U_i(g^s) = U_i(g^e)$. Player $i$ (looking forward towards $g^s$) cuts her link to her neighbor $j$ and we move back to step 2.1 with $g'' - ij$ replacing $g'$. If there is no $i \in F$ such that $d_i(g'') = 1$ and $i \notin I(g'')$, we move to step 2.3. **Step (2.3)** Take any $i \in F$ such that $d_i(g'') = 2$ and $i \notin I(g'')$. Player $i$ gets at most $(\beta(m) - c)$ from her link with a myopic player and at most $d_i(\beta(d_i) - c)$ from her links with the farsighted players since all farsighted players $j \in N_i(g'') \cap F$ have $d_j \geq d_i$. Since $d_i \leq m$ $(m \geq 2)$ we have that all players

[23] Instead of assuming a unique $d^* = \arg\max x\beta(x)$, Morrill (2011) imposes a stronger regularity condition for analyzing the network game with transfers: the social payoff function $x\beta(x)$ is single peaked.

$j \in N_i(g'')$ have $d_j \geq d_i$, and so $U_i(g'') \leq d_i(\beta(d_i) - c) < d^*(\beta(d^*) - c) = U_i(g^e)$. Player $i$ (looking forward towards $g^s$) cuts successively her links to $j$ and $k$ to obtain $g'' - ij - ik$ and we move back to step 2.1 with $g'' - ij - ik$ replacing $g'$. Her payoff along the sequence decreases. Hence, we do have $U_i(g'') < U_i(g^s) = U_i(g^e)$, $U_i(g'' - ij) < U_i(g^s) = U_i(g^e)$ and $U_i(g'' - ij - ik) < U_i(g^s) = U_i(g^e)$. If there is no $i \in F$ such that $d_i(g'') = 2$ and $i \notin I(g'')$, we move to step 2.4. **Step (2.4)** Take any $i \in F$ such that $d_i(g'') = 3$ and $i \notin I(g'')$. Player $i$ gets at most $2(\beta(m) - c)$ from her links with myopic players and at most $d_i(\beta(d_i) - c)$ from her links with the farsighted players since all players $j \in N_i(g'') \cap F$ have $d_j \geq d_i$. If $d_i \leq m$ then $U_i(g'') \leq d_i(\beta(d_i) - c) < d^*(\beta(d^*) - c) = U_i(g^e)$ since all players $j \in N_i(g'')$ have $d_j \geq d_i$. If $d_i \geq m+1$ then the condition, $m\beta(m) + (x-m)\beta(x) - xc < d^*(\beta(d^*) - c)$ for $x = m+1, ..., n-1$,[24] guarantees that $U_i(g'') < d^*(\beta(d^*) - c) = U_i(g^e)$. Player $i$ (looking forward towards $g^s$) cuts successively her links to $j$, $k$ and $l$ to obtain $g'' - ij - ik - il$ and we move back to step 2.1 with $g'' - ij - ik - il$ replacing $g'$. Her payoff along the sequence decreases. If there is no $i \in F$ such that $d_i(g'') = 3$ and $i \notin I(g'')$, we move to step 2.5. ... If there is no $i \in F$ such that $d_i(g'') = q - 1$ and $i \notin I(g'')$, we move to step 2.$q$. **Step (2.$q$)** Take any $i \in F$ such that $d_i(g'') = q - 1$ and $i \notin I(g'')$. Player $i$ gets at most $(q-2)(\beta(m) - c)$ from her links with myopic players and at most $d_i(\beta(d_i) - c)$ from her links with the farsighted players since all players $j \in N_i(g'') \cap F$ have $d_j \geq d_i$. If $d_i \leq m$ then $U_i(g'') \leq d_i(\beta(d_i) - c) < d^*(\beta(d^*) - c) = U_i(g^e)$ since all players $j \in N_i(g'')$ have $d_j \geq d_i$. If $d_i \geq m+1$ then the condition, $m\beta(m) + (x-m)\beta(x) - xc < d^*(\beta(d^*) - c)$ for $x = m+1, ..., n-1$, guarantees that $U_i(g'') < d^*(\beta(d^*) - c) = U_i(g^e)$. Player $i$ (looking forward towards $g^s$) cuts successively her links to her neighbors to obtain $g'' \setminus \{ij \mid j \in N_i(g'')\}$ and we move back to step 2.1 with $g'' \setminus \{ij \mid j \in N_i(g'')\}$ replacing $g'$. Her payoff along the sequence decreases. If there is no $i \in F$ such that $d_i(g'') = q - 1$ and $i \notin I(g'')$, we move to step 2.$q$+1. ... If there is no $i \in F$ such that $d_i(g'') = n-2$ and $i \notin I(g'')$, we move to step 2.$n$. **Step (2.$n$)** Take any $i \in F$ such that $d_i(g'') = n-1$ and $i \notin I(g'')$. Player $i$ gets $(n-1)(\beta(n-1) - c)$ from her links. Since $d^* < n - 1$, we have that $U_i(g'') = (n-1)(\beta(n-1) - c) < d^*(\beta(d^*) - c) = U_i(g^e)$. Player $i$ (looking forward towards $g^s$) cuts successively her links to her neighbors to obtain $g'' \setminus \{ij \mid j \in N_i(g'')\}$ and we move back to step 2.1 with $g'' \setminus \{ij \mid j \in N_i(g'')\}$ replacing $g'$. If there is no

[24] When the number of farsighted players is large, it might happen that the farsighted player with the smallest number of links has more links than the myopic players she is linked to. Remember that all myopic players are linked to each other. Hence, $m\beta(m)$ is the maximal gain she can obtain from her links to myopic players and $(x-m)\beta(x)$ is the gain she obtains from being linked to other farsighted players who have at least the same number of links than her. The condition $m\beta(m) + (x-m)\beta(x) - xc < d^*(\beta(d^*) - c)$ guarantees that this farsighted player prefers being in the strongly efficient network. Hence, she is ready to cut her links looking forward to some network where she gets the socially optimal payoff.

$i \in F$ such that $d_i(g'') = n-1$ and $i \notin I(g'')$, then the process ends. Since $n$ is finite this process stops after a finite number of steps. At the end of the process we reach a network $g'''$ where every farsighted player $i \in F$ is either isolated (i.e. $d_i = 0$) or she has exactly $d^*$ links and her neighbors are farsighted and have $d^*$ links too (i.e. $d_j = d^*$ for all $j \in N_i(g''') \subseteq F$ and $U_i(g^e) = U_i(g''') = U_i(g^s)$) and every myopic player has exactly $m-1$ links.

**3.** Start with $g'''$. Take any farsighted player $i$ such that $d_i = d^*$ (if there is no such player, then go directly to **5**). Player $i$ looking forward to $g^s$ ($i$ is indifferent between her current payoff and the end payoff at $g^s$) builds a link with some myopic player $j$ to form $g''' + ij$. In $g''' + ij$, we have $U_k(g^s) = U_k(g''') > U_k(g''' + ij)$ for all $k \in N_i(g''')$. Next one farsighted player $k \in N_i(g''')$ cuts her link with player $i$ to form $g''' + ij - ik$ looking forward to $g^s$. Next player $i$ who is farsighted cuts successively all her links, with her link $ij$ being the last one to be deleted. We reach the network $g''' \setminus \{ij \mid j \in N_i(g''')\}$ where player $i$ is isolated. Notice that $U_i(g''' + ij) > U_i(g^s) = d^*(\beta(d^*) - c) > U_i(g''' + ij - ik)$ since $d_j(g''' + ij - ik) \geq m > d^* = d_i(g''' + ij - ik)$, $d_l(g''' + ij - ik) = d^*$ for all $l \in N_i(g''' + ij - ik)$, $l \neq k$, and $d_k(g''' + ij - ik) = d^* - 1$.

**4.** We repeat the process from **2** with $g''' \setminus \{ij \mid j \in N_i(g''')\}$ replacing $g'$ until we reach the network $g^M$ where all farsighted players are isolated ($d_i = 0$ for all $i \in F$) and all myopic players have $m-1$ links.

**5.** From the network $g^M$, we build a sequence of networks $g_1, g_2, ..., g_K$ such that $g_1 = g^M$, $g_K = g^s$ and $|\#N_i(g_k) - \#N_j(g_k)| \leq 1$, $k = 1, ...K$, for all $i, j \in N(h)$, $h \in H(g^s)$ and $h$ is $d^*$-regular. It guarantees that along such a sequence, farsighted players who look forward towards $g^s$ do have incentives to build those links to form such $g^s$. Hence, $\{g^s\}$ satisfies (**ES**). □

Proposition 2.10 tells us that once we have enough farsighted players in the population, myopic and farsighted players may end up segregated with overconnected myopic players and farsighted players who obtain the socially optimal payoff. The next example illustrates Proposition 2.10.

**Example 2.3.** Take the degree-based utility model with $\beta(d_j) = (1/2)^{d_j}$, $0 < c < (1/2)^9$, $M = \{1, 2, 3, 4, 5, 6\}$ and $F = \{7, 8, 9, 10\}$. Thus, $m = 6$, $n - m = 4$, and we have $d^* = 1$ and $\overline{d} \geq 9$. Notice that the condition $m\beta(m) + (x - m)\beta(x) - xc = 6(1/2)^6 + (x-6)(1/2)^x - xc < d^*(\beta(d^*) - c) = (1/2) - c$ is clearly satisfied for $x = 7, 8, 9$. From Proposition 2.10 the singleton set $\{g\}$ is a myopic-farsighted stable set if the network $g$ is such that (i) there are no links between farsighted and myopic players, (ii) every myopic player is linked to all other myopic players, and (iii) farsighted players form regular components where each farsighted player has exactly

$d^*$ links. Figure 2.2 illustrates such a network where the six myopic players and the four farsighted players are fully segregated with myopic players being overconnected (all of them have five links) and farsighted players having exactly one link.

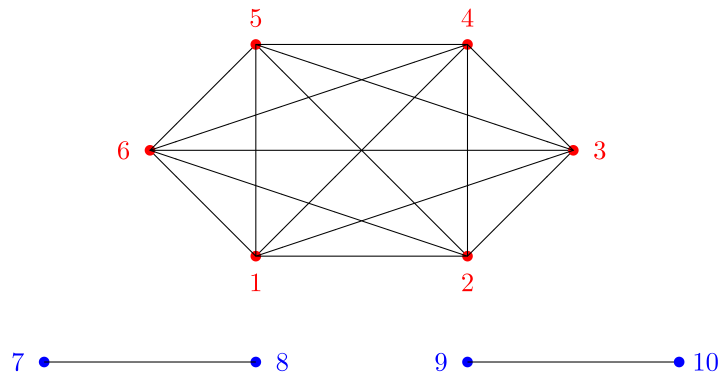


Figure 2.2: Degree-based utility with segregation.

When the cost for maintaining links increases, myopic players may stop having incentives to build links to all other players, i.e. $\overline{d}$ becomes lower than $n-1$. As a result, the path from some inefficient network $g \notin G^s$ to some network $g' \in G^s$ becomes more tedious. For instance, suppose $\beta(d_j) = (3/5)^{d_j}$, $c = 1/5$, $M = \{1,2,3,4\}$ and $F = \{5,6\}$. Thus, $m = 4$, $n - m = 2$, and we have $d^* = 1$ and $\overline{d} = 3$. Take the network $\{16, 56, 23, 24, 26, 34, 35, 45\}$ depicted in Figure 2.3. Player 1 who is myopic has no incentives to link to one of the other myopic players since they have exactly three links. Player 6 who is farsighted obtains more than the socially optimal payoff: $(3/5) + 2(3/5)^3 - 3c > (3/5) - c = U_6(g^s)$. However, the other farsighted player, who has also three links, is worse off than at $g^s$: $3(3/5)^3 - 3c < 3/5 - c = U_5(g^s)$. Player 5 then cuts successively all her links and becomes isolated. Next player 1 has now incentives to link successively with players 3 and 4. We reach the network $\{16, 13, 14, 23, 24, 26, 34\}$. Now player 6's current payoff is equal to $2(3/5)^3 - 2c < 3/5 - c$. Hence, player 6 cuts successively all her links and becomes isolated. Next player 1 adds a link to player 2. Finally, player 5 builds a link to player 6 and we reach the network $\{12, 13, 14, 23, 24, 34, 56\} \in G^s$.

Let $G^{SYM} = \{g \in \mathcal{G} \mid d_i(g) = d_j(g)$ for all $i, j \in F$, $d_k(g) = d_l(g)$ for all $l, k \in M\}$ be the set of symmetric networks where all myopic players have the same number of links and all farsighted players have the same number of links. Notice that $d_i(g)$ might be different than $d_k(g)$ for $i \in F$, $k \in M$. Since $\phi(g^s) \cap G^{SYM} = \emptyset$ for all $g^s \in G^s$, any set $\{g\}$ with $g \in G^{SYM} \setminus G^s$ violates (**ES**) and cannot be a myopic-farsighted stable set. Hence, all $g^s \in G^s$ are the only symmetric networks that can emerge in the long run as singleton myopic-farsighted stable sets.

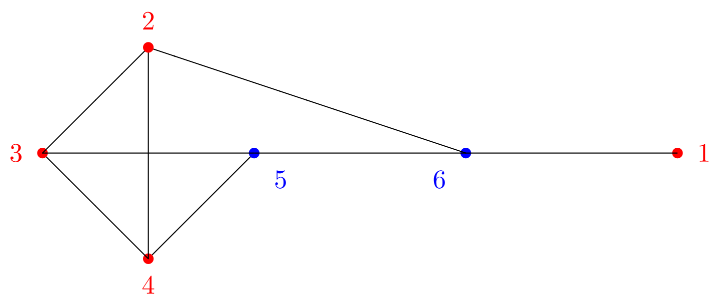


Figure 2.3: Degree-based utility: some inefficient network.

*Remark* 2.2. Consider the degree-based utility model with an even number of myopic players and an even number of farsighted players. Suppose $d^* = \arg\max x(\beta(x)-c)$, $m-1 > d^*$, $n-m > d^*$, $\overline{d} \geq n-1$ and $m\beta(m)+(x-m)\beta(x)-xc < d^*(\beta(d^*)-c)$ for $x = m+1, ..., n-1$. If $g \in G^{SYM} \setminus G^s$, then $\{g\}$ is not a myopic-farsighted stable set.

Remember that, when $n$ is even, the set of $d^*$-regular networks $G^{d^*} = \{g \in \mathcal{G} \mid d_i(g) = d^*$ for all $i \in N\}$ is the set of strongly efficient networks $E$, and $g^e$ denotes some strongly efficient network. Once all players are farsighted, the set $G^{d^*}$ consisting of all strongly efficient networks is a myopic-farsighted stable set.

**Proposition 2.11.** *Consider the degree-based utility model with an even number of players. Suppose all players are farsighted, $N = F$, and $d^* = \arg\max x(\beta(x)-c)$. The set $G^{d^*} = \{g \in \mathcal{G} \mid d_i(g) = d^*$ for all $i \in N\}$ is a myopic-farsighted stable set.*

*Proof.* We show that $G^{d^*} = \{g \in \mathcal{G} \mid d_i(g) = d^*$ for all $i \in N\}$ satisfies both internal stability (i.e. condition (**IS**) in Definition 4.3) and external stability (i.e. condition (**ES**) in Definition 4.3). Since $d^* = \arg\max x(\beta(x)-c)$, a network $g^e$ is strongly efficient if and only if $g^e$ is a $d^*$-regular network. So, $G^{d^*}$ is the set of strongly efficient networks. Take any network $g \notin G^{d^*}$. There is always some player $i$ such that $U_i(g^e) > U_i(g)$.

**IS.** Players obtain the same payoff in all networks in $G^{d^*}$: $U_i(g) = d^*(\beta(d^*)-c)$ for all $i \in N$, $g \in G^{d^*}$. Hence, for every $g, g' \in G^{d^*}$, it holds that $g' \notin \phi(g)$.

**ES.** Take any network $g \notin G^{d^*}$. We build in steps a myopic-farsighted improving path from $g$ to some $g^e \in G^{d^*}$. Remember that $I(g) = \{i \in N \mid d_i(g) = d^*$ and $d_j(g) = d^*$ for all $j \in N_i(g)\}$.

**1. 1. Step (1.1)** Take any $i \in F$ such that $d_i(g) = 1$ and $i \notin I(g)$. We do have $U_i(g) < U_i(g^e)$. Player $i$ (looking forward towards $g^e$) cuts her link to her neighbor

$j$ and we move back to step 1.1 with $g - ij$ replacing $g$. If there is no $i \in F$ such that $d_i(g) = 1$ and $i \notin I(g)$, we move to step 1.2. **Step (1.2)** Take any $i \in F$ such that $d_i(g) = 2$ and $i \notin I(g)$. Player $i$ (looking forward towards $g^e$) cuts successively her links to $j$ and $k$ to obtain $g - ij - ik$ and we move back to step 1.1 with $g - ij - ik$ replacing $g$. In the sequence, player $i$ first cuts all her links with players such that $d_j \leq \overline{d}$ (if any such $j$). It guarantees that her payoff along the sequence decreases or is negative. Hence we do have $U_i(g) < U_i(g^e)$, $U_i(g - ij) < U_i(g^e)$ and $U_i(g - ij - ik) < U_i(g^e)$. Notice that $U_i(g) \leq d_i(\beta(d_i) - c) < d^*(\beta(d^*) - c) = U_i(g^e)$ since all players $j \in N_i(g)$ have $d_j \geq d_i$. If there is no $i \in F$ such that $d_i(g) = 2$ and $i \notin I(g)$, we move to step 1.3. **Step (1.3)** Take any $i \in F$ such that $d_i(g) = 3$ and $i \notin I(g)$. Player $i$ (looking forward towards $g^e$) cuts successively her links to $j$, $k$ and $l$ to obtain $g - ij - ik - il$ and we move back to step 1.1 with $g - ij - ik - il$ replacing $g$. In the sequence, player $i$ first cuts all her links with players such that $d_j \leq \overline{d}$ (if any such $j$). It guarantees that her payoff along the sequence decreases or is negative. Notice that $U_i(g) \leq d_i(\beta(d_i) - c) < d^*(\beta(d^*) - c) = U_i(g^e)$ since all players $j \in N_i(g)$ have $d_j \geq d_i$. If there is no $i \in F$ such that $d_i(g) = 3$ and $i \notin I(g)$, we move to step 1.4. ... If there is no $i \in F$ such that $d_i(g) = q - 1$ and $i \notin I(g)$, we move to step 1.$q$. **Step (1.$q$)** Take any $i \in F$ such that $d_i(g) = q$ and $i \notin I(g)$. Player $i$ (looking forward towards $g^e$) cuts successively her links to her neighbors to obtain $g \setminus \{ij \mid j \in N_i(g)\}$ and we move back to step 1.1 with $g \setminus \{ij \mid j \in N_i(g)\}$ replacing $g$. In the sequence, player $i$ first cuts all her links with players such that $d_j \leq \overline{d}$ (if any such $j$). It guarantees that her payoff along the sequence decreases or is negative. Notice that $U_i(g) \leq d_i(\beta(d_i) - c) < d^*(\beta(d^*) - c) = U_i(g^e)$ since all players $j \in N_i(g)$ have $d_j \geq d_i$. If there is no $i \in F$ such that $d_i(g) = q$ and $i \notin I(g)$, we move to step 1.$q+1$. ... If there is no $i \in F$ such that $d_i(g) = n-2$ and $i \notin I(g)$ we move to step 1.$n-1$. **Step (1.$n-1$)** Take any $i \in F$ such that $d_i(g) = n - 1$ and $i \notin I(g)$. Notice that $U_i(g) = (n-1)(\beta(n-1) - c) < d^*(\beta(d^*) - c) = U_i(g^e)$ since all players $j \in N_i(g)$ have $d_j = n - 1$ and $i \notin I(g)$. Player $i$ (looking forward towards $g^e$) cuts successively her links to her neighbors to obtain $g \setminus \{ij \mid j \in N_i(g)\}$ and we move back to step 1.1 with $g \setminus \{ij \mid j \in N_i(g)\}$ replacing $g$. If there is no $i \in F$ such that $d_i(g) = n - 1$ and $i \notin I(g)$, then the process ends. Since $n$ is finite this process stops after a finite number of steps. At the end of the process we reach a network $g'$ where every farsighted player $i \in F$ is either isolated (i.e. $d_i = 0$) or she has exactly $d^*$ links and her neighbors too (i.e. $d_j = d^*$ for all $j \in N_i(g)$ and $U_i(g^e) = U_i(g')$).

**2.** (**2a**) $d^* = 1$. At $g'$ there is an even number of players who have just one link; so both $\#N(g')$ and $n - \#N(g')$ are even numbers. From $g'$, every player who has no

links builds exactly one link to another player who has no links and we reach some $g^e \in G^{d^*}$. (**2b**) $d^* \neq 1$. (i) At $g'$ some isolated player $i$ (i.e. $d_i(g') = 0$) forms a link with some player $j$ who has $d^*$ links looking forward to some $g^e \in G^{d^*}$. Player $i$ strictly prefers the end network $g^e$ while player $j$ is indifferent: $U_i(g^e) > U_i(g') = U_i(g^{\emptyset})$ and $U_j(g^e) = U_j(g') = d^*(\beta(d^*) - c)$. In $g' + ij$, we have $U_i(g^e) > U_i(g' + ij) > U_i(g')$ and $U_k(g^e) = U_k(g') > U_k(g' + ij)$ for all $k \in N_j(g')$. Next one $k \in N_j(g' + ij)$ $(k \neq i, j)$ cuts successively all her links looking forward to some $g^e \in G^{d^*}$. Each time she is cutting one of her links she is decreasing her current payoff and so she is always better off at the end network $g^e \in G^{d^*}$. Next player $i$ cuts her link to player $j$ and player $i$ is again isolated. In $g' + ij - \{kl \mid l \in N_k(g' + ij)\}$, we have $U_i(g^e) > U_i(g' + ij - \{kl \mid l \in N_k(g' + ij)\})$ since $d_i(g' + ij - \{kl \mid l \in N_k(g' + ij)\}) = 1 \neq d^*$ and $d_j(g' + ij - \{kl \mid l \in N_k(g' + ij)\}) = d^*$. We reach the network $g'' = g' + ij - \{kl \mid l \in N_k(g' + ij)\} - ij$ where players $k$ and $i$ are isolated.
**3.** We repeat the process from step **1** with $g''$ replacing $g$, and we proceed in this way until we reach the empty network $g^{\emptyset}$.
**4.** From the empty network $g^{\emptyset}$ we build a sequence of networks $g_1, g_2, ..., g_K$ such that $g_1 = g^{\emptyset}$, $g_K = g^e$ and $|\#N_i(g_k) - \#N_j(g_k)| \leq 1$, $k = 1, ...K$, for all $i, j \in N(h)$, $h \in H(g^e)$. Along such a sequence, farsighted players who look forward towards some $g^e$ do have incentives to build those links to form such $g^e$. Hence $G^{d^*}$ satisfies (**ES**). □

The next example shows that, once all players become farsighted (i.e. $N = F$), the set consisting of all strongly efficient networks is a myopic-farsighted stable set, but it is not necessarily the unique myopic-farsighted stable set.

**Example 2.4.** Take the degree-based utility model with $\beta(d_j) = (1/2)^{d_j}$, $1/16 < c < 1/8$, $N = F = \{1, 2, 3, 4\}$. We have $d^* = 1$ and $\overline{d} = 3$. From Proposition 2.11 the set of networks $G^{d^*}$ such that all players have exactly one link is a myopic-farsighted stable set. However, the set $\{\{12, 23, 34\}, \{12, 14, 34\}, \{14, 24, 23\}, \{14, 13, 23\}, \{13, 34, 24\}, \{13, 12, 24\}\}$ composed of asymmetric networks where the two central players obtain a higher payoff than in $g^e$ while the other two (loose-end) players get less than in $g^e$ is a (myopic-)farsighted stable set. Similarly, the set $\{\{13, 23, 24\}, \{13, 14, 24\}, \{12, 24, 34\}, \{12, 13, 34\}, \{14, 34, 23\}, \{14, 12, 23\}\}$ is a (myopic-)farsighted stable set.

## 2.5 Discussion

### 2.5.1 Evolution and dynamics

To study how networks evolve when myopic players may become farsighted over time, we start with a group of players who are initially unconnected to each other. Over time, pairs of players decide whether or not to form or cut links with each other. A link can be cut unilaterally but agreement by both players is needed to form a link. All players are initially myopic, and thus decide to form or cut links if doing so increases their current payoffs. The length of a period is sufficiently long so that the process can converge to some stable network. At the beginning of each period after the initial period, some myopic players become farsighted.[25] Depending on their positions in the network, the process either stays at the same network or evolves to another stable network.

Time is divided into periods and is modeled as a countable and infinite set, $T = \{1, 2, ..., t, ...\}$. We denote by $g(t)$ the network that exists at the end of period $t \in T$ and by $g(0)$ the initial network. The process of forming links starts from the empty network. Hence, $g(0) = g^{\emptyset}$. We denote by $M(t)$ ($F(t)$) the set of myopic (farsighted) players at the beginning of period $t \in T$. The population dynamics of players is described by the following sequence $\{M(t), F(t)\}_{t=1}^{\infty}$ where $M(t) = N \setminus F(t)$, $M(1) = N$, $M(t) \subset M(t-1)$ for $2 \leq t < \bar{t}$ and $M(t) = \emptyset$ for $t \geq \bar{t}$. A myopic-farsighted improving path in period $t \in T$ from a network $g(t-1)$ to a network $g(t) \neq g(t-1)$ is a finite sequence of graphs $g_1, \ldots, g_K$ with $g_1 = g(t-1)$ and $g_K = g(t)$ such that for any $k \in \{1, \ldots, K-1\}$ either (i) $g_{k+1} = g_k - ij$ for some $ij$ such that $U_i(g_{k+1}) > U_i(g_k)$ and $i \in M(t)$ or $U_j(g_K) > U_j(g_k)$ and $j \in F(t)$; or (ii) $g_{k+1} = g_k + ij$ for some $ij$ such that $U_i(g_{k+1}) > U_i(g_k)$ and $U_j(g_{k+1}) \geq U_j(g_k)$ if $i, j \in M(t)$, or $U_i(g_K) > U_i(g_k)$ and $U_j(g_K) \geq U_j(g_k)$ if $i, j \in F(t)$, or $U_i(g_{k+1}) \geq U_i(g_k)$ and $U_j(g_K) \geq U_j(g_k)$ (with one inequality holding strictly) if $i \in M(t), j \in F(t)$. We denote by $\phi_t(g)$ the set of all networks that can be reached from $g$ by a myopic-farsighted improving path in period $t$. We denote by $\overline{G}$ the set of networks that belong to some myopic-farsighted stable set, $\overline{G} = \{g \in G \mid G \subseteq \mathcal{G}$ is a myopic-farsighted stable set$\}$, and we suppose that $\overline{G} \neq \emptyset$ for all $M, F$ such that $N = M \cup F$.

Starting in period 1 from $g(0)$ with $M(1) = N$ and $F(1) = \emptyset$, the dynamic process will evolve to some $g(1)$ such that (i) there is a myopic-farsighted improving path from $g(0)$ to $g(1)$ and (ii) $g(1)$ belongs to some myopic-farsighted stable set,

[25] For instance, a myopic player may become faster farsighted when interacting in an environment composed mainly of farsighted players (e.g. when it belongs to a component with a majority of farsighted players).

i.e. $g(1) \in \overline{G}(M(1), F(1))$. At the very beginning of period 2 some myopic players become farsighted, $M(2) \subset N$ and $F(2) \neq \emptyset$. If $g(1)$ is no more stable (i.e. $g(1) \notin \overline{G}(M(2), F(2))$) then the dynamic process will evolve to some $g(2) \neq g(1)$ such that (i) $g(2) \in \phi_2(g(1))$ and (ii) $g(2) \in \overline{G}(M(2), F(2))$. Otherwise, it remains where it was, i.e. $g(2) = g(1)$. Given the population dynamics $\{M(t), F(t)\}_{t=1}^{\infty}$, we say that $\{g(t)\}_{t=1}^{\infty}$ is an evolution of stable networks if and only if (i) $g(t) \in \overline{G}(M(t), F(t))$ and (ii) if $g(t) \neq g(t-1)$ then $g(t) \in \phi_t(g(t-1))$.

Consider again the network formation with utility function $U$ such that $U_i(g) = \sum_{ij \in g} \alpha(d_j)$ for all $i \in N$ where $\alpha(d_j)$ exhibits positive convex externalities. Take $n - 1 \geq d^* > 1$. Starting from the empty network $g^{\emptyset}$ with a population consisting of only myopic players ($M(1) = N$), the dynamic process first remains at the empty network $g^{\emptyset}$. In fact, the empty network will persist until some period $t^*$ where $\#M(t^*) \geq d^* > \#M(t^* + 1)$. At period $t^* + 1$, there are now enough farsighted players within the population to dismantle $g^{\emptyset}$ and move the process towards the complete network $g^N$ that Pareto dominates all other networks and will persist forever.

Since players do not interact in the empty network, it is not excluded that more time would be needed for a myopic player to become farsighted. Hence, if $d^*$ is large, the empty network can persist many periods until the dynamic process moves away to the complete network. On the contrary, in the distance-based utility model with $b(1) - b(2) < c < b(1)$, it is likely that the dynamic process will evolve faster from one star network to another star network since myopic players in the center interact closely with all farsighted players. Starting from the empty network, the dynamic process first converges to some pairwise stable network. Once the number of myopic players who have become farsighted is large enough, the dynamic process evolves to a star network with some myopic player in the center. Such star network will be dismantled once the myopic player in the center of the star becomes farsighted. In this case, the process evolves next to another star network with one of the remaining myopic player in the center.

Finally, consider the network formation under the egalitarian utility function. Starting from the empty network $g^{\emptyset}$ with a population consisting of only myopic players ($M(1) = N$), the dynamic process first converges to some pairwise stable network $g \in G^{\emptyset} = P$. If this pairwise stable network is strongly efficient (i.e. if $g \in E$), this network will persist forever. Otherwise, some myopic players become farsighted ($M(2) \subsetneq M(1) = N$) and the dynamic process either remains where it was (if $g \in G^{\emptyset}_{|M(2),F(2)} \subseteq G^{\emptyset}_{|M(1),F(1)}$) or evolves to some network $g' \in G^{\emptyset}_{|M(2),F(2)}$ that Pareto dominates $g$ (if $g \in G^{\emptyset}_{|M(1),F(1)} \setminus G^{\emptyset}_{|M(2),F(2)}$). If this network is strongly efficient (i.e. if $g' \in E$), it will persist forever. Otherwise, some remaining myopic players become now farsighted. In the end, the dynamic process always reaches some strongly efficient network that will persist forever.

### 2.5.2 Coalitions

In the notion of myopic-farsighted stable set we only consider deviations by at most a pair of players at a time. It might be that some coalition of players could all be made better off by some complicated reorganization of their links, which is not accounted for under myopic-farsighted stable sets with pairwise deviations. Groupwise deviations make sense in situations where players have substantial information about the overall structure and potential payoffs and can coordinate their actions. Our definition of myopic-farsighted stable set can be extended to groupwise deviations.

A network $g'$ is obtainable from $g$ via deviations by group $S \subseteq N$ if (i) $ij \in g'$ and $ij \notin g$ implies $\{i, j\} \subseteq S$, and (ii) $ij \in g$ and $ij \notin g'$ implies $\{i, j\} \cap S \neq \emptyset$. A groupwise myopic-farsighted improving path from a network $g$ to a network $g' \neq g$ is a finite sequence of networks $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$, (i) $g_{k+1}$ is obtainable from $g_k$ via deviations by $S_k \subseteq N$, (ii) $U_i(g_{k+1}) \geq U_i(g_k)$ for all $i \in S_k \cap M$ and $U_j(g_K) \geq U_j(g_k)$ for all $j \in S_k \cap F$ (with one inequality holding strictly). For a given network $g$, let $\Phi(g)$ be the set of networks that can be reached by a groupwise myopic-farsighted improving path from $g$. A set of networks $G \subseteq \mathcal{G}$ is a myopic-farsighted stable set with groupwise deviations if: **(IS)** for every $g, g' \in G$ $(g \neq g')$, it holds that $g' \notin \Phi(g)$; and **(ES)** for every $g \in \mathcal{G} \setminus G$, it holds that $\Phi(g) \cap G \neq \emptyset$.

For any given network $g$, we have $\phi(g) \subseteq \Phi(g)$. Hence, if $\phi(g) \cap G \neq \emptyset$ for all $g \in \mathcal{G} \setminus G$, then $\Phi(g) \cap G \neq \emptyset$ for all $g \notin \mathcal{G} \setminus G$. Thus, if $\{g\}$ is a myopic-farsighted stable set, then it is a myopic-farsighted stable set with groupwise deviations.[26] However, if $G$ is a myopic-farsighted stable set, then $G$ may become unstable with groupwise deviations since $G$ might now violate (**IS**).

In the case of network formation under the egalitarian utility function, groupwise deviations imply that the set of strongly efficient networks $E$ is the unique myopic-farsighted stable whatever the number of farsighted and myopic players. Remember that all networks $g \in E$ Pareto dominates all networks $g' \in \mathcal{G} \setminus E$. Hence, $\Phi(g') \cap E \neq \emptyset$ for all $g' \in \mathcal{G} \setminus E$ and $\Phi(g) = \emptyset$ for all $g \in E$. Similarly, in the case of positive convex externalities, we have that $G = \{g^N\}$ is the unique myopic-farsighted stable with groupwise deviations. When there is no externality and the number of

[26] If $\{g\}$ was the unique myopic-farsighted stable set, then it is not necessarily the unique myopic-farsighted stable set with groupwise deviations since now we could have $\Phi(g) \neq \emptyset$ while $\phi(g) = \emptyset$.

players is even, the set of $d^*$-regular networks is the unique myopic-farsighted stable with groupwise deviations. Indeed, in a $d^*$-regular network compared to any other network, players are at least as well off with at least one of them being strictly better off.

In the distance-based utility model, for $b(1) - b(2) < c < b(1)$, the set consisting of all star networks where the center of the star is a myopic player, $\{g^{*i} \mid i \in M\}$, is still the unique myopic-farsighted stable set with groupwise deviations. However, we now only need that there is at least one farsighted player and one myopic player in the population to sustain this result. From any $g' \notin \{g^{*i} \mid i \in M\}$, we can build a groupwise myopic-farsighted improving path leading to some $g \in \{g^{*i} \mid i \in M\}$. First, $S = N$ deviates from $g'$ to form in one step a star network $g^{*j}$ with some farsighted player $j$ in the center. Next, player $j$ cuts all her links leading to the empty network $g^{\emptyset}$. Finally, $S = N$ deviates from $g^{\emptyset}$ to form in one step a star network $g^{*i}$ with some myopic player $i$ in the center. Obviously, $\Phi(g) = \emptyset$ for all $g \in \{g^{*i} \mid i \in M\}$.

### 2.5.3 Limited farsightedness

Pairwise stability requires that networks are immune to immediate deviations. On top of this requirement, one my look for networks that are also immune to deviations by myopic and farsighted players. A network $g \in \mathcal{G}$ is myopic-farsightedly pairwise stable if $\phi(g) = \emptyset$. The set of myopic-farsightedly pairwise stable networks is denoted by $P_{MF}$. When $N = F$ it reverts to Jackson (2008) set of farsightedly pairwise stable networks. Similar to pairwise stability, there is no guarantee that the set $P_{MF}$ is non-empty. From the proofs of Proposition 2.5 and Proposition 2.7, we have that, under the egalitarian utility function or in the presence of positive convex externalities, each network belonging to the unique myopic-farsighted stable set is myopic-farsightedly pairwise stable. Consider now the distance-based utility model with $b(1) - b(2) < c < b(1)$. From the proof of Proposition 2.8 we have that, if $n > \#F \geq 1 + b(2)/(b(2) - b(3))$ then, each network $g \in G^* = \{g^{*i} \mid i \in M\}$ is myopic-farsightedly pairwise stable, i.e. $P_{MF} = G^*$. So, there are no myopic-farsighted deviations from networks in $G^*$. In addition, myopic-farsighted deviations from networks outside $G^*$ to networks inside $G^*$ are credible since networks in $G^*$ are stable.

The notion of myopic-farsighted stable set assumes that each player is either myopic or farsighted. But, it could be that each player is nor fully myopic nor fully farsighted. Herings, Mauleon and Vannetelbosch (2019) propose the concept of a horizon-$K$ farsighted set to analyze which networks are going to emerge in the long run when players have an arbitrary homogeneous degree of farsightedness. A set of networks is a horizon-$K$ farsighted set if three conditions are satisfied: (i) deviations to networks outside the set are horizon-$K$ deterred, (ii) from any network outside the set there is a sequence of farsighted improving paths of length smaller than or equal to $K$ leading to some network in the set, and (iii) there is no proper subset satisfying the conditions (i) and (ii). There is no general relationship between the myopic-farsighted stable set and the horizon-$K$ farsighted set, except when all players are myopic. There is a unique horizon-1 farsighted set that consists of all pairwise networks and all networks belonging to the closed cycles. Hence, the horizon-1 farsighted set is equal to the union of all myopic-farsighted stable sets when all players are myopic.

## 2.6 Conclusion

We have adopted the notion of myopic-farsighted stable set to determine the networks that emerge when myopic and farsighted individuals decide with whom they want to form a link, according to some utility function that weighs the costs and benefits of each connection. We have provided conditions on the utility function that guarantee the existence and uniqueness of a myopic-farsighted stable set. We have shown that, under the egalitarian utility function or in the presence of positive convex externalities or in the case of no externality, the unique myopic-farsighted stable set consists of all pairwise stable networks when all players are myopic. When the population becomes mixed, the myopic-farsighted stable set refines the set of pairwise stable networks by eliminating some Pareto-dominated networks. In the end, when all players are farsighted, the unique myopic-farsighted stable set only consists of all strongly efficient networks. Hence, under the egalitarian utility function or in the presence of positive convex externalities or in the case of no externality, turning myopic players into farsighted players alleviates the tension between stability and efficiency. In addition, myopic players can only improve by becoming farsighted since the worst pairwise stable networks are progressively discarded.

It is important to understand what happens when myopic players interact with farsighted players since, in general, some networks that are nor stable when all players are myopic nor stable when all players are farsighted could emerge in the long run. In addition, turning myopic players into farsighted players might be costly for the society. Hence, a social planner would face a trade-off between the costs for increasing the number of farsighted players and the gains in terms of efficiency.

In the context of network formation with distance-based utilities (where links have positive externalities but diminishing with the distance), we have shown that, once the population of myopic and farsighted players is mixed, there is no tension between stability and efficiency. On the contrary, when all players are farsighted (or all players are myopic), a conflict is likely to arise. Hence, with distance-based utilities, once there are enough farsighted players in the population, there is no need for turning more myopic players into farsighted ones.

In the context of network formation with degree-based utilities (where links have negative externalities), we have shown that, in the case of a mixed population, segregation is likely to occur where myopic players tend to build too many links, while farsighted players coordinate for building the socially optimal number of links. Hence, with degree-based utilities, turning myopic players into farsighted ones improves *continuously* efficiency.

Finally, notice that farsighted players do better than myopic players under the different models we have studied (egalitarian utility function, distance-based utility, degree-based utility, positive convex externalities, no externality). But this is not always the case. For instance, in R&D networks, some myopic firms may obtain a higher profit than some farsighted firms in a myopic-farsighted stable set (see Mauleon, Sempere-Monerris and Vannetelbosch, 2018).

# Chapter 3

# Coalition-Proof Stable Networks

Joint work with Ana Mauleon and Vincent Vannetelbosch

**Abstract**

We propose the notion of coalition-proof stability for predicting the networks that could emerge when group deviations are allowed. A network is coalition-proof stable if there exists no coalition which has a credible group deviation. A coalition is said to have a credible group deviation if there is a profitable group deviation to some network and there is no subcoalition of the deviating players which has a subsequent credible group deviation. Coalition-proof stability is a coarsening of strong stability. There is no relationship between the set of coalition-proof stable networks and the set of networks induced by a coalition-proof Nash equilibrium of Myerson's linking game. Contrary to coalition-proof stability, coalition-proof Nash equilibria of Myerson's linking game tend to support unreasonable networks.

Keywords: networks; stability; group deviations; coalition-proofness; existence and efficiency; farsightedness.

JEL Classification: A14, C70, D20.

## 3.1 Introduction

The organization of players into networks plays an important role in the determination of the outcome of many social and economic interactions. Moreover, in many situations (R&D networks, free-trade networks, networks of buyers and sellers, criminal networks, ...) networks are neither fixed nor randomly determined but rather emerge through the decisions taken by the players.[1]

A first approach to analyze the networks that one might expect to emerge in the long run is the stability approach. It requires that players do not benefit from altering the structure of the network. Jackson and Wolinsky (1996) propose the notion of pairwise stability where a network is pairwise stable if no player benefits from severing one of her links and no two players benefit from adding a link between them. Pairwise stability only considers deviations involving a single link at a time. That is, link addition is bilateral (two players that would be involved in the link must agree to add the link), link deletion is unilateral (at least one player involved in the link must agree to delete the link), and network changes take place one link at a time. But, it might be that some group of players could all be made better off by some complicated reorganization of their links, which is not accounted for under pairwise stability. Hence, Jackson and van den Nouweland (2005) propose the notion of strong stability that allows for group deviations involving several links within some group of players at a time. Link addition is bilateral, link deletion is unilateral, and multiple link changes can take place at a time. Whether a pairwise deviation or a group deviation makes more sense depends on the setting within which network formation takes place.

A second approach to model network formation is by means of a noncooperative game. In Myerson's (1991) linking game, players choose simultaneously the links they wish to form and the formation of a link requires the consent of both players. Belleflamme and Bloch (2004) or Goyal and Joshi (2006) propose the notion of pairwise Nash stability: a network is pairwise Nash stable if there exists a pairwise Nash equilibrium of the Myerson's (1991) linking game that supports the network.[2] Pairwise Nash stability only allows for pairwise deviations. So, Dutta and Mutuswami (1997) propose the concepts of strong stability and weak stability. A network is strongly (weakly) stable if it corresponds to a strong (coalition-proof)

[1]Jackson (2008) and Goyal(2007) provide a comprehensive introduction to the theory of social and economic networks. Mauleon and Vannetelbosch (2016) give an overview of the solution concepts for solving network formation games.

[2]Pairwise Nash stability is a refinement of pairwise stability. Pairwise Nash stability requires that a network is immune both to the formation of a new link by any two players and to the deletion of any number of links by any player.

Nash equilibrium of the Myerson's (1991) linking game.[3]

In this paper, we adopt the first approach, i.e. the stability approach. A strongly stable network often fails to exist because networks can be classified as not stable while they rely on group deviations that are not credible. Hence, we propose the notion of coalition-proof stability for predicting the networks that could emerge in the long run. A network is said to be coalition-proof stable if there exists no coalition which has a credible group deviation. A coalition is said to have a credible group deviation if there is a profitable group deviation to some network and there is no subcoalition of the deviating players which has a subsequent credible group deviation. Coalition-proof stability is a coarsening of strong stability. In Belleflamme and Bloch (2004) model of market-sharing agreements, there is no strongly stable network while the empty network is the unique coalition-proof stable network.

More surprisingly, we show that there is no relationship between the set of coalition-proof stable networks and the set of networks induced by a coalition-proof Nash equilibrium of Myerson's linking game. In addition, coalition-proof stability often tends to predict the most plausible networks while some coalition-proof Nash equilibria of Myerson's linking game support unreasonable networks. For instance, in a model where network components compete for a loot, coalition-proof stability predicts the emergence of a network with a minimally winning component while there is no strongly stable network and coalition-proof Nash equilibria of Myerson's linking game sustain many more networks. The reason why coalition-proof Nash equilibria of Myerson's linking game support more networks and less reasonable ones has to do with the following drawback. If the deviation by a coalition involves the deletion of links with players outside the coalition, then a single deviating player who has just deleted a link with some player not in the deviating coalition can form again this link in a subsequent deviation without requiring the mutual consent of the other player. Coalition-proof stability overcomes such a drawback by requiring that this player belongs to the deviating coalition in the subsequent deviation.

Similarly to strong stability, a coalition-proof stable network may fail to exist. We then look for conditions on the utility function such that the existence of a coalition-proof stable network is guaranteed. We show that under a componentwise egalitarian utility function where players belonging to the same component get the same utility and there are no externalities across components, there always exists a coalition-proof stable network and coalition-proof stability coincides with strong stability.

[3]The definition of strong stability of Dutta and Mutuswami (1997) considers a deviation to be valid only if all members of a deviating coalition are strictly better off, while the definition of Jackson and van den Nouweland (2005) is slightly stronger by allowing for a deviation to be valid if some members are strictly better off and others are weakly better off.

Moreover, if the utility function is also top convex then both strong stability and coalition-proof stability single out the strongly efficient networks.

Up to now, we consider (strict) group deviations where a group of players deviate only if each of its members can be made (strictly) better off. Alternatively, we can look at weak group deviations where a group of players deviate only if at least one of its members is (strictly) better off while all other members are at least as well off. Although strong stability with weak group deviations refines strong stability with strict group deviations, we show that there is no relationship between coalition-proof stability with strict group deviations and coalition-proof stability with weak group deviations. However, if the network utility function is link-responsive (i.e. no player is indifferent to a change in her set of links), then both notions coincide.

Finally, there are situations where only pairwise deviations are feasible. In such situations, farsighted players may look beyond the immediate consequence of adding or deleting a link and anticipate the subsequent changes that will occur afterwards. Is coalition-proof stability with farsighted players but restricted to pairwise deviations equivalent to coalition-proof stability with group deviations? In general, the answer is no. Nevertheless, the set of coalition-proof farsightedly stable networks and the set of farsightedly stable networks coincide under the componentwise egalitarian utility function.

The paper is organized as follows. In Section 2 we introduce networks, pairwise stability, and strong stability, and we consider Jackson and Watts (2002) exchange networks model to illustrate the lack of credibility of some group deviations. In Section 3 we introduce the notion of coalition-proof stability. In Section 4 we compare coalition-proof stability with coalition-proof Nash equilibrium of the Myerson's linking game. In Section 5 we study the existence and efficiency of coalition-proof stable networks. In Section 6 we consider strict versus weak group deviations. In Section 7 we extend our notion of coalition-proof stability to farsighted players. In Section 8 we conclude.

## 3.2 Network formation

Let $N = \{1, ..., n\}$ be the finite set of players who are connected in some network relationship. The network relationships are reciprocal and the network is thus modeled as a non-directed graph. A network $g$ is a list of players who are linked to each other. We write $ij \in g$ to indicate that $i$ and $j$ are linked in the network $g$. Let $g^S$ be the set of all subsets of $S \subseteq N$ of size 2, so $g^N$ is the complete network. The set of all possible networks on $N$ is denoted by $\mathcal{G}$ and consists of all subsets of $g^N$. The network obtained by adding link $ij$ to an existing network $g$ is denoted $g + ij$ and the network obtained by cutting link $ij$ from an existing network $g$ is denoted $g - ij$. For any network $g$, we denote by $N(g) = \{i \mid \exists\, j \text{ such that } ij \in g\}$ the set of players who have at least one link in the network $g$. A path in a network $g$ between $i$ and $j$ is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = i$ and $i_K = j$. A non-empty network $h \subseteq g$ is a component of $g$, if for all $i \in N(h)$ and $j \in N(h) \setminus \{i\}$, there exists a path in $h$ connecting $i$ and $j$, and for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$.[4] We denote by $C(g)$ the set of components of $g$. A component $h$ of $g$ is minimally connected if $h$ has $\#N(h) - 1$ links (i.e. every pair of players in the component are connected by exactly one path). The partition of $N$ induced by $g$ is denoted by $\Pi(g)$, where $S \in \Pi(g)$ if and only if either there exists $h \in C(g)$ such that $S = N(h)$ or there exists $i \notin N(g)$ such that $S = \{i\}$.

A network utility function (or payoff function) is a mapping $u : \mathcal{G} \to \mathbb{R}^N$ that assigns to each network $g$ a utility $u_i(g)$ for each player $i \in N$. A network $g \in \mathcal{G}$ is strongly efficient relative to $u$ if it maximizes $\sum_{i \in N} u_i(g)$. A network $g \in \mathcal{G}$ Pareto dominates a network $g' \in \mathcal{G}$ relative to $u$ if $u_i(g) \geq u_i(g')$ for all $i \in N$, with strict inequality for at least one $i \in N$. A network $g \in \mathcal{G}$ is Pareto efficient relative to $u$ if it is not Pareto dominated and, a network $g \in \mathcal{G}$ is Pareto dominant if it Pareto dominates any other network.

A simple way to analyze the networks that one might expect to emerge in the long run is to examine a sort of equilibrium requirement that players do not benefit from altering the structure of the network. Jackson and Wolinsky (1996) define the notion of pairwise stability. A network is pairwise stable if no player benefits from severing one of their links and no other two players benefit from adding a link between them. Formally, a network $g$ is pairwise stable with respect to $u$ if and only if (i) for all $ij \in g$, $u_i(g) \geq u_i(g - ij)$ and $u_j(g) \geq u_j(g - ij)$, and (ii) for all $ij \notin g$, if $u_i(g) < u_i(g + ij)$ then $u_j(g) \geq u_j(g + ij)$.[5] Two networks $g$ and $g'$ are adjacent if they differ by one link. That is, $g'$ is adjacent to $g$ if $g' = g + ij$ or $g' = g - ij$ for some $ij$. A network $g'$ defeats $g$ if either $g' = g - ij$ with $u_i(g') > u_i(g)$ or $u_j(g') > u_j(g)$, or if $g' = g + ij$ with $u_i(g') > u_i(g)$ and $u_j(g') > u_j(g)$. Hence, a network is pairwise stable if and only if it is not defeated by another (necessarily adjacent) network. In the 3-player example of Figure 3.1 (Mauleon and Vannetelbosch, 2016), both the partial networks $g_1$, $g_2$ and $g_3$ and the complete network $g_7$ are pairwise stable.

[4] We use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion, and $\#$ refers to the notion of cardinality.

[5] The original definition of Jackson and Wolinsky (1996) allows for a pairwise deviation to be valid if one deviating player is better off and the other one is at least as well off.

The empty network $g_0$ is not pairwise stable because two players have incentives to link to each other and the star networks $g_4$, $g_5$ and $g_6$ are not pairwise stable since the peripheral players have incentives to add the missing link to form the complete network.

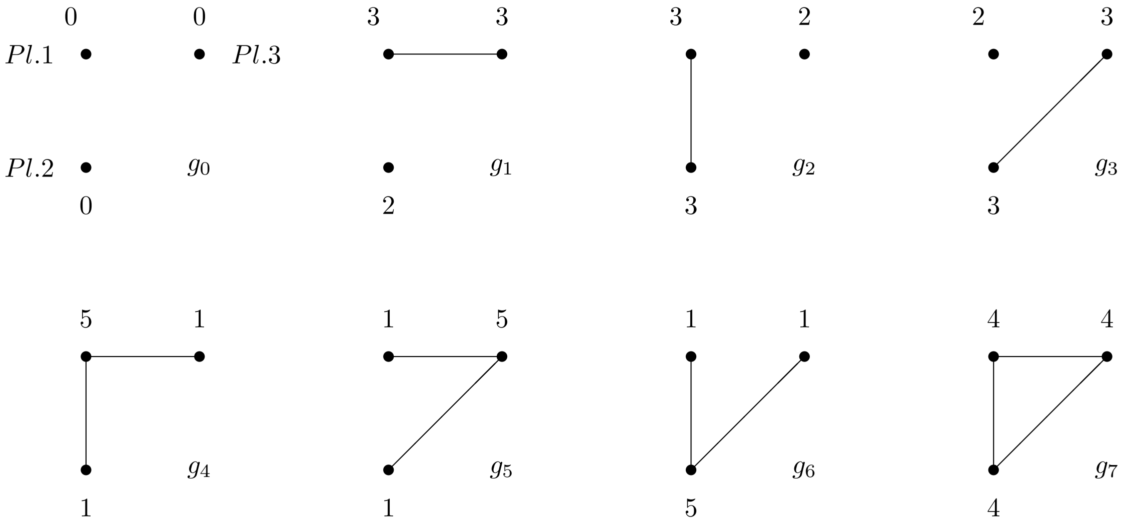


Figure 3.1: The networks that can be formed among three players with their utilities.

The notion of pairwise stability only considers deviations by at most a pair of players at a time. It might be that some group of players could all be made better off by some complicated reorganization of their links, which is not accounted for under pairwise stability. Group deviations make sense in situations where players have substantial information about the overall structure and potential payoffs and can coordinate their actions. Dutta and Mutuswami (1997) and Jackson and van den Nouweland (2005) propose alternative definitions of stability that allow for group deviations. The definition of strong stability of Dutta and Mutuswami (1997) considers a deviation to be valid only if all members of a deviating coalition are strictly better off, while the definition of Jackson and van den Nouweland (2005) is slightly stronger by allowing for a deviation to be valid if some members are strictly better off and others are weakly better off. Under the definition of Dutta and Mutuswami (1997), a network is strongly stable if it corresponds to a strong Nash equilibrium of Myerson's linking game.

We consider here a strict version of Jackson and van den Nouweland's (2005) notion of strong stability that refines the set of pairwise stable networks.

**Definition 3.1.** Coalition $S \subseteq N$ is said to have a group deviation from $g$ to $g'$ if

**(i)** $ij \in g'$ and $ij \notin g \Rightarrow \{i, j\} \subseteq S$,

**(ii)** $ij \in g$ and $ij \notin g' \Rightarrow \{i, j\} \cap S \neq \emptyset$,

**(iii)** $u_i(g') > u_i(g)$ for all $i \in S$.

A coalition $S$ is said to have a group deviation from the network $g$ to the network $g'$ if three conditions are satisfied. Condition (i) requires that any new links that are added can only be between players inside $S$. Condition (ii) requires that there must be at least one player belonging to $S$ for the deletion of a link. Condition (iii) requires that all members of $S$ are better off. This definition identifies possible profitable changes in a network that can be made by a coalition $S$.

**Definition 3.2.** A network $g$ is strongly stable if there exists no coalition $S \subseteq N$ which has a group deviation from $g$.

Let **SS** be the set of strongly stable networks. In the 3-player example of Figure 3.1, the complete network $g_7$ is the unique strongly stable network. However, there are situations where a pairwise stable network (and hence, a strongly stable network) fails to exist.

**Example 3.1** (Exchange networks; Jackson and Watts, 2002)**.** Four players get value from trading goods with each other. There are two goods. Players have the same utility function for the two goods, $u(x, y) = x \cdot y$. Players form first a network. Players then receive a random endowment which is independently and identically distributed: $(1, 0)$ with probability $1/2$ and $(0, 1)$ with probability $1/2$. Finally, trade flows without friction along any path and each connected component trades to a Walrasian equilibrium. Thus, $\{12, 23\}$ and $\{12, 23, 13\}$ lead to the same expected trades, but lead to different costs of links. Ignoring the costs of links, the player's expected utility is increasing and strictly concave in the number of other players that she is connected to: (i) the utility of being alone is 0; (ii) the expected utility of being connected to one player is $1/8$; (iii) the expected utility of being connected to two players is $1/6$; (iv) the expected utility of being connected to three players is $3/16$. Let $c = 5/96$ be the cost of maintaining a link. There is no pairwise or strongly stable network in Jackson and Watts exchange networks model with four players. The network $\{12, 34\}$ is defeated by $\{12, 23, 34\}$ which is defeated by $\{12, 23\}$ which is defeated by $\{12\}$ which is defeated by $\{12, 34\}$. See Figure 3.2.

Notice that the deviation by players 2 and 3 from $\{12, 34\}$ to $\{12, 23, 34\}$ might be questionable since at $\{12, 23, 34\}$ one of the two players has incentives to delete one of her links. For instance, player 3 has incentives to cut the link 34 to reach the network $\{12, 23\}$ where she gets a payoff of $11/96$ instead of $8/96$. Hence, the deviation from $\{12, 34\}$ to $\{12, 23, 34\}$ by players 2 and 3 is not credible because at

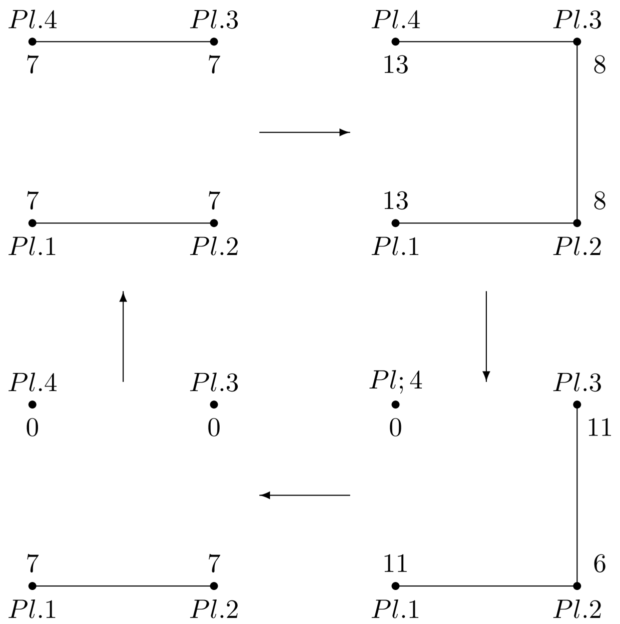


Figure 3.2: Exchange networks (all payoffs are in 96-th's).

$\{12, 23, 34\}$ one of the deviating players has a profitable deviation to $\{12, 23\}$ that does not involve other players.

## 3.3 Coalition-proof stability

Under the notion of strong stability, some networks are declared not stable meanwhile they rely on group deviations that are not credible. Hence, we now introduce the notion of coalition-proof stability (CPS) that checks for the credibility of group deviations.

**Definition 3.3.** Coalition $S \subseteq N$ is said to have a credible group deviation from $g$ if

**(i)** $g'$ is a group deviation from $g$ by $S$, and

**(ii)** there exists no subcoalition $T \subset S$ which has a credible group deviation from $g'$.

Notice the recursion in the definition of a credible group deviation. Each singleton coalition has a credible deviation if it has a deviation; each two-player coalition has a credible group deviation if it has a group deviation at which no player of the two has a credible deviation; each three-player coalition has a credible group deviation if it has a group deviation at which no player of the three and no two-player coalition among them have a credible group deviation; and so on.

**Definition 3.4.** A network $g$ is coalition-proof stable (CPS) if there exists no coalition $S \subseteq N$ which has a credible group deviation from $g$.

The concept of coalition-proof stability is weaker than that of strong stability: fewer group deviations are allowed since some are declared not credible because of their lack of internal consistency. Let **CPS** be the set of coalition-proof stable networks. In the exchange networks model, there is no strongly stable network. But, the profitable group deviation from $\{12, 34\}$ to $\{12, 23, 34\}$ by players 2 and 3 is not credible because at $\{12, 23, 34\}$ one of the deviating player has a profitable deviation to $\{12, 23\}$ that does not involve other players. Hence, $\{12, 34\}$ is a coalition-proof stable network.

**Example 3.2** (Market sharing agreements; Belleflamme and Bloch, 2004)**.** There are $n \geq 3$ firms and each firm $i$ has a home market and can be active in foreign markets. For any market $i$, let $n_i$ be the number of active firms on the market. Let $\pi_i^j(n_i)$ be the profit of firm $j$ on market $i$. Firms can sign bilateral market sharing agreements that refrain them from entering on the other firm's market. Let $g$ be a network of market sharing agreements: $ij \in g$ means that firms $i$ and $j$ are linked by a market sharing agreement and are not active on each other's market, while $ij \notin g$ means that firm $i$ is present on the market $j$ and firm $j$ on market $i$. On each market, active firms compete à la Cournot with zero marginal cost and a linear inverse demand given by $p = 10 - q$. Then, profits on markets are simply given by $\pi_i^j(n_i) = 100/(n_i + 1)^2$. The total payoff of firm $i$ is given by the sum of the profits firm $i$ gets on its home market and on all foreign markets for which it has not formed market sharing agreements:

$$u_i(g) = \pi_i^i(n_i) + \sum_{j:ij\notin g} \pi_j^i(n_j).$$

(i) We first argue that all networks $g \neq g^\emptyset$ are not strongly stable since any firm $i$ such that $n > n_i \geq n_j$ for all $j \in N$ has incentives to cut all its links.[6] Indeed, $u_i(g) = 100/(n_i{+}1)^2{+}\sum_{j:ij\notin g} 100/(n_j{+}1)^2$ and $u_i(g') = 100/(n{+}1)^2{+}\sum_{j:ij\notin g} 100/(n_j{+}1)^2{+}\sum_{k:ik\notin g',ik\in g} 100/(n_k + 1)^2$ with $g' = g \setminus \{jk \in g \mid j = i \text{ or } k = i\}$. Since $n > n_i \geq n_j$ for all $j \in N$, we have that $100(n - n_i)/(n_i + 2)^2 \leq \sum_{k:ik\notin g',ik\in g} 100/(n_k + 1)^2$ and $100/(n_i + 1)^2 < 100/(n + 1)^2 + 100(n - n_i)/(n_i + 2)^2$. Hence, $u_i(g) < u_i(g')$. In other words, firms having the fewer market sharing agreements among firms that do have market sharing agreements have incentives to cancel all its market sharing agreements. Since this deviation involves only a single firm, it is a credible one. Hence, all networks $g \neq g^\emptyset$ are not coalition-proof stable. (ii) We next argue that the empty network $g^\emptyset$ is not strongly stable since the grand coalition $N$ has a group

[6]In other words, the firm with less market sharing agreements (but at least one) has incentives to put an end to all its market sharing agreements.

deviation to the complete network $g^N$. Indeed, $u_i(g^\emptyset) = 100n/(n+1)^2 < u_i(g^N) = 100/4$ for all $i \in N$. However, any group deviation from $g^\emptyset$ to some $g$ is not credible since there is some $\{i\}$ who has a credible group deviation from $g$ as shown in (i). Hence, the empty network $g^\emptyset$ is the unique coalition-proof stable network.

**Proposition 3.1.** *In the market sharing networks model, there is no strongly stable network while the empty network $g^\emptyset$ is the unique coalition-proof stable network.*

## 3.4 Myerson's linking game

An alternative way to model network formation is Myerson's (1991) linking game $G = \langle N, (\Sigma_i)_{i\in N}, (U_i)_{i\in N}\rangle$where players choose simultaneously the links they wish to form and where the formation of a link requires the consent of both players. A strategy of player $i \in N$ is a vector $\sigma_i = (\sigma_{i1}, ..., \sigma_{ii-1}, \sigma_{ii+1}, ..., \sigma_{in})$ where $\sigma_{ij} \in \{0,1\}$ for each $j \in N\backslash\{i\}$. If $\sigma_{ij} = 1$, player $i$ wishes to form a link with player $j$. Let $\Sigma_i$ be the strategy set of player $i$ and $\Sigma$ be the set of strategy profiles. Given the strategy profile $\sigma = (\sigma_1, ..., \sigma_n)$, the network $g(\sigma)$ is formed where $ij \in g(\sigma)$ if and only if $\sigma_{ij} = 1$ and $\sigma_{ji} = 1$. The payoff function of player $i$ is given by $U_i(\sigma) = u_i(g(\sigma))$ for all $\sigma \in \Sigma$, with $g(\sigma) = \{ij \mid \sigma_{ij} = 1 \text{ and } \sigma_{ji} = 1\}$.[7]

**Definition 3.5** (Aumann, 1959)**.** A strategy profile $\sigma^* \in \Sigma$ is a strong Nash equilibrium of Myerson's linking game $\langle N, (\Sigma_i)_{i\in N}, (U_i)_{i\in N}\rangle$ if there is no $S \subseteq N$ and $\sigma \in \Sigma$ such that (i) $\sigma_i = \sigma_i^*$ for all $i \notin S$ and (ii) $U_i(\sigma) > U_i(\sigma^*)$ for all $i \in S$.

Let **SNE** $\equiv \{g(\sigma) \in \mathcal{G} \mid \sigma$ is a strong Nash equilibrium of Myerson's linking game $\langle N, (\Sigma_i)_{i\in N}, (U_i)_{i\in N}\rangle\}$ be the networks induced by a strong Nash equilibrium of Myerson's linking game. It corresponds to Dutta and Mutuswami's (1997) set of strongly stable networks.

**Proposition 3.2. SS = SNE**

*Proof.* ($\Leftarrow$) Suppose that $\sigma$ with $g(\sigma) = g$ is a strong Nash equilibrium of Myerson's linking game. Suppose on the contrary that $g$ is not strongly stable. That is, there is a group deviation by $S \subseteq N$ to $g'$ such that (i) $ij \in g'$ and $ij \notin g \Rightarrow \{i,j\} \subseteq S$, (ii) $ij \in g$ and $ij \notin g' \Rightarrow \{i,j\} \cap S \neq \emptyset$, (iii) $u_i(g') > u_i(g)$ for all $i \in S$. We now show that there is a group deviation by $S$ from $\sigma$ with $g(\sigma) = g$ to $\sigma'$ with

[7] Gilles and Sarangi (2010) extend Myerson's linking game to include additive link formation costs: if player $i$ attempts to form a link with player $j$ (i.e. $\sigma_{ij} = 1$), then player $i$ incurs a cost $c_{ij} \geq 0$ regardless of $\sigma_{ji}$. Bloch and Jackson (2006, 2007) compare pairwise stable networks with those based on the Nash equilibria of Myerson's linking game, and those based on equilibria of a link formation game where transfers are possible.

$g(\sigma') = g'$. Take (a) for all $j \notin S$, $\sigma'_j = \sigma_j$, (b) for all $i, j \in S$, $\sigma'_{ij} = \sigma'_{ji} = 1$ if and only if $ij \in g'$, (c) for all $i \in S$, for all $j \notin S$, $\sigma'_{ij} = 0$ if and only if $ij \notin g'$. Since $U_i(\sigma') = u_i(g(\sigma')) = u_i(g') > U_i(\sigma) = u_i(g(\sigma)) = u_i(g)$ for all $i \in S$, it then contradicts that $\sigma$ is a strong Nash equilibrium of Myerson's linking game. Thus, $g$ is strongly stable.

($\Rightarrow$) Suppose that $g$ is strongly stable. Take $\sigma$ such that, for all $i, j \in N$, $\sigma_{ij} = 1$ if and only if $ij \in g$. Suppose that $\sigma$ is not a strong Nash equilibrium of Myerson's linking game. That is, there is $S \subseteq N$ and $\sigma'$ with $g(\sigma') = g'$ such that (i) $\sigma'_i = \sigma_i$ for all $i \notin S$ and (ii) $U_i(\sigma') > U_i(\sigma)$ for all $i \in S$. Since $\sigma_{ij} = \sigma_{ji} = 0$ and $\sigma'_{ij} = \sigma'_{ji} = 1$ we have that $ij \in g'$ and $ij \notin g$ implies that $\{i, j\} \subseteq S$. Since $\sigma_{ij} = \sigma_{ji} = 1$ and $\sigma'_{ij} = 0$ or $\sigma'_{ji} = 0$ we have that $ij \in g$ and $ij \notin g' \Rightarrow \{i, j\} \cap S \neq \emptyset$. Since $U_i(\sigma') > U_i(\sigma)$ for all $i \in S$ we have that $u_i(g(\sigma')) = u_i(g') > u_i(g(\sigma)) = u_i(g)$ for all $i \in S$. So, there is a group deviation by $S$ from $g$ to $g'$. It then contradicts that $g$ is strongly stable. Thus, $\sigma$ with $g(\sigma) = g$ is a strong Nash equilibrium of Myerson's linking game. □

For the Myerson's linking game $G = \langle N, (\Sigma_i)_{i\in N}, (U_i)_{i\in N} \rangle$ and any fixed strategy profile $\sigma$, let $G^S_\sigma = \left\langle S, (\Sigma_i)_{i\in S}, (\widetilde{U}_i)_{i\in S} \right\rangle$ be the reduced Myerson's linking game for coalition $S$ given $\sigma$ where $\widetilde{U}_i(\sigma') = U_i(\sigma'_S, \sigma_{N\setminus S})$. The reduced game is obtained by fixing the strategies of all the players outside $S$ and defining the utility of every player given this fixed strategy choices.

**Definition 3.6** (Bernheim, Peleg, and Whinston, 1987)**.** A coalition-proof Nash equilibrium (CPNE) of the Myerson's linking game $G = \langle N, (\Sigma_i)_{i\in N}, (U_i)_{i\in N} \rangle$ is defined recursively. For $n = 1$, $\sigma^*_i$ is a coalition-proof Nash equilibrium (CPNE) if and only if $U_i(\sigma^*_i) \geq U_i(\sigma_i)$ for any $\sigma_i \in \Sigma_i$. Let $n > 1$ and assume that CPNE have been defined for all $m < n$. Then,

(i) $\sigma^*$ is self-enforcing for $G$ if and only if, for all $S \subsetneq N$, $\sigma^*_S$ is a CPNE of $G^S_{\sigma^*}$.

(ii) $\sigma^*$ is a CPNE if and only if it is self-enforcing and there does not exist another self-enforcing strategy $\sigma$ such that $U_i(\sigma) > U_i(\sigma^*)$ for all $i \in N$.

Let **CPNE** $\equiv \{g(\sigma) \in \mathcal{G} \mid \sigma$ is a coalition-proof Nash equilibrium of Myerson's linking game $\langle N, (\Sigma_i)_{i\in N}, (U_i)_{i\in N} \rangle\}$ be the networks induced by a coalition-proof Nash equilibrium of Myerson's linking game. It corresponds to Dutta and Mutuswami (1997) set of weakly stable networks.

**Example 3.3** (Contest networks)**.** Each component of a network is a team. Teams compete for winning the loot $B > 0$. The loot is divided among the winning team based on the network architecture. A team is winning only if the majority of players belong to the team. Within the winning team, the loot is divided equally among the players who have the most links. For any team $S \in \Pi(g)$ of connected players, let $\overline{d}(S) = \max_{i \in S} d_i$. Formally, the payoff of player $i \in S$, $S \in \Pi(g)$, is given by

$$u_i(g) = \begin{cases} B/\#\{j \in S \mid d_j = \overline{d}(S)\} - cd_i & \text{if } \#S > n/2 \text{ and } d_i = \overline{d}(S); \\ \\ -cd_i & \text{otherwise.} \end{cases}$$

In Figure 3.3 we depict the networks and the payoffs in the case of three players. In the empty network, there is no winner and all players get 0; in the partial networks, the team composed of the two linked players wins the loot and they share it equally ($B/2$); in the star networks, there is a single team and the player in the center gets the whole loot ($B$); in the complete network, the three players share equally the loot ($B/3$).

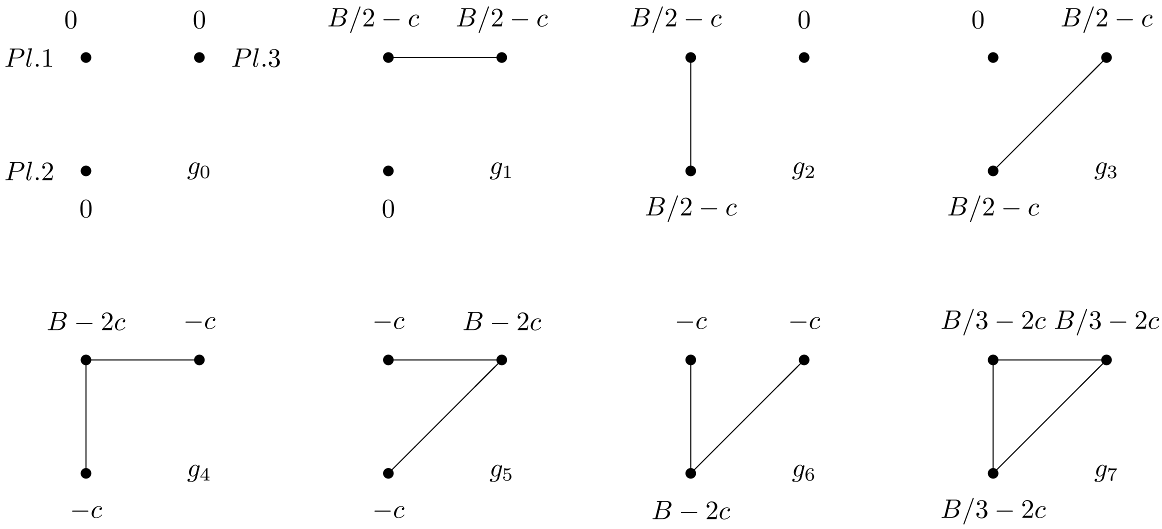


Figure 3.3: Contest networks among three players with their utilities.

In the contest networks model, coalition-proof Nash equilibria of the Myerson's linking game support unreasonable networks. Indeed, coalition-proof stability predicts the emergence of a network with a minimally winning component while there is no strongly stable network and coalition-proof Nash equilibria of Myerson's linking game sustain many more networks.

**Proposition 3.3.** *In the contest networks model with $B > n(n-1)c$ and $n \geq 6$,* $\mathbf{SS} = \emptyset$, $\mathbf{CPS} = \{g^{S^*} \mid (n+2)/2 \geq \#S^* > n/2\}$ *while* $\mathbf{CPNE} = \{g^S \mid \#S > n/2\}$.

*Proof.* (**a**) We first show that $\mathbf{SS} = \emptyset$ and $\mathbf{CPS} = \{g^{S^*} \mid (n+2)/2 \geq \#S^* > n/2\}$.

**(ia)** Take any $g$ such that there is some $i \in N(g)$ with $u_i(g) < 0$. In $g$, either $i$ belongs to a loosing component or $i$ belongs to the winning component but she has less links than some other member(s) of the winning component. Then, player $i$ has incentives to cut all her links and the deviation from $g$ to $g \setminus \{jk \in g \mid j = i \text{ or } k = i\}$ is credible. Hence, $g \notin \mathbf{SS}$ and $g \notin \mathbf{CPS}$ (and $g \notin \mathbf{CPNE}$).

Thus, the only candidates for being strongly stable or coalition-proof stable are networks such that players who have links belong to the winning component and have the same number of links: $g$ such that $\#C(g) = 1$, $\#N_i(g) = \#N_j(g)$ for all $i, j \in S \in \Pi(g)$, and $\#S > n/2$.

**(iia)** Take any $g$ such that $n \geq 3$, $\#C(g) = 1$ and $\#N_i(g) = \#N_j(g)$ for all $i, j \in S \in \Pi(g)$, $n \geq \#S > (n+2)/2 \geq \#S^* > n/2$. The members of any coalition $S^* \subsetneq S$ have incentives to deviate from $g$ to $g^{S^*}$. Moreover, this deviation is credible since $g^{S^*} \in \mathbf{CPS}$ as shown in (**va**). Hence, $g \notin \mathbf{SS}$ and $g \notin \mathbf{CPS}$.

**(iiia)** Take the empty network $g^{\emptyset}$. The members of any coalition $S^* \subsetneq S$ have incentives to deviate from $g^{\emptyset}$ to $g^{S^*}$. Moreover, this deviation is credible since $g^{S^*} \in \mathbf{CPS}$ as shown in (**va**). Hence, $g^{\emptyset} \notin \mathbf{SS}$ and $g^{\emptyset} \notin \mathbf{CPS}$.

Thus, the only candidates for being strongly stable or coalition-proof stable are networks $g$ such that $g \subseteq g^{S^*}$, $\#C(g) = 1$ and $\#N_i(g) = \#N_j(g)$ for all $i, j \in S^* \in \Pi(g)$.

**(iva)** Any non-empty network $g$ such that $n \geq 6$ (then $\#S^* \geq 4$), $g \subsetneq g^{S^*}$, $\#C(g) = 1$ and $\#N_i(g) = \#N_j(g)$ for all $i, j \in S^* \in \Pi(g)$ are neither strongly stable nor coalition-proof stable since two players $i$ and $j$ such that $i, j \in S^*$ and $ij \notin g$ have incentives to add this link to form $g + ij$ and to get $B/2 - cd_i$ by sharing together the entire loot $B$. Moreover, this is a credible group deviation for $S = \{i, j\}$. Hence, $g \notin \mathbf{SS}$ and $g \notin \mathbf{CPS}$.

**(va)** The network $g^{S^*} \notin \mathbf{SS}$ for $n \geq 6$ (then $\#S^* \geq 4$) since the members of coalition $S^*$ have a group deviation to the circle network among the members of $S^*$ ($g'$ such that $g' \subsetneq g^{S^*}$, $\#C(g') = 1$ and $\#N_i(g') = \#N_j(g') = 2$ for all $i, j \in S^* \in \Pi(g)$) where they get the same benefits than in $g^{S^*}$ but incur less costs. However, this group deviation from $g^{S^*}$ to the circle network $g'$ is not credible since there is a subcoalition $\{i, j\} \subsetneq S^*$ such that $ij \notin g'$ who has a credible deviation by adding the link $ij$ to $g'$ to form $g' + ij$ and to share together the entire loot $B$. Similarly, any group deviation from $g^{S^*}$ to $g''$ such that $g'' \subsetneq g^{S^*}$, $\#C(g'') = 1$ and $\#N_i(g'') = \#N_j(g'') = k$ for all $i, j \in S^* \in \Pi(g)$ with $2 < k < \#S^* - 1$ is not credible. Hence, $g^{S^*} \in \mathbf{CPS}$.

Thus, we have $\mathbf{SS} = \emptyset$ and $\mathbf{CPS} = \{g^{S^*} \mid (n+2)/2 \geq \#S^* > n/2\}$.

(**b**) We next show that $\mathbf{CPNE} = \{g^S \mid \#S > n/2\}$.

**(ib)** Take any $\sigma$ such that there is some $i \in N(g(\sigma))$ with $U_i(\sigma) = u_i(g(\sigma)) < 0$.

Then, $\sigma$ is neither a strong Nash equilibrium nor a coalition-proof Nash equilibrium of the Myerson's linking game since there is $\{i\}$ and $\sigma'$ with $\sigma'_j = \sigma_j$ for all $j \neq i$ and $\sigma'_i = (0,0,...,0,0)$ such that $U_i(\sigma) = u_i(g(\sigma)) < U_i(\sigma') = u_i(g(\sigma'))$. The deviation from $\sigma$ to $\sigma'$ is self-enforcing since $\{i\}$ is a singleton.

**(iib)** Take any $\sigma$ such that $\#C(g(\sigma)) = 1$, $\#N_i(g(\sigma)) = \#N_j(g(\sigma)) \neq \#S - 1$ for all $i, j \in S \in \Pi(g)$, $\#S > n/2$, and $\sigma_l = (0,0,...,0,0)$ for all $l \notin S$. Then, $\sigma$ is not a strong Nash equilibrium nor a coalition-proof Nash equilibrium of the Myerson's linking game since there is $\{i,j\} \subsetneq S$ and $\sigma'$ with $\sigma'_k = \sigma_k$ for all $k \neq i,j$, $\sigma'_i = \sigma_i$ except that $\sigma_{ij} = 0$ while $\sigma'_{ij} = 1$, $\sigma'_j = \sigma_j$ except that $\sigma_{ji} = 0$ while $\sigma'_{ji} = 1$ such that $U_i(\sigma) = u_i(g(\sigma)) < U_i(\sigma') = u_i(g(\sigma'))$ and $U_j(\sigma) = u_j(g(\sigma)) < U_j(\sigma') = u_j(g(\sigma'))$. This deviation from $\sigma$ to $\sigma'$ is self-enforcing since no player belonging to $\{i,j\}$ has an incentive to deviate from $\sigma'$ by cutting one of her links.

**(iiib)** Take any $\sigma$ such that $g(\sigma) = g^S$, $\#S > n/2$ and $\sigma_l = (0,0,...,0,0)$ for all $l \notin S$. Then, $\sigma$ is not a strong Nash equilibrium of the Myerson's linking game. In $\sigma$, we have $\sigma_{ij} = 1$ and $\sigma_{ji} = 1$ for all $i,j \in S$. There are profitable deviations from $\sigma$ to $\sigma'$ by coalition $S'$, $S' \cap S \neq \emptyset$, such that $g(\sigma') \subseteq g^{S'}$, $\#C(g(\sigma')) = 1$, $\#N_i(g(\sigma')) = \#N_j(g(\sigma')) < \#S - 1$ for all $i,j \in S'$, and $\#S' > n/2$. (a) If $S' = S$ then there is $\{i,j\} \subsetneq S'$ and $\sigma''$ with $\sigma''_k = \sigma'_k$ for all $k \neq i,j$, $\sigma''_i = \sigma'_i$ except that $\sigma'_{ij} = 0$ while $\sigma''_{ij} = 1$, $\sigma''_j = \sigma'_j$ except that $\sigma'_{ji} = 0$ while $\sigma''_{ji} = 1$ such that $U_i(\sigma') = u_i(g(\sigma')) < U_i(\sigma'') = u_i(g(\sigma''))$ and $U_j(\sigma') = u_j(g(\sigma')) < U_j(\sigma'') = u_j(g(\sigma''))$. The deviation from $\sigma'$ to $\sigma''$ is self-enforcing since no player belonging to $\{i,j\}$ has an incentive to deviate from $\sigma''$ by cutting one of her links. Hence, the first deviation by $S'$ from $\sigma$ to $\sigma'$ is not self-enforcing and $\sigma$ is a coalition-proof Nash equilibrium of the Myerson's linking game. (b) If $S' \neq S$ then there is $i \in S' \cap S$ and $j \in S \setminus S'$ and $\sigma''$ with $\sigma''_k = \sigma'_k$ for all $k \neq i,j$, $\sigma''_i = \sigma'_i$ except that $\sigma'_{ij} = 0$ while $\sigma''_{ij} = 1$, $\sigma''_j = \sigma'_j = \sigma_j$ with $\sigma_{ji} = 1$ such that $U_i(\sigma') = u_i(g(\sigma')) < U_i(\sigma'') = u_i(g(\sigma''))$. The deviation from $\sigma'$ to $\sigma''$ is self-enforcing since it involves only player $i$ and she has an incentive to deviate from $\sigma'$ by linking to player $j$. Hence, the first deviation by $S'$ from $\sigma$ to $\sigma'$ is not self-enforcing and $\sigma$ is a coalition-proof Nash equilibrium of the Myerson's linking game. So, any $\sigma$ such that $g(\sigma) = g^S$, $\#S > n/2$ and $\sigma_l = (0,0,...,0,0)$ for all $l \notin S$ is a coalition-proof Nash equilibrium of the Myerson's linking game.

**(ivb)** Take $\sigma$ such $g(\sigma) = g^{\emptyset}$. There is a deviation from $\sigma$ to $\sigma'$ such that $g(\sigma') = g^N$ by the grand coalition. Hence, $\sigma$ is not a strong Nash equilibrium of the Myerson's linking game. Moreover, this deviation is self-enforcing since any deviation from $\sigma'$ by any coalition $S \subsetneq N$ is not self-enforcing as shown in **(iiib)**. Hence, $\sigma$ such $g(\sigma) = g^{\emptyset}$ is not a coalition-proof Nash equilibrium of the Myerson's linking game.

So, $\sigma$ is a coalition-proof Nash equilibrium of the Myerson's linking game if and only if $g(\sigma) = g^S$, $\#S > n/2$ and $\sigma_l = (0, 0, ..., 0, 0)$ for all $l \notin S$. □

The contest networks model highlights a drawback of CPNE in the Myerson's linking game. If the deviation by a coalition involves the deletion of links with players outside the coalition, then a single deviating player who has just deleted a link with some player not in the deviating coalition can form again this link in a subsequent deviation without requiring the mutual consent of the other player. CPS overcomes such a drawback by requiring that this player belongs to the deviating coalition in the subsequent deviation. This drawback is the reason why coalition-proof Nash equilibria of Myerson's linking game sustain many more networks and less reasonable ones.

The above contest networks model seems to suggest that CPNE would be a coarsening of CPS. However, the next example shows that there is no relationship between both concepts. In Figure 3.4 we depict some networks and their payoffs for an example with four players. For all other network configurations, the four players get a payoff of $-10$. Solving this example we get that $g_1 \in$ **CPNE** and $g_0 \notin$ **CPNE** while $g_1 \notin$ **CPS** and $g_0 \in$ **CPS**. Intuitively, the group deviation by $\{1, 3\}$ from $\sigma^*$ where $\sigma^*_{12} = 0, \sigma^*_{13} = 1, \sigma^*_{14} = 1$, $\sigma^*_{2k} = 0, k = 1, 3, 4$, $\sigma^*_{31} = 1, \sigma^*_{32} = 0, \sigma^*_{34} = 1$, $\sigma^*_{41} = 1, \sigma^*_{42} = 0, \sigma^*_{43} = 1$ (with $g(\sigma^*) = g_1$) to $\sigma'$ where $\sigma'_{12} = 0, \sigma'_{13} = 1, \sigma'_{14} = 0$, $\sigma'_{2k} = 0, k = 1, 3, 4$, $\sigma'_{31} = 1, \sigma'_{32} = 0, \sigma'_{34} = 0$, $\sigma'_{41} = 1, \sigma'_{42} = 0, \sigma'_{43} = 1$ (with $g(\sigma') = g_2$) is not self-enforcing. Given $\sigma'$, player 3 has incentives to switch from $\sigma'_{31} = 1, \sigma'_{32} = 0, \sigma'_{34} = 0$ to $\sigma''_{31} = 0, \sigma''_{32} = 0, \sigma''_{34} = 1$ with $g(\sigma'') = g_3$. Hence, the group deviation from $\sigma$ where $\sigma_{12} = 1, \sigma_{13} = 0, \sigma_{14} = 0$, $\sigma_{21} = 1, \sigma_{23} = 0, \sigma_{24} = 1$, $\sigma_{3k} = 0, k = 1, 2, 4$, $\sigma_{41} = 0, \sigma_{42} = 1, \sigma_{43} = 0$ (with $g(\sigma) = g_0$) to $\sigma^*$ where $\sigma^*_{12} = 0, \sigma^*_{13} = 1, \sigma^*_{14} = 1$, $\sigma^*_{2k} = 0, k = 1, 3, 4$, $\sigma^*_{31} = 1, \sigma^*_{32} = 0, \sigma^*_{34} = 1$, $\sigma^*_{41} = 1, \sigma^*_{42} = 0, \sigma^*_{43} = 1$ (with $g(\sigma^*) = g_1$) becomes self-enforcing and so $g_0 \notin$ **CPNE** while $g_1 \in$ **CPNE**. But, the group deviation by $\{1, 3\}$ from $g_1$ to $g_2$ is credible. At $g_2$ neither $\{1\}$ nor $\{3\}$ has a deviation alone. Thus, the group deviation by $\{1, 3, 4\}$ from $g_0$ to $g_1$ is not credible since $\{1, 3\} \subsetneq \{1, 3, 4\}$ has a credible group deviation from $g_1$ to $g_2$. Hence, $g_0 \in$ **CPS** while $g_1 \notin$ **CPS**.

## 3.5 Existence and efficiency

Similarly to SS, a CPS network may fail to exist. Take Jackson and Wolinsky's (1996) coauthor model with three players. Payoffs for each possible network are given in Figure 3.5. The complete network $g_7$ is the unique pairwise stable network[8]

[8]From the exchange networks example and the coauthor example we observe that there is no relationship between pairwise stability and coalition-proof stability.

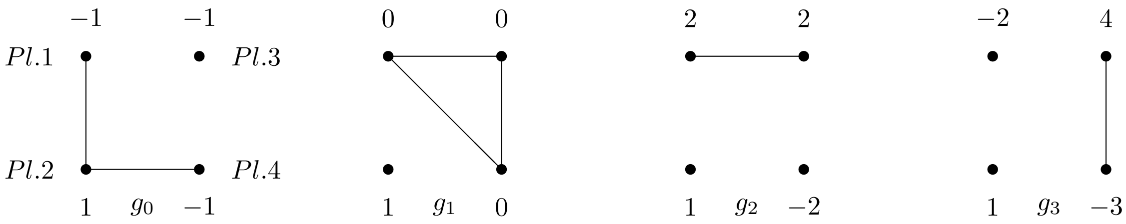


Figure 3.4: No relationship between CPS and CPNE.

but is not strongly stable since a coalition of players $\{i, j\}$ has a group deviation to the network $\{ij\}$ where they both get a payoff of 3 instead of 2.5. Moreover, this group deviation is credible since none of the deviating players has an incentive to cut the link afterwards. Consider now the group deviation by $\{i, j\}$ from $\{ik, kj\}$ to $\{ij, ik, kj\}$. This deviation is credible since neither $\{i\}$ nor $\{j\}$ has a deviation at $\{ij, ik, kj\}$. A similar reasoning holds for the group deviation by $\{i, j\}$ from $\{ik\}$ to $\{ij, ik\}$ and from $g^{\emptyset}$ to $\{ij\}$. Hence, there is no CPS network in the coauthor model with three players.

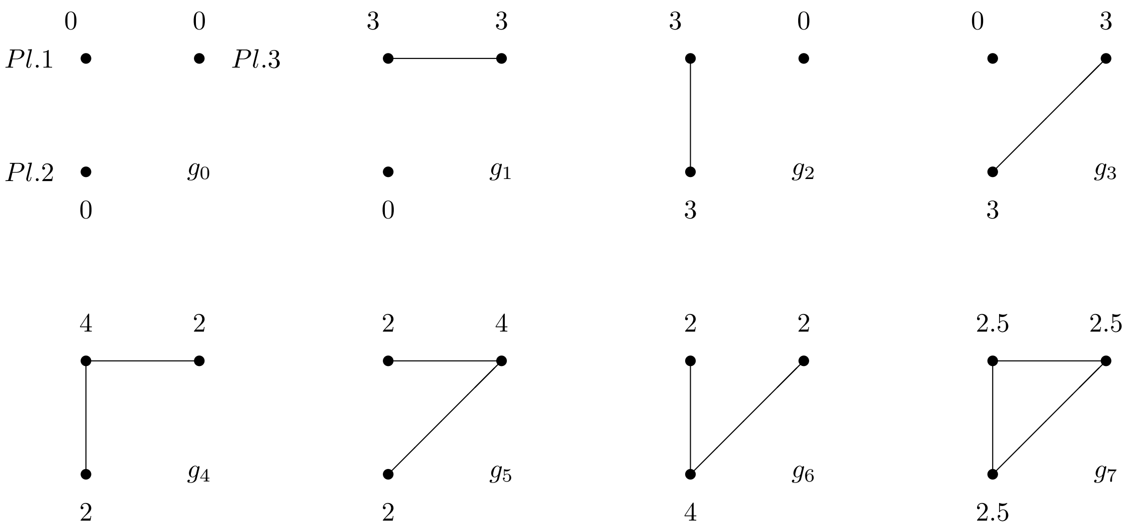


Figure 3.5: The co-author model with three players.

We now look for conditions on the utility function such that the existence of CPS or SS is guaranteed. Let

$$g(S) = \left\{ g \subseteq g^S, g \neq \varnothing \,\middle|\, \frac{\sum_{i \in N(g)} u_i(g)}{\#N(g)} \geq \frac{\sum_{i \in N(g')} u_i(g')}{\#N(g')} \; \forall g' \subseteq g^S,\, g' \neq \emptyset \right\}$$

be the set of networks with the highest average payoff out of those that can be formed by players in $S \subseteq N$. Suppose that $u$ is a 0-normalized componentwise egalitarian utility function such that (i) players belonging to the same component get the same strictly positive utility; (ii) players not belonging to any component get the zero utility; (iii) there are no externalities across components (i.e. payoffs of players belonging to a component in a given network do not depend on the structure of other components). Given a componentwise egalitarian utility function $u$ such that (i) $u_i(g) = u_j(g) \geq 0$ for all $i, j \in S \in \Pi(g)$;(ii) $u_i(g) = 0$ if there exists $S \in \Pi(g)$ such that $\#S = 1, i \in S$; (iii) $u_i(g) = u_i(h)$ with $h \in C(g)$ and $i \in N(h)$, find a network $\widehat{g}$ through the following algorithm due to Banerjee (1999). Pick some $h_1 \in g(N)$. Next, pick some $h_2 \in g(N \setminus N(h_1))$. At stage $k$ pick some $h_k \in g(N \setminus \cup_{l \leq k-1} N(h_l))$. Since $N$ is finite this process stops after a finite number $K$ of stages. The union of the components picked in this way defines a network $\widehat{g}$. We denote by $\widehat{G}$ the set of all networks that can be found through this algorithm.[9]

**Proposition 3.4.** *Take any 0-normalized componentwise egalitarian utility function $u$ such that* (**i**) $u_i(g) = u_j(g) \geq 0$ *for all* $i, j \in S \in \Pi(g)$; (**ii**) $u_i(g) = 0$ *if there exists* $S \in \Pi(g)$ *such that* $\#S = 1, i \in S$; (**iii**) $u_i(g) = u_i(h)$ *with* $h \in C(g)$ *and* $i \in N(h)$. *We have* **CPS** = **SS** = $\widehat{G}$.

*Proof.* (**i**) Take any $g \in \widehat{G}$ where $g = \cup_{k=1}^{K} h_k$ with $h_k \in g(N \setminus \cup_{l \leq k-1} N(h_l))$. Players belonging to $N(h_1)$ in $g$ will never engage in a group deviation since they can never be (strictly) better off than in $g$. Players belonging to $N(h_2)$ in $g$ will only engage in a group deviation if they can end up in some $h$ such that $u_i(h) > u_i(h_2)$. Suppose there exists some $h$ such that $u_i(h) > u_i(h_2)$. Since $h_2 \in g(N \setminus N(h_1))$ it follows that $N(h) \cap N(h_1) \neq \emptyset$. Given that players in $N(h_1)$ will never engage in a group deviation, players belonging to $N(h_2)$ can never end up (strictly) better off than in $g$. So, players belonging to $N(h_2)$ in $g$ will never engage in a group deviation. Players belonging to $N(h_k)$ in $g$ will only engage in a group deviation if they can end up in some $h$ such that $u_i(h) > u_i(h_k)$. Suppose there exists some $h$ such that $u_i(h) > u_i(h_k)$. Since $h_k \in g(N \setminus \cup_{l \leq k-1} N(h_l))$ it follows that $N(h) \cap \{\cup_{l \leq k-1} N(h_l)\} \neq \emptyset$. Given that players in $\cup_{l \leq k-1} N(h_l)$ will never engage in a group deviation, players belonging to $N(h_k)$ can never end up (strictly) better off than in $g$. So, players belonging to $N(h_k)$ in $g$ will never engage in a group deviation; and so on. Thus, **SS** $\supseteq \widehat{G}$ and **CPS** $\supseteq \widehat{G}$.

(**ii**) Take any $g' \notin \widehat{G}$. We show that there always exist a credible group deviation from $g'$.

(**Step** 1.) If there exists some $h_1 \in g(N)$ such that $h_1 \in C(g')$ then go to Step 2. Otherwise, pick some $h_1 \in g(N)$. In $g'$ all players are strictly worse off than

[9]More than one network may be picked up through this algorithm since players may be permuted or even be indifferent between components of different sizes.

the players belonging to $N(h_1)$. Then, we have that all members of $N(h_1)$ have a group deviation from $g'$ to $g'' = g'|_{N \setminus N(h_1)} \cup h_1$. Indeed, players who belong to $N(h_1)$ delete their links in $g'$ with players not in $N(h_1)$ and build the missing links of $h_1$. So, $g' \notin$ **SS**. Since $h_1 \in g(N)$, it is a credible group deviation. Indeed, there is no $S \subset N(h_1)$ that has a group deviation at $g'' = g'|_{N \setminus N(h_1)} \cup h_1$. So, $g' \notin$ **CPS**.

**(Step** 2.**)** If there exists some $h_2 \in g(N \setminus N(h_1))$ such that $h_2 \in C(g')$ then go to Step 3. Otherwise, pick some $h_2 \in g(N \setminus N(h_1))$. In $g'$ all the remaining players who are belonging to $N \setminus N(h_1)$ are strictly worse off than the players belonging to $N(h_2)$. Then, we have that all members of $N(h_2)$ have a group deviation from $g'$ to $g'' = g'|_{N \setminus N(h_2)} \cup h_2$. Indeed, players who belong to $N(h_2)$ delete their links in $g'$ with players not in $N(h_2)$ and build the missing links of $h_2$. So, $g' \notin$ **SS**. Since $h_2 \in g(N \setminus N(h_1))$, it is a credible group deviation. Indeed, there is no $S \subset N(h_2)$ that has a group deviation at $g'' = g'|_{N \setminus N(h_2)} \cup h_2$. So, $g' \notin$ **CPS**.

**(Step** $k$.**)** If there exists some $h_k \in g(N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$ such that $h_k \in C(g')$ then go to Step $k+1$. Otherwise, pick some $h_k \in g(N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$. In $g'$ all the remaining players who are belonging to $N \setminus \{N(h_1) \cup ... \cup N(k-1)\}$ are strictly worse off than the players belonging to $N(h_k)$. Then, we have that all members of $N(h_k)$ have a group deviation from $g'$ to $g'' = g'|_{N \setminus N(h_k)} \cup h_k$. Indeed, players who belong to $N(h_k)$ delete their links in $g'$ with players not in $N(h_k)$ and build the missing links of $h_k$. So, $g' \notin$ **SS**. Since $h_k \in g(N \setminus \{N(h_1) \cup ... \cup N(k-1)\}$, it is a credible group deviation. Indeed, there is no $S \subset N(h_k)$ that has a group deviation at $g'' = g'|_{N \setminus N(h_k)} \cup h_k$. So, $g' \notin$ **CPS**.

**(Step** $K$.**)** Pick some $h_K \in g(N \setminus \{N(h_1) \cup ... \cup N(K-1)\})$. In $g'$ all the remaining players who are belonging to $N \setminus \{N(h_1) \cup ... \cup N(K-1)\}$ are strictly worse off than the players belonging to $N(h_K)$. Then, we have that all members of $N(h_K)$ have a group deviation from $g'$ to $g'' = g'|_{N \setminus N(h_K)} \cup h_K$. Indeed, players who belong to $N(h_K)$ delete their links in $g'$ with players not in $N(h_K)$ and build the missing links of $h_K$. So, $g' \notin$ **SS**. Since $h_K \in g(N \setminus \{N(h_1) \cup ... \cup N(K-1)\}$, it is a credible group deviation. Indeed, there is no $S \subset N(h_K)$ that has a group deviation at $g'' = g'|_{N \setminus N(h_K)} \cup h_K$. So, $g' \notin$ **CPS**.

Thus, $g' \notin \widehat{G} \Rightarrow g' \notin$ **CPS** and $g' \notin \widehat{G} \Rightarrow g' \notin$ **SS**. It then follows from **(i)** that **SS** $= \widehat{G}$ and **CPS** $= \widehat{G}$. □

The network utility function $u$ is top convex if some strongly efficient network maximizes the per-capita sum of utilities among players. Let $\rho(u, S) = \max_{g \subseteq g^S} \sum_{i \in S} u_i(g)/\#S$. The network utility function $u$ is top convex if $\rho(u, N) \geq \rho(u, S)$ for all $S \subseteq N$. Suppose again that $u$ is such that (i) players belonging to the same component get the same utility and (ii) there are no externalities across components. If $u$ is also top convex then both strong stability and coalition-proof stability single out the strongly efficient networks, independently of strict or weak group deviations.

**Proposition 3.5.** *Take any componentwise egalitarian utility function $u$ such that* (**i**) $u_i(g) = u_j(g)$ *for all* $i, j \in S \in \Pi(g)$ *and* (**ii**) $u_i(g) = u_i(h)$ *with* $h \in C(g)$ *and* $i \in N(h)$. *If $u$ is top convex, then* $\mathbf{CPS} = \mathbf{SS} = E$.

*Proof.* Top convexity of $u$ implies that all components of a strongly efficient network must lead to the same per-capita sum of utilities (if some component led to a lower per-capita sum of utilities than the average, then another component would have to lead to a higher per-capita sum of utilities than the average which would contradict top convexity). Top convexity also implies that under a componentwise egalitarian utility function any $g \in E$ Pareto dominates all $g' \notin E$. Then, it is immediate that $E \subseteq \mathbf{SS}$ and $E \subseteq \mathbf{CPS}$, and $\{g'\} \cap \mathbf{SS} = \emptyset$ and $\{g'\} \cap \mathbf{CPS} = \emptyset$ for all $g' \in \mathcal{G} \setminus E$. Hence, $\mathbf{CPS} = \mathbf{SS} = E$. □

Grandjean, Mauleon, and Vannetelbosch (2011) show that when players are farsighted, the set of strongly efficient networks is the unique pairwise farsightedly stable set if and only if $u$ is top convex. So, strong stability or coalition-proof stability selects the networks that are stable when players are farsighted if $u$ is top convex.

## 3.6 Strict versus weak group deviations

Two different notions of a group deviation or move can be found in the game-theoretic literature. Up to now, we have considered (strict) group deviations where a group of players deviates only if each of its members can be made (strictly) better off. Alternatively, we could look at weak group deviations where a group of players deviates only if at least one of its members is (strictly) better off while all other members are at least as well off. Weak group deviations make sense when very small transfers among the deviating group of players are allowed.

**Definition 3.7.** Coalition $S \subseteq N$ is said to have a weak group deviation from $g$ to $g'$ if

(**i**) $ij \in g'$ and $ij \notin g \Rightarrow \{i, j\} \subseteq S$,

(**ii**) $ij \in g$ and $ij \notin g' \Rightarrow \{i, j\} \cap S \neq \emptyset$,

(**iii**) $u_i(g') \geq u_i(g)$ for all $i \in S$ and there is $j \in S$ such that $u_j(g') > u_j(g)$.

A coalition $S$ is said to have a weak group deviation from the network $g$ to the network $g'$ if three conditions are satisfied. Condition (i) requires that any new links that are added can only be between players inside $S$. Condition (ii) requires that there must be at least one player belonging to $S$ for the deletion of a link. Condition (iii) requires that some members of $S$ are better off and other members of $S$ are at least as well off.

**Definition 3.8** (Jackson and van den Nouweland, 2005)**.** A network $g$ is w-strongly stable if there exists no coalition $S \subseteq N$ which has a weak group deviation from $g$.

Let **wSS** be the set of w-strongly stable networks. It corresponds to Jackson and van den Nouweland (2005) set of strongly stable networks. Obviously, **wSS** $\subseteq$ **SS**.

**Definition 3.9.** Coalition $S \subseteq N$ is said to have a credible weak group deviation from $g$ if

**(i)** $g'$ is a weak group deviation from $g$ by $S$, and

**(ii)** there exists no subcoalition $T \subset S$ which has a weak credible group deviation from $g'$.

**Definition 3.10.** A network $g$ is w-coalition-proof stable if there exists no coalition $S \subseteq N$ which has a weak credible group deviation from $g$.

Let **wCPS** be the set of w-coalition-proof stable networks. The next two examples show that there is no relationship between wCPS and CPS whereas wSS is a refinement of SS. Take $N = \{1, 2\}$ with $u_1(g^\emptyset) = u_2(g^\emptyset) = 0$, $u_1(\{12\}) = 0$ and $u_2(\{12\}) = 1$. Then, **wCPS** $= \{\{12\}\}$ while **CPS** $= \{g^\emptyset, \{12\}\}$. In the example of Figure 3.6, we get **wCPS** $= \{g_0, g_7\}$ while **CPS** $= \{g_4, g_7\}$. The network $g_0$ is coalition-proof stable under weak group deviations but not under (strict) group deviations. The only profitable deviation from $g_0$ is to $g_4$ and it involves all players. But, under weak group deviations, this deviation is not credible since at $g_4$ players 2 and 3 have incentives to move to $g_7$. Hence, $g_0$ is coalition-proof stable under weak group deviations. However, at $g_4$ player 3 would block the deviation to $g_7$ under (strict) group deviations. Hence, the deviation from $g_0$ to $g_4$ is credible and $g_0$ is not coalition-proof stable under (strict) group deviations.

We now provide a condition on the utility function such that **wCPS** $=$ **CPS**. Let $L_i(g) = \{jk \in g \mid j = i \text{ or } k = i\}$ be the set of player $i$'s links and $L_i(g^N \setminus g) = \{ij \in g^N \mid j \neq i \text{ and } ij \notin g\}$ be the set of player $i$'s links not in $g$. So, $ij \notin g$ is equivalent to $ij \in L_i(g^N \setminus g)$. Ilkiliç and Ikizler (2019) introduce the property of link-responsiveness. Under link-responsiveness, no player is indifferent to a change

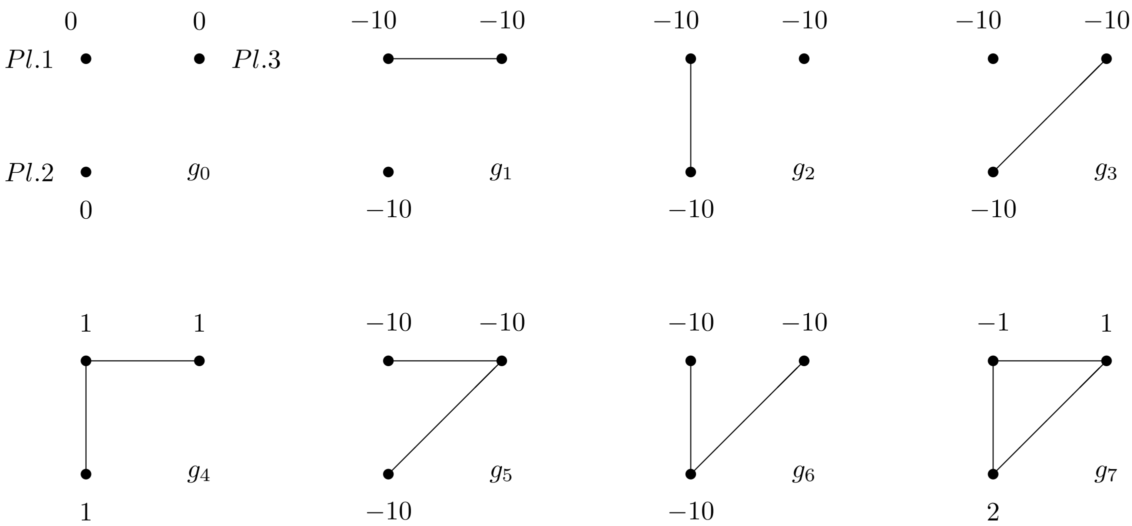


Figure 3.6: No relationship between wCPS and CPS.

in her set of links. Formally, the network utility function $u$ is link-responsive on $g$ if and only if we have $u_i(g + l' - l) \neq u_i(g)$, for all $i \in N$, and for all $l \subseteq L_i(g)$ and $l' \in L_i(g^N \setminus g)$ such that $g + l' - l \neq g$.

**Proposition 3.6.** *Take any link-responsive $u$. We have* **wSS** = **SS** *and* **wCPS** = **CPS**.

*Proof.* We first show that $S \subseteq N$ has a weak group deviation from $g$ to $g'$ if and only if $S \subseteq N$ has a (strict) group deviation from $g$ to $g'$. ($\Leftarrow$) If $S \subseteq N$ has a (strict) group deviation from $g$ to $g'$, $S \subseteq N$ has obviously a weak group deviation from $g$ to $g'$ (independently of link-responsiveness). ($\Rightarrow$) Suppose that $S \subseteq N$ has a weak group deviation from $g$ to $g'$. We have that (i) $ij \in g'$ and $ij \notin g \Rightarrow \{i, j\} \subseteq S$, (ii) $ij \in g$ and $ij \notin g' \Rightarrow \{i, j\} \cap S \neq \emptyset$, (iii) $u_i(g') \geq u_i(g)$ for all $i \in S$ and there is $j \in S$ such that $u_j(g') > u_j(g)$. (i) and (ii) implies that $L_i(g) \neq L_i(g')$ for all $i \in S$. By link-responsiveness, we have $u_i(g') \neq u_i(g)$ for all $i \in S$. Thus, $u_i(g') > u_i(g)$ for all $i \in S$ and $S \subseteq N$ has a (strict) group deviation from $g$ to $g'$. Hence, **wSS** = **SS**. From Definition 3.3 and Definition 3.9 it follows that $S \subseteq N$ has a credible weak group deviation from $g$ to $g'$ if and only if $S \subseteq N$ has a credible (strict) group deviation from $g$ to $g'$. Hence, **wCPS** = **CPS**. □

## 3.7 Coalition-proof farsightedly stable networks

There are situations where only pairwise deviations are feasible. Pairwise deviations involve a single link at a time: link addition is bilateral, link deletion is unilateral and network changes take place one link at a time. In such situations farsighted players may look beyond the immediate consequence of adding or deleting a link and anticipate the subsequent changes that will occur afterwards.[10] One raising question is whether or when coalition-proof stability with farsighted players but restricted to pairwise deviations is equivalent to coalition-proof stability with group deviations.

**Definition 3.11.** A farsighted improving path from a network $g$ to a network $g'$ for a coalition $S \subseteq N$ is a finite sequence of networks $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$ either

**(i)** $g_{k+1} = g_k - ij$ for some $ij$ such that $U_i(g_K) > U_i(g_k)$ and $i \in S$ or $U_j(g_K) > U_j(g_k)$ and $j \in S$; or

**(ii)** $g_{k+1} = g_k + ij$ for some $ij$ such that $U_i(g_K) > U_i(g_k)$, $U_j(g_K) > U_j(g_k)$ and $i, j \in S$.

If there exists a farsighted improving path from a network $g$ to a network $g'$ for a given coalition $S \subseteq N$, then we write $g \to_S g'$. The set of all networks that can be reached from a network $g \in \mathcal{G}$ for a given coalition $S \subseteq N$ by a farsighted improving path is denoted by $\phi_S(g)$, $\phi_S(g) = \{g' \in \mathcal{G} \mid g \to_S g'\}$.

**Definition 3.12.** Coalition $S \subseteq N$ is said to have a farsighted deviation from $g$ to $g'$ if $g' \in \phi_S(g)$.

**Definition 3.13.** A network $g$ is farsightedly stable if there exists no coalition $S \subseteq N$ which has a farsighted deviation from $g$.

**Definition 3.14.** Coalition $S \subseteq N$ is said to have a credible farsighted deviation from $g$ if

**(i)** $g'$ is a farsighted deviation from $g$ by $S$ (i.e. $g' \in \phi_S(g)$), and

**(ii)** there exists no subcoalition $T \subset S$ which has a credible farsighted deviation from $g'$.

**Definition 3.15.** A network $g$ is coalition-proof farsightedly stable if there exists no coalition $S \subseteq N$ which has a credible farsighted deviation from $g$.

Let **CPFS** be the set of coalition-proof farsightedly stable networks and let **FS** be the set of farsightedly stable networks. We now show that **CPFS** and **FS** coincide under the componentwise egalitarian utility function.

[10] Alternative notions of farsightedness for network formation are suggested by Dutta, Ghosal and Ray (2005), Herings, Mauleon and Vannetelbosch (2009, 2019), Page and Wooders (2009) among others.

**Proposition 3.7.** *Take any componentwise egalitarian utility function $u$ such that* (**i**) $u_i(g) = u_j(g)$ *for all* $i, j \in S \in \Pi(g)$ *and* (**ii**) $u_i(g) = u_i(h)$ *with* $h \in C(g)$ *and* $i \in N(h)$. *We have* **CPFS** $=$ **FS** $= \widehat{G}$.

*Proof.* (**i**) Take any $g \in \widehat{G}$ where $g = \cup_{k=1}^{K} h_k$ with $h_k \in g(N \setminus \cup_{l \leq k-1} N(h_l))$. Players belonging to $N(h_1)$ in $g$ who are looking forward will never engage in a move since they can never be strictly better off than in $g$ given the componentwise egalitarian utility function $u$. Players belonging to $N(h_2)$ in $g$ who are forward looking will only engage in a move if they can end up in some $h$ such that $u_i(h) > u_i(h_2)$. Suppose there exists some $h$ such that $u_i(h) > u_i(h_2)$. Since $h_2 \in g\,(N \setminus N(h_1))$ it follows that $N(h) \cap N(h_1) \neq \emptyset$. Given that players in $N(h_1)$ will never engage in a move, players belonging to $N(h_2)$ can never end up strictly better off than in $g$ under the componentwise egalitarian utility function $u$. So, players belonging to $N(h_2)$ in $g$ will never engage in a move. Players belonging to $N(h_k)$ in $g$ who are forward looking will only engage in a move if they can end up in some $h$ such that $u_i(h) > u_i(h_k)$. Suppose there exists some $h$ such that $u_i(h) > u_i(h_k)$. Since $h_k \in g\,(N \setminus \cup_{l \leq k-1} N(h_l))$ it follows that $N(h) \cap \{\cup_{l \leq k-1} N(h_l)\} \neq \emptyset$. Given that players in $\cup_{l \leq k-1} N(h_l)$ will never engage in a move, players belonging to $N(h_k)$ can never end up strictly better off than in $g$ under the componentwise egalitarian utility function $u$. So, players belonging to $N(h_k)$ in $g$ will never engage in a move; and so on. Thus, $\phi_S(g) = \emptyset$ for all $S \subseteq N$. Hence, **FS** $\supseteq \widehat{G}$ and **CPFS** $\supseteq \widehat{G}$.
(**ii**) Take any $g' \notin \widehat{G}$. We show that there always exist a credible farsighted deviation from $g'$ to some $g \in \widehat{G}$.

(**Step** 1.) If there exists some $h_1 \in g(N)$ such that $h_1 \in C(g')$ then go to Step 2 with $g_1 = g'$. Otherwise, two cases have to be considered. (**A**) There exists $h \in C(g')$ such that $h_1 \subsetneq h$ for some $h_1 \in g(N)$. Then, take $h_1 \in g(N)$ such that there does not exist $h_1' \in g(N)$ with $h_1 \subsetneq h_1' \subsetneq h$. From $g'$, let the players who belong to $N(h_1)$ and who look forward to $g \in \widehat{G}$ delete successively their links that are not in $h_1$ to reach $g_1 = g' - \{ij \mid i \in N(h_1) \text{ and } ij \notin h_1\}$. Along the sequence from $g'$ to $g_1$ all players who are moving always prefer the end network $g$ to the current network. (**B**) There does not exist $h \in C(g')$ such that $h_1 \subsetneq h$ with $h_1 \in g(N)$. Pick $h_1 \in g(N)$ such that there does not exist $h_1' \in g(N)$ with $h_1' \subsetneq h_1$. From $g'$, let the players who belong to $N(h_1)$ and who are looking forward to $g \in \widehat{G}$ such that $h_1 \in C(g)$ first delete successively their links not in $h_1$ and then build successively the links in $h_1$ that are not in $g'$ leading to $g_1 = g' - \{ij \mid i \in N(h_1) \text{ and } ij \notin h_1\} + \{ij \mid i \in N(h_1),\ ij \in h_1 \text{ and } ij \notin g'\}$. Along the sequence from $g'$ to $g_1$ all players who are moving always prefer the end network $g$ to the current network. Once $g_1$ and $h_1$ are formed, we move to Step 2.

**(Step** 2.**)** If there exists some $h_2 \in g(N \setminus N(h_1))$ such that $h_2 \in C(g_1)$ then go to Step 3 with $g_2 = g_1$. Otherwise, two cases have to be considered. (**A**) There exists $h \in C(g')$ such that $h_2 \subsetneq h$ for some $h_2 \in g(N \setminus N(h_1))$. Then, take $h_2 \in g(N \setminus N(h_1))$ such that there does not exist $h_2' \in g(N \setminus N(h_1))$ with $h_2 \subsetneq h_2' \subsetneq h$. From $g_1$ let the players who belong to $N(h_2)$ and who look forward to $g \in \widehat{G}$ such that $h_1 \in C(g)$ and $h_2 \in C(g)$ delete successively all their links that are not in $h_2$ to reach $g_2 = g_1 - \{ij \mid i \in N(h_2) \text{ and } ij \notin h_2\}$. Along the sequence from $g_1$ to $g_2$ all players who are moving always prefer the end network $g$ to the current network. (**B**) There does not exist $h \in C(g')$ such that $h_2 \subsetneq h$ with $h_2 \in g(N \setminus N(h_1))$. Pick $h_2 \in g(N \setminus N(h_1))$ such that there does not exist $h_2' \in g(N \setminus N(h_1))$ with $h_2' \subsetneq h_2$. From $g_1$ let the players who belong to $N(h_2)$ and who are looking forward to $g \in \widehat{G}$ such that $h_1 \in C(g)$ and $h_2 \in C(g)$ first delete successively their links not in $h_2$ and then build successively the links in $h_2$ that are not in $g_1$ leading to $g_2 = g_1 - \{ij \mid i \in N(h_2) \text{ and } ij \notin h_2\} + \{ij \mid i \in N(h_2),\ ij \in h_2 \text{ and } ij \notin g_1\}$. Along the sequence from $g_1$ to $g_2$ all players who are moving always prefer the end network $g$ to the current network. Once $g_2$ and $h_2$ are formed, we move to Step 3.

**(Step** $k$.**)** If there exists some $h_k \in g(N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$ such that $h_k \in C(g_{k-1})$ then go to Step $k+1$ with $g_k = g_{k-1}$. Otherwise, two cases have to be considered. (**A**) There exists $h \in C(g')$ such that $h_k \subsetneq h$ for some $h_k \in g(N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$. Then, take $h_k \in g(N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$ such that there does not exist $h_k' \in g(N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$ with $h_k \subsetneq h_k' \subsetneq h$. From $g_{k-1}$ let the players who belong to $N(h_k)$ and who look forward to $g \in \widehat{G}$ such that $h_1 \in C(g)$, $h_2 \in C(g)$, ..., $h_k \in C(g)$ delete successively their links not in $h_k$ to reach $g_k = g_{k-1} - \{ij \mid i \in N(h_k) \text{ and } ij \notin h_k\}$. Along the sequence from $g_{k-1}$ to $g_k$ all players who are moving always prefer the end network $g$ to the current network. (**B**) There does not exist $h \in C(g')$ such that $h_k \subsetneq h$ with $h_k \in g(N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$. Pick $h_k \in g(N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$ such that there does not exist $h_k' \in g(N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$ with $h_k' \subsetneq h_k$. From $g_{k-1}$ let the players who belong to $N(h_k)$ and who are looking forward to $g \in \widehat{G}$ such that $h_1 \in C(g)$, $h_2 \in C(g)$, ..., $h_k \in C(g)$ first delete successively their links not in $h_k$ and then build successively the links in $h_k$ that are not in $g_{k-1}$ leading to $g_k = g_{k-1} - \{ij \mid i \in N(h_k) \text{ and } ij \notin h_k\} + \{ij \mid i \in N(h_k),\ ij \in h_k \text{ and } ij \notin g_{k-1}\}$. Along the sequence from $g_{k-1}$ to $g_k$ all players who are moving always prefer the end network $g$ to the current network. Once $g_k$ and $h_k$ are formed, we move to Step $k+1$; and so on until we reach the network $g = \bigcup_{k=1}^{K} h_k$ with $h_k \in g(N \setminus \cup_{i \leq k-1} N(h_i))$.

Thus, we have build a farsightedly improving path from $g'$ to $g$. That is, $g \in \phi_S(g')$ for some $S \subseteq N$. Since $\phi_S(g) = \emptyset$ for all $S \subseteq N$, for all $g \in \widehat{G}$, there is no farsighted deviation from $g$. Hence, the farsighted deviation from $g' \notin \widehat{G}$ to $g \in \widehat{G}$ is credible. Thus, $g' \notin \widehat{G} \Rightarrow g' \notin$ **CPFS** and $g' \notin \widehat{G} \Rightarrow g' \notin$ **FS**. It then follows from **(i)** that **FS** $= \widehat{G}$ and **CPFS** $= \widehat{G}$ under any componentwise egalitarian utility function. □

Combining Proposition 3.4 with Proposition 3.7 we have that **CPFS** = **CPS** under any componentwise egalitarian utility function.

**Corollary 3.1.** Take any componentwise egalitarian utility function $u$ such that (**i**) $u_i(g) = u_j(g)$ for all $i, j \in S \in \Pi(g)$ and (**ii**) $u_i(g) = u_i(h)$ with $h \in C(g)$ and $i \in N(h)$. We have **CPFS** = **CPS**.

## 3.8 Conclusion

We have proposed the notion of coalition-proof stability for predicting the networks that could emerge when group deviations are allowed. A network is coalition-proof stable if there exists no coalition which has a credible group deviation. A coalition is said to have a credible group deviation if there is a profitable group deviation to some network and there is no subcoalition of the deviating players which has a subsequent credible group deviation. Obviously, coalition-proof stability is a coarsening of strong stability. But, there is no relationship between the set of coalition-proof stable networks and the set of networks induced by a coalition-proof Nash equilibrium of Myerson's linking game. Contrary to coalition-proof stability, coalition-proof Nash equilibria of Myerson's linking game often support unreasonable networks.

The concept of coalition-proof stability could be useful in the study of the formation of a network of bilateral free trade agreements. Goyal and Joshi (2006) show that global free trade, represented by the complete network, is pairwise stable, implying that global free trade, if reached, will prevail. However, the complete network is not the unique pairwise stable network. Is global free trade strongly stable or coalition-proof stable? Can global free trade be obtained from the empty network or any preexisting free trade network through coordination among some group of countries?[11]

[11] Mauleon, Song, and Vannetelbosch (2010) find that the asymmetry consisting of having unionized and non-unionized countries could impede the formation of the global free trade network. Recently, Zhang, Xue, and Zu (2013) complements the analysis of Goyal and Joshi (2006) by examining whether global free trade can result from a sequence of bilateral free trade agreements when countries are farsighted.

# Chapter 4

# Segregation and Assimilation in Friendship Networks with Myopic and Farsighted Agents

Joint work with Ana Mauleon and Vincent Vannetelbosch

**Abstract**

We reconsider de Marti and Zenou's (2017) model of friendship network formation where individuals belong to two different communities. Benefits from direct and indirect connections decay with distance while the costs of forming links depend on community memberships. Individuals are now either farsighted or myopic when deciding about the friendship links they want to form. When all individuals are myopic, many inefficient friendship networks (e.g. complete segregation) can arise. When the larger (smaller) community is farsighted while the smaller (larger) community is myopic, the friendship network where the myopic community is assimilated into the farsighted community is the unique stable network when inter-community costs are large. In fact, farsightedness helps the society to avoid ending up segregated. Once inter-community costs are small enough, the complete integration network becomes stable. Finally, when all individuals are farsighted, the friendship network where the smaller community ends up being assimilated into the dominant community is likely to arise.

Keywords: friendship networks; stable sets; myopic and farsighted players; assimilation; segregation.

JEL Classification: A14, C70, D20.

## 4.1 Introduction

Social networks or friendship networks are important in obtaining information on goods and services, like product information or information about job opportunities. Individuals are often regrouped into communities based on their ethnicity, religion, income, education, etc. (see e.g. de Marti and Zenou, 2017; Patacchini and Zenou, 2016). Besides belonging to different communities, individuals often differ in their degree of farsightedness, i.e., their ability to forecast how others will react to the decisions they make. Indeed, recent experiments on network formation provide evidence in favor of a mixed population consisting of both myopic and (limited) farsighted individuals (see Kirchsteiger, Mantovani, Mauleon, and Vannetelbosch, 2016; Teteryatnikova and Tremewan, 2020). The degree of farsightedness or the depth of reasoning is likely to be correlated with other relevant attributes such as education, income, age, etc. (see Mauersberger and Nagel, 2018).

The aim of this paper is to provide a theoretical study of how different degrees of farsightedness will affect the formation of friendship relationships when individuals can belong to various communities.[1] In particular, we are interested in addressing the following set of questions. What are the friendship network structures that may endogenously arise once individuals belonging to two different communities can be either myopic or farsighted in forming links? When do we observe integration, segregation or (partial) assimilation? Does farsightedness help to bridge communities and to more integrated societies? Are farsighted individuals more likely to be linked to others who have different characteristics? How might the network structure change if the dominant community is farsighted while the other one is myopic? Do myopic individuals end up assimilated to the dominant community? Are individual incentives to link adequate from a social welfare point of view? Does it improve efficiency if some individuals become farsighted? And if yes, whom?

To answer these questions we reconsider de Marti and Zenou's (2017) model of network formation where individuals belong to two different communities. Communities may be defined along with social categories such as ethnicity, religion, education, income, etc. In contrast to de Marti and Zenou (2017) where all individuals were myopic, we now allow the possibility of having a mixed population composed of both myopic and farsighted individuals. Myopic or farsighted individuals decide with whom they want to form a link, according to a utility function that weighs the costs and benefits of each connection. Farsighted individuals are able to anticipate

[1] Jackson (2008) and Goyal (2007) provide a comprehensive introduction to the theory of social and economic networks. Mauleon and Vannetelbosch (2016) give an overview of the solution concepts for solving network formation games. In Bramoullé, Galeotti and Rogers (2016), one can find the recent developments on the economics of networks.

that once they add or delete some links, other individuals could add or delete links afterwards. The benefits of a friendship connection decrease with distance in the network, while the cost of a link depends on the type of individuals involved. Two individuals from the same community face a low linking cost, while the cost of forming a friendship relationship between two individuals from different communities decreases with the rate of exposure of each of them to the other community.

We adopt the notion of the myopic-farsighted stable set to determine the friendship networks that emerge when some individuals are myopic while others are farsighted.[2] A myopic-farsighted stable set is the set of networks satisfying internal and external stability with respect to the notion of the myopic-farsighted improving path. That is, a set of networks is a myopic- farsighted stable set if there is no myopic-farsighted improving path between networks within the set and there is a myopic-farsighted improving path from any network outside the set to some network within the set. A myopic-farsighted improving path is simply a sequence of networks that can emerge when farsighted individuals form or delete links based on the improvement the end network offers relative to the current network while myopic individuals form or delete links based on the improvement of the resulting network offers relative to the current network.

When all individuals are myopic, de Marti and Zenou (2017) already show that many friendship networks can be stable. In the case of low intra-community costs, the complete integration is stable when inter-community costs are sufficiently low. For higher inter-community costs, the complete segregation becomes stable. They also point out that some asymmetric network configurations can be stable. For instance, the network in which both communities are fully intra-connected and where there is only one bridge link can be stabilized. In addition, we show that friendship networks where one community is fully or partially assimilated to the other community can also emerge in the long run.

What happens when the population is composed of both myopic and farsighted individuals? Suppose first that all members of one community are farsighted while all members of the other community are myopic. We show that, in the case of low intra-community costs, there is a single friendship network that emerges in the long run when inter-community costs are large enough: the friendship network where the myopic community ends up being assimilated into the farsighted community. Precisely, a singleton set consisting of the network where the myopic community

[2] Herings, Mauleon, and Vannetelbosch (2020) were the first to define the myopic-farsighted stable set for two-sided matching problems. This notion is extended to R&D network formation with pairwise deviations in Mauleon, Sempere-Monerris, and Vannetelbosch (2020) and to general network formation problems in Luo, Mauleon, and Vannetelbosch (2020).

is assimilated into the farsighted community is the unique myopic-farsighted stable set. Farsighted individuals are able to destabilize the complete segregated network by luring the myopic individuals with the prospect of forming a friendship network where the farsighted community is fully assimilated into the myopic community. From such friendship network, farsighted individuals are able to induce a switch towards the opposite fully assimilated network, the friendship network where the myopic community is fully assimilated into the farsighted community, where they achieve their best outcome. When inter-community costs are smaller, the complete integration network becomes again stable whatever the number of farsighted and myopic individuals within the population.

One may wonder if assimilated friendship networks are still stable once individuals from the myopic community become farsighted. We find that, when all the population is farsighted and intra-community costs are low, the friendship network where the smaller community is fully assimilated into the larger or dominant community is likely to emerge in the long run whatever the inter-community costs. However, the opposite fully assimilated network and the complete segregation network are very unlikely to arise. In addition, the complete segregation network is even Pareto-dominated by the friendship network where the smaller community is fully assimilated into the dominant community. In fact, in terms of efficiency, either the complete integration network or the network where the smaller community is fully assimilated into the dominant one is the optimal network structures when intra-community costs are low. Thus, for recovering efficiency, it is better to make individuals belonging to the dominant community farsighted instead of individuals of the smaller community.

In the case of intermediate intra-community costs, many friendship networks are again stable when all individuals are myopic. However, we show that if there are enough farsighted individuals, independently to which community they belong, then a star network with a myopic in the center will arise. In addition, star networks turn to be efficient networks for intermediate intra-community costs. Hence, a mixed population of farsighted and myopic individuals solves the tension between stability and efficiency.

We now turn to the related literature. There is an extensive literature using network models to explain the fact that individuals are more likely to be linked to individuals who have similar characteristics. Currarini, Jackson, and Pin (2009) develop a dynamic random matching model with a population formed by groups of different sizes and show that segregation in social networks results from the decisions of the individuals involved and/or from the ways in which individuals meet and interact. In equilibrium, individuals' behavior is totally homogeneous within the same group of individuals. Bramoullé, Currarini, Jackson, Pin, and Rogers (2012) develop a model of dynamic matching with both random meetings and network-based search. They show that majority and minority groups have different patterns of interactions and that relative homophily in the network is strongest when groups have equal size, and vanishes as groups have increasingly unequal sizes.[3]

Despite strong empirical evidence, few models of network formation with differentiated communities have studied the impact of social networks on the long-run integration outcome of minorities. Jackson and Rogers (2005) extend the Jackson and Wolinsky (1996)'s connection model by including two communities and assuming that the cost of linking two individuals from different communities is exogenous and independent of the behavior of the two individuals involved in the link. Johnson and Gilles (2000) add a geographical dimension to Jackson and Wolinsky (1996)'s connection model assuming that the cost of a link is proportional to the geographical distance between two individuals. As already mentioned, de Marti and Zenou (2017) model is a variation of the connection model where the cost of a link is endogenous and depends on the neighborhood structure of the two individuals involved in the link.

We go further on the related literature by considering the impact of a mixed population along two dimensions (community membership and degree of farsightedness) on the stability of friendship networks. That is, we analyze how the presence of farsighted individuals can affect the long-run integration outcome and under which circumstances this can lead either to a segregated society or to a society where one community is fully or partially assimilated into the other one. By doing so, we are the first to provide a theoretical network formation model that stabilizes in the long-run the efficient network structure where the smaller community ends up fully assimilated into the larger community.[4]

Another strand of the literature studies the role of social networks in the assimilation of immigrants, a hot debate in the United States and in Europe. There is strong evidence showing that family, peers, and communities affect assimilation

[3]Mele (2017) proposes a dynamic model of network formation that combines strategic and random network features. In each period an individual meets another individual and decides whether to form a new link, keep an existing link or do nothing. He shows that the model converges to a unique stationary equilibrium distribution over networks.

[4]Using data from the German Socio-Economic Panel for the period 1996 to 2011, Facchini, Patacchini, and Steinhardt (2015) find that first-generation migrants who have a German friend are more similar to German natives than migrants who do not. In addition, educational achievement is positively related to the likelihood of forming friendships with the majority group members. Similarly, from data of the European Community Household Panel (1994-2001), de Palo, Faini and Venturini (2007) find that more educated migrants tend to socialize more intensively with the majority community.

decisions (see e.g. Bisin, Patacchini, Verdier, and Zenou, 2016). In particular, there may be a conflict between an individual's assimilation choice and that of her peers and between an individual's assimilation choice and that of her family and community. Verdier and Zenou (2017) study the role of the immigrant network in the assimilation process of ethnic minorities. They show that, in an exogenous network, the more central minority individuals are located in the social network, the more they assimilate to the majority culture. By endogenizing the network structure, they show when the ethnic minority will integrate or not into the majority group.

The paper is organized as follows. In Section 2 we present de Marti and Zenou's (2017) model of friendship networks with two communities and we look at which networks are likely to arise when all individuals are myopic. In Section 3 we introduce the concept of myopic-farsighted stable sets. In Section 4 we provide a characterization of the myopic-farsighted stable sets when intra-community costs are low. In Section 5 we consider the case where intra-community costs are intermediate. Finally, in Section 6 we conclude.

## 4.2 Friendship networks with two communities

We consider de Marti and Zenou (2017) model of friendship networks where individuals belong to two different communities. Individuals benefit from direct and indirect connections to others, which can be interpreted as positive externalities. These benefits decay with the distance between individuals and the cost of forming links may depend on community memberships. The novelty is that individuals can now be either farsighted or myopic when deciding about the friendship links they want to form. In de Marti and Zenou (2017), all individuals were supposed to be myopic.

The set of individuals is denoted by $N = N^M \cup N^F$, where $N^M$ is the set of myopic individuals and $N^F$ is the set of farsighted individuals. Let $n$ be the total number of individuals and $n^M \geq 0$ ($n^F = n - n^M \geq 0$) be the number of myopic (farsighted) individuals. Moreover, the population is divided into two communities $N = N^B \cup N^G$, where $N^B$ is the *blue* community and $N^G$ is the *green* community. Each individual belongs to one of the two communities and the type of individual $i$ is denoted as $\tau(i) \in \left\{N^B, N^G\right\}$. We have $n = n^B + n^G$, where $n^B$ and $n^G$ denote, respectively, the number of $N^B$ individuals and the number of $N^G$ individuals in the population. Without loss of generality, the green community is the largest one: $n^B \leq n^G$.

A friendship network $g$ is a list of which pairs of individuals are linked to each other and $ij \in g$ indicates that $i$ and $j$ are linked under $g$. The complete network on the set of individuals $S \subseteq N$ is denoted by $g^S$ and is equal to the set of all subsets of $S$ of size 2. It follows in particular that the empty network is denoted by $g^{\emptyset}$. The set of all possible networks on $N$ is denoted by $\mathcal{G}$ and consists of all subsets of $g^N$. The network obtained by adding link $ij$ to an existing network $g$ is denoted $g + ij$ and the network that results from deleting link $ij$ from an existing network $g$ is denoted $g - ij$. Let $N(g) = \{i \mid \text{there is } j \text{ such that } ij \in g\}$ be the set of individuals who have at least one link in the network $g$. Let $N_i(g) = \{j \in N \mid ij \in g\}$ be the set of neighbors (or friends) of individual $i$ in $g$.[5] Let $n_i(g) = \#(N_i(g))$ be the number of neighbors (or friends) of individual $i$ in $g$. A path in a network $g$ between $i$ and $j$ is a sequence of individuals $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = i$ and $i_K = j$. A network $g$ is connected if for all $i \in N$ and $j \in N \setminus \{i\}$, there exists a path in $g$ connecting $i$ and $j$. A nonempty subnetwork $h \subseteq g$ is a component of $g$, if for all $i \in N(h)$ and $j \in N(h) \setminus \{i\}$, there exists a path in $h$ connecting $i$ and $j$, and for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$. A star network is a network such that there exists some individual $i$ (the center) who is linked to every other individual $j \neq i$ (the peripherals) and that contains no other links (i.e. $g$ is such that $N_i(g) = N \setminus \{i\}$ and $N_j(g) = \{i\}$ for all $j \in N \setminus \{i\}$).

A network utility function (or payoff function) is a mapping $U_i : \mathcal{G} \to \mathbb{R}$ that assigns to each network $g$ a utility $U_i(g)$ for each individual $i \in N$. A network $g \in \mathcal{G}$ is strongly efficient if $\sum_{i \in N} U_i(g) \geq \sum_{i \in N} U_i(g')$ for all $g' \in \mathcal{G}$. Preferences are given by

$$U_i(g) = \sum_{j \neq i} \delta^{t(i,j)} - \sum_{j \in N_i(g)} c_{ij}(g),$$

where $t(ij)$ is the number of links in the shortest path between $i$ and $j$ (setting $t(ij) = \infty$ if there is no path between $i$ and $j$), $0 < \delta < 1$ is the benefit from a connection that decreases with the distance of the relationship,[6] and $c_{ij}(g) > 0$ is the cost for individual $i$ of maintaining a direct link with $j$. The cost of forming one link may vary as a function of the type of individuals connected by such link.

**Definition 4.1** (de Marti and Zenou, 2017)**.** Given a network $g$, the rate of exposure of individual $i$ to their own community $\tau(i)$ is

$$e_i^{\tau(i)}(g) = \begin{cases} n_i^{\tau(i)}(g)/(n_i(g) - 1) & \text{if } 0 < n_i^{\tau(i)}(g) < n_i(g) \\ 0 & \text{if } n_i^{\tau(i)}(g) = 0 \end{cases} \tag{4.2.1}$$

[5]Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. Finally, # will refer to the notion of cardinality.

[6]It is similar to the connections model introduced by Jackson and Wolinsky (1996).

where $n_i^{\tau(i)}(g)$ is the number of $i$'s same-type friends in network $g$ while $n_i(g)$ is the total number of $i$'s friends in network $g$.

Let $c$ and $C$ be strictly positive parameters, $c > 0$ and $C > 0$. The cost for individual $i$ of maintaining a link with $j$, $c_{ij}(g)$, depends on whether $i$ and $j$ belong or not to the same community:

$$c_{ij}(g) = \begin{cases} c & \text{if } \tau(i) = \tau(j) \\ c + e_i^{\tau(i)}(g) \cdot e_j^{\tau(j)}(g) \cdot C & \text{if } \tau(i) \neq \tau(j) \end{cases}.$$

Such cost function assumes that it is less costly to interact with someone of the same type (intra-community cost) than with someone of a different type (inter-community cost). Notice that $C$ is not present in the cost of a link between individuals of the same community. But, $C$ becomes an additional cost when two individuals from different communities, having links with individuals of their own community, form a link between them. For instance, if a green individual has only green friends, then it will be more costly for her to interact with a blue individual that has mostly blue friends. However, the more similar the friendship composition of two individuals of different types, the easier it is for them to interact. If at least $i$ or $j$ has no friends of the same type (i.e., $e_i^{\tau(i)} = 0$ or $e_j^{\tau(j)} = 0$), then it is equally costly for them to interact with someone of the opposite type as with someone of the same type (i.e., the cost is $c$ in both cases).[7] In Figure 4.1 we depict a friendship network among seven individuals and two communities ($N^G = \{1, 2, 3, 4\}$, $N^B = \{5, 6, 7\}$) with a bridge link between both communities. Green individuals are represented by solid circles while blue individuals are represented by circles. For instance, green individual 4's payoff is equal to $4\delta + 2\delta^2 - 4c - C$ since $e_4^{\tau(4)} = 3/(4-1) = 1$ and $e_7^{\tau(7)} = 2/(3-1) = 1$.

We now describe some prominent network configurations in the case of friendship networks with communities. Let $g_{\text{assi,green}}$ denote the network where all members of the blue community are fully assimilated to the dominant (or larger) green community. That is, each green individual is linked to all other (green and blue) individuals while each blue individual is only linked to all green individuals. Formally, $g_{\text{assi,green}} = g^{N^G} \cup \{ij \mid i \in N^G, j \in N^B\}$. In Figure 4.2 we depict $g_{\text{assi,green}}$ for $N^G = \{1, 2, 3, 4\}$ and $N^B = \{5, 6\}$. Similarly, let $g_{\text{assi,blue}}$ denote the network where all members of the green community are fully assimilated to the smaller blue

[7]In the definition of the rate of exposure (see the expression (4.2.1)), we subtract 1 in the denominator because, when computing the cost of a given bridge link between communities, this bridge link is not included in the computation of the cost. What is relevant for the cost is the rate of exposure according to the rest of the connections of each of the two individuals involved in the bridge link.

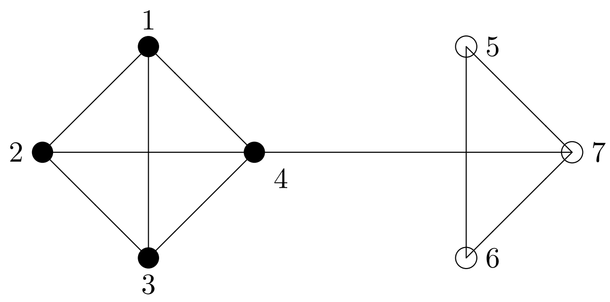


Figure 4.1: A bridge link between both communities. Greens are represented by solid circles while blues are represented by circles.

community. That is, each blue individual is linked to all other (green and blue) individuals while each green individual is only linked to all blue individuals. Formally, $g_{\text{assi,blue}} = g^{N^B} \cup \{ij \mid i \in N^B, j \in N^G\}$. In Figure 4.3 we depict $g_{\text{assi,blue}}$ for $N^G = \{1, 2, 3, 4\}$ and $N^B = \{5, 6\}$. Let $g_{\text{int}}$ denote the complete integration network where both communities are fully intra-connected and fully inter-connected: $g_{\text{int}} = g^N$ and is depicted in Figure 4.4. Let $g_{\text{seg}}$ denote the complete segregation network where both communities are fully intra-connected but isolated of each other: $g_{\text{seg}} = g^{N^G} \cup g^{N^B}$ and is depicted in Figure 4.5.

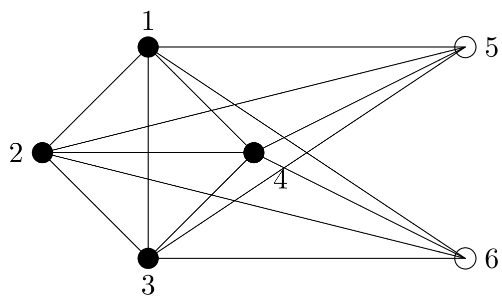


Figure 4.2: The blue community is fully assimilated within the green community.

de Marti and Zenou (2017) adopt the notion of pairwise stability, introduced by Jackson and Wolinsky (1996), to study the networks that will be formed at equilibrium. A network is pairwise stable if no individual benefits from deleting a link and no two individuals benefit from adding a link between them. Formally, a network $g \in \mathcal{G}$ is pairwise stable if (i) for all $ij \in g$, $U_i(g) \geq U_i(g - ij)$ and $U_j(g) \geq U_j(g - ij)$, (ii) for all $ij \notin g$, if $U_i(g) < U_i(g + ij)$ then $U_j(g) > U_j(g + ij)$.

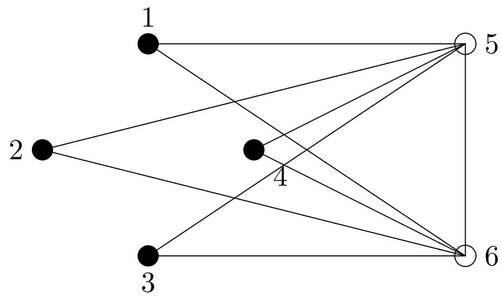


Figure 4.3: The green community is fully assimilated within the blue community.

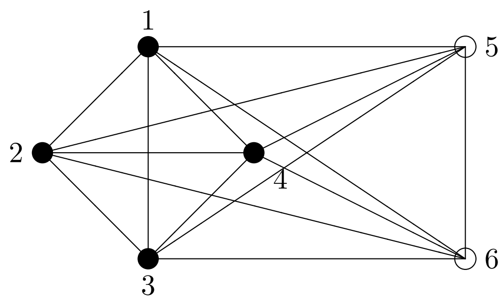


Figure 4.4: Both communities are fully integrated.

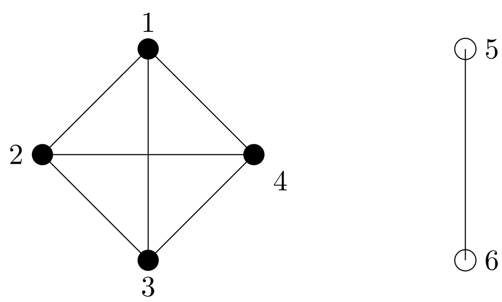


Figure 4.5: Both communities are segregated.

Let $P$ be the set of pairwise stable networks. Pairwise stability presumes that individuals are myopic: they do not anticipate that other individuals may react to their changes. Denote $\Delta \equiv \delta - \delta^2 - c$. De Marti and Zenou (2017) find necessary and sufficient conditions for the stability of the complete integration (segregation) network.

**Proposition 4.1** (de Marti and Zenou, 2017)**.** *Assume low intra-community costs,* $0 < \Delta$ *or* $c < \delta - \delta^2$.

**(i)** *The complete integration network* $g_{int} = g^N$ *is pairwise stable if and only if*

$$C < \frac{(n-2)^2(n-3)}{n^G(n^G-1)^2}\Delta;$$

**(ii)** *The complete segregation network* $g_{seg} = g^{N^G} \cup g^{N^B}$ *is pairwise stable*[8] *if and only if*

$$C > \Delta + n^B \cdot \delta^2.$$

We now show that friendship networks where one community is fully or partially assimilated to the other community can also emerge in the long run when intra-community costs are low. In Figure 4.6 (4.7) we depict a network where one blue (green) individual is assimilated to the green (blue) community, while the rest of blue (green) individuals are isolated. All the proofs not in the main text can be found in the appendix.

**Proposition 4.2.** *Assume low intra-community costs,* $0 < \Delta$ *or* $c < \delta - \delta^2$.

**(i)** *The network* $g_{assi,green} = g^{N^G} \cup \{ij \mid i \in N^G, j \in N^B\}$ *where all the blue community is fully assimilated to the green community is pairwise stable if and only if*

$$C > \frac{(n-2)}{(n^G-1)}\Delta;$$

**(ii)** *The network* $g_{assi,blue} = g^{N^B} \cup \{ij \mid i \in N^B, j \in N^G\}$ *where all the green community is fully assimilated to the blue community is pairwise stable if and only if*

$$C > \frac{(n-2)}{(n^B-1)}\Delta;$$

**(iii)** *Take any* $N^{B_1} \subsetneq N^B$ *such that* $1 \leq n^{B_1} \leq n^B - 2$. *The network* $g_{passi,green} = g^{N^G} \cup g^{N^B \setminus N^{B_1}} \cup \{ij \mid i \in N^G, j \in N^{B_1}\}$ *where* $n^{B_1}$ *blue individuals are*

[8]Notice that there is a typo in de Marti and Zenou's original condition: $C > \Delta + n^G\delta^2$ has to be replaced by $C > \Delta + n^B\delta^2$.

*assimilated to the green individuals and all other blue individuals are intra-connected and segregated is pairwise stable if and only if*

$$C > \frac{(n^G + n^{B_1} - 1)}{(n^G - 1)} \left(\Delta + (n^B - n^{B_1})\delta^2\right);$$

**(iv)** *Take any $N^{G_1} \subsetneq N^G$ such that $1 \leq n^{G_1} \leq n^G - 2$. The network $g_{passi,blue} = g^{N^B} \cup g^{N^G \setminus N^{G_1}} \cup \{ij \mid i \in N^B, j \in N^{G_1}\}$ where $n^{G_1}$ green individuals are assimilated to the blue individuals and all other green individuals are intra-connected and segregated is pairwise stable if and only if*

$$C > \begin{cases} \widehat{C}_1 & \text{if } n^{G_1} \leq \frac{1}{2}(n^G - n^B); \\ \widehat{C}_2 & \text{if } n^{G_1} > \frac{1}{2}(n^G - n^B); \end{cases}$$

*where*

$$\widehat{C}_1 = \max\left\{\frac{(n^B + n^{G_1} - 1)}{(n^B - 1)}\left(\Delta + (n^B + n^{G_1})\delta^2\right), \frac{(n^B + n^{G_1} - 2)}{(n^B - 1)}\left(\Delta + (n^G - n^{G_1})\delta^2\right)\right\};$$

$$\widehat{C}_2 = \frac{(n^B + n^{G_1} - 1)}{(n^B - 1)}\left(\Delta + (n^G - n^{G_1})\delta^2\right).$$

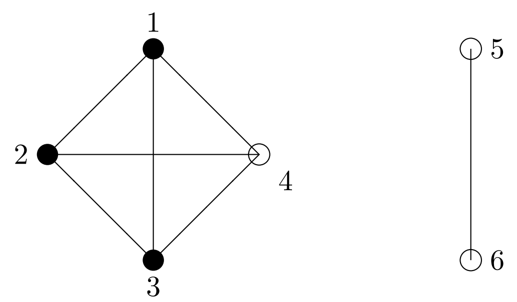


Figure 4.6: One blue individual is assimilated to the green community while the rest of blue individuals are segregated.

Proposition 2 of de Marti and Zenou (2017) points out that if intra-community costs are low, some asymmetric network configurations can also be pairwise stable: (i) the network in which both communities are fully intra-connected and where there is only one bridge link (see Figure 4.1), (ii) the network in which both communities are fully intra-connected, where each blue individual has one and only one bridge link, and where each green individual has at most one bridge link, and (iii) the network in which both communities are fully intra-connected and with a unique

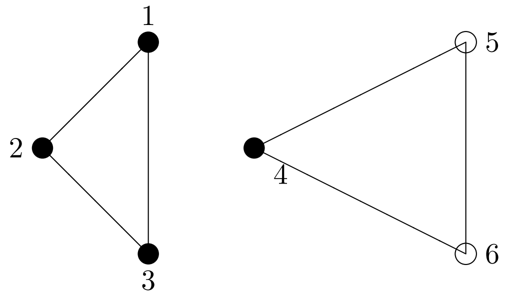


Figure 4.7: One green individual is assimilated to the blue community while the other green individuals are segregated.

blue individual connected with all green individuals. In the appendix we show that even more friendship networks can be pairwise stable. For instance, the network in which both communities are fully intra-connected and in which one green individual is linked to all blue individuals.

In terms of strong efficiency considerations, one might wonder which of the pairwise stable networks is better from a social point of view. de Marti and Zenou (2017) only compare the efficiency of the complete integrated network and the complete segregated network, and they conclude that, depending on the size of relative communities, one cannot plead for integrated or segregated socialization patterns a priori. We next compare in terms of strong efficiency the complete integrated network, the complete segregated network, and the networks with full or partial assimilation.

**Proposition 4.3.** *Assume low intra-community costs, $0 < \Delta$ or $c < \delta - \delta^2$. Let*

$$C^* = \frac{(n-2)^2}{2n^G(n^G-1)}\Delta.$$

**(i)** *If $C < C^*$, the complete integrated network $g_{int}$ is strongly efficient.*

**(ii)** *If $C > C^*$, the network $g_{assi,green}$ in which all blue individuals are fully assimilated into the dominant green community and all green individuals are fully inter- and intra-connected is strongly efficient.*

**(iii)** *The complete segregated network $g_{seg}$ is never strongly efficient for any value of $C$.*

Thus, contrary to de Marti and Zenou (2017), we obtain that the complete segregated network is never strongly efficient. Only the complete integrated network and the network in which the blue individuals are fully assimilated into the dominant green community are strongly efficient. Indeed, the efficiency of one or the other network depends on $C$, which affects the exposure effect that the formation of a new link has on the exposure rates of the individuals involved in it. The formation of a link between two individuals from different communities (the same community), has a positive (negative) exposure effect for the individuals involved in it because the decrease (increase) in the rate of exposure of each of these individuals to their own community will reduce (increase) their inter-community costs that are proportional to $C$. When $C$ is small enough (close to 0), the difference between the inter-community and the intra-community costs is negligible and then one can consider that the entire population belong to only one community. When this is the case, Proposition 2 in Jackson and Wolinsky (1996) is applicable and the complete integrated network is both pairwise stable and strongly efficient. When $C$ increases, the inter-community costs might overcome the benefits derived from connecting to the other community. When this is the case, it becomes preferable to avoid the inter-community costs, making efficient the network in which the blue individuals (without any link to other blue individuals) are fully assimilated into the dominant green community.

Proposition 3 and Proposition 5 in de Marti and Zenou (2017) provide conditions for the stability of some type of networks when intra-community costs are intermediate (i.e. $\delta - \delta^2 < c < \delta - \delta^3$ or $\delta - \delta^2 < c < \delta$): (i) the bipartite network in which all green individuals are linked to all blue individuals, and in which all blue individuals are linked to all green individual, (ii) the network with two disconnected star-shaped communities, (iii) the network where the star-shaped communities are connected through their central individuals, (iv) the network where the star-shaped communities are connected through their peripheral individuals, and (v) the network where the star-shaped communities are connected through their central individuals and through their peripheral individuals. However, all those networks are not efficient. In fact, a star network encompassing all individuals is pairwise stable and is strongly efficient.

**Proposition 4.4.** *Assume intermediate intra-community costs, $\delta - \delta^2 < c < \delta$. A star network is both pairwise stable and strongly efficient.*

Up to now, it has been assumed that all individuals were myopic in the friendship network formation. We next allow the population to include not only myopic individuals but also farsighted ones. Farsighted individuals are able to anticipate that once they add or delete some links, other individuals could add or delete links afterwards.

## 4.3 Myopic-farsighted stable sets

We adopt the notion of myopic-farsighted stable set introduced by Herings, Mauleon and Vannetelbosch (2020) to determine the networks that are stable when some individuals are myopic while others are farsighted.[9] A set of networks $G$ is said to be a myopic-farsighted stable set if it satisfies the following two types of stability. Internal stability: No network in $G$ is dominated by any other network in $G$. External stability: Every network not in $G$ is dominated by some network in $G$. A network $g'$ is said to be dominated by a network $g$ if there is a myopic-farsighted improving path from $g'$ to $g$.

A myopic-farsighted improving path is a sequence of distinct networks that can emerge when farsighted individuals form or delete links based on the improvement of the end network offers relative to the current network while myopic individuals form or delete links based on the improvement the resulting network offers relative to the current network. Since we only allow for pairwise deviations, each network in the sequence differs from the previous one in that either a new link is formed between two individuals or an existing link is deleted. If a link is deleted, then it must be that either a myopic individual prefers the resulting network to the current network or a farsighted individual prefers the end network to the current network. If a link is added between some myopic individual $i$ and some farsighted individual $j$, then the myopic individual $i$ must prefer the resulting network to the current network and the farsighted individual $j$ must prefer the end network to the current network.[10]

**Definition 4.2.** A myopic-farsighted improving path from a network $g$ to a network $g'$ is a finite sequence of distinct networks $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$ either

**(i)** $g_{k+1} = g_k - ij$ for some $ij$ such that $U_i(g_{k+1}) > U_i(g_k)$ and $i \in N^M$ or $U_j(g_K) > U_j(g_k)$ and $j \in N^F$; or

**(ii)** $g_{k+1} = g_k + ij$ for some $ij$ such that $U_i(g_{k+1}) > U_i(g_k)$ and $U_j(g_{k+1}) \geq U_j(g_k)$ if $i, j \in N^M$, or $U_i(g_K) > U_i(g_k)$ and $U_j(g_K) \geq U_j(g_k)$ if $i, j \in N^F$, or

[9] See Chwe (1994), Herings, Mauleon and Vannetelbosch (2009), Mauleon, Vannetelbosch and Vergote (2011), Ray and Vohra (2015, 2019), Roketskiy (2018) for definitions of the farsighted stable set when individuals are farsighted. Alternative notions of farsightedness for network formation are suggested by Dutta, Ghosal, and Ray (2005), Dutta and Vohra (2017), Herings, Mauleon, and Vannetelbosch (2019), Page, Wooders, and Kamat (2005), Page and Wooders (2009) among others.

[10] Along a myopic-farsighted improving path, myopic players do not care whether other players are myopic or farsighted. They behave as if all players are myopic and they compare their resulting network's payoff to their current network's payoff for taking a decision. However, farsighted players know exactly who is farsighted and who is myopic and they compare their end network's payoff to their current network's payoff for taking a decision.

$U_i(g_{k+1}) \geq U_i(g_k)$ and $U_j(g_K) \geq U_j(g_k)$ (with one inequality holding strictly) if $i \in N^M, j \in N^F$.

If there exists a myopic-farsighted improving path from a network $g$ to a network $g'$, then we write $g \to g'$. The set of all networks that can be reached from a network $g \in \mathcal{G}$ by a myopic-farsighted improving path is denoted by $\phi(g)$, $\phi(g) = \{g' \in \mathcal{G} \mid g \to g'\}$. When all individuals are myopic, our notion of myopic-farsighted improving path reverts to Jackson and Watts (2002) notion of improving path. When all individuals are farsighted, our notion of myopic-farsighted improving path reverts to Jackson (2008) and Herings, Mauleon, and Vannetelbosch's (2009) notion of farsighted improving path. A set of networks $G$ is a myopic-farsighted stable set if the following two conditions hold. Internal stability: for any two networks $g$ and $g'$ in the myopic-farsighted stable set $G$ there is no myopic-farsighted improving path from $g$ to $g'$ (and vice versa). External stability: for every network $g$ outside the myopic-farsighted stable set $G$ there is a myopic-farsighted improving path leading to some network $g'$ in the myopic-farsighted stable set $G$ (i.e. there is $g' \in G$ such that $g \to g'$).

**Definition 4.3.** A set of networks $G \subseteq \mathcal{G}$ is a myopic-farsighted stable set if: **(IS)** for every $g, g' \in G$, it holds that $g' \notin \phi(g)$; and **(ES)** for every $g \in \mathcal{G} \setminus G$, it holds that $\phi(g) \cap G \neq \emptyset$.

When all individuals are farsighted, the myopic-farsighted stable set is simply the farsighted stable set as defined in Herings, Mauleon, and Vannetelbosch (2009) or Ray and Vohra (2015). When all individuals are myopic, the myopic-farsighted stable set boils down to the pairwise CP vNM set as defined in Herings, Mauleon, and Vannetelbosch (2017) for two-sided matching problems.[11]

When all individuals are myopic, Jackson and Watts (2002) define the notions of cycle and closed cycle. A set of networks $\mathcal{C}$, form a cycle if for any $g \in \mathcal{C}$ and $g' \in \mathcal{C}$ there exists an improving path connecting $g$ to $g'$. A cycle $\mathcal{C}$ is a closed cycle if no network in $\mathcal{C}$ lies on an improving path leading to a network that is not in $\mathcal{C}$. Luo, Mauleon, and Vannetelbosch (2020) characterize the myopic-farsighted stable set when all individuals are myopic (i.e. $N = N^M$): a set of networks is a myopic-farsighted stable set if and only if it consists of all pairwise stable networks and one network from each closed cycle.

Similar to pairwise stability, one may alternatively look for networks that are immune to deviations by myopic and farsighted individuals. A network $g \in \mathcal{G}$

[11] The pairwise CP vNM set follows the approach by Page and Wooders (2009) who define the stable set with respect to path dominance, i.e. the transitive closure of $\phi$.

is myopic-farsightedly pairwise stable if $\phi(g) = \emptyset$. The set of myopic-farsightedly pairwise stable networks is denoted by $P_{MF}$. When $N = N^F$ it reverts to Jackson's (2008) set of farsightedly pairwise stable networks. Since $P_{MF} \subseteq P$, it is not surprising that the set $P_{MF}$ is often empty.

## 4.4 Low intra-community costs

Suppose now that the population of individuals is mixed in terms of their degree of farsightedness. We first show that if the intra- and inter-community costs are low, i.e. $c + n^G C < \delta - \delta^2$, then the complete integrated network is stable whatever the composition of the population in terms of farsightedness.

**Proposition 4.5.** *Assume low intra-community costs and low inter-community costs, $n^G C < \Delta$ or $c + n^G C < \delta - \delta^2$. The set $G = \{g_{int}\}$, where $g_{int} = g^N$, is a myopic-farsighted stable set.*

*Proof.* The set $G = \{g_{\text{int}}\}$, where $g_{\text{int}} = g^N$, satisfies (**IS**) in Definition 4.3 since it is a singleton set. We now show that it also satisfies (**ES**).
**ES.** Take any network $g \neq g_{\text{int}}$. Since $n^G C < \Delta$ or $c + n^G C < \delta - \delta^2$, it follows that $U_i(g + ij) > U_i(g)$ and $U_j(g + ij) > U_j(g)$ as well as $U_i(g^N) \geq U_i(g + ij) > U_i(g)$ and $U_j(g^N) \geq U_j(g + ij) > U_j(g)$. Hence, the sequence starting at $g_1 = g$, followed by $g_{k+1} = g_k + ij$ with $ij \in g^N \setminus g^k$, for $k = 1, 2...$, and ending at $g_K = g^N$, is a sequence along which $U_i(g_k + ij) > U_i(g_k)$, $U_j(g_k + ij) > U_j(g_k)$, $U_i(g^N) > U_i(g_k)$ and $U_j(g^N) > U_j(g_k)$. Thus, this sequence is a myopic-farsighted improving path from $g$ to $g^N$ whatever the composition of the population in terms of myopia and farsightedness (i.e. $N^M$ and $N^F$), and $G = \{g_{\text{int}}\}$ satisfies (**ES**). □

When all individuals are myopic each myopic-farsighted stable set contains all pairwise networks. Hence, many inefficient friendship networks can emerge in the long run when both communities are composed of only myopic individuals.

We next focus on three particular cases: (1) all individuals in the larger green community are farsighted, while all individuals in the smaller blue community are myopic; (2) all individuals in the larger green community are myopic, while all individuals in the smaller blue community are farsighted; (3) all individuals in both communities are farsighted.

### 4.4.1 Greens are farsighted, blues are myopic

We now show that if the dominant group (green community) is farsighted while the other group (blue community) is myopic, the friendship network where the blue individuals end up assimilated to the dominant green community is the unique stable network and is strongly efficient. Let $\overline{C}_1$ be the upper bound on the inter-community cost parameter $C$ such that a blue individual has no incentive to cut a link with another blue individual in the complete integrated network, and it is given by

$$\overline{C}_1 = \frac{(n-2)^2(n-3)}{n^G(n^G-1)^2}\Delta.$$

Thus, if $C > \overline{C}_1$, each myopic blue individual has an incentive to delete some link to another blue individual in the complete integrated network $g^N$.

**Proposition 4.6.** *Assume low intra-community costs,* $0 < \Delta$ *or* $c < \delta - \delta^2$ *and large inter-community costs,* $C > \overline{C}_1$*. Assume all individuals in the blue community are myopic,* $N^M = N^B$*, and all individuals in the green community are farsighted,* $N^F = N^G$*. Then, the set* $G = \{g_{assi,green}\}$*, where* $g_{assi,green} = g^{N^G} \cup \{ij \mid i \in N^G, j \in N^B\}$*, is the unique myopic-farsighted stable set.*

*Proof.* The set $G = \{g_{\text{assi,green}}\}$ satisfies (**IS**) in Definition 4.3 since it is a singleton set. We now show that it also satisfies (**ES**).
**ES.** Take any network $g \neq g_{\text{assi,green}}$. We build in steps a myopic-farsighted improving path from $g$ to $g_{\text{assi,green}}$.
**Step 0:** If $g$ is such that blue individuals have links among themselves, i.e., $g \cap g^{N^B} \neq \emptyset$ then go to Step 1. Otherwise, starting from $g$, green individuals first build all the missing links between green individuals to reach $g' = g \cup g^{N^G}$ looking forward to $g_{\text{assi,green}}$, where they obtain their highest possible payoff given $c < \delta - \delta^2$, $U_i(g_{\text{assi,green}}) = (n-1)(\delta - c)$. From $g'$ green individuals build all the missing links with blue individuals to finally reach $g'' = g' \cup \{ij \mid i \in N^G, j \in N^B\} = g_{\text{assi,green}}$. Since $c < \delta - \delta^2$ and $g'' \cap g^{N^B} = \emptyset$, blue individuals are assimilated to the green community in $g''$ and they are not affected by $C$ and so they have incentives to add the links with the green individuals.
**Step 1:** Starting in $g$, green individuals who are all farsighted ($N^F = N^G$) delete successively all the links (if any) they have with green individuals looking forward to $g_{\text{assi,green}}$, where they obtain their highest possible payoff given $c < \delta - \delta^2$, $U_i(g_{\text{assi,green}}) = (n-1)(\delta - c)$. We reach the network $g' = g \setminus g^{N^G}$ where there are no links between green individuals.
**Step 2:** From $g' = g \setminus g^{N^G}$, since $c < \delta - \delta^2$, blue individuals who are all myopic have incentives to build all the links with the green individuals. Green individuals who are looking forward $g_{\text{assi,green}}$ prefer the end network to the current one. We reach the network $g'' = g' \cup \{ij \mid i \in N^G, j \in N^B\}$ where all possible links between blue and green individuals are formed.

**Step 3:** From $g'' = g' \cup \{ij \mid i \in N^G, j \in N^B\}$, since $c < \delta - \delta^2$, blue individuals who are all myopic have incentives to build all the missing links between the blue individuals. We reach the network $g''' = g'' \cup g^{N^B}$ where all the green individuals are assimilated to the blue community and the blue community is fully intra-connected. In fact, $g''' = g_{\text{assi,blue}}$ and all green individuals prefer $g_{\text{assi,green}}$ to $g_{\text{assi,blue}}$.
**Step 4:** From $g''' = g'' \cup g^{N^B}$, green individuals who are all farsighted and look forward towards $g_{\text{assi,green}}$ build all the links between the green individuals to reach $g^N$.
**Step 5:** From the complete network $g^N$, since $C > \overline{C}_1$, blue individuals who are myopic have incentives to delete successively all the links they have with other blue individuals to finally reach the network $g_{\text{assi,green}} = g^N \setminus g^{N^B}$. The condition $C > \overline{C}_1$ guarantees that, along the myopic-farsighted improving path starting at $g_1 = g^N$, followed by $g_{k+1} = g_k - ij$ with $ij \in g_k$ and $i, j \in N^B$ for $k \geq 1$, and ending at $g_K = g^N \setminus g^{N^B} = g_{\text{assi,green}}$, all the blue individuals have myopic incentives to delete their links with other blue individuals. Indeed, consider a sequence starting at $g_1 = g^N$, followed by $g_{k+1} = g_k - ij$ with $i \in N^B, j \in N_i(g_k) \cap N^B$, for $k = 1, ...n^B - 1$. Along this sequence, a blue individual $i$ successively deletes all her links with the other blue individuals and she has incentives to cut her $k$th link to some blue individual if and only if

$$C > \Delta \frac{(n-2)(n-1-k)(n-2-k)}{n^G(n^G-1)^2}.$$

This condition is satisfied since $C > \overline{C}_1$ and

$$\overline{C}_1 = \Delta \frac{(n-2)^2(n-3)}{n^G(n^G-1)^2} \geq \Delta \frac{(n-2)(n-1-k)(n-2-k)}{n^G(n^G-1)^2}.$$

Farsighted green individuals obtain their highest possible payoff in $g_{\text{assi,green}}$ and myopic blue individuals have no incentive to delete any link nor to add a new link since $C > \overline{C}_1$ and $c < \delta - \delta^2$. Hence, $\phi(g_{\text{assi,green}}) = \emptyset$. So, since $\phi(g) \cap \{g_{\text{assi,green}}\} \neq \emptyset$ for all $g \neq g_{\text{assi,green}}$ and $\phi(g_{\text{assi,green}}) = \emptyset$, the set $G = \{g_{\text{assi,green}}\}$ is the unique myopic-farsighted stable set (any other set would violate (**IS**) and/or (**ES**)). □

Remark that since $\phi(g) \cap \{g_{\text{assi,green}}\} \neq \emptyset$ for all $g \neq g_{\text{assi,green}}$ and $\phi(g_{\text{assi,green}}) = \emptyset$, the network $g_{\text{assi,green}}$ is the unique myopic-farsightedly pairwise stable network, i.e., $P_{MF} = \{g_{\text{assi,green}}\}$.

### 4.4.2 Greens are myopic, blues are farsighted

However, when the dominant green community is myopic and the blue community is farsighted, a conflict between stability and efficiency can arise. Let $\overline{C}_2$ be the upper bound on the inter-community cost parameter $C$ such that a green individual has no incentive to delete a link with another green individual in the complete integrated network, and it is given by

$$\overline{C}_2 = \frac{(n-2)^2(n-3)}{n^B(n^B-1)^2}\Delta.$$

Thus, if $C > \overline{C}_2$, each myopic green individual has an incentive to delete some link to another green individual in the complete integrated network $g^N$.

**Proposition 4.7.** *Assume low intra-community costs,* $0 < \Delta$ *or* $c < \delta - \delta^2$ *and large inter-community costs,* $C > \overline{C}_2$*. Assume all individuals in the blue community are farsighted,* $N^F = N^B$*, and all individuals in the green community are myopic,* $N^M = N^G$*. Then, the set* $G = \{g_{assi,blue}\}$*, where* $g_{assi,blue} = g^{N^B} \cup \{ij \mid i \in N^B, j \in N^G\}$*, is the unique myopic-farsighted stable set.*

The proof of Proposition 4.7 is similar to the proof of Proposition 4.6 by just switching blue individuals for green ones and vice versa. For completeness, the proof of Proposition 4.7 can be found in the appendix. So, when $C$ is large enough ($C > \overline{C}_2$), the efficient network in which the farsighted blue individuals are fully assimilated into the green community[12] is not stable. Farsighted blue individuals stabilize the opposite network in which the myopic green individuals are fully assimilated into the blue community. Remark that the network $g_{\text{assi,blue}} = g^{N^B} \cup \{ij \mid i \in N^B, j \in N^G\}$ is the unique myopic-farsightedly pairwise stable network, i.e., $P_{MF} = \{g_{\text{assi,blue}}\}$.

### 4.4.3 Greens and blues are farsighted

**Proposition 4.8.** *Assume low intra-community costs,* $0 < \Delta$ *or* $c < \delta - \delta^2$ *and inter-community costs,* $C > 0$*. Assume all individuals are farsighted,* $N^F = N$*. Then, the set* $G = \{g_{assi,green}\}$ *is a myopic-farsighted stable set.*

*Proof.* The set $G = \{g_{\text{assi,green}}\}$ satisfies (**IS**) in Definition 4.3 since it is a singleton set. We now show that it also satisfies (**ES**).

**ES.** Take any network $g \neq g_{\text{assi,green}}$. We build in steps a myopic-farsighted improving path from $g$ to $g_{\text{assi,green}}$.

[12]Since $C^* < \overline{C}_1 < \overline{C}_2$ the network in which the farsighted blue individuals are fully assimilated into the green community is the efficient network.

**Step 0:** If $g$ is such that $g \cap g^{N^B} \neq \emptyset$ then go to Step 1. Otherwise, starting from $g$, green individuals first build all the missing links between green individuals to reach $g' = g \cup g^{N^G}$ looking forward to $g_{\text{assi,green}}$, where they obtain their highest possible payoff given $c < \delta - \delta^2$ and $C > 0$, $U_i(g_{\text{assi,green}}) = (n-1)(\delta - c)$. From $g'$ green individuals build all the missing links with blue individuals to finally reach $g'' = g' \cup \{ij \mid i \in N^G, j \in N^B\} = g_{\text{assi,green}}$. Since $c < \delta - \delta^2$ and $g'' \cap g^{N^B} = \emptyset$, blue individuals are assimilated to the green community in $g''$ and they are not affected by $C$ and so they have incentives to add the links with the green individuals looking forward to $g_{\text{assi,green}}$.
**Step 1:** Starting in $g$, green individuals who are all farsighted ($N^F = N$) delete successively all the links (if any) they have with green and blue individuals looking forward to $g_{\text{assi,green}}$, where they obtain their highest possible payoff given $c < \delta - \delta^2$ and $C > 0$, $U_i(g_{\text{assi,green}}) = (n-1)(\delta - c)$. We reach the network $g' = g \cap g^{N^B}$ where all the links involving green individuals in $g$ have been deleted. Thus, $g' \subseteq g^{N^B}$.
**Step 2:** From $g' = g \cap g^{N^B}$, since $n^G \geq n^B$, all blue individuals who are all farsighted prefer $g_{\text{assi,green}}$ to $g'$ and so are ready to delete all their links looking forward to $g_{\text{assi,green}}$. We reach the empty network $g^{\emptyset}$.
**Step 3:** From the empty network $g^{\emptyset}$ green individuals and blue individuals who are farsighted and look forward to $g_{\text{assi,green}}$ build all the links in $g^{N^G} \cup \{ij \mid i \in N^G, j \in N^B\}$ to finally reach the network $g_{\text{assi,green}}$. Since along the myopic-farsighted improving blue individuals have no links to other blue individuals, the payoffs of both green and blue individuals are not affected by $C$. So, each time they add a link they all prefer the end network $g_{\text{assi,green}}$ to the current network.
Hence, $\phi(g) \cap \{g_{\text{assi,green}}\} \neq \emptyset$ for all $g \neq g_{\text{assi,green}}$ and $G = \{g_{\text{assi,green}}\}$ satisfies (**ES**). □

Notice that if $n^B \leq n^G \leq 1 + n^B \frac{(\delta - c)}{\Delta}$, we can replicate the above proof (by just switching blue individuals for green ones and vice versa) to show that the set $G = \{g_{\text{assi,blue}}\}$ is a myopic-farsighted stable set. However, once $n^G > 1 + n^B \frac{(\delta - c)}{\Delta}$, the set $G = \{g_{\text{assi,blue}}\}$ is never a myopic-farsighted stable set because $\phi(g_{\text{seg}}) \cap \{g_{\text{assi,blue}}\} = \emptyset$. Moreover, the set $G = \{g_{\text{seg}}\}$ is never a myopic-farsighted stable set because $\phi(g_{\text{assi,green}}) \cap \{g_{\text{seg}}\} = \emptyset$. Thus, the complete segregation network $g_{\text{seg}}$ and the network $g_{\text{assi,blue}}$ in which all green individuals are fully assimilated into the smaller blue community are unlikely to emerge in the long run when all individuals are farsighted.

*Remark* 4.1. Assume low intra-community costs, $0 < \Delta$ or $c < \delta - \delta^2$ and inter-community costs, $C > 0$. Assume all individuals are farsighted, $N^F = N$.

(**i**) The set $G = \{g_{\text{assi,blue}}\}$ is never a myopic-farsighted stable set if $n^G > 1 +$

$n^B \frac{(\delta-c)}{\Delta}$.

(**ii**) The set $G = \{g_{\text{seg}}\}$ is never a myopic-farsighted stable set.

## 4.5 Intermediate intra-community costs

We now consider situations where intra-community costs are intermediate, i.e. $\delta - \delta^2 < c < \delta$. So, it becomes more expensive to build links with individuals from the same community. We denote by $g^{*i}$ the star network where individual $i$ is the center of the star.

**Proposition 4.9.** *Assume intermediate intra-community costs, $\delta - \delta^2 < c < \delta$, $N^F \neq \emptyset$ and $N^M \neq \emptyset$. If $\delta - \delta^2 < c + C < (\delta - \delta^2)(1 + \delta(n^F - 1))$, then the set $G^* = \left\{g^{*i} \mid i \in N^M\right\}$ is the unique myopic-farsighted stable set.*

*Proof.* We first show that $G^* = \left\{g^{*i} \mid i \in N^M\right\}$ satisfies both internal stability (i.e. condition (**IS**) in Definition 4.3) and external stability (i.e. condition (**ES**) in Definition 4.3).

**IS.** Farsighted individuals are peripherals in all networks in $G^*$ so that they always obtain the same payoff: $U_i(g) = \delta + (n-2)\delta^2 - c$ for all $i \in N^F$, $g \in G^*$. Myopic individuals who are peripherals have no incentive to delete their single link ($\delta + (n-2)\delta^2 - c > 0$) nor to add a new link to any other individual ($2\delta + (n-3)\delta^2 - 2c < \delta + (n-2)\delta^2 - c$ since $\delta - \delta^2 < c$). The center who is myopic has no incentive to delete one link since $c < \delta$. Hence, for every $g, g' \in G^*$, it holds that $g' \notin \phi(g)$.

**ES.** Take any network $g \notin G^*$. We build in steps a myopic-farsighted improving path from $g$ to some $g^{*i} \in G^*$.
**Step 1:** Starting in $g$, farsighted individuals delete all their links successively looking forward to some $g^{*i} \in G^*$, where they obtain their highest possible payoff given $\delta - \delta^2 < c$. Notice that if $g$ is a star network where the center is a farsighted individual, then the center starts by deleting all her links since only the center is better off in $g^{*i}$ compared to $g$ (and we go directly to Step 8). We reach a network $g^1$ where all farsighted individuals have no link and myopic individuals only keep the links to myopic individuals they had in $g$.
**Step 2:** From $g^1$, looking forward to $g^{*i} \in G^*$, farsighted individuals build a star network $g^{*jF}$ restricted to farsighted individuals with individual $j$ being the center (i.e. $g^{*jF}$ is such that $j \in N^F$, $N_j(g^{*jF}) = N^F \setminus \{j\}$ and $N_k(g^{*jF}) = \{j\}$ for all $k \in N^F \setminus \{j\}$), and we obtain $g^2 = g^1 \cup g^{*jF}$ where all farsighted individuals are still disconnected from the myopic individuals.

**Step 3:** From $g^2$, looking forward to $g^{*i} \in G^*$, the farsighted individual $j$ who is the center of $g^{*jF}$ adds a link to some myopic individual, say individual 1. Individual $j$ is better off in $g^{*i}$ compared to $g^2$, $\delta + (n-2)\delta^2 - c > (n - n^M - 1)(\delta - c)$, while individual 1 is better in $g^2 + j1$ if $c + C < \delta + \delta^2(n^F - 1)$. This last inequality holds since $c + C < (\delta - \delta^2)(1 + \delta(n^F - 1)) < \delta + \delta^2(n^F - 1)$.

**Step 4:** From $g^2 + j1$, looking forward to $g^{*i} \in G^*$, the farsighted individual $j$ adds a link successively to the myopic individuals who are neighbors of individual 1 (if any), say individual 2. Individual 2 who is myopic and linked to individual 1 has an incentive to add the link $j2$ if $\delta^2 + (n - n^M - 1)\delta^3 < \delta - c - C + (n - n^M - 1)\delta^2$. Thus, a sufficient condition for adding the link is

$$c + C < \delta - \delta^2 + (n - n^M - 1)(\delta^2 - \delta^3), \tag{4.5.1}$$

or

$$c + C < (\delta - \delta^2)(1 + \delta(n^F - 1)) \tag{4.5.2}$$

where $n - n^M$ is the number of farsighted individuals ($n^F$). In the network $g^2 + j1 + \{jl \mid l \in N_1(g^2 + j1) \cap N^M\}$, individual $j$ is (directly) linked to all other farsighted individuals, individual 1 and all neighbors of individual 1.

**Step 5:** From $g^2 + j1 + \{jl \mid l \in N_1(g^2 + j1) \cap N^M\}$, the myopic individuals who are neighbors of individual 1 and have just added a link to the farsighted individual $j$ delete their link successively with individual 1. They have incentives to do so since $\delta - \delta^2 < c < \delta$ and we reach $g^2 + j1 + \{jl \mid l \in N_1(g^2 + j1) \cap N^M\} - \{1l \mid l \in N_1(g^2 + j1) \cap N^M\}$.

**Step 6:** Next, looking forward to $g^{*i} \in G^*$, the farsighted individual $j$ adds a link successively to the myopic individuals who are neighbors of some $l \in N_1(g^2 + j1) \cap N^M$ and we proceed as in Step 4 and Step 5. We repeat this process until we reach a network $g^3$ where there is no myopic individual linked directly to the myopic neighbors of individual $j$ (i.e. $N_k(g^3) \cap N^M = \emptyset$ for all $k \in N_j(g^3) \cap N^M$).

**Step 7:** From $g^3$, individual $j$ adds a link to some myopic individual belonging to another component (if any) as in Step 3 and we proceed as in Step 4 to Step 6. We repeat this process until we end up with a star network $g^{*j}$ with individual $j$ (who is farsighted) in the center (i.e. $N_j(g^{*j}) = N \setminus \{j\}$ and $N_k(g^{*j}) = \{j\}$ for all $k \in N \setminus \{j\}$).

**Step 8:** From $g^{*j}$, looking forward to $g^{*i} \in G^*$, the farsighted individual $j$ deletes all her links successively to reach the empty network $g^{\emptyset}$. From $g^{\emptyset}$, myopic and farsighted individuals have both incentives (since $\delta > c$) to add links successively to build the star network $g^{*i} \in G^*$ where some myopic individual $i \in N^M$ is the center.

We now show that $G^*$ is the unique myopic-farsighted stable set. Farsighted individuals who are peripherals in all networks in $G^*$ obtain their highest possible payoff. Myopic individuals who are peripherals have no incentive to delete their single link nor to add a new link. The center who is myopic has no incentive to delete one link. Hence, $\phi(g) = \emptyset$ for every $g \in G^*$. Suppose that $G \neq G^*$ is another myopic-farsighted stable set. (1) $G$ does not include $G^*$: $G \not\supseteq G^*$. External stability would be violated since $\phi(g) = \emptyset$ for every $g \in G^*$. (2) $G$ includes $G^*$: $G \supsetneq G^*$. Internal stability would be violated since for every $g \in \mathcal{G} \setminus G^*$, it holds that $\phi(g) \cap G^* \neq \emptyset$. □

Thus, when intra-community costs are intermediate and the population is formed by myopic and farsighted individuals, the set of star networks with a myopic individual at the center of the star is both a myopic-farsighted stable set and strongly efficient. From the proof of Proposition 4.9 we get the characterization of the myopic-farsightedly pairwise stable networks: if $\delta - \delta^2 < c + C < (\delta - \delta^2)(1 + \delta(n^F - 1))$, then $P_{MF} = \left\{g^{*i} \mid i \in N^M\right\}$.

Once all individuals become farsighted (i.e. $N = N^F$), for $\delta - \delta^2 < c < \delta$ and for $C > 0$, every set consisting of a star network encompassing all individuals is a myopic-farsighted stable set

**Proposition 4.10.** *Assume intermediate intra-community costs, $\delta - \delta^2 < c < \delta$, and all individuals farsighted, $N = N^F$. If $g$ is a star network then $\{g\}$ is a myopic-farsighted stable set.*

*Proof.* Since each set is a singleton set, internal stability (**IS**) is satisfied. (**ES**) Take any network $g \neq g^{*i}$, we need to show that $\phi(g) \ni g^{*i}$. (i) Suppose $g \neq g^{*j}$ $(j \neq i)$. From $g$, looking forward to $g^{*i}$ (where they obtain their highest possible payoff), farsighted individuals ($\neq i$) delete all their links successively to reach the empty network. From $g^{\emptyset}$, farsighted individuals have incentives (since $\delta > c$) to add links successively to build the star network $g^{*i}$ with individual $i$ in the center. (ii) Suppose $g = g^{*j}$ $(j \neq i)$. From $g$, looking forward to $g^{*i}$, the farsighted individual $j$ deletes all her links successively to reach the empty network. From $g^{\emptyset}$, farsighted individuals have incentives (since $\delta > c$) to add links successively to build the star network $g^{*i}$ with individual $i$ in the center. □

While every set consisting of a star network is a myopic-farsighted stable set, there may be other myopic-farsighted stable sets. Nevertheless, every star network encompassing all individuals is strongly efficient.

## 4.6 Conclusion

We have reconsidered de Marti and Zenou (2017) model of friendship network formation where individuals belong to two different communities (greens and blues). We have added a second heterogeneity dimension: individuals can be either myopic or farsighted. Our main results for low intra-community costs are summarized in Figure 4.8. When all individuals are myopic many friendship networks (complete integration, complete segregation, (partial) assimilation, ...) can be pairwise stable and a tension between efficiency and stability may occur. Once the population becomes mixed in terms of farsightedness and myopia, most inefficient friendship networks tend to be destabilized. When the larger (smaller) community is farsighted while the smaller (larger) community is myopic, the friendship network where the myopic community is assimilated into the farsighted community emerges in the long run when inter-community costs are large enough. Once all individuals are farsighted, the friendship network where the smaller community ends up being assimilated into the dominant community is likely to arise. When inter-community costs are small enough, the complete integration is stable whatever the number of farsighted and myopic individuals in both communities.

| Blues \ Greens | Myopic | Farsighted |
| --- | --- | --- |
| Myopic | Segregation<br>Assimilation to Greens / Blues<br>Integration<br>+ many others | Assimilation to Greens (for $C > \overline{C}_1$)<br>Integration (for $n^G C < \Delta$) |
| Farsighted | Assimilation to Blues (for $C > \overline{C}_2$)<br>Integration (for $n^G C < \Delta$) | Assimilation to Greens (for $C > 0$)<br>Integration (for $n^G C < \Delta$) |

Figure 4.8: A summary of stable friendship networks with low intra-community costs.

What would happen if there are farsighted and myopic individuals in both communities when intra-community costs are low and inter-community costs are large? Take the friendship network $g = \{12, 13, 14, 23, 24, 34, 56\}$ depicted in Figure 4.6 with $N^F = \{2, 3, 4\}$, $N^M = \{1, 5, 6\}$, $N^G = \{1, 2, 3\}$ and $N^B = \{4, 5, 6\}$. There are no myopic-farsighted improving paths emanating from $g$ when inter-community costs are large; $\phi(g) = \emptyset$. Hence, this friendship network, where the farsighted blue individual is assimilated to the dominant green community, belongs to all myopic-farsighted stable sets (if they exist).[13] However, the complete segregated network $g' = \{12, 13, 23, 45, 46, 56\}$ will never occur since $g \in \phi(g')$ and $G \supseteq \{g, g'\}$ would violate internal stability. Indeed, the farsighted blue individual 4 has incentives to first delete her links in $g'$ and next build the links with all green individuals to form $g$. Providing a full-fledged characterization of the myopic-farsighted stable sets turns to be extremely hard, if not impossible. To summarize, depending on the costs for interacting, either a fully integrated society or a (partially) assimilated society is likely to arise in the long run. In addition, farsightedness seems to dampen the tension between efficiency and stability in friendship networks when individuals belong to different communities.

Notice that the degree of farsightedness of an individual is likely to be correlated with her level of education or grades at school. Hence, for future research, it would be interesting to confront our theoretical predictions with data. (i) In presence of only highly educated communities, is it likely that the smaller community ends up assimilated into the dominant one? (ii) In presence of a highly educated community and a low educated community, is it likely that the lower educated community ends up assimilated into the high educated one? (iii) Complete segregation mostly occurs when both communities are low educated. (iv) When one community has a large number of highly educated individuals while the other community has a low number of highly educated individuals, is it likely that the high educated individuals belonging to the less educated community end up assimilated into the more educated community?

## 4.7 Appendix

## .1 More pairwise stable friendship networks

We now show that any network where $n^G$ green individuals are fully intra-connected, $n^B$ blue individuals are fully intra-connected, and one green individual is linked to all blue individuals is pairwise stable for intermediate inter-community costs. In Figure 9 we depict such a network.

**Proposition .11.** *Assume low intra-community costs,* $0 < \Delta$ *or* $c < \delta - \delta^2$*. The network* $\widetilde{g} = g^{N^G} \cup g^{N^B} \cup \{ij \mid j \in N^B, i = \widetilde{i} \text{ with } \widetilde{i} \in N^G\}$ *where* $n^G$ *green individuals*

[13] Patacchini and Zenou (2016) look at friendship networks among US high-school students (Add Health data). They find that, for mixed schools, most of the white students have white friends while one part of the black students has mostly white friends and the other part have mostly black friends.

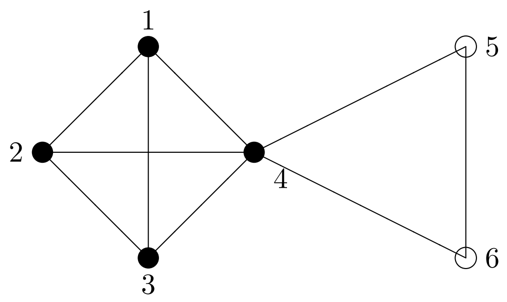


Figure 9: Both communities are fully intra-connected and one green individual is linked to all blue individuals.

*are fully intra-connected, $n^B$ blue individuals are fully intra-connected, and one green individual is linked to all blue individuals is pairwise stable if and only if*

$$\underline{C}_3 < C < \overline{C}_3$$

*where*

$$\underline{C}_3 = \frac{n^B}{n^B - 1}\Delta;$$
$$\overline{C}_3 = \min\left\{\frac{(n-2)(n-3)}{(n^G-1)(n^G-2)}\Delta, \frac{(n-2)(n-3)}{(n^B)(n^B-1)}\Delta, \frac{(n-2)}{(n^G-1)}\left(\Delta + \delta^2(1-\delta)(n^G-1)\right)\right\}.$$

*Proof.* Individual $\widetilde{i} \in N^G$ is the green individual who is linked to all individuals in $\widetilde{g}$.

**(i)** In $\widetilde{g}$ individual $\widetilde{i}$ has no incentive to cut a link with a blue individual if and only if

$$C < \frac{(n-2)(n-3)}{(n^G-1)(n^G-2)}\Delta.$$

In $\widetilde{g}$ any blue individual has no incentive to cut a link with the green individual $\widetilde{i}$ if and only if

$$C < \frac{(n-2)}{(n^G-1)}\left(\Delta + \delta^2(1-\delta)(n^G-1)\right).$$

Combining these two conditions, a link between $\widetilde{i}$ and a blue individual will not be deleted if and only if

$$C < \min\left\{\frac{(n-2)(n-3)}{(n^G-1)(n^G-2)}\Delta, \frac{(n-2)}{(n^G-1)}\left(\Delta + \delta^2(1-\delta)(n^G-1)\right)\right\}.$$

**(ii)** In $\widetilde{g}$ player $\widetilde{i}$ has no incentive to cut a link with a green individual if and only if

$$C < \frac{(n-2)(n-3)}{(n^B)(n^B-1)}\Delta.$$

Since $\Delta + n^B\delta^2(1-\delta) > 0$, any green individual $i \neq \widetilde{i}$ has no incentive to delete her link with $\widetilde{i}$. Moreover, since $0 < \Delta$, any green individual $i \neq \widetilde{i}$ has no incentive to delete her link with another green individual $j \neq \widetilde{i}$. Thus, a link between any two green individuals will not be deleted if and only if

$$C < \frac{(n-2)(n-3)}{(n^B)(n^B-1)}\Delta.$$

**(iii)** Since $0 < \Delta$, any blue individual has no incentive to delete her link with another blue individual.

**(iv)** In $\widetilde{g}$ any green individual $i \neq \widetilde{i}$ has no incentive to add a link with a blue individual if and only if

$$C > \frac{n^B}{n^B-1}\Delta.$$

In addition, any blue individual has no incentive to add a link to a green individual $i \neq \widetilde{i}$ if and only if

$$C > \frac{n^B(n-2)}{(n^B-1)(n-2)-(n^G-1)}\Delta.$$

Combining these two conditions, a link between any green individual $i \neq \widetilde{i}$ and a blue individual will not be added if and only if

$$C > \frac{n^B}{n^B-1}\Delta = \underline{C}_3.$$

From **(i)**, **(ii)**, **(iii)** and **(iv)** we have that $\widetilde{g}$ is pairwise stable if and only if $\underline{C}_3 < C < \overline{C}_3$, where

$$\overline{C}_3 = \min\left\{\frac{(n-2)(n-3)}{(n^G-1)(n^G-2)}\Delta, \frac{(n-2)(n-3)}{(n^B)(n^B-1)}\Delta, \frac{(n-2)}{(n^G-1)}\left(\Delta + \delta^2(1-\delta)(n^G-1)\right)\right\}.$$

□

We now show that any network where both communities are fully intra-connected, some blue individuals are assimilated to the green community and some green individuals are assimilated to the blue community is pairwise stable for intermediate inter-community costs. In Figure 10 we depict such a network.

**Proposition .12.** *Assume low intra-community costs,* $0 < \Delta$ *or* $c < \delta - \delta^2$. *The network* $\widehat{g} = g^{N^G \setminus N^{G_1}} \cup g^{N^B \setminus N^{B_1}} \cup \{ij \mid i \in N^{B_1}, j \in N^G\} \cup \{ij \mid i \in N^{G_1}, j \in N^B\}$, *where both communities are fully intra-connected,* $n^{B_1}$ *(*$1 \leq n^{B_1} < n^B$*) blue*

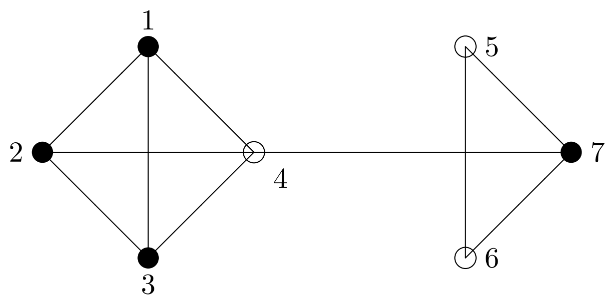


Figure 10: One green and one blue individual are assimilated to the other community.

*individuals are assimilated to the green community and* $n^{G_1}$ $(1 \leq n^{G_1} < n^G)$ *green individuals are assimilated to the blue community, is pairwise stable if and only if*

$$C > \max\left\{\frac{n^B(n^B + n^{G_1} - n^{B_1} - 2)}{(n^B - n^{B_1})(n^B - n^{B_1} - 1)}\Delta, \frac{n^G(n^G + n^{B_1} - n^{G_1} - 2)}{(n^G - n^{G_1})(n^G - n^{G_1} - 1)}\Delta, \right. \tag{.1.1}$$
$$\left. \frac{(n^G + n^{B_1} - n^{G_1} - 1)(n^B + n^{G_1} - n^{B_1} - 1)}{(n^G - n^{G_1} - 1)(n^B - n^{B_1} - 1)}(\Delta + \delta^2(1-\delta)(n^B - n^{B_1}))\right\}.$$

*Proof.* In $\widehat{g}$ the $n^{B_1}$ $(1 \leq n^{B_1} < n^B)$ blue individuals are fully assimilated to the green community (with payoff $n^G(\delta - c)$) and the $n^{G_1}$ $(1 \leq n^{G_1} < n^G)$ green individuals are fully assimilated to the blue community (with payoff $n^B(\delta - c)$), while the other $n^B - n^{B_1}$ blue individuals obtain $(n^B - n^{B_1} + n^{G_1} - 1)(\delta - c)$ and the other $n^G - n^{G_1}$ green individuals obtain $(n^G - n^{G_1} + n^{B_1} - 1)(\delta - c)$. Since $0 < \Delta$ or $c < \delta - \delta^2$, all individuals have no incentive to delete a link in $\widehat{g}$.

**(i)** In $\widehat{g}$ any green individual $i \in N^{G_1}$ has no incentive to add a link to another green individual $j \in N^G \setminus N^{G_1}$ if and only if

$$C > \frac{n^B(n^B + n^{G_1} - n^{B_1} - 2)}{(n^B - n^{B_1})(n^B - n^{B_1} - 1)}\Delta.$$

Since $\Delta + \delta^2(1-\delta)(n^B - n^{B_1}) > 0$, any green individual $j \in N^G \setminus N^{G_1}$ has always incentives to form a link with a green individual $i \in N^{G_1}$. Hence, a link between a green individual $i \in N^{G_1}$ and a green individual $j \in N^G \setminus N^{G_1}$ will not be formed in $\widehat{g}$ if and only if

$$C > \frac{n^B(n^B + n^{G_1} - n^{B_1} - 2)}{(n^B - n^{B_1})(n^B - n^{B_1} - 1)}\Delta.$$

**(ii)** In $\widehat{g}$ any blue individual $i \in N^{B_1}$ has no incentive to add a link to a blue individual $j \in N^B \setminus N^{B_1}$ if and only if

$$\frac{n^G(n^G + n^{B_1} - n^{G_1} - 2)}{(n^G - n^{G_1})(n^G - n^{G_1} - 1)}\Delta.$$

Since $\Delta + \delta^2(1-\delta)(n^G - n^{G_1}) > 0$, any blue individual $j \in N^B \setminus N^{B_1}$ has always incentives to form a link with a blue individual $i \in N^{B_1}$. Hence, a link between a blue individual $i \in N^{B_1}$ and a blue individual $j \in N^B \setminus N^{B_1}$ will not be formed in $\widehat{g}$ if and only if

$$\frac{n^G(n^G + n^{B_1} - n^{G_1} - 2)}{(n^G - n^{G_1})(n^G - n^{G_1} - 1)}\Delta.$$

**(iii)** In $\widehat{g}$ any blue individual $i \in N^{B_1}$ has no incentive to add a link to another blue individual $j \in N^{B_1}$ if and only if

$$C > \frac{n^G(n^G + n^{B_1} - n^{G_1} - 2)}{(n^G - n^{G_1})(n^G - n^{G_1} - 1)}\Delta.$$

**(iv)** In $\widehat{g}$ any green individual $i \in N^{G_1}$ has no incentive to add a link to another green individual $j \in N^{G_1}$ if and only if

$$C > \frac{n^B(n^B + n^{G_1} - n^{B_1} - 2)}{(n^B - n^{B_1})(n^B - n^{B_1} - 1)}\Delta.$$

**(v)** In $\widehat{g}$ any green individual $i \in N^G \setminus N^{G_1}$ has no incentive to add a link to a blue individual $j \in N^B \setminus N^{B_1}$ if and only if

$$C > \frac{(n^G + n^{B_1} - n^{G_1} - 1)(n^B + n^{G_1} - n^{B_1} - 1)}{(n^G - n^{G_1} - 1)(n^B - n^{B_1} - 1)}(\Delta + \delta^2(1-\delta)(n^B - n^{B_1})).$$

In $\widehat{g}$ any blue individual $j \in N^B \setminus N^{B_1}$ has no incentive to add a link to a green individual $i \in N^G \setminus N^{G_1}$ if and only if

$$C > \frac{(n^G + n^{B_1} - n^{G_1} - 1)(n^B + n^{G_1} - n^{B_1} - 1)}{(n^G - n^{G_1} - 1)(n^B - n^{B_1} - 1)}(\Delta + \delta^2(1-\delta)(n^G - n^{G_1})).$$

Hence, a link between a green individual $i \in N^G \setminus N^{G_1}$ and a blue individual $j \in N^B \setminus N^{B_1}$ will not be formed in $\widehat{g}$ if and only if

$$C > \min\left\{ \frac{(n^G + n^{B_1} - n^{G_1} - 1)(n^B + n^{G_1} - n^{B_1} - 1)}{(n^G - n^{G_1} - 1)(n^B - n^{B_1} - 1)}(\Delta + \delta^2(1-\delta)(n^B - n^{B_1})), \right. \quad (1.12)$$
$$\left. \frac{(n^G + n^{B_1} - n^{G_1} - 1)(n^B + n^{G_1} - n^{B_1} - 1)}{(n^G - n^{G_1} - 1)(n^B - n^{B_1} - 1)}(\Delta + \delta^2(1-\delta)(n^G - n^{G_1})) \right\}.$$

From **(i)**, **(ii)**, **(iii)**, **(iv)** and **(v)** we have that the network $\widehat{g}$ is pairwise stable if and only if

$$C > \max\left\{\frac{n^B(n^B + n^{G_1} - n^{B_1} - 2)}{(n^B - n^{B_1})(n^B - n^{B_1} - 1)}\Delta, \frac{n^G(n^G + n^{B_1} - n^{G_1} - 2)}{(n^G - n^{G_1})(n^G - n^{G_1} - 1)}\Delta, \right. \quad (.1.3)$$
$$\left. \frac{(n^G + n^{B_1} - n^{G_1} - 1)(n^B + n^{G_1} - n^{B_1} - 1)}{(n^G - n^{G_1} - 1)(n^B - n^{B_1} - 1)}(\Delta + \delta^2(1-\delta)(n^B - n^{B_1}))\right\}.$$

□

## .2 Proofs

**Proof of Proposition 4.2.** Assume low intra-community costs, $0 < \Delta$ or $c < \delta - \delta^2$.

**(i)** We show that the network $g_{\text{assi,green}} = g^{N^G} \cup \{ij \mid i \in N^G, j \in N^B\}$ where all the blue community is fully assimilated to the green community is pairwise stable if and only if

$$C > \frac{(n-2)}{(n^G - 1)}\Delta.$$

In $g_{\text{assi,green}}$ all green individuals get as payoff $(n-1)(\delta - c)$ and all blue individuals get as payoff $n^G(\delta - c) + (n^B - 1)\delta^2$. Since $0 < \Delta$, all blue individuals have no incentive to cut a link and all green individuals have no incentive to cut a link with a green or blue individual. In $g_{\text{assi,green}}$, any blue individual will not add a link to another blue individual if and only

$$C > \frac{(n-2)}{(n^G - 1)}\Delta.$$

**(ii)** We show that the network $g_{\text{assi,blue}} = g^{N^B} \cup \{ij \mid i \in N^B, j \in N^G\}$ where all the green community is fully assimilated to the blue community is pairwise stable if and only if

$$C > \frac{(n-2)}{(n^B - 1)}\Delta.$$

In $g_{\text{assi,blue}}$ all blue individuals get as payoff $(n-1)(\delta - c)$ and all green individuals get as payoff $n^B(\delta - c) + (n^G - 1)\delta^2$. Since $0 < \Delta$, all green individuals have no incentive to cut a link and all blue individuals have no incentive to cut a link with a green or blue individual. In $g_{\text{assi,blue}}$, any green individual will not add a link to another green individual if and only

$$C > \frac{(n-2)}{(n^B - 1)}\Delta.$$

**(iii)** Take any $N^{B_1} \subsetneq N^B$ such that $1 \leq n^{B_1} \leq n^B - 2$. We show that the network $g_{\text{passi,green}} = g^{N^G} \cup g^{N^B \setminus N^{B_1}} \cup \{ij \mid i \in N^G, j \in N^{B_1}\}$ where $n^{B_1}$ blue individuals are assimilated to the green individuals and all other blue individuals are intra-connected and segregated is pairwise stable if and only if

$$C > \frac{(n^G + n^{B_1} - 1)}{(n^G - 1)} \left(\Delta + (n^B - n^{B_1})\delta^2\right).$$

(iiia) In $g_{\text{passi,green}}$, all segregated blue individuals get as payoff $(n^B - n^{B_1} - 1)(\delta - c)$, all assimilated blue individuals get as payoff $(n^G)(\delta - c)$ and all green individuals get as payoff $(n^G + n^{B_1} - 1)(\delta - c)$. Since $0 < \Delta$, all individuals have no incentive to cut a link. In $g_{\text{passi,green}}$, any green individual will not add a link to a blue individual $j \in N^B \setminus N^{B_1}$ if and only if

$$C > \frac{(n^G + n^{B_1} - 1)}{(n^G - 1)} \left(\Delta + (n^B - n^{B_1})\delta^2\right).$$

In $g_{\text{passi,green}}$, any blue individual $j \in N^B \setminus N^{B_1}$ will not add a link to a green individual if and only if

$$C > \frac{(n^G + n^{B_1} - 1)}{(n^G - 1)} \left(\Delta + (n^G + n^{B_1})\delta^2\right).$$

Hence, by mutual consent, a link between a blue individual $j \in N^B \setminus N^{B_1}$ and a green individual will not be added in $g_{\text{passi,green}}$ if and only if

$$\begin{aligned} C &> \min\left\{\frac{(n^G + n^{B_1} - 1)}{(n^G - 1)} \left(\Delta + (n^G + n^{B_1})\delta^2\right), \frac{(n^G + n^{B_1} - 1)}{(n^G - 1)} \left(\Delta + (n^B - n^{B_1})\delta^2\right)\right\} \\ &= \frac{(n^G + n^{B_1} - 1)}{(n^G - 1)} \left(\Delta + (n^B - n^{B_1})\delta^2\right). \end{aligned} \tag{2.2.1}$$

(iiib) In $g_{\text{passi,green}}$, any blue individual $i \in N^{B_1}$ will not add a link to a blue individual $j \in N^B \setminus N^{B_1}$ if and only if

$$C > \frac{(n^G + n^{B_1} - 2)}{(n^G - 1)} \left(\Delta + (n^B - n^{B_1})\delta^2\right).$$

However, any blue individual $j \in N^B \setminus N^{B_1}$ has always incentives to add a link to a blue individual $i \in N^{B_1}$ since $0 < \Delta$. Hence, a link between a blue individual $i \in N^{B_1}$ and a blue individual $j \in N^B \setminus N^{B_1}$ will not be added in $g_{\text{passi,green}}$ if and only if

$$C > \frac{(n^G + n^{B_1} - 2)}{(n^G - 1)} \left(\Delta + (n^B - n^{B_1})\delta^2\right).$$

(iiic) In $g_{\text{passi,green}}$, any blue individual $i \in N^{B_1}$ will not add a link to another blue individual $j \in N^{B_1}$ if and only if

$$C > \frac{(n^G + n^{B_1} - 2)}{(n^G - 1)} \Delta.$$

From (iiia), (iiib) and (iiic), we have that $g_{\text{passi,green}}$ (with $1 \leq n^{B_1} \leq n^B - 2$) is pairwise stable if and only if

$$C > \frac{(n^G + n^{B_1} - 1)}{(n^G - 1)} \left(\Delta + (n^B - n^{B_1})\delta^2\right).$$

**(iv)** Take any $N^{G_1} \subsetneq N^G$ such that $1 \leq n^{G_1} \leq n^G - 2$. We show that the network $g_{\text{passi,blue}} = g^{N^B} \cup g^{N^G \setminus N^{G_1}} \cup \{ij \mid i \in N^B, j \in N^{G_1}\}$ where $n^{G_1}$ green individuals are assimilated to the blue individuals and all other green individuals are intra-connected and segregated is pairwise stable if and only if

$$C > \begin{cases} \widehat{C}_1 & \text{if } n^{G_1} \leq \frac{1}{2}(n^G - n^B); \\ \widehat{C}_2 & \text{if } n^{G_1} > \frac{1}{2}(n^G - n^B); \end{cases}$$

where

$$\widehat{C}_1 = \max\left\{\frac{(n^B + n^{G_1} - 1)}{(n^B - 1)} \left(\Delta + (n^B + n^{G_1})\delta^2\right), \frac{(n^B + n^{G_1} - 2)}{(n^B - 1)} \left(\Delta + (n^G - n^{G_1})\delta^2\right)\right\};$$
$$\widehat{C}_2 = \frac{(n^B + n^{G_1} - 1)}{(n^B - 1)} \left(\Delta + (n^G - n^{G_1})\delta^2\right).$$

(iva) In $g_{\text{passi,blue}}$, all segregated green individuals get as payoff $(n^G - n^{G_1} - 1)(\delta - c)$, all assimilated green individuals get as payoff $(n^B)(\delta - c)$ and all blue individuals get as payoff $(n^B + n^{G_1} - 1)(\delta - c)$. Since $0 < \Delta$, all individuals have no incentive to cut a link. In $g_{\text{passi,blue}}$, any blue individual will not add a link to a green individual $j \in N^G \setminus N^{G_1}$ if and only if

$$C > \frac{(n^B + n^{G_1} - 1)}{(n^B - 1)} \left(\Delta + (n^G - n^{G_1})\delta^2\right).$$

In $g_{\text{passi,blue}}$, any green individual $j \in N^G \setminus N^{G_1}$ will not add a link to a blue individual if and only if

$$C > \frac{(n^B + n^{G_1} - 1)}{(n^B - 1)} \left(\Delta + (n^B + n^{G_1})\delta^2\right).$$

Hence, by mutual consent, a link between a green individual $j \in N^G \setminus N^{G_1}$ and a blue individual will not be added in $g_{\text{passi,blue}}$ if and only if

$$C > \min\left\{\frac{(n^B + n^{G_1} - 1)}{(n^B - 1)} \left(\Delta + (n^B + n^{G_1})\delta^2\right), \frac{(n^B + n^{G_1} - 1)}{(n^B - 1)} \left(\Delta + (n^G - n^{G_1})\delta^2\right)\right\}.$$

(ivb) In $g_{\text{passi,blue}}$, any green individual $i \in N^{G_1}$ will not add a link to another green individual $j \in N^G \setminus N^{G_1}$ if and only if

$$C > \frac{(n^B + n^{G_1} - 2)}{(n^B - 1)} \left(\Delta + (n^G - n^{G_1})\delta^2\right).$$

However, any green individual $j \in N^G \setminus N^{G_1}$ has always incentives to add a link to a green individual $i \in N^{G_1}$ since $0 < \Delta$. Hence, a link between a green individual $i \in N^{G_1}$ and a green individual $j \in N^G \setminus N^{G_1}$ will not be added in $g_{\text{passi,blue}}$ if and only if

$$C > \frac{(n^B + n^{G_1} - 2)}{(n^B - 1)} \left(\Delta + (n^G - n^{G_1})\delta^2\right).$$

(ivc) In $g_{\text{passi,blue}}$, any green individual $i \in N^{G_1}$ will not add a link to another green individual $j \in N^{G_1}$ if and only if

$$C > \frac{(n^B + n^{G_1} - 2)}{(n^B - 1)}\Delta.$$

From (iva), (ivb) and (ivc), we have that $g_{\text{passi,blue}}$ (with $1 \leq n^{G_1} \leq n^G - 2$) is pairwise stable if and only if

$$C > \begin{cases} \widehat{C}_1 & \text{if } n^{G_1} \leq \frac{1}{2}(n^G - n^B); \\ \widehat{C}_2 & \text{if } n^{G_1} > \frac{1}{2}(n^G - n^B). \end{cases}$$

□

**Proof of Proposition 4.3.** Assume low intra-community costs, $0 < \Delta$ or $c < \delta - \delta^2$. In the complete segregated network $g_{\text{seg}}$, a green individual obtains $(n^G - 1)(\delta - c)$ as payoff, while a blue obtains $(n^B - 1)(\delta - c)$ as payoff. In the complete integrated network $g_{\text{int}}$, a green individual and a blue individual obtain, respectively,

$$(n-1)(\delta - c) - n^B \frac{n^G - 1}{n-2}\frac{n^B - 1}{n-2} C \text{ and } (n-1)(\delta - c) - n^G \frac{n^B - 1}{n-2}\frac{n^G - 1}{n-2} C$$

as payoff. In the network where the blue individuals are fully assimilated to the green community $g_{\text{assi,green}}$, a green individual obtains $(n-1)(\delta - c)$ as payoff, while a blue obtains $(n^G)(\delta - c) + (n^B - 1)\delta^2$ as payoff. In the network where the green individuals are fully assimilated to the blue community $g_{\text{assi,blue}}$, a blue individual obtains $(n-1)(\delta - c)$ as payoff, while a green obtains $(n^B)(\delta - c) + (n^G - 1)\delta^2$ as payoff. Since, $n^G \geq n^B$, the network $g_{\text{assi,blue}}$ is never better than the network $g_{\text{assi,green}}$ in terms of strong efficiency. Comparing the network $g_{\text{assi,green}}$ with the complete integrated network $g_{\text{int}}$, we have that the complete integrated network $g_{\text{int}}$

is better than the network $g_{\text{assi,green}}$ in terms of strong efficiency (i.e. sum of the payoffs of all individuals) if and only if

$$C < \frac{(n-2)^2}{2n^G(n^G-1)}\Delta = C^*.$$

In addition, we have that the network $g_{\text{assi,green}}$ is always better than the complete segregated network $g_{\text{seg}}$ in terms of strong efficiency: $n^G(n-1)(\delta-c)+n^B(n^G)(\delta-c)+n^B(n^B-1)\delta^2 > n^G(n^G-1)(\delta-c)+n^B(n^B-1)(\delta-c)$.

□

**Proof of Proposition 4.4.**

In a star network, the center gets $(n-1)(\delta-c)$ as payoff while the individuals at the periphery get $\delta+(n-2)\delta^2-c$ as payoff. Since $\delta-\delta^2<c<\delta$, individuals at the periphery of a star network get their highest possible payoff. Hence, they will not add a link between them nor they will cut a link with the center. Obviously, the center has no incentive to cut a link to a peripheral individual. Thus, any star network is pairwise stable. Jackson and Wolinsky (1996) show that a star network is strongly efficient for $\delta-\delta^2<c<\delta$ (and $C=0$). Hence, such star network is also strongly efficient for $\delta-\delta^2<c<\delta$ and $C>0$.

□

**Proof of Proposition 4.7.** The set $G=\{g_{\text{assi,blue}}\}$ satisfies (**IS**) in Definition 4.3 since it is a singleton set. We now show that it also satisfies (**ES**).

**ES.** Take any network $g \neq g_{\text{assi,blue}}$. We build in steps a myopic-farsighted improving path from $g$ to $g_{\text{assi,blue}}$.

**Step 0:** If $g$ is such that $g \cap g^{N^G} \neq \emptyset$ then go to Step 1. Otherwise, starting from $g$, blue individuals first build all the missing links between blue individuals to reach $g' = g \cup g^{N^B}$ looking forward to $g_{\text{assi,blue}}$, where they obtain their highest possible payoff given $c<\delta-\delta^2$, $U_i(g_{\text{assi,blue}})=(n-1)(\delta-c)$. From $g'$ blue individuals build all the missing links with green individuals to finally reach $g''=g' \cup \{ij \mid i \in N^B, j \in N^G\} = g_{\text{assi,blue}}$. Since $c<\delta-\delta^2$ and $g'' \cap g^{N^G}=\emptyset$, green individuals are assimilated to the blue community in $g''$ and they are not affected by $C$ and so they have incentives to add the links with the blue individuals.

**Step 1:** Starting in $g$, blue individuals who are all farsighted ($N^F=N^B$) delete successively all the links (if any) they have with blue individuals looking forward to

$g_{\text{assi,blue}}$, where they obtain their highest possible payoff given $c < \delta - \delta^2$, $U_i(g_{\text{assi,blue}}) = (n-1)(\delta - c)$. We reach the network $g' = g \setminus g^{N^B}$ where there are no links between blue individuals.

**Step 2:** From $g' = g \setminus g^{N^B}$, since $c < \delta - \delta^2$, green individuals who are all myopic have incentives to build all the links with the blue individuals. Blue individuals who are looking forward $g_{\text{assi,blue}}$ prefer the end network to the current one. We reach the network $g'' = g' \cup \{ij \mid i \in N^B, j \in N^G\}$ where all possible links between green and blue individuals are formed.

**Step 3:** From $g'' = g' \cup \{ij \mid i \in N^B, j \in N^G\}$, since $c < \delta - \delta^2$, green individuals who are all myopic have incentives to build all the missing links between the green individuals. We reach the network $g''' = g'' \cup g^{N^G}$ where all the blue individuals are assimilated to the green community and the green community is fully intra-connected. In fact, $g''' = g_{\text{assi,green}}$ and all blue individuals prefer $g_{\text{assi,blue}}$ to $g_{\text{assi,green}}$.

**Step 4:** From $g''' = g'' \cup g^{N^G}$, blue individuals who are all farsighted and look forward towards $g_{\text{assi,blue}}$ build all the links between the blue individuals to reach $g^N$.

**Step 5:** From the complete network $g^N$, since $C > \overline{C}_2$, green individuals who are myopic have incentives to delete successively all the links they have with other green individuals to finally reach the network $g_{\text{assi,blue}} = g^N \setminus g^{N^G}$. The condition $C > \overline{C}_2$ guarantees that, along the myopic-farsighted improving path starting at $g_1 = g^N$, followed by $g_{k+1} = g_k - ij$ with $ij \in g_k$ and $i, j \in N^G$ for $k \geq 1$, and ending at $g_K = g^N \setminus g^{N^G} = g_{\text{assi,blue}}$, all the green individuals have myopic incentives to delete their links with other green individuals. Indeed, consider a sequence starting at $g_1 = g^N$, followed by $g_{k+1} = g_k - ij$ with $i \in N^G, j \in N_i(g_k) \cap N^G$, for $k = 1, \ldots n^G - 1$. Along this sequence, a green individual $i$ successively deletes all her links with the other green individuals and she has incentives to cut her $k$th link to some green individual if and only if

$$C > \Delta \frac{(n-2)(n-1-k)(n-2-k)}{n^B(n^B-1)^2}.$$

This condition is satisfied since $C > \overline{C}_2$ and

$$\overline{C}_2 = \Delta \frac{(n-2)^2(n-3)}{n^B(n^B-1)^2} \geq \Delta \frac{(n-2)(n-1-k)(n-2-k)}{n^B(n^B-1)^2}.$$

Farsighted blue individuals obtain their highest possible payoff in $g_{\text{assi,blue}}$ and myopic green individuals have no incentive to delete any link nor to add a new link since $C > \overline{C}_2$ and $c < \delta - \delta^2$. Hence, $\phi(g_{\text{assi,blue}}) = \emptyset$. So, since $\phi(g) \cap \{g_{\text{assi,blue}}\} \neq \emptyset$

for all $g \neq g_{\text{assi,blue}}$ and $\phi(g_{\text{assi,blue}}) = \emptyset$, the set $G = \{g_{\text{assi,blue}}\}$ is the unique myopic-farsighted stable set (any other set would violate (**IS**) and/or (**ES**)).

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