Goodwillie calculus in the category of algebras over a chain operad

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Abstract. Goodwillie functor calculus is a method invented by Thomas Goodwillie to analyze functors that arise in Topology. This theory has some compelling similarities with differential calculus of Newton and Leibnitz in the sense that the method produces a tower for approximating a functor which plays the role of the Taylor series approximating a function. One of the major difficulties in this theory is that, Goodwillie Taylor series (or towers) are very abstract from their constructions and hence not easy to compute in general. The goal of this thesis is to produce an explicit approximation of functors between algebraic categories. Namely, we look at functors between the category of chain complexes and the category of algebras over a chain complex operad. We study properties on their tower of approximation, such as analyzing the difference between two consecutive terms of the Taylor tower. Moreover, we extend this approach to produce an explicit and computable description of the Taylor tower.

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Introduction

Calculus of homotopy functors is a method introduced by Goodwillie in the 1990's [Goo90, Goo92, Goo03] which gives a way to approximate homotopy functors by "polynomial functors". This yields to a Taylor tower to approximate those functors very analogous to the approximation of functors in one variable by their Taylor series. T. Goodwillie and its successors (N. Kuhn, G. Arone, M. Ching) have thoroughly developed this theory for functors in the category of topological spaces and/or spectra (which are a "stabilized" version of topological spaces). In this thesis, we develop this theory for more algebraic categories, such as the category of algebras over an operad. As an example of operad, there is the Lie operad: Lie. An algebra over this operad is exactly a differential graded Lie algebra.

Before explaining our results, let us recall the main ideas of Goodwillie calculus. We consider homotopy functors $F : Top_* \longrightarrow Top_*$ between the category of pointed topological spaces. These are covariant functors which preserve weak equivalences. Goodwillie calculus ([Goo90, Goo92, Goo03]) is basically meant to analyze certain notions on such functors F. Namely,

1. There is the notion of polynomial functor of degree n, that we will define later on (in Def 2.2), and which can briefly be thought as:

   - A polynomial functor of degree 0 is a constant functor (up to homotopy);
   - A polynomial functor of degree 1, or sometimes just called linear functor, is a functor which turns a homotopy pushout square into a homotopy pullback square. [1];
   - More generally, a polynomial functor of degree n generalizes this idea with higher cubical diagrams;
   - A polynomial functor of degree n is in particular a polynomial functor of degree $n+1$;

[1] A homotopy pushout(resp. pullback) is a variation of pushout (resp. pullback) which preserves weak equivalences

2. There is the notion of polynomial approximation of an arbitrary functor. In fact, given a functor F, one associates a polynomial functor P_nF of degree n, (for any n) and a natural transformation $p_nF : F \longrightarrow P_nF$. This construction comes with a universal property which says that: any natural transformation from F to any other polynomial functor G of degree n factors uniquely (up to homotopy) via p_nF. In other words, P_nF is the "best possible" polynomial approximation of F.

In Goodwillie calculus, we study polynomial functors and we construct approximation of arbitrary functors. The approximation of the functor F gives rise to a tower

$$F \longrightarrow ...P_nF \longrightarrow P_{n-1}F \longrightarrow ...P_0F \tag{1}$$

which converges to F in many cases when some nice connectedness properties are satisfied. This tower is called the "Taylor tower" of F and P_nF is seen as an analogue of the n-th polynomial approximation P_nf of an analytic function $f : \mathbb{R} \longrightarrow \mathbb{R}$.

Before motivating this analogy, we will make a very short detour to give some insight of a new category: the category of spectra Sp which is built out of Top_*. For this, remind first the suspension functor $\Sigma : Top_* \longrightarrow Top_*$ where the suspension of a pointed topological space $(X, *)$ is given by the homotopy pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma(X) \end{tikzcd}$$

For instance, the suspension of the circle is the 2-dimensional sphere

$$\Sigma(S^1) = S^2.$$

Formally, a spectrum $\mathbb{X} \in Sp$ consists of a sequence of spaces $\mathbb{X} = (X_0, X_1, ..., X_n, ...)$ along with continuous maps

$$\Sigma X_n \longrightarrow X_{n+1}$$

satisfying some properties (see [HSS00]). This definition might be technical, but a non expert reader can think of Sp as a category where the suspension functor $\Sigma : Sp \longrightarrow Sp$, defined similarly as in Top_*, has an inverse $\Sigma^{-1} : Sp \longrightarrow Sp$. A link between the categories Sp and Top_* is given by the functor

$$\begin{aligned} \Sigma^\infty : Top_* &\longrightarrow Sp \\ X &\longmapsto (X, \Sigma X, \Sigma^2 X, ..., \Sigma^k X, ...) \end{aligned}$$

This functor has a right adjoint generally denoted $\Omega^\infty : Sp \longrightarrow Top_*$.

To see the analogy between Goodwillie calculus and the classical Newton and Leibniz calculus for analytic functions, consider the "difference" D_nF between the polynomial approximations of F of degree n and of degree $n-1$. In homotopy theory, this means taking the homotopy fiber

$$D_nF(X) := \text{hofiber } (P_nF(X) \longrightarrow P_{n-1}F(X)),$$

that we call the "n-th layer of the tower". This is a categorical version of making (in classical calculus) the difference

$$d_nf(x) = P_nf(x) - P_{n-1}f(x) \tag{2}$$

which is the $n - th$ term of the Taylor series. It is well known that this $n - th$ term is given by the formula

$$d_nf(x) = \frac{f^{(n)}(0)}{n!}x^n \tag{3}$$

The following formula which appears in Goodwillie's results ([Goo03, Thm 2.1, Thm 3.5]) analyses more precisely the functor D_nF.

Theorem 0.1 (Goodwillie)**.** There is a spectrum $\partial_nF \in Sp$, with an action of the symmetric group Σ_n, and an equivalence (natural in X)

$$D_nF(X) \simeq \Omega^\infty(\partial_nF \underset{h\Sigma_n}{\wedge} (\Sigma^\infty X)^{\otimes n}) \tag{4}$$

where $(...)_{h\Sigma_n}$ is the space of (homotopy) orbits with respect to the Σ_n action, also called the Borel construction associated to the diagonal action of Σ_n on the two factors.

Surprisingly, the formulas (2) and (4) are very similar in the sens that: ∂_nF plays the role of the coefficient $f^{(n)}(0)$; the input $(\Sigma^\infty X)^{\otimes n}$ plays the role of x^n while the orbit over the set of permutations Σ_n plays the role of the division by the number of permutations $n!$. Because of this striking analogy, we say that: the Σ_n-spectrum ∂_nF is the $n - th$ derivative of F.

Even if Theorem 0.1 gives a fairly explicit description of the layers $D_nF(X)$, this is not enough to completely describe the Taylor tower (1). One of the natural and non trivial question is then the following:

Question 1. Can we recover the Taylor tower $\{P_nF\}_n$ of F from the sequence $\partial_*F = (\partial_0F, \partial_1F, ..., \partial_n, ..)$ of its derivatives?

In classical calculus, this is a trivial question as we know that

$$P_nf(x) = \sum_{k=0}^{n} \frac{f^k(0)}{k!}x^k.$$

But in functor calculus, P_nF is not simply given by $P_{n-1}F$ and D_nF. Recently, G. Arone and M. Ching came out with an optimistic refinement of Question 1. by asking the following:

Question 2 ([AC11])**.** Which additional structure should we endow on the sequence $\partial_*F = (\partial_0F, \partial_1F, ..., \partial_n, ..)$ in order to recover the Taylor tower $\{P_nF\}_n$ of F?

They started their analysis on a result of Ching ([Chi05]) which says that: given the identity functor $Id : Top_* \longrightarrow Top_*$, then the sequence $\partial_* Id$ is a "monoid" in the category of symmetric sequences. In other words, $\partial_* Id$ is an operad. These authors investigated in their article ([AC11, Remarks 17.27, 18.14, 19.3.]) the influence of the $\partial_* Id$-module structure on $\partial_* F$ (for an arbitrary functor F) and they have built a so-called "fake" tower $\Phi_* F$. This fake tower is equivalent to the Taylor tower $P_* F$ up to a certain "Tate spectrum" which vanishes in characteristic 0.

In this thesis, we follow the Arone-Ching strategy to investigate the Taylor towers of functors defined on algebraic categories, more precisely on the categories of algebras over operads in chain complexes. In that setting and when the underlying field is of characteristic 0, we have been able to get very explicit models for both the derivatives $\partial_* F$ and the Taylor tower $\{P_n F\}_{n \geq 0}$.

Goodwillie, Arone-Ching study functors $F : \mathcal{C} \longrightarrow \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are each either $Top_*, sSet$(category of simplicial sets) or Sp. In our research, we make the following replacement

$Top_* \rightsquigarrow \mathrm{Alg}_{\mathcal{O}}$: The category of algebras over the operad $\mathcal{O}$;

$Sp \rightsquigarrow Ch$: The category of $\mathbb{Z}$-graded chain complexes over a field $\Bbbk$.

A typical example of such operad $\mathcal{O}$ is the Lie operad, $\mathcal{O} = Lie$ and an algebra over Lie is exactly the differential graded Lie algebra $\mathrm{Alg}_{Lie} = DGL$.

On the other hand, our transposition is relevant as in rational homotopy theory, the rational localization of Top_* (resp. Sp) is equivalent to DGL (resp. Ch). However, note that the equivalences between these algebraic and topological categories are only given by zig-zag. Hence our transposition is far from relying in a sort of transfer structure theorem.

The idea of extending Goodwillie's constructions to other categories is almost old as the theory itself. One of the early contributors is Kuhn ([Kuh07]). Among the other published papers in this sense , we can quote: Walter [Wal06], Pereira [Per13], G.Biedermann and O. Röndigs [BR14], D. Barnes and R. Eldred [BE16], J. Lurie [Lur17, § 6].

As we stated above, one can see DGL as the category Alg_{Lie} of algebras over the "Lie operad". In this process, the object called "Lie operad" is a sequence of chain complexes which has in arity two the Lie bracket $[x_1, x_2]$, and more generally in arbitrary arity n live all the n-length brackets $[x_1, ...[x_{n-1}, x_n]]$. In other words, the Lie operad is meant to encode all the operations that one can define on a Lie algebra. Likewise, the commutative operad "Com" governs all the operations that one can define on a commutative algebra. Finally, note that these operads "Com" and "Lie" are monoid in a certain category, thus it will make sense to consider modules over Lie and modules over Com.

Summary of results

In this thesis, we study Goodwillie theory in $\mathrm{Alg}_{\mathcal{O}}$: The category of algebras over an arbitrary operad $\mathcal{O}$ of Ch_+(the category of non-negatively graded chain complexes). However, in this introduction, as we avoid a formal definition of operads, we present our results in the explicit case of the Lie operad as people not familiar with operads could follow. Hence, our statements are all given on $\mathrm{Alg}_{Lie} = DGL$.

Among the authors who studied the extension of Goodwillie calculus, we make a focus on Kuhn's research who formulated in [Kuh07] a general model category requirement for running Goodwillie's arguments to an arbitrary homotopy functor $F : \mathcal{C} \longrightarrow \mathcal{D}$. By investigation, our algebraic categories Ch, Ch_+, DGL meet Kuhn's requirements. Therefore we get freely (using the Kuhn's or Goodwillie's construction) the Taylor tower for functors. However, we don't have yet a formula for D_nF as in Equation (4), hence the object ∂_*F is still meaningless. This is because, in Kuhn's argument, he will additionally require that the categories $\mathcal{C}$ and $\mathcal{D}$ are tensored over $sSet$. This is not our case since there is not a genuine tensoring of the category Ch over sSets.

In the first step of our research, we have conducted a preliminary investigation in other to show that in this algebraic setting, we have a formula for the functors D_nF similar to Equation (4).

Theorem A (Theorem 2.22)**.** We assume char($\Bbbk$)=0. Let $\mathcal{C}, \mathcal{D} = Ch, Ch_+$ or DGL and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a homotopy functor. There is a chain complex $\partial_n F \in Ch$, with an action of the symmetric group Σ_n, such that we form an equivalence (natural in L)

$$D_nF(L) \simeq \Omega^\infty(\partial_n F \underset{h\Sigma_n}{\otimes} (\Sigma^\infty L)^{\otimes n}) \tag{5}$$

Formula (5) is completely analogous to Goodwillie's formula (4) , but we need to explain the meaning of Σ^∞ and Ω^∞ in our algebraic setting:

- The functor $\Sigma^\infty : DGL \longrightarrow Ch$ associates to any Lie-algebra L, its derived indecomposable part. For instance if $L = \mathbb{L}_V$ is the free Lie algebra generated by a chain complex V, then $\Sigma^\infty(L) \simeq V$. More generally, $\Sigma^\infty L$ is equivalent to the Chevalley-Eilenberg functor applied to the DGL L(see [**?**]) :
$$\Sigma^\infty L \simeq C_*(L) = (\Lambda(sL), d).$$
- The functor $\Omega^\infty : Ch \longrightarrow DGL$ associates to a $\mathbb{Z}$-graded chain complex (V, d), the $\mathbb{N}$-graded chain complex
$$V' = [(\ker\ (V_0 \xrightarrow{d_0} V_{-1}), V_1, V_2, ..., V_k, ...), d]$$
endowed with a trivial Lie-bracket.

Theorem A was originally proved by Walter [Wal06] in the special case of the Lie operad. In this thesis, we generalize this to the case of any $\mathcal{O}$-algebra, where $\mathcal{O}$ is a reduced operad.

Now that we have this result, we can reconsider Question 1 and thus Question 2 in our context. We arrive now at the most important result of this thesis which is an explicit description of the Taylor tower $\{P_nF\}_n$ for homotopy functors $F : DGL \longrightarrow Ch$ when the ground field is of characteristic 0.

We first state the result before describing some of its ingredients.

Theorem B (Theorem 5.35)**.** We assume char($\Bbbk$)=0. Let $F : DGL \longrightarrow Ch$ be a simplicial and finitary functor and $P = B(Com)^\vee$. Then the Taylor tower of F is given by

$$P_nF(L) \simeq B(\partial_{*\leq n}F, P, L) \tag{6}$$

where $\partial_{*\leq n}F$ is the truncated symmetric sequence of chain complexes

$$\partial_{*\leq n}F := (\partial_0 F, \partial_1 F, ..., \partial_n F, 0, ..., 0, ...).$$

To make sense, we need to explain the various ingredients in this theorem. In a nutshell:

- A finitary functor is a functor which commutes with directed or filtered colimits;

- Com is the commutative operad;

- $B(Com)^\vee$ is the linear dual of its bar-construction. In fact $B(Com)^\vee = Lie_\infty$ which is a cofibrant replacement of the Lie operad;

- $B(\partial_{*\leq n}F, P, L)$ is the bar construction with coefficients that we will explain below;

- A simplicial and finitary functor is a technical hypothesis on functor which is valid in more interesting homotopy functors.

In Theorem B, we need a structure of right $B(Com)^\vee$-module on $\partial_* F$ which is not explicit in Theorem A. This right module structure will be the consequence of Theorem C below, but first let us explicitly explain what we mean by the bar-construction. Namely, we will explain the chain complex $B(Lie)$ called " bar-construction " on the Lie operad (cf § 1.7). This is roughly speaking a combinatorial object made of colored non planar trees where a vertex of the tree with two entries is colored with the Lie bracket : $[x_1, x_2]$; and in general a vertex of the tree with n entries is colored with an n^{th} iteration of the Lie bracket: $[x_1...[x_{n-1}, x_n]]$. Here below is an example of a tree T which lives in $B(Lie)(5)$. The number 5 refers to the set of index $\{1, 2, 3, 4, 5\}$ which is used to label the leaves of the tree.

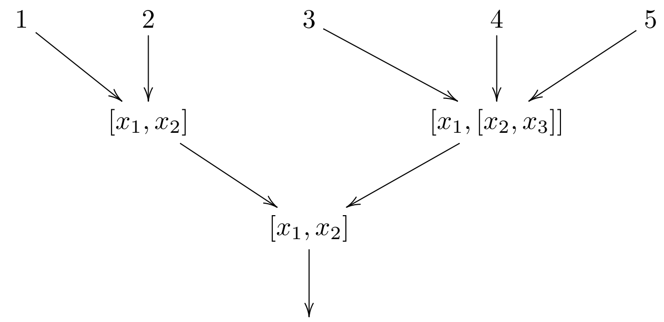


Fig 1. The tree T.

There is a natural decomposition map on the bar construction which consists of un-grafting trees. For instance the above tree T of $B(Lie)(5)$ could be decomposed into a tree $T_1 \in B(Lie)(2)$ which is the root sub-tree of T, the tree $T_2 \in B(Lie)(2)$ and $T_3 \in B(Lie)(3)$ which was both grafted on T_1. This is shown here below in Fig 2.

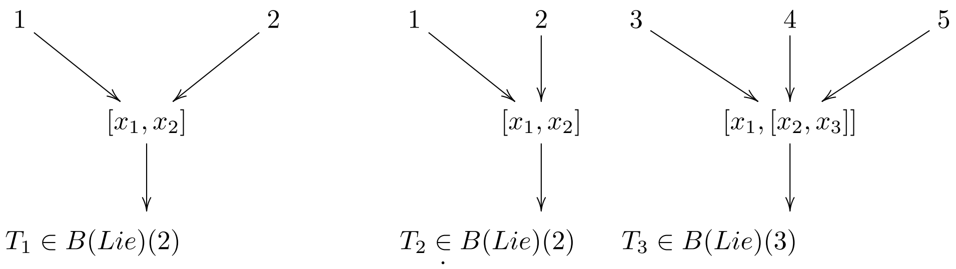


Fig 2. Decomposition of T

This decomposition of the tree T is encoded by the formula:

$$B(Lie)(5) \longrightarrow B(Lie)(2) \otimes B(Lie)(2) \otimes B(Lie)(3) \tag{7}$$

More generally, this operation looks like this :

$$B(Lie) \longrightarrow B(Lie) \circ B(Lie) \tag{8}$$

where the right hand side $B(Lie) \circ B(Lie)$ of Equation (8) generalizes the right hand side of Equation (7) as we consider all possible decompositions.

More generally, there is the notion of bar construction with coefficients $B(R, Lie, L)$, provided a right Lie-module R and a left Lie-module L. It is also defined with trees and the only change resides on the way one labels the vertices on a tree. This bar construction also has a decomposition.

The conclusion of this detour is the following:

- The notion of bar construction $B(Lie)$ is defined with trees and is actually a cooperad (with the decomposition (8)). Therefore, its linear dual $B(Lie)^\vee$ is an operad;

- There is also the bar construction with coefficients $B(R, Lie, L)$ which is a chain complex with some extra that will appear latter.

This bar-construction exists for any operad, and not only Lie. In particular $B(Com)$ for the commutative operad.

Theorem C (Theorem 5.10)**.** We assume char($\Bbbk$)=0. Given a simplicial functor $F : DGL \longrightarrow Ch$, there is a filtered diagram $\mathcal{B}_F$ of right Com-modules R such that we get a quasi-isomorphism

$$\partial_* F \simeq \operatorname*{colim}_{R \in \mathcal{B}_F} B(R, Com, \mathbb{I})^\vee \tag{9}$$

In this expression,

- Com is the "commutative" operad which governs the operations on a commutative algebra.
- $\mathbb{I}$ is the trivial symmetric sequence $\mathbb{I} = (0, \Bbbk, 0, ..., 0, ...)$. It is always a left Com-module.

The statement of Theorem B might not tell you too much because it doesn't specify R or $\mathcal{B}_F$, but let us first say that when F is a finite filtered colimit of representable functors (see below), then we can take

$$R = Nat(F\Omega^\infty I, I^{\otimes *}) = \{Nat(F\Omega^\infty I, I^{\otimes n})\}_n$$

where

- the functor $F\Omega^\infty I$ is the composite

$$Ch_+ \xrightarrow{I} Ch \xrightarrow{\Omega^\infty} DGL \xrightarrow{F} Ch;$$

- the functor $I^{\otimes n}$ is given by

$$I^{\otimes n} : Ch_+ \longrightarrow Ch, V \longmapsto V^n;$$

- $Nat(F\Omega^\infty I, I^{\otimes *})$ is the chain complex of natural transformations.

In this case, $\mathcal{B}_F = \{R\}$ is the category with a single object.

Using the decomposition of the bar construction (in the general case) analogous to Equation (8) where Lie is replaced by Com, and then taking the dual, one can deduce that the derivative $\partial_* F$ is a right module over the monoid $P = B(Com)^\vee$ which is actually equivalent to the Lie operad. In other words, we get the following:

Corollary B-1. *(Corollary 5.11) Using Equation* (9)*, we can endow $\partial_* F$ with the structure of a $B(Com)^\vee$-right module.*

We still need to explain what we mean by "simplicial functors" in Theorem B and in Theorem C.

Functors that we study in Goodwillie calculus are always "homotopy functor" in the sense that they send weak equivalences to weak equivalences (in topological spaces this mean weak homotopy equivalences and in categories related to chain complexes it means quasi-isomorphisms, that is morphisms that induce an isomorphism in homology).

Actually many (and maybe even all) such homotopy functors are simplicial functors (for which we give a special definition in our setting). The key point on these functors is that they can be built out of "representable functors" which have the form:

$$N\Bbbk[Hom_{DGL}(\mathbb{L}_V, - \otimes Apl_\bullet)] \tag{10}$$

where

- $\mathbb{L}_V$ is a quasi-free differential graded Lie algebra;
- $Apl_\bullet$ is the simplicial commutative algebra of polynomial of differential forms;
- For any Lie algebras L then

  $$\Bbbk[Hom_{DGL}(\mathbb{L}_V, L \otimes Apl_\bullet)]$$

  is the free simplicial chain complex generated by the simplicial set

  $$Hom_{DGL}(\mathbb{L}_V, L \otimes Apl_\bullet).$$

- $N : sVect_\Bbbk \longrightarrow Ch$ is the normalization functor;

Remark 0.2. *The results we have presented in this section was given when the ground field is of characteristic* 0. *This is a technical requirement as many constructions such as the homotopy pushouts and pullback formulas was given under this restriction. However the result of Theorem A can be extended to any characteristics (when $\mathcal{D} = Ch$) if we put now restrictions on the operad $\mathcal{O}$ (e.g. like being cofibrant as a symmetric sequence).*

Furthermore, the restriction of Theorem B and Theorem C in 0 *characteristic is due to a serious obstruction.*

Perspectives. In this thesis, we give in Theorem B an explicit description of the Taylor tower of simplicial functors $F : DGL \longrightarrow Ch$. We conjecture that Theorem C could be extended to simplicial functors $F : DGL \longrightarrow DGL$ but we have not yet the proof.

Some evidence of our conjecture is that, the identity functor $Id : DGL \longrightarrow DGL$ is a simplicial functor and its derivatives are given by:

$$\partial_* Id = Lie$$

We know that the Taylor tower of Id is given by (in [KP17])

$$P_n Id(L) \simeq B(Lie_{*\leq n}, P, L)$$

So the question is:

Can we extend Theorem C to simplicial functors $F : DGL \longrightarrow DGL$?

Outline. This thesis has the following outline:

- The first chapter gives the classical definitions and constructions on algebraic categories. Namely, we remind the categories Ch, Ch_+, and $\mathrm{Alg}_{\mathcal{O}}$ (The reader can find their location in the Index). We remind some notions on operads and cooperads such as the cobar-bar duality. We explore the homotopy theory of the category $\mathrm{Alg}_{\mathcal{O}}$. More precisely, we study homotopy limits and homotopy colimits. Then we construct an explicit characterization of homotopy pushouts and homotopy pullbacks in this category. In application, we get an explicit model for the loop and suspension functors. At the end of the chapter, we give a short resume which contents the objects that we will need in the next chapters.

- In the second chapter, we introduce Goodwillie calculus in algebraic context. In particular, we construct the Taylor tower and study the derivative through cross-effect and multilinearization. The goal of this chapter is to prove Theorem A. This theorem is the foundation of our functor calculus approach as we deduce from it the notion of derivatives $\partial_* F$, for any homotopy functors F. At the end of this chapter, we compute explicitly the derivatives of many interesting functors. In particular of $Id : \mathrm{Alg}_{\mathcal{O}} \longrightarrow \mathrm{Alg}_{\mathcal{O}}, \Sigma^\infty\Omega^\infty : Ch \longrightarrow Ch$ and some representable functors in Ch.

  In the rest of the thesis, we will now try to answer to Question 2 raised in this introduction.

- The goal of chapter 3 is to construct enriched categories (associated to $\mathrm{Alg}_{\mathcal{O}}$) that will permit us to get additional properties on functors $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$. Namely we will build a simplicial category $\mathrm{Alg}'_{\mathcal{O}}$ whose objects are the same as the category $\mathrm{Alg}_{\mathcal{O}}$. People familiar with this theory know from Hinich's work that the category $\mathrm{Alg}_{\mathcal{O}}$ (or DGL in particular) is itself enriched over simplicial sets. But, for some technical reasons, this is not the enrichment that we consider. We further deduce from $\mathrm{Alg}'_{\mathcal{O}}$ a category $\widetilde{\mathrm{Alg}}_{\mathcal{O}}$ which is enriched over Ch. On the other hand, the category Ch is enriched over itself. We will then consider enriched functors $\widetilde{F} : \widetilde{\mathrm{Alg}}_{\mathcal{O}} \longrightarrow Ch$. These enriched functors will uniquely induce each a real functor $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$ that is our focus.

- Chapter 4 appears as the next step after the foundation of Chapter 3. In this chapter, we describe the model structure on the category of enriched functors. This is in fact a cofibrantly generated model structure, so it permits a cellular decomposition of enriched functors. In other words, we show that simplicial functors are filtered homotopy colimits of representable functor as (10).

- Chapter 5 has two important parts. We first prove Theorem B using the cellular decomposition of functors introduced in Chapter 4. We then have all the ingredients to state and prove Theorem C in the second part.

  Finally in the last section of this chapter (§5.5) we compute as example the Taylor tower of two functors: the representable functor and the forgetful functor
$$\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, -), IU(-) : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch.$$

CHAPTER 1

Algebraic categories

1.1 Background on chain complexes

We consider that chain complexes are over a field $\Bbbk$ of any characteristics. The purpose of this section is to fix conventions and review basic properties which are the background of our constructions. A summary of the most important construction in this first chapter is given in section 1.14 and the hasty reader could jump there to have an overview of the chapter.

In this thesis, we denote by Ch the category of $\mathbb{Z}$- graded chain complexes over $\Bbbk$. Objects in this category are pairs (V, d) where V is a graded vector space $V = \oplus_{n\in\mathbb{Z}} V_n$, and $d : V \longrightarrow V$ (called the differential) which is a morphism of graded vector spaces which decreases the degree by 1 and satisfies the equation $d^2 = 0$. Morphisms in this category are degree 0 maps $f : V \longrightarrow V$ which are compatible with the differentials. This category has a symmetric monoidal structure. The tensor product of chain complexes $V, W \in Ch$ is defined by:

$$(V \otimes W)_n := \bigoplus_{p+q=n} V_p \otimes W_q$$

with the differential such that: $\forall x \otimes y \in V_p \otimes W_q,\ d(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes d(y)$. The unit of the monoid $- \otimes -$, which we denote abusively $\Bbbk$, is the chain complex having $\Bbbk$ in degree 0 and is trivial in all other degrees. The switch morphism $T : V \otimes W \longrightarrow W \otimes V$ involves the Koszul sign: $T(x \otimes y) = (-1)^{pq} y \otimes x$.

The tensor product $- \otimes -$ has a right adjoint $\underline{hom}(-, -)$ given by:

$$\underline{hom}(V, W) := \bigoplus_{i\in\mathbb{Z}} \underline{hom}^i(V, W)$$

where $\underline{hom}^i(V, W)$ denotes the vector space of morphisms $f : V_* \longrightarrow V_{*+i}$ of degree i.

Similarly We denote by Ch_+, the sub-category of Ch which consist of non negatively graded chain complexes. There is a natural adjunction between these two categories given by

$$I : Ch_+ \rightleftarrows Ch : red_0$$

where I is the inclusion functor defined by $I(V)_t := \begin{cases} V_t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$ and where for any chain complex C_*, $red_0(C_*)_t := \begin{cases} C_t & \text{if } t > 0 \\ ker(d_0) & \text{if } t = 0 \end{cases}$

Finally, we denote by Ch_- the sub-category of Ch which consist of non positively graded chain complexes.

In this thesis, when we will say chain complexes, we will mean either Ch or Ch_+ depending on the context, and we will be more precise if needed.

Twisted chain complex

Let (V, d_V) be a chain complex. A *twisting homomorphism* of degree -1, $d : V \longrightarrow V$ is a morphism of graded vector space of degree -1 which is added to the internal differential d_V to produce a new differential $d_V + d : V \longrightarrow V$ on V. The equation $(d_V + d)^2 = 0$ is equivalent to the equation

$$d_V(d) + d^2 = 0$$

in $Hom_{Ch_+}(V, V)$, with $d_V(d) := d_V d + d d_V$. This later equation will be called the *equation of twisting homomorphism* and the new chain complex $(V, d_V + d)$ is called the twisted chain complex associated to $(V, d_V + d)$.

Model category structure on Ch_+

The category Ch_+ is a cofibrantly generated model category (for instance see [Qui67, II p. 4.11, Remark 5], [DS95, Thm 7.2]):

- weak equivalences are quasi-isomorphisms;
- fibrations are the morphisms that are surjective in degree > 0;
- cofibrations are monomorphism with degreewise projective cokernels.

Since $\Bbbk$ is a field, all objects are cofibrant and fibrant in this model category. In addition, this category is proper closed.

Model category structure on Ch

The category Ch is a cofibrantly generated model category (for instance see [HPS97, remark after Thm. 9.3.1]):

- weak equivalences are quasi-isomorphisms;
- fibrations are surjections and
- cofibrations are morphisms having the left lifting property with respect to trivial fibrations.

In this model category, the cofibrations, that we do not describe explicitly, are in particular degreewise split injections.

1.2 Dold-Kan correspondence

The category of non negatively graded chain complexes Ch_+ is Quillen equivalent to the category $sVect_{\Bbbk}$ of simplicial vector spaces over $\Bbbk$. More precisely there are two functors:

$$\begin{aligned} N : sVect_{\Bbbk} &\longrightarrow Ch_+ \\ A_\bullet &\longmapsto NA_\bullet \end{aligned}$$

is the normalization functor defined as follows: $(NA_\bullet)_n$ consists of the subgroup of A_n that is killed by the face maps $d_i, i < n$. The differential $(NA)_n \longrightarrow (NA)_{n-1}$ is given by $(-1)^n d_n$.

$$\begin{aligned} \Gamma : Ch_+ &\longrightarrow sVect_{\Bbbk} \\ V &\longmapsto Hom_{Ch_+}(N\Bbbk\triangle^\bullet, V) \end{aligned}$$

Theorem 1.1. *([Dol58, Thm1.9], [SS03, §4]) The pair (N, Γ) and (Γ, N) are Quillen equivalences with respect to these model categories.*

1.3 Operads of chain complexes

1.3.1 Symmetric sequences

We give here the definition of a symmetric sequence along with the monoidal structure on the category of symmetric sequences. We refer to [Chi12, § 2.] for more details on this topic. We denote by $FinSet$ the category whose objects are finite sets and whose morphisms are bijections. Let $FinSet_0$ be the subcategory of $FinSet$ whose objects are the finite sets $\underline{r} := \{1, ..., r\}$ for $r \geq 0$ (with $\underline{0}$ the empty set), and whose morphisms are bijections.

In this thesis, we will only consider symmetric sequences and operads on Ch or Ch_+ and therefore we specialize our definition to that context.

Definition 1.2 (Symmetric sequence)**.** *Let $\mathcal{C} = Ch$ or Ch_+.*

1. *A symmetric sequence in the category $\mathcal{C}$ is a functor $M : FinSet \longrightarrow \mathcal{C}$. We denote the category of all symmetric sequences in $\mathcal{C}$ by $[FinSet, \mathcal{C}]$ (in which morphisms are natural transformations).*
2. *The composition $M \circ L$, of the two symmetric sequences M and L, is defined by:*

$$(M \circ L)(J) := \overset{\infty}{\underset{r=0}{\oplus}} [\underset{J=J_1 \amalg \ldots \amalg J_r}{\oplus} M(\underline{r}) \otimes L(J_1) \ldots \otimes L(J_r)]_{\Sigma_r}. \tag{1.1}$$

where Σ_r acts diagonally by permuting the sets $J_1, ..., J_r$ on one hand and on the other hand using the internal action on $M(\underline{r})$.

3. The unit symmetric sequence $\mathbb{I}$ is given by

$$\mathbb{I}(J) = \Bbbk, \text{ if } |J| = 1, \text{ and } \mathbb{I}(J) = 0 \text{ otherwise;}$$

Any symmetric sequence $M : FinSet \longrightarrow \mathcal{C}$ is determined, up to canonical isomorphism, by its restriction $M : FinSet_0 \longrightarrow \mathcal{C}$. This restriction consists of the sequence $M(\underline{0}), M(\underline{1}), M(\underline{2}), ...$ of objects in $\mathcal{C}$, together with an action of the symmetric group Σ_r on $M(\underline{r})$, hence the name "symmetric sequence."

The category $(Ch_+, \otimes, \Bbbk)$ (resp. $(Ch, \otimes, \Bbbk)$) is closed symmetric monoidal since the tensor product $\otimes$ has a right adjoint, so it preserves all colimits. Therefore $([FinSet, Ch_+], \circ, \mathbb{I})$ (resp. $([FinSet, Ch], \circ, \mathbb{I})$) is a monoidal category.

1.3.2 Operads

Let $\mathcal{C} = Ch_+$ or Ch. An operad in $\mathcal{C}$ is a monoid in $([Finset, \mathcal{C}], \circ, \mathbb{I})$.

The unit operad is the symmetric sequence $\mathbb{I}$ of Definition 1.2 with the only multiplication $\mathbb{I}(\underline{1}) \otimes \mathbb{I}(\underline{1}) \longrightarrow \mathbb{I}(\underline{1})$ which is the multiplication in $\mathbb{I}(\underline{1}) = \Bbbk$.

An augmented operad is an operad $\mathcal{O}$ together with a morphism of operads $\varepsilon : \mathcal{O} \longrightarrow \mathbb{I}$.

A reduced operad is an operad $\mathcal{O}$ such that $\mathcal{O}(\underline{0}) = 0$ and $\mathcal{O}(\underline{1}) = \Bbbk$. Note that a reduced operad is automatically augmented. The category of augmented operads in $\mathcal{C}$ is denoted $Op_{\mathcal{C}}$.

1.4 Algebra over an operad

Let $\mathcal{O}$ be an augmented operad in Ch_+. An $\mathcal{O}-$algebra consists of a chain complex (X, d) together with structure maps, for any $n \geq 0$:

$$m_n : \mathcal{O}(n) \underset{\Sigma_n}{\otimes} X^{\otimes n} \longrightarrow X,$$

satisfying the appropriate compatibility conditions, and where the symmetric group Σ_n acts diagonally by (on one hand) its usual action on $\mathcal{O}(n)$ and by (on other hand) permuting the factors of $X^{\otimes n}$.

Maps of $\mathcal{O}-$algebras are given by chain complex morphisms $f : X \longrightarrow X'$ which are degree 0 and preserve the $\mathcal{O}-$algebra structures of X and X'. The category of $\mathcal{O}$-algebra is denoted $\mathrm{Alg}_{\mathcal{O}}$.

Free $\mathcal{O}$-algebra

A free $\mathcal{O}$-algebra is an algebra of the form $(\mathcal{O}(V), d_0)$ where

- $\mathcal{O}(V) = \underset{n\geq 0}{\oplus} \mathcal{O}(n) \underset{\Sigma_n}{\otimes} V^{\otimes n}$ and
- the differential $d_0 : \mathcal{O}(V) \longrightarrow \mathcal{O}(V)$ is induced in the usual way by the differentials $\{d : \mathcal{O}(n) \longrightarrow \mathcal{O}(n)\}_n$ and $d : V \longrightarrow V$.

There is an obvious forgetful functor $U : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch_+$ which is the right adjoint of the functor

$$\begin{aligned} \mathcal{O}(-) : Ch_+ &\longrightarrow \mathrm{Alg}_{\mathcal{O}} \\ V &\longmapsto \mathcal{O}(V). \end{aligned}$$

Quasi-free $\mathcal{O}$-algebra

A quasi-free $\mathcal{O}$-algebra is an algebra of the form $(\mathcal{O}(V), d, m)$, where the differential $d = d_0 + \delta$ consists of:

- A linear part $d_0 : \mathcal{O}(V) \longrightarrow \mathcal{O}(V)$ defined as in the above free case;
- A non-linear part $\delta = d_1 + d_2 + ... : \mathcal{O}(V) \longrightarrow \mathcal{O}(V)$ induced by derivation on the restrictions $d_i : V \longrightarrow \mathcal{O}(i) \underset{\Sigma_i}{\otimes} V^{\otimes i}$.

Literally a quasi-free algebra is a twisted free $\mathcal{O}$-algebra $(\mathcal{O}(V), d_0)$ with the twisting homomorphism δ which respects the $\mathcal{O}$-algebra structure. They are called "almost free $\mathcal{O}$-algebras" in [GJ94].

Model category structure on $\mathrm{Alg}_{\mathcal{O}}$

The adjunction between the free and forgetful functors

$$\mathcal{O}(-) : Ch_+ \rightleftarrows \mathrm{Alg}_{\mathcal{O}} : U$$

permits to define the projective model structure on $\mathrm{Alg}_{\mathcal{O}}$ (see [GJ94, Thm 4.4]). Namely weak equivalences(resp. fibrations) of $\mathrm{Alg}_{\mathcal{O}}$ are equivalences (resp. fibrations) in the underlined category Ch_+. The cofibrations are morphisms having the right lifting property with respect to acyclic fibrations. In particular, cofibrant $\mathcal{O}$-algebras are retract of quasi-free algebras.

1.5 Module over an operad

A right (resp. left) module over an operad $\mathcal{O}$ consists of a symmetric sequence R (resp. L) together with a structure map

$$R \circ \mathcal{O} \longrightarrow R \text{ (resp. } \mathcal{O} \circ L \longrightarrow L)$$

satisfying usual associativity as unit axioms.

A morphism of right (resp. left) $\mathcal{O}$-modules $f : R \longrightarrow R'$ (resp. $f : L \longrightarrow L'$) is a morphism of symmetric sequences which is compatible with the right (resp. left) module structure.

The category of right (resp. left) $\mathcal{O}$-module is denoted by $\mathcal{O}$-mod (resp. mod-$\mathcal{O}$).

Left module generated by $\mathcal{O}$-algebras

Given an $\mathcal{O}$-algebra X, there is an associated left $\mathcal{O}$-module $\widehat{X}$ concentrated in arity 0 defined as follows

$$\begin{cases} \widehat{X}(\underline{0}) = X & ; \\ \widehat{X}(\underline{n}) = 0 & \text{if } n > 0. \end{cases}$$

The left module structure map

$$m : \mathcal{O} \circ \widehat{X} \longrightarrow \widehat{X}$$

is induced uniquely by the $\mathcal{O}$-algebra structure maps $m_n : \mathcal{O}(n) \underset{\Sigma_n}{\otimes} X^{\otimes n} \longrightarrow X$. This defines an embedding functor $\widehat{-} : \mathrm{Alg}_{\mathcal{O}} \longrightarrow \mathcal{O}$-mod.

1.6 Cooperad of chain complexes

The notion of cooperad in Ch_+ is dual to the notion of operad in Ch_+. The dual notion consists of considering the opposite category $((Ch_+)^{op}, \otimes, \mathbb{I}_{Ch})$. We define the dual composition product $\widehat{\circ}$ of two symmetric sequences by replacing the coproduct in the Definition 1.2 (in Equation (1.1)) with a product. That is

$$(M \widehat{\circ} L)(J) := \prod_{r=0}^{\infty} [\prod_{J = J_1 \amalg ... \amalg J_r} M(\underline{r}) \otimes L(J_1) ... \otimes L(J_r)]_{\Sigma_r}. \tag{1.2}$$

where Σ_r acts by permuting the sets $J_1, ..., J_r$ and on $M(\underline{r})$ in the usual way.

Note that if the symmetric sequence L is connected ($L(0) = 0$), then the external product(over r) in $(M \widehat{\circ} L)(J)$ of Equation (1.2) is always a finite product. Since finite products and direct sums are equivalent in the underlying category Ch_+, we will have the isomorphism

$$M \widehat{\circ} L \cong M \circ L.$$

Definition 1.3 (Cooperad)**.** *1. A cooperad in Ch_+ is a triple (Q, m^c, η^c), where Q is a symmetric sequence in Ch_+ together with maps (of chain complexes)*

$$m^c : Q \longrightarrow Q \widehat{\circ} Q \text{ and } \eta^c : Q \longrightarrow \mathbb{I}$$

satisfying the co-associativity, the left and right co-unit condition.

2. *A coaugmented cooperad is a cooperad Q together with a morphism of cooperads $\varepsilon^c : \mathbb{I} \longrightarrow Q$ from the trivial cooperad.*
3. *A coaugmented cooperad Q is connected when $\widetilde{Q} := coker(\varepsilon^c)$ is concentrated strictly in positive degree.*
4. *A cooperad Q is reduced if $Q(0) = 0$ and $Q(1) = \mathbb{k}$.*

Note that reduced cooperads are automatically coaugmented.

In this paper, we only consider connected coaugmented cooperads. A morphism of connected coaugmented cooperads $f : Q \longrightarrow Q'$ is a morphism of symmetric sequences which is compatible with the product $\widehat{\circ}$. The category of connected coaugmented cooperads is denoted $coOp_{Ch_+}$.

1.7 Coalgebra over a cooperad

Another dual analogy with operads is the notion of the coalgebra over a cooperad. That is, any chain complex Y together with a structure map, $\forall n$, $m^c_n : Y \longrightarrow Q(n) \underset{\Sigma_n}{\otimes} Y^n$ satisfying the appropriate compatibility conditions. The maps of Q-coalgebras are degree 0 chain complex morphisms $f : Y \longrightarrow Y'$ which preserves the structures of Y and Y'. One denotes the category of Q-coalgebras by coAlg_Q.

Model category on $\mathbf{coAlg}_Q$

One assume now that the cooperad Q is connected. One consider the canonical adjunction:

$$U : \mathrm{coAlg}_Q \rightleftarrows Ch_+ : Q(-)$$

where $U : \mathrm{coAlg}_Q \longrightarrow Ch_+$ is the forgetful functor and $Q(-) : Ch_+ \longrightarrow \mathrm{coAlg}_Q$ is the co-free functor.

We use this adjunction to define an injective model structure on coAlg_Q (see [GJ94, Thm 4.7]). Namely weak equivalences(resp. cofibrations) of coAlg_Q are weak equivalences(resp. cofibrations) in the underlined category weak Ch_+. The fibrations are morphisms having the left lifting property with respect to acyclic cofibrations.

1.8 Bar construction

In this section:

- We define colored trees. These are non planar trees whose vertices and leaves are colored using symmetric sequences. The notion of "Tree" is fundamental to define the bar construction and cobar constructions.

- We define the two sided bar construction on an augmented operad $\mathcal{O}$, and denoted $B(R, \mathcal{O}, L)$, provided a right $\mathcal{O}$-module R and a left $\mathcal{O}$-module L. This is a symmetric sequence of chain complexes.

- We raise the fact that when $R = L = \mathbb{I}$, the bar construction $B(\mathcal{O}) := B(\mathbb{I}, \mathcal{O}, \mathbb{I})$ is a cooperad.

- Using the two sided bar construction notion, we define the bar construction $B(\mathcal{O}, X)$ (on a given $\mathcal{O}$-algebra X). This will be a $B(\mathcal{O})$-coalgebra.

1.8.1 Tree

We discuss in this part colored trees. We will illustrate two kinds which are different in terms the level of coloring. Finally, we will define a combinatorial object which is obtained by coloring the "space" of trees with a single symmetric sequences.

J-tree

Let J be a finite set. A $J-$tree is an abstract oriented tree with one outgoing edge at the bottom, and ingoing edges on the top indexed by J. These ingoing and outgoing edges are the *external edges* of the tree. The other edges are called *internal* edges. The structure of a J-tree T is defined by a set of vertices $V(T)$, a set of edges $E(T)$, together with a source map $s : E(T) \longrightarrow V(T) \amalg J$ and a target map $t : E(T) \longrightarrow V(T)$ such that given an edge $v \xrightarrow{e} w \in E(T)$, $s(e) := v$ and $t(e) := w$.

The sources of the ingoing edges are not considered as vertices. They are called leaves and labeled by elements of J. The source of the outgoing edge is an internal vertex called the root ot T. We write $in(v) \subset E(T)$ for the set of edges of T whose target is the vertex v. The inputs of the vertex $v \in V(T)$ is the set $J_v \subset V(T) \amalg J$ formed either by the sources of the edges of $in(v)$ or by leaf indexes.

$$\begin{tikzcd}
j_1 \arrow[dr, "e_1"'] & j_2 \arrow[d, "e_2"] & j_3 \arrow[dr, "e_3"'] & j_4 \arrow[d, "e_4"] & j_5 \arrow[dl, "e_5"] \\
& v_1 \arrow[dr, "e_6"'] & & v_2 \arrow[dl, "e_7"] & \\
& & r \arrow[d, "e_8"] & & \\
& & {} & &
\end{tikzcd}$$

In the above tree, $J = \{j_1, j_2, j_3, j_4, j_5\}$; r is the root, $inv(v_1) = \{e_1, e_2\}$ and $J_{v_1} = \{j_1, j_2\}$.

The set of $J-$trees, denoted by $\theta(J)$, is equipped with a natural groupoid structure. Formally, an isomorphism of $J-$trees $\theta : T' \longrightarrow T$ is defined by bijections $\theta_V : V(T') \longrightarrow V(T)$ and $\theta_E : E(T') \longrightarrow E(T)$ preserving the source and target of edges. In other word, $\theta(J)$ is the groupoid of $J-$labeled trees and non-planar isomorphisms.

$(J - R - M - L)$-tree

Let J be a finite set, and R, M, L three symmetric sequences on chain complexes.

A $(J - R - M - L)$-tree is a colored tree $T(y, c, x)$ whose:

- The root r is labeled by $y \in R(J_r)$;

- Each internal vertex $v \in V(T)$ is labeled by $c_v \in M(J_v)$ and $c := \bigotimes_{v \in V(T)} c_v$;

- Finally, the leaves $\{l_j, j \in \underline{k}\}$ of T are labeled by a partition $J = J_1 \amalg \ldots \amalg J_k : \forall j, x_j \in L(J_j)$ and $x := \bigotimes_{j \in \underline{k}} x_j$.

1.8.2 Two sided bar construction

The bar construction is a combinatorial object which is in operad theory an analogue to free modules or free groups in classical algebra. We define the bar construction using trees.

Given a symmetric sequence M and a tree T, we can always define a new symmetric sequence by labeling the vertices of the tree T with the elements of M. The *free object* associated to M and denoted by $F(M)$ consists of: chain complexes $(F(M)(J), \partial_0)$, for any finite set J, defined as

$$F(M)(J) = \bigoplus_{T \in \theta(J)} T(M)/\equiv$$

where $T(M) = \bigotimes_{v \in V(T)} M(J_v)$, and the equivalence classes are made of non planar isomorphisms of $J-$trees. The differential ∂_0 is induced naturally by the differentials of the chain complexes $(M(J_v), \partial_{J_v})$.

A bijection $\theta : J \longrightarrow J'$ gives an isomorphism $F(M)(J) \longrightarrow F(M)(J')$ by relabeling the leaves of the underlined trees. In this way $F(M)$ becomes a symmetric sequence in chain complexes.

Definition 1.4 (Two sided bar construction)**.** *Let $\mathcal{O}$ be an augmented operad, R be a right $\mathcal{O}$-module and L be a left $\mathcal{O}$-module. The two sided bar construction $B(R, \mathcal{O}, L)$ is the symmetric sequence of chain complexes given by: for any finite set J,*

$$B(R, \mathcal{O}, L)(J) := (R \circ F(s\widetilde{\mathcal{O}}) \circ L(J), \partial_0 + \partial), \text{ with } \widetilde{\mathcal{O}} = ker\ \varepsilon.$$

The differential ∂_0 is induced in the natural way by the differentials of the chain complexes $\{(R(J'), d_{J'})\}_{J' \subseteq J}$, $\{(\mathcal{O}(J'), d_{J'})\}_{J' \subseteq J}$, and $\{(L(J'), d_{J'})\}_{J' \subseteq J}$. The second differential $\partial = \partial_R + \partial_{\mathcal{O}} + \partial_L$ of this complex is the derivation which integrates the structure morphisms: $m_R : R \circ \mathcal{O} \longrightarrow R, m_L : \mathcal{O} \circ L \longrightarrow L$, and $m_{\mathcal{O}} : \mathcal{O} \circ \mathcal{O} \longrightarrow \mathcal{O}$ (for explicit description, see [Fre04, § 4.4.3.]).

The example in Fig. 1 below shows how ∂ is applied to a $J-R-s\widetilde{\mathcal{O}}-L$-tree:

Let $x_j \in L(J_j)$ for $j = 1, 2, 3, 4$ and $J_1 \amalg J_2 \amalg J_3 \amalg J_4 = J$ and $c_1, c_2 \in \widetilde{\mathcal{O}}(\underline{2})$ and $y \in R(\underline{2})$;

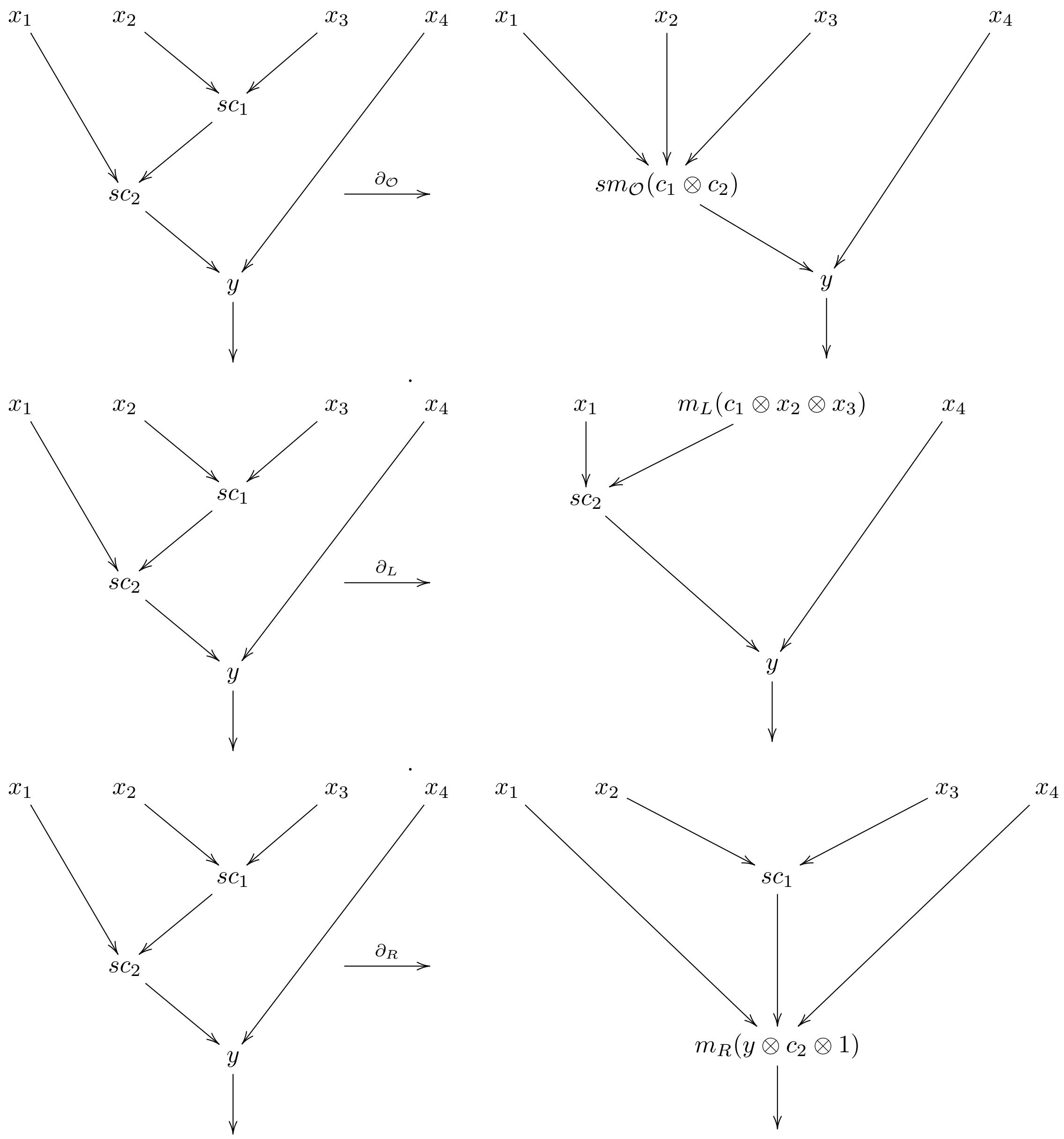


Fig 1.

Decomposition of the two sided bar construction

Let $\mathcal{O}$ be an augmented operad, R be a right $\mathcal{O}$-module and L be a left $\mathcal{O}$-module. The two sided bar construction $B(R, \mathcal{O}, L)$ has a natural decomposition map

$$B(R, \mathcal{O}, L) \longrightarrow B(R, \mathcal{O}, \mathbb{I}) \widehat{\circ} B(\mathbb{I}, \mathcal{O}, L) \tag{1.3}$$

that we intend to describe in this part. Roughly speaking, this decomposition consists of ungrafting trees.

Using the notations on Trees introduced in Section 1.8.1, an element in the summand $B(R, \mathcal{O}, L) = R \circ F(s\widetilde{\mathcal{O}}) \circ L(J)$ can be presented as a colored $(J - R - s\widetilde{\mathcal{O}} - L)$-tree $T(y, c, x)$ together with a partition $J = J_1 \amalg ... \amalg J_k$ where, the root r of T is labeled by $y \in R(J_r)$, each internal vertex $v \in V(T)$ is labeled by $c_v \in s\widetilde{\mathcal{O}}(J_v)$ and $c := \underset{v \in V(T)}{\otimes} c_v$; finally the leaves $\{l_j, j \in \underline{k}\}$ of T are labeled by a partition $J = J_1 \amalg ... \amalg J_k$ with $x_j \in L(J_j)$, and $x := \underset{j \in \underline{k}}{\otimes} x_j$.

A tree colored tree $T(y, c, x)$ can be seen as $(I_i - \mathbb{I} - s\widetilde{\mathcal{O}} - L)$-trees: $T_i(1, c_i, x_i), i \in \{1, ..., n\}$, grafted each on a single $(\underline{n} - R - s\widetilde{\mathcal{O}} - \mathbb{I})$-tree $T'(y, c, 1)$ with exactly n leaves, where:

- the leaves of a tree $T_i(1, c_i, x_i)$ are labeled by a partition $I_i = J_{1n_i} \amalg ... \amalg J_{\alpha n_i}$
- $c_1 \otimes ... \otimes c_n = c$;
- $x_1 \otimes ... \otimes x_n = x$;
- 1 is the unit in the field $\mathbb{k}$.

In other words, we have

$$T(y, c, x) = T'(y, c, 1)[T_1(1, c_1, x_1), ..., T_n(1, c_n, x_n)]. \tag{1.4}$$

The right hand side of Equation (1.4) is isomorphic to

$$T'(y, c, 1) \otimes T_1(1, c_1, x_1) \otimes ... \otimes T_n(1, c_n, x_n)$$

which lives in $B(R, \mathcal{O}, \mathbb{I}) \hat{\circ} B(\mathbb{I}, \mathcal{O}, L)(J)$.

The morphism of Equation (1.3) is defined naturally by means of this identification.

1.8.3 Bar construction on $\mathcal{O}$

This is a two sided bar construction on $\mathcal{O}$ when we consider trivial the left and right $\mathcal{O}$-modules.

Definition 1.5 (bar construction)**.** *Let $\mathcal{O}$ be an operad. If $R = L = \mathbb{I}$, the bar construction $B(\mathcal{O})$ is obtained from Definition 1.4: for any finite set J,*

$$B(\mathcal{O})(J) := (F(s\widetilde{\mathcal{O}})(J), \partial_0 + \partial), \text{ with } \widetilde{\mathcal{O}} = \ker \varepsilon.$$

The more important thing to know about this simple case is that we actually get a cooperad structure map on the bar construction $B(\mathcal{O})$, obtained by replacing $L = R = \mathbb{I}$ in Equation (1.3). This is proved in [[GJ94, §1.7]].

Lemma 1.6. *Let $\mathcal{O}$ be an augmented operad. Then the bar construction $B(\mathcal{O})$ is a connected coaugmented cooperad.*

1.8.4 Bar construction on $\mathcal{O}$-algebras

We define the bar construction $B(\mathcal{O}, X)$ on an $\mathcal{O}$-algebra X using the two sided bar construction. For that, we consider the embedding functor

$$\widehat{-} : \mathrm{Alg}_{\mathcal{O}} \longrightarrow \mathcal{O}\text{-mod}$$

(defined in Section 1.5) from $\mathcal{O}$-algebras into the category of left $\mathcal{O}$-modules.

Definition 1.7 (Bar construction on algebras)**.** *Let X be an algebra over an augmented operad $\mathcal{O}$. We define the bar construction on $\mathcal{O}$ with coefficient in X as the chain complex:*

$$B(\mathcal{O}, X) := (B(\mathbb{I}, \mathcal{O}, \hat{X})(0), \partial_0 + \partial)$$

The bar construction $B(\mathcal{O}, X)$ is not only a chain complex. It is actually a $B(\mathcal{O})$-coalgebra. The coalgebra structure map is obtained by replacing $L = \hat{X}$ and $R = \mathbb{I}$ in Equation (1.3).

Lemma 1.8. *Let $\mathcal{O}$ be an augmented operad and let X be an $\mathcal{O}$-algebra. Then the bar construction $B(\mathcal{O}, X)$ is a coalgebra over the cooperad $B(\mathcal{O})$.*

On the other hand, note that $B(\mathcal{O}, X)$ is the infinite sum

$$B(\mathcal{O}, X) = \bigoplus_{n \geq 0} B(\mathcal{O})(n) \underset{\Sigma_n}{\otimes} X^{\otimes n}.$$

1.9 Cobar construction

Let $(Q, Q \xrightarrow{m^c} Q \circ Q, Q \xrightarrow{\eta^c} \mathbb{I}, \mathbb{I} \xrightarrow{\varepsilon^c} Q)$ be a connected coaugmented cooperad in Ch_+, and denote $\widetilde{Q} := coker(\varepsilon^c)$.

We consider the cobar construction of Q with coefficients in a right Q-comodule R and a left Q-comodule L, whose the definition is dual to that of the bar construction in Definition 1.4. We also indicate Fresse ([Fre04, § 4.7.1]) as a reference for this part.

Definition 1.9 (cobar construction)**.** *We consider the above notations.*

1. *The cobar construction $B^c(R, Q, L)$ is the symmetric sequence: for any finite set J, sequence:*

   $$B^c(R, Q, L)(J) := (R \circ F(s^{-1}\widetilde{Q}) \circ L(J), \partial_0 + \partial^*),\ \textit{with}\ \widetilde{Q} = coker(\varepsilon^c).$$

   The differential ∂_0 is induced in the usual way and $\partial^* = \partial^R + \partial^Q + \partial^L$ is the dual of the twisting differential ∂ in Definition 1.4.

2. *When $L = R = \mathbb{I}$, then $B^c(R, Q, L)$ will simply be denoted $B^c(Q)$.*

In this definition, the cooperad Q needs to be connected to avoid the case where $F(s^{-1}\widetilde{Q})$ has elements in negative degree.

1.10 Cobar-Bar adjunction on operads

The bar and construction that we have defined in the previous sections respectively on operad and cooperads and not just dual in their construction. They are actually adjoint functors.

We are now ready to state the next theorem which gives a duality between the bar construction and the cobar construction.

Theorem 1.10. *[GJ94, Theorem 2.17] The functors*

$$B^c(-) : coOp_{Ch_+} \rightleftarrows Op_{Ch_+} : B(-)$$

between the categories of connected coaugmented cooperads and the category of augmented operads, form and adjoint pair $B^c(-) \vdash B(-)$.

In addition, it is proved in [GK95, Theorem 3.2.16] that the unit $\eta : Q \longrightarrow BB^c(Q)$ and the counit $\varrho : B^cB(\mathcal{O}) \longrightarrow \mathcal{O}$ of this adjunction are quasi-isomorphisms.

We end this part by reminding a useful homotopy invariance property of the bar construction carried by the bar-cobar adjunction. More precisely, under the quasi-isomorphism $B^c(B(\mathcal{O})) \underset{\simeq}{\xrightarrow{\varrho}} \mathcal{O}$, we deduce that any $\mathcal{O}$-algebra X has a natural $B^c(B(\mathcal{O}))$-algebra structure . We use this assumption to state the next lemma.

Lemma 1.11. *Let $X \in Alg_{\mathcal{O}}$. The morphism of $B(\mathcal{O})$-coalgebras*

$$B(\varrho, X) : B(B^c(B(\mathcal{O})), X) \longrightarrow B(\mathcal{O}, X)$$

is a weak equivalence.

Proof. We form the following diagram

$$\begin{tikzcd}
B(B^c(B(\mathcal{O})), X) = BB^cB(\mathcal{O}) \circ \widehat{X} \arrow[r, "B(\varrho)"] & B(\mathcal{O}, X) = B(\mathcal{O}) \circ \widehat{X} \\
B(\mathcal{O}, X) = B(\mathcal{O}) \circ \widehat{X} \arrow[u, "\eta \circ \widehat{X}"] \arrow[ur, "Id"] &
\end{tikzcd}$$

where $\eta : B(\mathcal{O}) \longrightarrow BB^cB(\mathcal{O})$ is the unit of the cobar-bar adjunction (B^c, B) applied to the cooperad $Q = B(\mathcal{O})$.

Since η is a quasi-isomorphism, it follows that $B(\varrho)$ is also a quasi-isomorphism. $\square$

1.11 Cobar-Bar adjunction on $\mathcal{O}$-algebras

We have defined in Section 1.8.4 the bar construction functor

$$B(\mathcal{O}, -) : \mathrm{Alg}_{\mathcal{O}} \longrightarrow \mathrm{coAlg}_{B(\mathcal{O})}$$

between the category of $\mathcal{O}$-algebras and the category of $B(\mathcal{O})$-coalgebras. In this section, we construct a left adjoint of $B(\mathcal{O}, -)$ which is actually a left Quillen equivalence

$$\Omega_{\mathcal{O}}(-) : \mathrm{coAlg}_{B(\mathcal{O})} \longrightarrow \mathrm{Alg}_{\mathcal{O}}$$

Note that the functor $\Omega_{\mathcal{O}}(-)$ that we we define, which is also called "cobar construction" in the literature is different from the functor B^c given in Definition 1.9 which is mainly used for modules (symmetric sequences). In this section, when we will say cobar construction on a coalgebra, we always mean the functor $\Omega_{\mathcal{O}}(-)$.

1.11.1 Cobar construction on Q-coalgebras

Let Q be a connected cooperad on chain complexes and Y a Q-coalgebra. We define here the cobar construction with a "twisting cochain" as it appears in [GJ94] and [LV12, § 11.2.8.].

We consider to have from now a reduced operad $\mathcal{O}$ such that $B^c(Q) \xrightarrow{\simeq} \mathcal{O}$. This later morphism induces a degree 0 morphism $s^{-1}\widetilde{Q} \longrightarrow \widetilde{\mathcal{O}}$ which gives a morphism $\pi : \widetilde{Q} \longrightarrow \widetilde{\mathcal{O}}$ of degree -1. The morphism π is generally named in the literature *twisting cochain* (see [GJ94, def 2.16]).

We use π to define the composite

$$w : Y \xrightarrow{m_Y^c} Q(Y) \longrightarrow \widetilde{Q}(Y) \xrightarrow{\pi(Y)} \widetilde{\mathcal{O}}(Y) \longrightarrow \mathcal{O}(Y)$$

The derivation $d_w : \mathcal{O}(Y) \longrightarrow \mathcal{O}(Y)$ of degree -1 associated to this w satisfies the equation of twisting homomorphism $d(w) + d_w.w = 0$ on Y. This is equivalent to say that $(\mathcal{O}(Y), d + d_w)$ is a quasi-free $\mathcal{O}$-algebra.

Definition 1.12 (cobar construction on a Q-coalgebra)**.** *The cobar construction on Y, associated to the twisting cochain $\pi : \widetilde{Q} \longrightarrow \widetilde{\mathcal{O}}$, and denoted $\Omega_\pi(Q, Y)$ is the quasi-free $\mathcal{O}$-algebra*

$$\Omega_\pi(Q, Y) = (\mathcal{O}(Y), d + d_w)$$

where d is the internal differential of $\mathcal{O}(Y)$ induced by the complexes $\mathcal{O}$ and Y.

Notation 1. *When $Q = B(\mathcal{O})$ and π is the natural projection (of degree -1) $\widetilde{B(\mathcal{O})} \longrightarrow \widetilde{\mathcal{O}}$, then the cobar construction $\Omega_\pi(Q, Y)$ will simply be denoted $\Omega_{\mathcal{O}}(Y)$.*

We form the cobar-bar adjoint pair

$$\Omega_{\mathcal{O}}(-) : \mathrm{coAlg}_{B(\mathcal{O})} \rightleftarrows \mathrm{Alg}_{\mathcal{O}} : B(\mathcal{O}, -)$$

whose the unit and co-unit functors have the following property:

Theorem 1.13 ([GJ94], Theorem 2.19)**.** *Given an $\mathcal{O}$-algebra X and a $B(\mathcal{O})$-coalgebra Y, the co-unit $\Omega_{\mathcal{O}}(B(\mathcal{O}, X)) \longrightarrow X$ and the unit $Y \longrightarrow B(\mathcal{O}, \Omega_{\mathcal{O}}(Y))$ are weak equivalences.*

With the model structure defined on $\mathrm{coAlg}_{B(\mathcal{O})}$ and $\mathrm{Alg}_{\mathcal{O}}$, we can see that the cobar-bar adjunction is actually a Quillen pair, and Theorem 1.13 completes in proving that this adjunction is a Quillen equivalence.

1.11.2 Cofibrant replacement in $\mathbf{Alg}_{\mathcal{O}}$

In this part, we use the co-unit of the cobar-bar adjunction on algebras to construct a functorial cofibrant replacement of any $\mathcal{O}$-algebra. We raise the reader's attention on the following:

- The operad $\mathcal{O}$ is now a reduced operad;
- The field $\mathbb{k}$ must satisfy one of the following conditions:
  (a) The field $\mathbb{k}$ is a field of characteristic 0 with no other restriction of the operad $\mathcal{O}$;
  (b) The field $\mathbb{k}$ is of any characteristic, but the operad $\mathcal{O}$ and cooperad $B(\mathcal{O})$ must be Σ_*-cofibrant, to mean cofibrant objects in the category of symmetric sequences.
- We are not saying that we have a Quillen equivalence between algebras and coalgebras in this case (at least, it not clearly stated in our reference papers). However given a reduced operad $\mathcal{O}$, which is in particular an augmented operad, it make sens to consider in this case the adjunction counit $\varrho : B^c B(\mathcal{O}) \longrightarrow \mathcal{O}$ (in Theorem 1.10). On the other hand, given an $\mathcal{O}$-algebra X, it also make sens to consider the counit $\Omega_{\mathcal{O}}(B(\mathcal{O}, X)) \longrightarrow X$ (in Theorem 1.13).

The results we use in this section was given for chain complexes over a ring in their original versions. Since we are only working on a field $\mathbb{k}$, we have rephrased these results in our context to keep our notations and assumptions. An interested reader can check out the references provided for more complete statements.

Proposition 1.14. *([Fre04, Prop 3.1.12.]) We assume that the ground field $\mathbb{k}$ is of any characteristic. Let $\mathcal{O}$ be a reduced operad. Then the cobar-bar adjunction unit $\varrho : B^c B(\mathcal{O}) \longrightarrow \mathcal{O}$ is a quasi-isomorphism.*

Theorem 1.15. *([Fre09, Thm 4.2.4]) Let $\mathcal{O}$ be any Σ_*-cofibrant operad. Let Q be any Σ_*- cofibrant reduced cooperad ($Q(0) = 0$ and $Q(1) = \mathbb{k}$) together with a twisting cochain $\theta : \widetilde{Q} \longrightarrow \widetilde{\mathcal{O}}$ associated to a weak equivalence $\phi_\theta : B^c(Q) \xrightarrow{\simeq} \mathcal{O}$.*

If X is a $\mathcal{O}$-algebra which is cofibrant as a chain complex, then the cobar-bar co-unit

$$\Omega_{\mathcal{O}}(B(\mathcal{O}, X)) \longrightarrow X$$

defines a weak equivalence and $X^c := \Omega_{\mathcal{O}}(B(\mathcal{O}, X))$ forms a cofibrant replacement of X in the category $Alg_{\mathcal{O}}$.

Note that when we consider $Q = B(\mathcal{O})$ and $\phi_\theta = \varrho$ (in Proposition 1.14), then Theorem 1.15 generalizes Theorem 1.13 which is only true in characteristic 0.

We end this section by defining the cofibrant replacement functor

$$\begin{aligned} (-)^c : \mathrm{Alg}_{\mathcal{O}} &\longrightarrow \mathrm{Alg}_{\mathcal{O}} \\ X &\longmapsto \Omega_{\mathcal{O}}(B(\mathcal{O}, X)) \end{aligned}$$

1.12 Definition of the functors Ω^∞ and Σ^∞

We define in this section two functors:

$$\Omega^\infty : Ch \longrightarrow \mathrm{Alg}_{\mathcal{O}} \text{ and}$$
$$\Sigma^\infty : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$$

which have the following properties:

- The functors Σ^∞ and Ω^∞ are homotopy invariant functors;

- These functors do not form adjoint pair but the composite $\Sigma^\infty\Omega^\infty$ is a comonad.

A reader familiar with homotopy theory in topological spaces might be interested with the relation between these functors and the usual ones defined between the category Top_* of topological spaces and the category Sp of spectra. In fact these functors are used in describing the loop and the suspension functors in the category $\mathrm{Alg}_{\mathcal{O}}$. For more detail, we refer to Proposition 1.25 and Corollary 1.29.

Let $\mathcal{O}$ be a reduced operad. The natural augmentation $\varepsilon : \mathcal{O} \longrightarrow \mathbb{I}$ is used to define a functor:

$$\mathbb{I} \underset{\mathcal{O}}{\circ} - : \mathrm{Alg}_{\mathcal{O}} \longrightarrow \mathrm{Alg}_{\mathbb{I}} = Ch_+$$

given by $\mathbb{I} \underset{\mathcal{O}}{\circ} X := colim_{Ch_+}(\mathbb{I}(\mathcal{O}(X)) \rightrightarrows \mathbb{I}(X))$

The first map of this colimit is produced by the multiplication $\mathbb{I} \circ \mathcal{O} \longrightarrow \mathbb{I}$ which is in fact the augmentation ε; The second map is given by the $\mathcal{O}$-algebra structure map $\mathcal{O}(X) \longrightarrow X$.

We define the abelianization functor as its composite with the forgetful functor

$$(-)^{ab} : \mathrm{Alg}_{\mathcal{O}} \xrightarrow{\mathbb{I} \underset{\mathcal{O}}{\circ} -} Ch_+ \overset{I}{\hookrightarrow} Ch$$

where I is the inclusion functor defined by $I(V)_t := \begin{cases} V_t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$

Lemma 1.16. *The abelianization functor has a right Quillen adjoint functor given by:*

$$\Omega^\infty : Ch \xrightarrow{red_0} Ch_+ \xrightarrow{(-)_{triv}} Alg_{\mathcal{O}}$$

where for any chain complex C_*, $red_0(C_*)_t := \begin{cases} C_t & if\, t > 0 \\ ker(d_0) & if\, t = 0 \end{cases}$

and $(-)_{triv}$ is the functor which assigns to any non negative graded chain complex the trivial $\mathcal{O}$-algebra structure.

Proof. It will be sufficient to prove the adjunctions

$$\mathbb{I} \underset{\mathcal{O}}{\circ} - \dashv (-)_{triv} \text{ and } I \dashv red_0$$

The second adjunction is straightforward. For the first adjunction, we define the map

$$\begin{aligned}\gamma : Hom_{Ch_+}(\mathbb{I} \underset{\mathcal{O}}{\circ} X, W) &\longrightarrow Hom_{\mathrm{Alg}_{\mathcal{O}}}(X, (W)_{triv})\\ f &\longmapsto (X \longrightarrow \mathbb{I} \underset{\mathcal{O}}{\circ} X \xrightarrow{f} f)\end{aligned}$$

One can check easily that this map is well defined. The inverse of this map is defined as follows:

$$\gamma' : Hom_{\mathrm{Alg}_{\mathcal{O}}}(X, (W)_{triv}) \longrightarrow Hom_{Ch_+}(\mathbb{I} \underset{\mathcal{O}}{\circ} X, W)$$

Let $h \in Hom_{\mathrm{Alg}_{\mathcal{O}}}(X, (W)_{triv})$. We have by definition of algebra structure the relation $\varepsilon(1_W) \circ \mathcal{O}(h) = h \circ m$, where $m : \mathcal{O} \circ \mathcal{O} \longrightarrow \mathcal{O}$ is the operad multiplication.

On the other hand, we have $\mathbb{I}(h) \circ \varepsilon(1_X) = \varepsilon(1_W) \circ \mathcal{O}(h)$. Therefore, we deduce the relation $h \circ m = \mathbb{I}(h) \circ \varepsilon(1_X)$. Thus using the universal property of the co-equalizers, there is a unique map $f : \mathbb{I} \underset{\mathcal{O}}{\circ} X \longrightarrow W$ such that h is the composite $h : X \longrightarrow \mathbb{I} \underset{\mathcal{O}}{\circ} X \xrightarrow{f} W$. One can check easily that $\gamma\gamma' = Id$ and that $\gamma'\gamma = Id$. This part proves the adjunction.

Now to prove that the pair $(-)^{ab} \dashv \Omega^\infty$ is a Quillen par, it is sufficient to prove that Ω^∞ preserves fibrations (surjections) and acyclic fibrations (quasi-isomorphic surjections). This is again straightforward since the model structure defined on the category $\mathrm{Alg}_{\mathcal{O}}$ is the projective one induced by the model structure of Ch_+, and the functor red_0 is a homotopy functor. □

We then deduce from the above analysis that we have an adjunction pair

$$(-)^{ab} : \mathrm{Alg}_{\mathcal{O}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} Ch : \Omega^\infty.$$

The functor $(-)^{ab}$ does not preserves quasi-isomorphisms in general, apart from preserving quasi-isomorphisms between quasi-free algebras (since they are cofibrant objects in $\mathrm{Alg}_{\mathcal{O}}$), Its derived associate functor is what is called in the literature Quillen homology.

Definition 1.17 (Quillen homology)**.** *If X is an $\mathcal{O}$-algebra, the* Quillen homology *$TQ(X)$ of X is the $\mathcal{O}$-algebra $\mathbb{I} \overset{h}{\underset{\mathcal{O}}{\circ}} X$.*

We will give in the next lines an explicit model of the functor $TQ(-)$ which we will need to define Σ^∞.

We have defined in Section 1.11.2 the cofibrant replacement functor $(-)^c : X \longmapsto \Omega_{\mathcal{O}}(B(\mathcal{O}, X))$. Using this expression, we make the following computation:

$$\begin{aligned}TQ(X) &\simeq \mathbb{I} \underset{\mathcal{O}}{\circ} X^c\\ &= \mathbb{I} \underset{\mathcal{O}}{\circ} (\mathcal{O}(B(\mathcal{O}, X)))\\ &= UB(\mathcal{O}, X),\end{aligned}$$

where $U : \mathrm{Coalg}_{B(\mathcal{O})} \longrightarrow Ch_+$ is the forgetful functor. Under this last quasi-isomorphism, we will consider the functor $UB(\mathcal{O}, -)$ as our explicit model for the functor $TQ(-)$ and we will denote by Σ^∞ the composite:

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}} \arrow[r, "B(\mathcal{O},-)"] \arrow[rrr, bend right, "\Sigma^\infty"'] & \mathrm{Coalg}_{B(\mathcal{O})} \arrow[r, "U"] & Ch_+ \arrow[r, hook, "I"] & Ch \end{tikzcd}$$

Roughly speaking, Σ^∞ a the homotopy invariant version of the abelianization functor, thus it is not adjoint to the functor Ω^∞. However, there is the following important property:

Proposition 1.18. *The composite $T = \Sigma^\infty \Omega^\infty : Ch \longrightarrow Ch$ is a comonad.*

Proof. To prove the result, it will be sufficient to prove that T is the composite of a true right and left adjoint functors. For this, we consider two adjunctions

$$\begin{tikzcd} \mathrm{coAlg}_{B(\mathcal{O})} \arrow[r, shift left, "U"] & Ch_+ \arrow[l, shift left, "B(\mathcal{O},(-)_{triv})"] \arrow[r, shift left, "I"] & Ch \arrow[l, shift left, "red_0"] \end{tikzcd} ,$$

where the top functors are each left adjoint functor and the bottom functors are each right adjoint functors. We then observe that the associate comonad is $IUB(\mathcal{O}, (-)_{triv})red_0 \cong \Sigma^\infty\Omega^\infty$. □

Note that the comonad structure map $T \longrightarrow TT$ on T explained in Proposition 1.18 is essentially given by the cooperad coproduct

$$B(\mathcal{O}) \longrightarrow B(\mathcal{O}) \circ B(\mathcal{O}).$$

We can extend the construction of the functors Σ^∞ and Ω^∞ to other categories as follows:

- $\Sigma^\infty := I : Ch_+ \longrightarrow Ch$;
- $\Sigma^\infty = Id : Ch \longrightarrow Ch$;
- $\Omega^\infty = red_0- : Ch \longrightarrow Ch_+$;
- $\Omega^\infty = Id : Ch \longrightarrow Ch$.

1.13 Homotopy limits and colimits in $\mathrm{Alg}_{\mathcal{O}}$

The purpose of this section is to remind a brief notion of homotopy limits and colimits, and give their explicit description in $\mathrm{Alg}_{\mathcal{O}}$ in terms of holims and hocolims in chain complexes.

Let $\mathcal{C}$ and $\mathcal{D}$ be any of the categories $\mathrm{Alg}_{\mathcal{O}}, \mathrm{coAlg}_{B(\mathcal{O})}$ and Ch_+. These categories are complete and cocomplete. The authors of [DHKS04] proved, in a general argument for complete and cocomplete model categories, that holims and hocolims always exists in $\mathcal{C}$ and are homotopical unique (see [DHKS04, 19.2]). More explicitly, given a small category J, and an J-diagram D in $\mathcal{C}$, they replace D through a functor $D \longmapsto D_{vf}$ (resp. $D \longmapsto D_{vc}$) which associate a so called "virtually fibrant replacement" (resp "virtually cofibrant replacement")

D_{vf} (resp. D_{vc}) such that there is a map $D \xrightarrow{\simeq} D_{vf}$ (resp. $D_{vc} \xrightarrow{\simeq} D$) natural in D. These replacement functors have the following properties(see [DHKS04, 20.5]):

1. The limit (resp. colimit) functor in $\mathcal{C}$ turns an objectwise weak equivalence between two virtually fibrant (resp. cofibrant) diagrams into a weak equivalence between fibrant (resp. cofibrant) objects in $\mathcal{C}$;

2. Any Quillen adjoint pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ induces the adjoint pair $F^J : \mathcal{C}^J \rightleftarrows \mathcal{D}^J : G^J$ which has the following properties

   (a) F^J preserves virtual cofibrancy and weak equivalence between virtually cofibrant diagrams.

   (b) G^J preserves virtual fibrancy and weak equivalence between virtually fibrant diagrams.

According to this vocabulary we can now set the definition of holims and hocolims:

Definition 1.19. *Given an J-diagram D in $\mathcal{C}$,*

$$holim_{\mathcal{C}}(D) := lim_{\mathcal{C}}(D)_{vf} \text{ and } hocolim_{\mathcal{C}}(D) := colim_{\mathcal{C}}(D)_{vc}.$$

In practice, Dwyer-Spalinski explains, in [DS95, § 10.], that computing the homotopy pullback of a diagram $X \xrightarrow{g} Z \xleftarrow{f} Y$ in a model category $\mathcal{C}$ involves replacing Z by a fibrant object and replacing the maps $X \xrightarrow{g} Z$ and $Y \xrightarrow{f} Z$ by fibrations.

Dually, computing the homotopy pushout of a diagram $D : X \xleftarrow{g} Z \xrightarrow{f} Y$ in a model category $\mathcal{C}$ involves replacing Z by a cofibrant object and replacing the maps $Z \xrightarrow{g} X$ and $Z \xrightarrow{f} Y$ by cofibrations (see [DS95, § 10.]).

These process of computing homotopy pullbacks and pushouts are simplified in right and left proper model categories:

Definition 1.20 (Proper model category)**.** *1. A model category $\mathcal{C}$ is right proper when a pullback of a fibration over an equivalence is an equivalence. In other words, consider the pullback diagram:*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

If $X \longrightarrow Z$ is a fibration and $Y \longrightarrow Z$ is an equivalence, then $P \longrightarrow X$ is an equivalence.

2. A model category $\mathcal{C}$ is left proper when a pushout of a cofibration over an equivalence is an equivalence.

Lemma 1.21. *If $\mathcal{C}$ is a right proper model category, then the homotopy pullback of a diagram $X \xrightarrow{g} Z \xleftarrow{f} Y$ can be computed by replacing Z by a fibrant object and replacing at least one of the maps f and g by a fibration.*

Proof. We consider a diagram $D : X \xrightarrow{g} Z \xleftarrow{f} Y$ and we assume that Z is a fibrant object and that the map f is a fibration. Using the above Dwyer-Spalinski's argument, we want to show that the limit of D is equivalent to the homotopy colimit of D. More precisely, considering the factorization

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\simeq"'] & & Z \ , \\ & X' \arrow[ur, two heads, "g'"'] & \end{tikzcd}$$

we form the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\simeq"'] & Z \arrow[d, "="] & Y \arrow[l, two heads, "f"'] \arrow[d, "="] \\ X' \arrow[r, two heads, "g'"'] & Z & Y \arrow[l, two heads, "f"] \end{tikzcd}$$

We want to show that the limit of the two horizontal diagrams are equivalent.

To prove this, we consider the diagram

$$\begin{tikzcd} X' \underset{Z}{\times} Y \arrow[r] \arrow[d, two heads] & X \underset{Z}{\times} Y \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads] \\ X' \arrow[r, "\simeq"'] & X \arrow[r, "g"'] & Z \end{tikzcd}$$

The outer square is a pullback by construction. Therefore, using pullback properties, we deduce that the most left square is also a pullback. In addition the category $\mathcal{C}$ is right proper, so we deduce that the map $X' \underset{Z}{\times} Y \longrightarrow X \underset{Z}{\times} Y$ is an equivalence. $\square$

Dually, we have the next lemma for homotopy pushout diagrams in left proper model categories.

Lemma 1.22. *If $\mathcal{C}$ is a left proper model category, then the homotopy pushout of a diagram $X \xleftarrow{g} Z \xrightarrow{f} Y$ can be computed by replacing Z by a cofibrant object and replacing at least one of the maps f and g by a cofibration.*

Remark 1.23.

1. *The category Ch_+ is left and right proper;*
2. *Since the model structure on the category $Alg_{\mathcal{O}}$ is a projective one induced by the model structure of Ch_+, we deduce that $Alg_{\mathcal{O}}$ is right proper;*
3. *Since the model structure on the category $coAlg_{B(\mathcal{O})}$ is an injective one induced by the model structure of Ch_+, we deduce that $coAlg_{B(\mathcal{O})}$ is left proper;*
4. *Even if the category $Alg_{\mathcal{O}}$ is not left proper, using a Reedy's result ([Hir03, Prop 13.1.2]), to compute the homotopy pushout of a diagram $X \xleftarrow{g} Z \xrightarrow{f} Y$ in $Alg_{\mathcal{O}}$ where all the algebras X, Y and Z are cofibrant, we can follow the conclusion of Lemma 1.22.*

1.13.1 Model for the loop of an $\mathcal{O}$-algebra

In any model category, the suspension of an object is defined as the homotopy pullback

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

The goal of this section is to give an explicit model for the loop of $\mathcal{O}$-algebra.

Theorem 1.24. *Let $\Bbbk$ be a field of characteristic 0. Given $X \in Alg_{\mathcal{O}}$, there is a weak equivalence of $\mathcal{O}$-algebras*

$$\Omega X \simeq (red_0 s^{-1} X)_{triv}$$

An important consequence of this theorem is that any loop in Alg$_{\mathcal{O}}$ has a trivial $\mathcal{O}$-algebra structure. More precisely,

Proposition 1.25. *Let $\Bbbk$ be a field of characteristic 0. If Y is an $\mathcal{O}$-algebra such that $Y \simeq \Omega X$ then there is a weak equivalence of $\mathcal{O}$-algebras*

$$Y \simeq \Omega^\infty U Y$$

Proof. From Theorem 1.24, we deduce that $Y \simeq \Omega^\infty s^{-1} X$. When we apply the forgetful functor U, we get the quasi-isomorphism in chain complexes $UY \simeq U\Omega^\infty s^{-1}X$. We apply again the functor Ω^∞ and get the $\mathcal{O}$-algebra weak equivalences

$$\Omega^\infty UY \simeq \Omega^\infty U \Omega^\infty s^{-1} X \cong \Omega^\infty s^{-1} X \simeq Y.$$

□

The rest of this part is dedicated to the proof of Theorem 1.24. We will produce in fact an explicit model for homotopy pullbacks in Alg$_{\mathcal{O}}$.

As observed in Remark 1.23-(2) and in Lemma 1.21, the homotopy pullback of a diagram $D : X \xrightarrow{g} Z \xleftarrow{f} Y$ in Alg$_{\mathcal{O}}$ is calculated using observations in the underlined category Ch_+. Since any $\mathcal{O}$-algebra is fibrant, it reduces to replace the map f by a fibration (to mean a surjection in positive degrees). The replacement of f is done through a its factorization by an acyclic map followed by a fibration. In this process, we need to construct a new $\mathcal{O}$-algebra associated to Z called "path object".

Construction of path objects in Alg$_{\mathcal{O}}$

Let $\mathcal{I} = (\wedge(t, dt), d)$ be the free differential graded commutative algebra generated by the element t in degree 0 and dt in degree -1, with differential d given by $d(t) = dt$ and $d(dt) = 0$. It is useful to notice that an element α of $\mathcal{I}$ has the form $\alpha = P(t) + Q(t)dt$ with $P, Q \in \Bbbk[t]$.

There are natural commutative algebra maps $s_0 : \Bbbk \longrightarrow \mathcal{I}$ and $p_0, p_1 : \mathcal{I} \longrightarrow \Bbbk$ defined as: $\forall(\alpha = P(t) + Q(t)dt \in \mathcal{I})$ and $k \in \Bbbk$,

$$p_0(\alpha) := P(0),\ p_1(\alpha) := P(1) \text{ and } s_0(k) = k$$

s_0 is a quasi-isomorphism and $p_0 s_0 = p_1 s_0 = 1_{\Bbbk}$.

For any $\mathcal{O}$-algebra Z, there is a natural $\mathcal{O}$-algebra structure on $\mathcal{I} \otimes Z$ (see [Liv99, §2.4]) given by : If $a \in \mathcal{O}(n), \alpha_i \otimes x_i \in \mathcal{I} \otimes Z$, for $1 \leq i \leq n$,

$$m(a \otimes \alpha_1 \otimes x_1 \otimes ... \otimes \alpha_n \otimes x_n) := \pm \alpha_1 ... \alpha_n \otimes m_Z(a \otimes x_1 \otimes ... \otimes x_n \otimes);$$

One then get the factorization in $\mathcal{O}$-algebras (unbounded algebras)

$$\begin{tikzcd} Z \arrow[r, "s_0 \otimes Z"] & \mathcal{I} \otimes Z \arrow[r, shift left, "p_1 \otimes Z"] \arrow[r, shift right, "p_0 \otimes Z"'] & Z \end{tikzcd}$$

which yield to the diagram in $\mathrm{Alg}_{\mathcal{O}}$:

$$\begin{tikzcd} Z \arrow[r, "s_0^Z"] & red_0(\mathcal{I} \otimes Z) \arrow[r, shift left, "p_1^Z"] \arrow[r, shift right, "p_0^Z"'] & Z \end{tikzcd}$$

One can prove that p_0^Z and p_1^Z are trivial surjections in positive degrees.

Definition 1.26 (path object)**.** *A path object associated to an $\mathcal{O}$-algebra Z is the $\mathcal{O}$-algebra $Z^{\mathcal{I}} := red_0(\mathcal{I} \otimes Z)$ together with the $\mathcal{O}$-algebra morphisms p_0^Z, p_1^Z and s_0^Z.*

Construction of homotopy pullbacks in $\mathbf{Alg}_{\mathcal{O}}$

Let us consider the commutative diagram in $\mathrm{Alg}_{\mathcal{O}}$:

$$\begin{tikzcd} Y \arrow[dr, bend right, "f"'] \arrow[drr, dotted, "(s_0^Z f, Y)"] \arrow[drrr, bend left, "1_Y"] & & & \\ & Z \arrow[dr, bend right, "s_0^Z"'] & Z^{\mathcal{I}} \underset{Z}{\times} Y \arrow[r, "\simeq", "\pi_2"'] \arrow[d, "\pi_1"] & Y \arrow[d, "f"] \\ & & Z^{\mathcal{I}} \arrow[r, "p_0^Z", "\simeq"'] & Z \end{tikzcd}$$

where the square in the middle is a pullback. From the left triangle, we build the following factorization of f :

$$\begin{tikzcd} Y \arrow[r, "(s_0^Z f, Y)", "\simeq"'] \arrow[dr, "f = p_1^Z s_0^Z f"'] & Z^{\mathcal{I}} \underset{Z}{\times} Y \arrow[d, "p_1^Z \pi_1"] \\ & Z \end{tikzcd} \tag{1.5}$$

Lemma 1.27. *The morphism $p_1^Z \pi_1 : Z^{\mathcal{I}} \underset{Z}{\times} Y \longrightarrow Z$ is a fibration of $\mathcal{O}$-algebras.*

Proof. It is sufficient to show that the induced map in Ch_+ is a surjection in positive degrees. We consider and element $z \in Z$ with degree $|z| \geq 1$. Then the pair $(zt, 0)$ is an element of $Z^{\mathcal{I}} \underset{Z}{\times} Y$ as $p_0^Z(zt) = 0$. On the other hand, $p_1^Z \pi_1(zt, 0) = z$. □

We use the above factorization to replace f in a diagram $D : X \xrightarrow{g} Z \xleftarrow{f} Y$ by the fibration $p_1^Z \pi_1$.

Proposition 1.28. *Given an $\mathcal{O}-$algebra diagram $D : X \xrightarrow{g} Z \xleftarrow{f} Y$, a homotopy pullback of D is the $\mathcal{O}-$algebra $P_D = X \underset{Z}{\times} Z^{\mathcal{I}} \underset{Z}{\times} Y$, namely*

$$P_D = lim_{Alg_{\mathcal{O}}}(X \xrightarrow{g} Z \xleftarrow{p_1^Z \pi_1} Z^{\mathcal{I}} \underset{Z}{\times} Y).$$

Proof. This follows from the factorization (1.5) and Lemma 1.21. □

Proof of Theorem 1.24. We consider the map

$$\begin{aligned} \Phi : (red_0 s^{-1} X)_{triv} &\longrightarrow \Omega X \\ s^{-1}x &\longmapsto (0, dt \otimes x, 0) \end{aligned}$$

Our objective is to prove that Φ is well defined and is a weak equivalence.

We first prove that Φ is a map of $\mathcal{O}$-algebras. Namely let $x_1, ..., x_n \in X$, and $a \in \mathcal{O}(n), (n \geq 2)$, then

$$\begin{aligned} m_{\Omega X}(a \otimes \Phi(s^{-1}x_1) \otimes ... \otimes \Phi(s^{-1}x_n)) &= (0, dt^n \otimes m_X(a \otimes x_1 \otimes ... \otimes x_n), 0) \\ &= 0 \text{ (since } dt^n = 0) \end{aligned}$$

This computation proves that Φ is a map of $\mathcal{O}$-algebras as the $\mathcal{O}$-algebra structure on $(red_0 s^{-1} X)_{triv}$ is trivial. It is obvious that the map Φ commutes with the differentials of the two complexes.

Now we prove by hand that $H_*(\Phi)$ is injective and surjective. Let us take

$$\overline{x} = a_0 + \sum_{l \geq 1} t^l a_l + \sum_{k \geq 0} t^k dt b_k \in X^{\mathcal{I}} \text{ such that } (0, \overline{x}, 0) \in \Omega X \cap Ker d$$

where for each l and k, $a_l, b_k \in X$;

$$\begin{aligned} (0, \overline{x}, 0) \in \Omega X &\Longleftrightarrow p_1^X(\overline{x}) = 0 = p_0^X(\overline{x}) \\ &\Longleftrightarrow a_0 = 0 = \sum_{l \geq 1} a_l \end{aligned}$$

One can also see that

$$d\overline{x} = 0 \Longleftrightarrow \forall l \geq 1, da_l = 0 \text{ and } \sum_{l \geq 1} l t^{l-1} dt a_l = \sum_{k \geq 0} t^k dt d b_k$$

This last equality implies that $\forall l \geq 1, a_l = \frac{1}{l} db_{l-1}$ and thus $\sum\limits_{l\geq 1} \frac{1}{l} db_{l-1} = 0$ One then get:

$$\begin{aligned}
\overline{x} &= \sum_{l\geq 1} \frac{1}{l} t^l db_{l-1} + \sum_{l\geq 1} t^{l-1} dt b_{l-1} \\
&= \sum_{l\geq 1} \frac{1}{l}(t^l db_{l-1} + l t^{l-1} dt b_{l-1}) \\
&= d(\sum_{l\geq 1} \frac{1}{l} t^l b_{l-1}) \\
&= d(\sum_{l\geq 1} \frac{1}{l} t^l b_{l-1} - t \sum_{l\geq 1} \frac{1}{l} b_{l-1} + t \sum_{l\geq 1} \frac{1}{l} b_{l-1}) \\
&= d(\sum_{l\geq 1} \frac{1}{l} t^l b_{l-1} - t \sum_{l\geq 1} \frac{1}{l} b_{l-1}) + d(t \sum_{l\geq 1} \frac{1}{l} b_{l-1})
\end{aligned}$$

One can see that $\sum\limits_{l\geq 1} \frac{1}{l} t^l b_{l-1} - t \sum\limits_{l\geq 1} \frac{1}{l} b_{l-1} \in \Omega X$ and that $d(t \sum\limits_{l\geq 1} \frac{1}{l} b_{l-1}) = dt \otimes \sum\limits_{l\geq 1} \frac{1}{l} b_{l-1}$, therefore

$$[\overline{x}] = [dt \otimes \sum_{l\geq 1} \tfrac{1}{l} b_{l-1}] = H_*(\Phi)([s^{-1} \sum_{l\geq 1} \tfrac{1}{l} b_{l-1}])$$

This implies that $H_*(\Phi)$ is surjective.

To prove that $H_*(\Phi)$ is injective, let's take $[s^{-1}x] \in (red_0 s^{-1} X)_{triv}$ such that $H_*(\Phi)([s^{-1}x]) = 0$. This implies that $dtx = d\overline{x}$, for a given $\overline{x} \in \Omega X$. As before we set $\overline{x} = \sum\limits_{l\geq 1} t^l a_l + \sum\limits_{k\geq 0} t^k dt b_k$, with $\sum\limits_{l\geq 1} a_l = 0$. An easy comparison on the degree of the polynomials proves that

$$\begin{aligned}
dtx = \sum_{l\geq 1} l t^{l-1} dt a_l + \sum_{l\geq 1} t^l dt da_l - \sum_{k\geq 0} t^k dt db_k &\iff x = a_1 - db_0 \text{ and } \forall l \geq 2, a_l = \frac{1}{l} db_{l-1} \\
&\implies x = -\sum_{l\geq 2} \frac{1}{l} db_{l-1} - db_0 = d(-\sum_{l\geq 1} \frac{1}{l} b_{l-1})
\end{aligned}$$

this means that $[s^{-1}x] = 0$ and proves that $H_*(\Phi)$ is injective.

□

1.13.2 Model for the suspension of an $\mathcal{O}$-algebra

In any model category, the suspension of an object is defined as a homotopy pushout

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \Sigma X
\end{tikzcd}$$

The goal of this section is to give an explicit model for the suspension of $\mathcal{O}$-algebras, which holds when $\Bbbk$ is of any characteristic.

Theorem 1.29. *Given $Z \in Alg_{\mathcal{O}}$, there is a weak equivalence of $\mathcal{O}$-algebras*

$$\Sigma Z \simeq \mathcal{O}(sUB(\mathcal{O}, Z))$$

In particular, a suspension of an $\mathcal{O}$-algebra is always equivalent to a free $\mathcal{O}$-algebra.

The rest of this part is dedicated to the proof of Theorem 1.29. We will produce in fact an explicit model for homotopy pushouts in $\mathrm{Alg}_{\mathcal{O}}$. Our approach is a dual version of homotopy pullback. More precisely, we would like to compute the homotopy pushout of a diagram $D : X \xleftarrow{g} Z \xrightarrow{f} Y$ in $\mathrm{Alg}_{\mathcal{O}}$. Since this category is not left proper, we replace this diagram by an equivalent one

$$D^c : X^c \xleftarrow{g^c} Z^c \xrightarrow{f^c} Y^c$$

where

$$(-)^c : Z \longmapsto Z^c := \Omega_{\mathcal{O}}(B(\mathcal{O}, Z))$$

is the cofibrant replacement functor defined in Section 1.11.2. Note that all the objects in the diagram D^c are now cofibrant, therefore using Remark 1.23, we can use the result of Lemma 1.22, which means that we only have to take a factorization of the map f^c by a cofibration followed by an equivalence. For this factorization of f^c, we need to define a new $\mathcal{O}$-algebra associated to Z^c called " cylinder object".

Construction of a cylinder of a quasi-free $\mathcal{O}$-algebra

We assume that $\mathbb{k}$ is a field of characteristic 0. We give in this part the construction of a cylinder of a quasi-free $\mathcal{O}$-algebra in the same line that the definition for differential graded Lie algebras in [Tan83, II.5.], and for closed DGL's in [BFMT16, § 5.].

Let $(\mathcal{O}(V), d)$ be a quasi-free $\mathcal{O}$-algebra, and let V' be a copy of V. We define :

- $\mathcal{O}(V)\widehat{\otimes}\mathcal{I} := (\mathcal{O}(V \oplus V' \oplus sV'), D)$, where: $(sv')_n = v'_{n-1}$, $Dv' = 0$, $Dsv' = v'$, $Dv = dv$.
- $\lambda_0 : (\mathcal{O}(V), d) \longrightarrow \mathcal{O}(V)\widehat{\otimes}\mathcal{I}$ the canonical injection;
- $p : \mathcal{O}(V)\widehat{\otimes}\mathcal{I} \longrightarrow (\mathcal{O}(V), d)$ is the $\mathcal{O}$-algebra morphism given by:
  $p(v) = v$; $p(v') = p(sv') = 0$; p is a quasi-isomorphism since $\mathcal{O}(V' \oplus sV')$ is acyclic.
- $i : \mathcal{O}(V)\widehat{\otimes}\mathcal{I} \longrightarrow \mathcal{O}(V)\widehat{\otimes}\mathcal{I}$ is the degree $+1$ $\mathcal{O}$-algebra derivation given by: $i(v) = sv'$; $i(sv') = i(v') = 0$;
- The $\mathcal{O}$-algebra derivation of degree 0, $\theta = Di+iD$ verifies $\theta D = D\theta, \theta(v') = \theta(sv') = 0$. We have the induced automorphism of $\mathcal{O}$-algebras $e^\theta = \sum\limits_{n\geq 0} \frac{\theta^n}{n!}$ (with inverse $e^{-\theta}$).

The automorphism e^θ is well defined for the following reason: let $v \in V_n$. We write down explicitly the differential d of $(\mathcal{O}(V), d)$ by $d = d_1 + d_2 + ...$, where $d_k v \in \mathcal{O}(k) \underset{\Sigma_k}{\otimes} V^{\otimes k}$, for any given k. Computation gives that $\theta^2(v) = \theta i(d_2 v + d_3 v + ...) \in \mathcal{O}(V_{<n})\widehat{\otimes}\mathcal{I}$. Therefore we deduce inductively that for any $x \in \mathcal{O}(V) \otimes I$, there always exist an integer n_x such that $\theta^{n_x}(x) = 0$.

- We define the second injection $\lambda_1 : (\mathcal{O}(V), d) \longrightarrow \mathcal{O}(V)\widehat{\otimes}\mathcal{I}$ by, $\lambda_1(v) = e^{\theta}(v)$.

The couple $(\mathcal{O}(V)\widehat{\otimes}\mathcal{I}, \lambda_0, \lambda_1, p)$ forms a cylinder of $(\mathcal{O}(V), d)$.

Construction of homotopy pushouts in $\mathbf{Alg}_{\mathcal{O}}$

Let Z be an $\mathcal{O}$-algebra and we have $Z^c := \Omega_{\mathcal{O}}(B(\mathcal{O}, Z))$. The cylinder object defined above and associated to Z^c will be simply denoted

$$Z^c\widehat{\otimes}\mathcal{I} := (\mathcal{O}(V \oplus V' \oplus sV'), D_1),$$

where $V = B(\mathcal{O}, Z)$.

Let $Z \xrightarrow{f} Y$ in $\mathrm{Alg}_{\mathcal{O}}$, we apply the functor $(-)^c$ to get the weakly equivalent morphism $Z^c \xrightarrow{f^c} Y^c$. Let us consider the commutative diagram in $\mathrm{Alg}_{\mathcal{O}}$:

$$\begin{tikzcd}
Z^c \arrow[r, "f^c"] \arrow[d, "\lambda_0"', "\simeq"] & Y^c \arrow[d, "j_2"', "\simeq"] \arrow[rdd, bend left, "1_{Y^c}"] & \\
Z^c\widehat{\otimes}\mathcal{I} \arrow[r, "j_1"] \arrow[rd, bend right, "p"'] & Z^c\widehat{\otimes}\mathcal{I} \coprod\limits_{Z^c} Y^c \arrow[rd, dotted, "f^c p \amalg 1_{Y^c}"] \arrow[r, phantom, "="] & \\
 & Z^c \arrow[r, bend right, "f^c"'] & Y^c
\end{tikzcd}$$

where the square in the middle is a pushout. From the lower triangle, we can then build the following factorization of f^c :

$$\begin{tikzcd}
Z^c \arrow[r, "j_1\lambda_1"] \arrow[rd, bend right, "f^c"'] & Z^c\widehat{\otimes}\mathcal{I} \coprod\limits_{Z^c} Y^c \arrow[d, "f^c p \amalg 1_{Y^c}", "\simeq"'] \\
 & Y^c
\end{tikzcd}$$

We use this later factorization to replace f^c in the diagram $D^c :\ X^c \xleftarrow{g^c} Z^c \xrightarrow{f^c} Y^c$ by the cofibration $\pi_1\lambda_1$.

Proposition 1.30. *We assume that char($\Bbbk$)=0. Given a $\mathcal{O}$-algebra diagram $D :\ X \xleftarrow{g} Z \xrightarrow{f} Y$, a homotopy pushout of D is given by $C_D = X^c \coprod\limits_{Z^c} Z^c\widehat{\otimes}\mathcal{I} \coprod\limits_{Z^c} Y^c$. Namely*

$$C_D = \ colim_{Alg_{\mathcal{O}}}(\ X^c \xleftarrow{g^c} Z^c \xrightarrow{\pi_1 i_1} Z^c\widehat{\otimes}\mathcal{I} \coprod\limits_{Z^c} Y^c\ ).$$

Proof. This is a dual analogue of the proof of Proposition 1.28. $\square$

Remark 1.31. *If* $D: \ X \xleftarrow{g} Z \xrightarrow{f} Y$ *is a diagram of quasi-free* $\mathcal{O}$-*algebras, then we don't need the cofibrant replacement functor* $(-)^c$ *in the construction, and we have*

$$C_D = X \coprod_Z Z \widehat{\otimes} \mathcal{I} \coprod_Z Y.$$

In the particular case of computing the suspension ΣX of an $\mathcal{O}$-algebra X, we can simply apply the homotopy pushout model in Proposition 1.30. We will show roughly that the suspension of an $\mathcal{O}$-algebra is a free $\mathcal{O}$-algebra.

Proposition 1.32. *We assume that char(*$\Bbbk$*)=0. Let* $(\mathcal{O}(V), d)$ *be a quasi-free algebra with the notation for the differential:* $d = d_1 + d_2 + ...$ *Then* $\Sigma(\mathcal{O}(V), d) \simeq (\mathcal{O}(sV'), D_1)$*, where* $D_1(sv') := -sd_1v'$ *and* V' *is a copy of* (V, d).

Proof. We set for short $Z = (\mathcal{O}, d)$;

In Proposition 1.30, we have proved that $(0 \coprod_Z Z \widehat{\otimes} \mathcal{I} \coprod_Z 0, D) \simeq \Sigma Z$.

Since $(e^{\theta})^{ab}(v) = v' + sd_1v'$, we deduce that in $(0 \coprod_Z Z \widehat{\otimes} \mathcal{I} \coprod_Z 0)^{ab}$,

$$\begin{aligned}
[Dsv'] &= [v'] \\
&= [v' + sd_1v - sd_1v'] \\
&= [-sd_1v']
\end{aligned}$$

Now we consider the morphism of $\mathcal{O}$-algebras

$$\psi : (\mathcal{O}(sV'), D_1) \longrightarrow (0 \coprod_Z Z \widehat{\otimes} \mathcal{I} \coprod_Z 0, D)$$

given by $\psi(sv') = [sv']$.

This is a well defined chain complex morphism since $[D\psi(sv')] = \psi(D_1(sv'))$ and in addition $B(\mathcal{O}, \psi) \simeq \psi^{ab}$ is a quasi-isomorphism, therefore ψ is a quasi-isomorphism.

$\square$

Remark 1.33. *The result of Proposition 1.32 holds in general when the ground field* $\Bbbk$ *is of any characteristic. In fact, we have the following pushout diagram*

$$\begin{tikzcd}
\mathcal{O}(V) \arrow[r] \arrow[d] & \mathcal{O}(V \oplus sV) \simeq 0 \arrow[d] \\
0 \arrow[r] & \mathcal{O}(sV)
\end{tikzcd}$$

This is also a homotopy pushout diagram, thus we deduce that $\Sigma\mathcal{O}(V) \simeq \mathcal{O}(sV)$.

Proof of Theorem 1.29. We make the following computation

$$\begin{aligned}
\Sigma Z &\simeq \Sigma B^c(B(\mathcal{O}), B(\mathcal{O}, X)) && \text{(Using Thm 1.15)} \\
&\simeq \mathcal{O}(sUB(\mathcal{O}, Z)) && \text{(Using Proposition 1.32 and Remark 1.33)}
\end{aligned}$$

$\square$

1.13.3 Filtered colimits in $\mathbf{Alg}_{\mathcal{O}}$

In this section we remind two important properties of filtered colimits in $\mathrm{Alg}_{\mathcal{O}}$. These are colimits of filtered diagrams of $\mathcal{O}$-algebras.

Proposition 1.34. *([Fre17, Prop 1.3.6-(a)]) Let $\mathcal{O}$ be an operad on Ch_+ (resp. Ch). The forgetful functor $U : Alg_{\mathcal{O}} \longrightarrow Ch_+$ (resp. $U : Alg_{\mathcal{O}} \longrightarrow Ch$) preserves filtered colimits.*

Roughly speaking, this result says that filtered colimits in $\mathrm{Alg}_{\mathcal{O}}$ are computed in the ground category Ch_+ or Ch.

Lemma 1.35. *In $Alg_{\mathcal{O}}$, filtered homotopy colimits commute with finite homotopy limits.*

Proof. Let $\mathcal{J}$ be a right filtered diagram and $\mathcal{K}$ be a small category. Consider a functor $F : \mathcal{J} \times \mathcal{K} \longrightarrow \mathrm{Alg}_{\mathcal{O}}$.

The goal of this proof is to show that the canonical morphism

$$\mathrm{colim}_{\ \mathcal{J}}\ \lim_{\mathcal{K}} F \xrightarrow{\ \varrho\ } \lim_{\mathcal{K}} \mathrm{colim}_{\ \mathcal{J}} F$$

is an isomorphism.

It is sufficient to prove that the morphism of chain complexes $U\varrho$ is an isomorphism, where $U : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch_+ \hookrightarrow Ch$ is the forgetful functor. On the other hand, we have

$$U \mathrm{colim}_{\ \mathcal{J}}\ \lim_{\mathcal{K}} F \cong \mathrm{colim}_{\ \mathcal{J}}\ \lim_{\mathcal{K}} UF$$

since U commutes with filtered colimit in $\mathrm{Alg}_{\mathcal{O}}$ and U commutes with limits in $\mathrm{Alg}_{\mathcal{O}}$ as a right adjoint functor.

Therefore the proof reduces to proving that

$$\mathrm{colim}_{\ \mathcal{J}}\ \lim_{\mathcal{K}} UF \xrightarrow{\ U\varrho\ } \lim_{\mathcal{K}} \mathrm{colim}_{\ \mathcal{J}} UF$$

is an objectwise isomorphism of chain complexes and this is known. $\square$

1.14 Resume of chapter 1

In this chapter, we gave some preliminaries on the algebraic categories of respectively $\mathbb{Z}$-graded and $\mathbb{N}$-graded chain complexes: $\mathcal{C} = Ch, Ch_+$; the category of symmetric sequences: $[FinSet, \mathcal{C}]$; the category of operads: $Op_{\mathcal{C}}$, the category of Q-coalgebras: coAlg_Q, where Q is a cooperad; the categories of respectively $\mathcal{O}$-algebras and right $\mathcal{O}$-modules: $\mathrm{Alg}_{\mathcal{O}}$, $\mathcal{O}$-mod, where $\mathcal{O}$ is a reduced operad.

In this section, we remind a couple of functors and adjunctions on a checking list, so that the reader could feel more comfortable with our notations in the upcoming chapters.

We have the adjunctions

(a)
$$\mathcal{O}(-) : Ch_+ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Alg}_{\mathcal{O}} : U.$$

where $\mathcal{O}(V)$ is the free $\mathcal{O}$-algebra generated by the chain complex V and U is the forgetful functor.

Dually, given a cooperad Q, we have the adjunction

(b)

$$U : \mathrm{coAlg}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} Ch_+ : Q(-).$$

where $Q(V)$ is the cofree Q-coalgebra generated by V.

(c) We have a straightforward adjunction between $\mathbb{Z}$-graded and $\mathbb{N}$-graded chain complexes:

$$I : Ch_+ \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} Ch : red_0.$$

(d) Another important adjunction is

$$(-)^{ab} : \mathrm{Alg}_{\mathcal{O}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} Ch : (red_0-)_{triv}.$$

where $(-)^{ab}$ consists in killing the decomposable and $(red_0V)_{triv}$ is the chain complex red_0V with a trivial $\mathcal{O}$-algebra structure.

Note that all the above functors are homotopy functors except the functor $(-)^{ab}$. A "homotopy" version of $(-)^{ab}$ is the functor Σ^∞ defined below.
We have also defined various bar constructions:

(e) The two sided bar construction $B(R, \mathcal{O}, L)$, provided a left (resp. right) $\mathcal{O}$-module L (resp. R) which is a symmetric sequence;

(f) The bar construction on an operad $B(\mathcal{O}) := B(\mathbb{I}, \mathcal{O}, \mathbb{I})$. This symmetric sequence is a cooperad and the defined functor $B(-)$ has a left adjoint called the cobar construction $B^c(-)$. In other words, we have the adjunction

$$B^c(-) : coOp_{Ch_+} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} Op_{Ch_+} : B(-).$$

(g) The bar construction on an $\mathcal{O}$-algebra X which is the chain complex $B(\mathcal{O}, X) := B(\mathbb{I}, \mathcal{O}, \widehat{X})(0)$. This chain complex is actually a $B(\mathcal{O})$-coalgebra and the defined functor $B(\mathcal{O}, -)$ has a left adjoint called the cobar construction $\Omega_{\mathcal{O}}(-)$. In other worlds, we have the adjunction

$$\Omega_{\mathcal{O}}(-) : \mathrm{coAlg}_{B(\mathcal{O})} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Alg}_{\mathcal{O}} : B(\mathcal{O}, -).$$

(h) The counit of this cobar-bar adjunction is a good cofibrant replacement functor $(-)^c = \Omega_{\mathcal{O}}(B(\mathcal{O}, -))$ on $\mathcal{O}$-algebras.

We have three versions of loop and suspension functors Ω and Σ, defined with homotopy pushouts and homotopy pullbacks respectively.

$\mathcal{C}$	$\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$	$\Omega : \mathcal{C} \longrightarrow \mathcal{C}$
Ch	s	s^{-1}
Ch_+	s	$red_0 s^{-1} I$
$\mathrm{Alg}_{\mathcal{O}}$	$\Sigma X \simeq \mathcal{O}(sB(\mathcal{O}, X)$	$(red_0 s^{-1} I-)_{triv}$ (if char($\mathbb{k}$)=0)

We have thee versions of the functors Σ^∞ and Ω^∞ given as follows:

$\mathcal{C}$	$\Sigma^\infty : \mathcal{C} \longrightarrow Ch$	$\Omega^\infty : Ch \longrightarrow \mathcal{C}$
Ch	Id	Id
Ch_+	s	red_0
$\mathrm{Alg}_{\mathcal{O}}$	$IUB(\mathcal{O}, -)$	$(red_0-)_{triv}$

The functors

$$\Sigma^\infty : \mathrm{Alg}_{\mathcal{O}} \underset{\longleftarrow}{\overset{\longrightarrow}{}} Ch : \Omega^\infty.$$

do not form a strict adjunction as we have reminded that Ω^∞ is the right adjoint of $(-)^{ab}$. However, we have the following facts:

(i) The functors Σ^∞ and $(-)^{ab}$ coincide on cofibrant $\mathcal{O}$-algebras;

(j) The composite $\Sigma^\infty\Omega^\infty : Ch \longrightarrow Ch$ is a comonad.

(k) In $\mathrm{Alg}_{\mathcal{O}}$ when char($\mathbb{k}$)=0, if $Y \simeq \Omega(X)$ then $Y \simeq \Omega^\infty(UX)$.

This literally says that, in characteristic 0, any loop space of $\mathcal{O}$-algebra has a trivial algebra structure.

CHAPTER 2

Calculus of functors

In all this chapter, we assume that $\mathcal{C}, \mathcal{D} = Ch, Ch_+$ or $\mathrm{Alg}_{\mathcal{O}}$.

In this chapter, we discuss the basics of functor calculus for functors $F : \mathcal{C} \longrightarrow \mathcal{D}$. More precisely, we approximate a functor F with a sequence $\{P_n F\}_n$ of so-called polynomial functors. This is a general construction due to Goodwillie in the cases $\mathcal{C}, \mathcal{D} = Top$ (category of topological spaces) or Sp (category of spectra). Kuhn showed in his research (see [Kuh07]) that Goodwillie's constructions of the approximation work in many other categories and among those, our categories of interest here.

The approximation of functors gives rise to a "Taylor tower"

$$\longrightarrow P_n F \longrightarrow P_{n-1} F \longrightarrow \dots \longrightarrow P_0 F. \tag{2.1}$$

A first step in the understanding of this tower is the study of the difference between two consecutive terms also called the "homogeneous part of the tower":

$$D_n F = \mathrm{hofib}(P_n F \longrightarrow P_{n-1} F). \tag{2.2}$$

Goodwillie obtained a concrete description of $D_n F$ in Top and Sp (see formula (0.1) in the introduction).

In this chapter, we study an analogous description in our algebraic setting. More precisely, we show in Theorem 2.22 that there always exists a chain complex, $\partial_n F$, called "Goodwillie derivatives" with an action of the symmetric group on n letters Σ_n such that

$$D_n F(X) \simeq \Omega^\infty(\partial_n F \underset{h\Sigma_n}{\otimes} (\Sigma^\infty X)^{\otimes n}). \tag{2.3}$$

Our formula for $D_n F$ generalizes the result of Walter in [Wal06] who proved that formula when $\mathcal{C}, \mathcal{D} = Ch, Ch_+$ or DGL. At the end of this chapter, we will use the formula for the derivatives to deduce the following derivatives: $\forall n$,

- $\partial_n(Id : \mathrm{Alg}_{\mathcal{O}} \longrightarrow \mathrm{Alg}_{\mathcal{O}}) \simeq \mathcal{O}(n)$ (in Proposition 2.41);
- $\partial_n(\Sigma^\infty\Omega^\infty : Ch \longrightarrow Ch) \simeq B(\mathcal{O})(n)$ (in Proposition 2.42);
- $$\partial_n(\underline{hom}(E, I^{\otimes k})^{\Sigma_k} : Ch \longrightarrow Ch) \simeq \begin{cases} 0 & \text{if } n \neq k; \\ \underline{hom}(E, \Bbbk) & \text{if } n = k. \end{cases}$$
  (in Proposition 2.43);
- $\partial_n(N\Bbbk Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, -) : Ch_+ \longrightarrow Ch) \simeq \underline{hom}(V, \Bbbk)^{\otimes n}$ (in Proposition 2.44).

These four derivatives will be a key ingredient in our description of the Taylor tower in Chapter 5.

The chapter has the following guidelines:

- In §2.1, we fix the terminologies in the functor category.
- In §2.2, we construct the tower of polynomial approximation $\{P_nF\}_n$ of a given functor $F : \mathcal{C} \longrightarrow \mathcal{D}$. This includes the notion of "polynomial" functors that we introduce at the beginning of this section.
- In §2.3, we analyze the functor D_nF. In other words, we establish the above Equation (2.3). Our construction mimics the Goodwillie's argument in the sense that we construct ∂_*F by "multi-linearizing" the "cross-effect". We also discuss these notions in the section. Finally, we introduce the notion of "co-cross-effect" which is used in practice to compute ∂_*F.
- In §2.4, we compute the Goodwillie derivatives of a couple of functors.

2.1 Functor category

In this section, we put hypothesis on the functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ we are interested in Goodwillie calculus. These functors should respect the model structure of the categories $\mathcal{C}$ and $\mathcal{D}$.

Definition 2.1 (Homotopy functor)**.**

1. *The functor F is reduced if $F(0) \simeq 0$;*
2. *F is a homotopy functor if it preserves weak equivalences.*
3. *F is finitary if it preserves filtered homotopy colimits.*

In this chapter, the functors F that we consider are always homotopy functors. Since the category $\mathcal{C}$ is not small in the three cases, we do not impose a model structure on the class of such functors. However we will use the following terminology:

Definition 2.2. *1. A natural transformation $F \longrightarrow G$ is a weak equivalence if $F(X) \longrightarrow G(X)$ is a weak equivalence for all $X \in \mathcal{C}$;*

2. *Given a diagram of functors* $D = \{F_\alpha\}_\alpha$*, we call hocolim* D *the functor defined by:* $\forall X \in \mathcal{C}$,

$$hocolim\ D(X) := \underset{\alpha}{hocolim}_{\mathcal{C}}(F_\alpha(X))$$

3. *Dually for the homotopy limit. In particular, a diagram of functors*

$$H \longrightarrow F \longrightarrow G$$

is a (homotopy) fiber sequence if

$$H(X) \longrightarrow F(X) \longrightarrow G(X)$$

is a (homotopy) fiber sequence for all $X \in \mathcal{C}$.

2.2 Polynomial functors

The goal of this section is to define n-excisive functors also known as polynomial functors of degree $\leq n$. Roughly speaking, a polynomial functor of degree ≤ 1 is a covariant homotopy functor that sends homotopy pushout squares to homotopy pullback squares. The generalization in higher degree involves the notion of "strongly" (co)-cartesian cube that we now define.

2.2.1 Cubical (co)-cartesian cubes

Definition 2.3.

1. *A n-cube in* $\mathcal{C}$ *is a functor* $\mathcal{X} : \mathcal{P}(\underline{n}) \longrightarrow \mathcal{C}$*, where* $\mathcal{P}(\underline{n})$ *is the poset of subsets of* $\underline{n} := \{1, ..., n\}$.

2. $\mathcal{X}$ *is Cartesian if the natural map*

$$\mathcal{X}(\emptyset) \longrightarrow \underset{T \in \mathcal{P}(\underline{n}) - \{\emptyset\}}{holim} \mathcal{X}(T)$$

is a weak equivalence.

3. $\mathcal{X}$ *is co-Cartesian if the natural map*

$$\underset{T \in \mathcal{P}(\underline{n}) - \{\underline{n}\}}{hocolim} \mathcal{X}(T) \longrightarrow \mathcal{X}(\underline{n})$$

is a weak equivalence.

4. $\mathcal{X}$ *is a strongly co-Cartesian if* $\mathcal{X}\mid_{\mathcal{P}(T)}: \mathcal{P}(T) \longrightarrow \mathcal{C}$ *is co-Cartesian for all* $\underline{2} \subseteq T \subseteq \underline{n}$.

Example 2.4. *We define the strongly co-Cartesian n-cube* $\mathcal{X} = \mathcal{S}^*(X_1, ..., X_n)$*, for objects* $X_1, ..., X_n$ *in* $\mathcal{C}$*, as follows:*

$$\forall T \subseteq [n],\ \mathcal{X}(T) := \coprod_{i \in T} X_i \text{ (in particular } \mathcal{X}(\emptyset) = 0).$$

The maps in the cube $\mathcal{X}$ are inclusions of the form $X_i \longrightarrow X_i \amalg X_j$ (since $\mathcal{C}$ is a pointed category).

For instance when $n = 2$,

$$\mathcal{X} = \mathcal{S}^*(X_1, X_2) = \left\{ \begin{tikzcd} 0 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & X_1 \amalg X_2 \end{tikzcd} \right\}$$

Definition 2.5 ([Kuh07], 4.6)**.** *Let $X \in \mathcal{C}$ and T be a finite set. We define the joint $X * T$, of X and T, to be the homotopy cofiber of the folding map*

$$X * T = hocof\,(\coprod_T X \xrightarrow{\nabla} X)$$

Example 2.6.

- $X * \underline{0} = hocof\,(0 \longrightarrow X) \simeq X$;
- $X * \underline{1} = hocof\,(X \longrightarrow X) \simeq 0$;
- $$\begin{aligned} X * \underline{2} &= hocof\,(X \amalg X \xrightarrow{\nabla} X) \\ &\simeq hopo\ (0 \longleftarrow X \longrightarrow 0) \\ &\simeq \Sigma X \end{aligned}$$

  Hence for $X \in Alg_{\mathcal{O}}$,

  $$X * \underline{2} \simeq \Sigma\Omega_{\mathcal{O}}(B(\mathcal{O}, X)) \simeq \Omega_{\mathcal{O}}(sUB(\mathcal{O}, X)).$$

We can use the joint of Definition 2.5 to define natural strongly co-Cartesian cubes.

Proposition 2.7. *([Wal06, lemma 7.1.4]) Given $X \in \mathcal{C}$ and $n \geq 1$, the n-cube*

$$\begin{aligned} \chi_n(X) : \mathcal{P}(n) &\longrightarrow \mathcal{C} \\ T &\longmapsto X * T \end{aligned}$$

is a strongly co-Cartesian.

Proof. If T, R, S are disjoint subsets in $\underline{n}$ then we have a homotopy pushout

$$\begin{tikzcd} \coprod_T X \arrow[r] \arrow[d] & \coprod_{T\cup S} X \arrow[d] \\ \coprod_{T\cup R} X \arrow[r] & \coprod_{T\cup S\cup R} X \end{tikzcd}$$

which induces the homotopy pushout

$$\begin{tikzcd}
X * T \arrow[r] \arrow[d] & X * (T \cup S) \arrow[d] \\
X * (T \cup R) \arrow[r] & X * (T \cup S \cup R)
\end{tikzcd}$$

Therefore every 2-face of the cube $\chi_n(X)$ is a homotopy pushout which implies that $\chi_n(X)$ is strongly co-Cartesian. $\square$

2.2.2 Polynomial functors

In this short part, we define "excisive" functors which are also called "polynomial" functors in many places in the literature.

Definition 2.8 (n-excisive functor)**.**

1. *A homotopy functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is n-excisive if whenever $\mathcal{X}$ is a strongly co-Cartesian $n+1$-cube in $\mathcal{C}, F(\mathcal{X})$ is a cartesian cube in $\mathcal{D}$;*

2. *A homotopy functor $F : \mathcal{C}^{\times n} \longrightarrow \mathcal{D}$ is multilinear if it is 1-excisive and reduced in each variables.*

There are several properties for excisive functors and the next lemma will be often used in this thesis.

Lemma 2.9. *Given a fiber sequence*

$$F \longrightarrow G \longrightarrow H$$

of functors $\mathcal{C} \longrightarrow \mathcal{D}$, if any two of the functors are n-excisive, so is the third.

Proof. We give the proof in the particular case when F and H are n-excisive. The other cases follow the same idea. Let $\mathcal{X}$ be a strongly co-cartesian n-cube in $\mathcal{C}$. We have

$$F(\mathcal{X}) \simeq \text{ hofib}(G(\mathcal{X}) \longrightarrow H(\mathcal{X})) \tag{2.4}$$

When we apply the total homotopy fiber functor (thofib) to the left and to the right hand side of Equation (2.4), and since hofib commutes with thofib, we get

$$\text{thofib}(F(\mathcal{X})) \simeq \text{ hofib}(\text{thofib}(G(\mathcal{X})) \longrightarrow \text{thofib}(H(\mathcal{X})))$$

Now since F and H are n-excisive, it follows that $\text{thofib}(F(\mathcal{X})) \simeq 0 \simeq \text{thofib}(H(\mathcal{X}))$. Therefore $\text{thofib}(G(\mathcal{X})) \simeq 0$ and since G has values in $\mathcal{D}$ whose underlying category is Ch, we deduce using the long sequence argument that $G(\mathcal{X})$ is a n-cartesian cube. $\square$

2.2.3 Polynomial approximation and the Taylor tower.

Many functors are not excisive and a trivial example is the identity functor

$$Id : \mathrm{Alg}_{\mathcal{O}} \longrightarrow \mathrm{Alg}_{\mathcal{O}}.$$

In this section, we build the n-polynomial approximation of any homotopy functor F. We follow the lines of [Kuh07, § 4 and § 5](and implicitly [Goo03]). The idea of the construction is that, since the cube

$$\chi_n(X) : \underline{n} \supset T \longmapsto X * T \tag{2.5}$$

is strongly co-cartesian (by Proposition 2.7), we replace F by a functor which by design sends the strongly co-cartesian cube $\chi_n(X)$ to a cartesian cube. It turned out that this guarantee that P_nF is n-excisive and is the best approximation. This is the content of the next definition and properties.

Definition 2.10.

1. *We define a functor* $T_nF : \mathcal{C} \longrightarrow \mathcal{D}$ *by:*

$$T_nF(X) := \underset{T \in \mathcal{P}(\underline{n+1})-\{\emptyset\}}{holim} F(\chi_n(X)(T))$$

This comes with a natural map $t_nF : F(X) = F(\chi_n(X)(\emptyset)) \longrightarrow T_nF(X)$;

2. *We write* T_n^iF *to denote the functor defined inductively out of* T_nF *by*

$$T_n^{i+1}F := T_n(T_n^iF)$$

3. *We define a functor* $P_nF : \mathcal{C} \longrightarrow \mathcal{D}$ *by:*

$$P_nF := hocolim\ (F \xrightarrow{t_nF} T_nF \xrightarrow{T_n(t_nF)} T_n^2(F) \xrightarrow{T_n^2(t_nF)} ...).$$

This comes with a natural map $p_nF : F \longrightarrow P_nF$.

Example 2.11. *If the functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *is homotopy and reduced, then*

$$\begin{aligned} T_1F(X) &= holim\ (F(X * \underline{1}) \longrightarrow F(X * \underline{2}) \longleftarrow F(X * \underline{1})) \\ &\simeq holim\ (0 \longrightarrow F(\Sigma X) \longleftarrow 0) \\ &\simeq \Omega F(\Sigma X). \end{aligned}$$

Therefore inductively we get

$$P_1F \simeq \underset{p \to \infty}{hocolim}\ \Omega^p F \Sigma^p.$$

Remark 2.12. *By construction, the functor* P_n- *is basically a filtered homotopy colimit. Since filtered colimits commute with finite limits (see Lemma 1.35), we deduce that* P_n- *preserves fiber sequences.*

Theorem 2.13. *[Goo03]*

1. *The functor P_nF is n-excisive;*

2. *The natural transformation $p_nF : F \longrightarrow P_nF$ is homotopy universal in the sense that any natural map*

$$u : F \longrightarrow G,$$

where G is n-excisive, factors uniquely (up to homotopy) through p_nF.

Remark 2.14. *The properties* (1) *and* (2) *of Theorem 2.13 say in other words that P_nF is the "best possible" n-excisive approximation of F.*

Proposition 2.15.

1. *If F is n-excisive, then t_nF is a weak equivalence;*

2. *If F is n-excisive, then p_nF is a weak equivalence;*

The inclusion of categories $\mathcal{P}(\underline{n}) \longrightarrow \mathcal{P}(\underline{n+1})$ induces a map

$$T_nF \longrightarrow T_{n-1}F$$

which extends formally to give a map

$$q_nF : P_nF \longrightarrow P_{n-1}F.$$

Definition 2.16. *([Goo03, 1.13]) The Taylor tower of F is the tower of excisive approximations*

$$\begin{tikzcd}
& \vdots \arrow[d, "q_{n+1}F"] \\
& P_nF \arrow[d, "q_nF"] \\
& P_{n-1}F \arrow[d, "q_{n-1}F"] \\
& \vdots \arrow[d, "q_2F"] \\
& P_1F \arrow[d, "q_1F"] \\
F \arrow[r, "p_0F"'] \arrow[ur, "p_1F"'] \arrow[uuur, "p_{n-1}F"'] \arrow[uuuur, "p_nF"] & P_0F
\end{tikzcd}$$

2.2.4 Homogeneous functors

Definition 2.17 (homogeneous functors)**.** *F is called n-homogeneous if*

- *F is n-excisive and*

- $P_{n-1}F \simeq 0$.

Proposition 2.18. *[Goo03, Prop 1.17] The functor*

$$D_nF := hofib(P_nF \overset{q_nF}{\longrightarrow} P_{n-1}F)$$

is n-homogeneous.

There is a delooping of homogeneous functors. This was originally proved by Goodwillie in [Goo03, Lemma 2.2] and repeated by Kuhn in [Kuh07, Lemma 5.7] who quoted again Goodwillie's paper for the proof.

Lemma 2.19. *([Goo03, Lemma 2.2]) Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a homotopy and reduced functor. There exists a n-homogeneous functor $R_nF : \mathcal{C} \longrightarrow \mathcal{D}$ fitting into a fiber sequence of functors*

$$P_nF \longrightarrow P_{n-1}F \longrightarrow R_nF.$$

Remark 2.20. *If in addition the functor F in Lemma 2.19 is n-homogeneous, then we have*

$$F \simeq P_nF \simeq \Omega R_nF.$$

Therefore, in the particular case that char($\mathbb{k}$)=0, $\mathcal{D} = Alg_{\mathcal{O}}$, we can rewrite F using Proposition 1.25 as follows:

$$F \simeq \Omega^{\infty} IUF,$$

where $U : Alg_{\mathcal{O}} \longrightarrow Ch_+$ is the forgetful functor. This means literally that if $F : \mathcal{C} \longrightarrow Alg_{\mathcal{O}}$ is n -homogeneous, then $F(X)$ always has a trivial $\mathcal{O}$-algebra structure.

Finally, we end this section with a result of Goodwillie which shows that every multilinear functor produces naturally a homogeneous functor.

Lemma 2.21. *([Goo03, Lemma 3.1]) If $F : \mathcal{C}^{\times n} \longrightarrow \mathcal{D}$ is multilinear (1-excisive and reduced in each variable), then the functor $F \circ \triangle : \mathcal{C} \longrightarrow \mathcal{D}$ is n-homogeneous. Here $\triangle : \mathcal{C} \longrightarrow \mathcal{C}^{\times n}$ is the diagonal map.*

2.3 Characterization of homogeneous functors

The goal of this part is to give an explicit description of the functor D_nF. In other words, the main result is the following:

Theorem 2.22. *Let $\mathcal{C}, \mathcal{D} = Ch, Ch_+$ or $Alg_{\mathcal{O}}$, and $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a homotopy functor. We assume that char($\mathbb{k}$)=0. There is an unbounded chain complex ∂_nF with an action of the symmetric group Σ_n such that if F is either finitary or X is finite, then we have a weak equivalence*

$$D_nF(X) \simeq \Omega^{\infty}(\partial_nF \otimes (\Sigma^{\infty}X)^{\otimes n})_{h\Sigma_n}$$

This result gives a very good understanding of the layers of the Taylor tower described in Definition 2.16. We get in fact a monomial expression of D_nF which roughly speaking depends on a single and constant coefficient $\partial_n F$. That is essentially the reason why P_*F is called "Taylor tower", making the analogy with the classical calculus of functions.

Since any arbitrary n-homogeneous functor H is equivalent to D_nH, we can claim that any homogeneous functors has a monomial shape as in Theorem 2.22.

To prove the theorem, we will mimic Goodwillie's constructions. In fact, we will multi-linearize homogeneous functors using the "cross-effect". In this process we will need to discuss the notion of "cross-effect" and discuss the interaction between the cross-effect and homogeneous functors.

2.3.1 Cross-effect

We define in this part the n-th cross-effect of functors which is a fundamental tool to study the layers D_nF of the Taylor tower. To motivate this notion, we first recall an analogy in classical calculus.

Let $f(x)$ be a function of one variable. The n-th cross-effect of the function is defined as

$$cr_n f(x_1, ..., x_n) := \sum_{I \subset \underline{n}} (-1)^{n-|I|} f(\sum_{i \in I} x_i)$$

for example

$$cr_1 f(x_1) = f(x_1) - f(0);$$

$$cr_2 f(x_1, x_2) = f(x_1 + x_2) - f(x_1) - f(x_2) + f(0).$$

When this function is polynomial, its degree n terms is closely related to its cross-effect. In fact, the following properties are easy to establish:

1. If f is polynomial of degree $\leq n$,

- $cr_n f$ is a n-multilinear function;

- $cr_n f = 0$ if and only if $deg(f) \leq n - 1$.

2. If h is homogeneous of degree n, then

$$h(x) = \frac{cr_n h(x, ..., x)}{n!}.$$

We will develop this analogy for the cross-effect of functors and study their properties.

There are two equivalent ways to define the cross effect associated to a functor. One can define it as a homotopy fiber (hofib) and one can also define it as a total homotopy fiber (thofib). These definitions are reported here bellow.

Definition 2.23 (Cross-effects)**.** *We define $cr_n F : \mathcal{C}^{\times n} \longrightarrow \mathcal{D}$, the n^{th} cross-effect of F, to be the functor of n variables given by*

$$cr_n F(X_1, ..., X_n) = \ hofib\{F(\coprod_{i \in \underline{n}} X_i) \longrightarrow \underset{T \in \mathcal{P}_0(n)}{holim} F(\coprod_{i \notin T} X_i)\}$$

This is equivalent to define the n^{th} cross-effect of F as:

$$cr_n F(X_1, ..., X_n) = \text{ thofib}(T \supseteq \underline{n} \mapsto F(\underset{i \notin T}{\amalg} X_i)). \tag{2.6}$$

Example 2.24. *The first cross-effect of F is*

$$\begin{aligned} cr_1 F(X_1) &:= thofib(F(X_1) \longrightarrow F(0)) \\ &= hofib(F(X_1) \longrightarrow F(0)) \end{aligned}$$

In other words, $cr_1 F$ is the "reduction" of the functor F. When F is already reduced, then $cr_1 F \simeq F$.

The second cross-effect of $F : \mathcal{C} \longrightarrow \mathcal{D}$ is

$$\begin{aligned} cr_2 F(X_1, X_2) &:= \mathit{thofib} \left\{ \begin{array}{ccc} F(X_1 \amalg X_2) & \longrightarrow & F(X_2) \\ \downarrow & & \downarrow \\ F(X_1) & \longrightarrow & F(0) \end{array} \right\} \\ &= \mathit{hofib} \left(\mathit{hofib} \left\{ \begin{array}{c} F(X_1 \amalg X_2) \\ \downarrow \\ F(X_1) \end{array} \right\} \longrightarrow \mathit{hofib} \left\{ \begin{array}{c} F(X_2) \\ \downarrow \\ F(0) \end{array} \right\} \right) \end{aligned}$$

Proposition 2.25. *If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a $n-$excisive functor such that $cr_n F \simeq 0$, then F is $(n-1)-$excisive.*

Proof. (i) We define the n-cube $\mathcal{X} = \mathcal{S}^*(X_1, ..., X_n)$, for objects $X_1, ..., X_n$ in $\mathcal{C}$, as follows:

$$\begin{aligned} &\forall T \subseteq [n], \mathcal{X}(T) := \underset{i \in T}{\amalg} X_i \\ &\mathcal{X}(\emptyset) = 0 \end{aligned}$$

and the maps in the cube $\mathcal{X}$ are inclusions. We associate to this cube $\mathcal{X}$ the n-cube $\mathcal{S}(X_1, ..., X_n)$ which has the same objects with $\mathcal{X}$, but where the inclusions are reversed to the projections. Let $U : \mathcal{D} \longrightarrow Ch$ be the forgetful functor when $\mathcal{D} = \mathrm{Alg}_{\mathcal{O}}$ and be the identity functor when $\mathcal{D} = Ch_+$. We make the following computations:

$$\begin{aligned} U \text{ thofib } F(\mathcal{X}) \cong \text{ thofib } UF(\mathcal{X}) &= \text{ thofib } UF(\mathcal{S}^*(X_1, ..., X_n)) \\ &= \Omega^n \text{ thofib } UF(\mathcal{S}(X_1, ..., X_n)) \\ &= \Omega^n cr_n(UF)(X_1, ..., X_n) \\ &= \Omega^n U cr_n F(X_1, ..., X_n) \simeq 0, \end{aligned}$$

One will then conclude from these that thofib $F(\mathcal{X}) \simeq 0$(or equivalently that $F(\mathcal{X})$ is cartesian) for all strongly co-cartesian cubes $\mathcal{X}$ in which $\mathcal{X}(\emptyset) = 0$, since any such cube $\mathcal{X}$ is naturally equivalent to $\mathcal{S}^*(\mathcal{X}(\{1\}), ..., \mathcal{X}(\{n\}))$(see [Goo92, Proposition 2.2]).

(ii) Let $\forall T \subseteq [n]$, and $a, b \in [n]$. Given an arbitrary strongly co-cartesian n-cube $\mathcal{X}$ in $\mathcal{C}$, put

$$\mathcal{X}'(T) = hocolim(0 \longleftarrow \mathcal{X}(\emptyset) \longrightarrow \mathcal{X}(T))$$

We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{X}(\emptyset) \arrow[r] \arrow[d] & \mathcal{X}(T) \arrow[r] \arrow[d] & \mathcal{X}(T \cup \{a\}) \arrow[d] \\
0 \arrow[r] & \mathcal{X}'(T) \arrow[r] & \mathcal{X}'(T \cup \{a\})
\end{tikzcd}$$

where the largest square is a homotopy pushout along with the most left square. It then follows that the most right square is also a homotopy pushout and therefore that the following square is a homotopy pushout:

$$\begin{tikzcd}
\mathcal{X}(T) \arrow[r] \arrow[d] & \mathcal{X}(T \cup \{a\}) \arrow[d] \\
\mathcal{X}'(T) \arrow[r] & \mathcal{X}'(T \cup \{a\})
\end{tikzcd}$$

and therefore it follows that

$$\begin{tikzcd}
\mathcal{X}'(T) \arrow[r] \arrow[d] & \mathcal{X}'(T \cup \{a\}) \arrow[d] \\
\mathcal{X}'(T \cup \{b\}) \arrow[r] & \mathcal{X}'(T \cup \{a, b\})
\end{tikzcd}$$

is a homotopy pushout diagram. This proves that the n-cube $\mathcal{X}'$ is strongly co-cartesian and that the map $\mathcal{X} \longrightarrow \mathcal{X}'$ is a strongly co-cartesian $n+1$-cube. F is n-excisive, thus $F(\mathcal{X}) \longrightarrow F(\mathcal{X}')$ is cartesian. In addition since $\mathcal{X}'(\emptyset) = 0$, we deduce from (i) that $F(\mathcal{X}')$ is cartesian and conclude that $F(\mathcal{X})$ is also cartesian.

□

Proposition 2.26. *([Goo03, Prop 3.3]) If F is n-excisive, then $0 \leq m \leq n$, $cr_{m+1}F$ is $(n-m)$-excisive in each variable. In particular, if F is n-excisive then $cr_n F$ is multilinear, and if F is $(n-1)$-excisive then $cr_n F \sim 0$.*

Remark 2.27. *The cross-effect commutes with fiber sequences, thus in particular, we have the fiber sequence*

$$cr_n D_n F \longrightarrow cr_n P_n F \longrightarrow cr_n P_{n-1} F$$

Since $P_{n-1}F$ is $(n-1)$-excisive, Proposition 2.26 says that $cr_n P_{n-1} F \simeq 0$. Therefore we obtain the natural equivalence

$$cr_n D_n F \xrightarrow{\simeq} cr_n P_n F.$$

2.3.2 Cross-effect and homogeneous functors

Proposition 2.28. *Let F and G be two $n-$homogeneous functors $\mathcal{C} \longrightarrow \mathcal{D}$, and a natural transformation $F \xrightarrow{J} G$. If $cr_n(J) : cr_nF \longrightarrow cr_nG$ is an equivalence, then so is J.*

Proof. Let $H = \operatorname{hofib}(F \xrightarrow{J} G)$. Since the functor P_*- preserves fiber sequences, we see that $P_nH \simeq H$ and that $P_{n-1}H \simeq 0$. In particular, the functor H is n-excisive. On the other hand, since the cross effect commutes with fiber sequences (in fact holims commute with themselves), we have

$$cr_nH \cong \operatorname{hofib}(cr_nF \xrightarrow{cr_nJ} cr_nG) \simeq 0.$$

Therefore, the functor H satisfies all the hypothesis of Proposition 2.25, thus H is $n-1$-excisive . Hence we have $H \simeq P_{n-1}H \simeq 0$.

Finally, we deduce from the long exact sequence obtained from the homotopy fiber sequence (of J) that J is a weak equivalence.

□

Proposition 2.29. *Let $\mathcal{D} = Ch_+$ or Ch and $H : \mathcal{C} \longrightarrow \mathcal{D}$ be a n-homogeneous functor. Then there is a weak equivalence (natural in X)*

$$((cr_nH) \circ \triangle(X))_{h\Sigma_n} \simeq H(X)$$

given one of the two hypothesis below:

1. *If $\mathcal{D} = Ch_+$, and char($\Bbbk$)= 0 ;*
2. *If $\mathcal{D} = Ch$, and $\Bbbk$ is a field of any characteristic.*

Proof. We consider the composite (of natural transformations)

$$J_H : ((cr_nH) \circ \triangle(X))_{h\Sigma_n} \longrightarrow (H(\coprod_{\underline{n}} X))_{h\Sigma_n} \longrightarrow H(X)$$

The goal in this proof is to show that J_H is an equivalence. We start by making the following remark: the functor cr_nH is multilinear (see Proposition 2.26). Thus the composite $cr_nH \circ \triangle$ is again n-homogeneous (see Lemma 2.21);

Therefore, if cr_nJ_H is an equivalence, by applying Proposition 2.28, we will deduce the result. We will now prove that cr_nJ_H is an equivalence.

We set $L(X) = cr_n(H)(X, ..., X)_{h\Sigma_n}$ and we make the following computations:

$$\begin{aligned} cr_nL(X_1, ..., X_n) &= \operatorname{thofib}(\underline{n} - T \mapsto L(\coprod_T X_i)) && (2.7)\\ &= \operatorname{thofib}(\underline{n} - T \mapsto cr_nH(\coprod_T X_i, ..., \coprod_T X_i)_{h\Sigma_n}) && (2.8)\\ &= \operatorname{thofib}(\chi)_{h\Sigma_n}, && (2.9) \end{aligned}$$

where

- The cube χ is defined by $\chi : \underline{n} - T \mapsto cr_n(H)(\coprod_T X_i, ..., \coprod_T X_i)$;

- Equation (2.9) is justified in the following cases: - When $\mathcal{D} = Ch$, the homotopy orbit $(-)_{h\Sigma_n}$ which is essentially a colimit commutes with thocofib (total homotopy cofiber) and thocofib is equivalent in Ch to thofib;
  - When $\mathcal{D} = Ch_+$ and $\Bbbk$ is in characteristic 0, then homotopy orbit $(-)_{h\Sigma_n}$ are equivalent to homotopy fixed points $(-)^{h\Sigma_n}$ and this later is essentially a limit, so commutes with thofib.

On the other hand, the functor cr_nH is multilinear (see Proposition 2.26). We deduce the weak equivalence (natural in T)

$$\chi(\underline{n} - T) \xrightarrow{\simeq} \prod_{\pi:\underline{n}\to T} cr_n(H)(X_{\pi(1)}, ..., X_{\pi(n)}). \tag{2.10}$$

Let's consider the map $\pi : \underline{n} \to \underline{n}$ and consider the cube $\mathcal{Y}_\pi$ defined by:

$$\mathcal{Y}_\pi(\underline{n} - T) = \begin{cases} cr_n(H)(X_{\pi(1)}, ..., X_{\pi(n)}) & \text{if } \pi(\underline{n}) \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

The morphism (2.10) is equivalent to $\chi(\underline{n} - T) \xrightarrow{\simeq} \prod_{\pi:\underline{n}\to\underline{n}} \mathcal{Y}_\pi(\underline{n} - T)$.

- If π is not a permutation and then not a surjection, we can find an element $s \notin \pi(\underline{n})$. All the maps $\mathcal{Y}_\pi(\underline{n} - T) \longrightarrow \mathcal{Y}_\pi(\underline{n} - T \cup \{s\})$ are isomorphisms, so $\mathcal{Y}_\pi$ is cartesian. Hence $thofib(\mathcal{Y}_\pi) \simeq 0$;
- If π is a permutation, $thofib(\mathcal{Y}_\pi) \cong \mathcal{Y}_\pi(\underline{n}) = cr_nH(X_{\pi(1)}, ..., X_{\pi(n)})$.

Therefore $thofib(\chi) \xrightarrow{\simeq} \prod_{\pi\in\Sigma_n} cr_nH(X_{\pi(1)}, ..., X_{\pi(n)})$. Thus

$$\begin{aligned} thofib(\chi)_{h\Sigma_n} &\xrightarrow{\simeq} (\prod_{\pi\in\Sigma_n} cr_nH(X_{\pi(1)}, ..., X_{\pi(n)}))_{h\Sigma_n} \\ &\xrightarrow{\simeq} cr_nH(X_1, ..., X_n). \end{aligned}$$

□

2.3.3 Multi-linearizing the cross-effect

In this section, we will consider n-variable functors and in particular, those which are linear in each variable. The cross-effect cr_nF of a functor F is a n-variable functor which is reduced in each variable. In this part, we will multilinearize cr_nF by applying the first term functor P_1-, of the Taylor tower, to each variable of its variables.

Definition 2.30. *1. The functor $L_nF : \mathcal{C}^n \longrightarrow \mathcal{D}$ is obtained from cr_nF by*

$$L_nF(X_1, ..., X_n) \simeq \underset{p_i\to\infty}{hocolim}\, \Omega^{p_1+\dots+p_n} cr_nF(\Sigma^{p_1}X_1, ..., \Sigma^{p_n}X_n)$$

2. The functor $\triangle_nF : \mathcal{C} \longrightarrow \mathcal{D}$ is obtained from L_nF by:

$$\triangle_nF = (L_nF) \circ \triangle$$

where $\triangle : \mathcal{C} \longrightarrow \mathcal{C}^{\times n}$ is the diagonal map. The symmetric group Σ_n acts on $\triangle_n F$ by permuting its n entries of the cross effect $cr_n F$.

Remark 2.31. *1. When $\mathcal{D} = Alg_{\mathcal{O}}$, the filtered homotopy colimit in the definition of the functor $L_n F$ can be seen, using Proposition 1.34, as a homotopy colimit in the underlying category of chain complexes ;*

2. The functor $L_n F$ of Definition 2.30 can also be seen as the multilinearization of F. That is:

(a) The functor obtained by applying the first Taylor approximation functor P_1 to each variable position of the multi-variable functor $cr_n F$. For instance,

i. $L_1 F = P_1 F$ (see Example 2.11);

ii. $L_2 F(X, Y) = P_1(Y \longmapsto P_1(X \longmapsto cr_2(X, Y)))$;

iii. and so on.

(b) The functor $L_n F$ is multilinear(1-excisive and reduced on each variable) by construction.

3. The functor $\triangle_n F$ is n-homogeneous using Lemma 2.21.

Lemma 2.32. *There is a natural weak equivalence*

$$P_n(L_n F \circ \triangle) \simeq L_n(P_n F) \circ \triangle.$$

Proof. One make the following observation:

$$\begin{aligned}
T_n(L_n F \circ \triangle)(X) := & \underset{T \in \mathcal{P}_0(\underline{n+1})}{\operatorname{holim}} \ \underset{p_i \to \infty}{\operatorname{hocolim}} \ \Omega^{p_1 + \dots + p_n} cr_n F(\Sigma^{p_1}(X * T), ..., \Sigma^{p_n}(X * T)) && (2.11) \\
\simeq & \underset{T \in \mathcal{P}_0(\underline{n+1})}{\operatorname{holim}} \ \underset{p_i \to \infty}{\operatorname{hocolim}} \ \Omega^{p_1 + \dots + p_n} cr_n F((\Sigma^{p_1} X) * T, ..., (\Sigma^{p_n} X) * T) && (2.12) \\
= & \underset{T \in \mathcal{P}_0(\underline{n+1})}{\operatorname{holim}} \ \underset{p_i \to \infty}{\operatorname{hocolim}} \ \Omega^{p_1 + \dots + p_n} \ \operatorname{thofib}(A \supseteq \underline{n} \mapsto F(\coprod_{\underline{n} - A} ((\Sigma^{p_j} X) * T)) && (2.13) \\
\simeq & \underset{T \in \mathcal{P}_0(\underline{n+1})}{\operatorname{holim}} \ \underset{p_i \to \infty}{\operatorname{hocolim}} \ \Omega^{p_1 + \dots + p_n} \ \operatorname{thofib}(A \supseteq \underline{n} \mapsto F((\coprod_{\underline{n} - A} \Sigma^{p_j} X) * T) && (2.14) \\
\simeq & \underset{p_i \to \infty}{\operatorname{hocolim}} \ \Omega^{p_1 + \dots + p_n} \ \operatorname{thofib}(A \supseteq \underline{n} \mapsto T_n F(\coprod_{\underline{n} - A} \Sigma^{p_j} X)) && (2.15) \\
= & \underset{p_i \to \infty}{\operatorname{hocolim}} \ \Omega^{p_1 + \dots + p_n} cr_n(T_n F)(\Sigma^{p_1} X, ..., \Sigma^{p_n} X) && (2.16) \\
= & L_n(T_n F) \circ \triangle(X) && (2.17)
\end{aligned}$$

where

Equation (2.12) is due to the isomorphism $\Sigma^{p_j}(X * T) \cong (\Sigma^{p_j} X) * T$, for each j;

Equation (2.14) is due to the isomorphism $\coprod_{\underline{n}-T}(\Sigma^{p_j} X * T) \cong (\coprod_{\underline{n}-T}\Sigma^{p_j} X) * T$, for each $T \subseteq \underline{n}$;

Equation (2.15) is because finite holims commute with filtered colimits (see Lemma 1.35), and holims commute with loops Ω and total fibers.

One also deduce from this observation steps that the following square is commutative

$$\begin{tikzcd}
L_n F \circ \triangle \arrow[r, "="] \arrow[d, "t_n L_n F \circ \triangle"'] & L_n F \circ \triangle \arrow[d, "L_n t_n F \circ \triangle"] \\
T_n(L_n F \circ \triangle) \arrow[r, "\simeq"] & L_n(T_n F) \circ \triangle
\end{tikzcd}$$

Thus we can deduce by induction on the iterations from this square that

$$P_n(L_n F \circ \triangle) \simeq (L_n P_n F) \circ \triangle.$$

$\square$

Given the diagonal (homogeneous) functor $\triangle_n F : \mathcal{C} \longrightarrow \mathcal{D}$, associated to a homotopy functor F, we can always deduce a homogeneous functor $\widehat{\triangle}_n F : \mathcal{C} \longrightarrow Ch$ with values in chain complexes.

Definition 2.33. *Given a functor $F : \mathcal{C} \longrightarrow Ch_+$, we define the functor $\widehat{L}_n F : \mathcal{C}^n \longrightarrow Ch_+$ by*

$$\widehat{L}_n F(X_1, ..., X_n) \simeq \underset{p_i \to \infty}{hocolim}\ {}_{Ch}(s^{-p_1 - ... - p_n} I cr_n F(\Sigma^{p_1} X_1, ..., \Sigma^{p_n} X_n))$$

where $I : Ch_+ \longrightarrow Ch$ is the inclusion functor.

Definition 2.34. *We define a functor $\widehat{\triangle}_n F : \mathcal{C} \longrightarrow Ch$ as follows:*

1. *When $\mathcal{D} = Alg_{\mathcal{O}}$, then*
$$\widehat{\triangle}_n F = (\widehat{L}_n UF) \circ \triangle$$
where $U : Alg_{\mathcal{O}} \longrightarrow Ch_+$ is the forgetful functor and ;
2. *When $\mathcal{D} = Ch_+$, then*
$$\widehat{\triangle}_n F = (\widehat{L}_n F) \circ \triangle$$
3. *When $\mathcal{D} = Ch$, then $\widehat{\triangle}_n F := \triangle_n F$.*

Note that the functor $\widehat{\triangle}_n F$ is n-homogeneous for the same reason as $\triangle_n F$ in Remark 2.31.

Lemma 2.35. *We assume char($\mathbb{k}$)=0 and $\mathcal{D} = Alg_{\mathcal{O}}$. Then there is a weak equivalence of $\mathcal{O}$-algebras: for any $X \in \mathcal{C}$,*

$$\triangle_n F(X) \simeq \Omega^\infty \widehat{\triangle}_n F(X).$$

Proof. The functor $\triangle_n F$ is n-homogeneous, thus using Lemma 1.25 we have the equivalence $\triangle_n F(X) \simeq \Omega^\infty IU\triangle_n F(X)$ where $U : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch_+$ denotes the forgetful functor. Now it remains to compute $U\triangle_n F(X)$.

$$\begin{aligned} U\triangle_n F(X) &\simeq \operatorname*{hocolim}_{p_i\to\infty}{}_{Ch_+}[red_0 s^{-p_1-\ldots-p_n} Icr_n UF(\Sigma^{p_1}X, ..., \Sigma^{p_n}X)] \\ &\simeq red_0 \operatorname*{hocolim}_{p_i\to\infty}{}_{Ch}[s^{-p_1-\ldots-p_n} Icr_n UF(\Sigma^{p_1}X, ..., \Sigma^{p_n}X)] \\ &\simeq red_0 \widehat{\triangle}_n F(X) \end{aligned}$$

This last equivalence is justified by the fact that the functor red_0 commutes with filtered colimits. Finally, by applying $\Omega^\infty I$ to the last equation, we obtain

$$\Omega^\infty IU\triangle_n F(X) \simeq \Omega^\infty Ired_0\widehat{\triangle}_n F(X) = \Omega^\infty \widehat{\triangle}_n F(X).$$

□

2.3.4 Proof of Theorem 2.22

The key ingredient behind the proof of Theorem 2.22 is the following result:

Theorem 2.36. *If char($\Bbbk$)=0, then there is a weak equivalence*

$$D_n F(X) \simeq \Omega^\infty(\widehat{\triangle}_n F(X)_{h\Sigma_n}).$$

where $(-)_{h\Sigma_n}$ denotes the homotopy orbits. When $\mathcal{D} = Ch$ then this result holds when the ground field $\Bbbk$ is of any characteristic.

The straight consequence of this result is that, we can write $D_n F(X)$ in terms of the homotopy indecomposable $\Sigma^\infty X$.

Corollary 2.37. *If char($\Bbbk$)=0 and $\mathcal{C} = Alg_{\mathcal{O}}$, then Then there is a weak equivalence*

$$D_n F(X) \simeq \Omega^\infty H(red_0\Sigma^\infty X)$$

where $H : Ch_+ \longrightarrow Ch$ is the n-homogeneous functor given by:

$$H(V) := \widehat{\triangle}_n(F\mathcal{O}(-))(V)_{h\Sigma_n}.$$

When $\mathcal{D} = Ch$ then this result holds when the ground field $\Bbbk$ is of any characteristic.

Proof. The functor H is n-homogeneous since it is the n-th stabilization of the cross effect of $F\mathcal{O}(-)$.

Let X be an algebra over the operad $\mathcal{O}$ and $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow \mathcal{D}$ be a homotopy and reduced functor. We observe that

$$\begin{aligned} \widehat{\triangle}_n F(X) &\simeq \Omega^n(\widehat{\triangle}_n F)(\Sigma X) && (\text{ since } \widehat{L}_n UF \text{ is n-multilinear }) \\ &\simeq \Omega^n(\widehat{\triangle}_n F)(\mathcal{O}(sUB(\mathcal{O}, X))) && (\text{ since } \Sigma X \simeq \mathcal{O}(sUB(\mathcal{O}, X)) \text{ from Corollary 1.29}) \\ &\simeq \Omega^n \widehat{\triangle}_n(F\mathcal{O}(-))(sUB(\mathcal{O}, X)) && (\text{ since } \mathcal{O}(-) \text{ commutes with coproducts}) \\ &\simeq \widehat{\triangle}_n(F\mathcal{O}(-))(UB(\mathcal{O}, X)) && (\text{ since } \widehat{L}_n(UF\mathcal{O}(-)) \text{ is n-multilinear }) \\ &\simeq \widehat{\triangle}_n(F\mathcal{O}(-))(red_0\Sigma^\infty X) && (\text{ since } red_0 I = Id_{Ch_+}) \end{aligned}$$

One deduce from this observation that $(\widehat{\triangle}_n F(X))_{h\Sigma_n} \simeq \widehat{\triangle}_n(F\mathcal{O}(-))(red_0\Sigma^\infty X)_{h\Sigma_n}$. Using Theorem 2.36, we obtain the quasi-isomorphism

$$D_nF(X) \simeq \Omega^\infty(\widehat{\triangle}_n(F\mathcal{O}(-))(red_0\Sigma^\infty X)_{h\Sigma_n}) \tag{2.18}$$

□

To prove Theorem 2.22 , we finally need the next lemma.

Lemma 2.38. *Let $\mathcal{C} = Ch_+$ or Ch. Let $L_r : \mathcal{C}^{\times r} \longrightarrow Ch$ be a r-multilinear functor. Then for any chain complexes $V_1, ..., V_r$ and finite chain complexes $W_1, ..., W_r$, there is a zig-zag of quasi-isomorphisms*

$$W_1 \otimes ... \otimes W_r \otimes L_r(V_1, ..., V_r) \simeq L_r(W_1 \otimes V_1, ..., W_r \otimes V_r).$$

Proof. 1. We first consider the case $r = 1$ and we want to construct a zig-zag of quasi-isomorphisms $W \otimes L_1(V) \simeq L_1(W \otimes V)$, for a given chain complex V and a finite chain complex W.

Let us consider the following commutative diagram

$$\begin{tikzcd}
L_1(sV \oplus V) \arrow[d] & 0 \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & L_1(sV) \oplus s^{-1}L_1(sV) \arrow[d] \\
L_1(sV) & L_1(sV) \arrow[l, "="'] \arrow[r, "="] & L_1(sV) \\
L_1(0) \arrow[u] & 0 \arrow[l, "\simeq"'] \arrow[r, "="] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

A homotopy limit functor applied on each column gives the zig-zag of quasi-isomorphisms

$$L_1(V) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} s^{-1}L_1(sV) \tag{2.19}$$

where the homotopy limit result of the first column in due to the fact that the functor L_1 is linear, which induces the pullback diagram

$$\begin{tikzcd}
L_1(V) \arrow[r] \arrow[d] & L_1(sV \oplus V) \arrow[d] \\
0 \arrow[r] & L_1(sV)
\end{tikzcd}$$

The zig-zag of Equation (2.19) can also be re-written as:

$$sL_1(V) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} L_1(sV)$$

which is then equivalent to

$$\Bbbk u \otimes L_1(V) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} L_1(\Bbbk u \otimes V)$$

for a given homogeneous element u of degree 1. One deduce inductively from this construction that ,$\forall n \geq 0$, we have

$$(\Bbbk u)^n \otimes L_1(V) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} L_1((\Bbbk u)^n \otimes V)$$

Therefore , given any homogeneous element u of an arbitrary degree, we have a zig-zag of quasi-isomorphisms

$$\alpha_u : \Bbbk u \otimes L_1(V) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} L_1(\Bbbk u \otimes V)$$

If $W = (\Bbbk u \oplus \Bbbk v, d)$ is a chain complex with 2 generators, we set $\alpha_u + \alpha_v$ to be the composite

$$\alpha_u+\alpha_v :\ W \otimes L_1(V) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} L_1(\Bbbk u \otimes V) \oplus L_1(\Bbbk v \otimes V) \xrightarrow{\simeq} L_1(W \otimes V)\ ,$$

where the last quasi-isomorphism is due to the fact that L_1 is linear. We generalize this construction inductively on the number of generators to any arbitrary finite chain complex W.

2. In the case that $r = 2$, let L_{2,V_1} be the linear functor $V_2 \longmapsto L_2(V_1, V_2)$; One have:

$$\begin{aligned} W_1 \otimes W_2 \otimes L_2(V_1, V_2) &\xrightarrow{\cong} W_1 \otimes (W_2 \otimes L_{2,V_1}(V_2)) \\ &\xleftarrow{\simeq} \bullet \xrightarrow{\simeq} W_1 \otimes L_{2,V_1}(W_2 \otimes V_2) \\ &\xrightarrow{\cong} W_1 \otimes L_{2,W_2\otimes V_2}(V_1) \\ &\xleftarrow{\simeq} \bullet \xrightarrow{\simeq} L_{2,W_2\otimes V_2}(W_1 \otimes V_1) = L_2(W_1 \otimes V_1, W_2 \otimes V_2) \end{aligned}$$

Again, we generalize this argument inductively to any arbitrary r.

□

Proof of Theorem 2.22. We give the proof in the three different cases.

1. When $\mathcal{C} = \mathrm{Alg}_{\mathcal{O}}$, we define the Σ_n-chain complex

$$\partial_n F := \widehat{\triangle}_n(F\mathcal{O}(-))(\Bbbk)$$

where $F\mathcal{O}(-)$ is the composite

$$Ch_+ \xrightarrow{\mathcal{O}(-)} \mathrm{Alg}_{\mathcal{O}} \xrightarrow{F} \mathcal{D}.$$

Here $\Bbbk$ is seen as a chain complex concentrated in degree 0 and the construction $\widehat{\triangle}_n(-)$ appears in Definition 2.34.

On the other hand, by Corollary 2.37, if we set $V := red_0\Sigma^\infty X \in Ch_+$,

$$D_nF(X) \simeq \Omega^\infty(\widehat{\triangle}_n(F\mathcal{O}(-))(V)_{h\Sigma_n})$$

Since $\widehat{\triangle}_n(F\mathcal{O}(-))(V)_{h\Sigma_n} = \widehat{L}_n(F\mathcal{O}(-))(V, ..., V)_{h\Sigma_n}$ and that $\widehat{L}_n(F\mathcal{O}(-))$ is multilinear, we deduce using Lemma 2.38, the Σ_n-equivariant zig-zag of quasi-isomorphisms (for $\Sigma^\infty X$ finite) :

$$(\Sigma^\infty X)^{\otimes n} \otimes \widehat{\triangle}_n(F\mathcal{O}(-))(\Bbbk) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} \widehat{\triangle}_n(F\mathcal{O}(-))(\Sigma^\infty X)$$

Therefore we deduce the quasi-isomorphism

$$(\Sigma^\infty X)^{\otimes n} \underset{h\Sigma_n}{\otimes} \widehat{\triangle}_n(F\mathcal{O}(-))(\Bbbk) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} \widehat{\triangle}_n(F\mathcal{O}(-))(\Sigma^\infty X)_{h\Sigma_n}.$$

In addition, if F is finitary then $\widehat{L}_n(F\mathcal{O}(-))$ is finitary on each variable. In this case for any arbitrary algebra X, we rewrite $\Sigma^\infty X$ as a filtered colimit of its finite subcomplexes and then apply again Lemma 2.38.

2. When $\mathcal{C} = Ch_+$, this is a particular case of $\mathrm{Alg}_{\mathcal{O}}$ when $\mathcal{O} = \mathbb{I}$.

3. When $\mathcal{C} = Ch$, we have from Theorem 2.36 the equivalence

$$D_nF(V) \simeq \Omega^\infty(\widehat{\triangle}_nF(V)_{h\Sigma_n}).$$

We know that $\widehat{\triangle}_nF(V) = \widehat{L}_nF(V, ..., V)$ and since L_nF is multilinear, we use again Lemma 2.38 to deduce the weak equivalence

$$V^{\otimes n} \underset{h\Sigma_n}{\otimes} \widehat{\triangle}_nF(\Bbbk) \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} \widehat{\triangle}_n(F(-))(V)_{h\Sigma_n}.$$

□

Now it remains to prove Theorem 2.36.

Proof of Theorem 2.36. 1. Assume that $\mathcal{D} = \mathrm{Alg}_{\mathcal{O}}$. The functor D_nF is n-homogeneous, thus using Lemma 1.25 we have the equivalence $D_nF(X) \simeq \Omega^\infty IUD_nF(X)$. By applying Proposition 2.29 to the functor $H = UD_nF$, we have the equivalence

$$cr_n(UD_nF) \circ \triangle_{h\Sigma_n} \xrightarrow{\simeq} UD_nF \tag{2.20}$$

We then deduce the equivalence

$$D_nF(X) \simeq \Omega^\infty(Icr_n(UD_nF) \circ \triangle_{h\Sigma_n}) \tag{2.21}$$

On the other hand, the functor D_nF is n homogeneous, thus cr_nD_nF is multilinear. The natural map

$$p_1...p_1cr_nP_nF : cr_n(UD_nF) \longrightarrow L_n(UD_nF), \tag{2.22}$$

which consists of applying P_1 to any variable of cr_nD_nF, is an equivalence.

The combination of Equations (2.21) and Equations (2.22) gives the equivalences

$$\begin{aligned} D_nF(X) &\simeq \Omega^\infty(IL_n(UD_nF) \circ \triangle_{h\Sigma_n}) \\ &\simeq \Omega^\infty(IL_n(UP_nF) \circ \triangle_{h\Sigma_n}) && (\text{since } cr_nP_nF \simeq cr_nD_nF) \\ &\simeq \Omega^\infty(IP_n(\triangle_nUF)_{h\Sigma_n}) && (\text{using Lemma 2.32}) \\ &\simeq \Omega^\infty(I(\triangle_nUF)_{h\Sigma_n}) && (\text{since } \triangle_nF \text{ is n excisive}) \end{aligned}$$

In addition, we know from Lemma 2.35 that

$$\triangle_n F(X) \simeq \Omega^\infty \widehat{\triangle}_n F(X) \tag{2.23}$$

By applying U to Equation (2.23), we get $\triangle_n UF(X) \simeq red_0 \widehat{\triangle}_n F(X)$. We then conclude

$$\begin{aligned} D_n F(X) &\simeq \Omega^\infty (I(red_0 \widehat{\triangle}_n F(X))_{h\Sigma_n}) &(2.24)\\ &\simeq \Omega^\infty (Ired_0(\widehat{\triangle}_n F(X))_{h\Sigma_n}) &(2.25)\\ &\simeq \Omega^\infty (\widehat{\triangle}_n F(X)_{h\Sigma_n}) &(2.26)\end{aligned}$$

2. The cases $\mathcal{D} = Ch_+$ and Ch use an analogous argument used in the first case.

□

2.3.5 Co-cross-effect

In the particular cases where $\mathcal{C} = Ch_+$ or Ch and $\mathcal{D} = Ch$, we can also describe the cross-effect of a functor F using the total homotopy cofiber (thocofib) of a certain cube. This dual construction, also called the "co-cross-effect", was considered by McCarthy [McC01, 1.3] in studying dual calculus, and the equivalence between the cross-effect and co-cross-effect was proved by Ching [Chi10, Lemma 2.2] for functors with values in spectra.

Let $W_1, ..., W_n \in \mathcal{C}$, we associate the n-cube $\mathcal{X}$ in $\mathcal{C}$ defined as follows:

- $T \subseteq \underline{n}, \mathcal{X}(T) := \underset{i \in T}{\oplus} W_j$;
- For $T \subsetneq \underline{n}$ and $j \in \underline{n} \backslash T$, the map $\mathcal{X}(T) \longrightarrow \mathcal{X}(T \cup \{j\})$ (in the cube) is induced by the inclusion

$$\begin{aligned} \underset{i\in T}{\oplus} W_i &\longrightarrow (\underset{i\in T}{\oplus} W_i) \oplus W_j \\ x &\longmapsto (x, 0) \end{aligned}$$

Definition 2.39 (Co-cross-effects)**.** *Let $\mathcal{C} = Ch_+$ or Ch and $F : \mathcal{C} \longrightarrow Ch$ be a homotopy functor. The n^{th} co-cross effect of F is the functor $cr^n F : \mathcal{C}^{\times n} \longrightarrow Ch$ which computes the homotopy total fiber of $F(\mathcal{X})$. That is:*

$$cr^n F(W_1, ..., W_n) := \ hocofib\{\underset{T \subsetneq \underline{n}}{holim} F(\underset{i\in T}{\oplus} W_i) \longrightarrow F(W_1 \oplus ... \oplus W_n)\}.$$

Similarly to Equation (2.6), we can equivalently define the co-cross-effect as the total homotopy cofiber (thocofib):

$$cr^n F(W_1, ..., W_n) = \ \text{thocofib}(T \supseteq \underline{n} \mapsto F(\underset{i\in T}{\oplus} W_i)) \tag{2.27}$$

Lemma 2.40. *([Chi10, Lemma 2.2]) Let $\mathcal{C} = Ch_+$ or Ch and $F : \mathcal{C} \longrightarrow Ch$ be a homotopy functor. Then the n^{th} cross-effect of F is equivalent to the n^{th} co-cross-effect of F. That is:*

$$cr_n F(W_1, ..., W_n) \xrightarrow{\simeq} cr^n F(W_1, ..., W_n)$$

Proof. Since Ch is a stable category and that in $\mathcal{C}$, finite products and finite coproducts are isomorphic, we simply mimic Ching's proof. $\square$

2.4 Examples: Computing the Goodwillie derivatives

In this section, we show how we compute the Goodwillie derivatives for a couple of interesting functors.

Proposition 2.41. *The Goodwillie derivative of the identity functor $Id : Alg_{\mathcal{O}} \longrightarrow Alg_{\mathcal{O}}$ is given by:*

$$\partial_* Id \simeq \mathcal{O}.$$

Proof. A straight computation gives the result.

$$\begin{aligned}
\partial_n Id \simeq & \operatorname*{hocolim}_{p_i \to \infty} s^{-p_1 - \dots - p_n} cr_n I(\mathcal{O}(\Sigma^{p_1}\mathbb{k}), ..., \mathcal{O}(\Sigma^{p_n}\mathbb{k})) \\
= & \operatorname*{hocolim}_{p_i \to \infty} s^{-p_1 - \dots - p_n} thofib(\underline{n} - T \mapsto \mathcal{O}(\bigoplus_{i \in T} \Sigma^{p_i}\mathbb{k})) \\
\cong & \operatorname*{hocolim}_{p_i \to \infty} s^{-p_1 - \dots - p_n} \mathcal{O}(\bigoplus_{i \in \underline{n}} \Sigma^{p_i}\mathbb{k}) \\
= & \operatorname*{hocolim}_{p_i \to \infty} s^{-p_1 - \dots - p_n} \bigoplus_{r \geq 0} (\mathcal{O}(r) \otimes_{\Sigma_r} (\bigoplus_{i \in \underline{n}} s^{p_i}\mathbb{k})^{\otimes r}) \\
= & \mathcal{O}(n) \otimes_{\Sigma_n} (\mathbb{k})^{\otimes n} \cong \mathcal{O}(n).
\end{aligned}$$

$\square$

Proposition 2.42. *The Goodwillie derivative of the comonad $\Sigma^\infty \Omega^\infty : Ch \longrightarrow Ch$ is given by:*

$$\partial_* \Sigma^\infty \Omega^\infty \simeq B(\mathcal{O}).$$

Proof. A straight computation gives the result.

$$\begin{aligned}
\partial_n \Sigma^\infty \Omega^\infty \simeq & \operatorname*{hocolim}_{p_i \to \infty} s^{-p_1 - \dots - p_n} cr_n \Sigma^\infty \Omega^\infty (\Sigma^{p_1}\mathbb{k}, ..., \Sigma^{p_n}\mathbb{k}) \\
= & \operatorname*{hocolim}_{p_i \to \infty} s^{-p_1 - \dots - p_n} \mathrm{thofib}\ (\underline{n} - T \mapsto B(\mathcal{O})(\bigoplus_{i \in T} \Sigma^{p_i}\mathbb{k})) \\
\cong & \operatorname*{hocolim}_{p_i \to \infty} s^{-p_1 - \dots - p_n} B(\mathcal{O})(\bigoplus_{i \in \underline{n}} \Sigma^{p_i}\mathbb{k}) \\
= & B(\mathcal{O})(n) \otimes_{\Sigma_n} (\mathbb{k})^{\otimes n} \cong B(\mathcal{O})(n).
\end{aligned}$$

$\square$

The next two examples consist of computing the derivatives of two particular functors which will be particularly important and used in chapter 5.

Proposition 2.43. *Let E be an unbounded chain complex with a Σ_n action. We define the functor*

$$\begin{aligned}
\underline{hom}(E, I^{\otimes n})^{\Sigma_n} : Ch & \longrightarrow Ch \\
W & \longmapsto \underline{hom}(E, W^{\otimes n})^{\Sigma_n}
\end{aligned}$$

Then

$$\partial_k \underline{hom}(E, I^{\otimes n})^{\Sigma_n} \simeq \begin{cases} 0 & \text{if } k \neq n; \\ \underline{hom}(E, \Bbbk) & \text{if } k = n. \end{cases}$$

Proof. In fact we use Lemma 2.40 to obtain the quasi-isomorphism:

$$\begin{tikzcd} cr_n(\underline{hom}(E, I^{\otimes n})^{\Sigma_n})(W_1, ..., W_n) \arrow[r, "\simeq"] & \underline{hom}(E, cr_n(I^{\otimes n})(W_1, ..., W_n))^{\Sigma_n} \arrow[d, "\simeq"] \\ & \underline{hom}(E, \text{thocofib } (\underline{n} \supseteq T \mapsto (\bigoplus_{i \in T} W_i)^{\otimes n})^{\Sigma_n}) \end{tikzcd}$$

On the other hand the maps in the cube $\underline{n} \supseteq T \mapsto (\bigoplus_{i \in T} W_i)^{\otimes n}$ are inclusions, therefore the homotopy cofiber is a strict cofiber. Computation gives:

$$\begin{aligned} \text{thocofib } (\underline{n} \supseteq T \mapsto (\bigoplus_{i \in T} W_i)^{\otimes n}) &\simeq \text{tcofib } (\underline{n} \supseteq T \mapsto (\bigoplus_{i \in T} W_i)^{\otimes n}) \\ &\simeq \bigoplus_{\sigma \in \Sigma_n} W_{\sigma(1)} \otimes ... \otimes W_{\sigma(n)} \end{aligned}$$

We then deduce:

$$\begin{aligned} cr_n(\underline{hom}(E, (-)^{\otimes n})^{\Sigma_n})(W_1, ..., W_n) &\simeq \underline{hom}(E, \bigoplus_{\sigma \in \Sigma_n} W_{\sigma(1)} \otimes ... \otimes W_{\sigma(n)})^{\Sigma_n} \\ &\simeq \underline{hom}(E, W_1 \otimes ... \otimes W_n) \end{aligned}$$

Now when we consider each $W_i = \Bbbk s^{p_i}$ and apply $\underset{p_i \to \infty}{\text{hocolim}}\, s^{-p_i}-$ to the cross effect, we get the result.

A similar computation by hand shows that $\partial_k \underline{hom}(E, I^{\otimes n})^{\Sigma_n} \simeq 0$ when $k \neq n$.

$\square$

Let V be a finite non negatively graded chain complex. By finite, we mean of finite dimension in each degree and bounded above. We define the functor

$$N\Bbbk Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, -) : Ch_+ \longrightarrow Ch_+$$

where,

- $N : sAb \longrightarrow Ch_+$ is the normalization functor;
- $\Bbbk Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, W)$ denotes the free simplicial $\Bbbk$-vector space generated by the simplicial set $Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, W)$.

Proposition 2.44. *We assume that the ground field $\Bbbk$ is of characteristic 0. Let $V \in Ch_+$. Then we have the quasi-isomorphism (in Ch)*

$$\partial_n N\Bbbk Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, -) \simeq \underline{hom}(V, \Bbbk)^{\otimes n}$$

Before we give the proof of Proposition 2.44, we remind the following fact which seems to be a classical construction: Let $p \in \mathbb{N}$, A be a simplicial $\Bbbk$-vector space and consider the following notations:

- We write S^1 to mean the simplicial model of the circle. S^1 is naturally considered as pointed and S^p denotes the smash product $S^p = (S^1)^{\wedge p}$.
- If $(X_\bullet, *)$ is a pointed simplicial set, then $\widetilde{\Bbbk} X_\bullet := \Bbbk X_\bullet / \Bbbk *$.
- We write $A[p]$ to mean the simplicial $\Bbbk$-vector space given level-wise by $A[p] := \widetilde{\Bbbk}[S^p] \otimes A$ which is in other word the p^{th}-suspension of A;

$A[p]$ is a p-connected Kan complex (as any simplicial abelian group), thus the Hurewicz map $A[p] \xrightarrow{h} \widetilde{\Bbbk}A[p]$, which is in fact induced by the unit of the adjoint pair $\Bbbk(-) : sVect_{\Bbbk} \rightleftarrows sSet : U$, is $2p$-connected. The Hurewicz theorem stated on this current form appears in [GJ99, Chap III, Thm 3.7] for abelian groups, and the rational Hurewicz case appears in [KK04].

In addition, considering the natural projection

$$\begin{aligned} l : \widetilde{\Bbbk}A[p] &\longrightarrow A[p] \\ \underset{i}{\oplus} x_i &\longmapsto \underset{i}{\Sigma} x_i \end{aligned}$$

since the composite $A[p] \xrightarrow{h} \widetilde{\Bbbk}A[p] \xrightarrow{l} A[p]$ is the identity, we deduce that the map l is also $2p$-connected. Therefore the map

$$\Omega^p \widetilde{\Bbbk}A[p] \xrightarrow{\Omega^p(l)} \Omega^p A[p]$$

is p-connected and the map

$$\underset{p\to\infty}{\operatorname{hocolim}}\ \Omega^p \widetilde{\Bbbk}A[p] \longrightarrow \underset{p\to\infty}{\operatorname{hocolim}}\ \Omega^p A[p]$$

is a weak equivalence of simplicial abelian groups. Now using the fact that the functor N is in the same time a Quillen left and right functor in the Dold Kan correspondence we deduce the quasi-isomorphism

$$\underset{p\to\infty}{\operatorname{hocolim}}\ \Omega^p N\widetilde{\Bbbk}A[p] \longrightarrow \underset{p\to\infty}{\operatorname{hocolim}}\ \Omega^p NA[p]. \tag{2.28}$$

Proof of Lemma 2.44. We use Lemma 2.40 to obtain the quasi-isomorphism:

$$\begin{array}{c} cr_n(N\Bbbk Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, -))(W_1, ..., W_n) \\ \Big\downarrow \simeq \\ \text{thocofib } (\underline{n} \supseteq T \mapsto N\Bbbk Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, \underset{i\in T}{\oplus} W_i)) \end{array}$$

On the other hand the functors $N : sAb \longrightarrow Ch_+$ and $\Bbbk(-) : sSet \longrightarrow sAb$ are left Quillen functors, we therefore have the equivalences

$$\begin{aligned} \text{thocofib } (N\Bbbk Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, \underset{i\in T}{\oplus} W_i)) &\simeq N\Bbbk \text{ thocofib}(Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, \underset{i\in T}{\oplus} W_i)) \\ &\simeq N\Bbbk \text{ thocofib}(\underset{i\in T}{\oplus} Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, W_i)) \end{aligned}$$

Since the maps in the $\underline{n}$-cube of pointed simplicial sets

$$T \longmapsto \underset{i\in T}{\oplus} Hom_{Ch_+}(V\otimes N\Bbbk\triangle^\bullet, W_i)$$

are inclusions, the total homotopy colimit is the strict total cofiber (tcofib), and computation shows (inductively) that

$$\text{tcofib}\,(\underset{i\in T}{\oplus} Hom_{Ch_+}(V\otimes N\Bbbk\triangle^\bullet, W_i)) \cong \tag{2.29}$$

$$N\Bbbk(Hom_{Ch_+}(V\otimes N\Bbbk\triangle^\bullet, W_1)\wedge ...\wedge Hom_{Ch_+}(V\otimes N\Bbbk\triangle^\bullet, W_n)) \cong \tag{2.30}$$

$$N(\widetilde{\Bbbk}Hom_{Ch_+}(V\otimes N\Bbbk\triangle^\bullet, W_1)\otimes ...\otimes \widetilde{\Bbbk}Hom_{Ch_+}(V\otimes N\Bbbk\triangle^\bullet, W_n)) \tag{2.31}$$

We then conclude the quasi-isomorphism:

$$cr_n(\widetilde{Ch}_+(V,-))(W_1,...,W_n)\simeq N\widetilde{\Bbbk}Hom_{Ch_+}(V\otimes N\Bbbk\triangle^\bullet, W_1)\otimes ...\otimes N\widetilde{\Bbbk}Hom_{Ch_+}(V\otimes N\Bbbk\triangle^\bullet, W_n) \tag{2.32}$$

If V is bounded below degree k, we have

$$\begin{aligned} Hom_{Ch_+}(V\otimes N\Bbbk\triangle^\bullet, s^{p+k}\Bbbk) &\cong Hom_{Ch_+}(N\Bbbk\triangle^\bullet, \underline{hom}(V, s^{p+k}\Bbbk)) && (2.33)\\ &\cong Hom_{Ch_+}(N\Bbbk\triangle^\bullet, \underline{hom}(V,\Bbbk)\otimes s^{p+k}\Bbbk) && (2.34)\\ &\underset{\simeq}{\longleftarrow} Hom_{Ch_+}(N\Bbbk\triangle^\bullet, \underline{hom}(V,\Bbbk)\otimes s^{k}\Bbbk)[p] && (2.35)\end{aligned}$$

where the weak equivalence (2.35) is given by the weak equivalence of simplicial vector spaces

$$\begin{array}{c} Hom_{Ch_+}(N\Bbbk\triangle^\bullet, \underline{hom}(V,\Bbbk)\otimes s^k\Bbbk)\otimes Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^p\Bbbk)\\ \Big\downarrow \simeq \\ Hom_{Ch_+}(N\Bbbk\triangle^\bullet, \underline{hom}(V,\Bbbk)\otimes s^{p+k}\Bbbk)\end{array}$$

defined in [SS03, (2.8), p 295].

Now, when we replace A in the map (2.28) with $Hom_{Ch_+}(N\Bbbk\triangle^\bullet, \underline{hom}(V,\Bbbk)\otimes s^k\Bbbk)$ and compose it with $\Omega^k(-)$, we get the quasi-isomorphisms

$$\begin{array}{c} \underset{p\to\infty}{\text{hocolim}}\ \Omega^p N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, \underline{hom}(V,\Bbbk)\otimes s^p\Bbbk)\\ \Big\uparrow \simeq \\ \underset{p\to\infty}{\text{hocolim}}\ \Omega^{p+k} N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, \underline{hom}(V,\Bbbk)\otimes s^k\Bbbk)[p]\\ \Big\downarrow \simeq \\ \underset{p\to\infty}{\text{hocolim}}\ \Omega^{p+k} NHom_{Ch_+}(N\Bbbk\triangle^\bullet, \underline{hom}(V,\Bbbk)\otimes s^k\Bbbk)[p]\\ \Big\downarrow \simeq \\ \underset{p\to\infty}{\text{hocolim}}\ \Omega^{p+k}\underline{hom}(V,\Bbbk)\otimes s^{p+k}\simeq \underline{hom}(V,\Bbbk)\end{array}$$

where the last equivalence is induced by the isomorphism of the Dold Kan equivalence

$$NHom_{Ch_+}(N\Bbbk\triangle^\bullet, \underline{hom}(V,\Bbbk)\otimes s^k\Bbbk) \cong \underline{hom}(V,\Bbbk)\otimes s^k\Bbbk$$

Using this above equivalence, we consider the specific case $W_i = s^{p_i}\Bbbk$ in Equation (2.32) and apply the functor $\operatorname{hocolim}_{p_i\to\infty}$ to the left-hand and right-hand side of this same equation, we get the quasi-isomorphism

$$\partial_n N\Bbbk Hom_{Ch_+}(V\otimes N\Bbbk\triangle^\bullet, -) \simeq \underline{hom}(V,\Bbbk)^{\otimes n}.$$

□

CHAPTER 3

Simplicial categories

Let $\mathcal{O}$ be a reduced operad in Ch_+. Our goal in this thesis is to study Goodwillie calculus for homotopy functors $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$ (or other analogous functors). An approach for this is to describe the derivatives of F. We have shown (in Theorem 2.22) that these derivatives are given by the formula

$$\partial_n F = \widehat{\triangle}_n F(\mathcal{O}(\Bbbk))$$

where $\widehat{\triangle}_n F$ is the stabilization of the cross effect on F. This formula only gives the structure of a symmetric sequences on the derivatives $\partial_* F$.

In order to re-construct the Taylor tower $\{P_n F\}_{n \geq 1}$ from the derivatives of F, we will need more structure on $\partial_* F$, more precisely the structure of left $\mathcal{O}$-module. We will build that structure in Chapter 5 and our strategy will be to first do it in the special case when F is a representable functor and then infer it for filtered homotopy colimits of those. To get that many homotopy functors are actually filtered colimits of representable functors, we will introduce in Chapter 4 a model category structure on the category of functor $[\mathcal{C}, Ch]$ where the cofibrant generators (or "cells") are exactly the representable functors.

To do this properly, we need to consider homotopy functors which are actually sort of simplicial functors, or more precisely enriched over Ch. This passes through a suitable Ch-enrichment of algebraic categories Ch_+ and $\mathrm{Alg}_{\mathcal{O}}$ which is not completely straightforward. The goal of this chapter is to describe precisely these Ch-enriched structures on Ch_+, $\mathrm{Alg}_{\mathcal{O}}$ and Ch (the later being the classical one).

The Ch-enriched categories that we will build are not genuine enrichment of the categories Ch_+ and $\mathrm{Alg}_{\mathcal{O}}$ in the sense that the discretization of our enriched categories is actually as linearization of the set-categories Ch_+ and $\mathrm{Alg}_{\mathcal{O}}$. To emphasize this distinction we will denote our enriched categories $\widetilde{\mathrm{Alg}_{\mathcal{O}}}$ and $\widetilde{Ch}_+$ (as well as $\widetilde{Ch}$ although the discretization of $\widetilde{Ch}$ is the standard Ch).

Our strategy relies of the work of Hinich who proved in [Hin97, § 4.8] that the category $\mathrm{Alg}_{\mathcal{O}}$ is a simplicial model category with the enriched hom functor

$$Map(X,Y) := Hom_{\mathrm{Alg}_{\mathcal{O}}}(X, Y \otimes Apl_{\bullet}) \tag{3.1}$$

where $\forall n \geq 0$, Apl_n is the (commutative) algebra of the polynomial of differential forms. This enrichment is not sufficient for us since it is not yet suitable for homotopy theory. In fact, the representable $Map(X,-)$ that arises from this enrichment does not preserves weak equivalences. We will then replace this hom set with

$$Map(X,Y) := Hom_{\mathrm{Alg}_{\mathcal{O}}}(X^c, Y \otimes Apl_{\bullet}) \tag{3.2}$$

where X^c is a well chosen cofibrant replacement of X.

This will define a category enriched over simplicial sets and closely related to $\mathrm{Alg}_{\mathcal{O}}$. At this point, we are not done yet. We would like an enrichment over chain complexes and the hom set of Equation (3.2) is not abelian, so we can not directly use the normalization functor N (of Dold Kan § 1.2) to get to chain complex. Therefore we will replace the hom set of Equation (3.2) with

$$Map(X,Y) := N\Bbbk Hom_{\mathrm{Alg}_{\mathcal{O}}}(X^c, Y \otimes Apl_{\bullet}) \tag{3.3}$$

This will define the Ch-enriched category that we will consider. After these constructions, we will consider Ch-enriched functors $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$ that we will generally named "simplicial" functors. This name does not mean that these are simplicially enriched functors in the strict sense, but it just keep encoding the fact that the Ch-enrichment is after-all due to Equation (3.1) which endows $\mathrm{Alg}_{\mathcal{O}}$ with a strict simplicial enrichment property.

We go beyond this discussion in this chapter and make an analogous discussion for the category Ch_+.

The chapter has the following guidelines:

- In §3.1, we define the simplicial structure on $\mathrm{Alg}_{\mathcal{O}}$ based on the Hinich's construction; We define a simplicial enriched category $\mathrm{Alg}'_{\mathcal{O}}$ and deduce the construction of a Ch-category $\widetilde{\mathrm{Alg}}_{\mathcal{O}}$;

- In §3.2, we make an analogous construction as in §3.1 to define the simplicial enriched category Ch'_+ and a Ch-category $\widetilde{Ch}_+$.

- In §3.3, we remind the enrichment of the Ch-category $\widetilde{Ch} = Ch$;

- In §3.4, we describe the discrete categories associated to the categories $\widetilde{\mathrm{Alg}}_{\mathcal{O}}$, $\widetilde{Ch}_+$ and $\widetilde{Ch}$.

- In §3.5, we define simplicial functors. These are Ch-enriched functors compatible with the Ch-enrichments of §3.1, §3.2 and §3.3.

- Finally in section §3.6, we give example of simplicial functors.

3.1 The Ch-enriched category $\widetilde{\mathbf{Alg}}_{\mathcal{O}}$

In this section we use the Hinich's construction of simplicial structure on $\mathrm{Alg}_{\mathcal{O}}$ to define a simplicial category $\mathrm{Alg}'_{\mathcal{O}}$. We will take the normalized functor $N : sVect_{\mathbb{k}} \longrightarrow Ch_+$ and the free abelian functor $\mathbb{k}(-) : sSet \longrightarrow sVect_{\mathbb{k}}$ to deduce the Ch-enriched category $\widetilde{\mathrm{Alg}}_{\mathcal{O}}$.

3.1.1 The Hinich's simplicial structure on $\mathbf{Alg}_{\mathcal{O}}$

In the Hinich's paper [Hin97, § 4.8], the based category is Ch and here in our case, the operad $\mathcal{O}$ along with $\mathcal{O}$-algebras are constructed over Ch_+. However his construction still applies to this context as we explain bellow:

There are natural inclusions:

$$\begin{aligned} I : Ch_+ &\longrightarrow Ch \\ V &\longmapsto I(V) \end{aligned}$$

where $I(V)_k := V_k$ if $k \geq 0$ and $I(V)_k := 0$ if $k < 0$.

$$\begin{aligned} \iota : Op_{Ch_+} &\longrightarrow Op_{Ch} \\ \mathcal{O} &\longmapsto \iota\mathcal{O} := \{I(\mathcal{O}(n))\}_n \end{aligned}$$

$$\begin{aligned} \iota : \mathrm{Alg}_{\mathcal{O}} &\longrightarrow \mathrm{Alg}_{\iota\mathcal{O}} \\ (X, m_X) &\longmapsto \iota X = (I(X), I(m_X)) \end{aligned}$$

There is the adjunction

$$\iota : \mathrm{Alg}_{\mathcal{O}} \rightleftarrows \mathrm{Alg}_{\iota\mathcal{O}} : \kappa$$

where $\kappa(\mathbb{X}, m_{\mathbb{X}}) := (red_0(\mathbb{X}), red_0(m_{\mathbb{X}}))$.

Note that the unit of this adjunction $1 \longrightarrow red_0\iota$ is an isomorphism. We finally get the bijections

$$\begin{aligned} Hom_{\mathrm{Alg}_{\mathcal{O}}}(X, Y) &\cong Hom_{\mathrm{Alg}_{\mathcal{O}}}(X, red_0\iota Y) \\ &\cong Hom_{\mathrm{Alg}_{\iota\mathcal{O}}}(\iota X, \iota Y) \end{aligned}$$

We now remind the Hinich's simplicial enrichment of $\mathrm{Alg}_{\iota\mathcal{O}}$ and deduce the simplicial enrichment on $\mathrm{Alg}_{\mathcal{O}}$ by the means of these above bijections. We define the simplicial commutative differential graded algebra $Apl_{\bullet} = \{Apl_n\}_n$ by:

- The chain algebra Apl_n is defined by

$$Apl_n := \frac{\wedge(t_0,\ldots,t_n,dt_0,\ldots,dt_n)}{(\Sigma t_i - 1, \Sigma dt_i)}$$

  where $|t_i| = 0$ and $|dt_i| = -1$

- The face and degeneracy morphisms are the unique chain algebra morphisms $\partial_i : Apl_{n+1} \longrightarrow Apl_n$ and $s_j : Apl_n \longrightarrow Apl_{n+1}$ satisfying

$$\partial_i : t_k \mapsto \begin{cases} t_k & k < i; \\ 0 & k = i \\ t_{k-1} & k > i \end{cases}$$

and

$$s_j : t_k \mapsto \begin{cases} t_k & k < j; \\ t_k + t_{k+1} & k = j \\ t_{k+1} & k > j \end{cases}$$

- the multiplication $m : Apl_\bullet \otimes Apl_\bullet \longrightarrow Apl_\bullet$ is defined level-wise by concatenation.

Let $\mathbb{Y} \in \mathrm{Alg}_{\iota\mathcal{O}}$. Note that Apl_n being a commutative differential graded $\Bbbk$-algebra, the tensor product $\mathbb{Y} \otimes Apl_n$ admits a natural $\iota\mathcal{O}$-algebra structure. We define the bi-functor

$$\begin{aligned} \mathrm{Alg}_{\iota\mathcal{O}}^{op} \times \mathrm{Alg}_{\iota\mathcal{O}} &\longrightarrow sSet \\ (\mathbb{X}, \mathbb{Y}) &\longmapsto Hom_{\mathrm{Alg}_{\iota\mathcal{O}}}(\mathbb{X}, \mathbb{Y} \otimes Apl_\bullet) \end{aligned}$$

This gives a simplicial enrichment structure on $\mathrm{Alg}_{\iota\mathcal{O}}$ (see [Hin97, § 4.8]). We then deduce the bi-functor

$$\begin{aligned} Map(-,-) : \mathrm{Alg}_{\mathcal{O}}^{op} \times \mathrm{Alg}_{\mathcal{O}} &\longrightarrow sSet \\ (X, Y) &\longmapsto Hom_{\mathrm{Alg}_{\mathcal{O}}}(X, red_0(Y \otimes Apl_\bullet)) \end{aligned}$$

which gives a simplicial enrichment on $\mathrm{Alg}_{\mathcal{O}}$.

3.1.2 A homotopy friendly simplicial category $\mathrm{Alg}'_{\mathcal{O}}$

Given an $\mathcal{O}$-algebra X, the representable functor provided by Hinich's constructions in § 3.1.1

$$Map(X, -) : \mathrm{Alg}_{\mathcal{O}} \longrightarrow sSet$$

is not a homotopy functor in general. However the bi-functor $Map(-,-)$ satisfies the pushout axiom (see [Hin97, § 4.8.4]) and a consequence is the following property:

(P) : If X is cofibrant, then the representable functor $Map(X, -)$ is a homotopy functor.

Since we intend to use all this development in Functor Calculus where we only use homotopy functors, we will now replace X by a particular cofibrant replacement denoted X^c. Recall that cofibrant replacements in $\mathrm{Alg}_{\mathcal{O}}$ are given by the cobar-bar adjunction (see § 1.11.2). Namely for $X \in \mathrm{Alg}_{\mathcal{O}}$,

$$X^c := \Omega_{\mathcal{O}}(B(\mathcal{O}, X)).$$

The goal of this section is to define a simplicial category $\mathrm{Alg}'_{\mathcal{O}}$:

Proposition 3.1 (Simplicial enriched category $\mathrm{Alg}'_{\mathcal{O}}$). *There is a sSet-category (or simplicial enriched category), denoted $Alg'_{\mathcal{O}}$, whose:*

- *Objects are $\mathcal{O}$-algebras;*

- *The enriched hom functor is given by: $\forall X, Y \in Alg_{\mathcal{O}}$,*

$$Map(X,Y) := Hom_{Alg_{\mathcal{O}}}(X^c, red_0(Y \otimes Apl_\bullet));$$

- *The composition is γ and the unit is $\eta_\bullet$ given respectively in Equation* (3.5) *and Equation* (3.4).

The rest of this section is devoted to the proof of Proposition 3.1. We will first construct the maps γ and $\eta_\bullet$ and then prove that they satisfy all the enrichment conditions.

(I)- There is a natural morphism of simplicial vector spaces

$$\sigma : UB(\mathcal{O}, red_0(Y \otimes Apl_\bullet)) \longrightarrow red_0(IUB(\mathcal{O}, Y) \otimes Apl_\bullet)$$

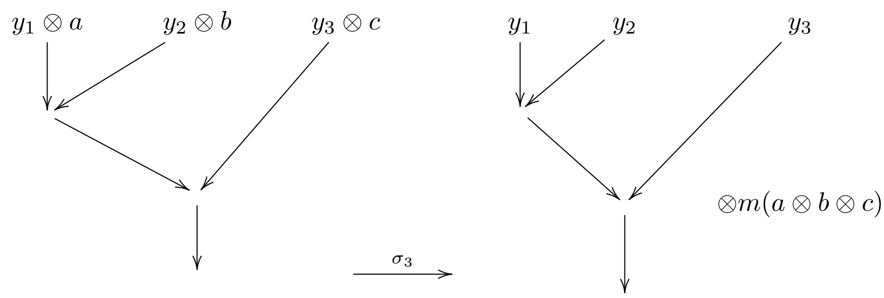
where $U : \mathrm{coAlg}_{B(\mathcal{O})} \longrightarrow Ch_+$ is the forgetful functor and σ is given by the maps:

$$\begin{aligned}\sigma : B(\mathcal{O})(\underline{n}) \underset{\Sigma_n}{\otimes} (red_0(Y \otimes Apl_\bullet))^{\otimes n} &\hookrightarrow red_0(B(\mathcal{O})(\underline{n}) \underset{\Sigma_n}{\otimes} Y^{\otimes n} \otimes Apl_\bullet^{\otimes n})\\ &\overset{1\otimes m^n}{\longrightarrow} red_0(B(\mathcal{O})(n) \underset{\Sigma_n}{\otimes} Y^{\otimes n} \otimes Apl_\bullet)\\ &\longrightarrow red_0(B(\mathcal{O}, Y) \otimes Apl_\bullet)\end{aligned}$$

where m^n is the n^{th} iteration of the multiplication

$$m : Apl_\bullet \otimes Apl_\bullet \longrightarrow Apl_\bullet.$$

The example below gives the construction of σ_3 on a tree in $B(\mathcal{O})(\underline{3})$: given $y_1, y_2, y_3 \in Y$ and $a, b, c \in Apl_k$,



(II)- Similarly as for σ, we define a map

$$\rho : \Omega_{\mathcal{O}}B(\mathcal{O}, red_0(Y \otimes Apl_\bullet)) \longrightarrow red_0(\Omega_{\mathcal{O}}B(\mathcal{O}, Y) \otimes Apl_\bullet)$$

which is obtained with the iteration of the multiplication

$$m : Apl_\bullet \otimes Apl_\bullet \longrightarrow Apl_\bullet.$$

(III)- Given a $B(\mathcal{O})$-coalgebra map $g : B(\mathcal{O}, Y) \longrightarrow B(\mathcal{O}, red_0(Z \otimes Apl_\bullet))$, we define the morphism of $B(\mathcal{O})$-coalgebras:

$$\xi(g) : B(\mathcal{O}, red_0(Y \otimes Apl_\bullet)) \longrightarrow B(\mathcal{O}, red_0(Z \otimes Apl_\bullet))$$

to be the adjoint of the composite

$$\begin{aligned}\Omega_{\mathcal{O}}B(\mathcal{O}, red_0(Y \otimes Apl_\bullet)) &\xrightarrow{\rho} red_0(\Omega_{\mathcal{O}}B(\mathcal{O}, Y) \otimes Apl_\bullet) \\ &\xrightarrow{\widetilde{g}} red_0(red_0(Z \otimes Apl_\bullet) \otimes Apl_\bullet) \\ &\longrightarrow red_0(Z \otimes Apl_\bullet \otimes Apl_\bullet) \\ &\xrightarrow{m} red_0(Z \otimes Apl_\bullet)\end{aligned}$$

where $\widetilde{g} : \Omega_{\mathcal{O}}B(\mathcal{O}, Y) \longrightarrow red_0(Z \otimes Apl_\bullet)$ is the adjoint of g and m is the multiplication

$$m : Apl_\bullet \otimes Apl_\bullet \longrightarrow Apl_\bullet.$$

(IV)- Now ,$\forall X \in \mathrm{Alg}_{\mathcal{O}}$, let

$$\eta_\bullet : S^0 \longrightarrow Hom_{\mathrm{coAlg}_{B(\mathcal{O})}}(B(\mathcal{O}, X), B(\mathcal{O}, red_0(X \otimes Apl_\bullet)))$$

be the morphism of simplicial sets defined aritywise by: $\forall n \geq 0$,

$$\eta_n(*) = B(1 \otimes \varepsilon_n) : B(\mathcal{O}, X) \cong B(\mathcal{O}, red_0(X \otimes \Bbbk)) \longrightarrow B(\mathcal{O}, red_0(X \otimes Apl_n)) \tag{3.4}$$

where $\varepsilon_\bullet : \Bbbk \longrightarrow Apl_\bullet$ is the unit of $Apl_\bullet$.

(V)- Define a map of sets

$$\gamma : Map(X_1, X_2) \otimes Map(X_2, X_3) \longrightarrow Map(X_1, X_3) \tag{3.5}$$

on generators $f_i \in Hom_{\mathrm{coAlg}_{B(\mathcal{O})}}(B(\mathcal{O}, X_i), B(\mathcal{O}, red_0(X_{i+1} \otimes Apl_\bullet)))$, $i = 1, 2$ by:

$$\gamma(f_1 \otimes f_2) := \xi(f_2) \circ f_1$$

Lemma 3.2. *The map γ of Equation* (3.5) *is a morphism of simplicial sets.*

Proof. To prove that γ is well defined, we have to prove that it is compatible with the face and degeneracy maps. Let d_i be the i^{th} face of

$$Hom_{\mathrm{coAlg}_{B(\mathcal{O})}}(B(\mathcal{O}, X_1), B(\mathcal{O}, red_0(X_3 \otimes Apl_\bullet)))$$

induced by $\partial_i : Apl_{k+1} \longrightarrow Apl_k$.

By hypothesis, ∂_i commutes with the product map m of $Apl_\bullet$. Thus we have

$$m(\partial_i \otimes \partial_i) = \partial_i m.$$

Thus a straight computation gives

$$\begin{aligned} d_i(\gamma(f_2 \otimes f_1)) =& B(\mathcal{O})(1 \otimes \partial_i)\xi(f_2) \circ f_1 \qquad (3.6)\\ =& \xi(d_i f_2) \otimes d_i f_1 \qquad (3.7)\end{aligned}$$

A similar argument can be made for degeneracy maps.

□

Proof of Proposition 3.1. We need in fact to prove that the couple $(\gamma, \eta_\bullet)$ satisfies the associativity and the unit axioms (see [Bor94, § 6.2.1]) .

1. For the associativity of γ, we consider the algebras $X_1, X_2, X_3, X_4 \in \mathrm{Alg}_{\mathcal{O}}$, and morphisms $f_i \in Hom_{\mathrm{coAlg}_{B(\mathcal{O})}}(B(\mathcal{O}, X_i), B(\mathcal{O}, red_0(X_{i+1} \otimes Apl_\bullet))), i = 1, 2, 3$. We want to show that

$$\gamma(f_3 \otimes \gamma(f_2 \otimes f_1)) = \gamma(\gamma(f_3 \otimes f_2) \otimes f_1)$$

   This comes straightforward from computations. In fact One can check that
$$\xi(f_3) \circ \xi(f_2) = \xi(\xi(f_3) \circ f_2),$$
   which is itself due to the associativity of

$$m : Apl_\bullet \otimes Apl_\bullet \longrightarrow Apl_\bullet$$

2. For the unit axiom, let $f \in Hom_{\mathrm{coAlg}_{B(\mathcal{O})}}(B(\mathcal{O}, X), B(\mathcal{O}, red_0(Y \otimes Apl_\bullet)))$. Computation shows that $\xi(\eta_\bullet) = Id$, thus

$$\gamma(\eta_\bullet \otimes f) = \xi(\eta_\bullet) \circ f = f.$$

   On the other hand, since Im $(\eta_\bullet) = B(\mathcal{O}, X)$ we deduce that

$$\gamma(f \otimes \eta_\bullet) = \xi(f) \circ \eta_\bullet = f.$$

□

3.1.3 The Ch-enriched category $\widetilde{\mathbf{Alg}}_{\mathcal{O}}$

Corollary 3.3 (Ch-enriched category $\widetilde{\mathrm{Alg}}_{\mathcal{O}}$)**.** *There is a Ch-category (or Chain complex enriched category), denoted $\widetilde{Alg}_{\mathcal{O}}$, whose:*

- *Objects are $\mathcal{O}$-algebras;*
- *The enriched hom functor, denoted $\widetilde{Alg}_{\mathcal{O}}(-,-)$ is given by : $\forall X, Y \in Alg_{\mathcal{O}}$,*

$$\widetilde{Alg}_{\mathcal{O}}(X,Y) := N\Bbbk Hom_{Alg_{\mathcal{O}}}(\Omega_{\mathcal{O}}(B(\mathcal{O},X), red_0(Y \otimes Apl_\bullet));$$

- *The composition and the unit are deduced from γ and $\eta_\bullet$.*

Proof. The result follows from the proof of Proposition 3.1. More precisely, since the functor $N : sAb \longrightarrow Ch_+$ is monoidal (see [SS03]) and that the free abelian functor $\Bbbk(-) : sSet \longrightarrow sAb$ is also monoidal, we deduce that $N(\Bbbk\gamma) : \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,Y) \otimes \widetilde{\mathrm{Alg}}_{\mathcal{O}}(Y,Z) \longrightarrow \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,Z)$ along with the unit $N(\Bbbk\eta_\bullet) : \Bbbk \longrightarrow \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,X)$ satisfy the enrichment properties stated in [Bor94, § 6.2.1]. □

3.2 The Ch-enriched category $\widetilde{Ch}_+$

One can see in particular a non negatively graded chain complex as an algebra over the trivial operad $\mathcal{O} = \mathbb{I} = (0, \Bbbk, 0, ..., 0, ...)$. In that case the functor, $\forall V, W \in Ch_+$,

$$Map(V,W) := Hom_{Ch_+}(V, red_0(W \otimes Apl_\bullet))$$

can be taken (using Proposition 3.1) to define a $sSet$-category (or a simplicial enriched category) associated to Ch_+.

Note that $W \otimes Apl_\bullet$ (resp. $W \otimes N\Bbbk\triangle^\bullet$) is a simplicial (resp. cosimplicial) frame associated to W. We refer to [Fre17, §3.2] for discussion on the framing construction. Technically, that is a notion often used when the category of interest (Ch in our case) is not tensored over the base monoidal category ($sSet$ in our case).

We then use [Fre17, Thm 3.2.15] to claim the existence of a zig-zag of quasi-isomorphisms

$$Hom_{Ch}(I(V) \otimes N\Bbbk\triangle^\bullet, I(W)) \xrightarrow{\simeq} \bullet \xleftarrow{\simeq} Hom_{Ch}(I(V), I(W) \otimes Apl_\bullet) \quad (3.8)$$

Using the quasi isomorphisms (3.8), we make the following computation

$$\begin{aligned} Hom_{Ch_+}(V, red_0(I(W) \otimes Apl_\bullet)) &\cong Hom_{Ch}(I(V), I(W) \otimes Apl_\bullet) \\ &\simeq Hom_{Ch}(I(V) \otimes N\Bbbk\triangle^\bullet, I(W)) \\ &\cong Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, W) \end{aligned}$$

We will then define a different simplicial enriched category associated to Ch_+ which is more handy but equivalent up to homotopy to the one we get from Proposition 3.1.

Proposition 3.4 ($sSet$-enriched and Ch-enriched categories Ch'_+ and $\widetilde{Ch}_+$)**.**

1. *There is a sSet-category (or simplicial enriched category), denoted Ch'_+, whose:*

    - *Objects are objects in Ch_+;*
    - *The enriched hom functor is given by: $\forall V, W \in Ch_+$,*

$$Map(V,W) := Hom_{Ch_+}(N(\Bbbk\triangle^\bullet) \otimes V, W);$$

2. There is a Ch-category (or Chain complex enriched category), denoted $\widetilde{Ch}_+$, whose:

- *Objects are objects in Ch_+;*
- *The enriched hom functor, denoted $\widetilde{Ch}_+(-,-)$ is given by : $\forall V, W \in Ch_+$,*

$$\widetilde{Ch}_+(V,W) := N\Bbbk Hom_{Ch_+}(N(\Bbbk\triangle^\bullet) \otimes V, W);$$

Before we prove Proposition 3.4 , we first define the composition map corresponding to $Map(-,-)$. We define the map

$$\varphi : Map(V_1, V_2) \otimes Map(V_2, V_3) \longrightarrow Map(V_1, V_3) \tag{3.9}$$

on generators $f_i \in Hom_{Ch_+}(V_i \otimes N(\Bbbk\triangle^\bullet), V_{i+1})$, $i = 1, 2$ by: $\varphi(f_1 \otimes f_2)$ is the composite:

$$\begin{tikzcd}
V_1 \otimes N(\Bbbk\triangle^\bullet) \arrow[r, "1\otimes\triangle"] & V_1 \otimes (N(\Bbbk\triangle^\bullet) \otimes N(\Bbbk\triangle^\bullet)) \cong (V_1 \otimes N(\Bbbk\triangle^\bullet)) \otimes N(\Bbbk\triangle^\bullet) \arrow[d, "f_1\otimes 1"] & \\
& V_2 \otimes N(\Bbbk\triangle^\bullet) \arrow[r, "f_2"] & V_3
\end{tikzcd}$$

Lemma 3.5. *The map φ of Equation* (3.9) *is a morphism of simplicial sets.*

Proof. To prove that this is well defined, we make the following computation: Let d_i be the i^{th} face map of $Hom_{Ch_+}(V_1 \otimes N(\Bbbk\triangle^\bullet), V_2)$ given by the i^{th} coface map $d^i : N(\Bbbk\triangle^k) \longrightarrow N(\Bbbk\triangle^{k+1})$. Then we have the following commutative diagram (due to the fact that $N(\Bbbk\triangle^\bullet)$ is a co-simplicial coalgebra)

$$\begin{tikzcd}
V_1 \otimes N(\Bbbk\triangle^k) \arrow[r, "1\otimes d^i"] \arrow[d, "1\otimes\triangle"'] & V_1 \otimes N(\Bbbk\triangle^{k+1}) \arrow[d, "1\otimes\triangle"] & & \\
V_1 \otimes N(\Bbbk\triangle^k) \otimes N(\Bbbk\triangle^k) \arrow[r, "1\otimes d^i\otimes d^i"'] & V_1 \otimes N(\Bbbk\triangle^{k+1}) \otimes N(\Bbbk\triangle^{k+1}) \arrow[r, "f_1\otimes 1"'] & V_2 \otimes N(\Bbbk\triangle^{k+1}) \arrow[r, "f_2"'] & V_3
\end{tikzcd}$$

We can then translate this into the following computation

$$\begin{aligned}
d_i(\varphi(f_1 \otimes f_2)) =& d_i(f_2(f_1 \otimes 1)(1 \otimes \triangle)) \\
&= f_2(f_1 \otimes 1)(1 \otimes \triangle)(1 \otimes d^i) \\
&= f_2(f_1 \otimes 1)(1 \otimes d^i \otimes d^i)(1 \otimes \triangle) \\
&= f_2((f_1(1 \otimes d^i)) \otimes d^i)(1 \otimes \triangle) \\
&= f_2(d_i(f_1) \otimes d^i)(1 \otimes \triangle) \\
&= d_i(f_2)(d_i(f_1) \otimes 1)(1 \otimes \triangle) \\
&= \varphi(d_i(f_1) \otimes d_i(f_2))
\end{aligned}$$

A similar argument can be made for degeneracy maps. □

We define the unit map associated to a chain complex V as follows: The map

$$\eta_\bullet : S^0 \longrightarrow Hom_{Ch_+}(V \otimes N(\Bbbk\triangle^\bullet), V)$$

is the morphism of simplicial sets defined aritywise by: $\forall n \geq 0$,

$$\eta_k(1) = 1 \otimes \eta^k : V \otimes N(\Bbbk\triangle^k) \longrightarrow V \otimes \Bbbk \cong V$$

where $\eta^\bullet : N(\Bbbk\triangle^\bullet) \longrightarrow \Bbbk$ is the co-unit of the coalgebra $N(\Bbbk\triangle^\bullet)$.

Proof of Proposition 3.4. 1. We need now to prove that the couple (φ, η) satisfies the associativity and the unit axioms (see [Bor94, § 6.2.1]) .

(a) For the associativity of φ, we consider the algebras $V_1, V_2, V_3, V_4 \in Ch_+$, and morphisms $f_i \in Hom_{Ch_+}(V_i \otimes N(\Bbbk\triangle^\bullet), V_{i+1}), i = 1, 2, 3$. We construct the following commutative diagram

$$\begin{tikzcd}
V_1 \otimes N(\Bbbk\triangle^\bullet) \arrow[r, "1\otimes\triangle"] \arrow[d, "1\otimes\triangle"] & V_1 \otimes N(\Bbbk\triangle^\bullet) \otimes N(\Bbbk\triangle^\bullet) \arrow[d, "1\otimes\triangle\otimes 1"] & \\
V_1 \otimes N(\Bbbk\triangle^\bullet) \otimes N(\Bbbk\triangle^\bullet) \arrow[r, "1\otimes 1\otimes\triangle"'] \arrow[d, "f_1\otimes C^\bullet"] & V_1 \otimes N(\Bbbk\triangle^\bullet) \otimes N(\Bbbk\triangle^\bullet) \otimes N(\Bbbk\triangle^\bullet) \arrow[d, "f_1\otimes 1\otimes 1"] & \\
V_2 \otimes N(\Bbbk\triangle^\bullet) \arrow[r, "1\otimes\triangle"'] & V_2 \otimes N(\Bbbk\triangle^\bullet) \otimes N(\Bbbk\triangle^\bullet) \arrow[d, "f_2\otimes 1"'] & \\
 & V_3 \otimes N(\Bbbk\triangle^\bullet) \arrow[r, "f_3"'] & V_4
\end{tikzcd}$$

where the upper diagram is commutative since the coproduct $\triangle$ is coassociatif. We then deduce that

$$f_3(f_2 \otimes 1)(1 \otimes \triangle)(f_1 \otimes 1)(1 \otimes \triangle) = f_3((f_2(1 \otimes \triangle)) \otimes 1)(1 \otimes \triangle)$$

and this proves that the composition map φ is associative.

(b) For the unit axiom, let $f \in Hom_{Ch_+}(V \otimes N(\Bbbk\triangle^\bullet), W)$. The left and right co-unit diagrams of the coalgebras $N(\Bbbk\triangle^k), k \geq 0$, lead to that

$$\varphi(\eta_\bullet \otimes f) = f \text{ and } \varphi(f \otimes \eta_\bullet) = f.$$

2. This follows from 1. using the fact that the functors $\Bbbk- : sSet \longrightarrow sAb$ and $N : sAb \longrightarrow Ch_+$ are monoidal.

$\square$

3.3 The Ch-enriched category $\widetilde{Ch}$

The category Ch is monoidal with the graded tensor product $- \otimes -$, and closed with the internal hom functor $\underline{hom}(V, W)$. It is then straightforward that the category Ch, which we often denotes $\widetilde{Ch}$, is enriched over itself with the enrichment functor given by $\underline{hom}(-, -)$ which we often denote $\widetilde{Ch}(-, -)$.

3.4 Discretization of Ch-enriched categories

To any Ch-enriched category $\widetilde{\mathcal{C}}$, one can associate an underlying category $\widetilde{\mathcal{C}}_0$ which has the same objects of $\widetilde{\mathcal{C}}$. The set of morphisms in $\widetilde{\mathcal{C}}_0$ between $X, Y \in \widetilde{\mathcal{C}}_0$ is defined by

$$Hom_{\widetilde{\mathcal{C}}_0}(X,Y) := Hom_{Ch}(\Bbbk, \widetilde{\mathcal{C}}(X,Y))$$

which corresponds exactly to the 0-cycles in $\widetilde{\mathcal{C}}(X,Y)$.

3.4.1 Discretization of $\widetilde{\mathbf{Alg}}_{\mathcal{O}}$

The set of morphisms between $X, Y \in (\widetilde{\mathrm{Alg}}_{\mathcal{O}})_0$ is given by

$$Hom_{(\widetilde{\mathrm{Alg}}_{\mathcal{O}})_0}(X,Y) := Hom_{Ch}(\Bbbk, N\Bbbk Hom_{\mathrm{Alg}_{\mathcal{O}}}(X^c, red_0(Y \otimes Apl_\bullet)))$$

Since $\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,Y)$ is concentrated in non-negative degrees, it is clear that this set is exactly the set of 0 degree elements of $\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,Y)$ and we have

$$Hom_{(\widetilde{\mathrm{Alg}}_{\mathcal{O}})_0}(X,Y) = \Bbbk Hom_{\mathrm{Alg}_{\mathcal{O}}}(X^c, Y) \tag{3.10}$$

Therefore the underlying category associated to the Ch-category $\widetilde{\mathrm{Alg}}_{\mathcal{O}}$ is not the usual category $\mathrm{Alg}_{\mathcal{O}}$ but some linearization of it (after taking the cofibrant replacement of the source).

Remark 3.6. *Given $X, Y \in Alg_{\mathcal{O}}$, there is an injection (morphism of sets)*

$$Hom_{Alg_{\mathcal{O}}}(X^c, Y) \longrightarrow Hom_{Ch}(\Bbbk, \widetilde{Alg}_{\mathcal{O}}(X,Y))$$

which sends any morphism of $Hom_{Alg_{\mathcal{O}}}(X^c, Y)$ to its corresponding generator in

$$\Bbbk Hom_{Alg_{\mathcal{O}}}(X^c, red_0(Y \otimes Apl_0)) = \Bbbk Hom_{Alg_{\mathcal{O}}}(X^c, Y).$$

On the other hand, using the cobar-bar adjunction co-unit

$$X^c := \Omega_{\mathcal{O}}(B(\mathcal{O}, X) \longrightarrow X,$$

we get the morphism of sets $Hom_{Alg_{\mathcal{O}}}(X,Y) \longrightarrow Hom_{Alg_{\mathcal{O}}}(X^c, Y)$. We therefore deduce the morphism of sets

$$\vartheta : Hom_{Alg_{\mathcal{O}}}(X,Y) \longrightarrow Hom_{(\widetilde{Alg}_{\mathcal{O}})_0}(X,Y) \tag{3.11}$$

Equation (3.11) defines in other words a functor:

$$\vartheta : \mathrm{Alg}_{\mathcal{O}} \longrightarrow (\widetilde{\mathrm{Alg}}_{\mathcal{O}})_0$$

where $(\widetilde{\mathrm{Alg}}_{\mathcal{O}})_0$ is the underlying category of $\widetilde{\mathrm{Alg}}_{\mathcal{O}}$ and ϑ is the identity on objects.

3.4.2 Discretization of $\widetilde{Ch}_+$

As in § 3.4.1, the set of morphisms between $V, W \in (\widetilde{Ch}_+)_0$ is given by

$$Hom_{(\widetilde{Ch}_+)_0}(V, W) := \Bbbk Hom_{Ch_+}(V, W)$$

Therefore the underlying category associated to the Ch-category $\widetilde{Ch}_+$ is not the usual category Ch_+ but some linearization of it. However, as in § 3.4.1 there is natural injection (morphism of sets): given $V, W \in Ch_+$,

$$\vartheta : Hom_{Ch_+}(V, W) \longrightarrow Hom_{(\widetilde{Ch}_+)_0}(V, W) \tag{3.12}$$

which sends any morphism of $Hom_{Ch_+}(V, W)$ to its corresponding generator in $\Bbbk Hom_{Ch_+}(N(\Bbbk\triangle^0) \otimes V, W) = \Bbbk Hom_{Ch_+}(V, W)$.

Equation (3.12) defines in other words a functor:

$$\vartheta : Ch_+ \longrightarrow (\widetilde{Ch}_+)_0$$

where $(\widetilde{Ch}_+)_0$ is the underlying category of $\widetilde{Ch}_+$ and ϑ is the identity on objects.

The functor ϑ has a left inverse

$$\zeta : (\widetilde{Ch}_+)_0 \longrightarrow Ch_+ \tag{3.13}$$

which is the identity on objects and whose, on morphisms

$$\zeta : Hom_{(\widetilde{Ch}_+)_0}(V, W) \longrightarrow Hom_{Ch_+}(V, W) \tag{3.14}$$

is the natural surjection of vector spaces.

3.4.3 Discretization of $\widetilde{Ch}$

The set of morphisms between $V, W \in (\widetilde{Ch})_0$ is given by

$$Hom_{(\widetilde{Ch})_0}(V, W) := Hom_{Ch}(V, W)$$

In this case, the underlying category associated to $\widetilde{Ch}$ is the category Ch. In this case, the functors

$$\vartheta : Ch \longrightarrow (\widetilde{Ch})_0 \tag{3.15}$$

$$\zeta : Hom_{(\widetilde{Ch})_0}(V, W) \longrightarrow Hom_{Ch}(V, W) \tag{3.16}$$

are in each case the identity functor.

3.5 Simplicial functors

Definition 3.7 (simplicial functors)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be either $Alg_{\mathcal{O}}$, Ch_+ or Ch. A simplicial functor $\widetilde{F} : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{D}}$ is a Ch-functor between the Ch-enriched categories $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$. Namely, this consists of giving:*

1. *for every* $X \in \mathcal{C}$, *an object* $\widetilde{F}(X) \in \widetilde{\mathcal{D}}$;
2. *for every pair* $X, Y \in \mathcal{C}$, *a morphism of chain complexes*

$$\widetilde{\mathcal{C}}(X,Y) \xrightarrow{\widetilde{F}_{X,Y}} \widetilde{\mathcal{D}}(\widetilde{F}(X), \widetilde{F}(Y))$$

satisfying the composition and the unit axioms (see [Bor94, 6.2.3] or [Kel05, §1.2 (1.5), (1.6)]).

Instead of calling such functors Ch-functors as in our reference papers, we decided to change the name to "simplicial functors" in order to encode the fact that the enrichment on $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ (when these are either $\mathrm{Alg}_{\mathcal{O}}$ or Ch_+) are induced by a simplicial structure (see Proposition 3.1 and Proposition 3.4).

We will see in what follows that simplicial functors induce functors in the usual sense.

Definition 3.8 (Functors associated to simplicial functors)**.** *Let* $\mathcal{C} = Alg_{\mathcal{O}}$, Ch_+ *or* Ch. *and* $\mathcal{D} = Ch_+$ *or* Ch. *A simplicial functor* $\widetilde{F} : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{D}}$ *induces a functor (in the classical sense)* $F : \mathcal{C} \longrightarrow \mathcal{D}$. *This is a functor whose*

- *on objects* $X \in \mathcal{C}$, $F(X) := \widetilde{F}(X)$;
- *on morphisms* $X \xrightarrow{f} Y$, $F_{X,Y}(f) := \zeta Hom_{Ch}(\mathbb{k}, \widetilde{F}_{X,Y})\vartheta(f)$

where,

- $Hom_{\mathcal{C}}(X,Y) \xrightarrow{\vartheta} \widetilde{\mathcal{C}}(X,Y)$ *is given in Equation* (3.12) *when* $\mathcal{C} = Ch_+$, *in Equation* (3.11) *when* $\mathcal{C} = Alg_{\mathcal{O}}$, *and in Equation* (3.15) *when* $\mathcal{C} = Ch$;
- $\zeta : Hom_{Ch}(\mathbb{k}, \widetilde{\mathcal{D}}(\widetilde{F}(X), \widetilde{F}(Y))) \longrightarrow Hom_{\mathcal{D}}(\widetilde{F}(X), \widetilde{F}(Y))$ *is the natural surjection defined in Equation* (3.14) *when* $\mathcal{D} = Ch_+$ *and in Equation* (3.16) *when* $\mathcal{D} = Ch$.

of vector spaces that make sense in the two cases.

Remark 3.9. *If* $\widetilde{F} : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{D}}$ *is a simplicial functor with* $\mathcal{D} = Ch_+$ *or* Ch *then the associated functor* F *given in Definition 3.8 is defined by the commutative diagram:*

$$\begin{tikzcd}
\widetilde{\mathcal{C}}_0 \arrow[r, "\widetilde{F}_0"] & \widetilde{\mathcal{D}}_0 \arrow[d, "\zeta"] \\
\mathcal{C} \arrow[u, "\vartheta"] \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

Conversely, we can say that a classical functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ *is a " simplicial functor" when* $\widetilde{F}$ *is well identified in such a way that the above diagram commutes. In that sense, we will see a simplicial functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *as a usual functor with an additional structure (given by this enrichment).*

Notation 2. *Based on Remark 3.9, we will abusively use the same notation for simplicial functors* $\widetilde{F}$ *and their associated functor* F *when there is no confusion in the context.*

3.6 Examples of simplicial functors

Example 3.10 (Representable functor)**.** *Let $\mathcal{C}$ be either $Alg_{\mathcal{O}}$, Ch_+ or Ch and $Z \in \mathcal{C}$. Then the representable functor $\widetilde{\mathcal{C}}(Z,-) : \widetilde{\mathcal{C}} \longrightarrow Ch$ is a simplicial functor. In fact, the adjoint of the composition morphism*

$$\widetilde{\mathcal{C}}(X,Y) \otimes \widetilde{\mathcal{C}}(Z,X) \longrightarrow \widetilde{\mathcal{C}}(Z,Y)$$

gives a morphism $\widetilde{\mathcal{C}}(X,Y) \xrightarrow{\widetilde{\mathcal{C}}(Z,-)_{X,Y}} \underline{hom}(\widetilde{\mathcal{C}}(Z,X), \widetilde{\mathcal{C}}(Z,Y))$.

Since the composition map is associative and satisfies the unit axiom, it follows that the above maps satisfy the composition and the unit axiom.

Example 3.11 (Inclusion functor)**.** *The inclusion functor $I : Ch_+ \longrightarrow Ch$ is simplicial. More precisely, I induces a simplicial functor $\widetilde{I} : \widetilde{Ch_+} \longrightarrow Ch$ given by:*

- $\widetilde{I}(V) := I(V) = V$;
- *The simplicial structure on $\widetilde{I}$ is given by the morphism of chain complexes*

$$\begin{aligned} \widetilde{I}_{X,Y} : N\Bbbk Hom_{Ch_+}(N(\Bbbk\triangle^\bullet) \otimes V, W) &\longrightarrow NHom_{Ch_+}(N(\Bbbk\triangle^\bullet) \otimes V, W) \\ &\xrightarrow{\cong} NHom_{Ch_+}(N(\Bbbk\triangle^\bullet), red_0\underline{hom}(V,W)) \\ &\longrightarrow red_0\underline{hom}(V,W) \\ &\longrightarrow \underline{hom}(V,W) \end{aligned}$$

where the first map is the natural projection $\bigoplus_i f_i \longmapsto \sum_i f_i$ and the second morphism is given by adjunction between the functors $- \otimes -$ and $\underline{hom}(-,-)$ and the last one is given by the Dold Kan correspondence.

Example 3.12 (Tensor product of simplicial functors)**.** *Let $\mathcal{C} = Ch_+$ or $Alg_{\mathcal{O}}$. If $\widetilde{F}, \widetilde{G} : \widetilde{\mathcal{C}} \longrightarrow Ch$ are two simplicial functors, then the tensor product $\widetilde{F} \otimes \widetilde{G} : \widetilde{\mathcal{C}} \longrightarrow Ch$, which is defined level-wise by $\widetilde{F} \otimes \widetilde{G}(X) := \widetilde{F}(X) \otimes \widetilde{G}(X)$, is a simplicial functor. In fact there is a natural map*

$$\widetilde{\mathcal{C}}(X,Y) \xrightarrow{\triangle} \widetilde{\mathcal{C}}(X,Y) \otimes \widetilde{\mathcal{C}}(X,Y)$$

induced by the diagonal map of simplicial sets

$$Map(X,Y) \xrightarrow{\triangle} Map(X,Y) \times Map(X,Y)$$

The adjoint of the map

$$\begin{aligned} \widetilde{F}(X) \otimes \widetilde{G}(X) \otimes \widetilde{\mathcal{C}}(X,Y) &\xrightarrow{\triangle} \widetilde{F}(X) \otimes \widetilde{G}(X) \otimes \widetilde{\mathcal{C}}(X,Y) \otimes \widetilde{\mathcal{C}}(X,Y) \\ &\xrightarrow{\widetilde{F}_{X,Y} \otimes \widetilde{G}_{X,Y}} \widetilde{F}(Y) \otimes \widetilde{G}(Y) \end{aligned}$$

gives the morphism of chain complexes

$$(\widetilde{F} \otimes \widetilde{G})_{X,Y} : \widetilde{\mathcal{C}}(X,Y) \longrightarrow \underline{hom}(F(X) \otimes G(X), F(Y) \otimes G(Y))$$

which we can check satisfies the composition and the unit axiom.

Example 3.13. *The combination of Example 3.11 and Example 3.12 produces simplicial functors,* $\forall n \geq 0$,

$$\begin{aligned} I^{\otimes n} : Ch_+ &\longrightarrow Ch \\ V &\longmapsto V^{\otimes n} \end{aligned}$$

Example 3.14. *If M is a right $\mathcal{O}$-module, then we define the functor*

$$\begin{aligned} B(M, \mathcal{O}, -) : Alg_{\mathcal{O}} &\longrightarrow Ch \\ X &\longmapsto B(M, \mathcal{O}, X) := \bigoplus_{\underline{n}} (B(M, \mathcal{O}, \widehat{X})(\underline{n}), \partial_0 + \partial), \end{aligned}$$

where $\widehat{X} = (X, 0, ..., 0, ...)$ is the left $\mathcal{O}$-module associated to X.

The functor $B(M, \mathcal{O}, -)$ is a simplicial functor. In fact, we consider the following composite of morphism of simplicial sets:

$$Hom_{coAlg_{B(\mathcal{O})}}(B(\mathcal{O}, X), B(\mathcal{O}, red_0(Y \otimes Apl_\bullet))) \longrightarrow \tag{3.17}$$

$$Hom_{Ch^+}(\underset{n}{\oplus} M(n) \underset{\Sigma_n}{\otimes} B(\mathcal{O}, X)^{\otimes n}, \underset{n}{\oplus} M(n) \underset{\Sigma_n}{\otimes} B(\mathcal{O}, red_0(Y \otimes Apl_\bullet))^{\otimes n}) \longrightarrow \tag{3.18}$$

$$Hom_{Ch^+}(\underset{n}{\oplus} M(n) \underset{\Sigma_n}{\otimes} B(\mathcal{O}, X)^{\otimes n}, \underset{n}{\oplus} M(n) \underset{\Sigma_n}{\otimes} red_0 B(\mathcal{O}, Y)^{\otimes n} \otimes Apl_\bullet^{\otimes n}) \longrightarrow \tag{3.19}$$

$$Hom_{Ch^+}(\underset{n}{\oplus} M(n) \underset{\Sigma_n}{\otimes} B(\mathcal{O}, X)^{\otimes n}, \underset{n}{\oplus} M(n) \underset{\Sigma_n}{\otimes} red_0 B(\mathcal{O}, Y)^{\otimes n} \otimes Apl_\bullet) = \tag{3.20}$$

$$Hom_{Ch^+}(B(M, \mathcal{O}, \widehat{X}), red_0(B(M, \mathcal{O}, \widehat{Y}) \otimes Apl_\bullet)) \tag{3.21}$$

where:

- *The map* (3.17) *is induced by the morphisms* $f \longmapsto f^{\otimes n}$;

- *The map* (3.18) *is induced by the previously defined map*

$$\sigma : B(\mathcal{O}, red_0(Y \otimes Apl_\bullet)) \longrightarrow red_0(B(\mathcal{O}, Y) \otimes Apl_\bullet);$$

- *The map* (3.19) *is induced by the product* $m : Apl_\bullet \otimes Apl_\bullet \longrightarrow Apl_\bullet$;

By applying $N\Bbbk-$ to this composition, we get the morphism of chain complexes

$$\widetilde{B}(M,\mathcal{O},-)_{X,Y} : Alg_{\mathcal{O}}(X,Y) \longrightarrow N\Bbbk Hom_{Ch^+}(B(M,\mathcal{O},\widehat{X}), red_0(B(M,\mathcal{O},\widehat{Y})\otimes Apl_\bullet)) \tag{3.22}$$

$$\longrightarrow NHom_{Ch^+}(B(M,\mathcal{O},\widehat{X}), red_0(B(M,\mathcal{O},\widehat{Y})\otimes Apl_\bullet)) \tag{3.23}$$

$$\xrightarrow{\cong} NHom_{Ch^+}(N\Bbbk\triangle^\bullet, \underline{hom}(B(M,\mathcal{O},\widehat{X}), B(M,\mathcal{O},\widehat{Y}))) \tag{3.24}$$

$$\longrightarrow \underline{hom}(B(M,\mathcal{O},\widehat{X}), B(M,\mathcal{O},\widehat{Y})) \tag{3.25}$$

where :

- *Equation* (3.23) *is induced by the natural projection(morphism of chain complexes)*

$$\begin{aligned} \Bbbk Hom_{Ch_+}(V,W) &\longrightarrow Hom_{Ch_+}(V,W) \\ \bigoplus_i f_i &\longmapsto \sum_i f_i \end{aligned}$$

- *Equation* (3.25) *is given by the Dold-Kan correspondence.*

One can check that $\widetilde{B}(M,\mathcal{O},-)_{X,Y}$ satisfies the composition and the unit axiom, and that $\widetilde{B}(M,\mathcal{O},-)_{X,Y}$ and $B(M,\mathcal{O},-)_{X,Y}$ fit into the diagram of Remark 3.9.

Example 3.15. *We consider the Ch-category $\mathcal{C} = \widetilde{Ch}_+ \times \widetilde{Ch}_+$ associated to the cartesian product $Ch_+ \times Ch_+$, with the enrichment hom functor given by: $X = (V_1, V_2), Y = (W_1, W_2) \in Ch_+ \times Ch_+$,*

$$\widetilde{Ch}_+ \times \widetilde{Ch}_+((V_1,V_2),(W_1,W_2)) := \widetilde{Ch}_+(V_1,W_1)\otimes\widetilde{Ch}_+(V_2,W_2)$$

The composition (resp. the unit) is given by the product of two copies of the composition (resp. unit) on $\widetilde{Ch}_+$.

The functors

$$\begin{aligned} \triangle : Ch_+ &\longrightarrow Ch_+ \times Ch_+ \\ V &\longmapsto (V,V) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \Pi : Ch_+ \times Ch_+ &\longrightarrow Ch_+ \\ (V,W) &\longmapsto V\oplus W \end{aligned}$$

are simplicial functors. More precisely, there are morphisms of chain complexes

$$\widetilde{\Pi}_{X,Y} : \widetilde{Ch}_+(V_1,W_1)\otimes\widetilde{Ch}_+(V_2,W_2) \longrightarrow \widetilde{Ch}_+(V_1\oplus V_2, W_1\oplus W_2)$$

induced by the morphisms of simplicial sets

$$\begin{gathered} Hom_{Ch_+}(V_1\otimes N\Bbbk\triangle^\bullet, W_1)\times Hom_{Ch_+}(V_2\otimes N\Bbbk\triangle^\bullet, W_2) \longrightarrow \\ Hom_{Ch_+}((V_1\oplus V_2)\otimes N\Bbbk\triangle^\bullet, W_1\oplus W_2) \end{gathered}$$

which is itself induced by the universal property of the pushout. One can check that the maps $\widetilde{\Pi}_{X,Y}$ satisfy the composition and the unit axioms, and that $\widetilde{\Pi}_{X,Y}$ and $\Pi_{X,Y}$ fit into the diagram of Remark 3.9.

On the other hand, for the functor $\triangle$, we have the morphisms of chain complexes:

$$\widetilde{\triangle}_{V,W} : \widetilde{Ch}_+(V,W) \longrightarrow \widetilde{Ch}_+(V,W) \otimes \widetilde{Ch}_+(V,W)$$

which is induced by the diagonal map of simplicial sets

$$\begin{aligned} Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, W) &\longrightarrow Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, W) \times Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, W) \\ f &\mapsto (f,f) \end{aligned}$$

As in the case of Π, *we can check that the maps* $\widetilde{\triangle}_{V,W}$ *defines an extension of* $\triangle$ *in the sense of Remark 3.9.*

CHAPTER 4

The model category of simplicial functors

Let $\mathcal{O}$ be a reduced operad in Ch_+ and let $\mathcal{C} = \mathrm{Alg}_{\mathcal{O}}$, Ch_+ or Ch. We are interested in studying the collection of simplicial functors $\widetilde{F} : \widetilde{\mathcal{C}} \longrightarrow Ch$ with appropriate morphisms, viewed as a category. The key fact of this chapter is that we can endow this category with the structure of cofibrantly generated model category for which the cofibrant generators are the representable functor. This fact will be of a fundamental importance in Chapter 5 because we will infer the Goodwillie calculus of simplicial functors from the special case of representable functors.

However we cannot consider the whole category of such functors without incurring set theoretic problems. We will then restrict functors to the subcategory of $\mathcal{C}$ which consists of finite objects. By finite object, we mean objects of the subcategory $\mathcal{C}^{fin}$ that we defined bellow:

- Ch^{fin} (resp. Ch_+^{fin}) is the subcategory of Ch (resp. Ch_+) which consists of chain complexes V_* such that $\forall n$, dim $V_n < +\infty$.

- $\mathrm{Alg}_{\mathcal{O}}^{fin}$ is the subcategory of $\mathrm{Alg}_{\mathcal{O}}$ which consists of finitely generated $\mathcal{O}$-algebras.

In this chapter we define the Ch-category $[\mathcal{C}^{fin}, Ch]$ whose objects are simplicial functors. We show that the underlying category $[\mathcal{C}^{fin}, Ch]_0$ has a cofibrantly generated model structure. Since any cofibrant replacement functor in this category has a cellular decomposition due to the small object argument, it follows a cellular resolution for any simplicial functor. As a straight consequence of this construction, we show that any simplicial functor in $[\mathcal{C}^{fin}, Ch]_0$ is a homotopy functor. Note that it seems that the converse is also true: any homotopy functor should be equivalent to a simplicial functor by some results due to S. Schwede, but we will not prove this.

The chapter has the following guidelines:

- In §4.1, we describe the functor category $[\mathcal{C}^{fin}, Ch]$ which is the enriched category of simplicial functors over chain complexes;

- In §4.2, we define a cofibrantly generated model structure on the category $[\mathcal{C}^{fin}, Ch]_0$ underlying $[\mathcal{C}^{fin}, Ch]$.

- Section §4.3 is devoted to the cellular decomposition of simplicial functors. We give properties of presented cell functors. These properties will be very useful in Chapter 5. For instance they will be used to describe an extra structure on the Goodwillie derivatives of simplicial functors $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$.

- In §4.4, we show that simplicial functors to preserve weak equivalences.

- Finally in §4.5, we compute the derived enriched natural transformation between two tensor powers of the inclusion functor $I : Ch_+^{fin} \longrightarrow Ch$. Namely we show that

  $$Nat(Q(I^{\otimes n}), I^{\otimes m}) \simeq \begin{cases} \Bbbk[\Sigma_n] & \text{if } n = m; \\ 0 & \text{if } n \neq m. \end{cases}$$

  where Q is the cofibrant replacement functor.

  This formula will be useful in Chapter 5 as to prove for instance that the derivative $\partial_*(I^{\otimes n})$ is of finite type.

4.1 Functor category $[\mathcal{C}^{fin}, Ch]$

Let $\mathcal{C} = \mathrm{Alg}_{\mathcal{O}}$, Ch_+ or Ch. In this section, we define explicitly the functor category $[\mathcal{C}^{fin}, Ch]$ (objects and morphisms). This is a particular case of a general notion (in enriched categories) which is discussed in [Kel05, § 2.3].

In our case, the objects of this category are simplicial functors and morphisms are enriched natural transformations. We are more explicit about what we mean in the next definitions.

Definition 4.1 (Construction of $Nat(-,-)$)**.** *Let $\widetilde{F}, \widetilde{G} : \widetilde{\mathcal{C}}^{fin} \longrightarrow Ch$ be two simplicial functors. We define the chain complex of natural transformations from $\widetilde{F}$ to $\widetilde{G}$ by the formula*

$$Nat(\widetilde{F}, \widetilde{G}) := lim\Big(\prod_{X \in \mathcal{C}^{fin}} \underline{hom}(\widetilde{F}(X), \widetilde{G}(X)) \rightrightarrows \prod_{X,Y \in \mathcal{C}^{fin}} \underline{hom}(\widetilde{\mathcal{C}}(X,Y) \otimes \widetilde{F}(X), \widetilde{G}(Y))\Big)$$

One of the maps in this equalizer

$$\underline{hom}(\widetilde{F}(Y), \widetilde{G}(Y)) \otimes \widetilde{\mathcal{C}}(X,Y) \otimes \widetilde{F}(X) \longrightarrow \widetilde{G}(Y)$$

is obtained using the enriched structure morphism of F

$$\widetilde{\mathcal{C}}(X,Y) \xrightarrow{\widetilde{F}_{X,Y}} \underline{hom}(\widetilde{F}(X), \widetilde{F}(Y))$$

and the other map

$$\underline{hom}(\widetilde{F}(X), \widetilde{G}(X)) \otimes \widetilde{\mathcal{C}}(X, Y) \otimes \widetilde{F}(X) \longrightarrow \widetilde{G}(Y)$$

is obtained using the enriched structure morphism of $\widetilde{G}$

$$\widetilde{\mathcal{C}}(X, Y) \overset{\widetilde{G}_{X,Y}}{\longrightarrow} \underline{hom}(\widetilde{G}(X), \widetilde{G}(Y))$$

Definition 4.2 (Functor category)**.** *We denote by* $[\mathcal{C}^{fin}, Ch]$ *the Ch-category whose*

- *objects are simplicial functors* $\widetilde{F} : \widetilde{\mathcal{C}}^{fin} \longrightarrow Ch$;
- *hom-object between two simplicial functors* $\widetilde{F}, \widetilde{G} : \widetilde{\mathcal{C}}^{fin} \longrightarrow Ch$ *is the chain complex* $Nat(\widetilde{F}, \widetilde{G})$.

In general, any enriched category has an underlying category (see [Kel05, § 1.3]). In our case here for the functor category $[\mathcal{C}^{fin}, Ch]$, the underlying category, denoted $[\mathcal{C}^{fin}, Ch]_0$, has the same objects with $[\mathcal{C}^{fin}, Ch]$ but morphisms are simplicial natural transformations, these are elements of the set obtained by applying the functor $Hom_{Ch}(\Bbbk, -)$ to $Nat(\widetilde{F}, \widetilde{G})$. Equivalently, we can define simplicial natural transformations as it appears in [Bor94, Prop 6.2.8] or [Kel05, §1.2 (1.7)]:

Definition 4.3 (Simplicial natural transformations)**.** *Let* $\mathcal{C} = Alg_{\mathcal{O}}$, Ch_+ *or* Ch. *A simplicial natural transformation* $\alpha : \widetilde{F} \longrightarrow \widetilde{G}$ *between two simplicial functors* $\widetilde{F}, \widetilde{G} : \widetilde{\mathcal{C}} \longrightarrow Ch$ *is a family of morphisms* $\alpha_X : \widetilde{F}(X) \longrightarrow \widetilde{G}(X)$, $\forall X \in \mathcal{C}$, *such that the following diagram commutes:*

$$\begin{tikzcd}
\widetilde{\mathcal{C}}(X,Y) \arrow[r, "\widetilde{F}_{X,Y}"] \arrow[d, "\widetilde{G}_{X,Y}"'] & \underline{hom}(\widetilde{F}(X), \widetilde{F}(Y)) \arrow[d, "\underline{hom}(1,\alpha_Y)"] \\
\underline{hom}(\widetilde{G}(X), \widetilde{G}(Y)) \arrow[r, "\underline{hom}(\alpha_X,1)"'] & \underline{hom}(\widetilde{F}(X), \widetilde{G}(Y))
\end{tikzcd}$$

We denote by $Nat(\widetilde{F}, \widetilde{G})_0$ *the set of simplicial natural transformations between* $\widetilde{F}$ *and* $\widetilde{G}$.

We now define explicitly the category $[\mathcal{C}^{fin}, Ch]_0$.

Definition 4.4 (Category of simplicial functors)**.** *We denote by* $[\mathcal{C}^{fin}, Ch]_0$ *the category of simplicial functors* $\widetilde{F} : \widetilde{\mathcal{C}}^{fin} \longrightarrow Ch$ *and whose morphisms* $\widetilde{F} \longrightarrow \widetilde{G}$ *are simplicial natural transformations.*

Note that the category $\widetilde{\mathcal{C}}^{fin}$ is skeletally small, the collection of simplicial natural transformations between two simplicial functors is a set. Therefore $[\mathcal{C}^{fin}, Ch]_0$ is a category in the usual sense.

4.2 Model structure on $[\mathcal{C}^{fin}, Ch]_0$

To define a model category structure on the category of simplicial functors $[\mathcal{C}^{fin}, Ch]_0$, we simply follow the guidelines of [Hir03, 11.6.1]. Note that even if we restrict our argument to $\mathcal{C} = Alg_{\mathcal{O}}$, Ch_+ or Ch, the model structure that we define here below works (for an analogous argument) whenever $\mathcal{C}$ is an arbitrary small Ch-category. The main result of this section is the following:

Proposition 4.5 (Projective model structure on simplicial functors)**.** *The category $[\mathcal{C}^{fin}, Ch]_0$ is cofibrantly generated with the following properties:*

- *A (simplicial) natural transformation $\widetilde{F} \longrightarrow \widetilde{G}$ is a weak equivalence (resp. fibration) if and only if $\forall X \in \mathcal{C}^{fin}, \widetilde{F}(X) \longrightarrow \widetilde{G}(X)$ is a quasi-isomorphism (resp. fibration).*
- *The generating cofibrations(resp. trivial cofibrations) are of the form*

$$V_0 \otimes \widetilde{\mathcal{C}}(X, -) \xrightarrow{r \otimes 1} V_1 \otimes \widetilde{\mathcal{C}}(X, -)$$

where $V_0 \xrightarrow{r} V_1$ is a generating cofibration (resp. trivial cofibration) in chain complexes (these are described in [Hov99, § 2.1.])

The rest of this section is dedicated to the proof of Proposition 4.5.Roughly speaking, we will say that the proof comes from the "Strong" enriched Yoneda lemma and the Kan's Theorem.

Lemma 4.6 (Strong enriched Yoneda Lemma)**.** *Let $V \in Ch$, $X \in \mathcal{C}^{fin}$ and $\widetilde{G} \in [\mathcal{C}^{fin}, Ch]$. Then there is an isomorphism*

$$Nat(V \otimes \widetilde{\mathcal{C}}(X, -), \widetilde{G}) \cong \underline{hom}(V, \widetilde{G}(X))$$

Proof. A map from the left to the right is given by the composite

$$\Gamma : Nat(V \otimes \widetilde{\mathcal{C}}(X, -), \widetilde{G}) \longrightarrow \underline{hom}(V \otimes \widetilde{\mathcal{C}}(X, X), \widetilde{G}(X)) \longrightarrow \underline{hom}(V, \widetilde{G}(X))$$

where the first map is the natural projection(from the limit) and the second map uses the unit $\Bbbk \longrightarrow \widetilde{\mathcal{C}}(X, X)$.

The inverse of this map, denoted Γ', is given by the family $\{\Gamma'_Y\}_{Y \in \mathcal{C}}$ whose the element $\Gamma'_Y (\forall Y \in \mathcal{C}\ )$ is the adjoint of the map:

$$\begin{aligned} \underline{hom}(V, \widetilde{G}(X)) \otimes V \otimes \widetilde{\mathcal{C}}(X, Y) &\xrightarrow{1 \otimes \widetilde{G}_{X,Y}} \underline{hom}(V, \widetilde{G}(X)) \otimes V \otimes \underline{hom}(\widetilde{G}(X), \widetilde{G}(Y)) \\ &\longrightarrow \widetilde{G}(X) \otimes \underline{hom}(\widetilde{G}(X), \widetilde{G}(Y)) \\ &\longrightarrow \widetilde{G}(Y) \end{aligned}$$

where the second and the third maps are evaluation maps.

The naturality in the family $\{\Gamma'_Y\}_{Y \in \mathcal{C}}$ comes from the composition axiom that satisfy the maps $\widetilde{G}_{Y,Y'}$ and $V \otimes \widetilde{\mathcal{C}}(X, -)_{Y,Y'}$. We deduce from this that $\Gamma' : \underline{hom}(V, \widetilde{G}(X)) \longrightarrow Nat(V \otimes \widetilde{\mathcal{C}}(X, -), \widetilde{G})$ is well defined.

One can check that the maps Γ and Γ' are inverse. $\square$

Theorem 4.7 (D. M. Kan)**.** *Let $\mathcal{M}$ be a cofibrantly generated model category with cofibrations I and generating cofibrations J. Let $\mathcal{N}$ be a category that is closed under small limits and colimits, and let $\mathcal{F} : \mathcal{M} \rightleftarrows \mathcal{N} : U$ be a pair of adjoint functors. If we let $\mathcal{F}I = \{\mathcal{F}u/u \in I\}$ and $\mathcal{F}I = \{\mathcal{F}v/v \in I\}$ and if*

1. *both $\mathcal{F}I$ and $\mathcal{F}I$ permits the small object argument and*
2. *U takes relative $\mathcal{F}J$-cell complexes to weak equivalences,*

then there is a cofibrantly generated model category structure on $\mathcal{N}$ in which $\mathcal{F}I$ is a set of generating cofibrations, $\mathcal{F}I$ is a set of generating trivial cofibrations, and the weak equivalences are the maps that U takes into a weak equivalence in $\mathcal{M}$. Furthermore, with respect to this model category structure, (F, U) is a Quillen pair.

Sketch of the proof of Proposition 4.5. We only give the sketch of the proof as this is the enriched version of [Hir03, 11.6.1.]. The argument is based on a transfer of the model structure using the above Kan's Theorem 4.7. We will first define two categories.

1. Let $\mathcal{C}^{dics}$ be the discrete category whose objects are objects X of $\mathcal{C}^{fin}$ and morphisms $Hom_{\mathcal{C}^{disc}}(X, Y) := \emptyset$;
2. Let $[\mathcal{C}^{dics}, Ch]_0$ be the category where objects are functors $D : \mathcal{C}^{dics} \longrightarrow Ch$ and morphisms $Hom_{[\mathcal{C}^{dics}, Ch]_0}(D, D') := \prod_{X \in \mathcal{C}^{fin}} Hom_{Ch}(DX, D'X)$

One can equivalently write $[\mathcal{C}^{dics}, Ch]_0 \cong \prod_{X \in \mathcal{C}^{fin}} Ch$ and it is proved in [Hir03, Prop 11.1.10] that this category has a cofibrantly generated model structure with generating (resp. trivial) cofibrations I_{disc} (resp. J_{disc}) where

$$\begin{aligned} I_{disc} &= \bigcup_{X \in \mathcal{C}^{fin}} I_{Ch} \times \prod_{Y \in \mathcal{C}^{fin}, Y \neq X} 1_\phi \\ J_{disc} &= \bigcup_{X \in \mathcal{C}^{fin}} J_{Ch} \times \prod_{Y \in \mathcal{C}^{fin}, Y \neq X} 1_\phi \end{aligned}$$

where I_{Ch} (resp. J_{Ch}) denote the set of generating (resp. trivial) cofibration in Ch, and $1_\phi : 0 \longrightarrow 0$ is the identity (trivial) map of chain complexes.

We define the functor $\mathcal{F} : [\mathcal{C}^{dics}, Ch]_0 \longrightarrow [\mathcal{C}^{fin}, Ch]_0$ as follows:

- on objects $\mathcal{F}(D) := \bigoplus_{X \in \mathcal{C}^{fin}} D(X) \otimes \widetilde{\mathcal{C}}(X, -)$;
- on morphisms, the map of sets $Hom_{[\mathcal{C}^{dics}, Ch]_0}(D, D') \longrightarrow Nat(\mathcal{F}(D), \mathcal{F}(D'))_0$ is given level-wise ($\forall X \in \mathcal{C}$) by the morphisms:

$$(DX \xrightarrow{r} D'X) \longmapsto (DX \otimes \widetilde{\mathcal{C}}(X, -) \xrightarrow{r \otimes 1} D'X \otimes \widetilde{\mathcal{C}}(X, -))$$

Using the above enriched Yoneda lemma, we make the following computation

$$\begin{aligned} Nat(\mathcal{F}(D), \widetilde{G})_0 &\cong \prod_{X \in \mathcal{C}^{fin}} Nat(DX \otimes \widetilde{\mathcal{C}}(X, -), \widetilde{G}) \\ &\cong \prod_{X \in \mathcal{C}^{fin}} Hom_{Ch}(DX, \widetilde{G}(X)) = Hom_{[\mathcal{C}^{dics}, Ch]_0}(D, U\widetilde{G}) \end{aligned}$$

where $U : [\mathcal{C}^{fin}, Ch]_0 \longrightarrow [\mathcal{C}^{dics}, Ch]_0$ is the functor which forgets the enriched hom functor of $\widetilde{\mathcal{C}}$. One therefore deduce the adjunction

$$\mathcal{F} : [\mathcal{C}^{dics}, Ch]_0 \rightleftarrows [\mathcal{C}^{fin}, Ch]_0 : U$$

We will now apply the Kan's Theorem to this adjunction. Note that

$$\mathcal{F}I_{disc} = \{V_0 \otimes \widetilde{\mathcal{C}}(X,-) \xrightarrow{r\otimes 1} V_1 \otimes \widetilde{\mathcal{C}}(X,-) | r \in I_{Ch}\}$$
$$\mathcal{F}J_{disc} = \{V_0 \otimes \widetilde{\mathcal{C}}(X,-) \xrightarrow{r\otimes 1} V_1 \otimes \widetilde{\mathcal{C}}(X,-) | r \in J_{Ch}\}$$

One can check using the same argument with Hirschhorn (in [Hir03, proof of Theorem 6.1]) that $\mathcal{F}I_{disc}$ and $\mathcal{F}J_{disc}$ permits the small object argument. In addition, One can also check that the limits and the colimits in $[\mathcal{C}^{fin}, Ch]_0$ are computed level-wise. It then follows that $\mathcal{F}J_{disc}$-cell complexes are obtained by pushouts of J_{Ch}-cell. These are then level-wise weak equivalences. This proves that U takes $\mathcal{F}J_{disc}$-cell complexes to weak equivalences.

□

4.3 Cellular decomposition of simplicial functors

Definition 4.8 (Cell functors)**.**

- *A cell functor in $[\mathcal{C}^{fin}, Ch]_0$ is a cell complex with respect to the generating cofibrations described in Proposition 4.5.*
- *A presented cell functor $\widetilde{F}$ is a cell functor together with a sequence of functors*

$$0 = \widetilde{F}_0 \longrightarrow \widetilde{F}_1 \longrightarrow ... \longrightarrow \widetilde{F}$$

such that $\widetilde{F} = \underset{i}{colim} \widetilde{F}_i$, and there are pushouts of the form

$$\begin{tikzcd} \bigoplus\limits_{\alpha\in A_i} V_0^\alpha \otimes \widetilde{\mathcal{C}}(X_\alpha,-) \arrow[r] \arrow[d] & \widetilde{F}_i \arrow[d] \\ \bigoplus\limits_{\alpha\in A_i} V_1^\alpha \otimes \widetilde{\mathcal{C}}(X_\alpha,-) \arrow[r] & \widetilde{F}_{i+1} \end{tikzcd}$$

where A_i is an indexing set, and each $V_0^\alpha \longrightarrow V_1^\alpha$ is a generating cofibration of chain complexes, and each X_α is an object in $\mathcal{C}^{fin}$.

Remark 4.9. *If $\widetilde{F} : \widetilde{\mathcal{C}}^{fin} \longrightarrow Ch$ is a cell functor, then $\forall X \in \mathcal{C}^{fin}$, $\widetilde{F}(X)$ has a cell structure. The cells are in $1-1$ correspondence with pairs (α, ε), where α is one of the cells*

$$V_1^\alpha \otimes \widetilde{\mathcal{C}}(X_\alpha, -) \longrightarrow \widetilde{F}$$

of $\widetilde{F}$ *and* ε *corresponds to a non-degenerate simplex* $X_\alpha \otimes \triangle^k \longrightarrow X$ *in* $\mathcal{C}(X_\alpha, X)$.

Definition 4.10 (Cofibrant replacement for functors)**.** *Given* $\widetilde{F} \in [\mathcal{C}^{fin}, Ch]$, *the small object argument (see [Hir03, § 10.5.16]) determines the cofibrant replacement* $Q\widetilde{F}$ *of* $\widetilde{F}$. $Q\widetilde{F}$ *comes equipped with a natural cell structure*

$$0 = (Q\widetilde{F})_0 \longrightarrow (Q\widetilde{F})_1 \longrightarrow ... \longrightarrow (Q\widetilde{F})_i \longrightarrow ... \longrightarrow (Q\widetilde{F})$$

in which the $i+1$ *cells are* $1-1$ *correspondence with commutative diagrams of the form*

$$\begin{array}{ccc} V_0 \otimes \widetilde{\mathcal{C}}(X, -) & \longrightarrow & (Q\widetilde{F})_i \\ \downarrow & & \downarrow \\ V_1 \otimes \widetilde{\mathcal{C}}(X, -) & \longrightarrow & \widetilde{F} \end{array}$$

Lemma 4.11. *If* $\widetilde{F} : \widetilde{Alg}_{\mathcal{O}}^{fin} \longrightarrow Ch$ *is a presented cell functor, then the functor*

$$\widetilde{F}\Omega^\infty I : \widetilde{Ch}_+^{fin} \longrightarrow Ch$$

is equivalent to a presented cell functor in $[Ch_+^{fin}, Ch]_0$ *in which the cells correspond 1-1 with the cells of* $\widetilde{F}$.

Proof. We choose the following presentation of $\widetilde{F}$:

$$0 = \widetilde{F}_0 \longrightarrow \widetilde{F}_1 \longrightarrow ... \longrightarrow \widetilde{F}_i \longrightarrow ...$$

with the attaching cells(pushout diagrams) of the form:

$$\begin{array}{ccc} \bigoplus_\alpha W_0^\alpha \otimes \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X_\alpha, -) & \longrightarrow & \widetilde{F}_i \\ \downarrow & & \downarrow \\ \bigoplus_\alpha W_1^\alpha \otimes \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X_\alpha, -) & \longrightarrow & \widetilde{F}_{i+1} \end{array}$$

where $X_\alpha \in \mathrm{Alg}_{\mathcal{O}}^{fin}$. When we pre-compose this diagram with Ω^∞, we make the following observation: $\forall V \in Ch_+$,

$$\begin{aligned} \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X_\alpha, \Omega^\infty I(V)) =& N\Bbbk Hom_{\mathrm{Alg}_{\mathcal{O}}}(\Omega_{\mathcal{O}}(B(\mathcal{O}, X_\alpha)), red_0(\Omega^\infty I(V) \otimes Apl_\bullet)) & (4.1) \\ \cong& N\Bbbk Hom_{\mathrm{coAlg}_{B(\mathcal{O})}}(B(\mathcal{O}, X_\alpha), B(\mathcal{O})(red_0(I(V) \otimes Apl_\bullet))) & (4.2) \\ \cong& N\Bbbk Hom_{Ch_+}(UB(\mathcal{O}, X_\alpha), red_0(I(V) \otimes Apl_\bullet)) & (4.3) \\ \simeq& N\Bbbk Hom_{Ch_+}(UB(\mathcal{O}, X_\alpha) \otimes N\Bbbk\triangle^\bullet, V) & (4.4) \end{aligned}$$

where

- $U : \mathrm{coAlg}_{B(\mathcal{O})} \longrightarrow Ch_+$ is the forgetful functor and is the left adjoint of the co-free functor $B(\mathcal{O})(-) : Ch_+ \longrightarrow \mathrm{coAlg}_{B(\mathcal{O})}$;

- The quasi-isomorphism (4.4) is proved in §3.2.

Since X_α is a finite $\mathcal{O}$-algebra, then $B(\mathcal{O}, X_\alpha)$ is a finite chain complex. Therefore, to define the cell structure on $\widetilde{F}\Omega^\infty I$, we simply set

$$(\widetilde{F}\Omega^\infty I)_i := \widetilde{F}_i\Omega^\infty I.$$

$\square$

Definition 4.12 (Subcomplex of cell functors)**.** *Let $\widetilde{F} : \widetilde{\mathcal{C}}^{fin} \longrightarrow Ch$ be a presented cell functor.*

- *A subcomplex $\widetilde{C}$ of $\widetilde{F}$ is a cell functor in $[\mathcal{C}^{fin}, Ch]$ with a presentation*

$$0 = \widetilde{C}_0 \longrightarrow \widetilde{C}_1 \longrightarrow ... \longrightarrow \widetilde{C}_i \longrightarrow ... \longrightarrow \widetilde{C}$$

where $\forall i \geq 0$, there is a monomorphism $\widetilde{C}_i \longrightarrow \widetilde{F}_i$ so that each cell α of degree i in $\widetilde{C}_i$ is obtained by an equivalent cell of degree i in $\widetilde{F}_i$ via a factorization of the attaching map

$$\begin{tikzcd} V_0^\alpha \otimes \widetilde{\mathcal{C}}(X_\alpha, -) \arrow[rr] \arrow[dr] & & \widetilde{F}_{i-1} \\ & \widetilde{C}_i \arrow[ur] & \end{tikzcd}$$

In other words, the subcomplex $\widetilde{C}$ is a subset of the cells of $\widetilde{F}$.

- *A subcomplex $\widetilde{C}$ of $\widetilde{F}$ is finite if it has finitely many cells. The finite subcomplexes of $\widetilde{F}$ form a partially ordered set (under inclusion) which we denote $Sub(\widetilde{F})$.*

This section has two independent results on presented cell functors. We start with the first one which shows that we always have the a certain restriction property for natural transformations between cell functors:

Proposition 4.13. *Let $\widetilde{F}, \widetilde{G}, \widetilde{H} : Ch \longrightarrow Ch$ be presented cell functors, and $\alpha : \widetilde{H} \longrightarrow \widetilde{F}\widetilde{G}$ be a natural transformation. If E is a finite subcomplex of H, then there exist finite subcomplexes $C \in Sub(\widetilde{F})$ and $D \in Sub(\widetilde{G})$ such that α restricts to $E \longrightarrow CD$. We then have the commutative diagram*

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\alpha_{|E}"] & \widetilde{H} \arrow[d, "\alpha"] \\ CD \arrow[r] & \widetilde{F}\widetilde{G} \end{tikzcd}$$

To prove this result, we will need the following two intermediate lemmas.

Lemma 4.14. *If $\widetilde{F} : \widetilde{\mathcal{C}}^{fin} \longrightarrow Ch$ is a presented cell functor, then any cell*

$$V_1 \otimes \widetilde{\mathcal{C}}(X, -) \longrightarrow \widetilde{F}$$

factors through a finite subcomplex of $\widetilde{F}$.

Proof. Since $\widetilde{F} = \underset{i}{hocolim} \widetilde{F}_i$, we will give the proof by induction on i. Let us assume that we have the following property: any j cell, $(j \leq i)$, $V_1 \otimes \widetilde{\mathcal{C}}(X, -) \longrightarrow \widetilde{F}_j \longrightarrow \widetilde{F}$ factors through a finite subcomplex C of $\widetilde{F}$

$$V_1 \otimes \widetilde{\mathcal{C}}(X, -) \longrightarrow C \longrightarrow \widetilde{F}.$$

Now consider the following single $i+1$ cell attachment pushout:

$$\begin{tikzcd}
V_0 \otimes \widetilde{\mathcal{C}}(X, -) \arrow[r] \arrow[d] & \widetilde{F}_i \arrow[d] \\
V_1 \otimes \widetilde{\mathcal{C}}(X, -) \arrow[r] & \widetilde{F}_{i+1}
\end{tikzcd}$$

The $i+1$ attachment map $V_0 \otimes \widetilde{\mathcal{C}}(X, -) \longrightarrow \widetilde{F}_i$ of the cell α is equivalent via the (enriched) Yoneda lemma to the map $V_0 \longrightarrow \widetilde{F}_i(X)$, and since V_0 is of finite dimension, there is a natural factorization ,

$$\begin{tikzcd}
V_0 \arrow[rr] \arrow[dr] & & \widetilde{F}_i(X) \\
& A \arrow[ur, hook] &
\end{tikzcd}$$

with A finite.

Each cell of A corresponds to a cell in $\widetilde{F}$ which is of course of degree at most i. By the induction hypothesis, each of these cells is contained in a finite subcomplex of $\widetilde{F}$. Taking the union of all these gives a finite subcomplex C of $\widetilde{F}$ such that $A \subseteq C(X)$. This is adjoint to

$$V_0 \otimes \widetilde{\mathcal{C}}(X, -) \longrightarrow C \longrightarrow \widetilde{F}$$

We then set $C \oplus \alpha$ as the finite subcomplex which contains α. □

Lemma 4.15. *Let $C : Ch^{fin} \longrightarrow Ch$ be a cell functor, W be a cell chain complex, and A be a finite chain complex. Then any morphism $A \longrightarrow C(W)$ factors as*

$$A \longrightarrow C(L) \longrightarrow C(W)$$

where L is a finite subcomplex of W.

Proof. Since A is finite and $C(W)$ has a cell structure (see Remark 4.9), then there exists a finite subcomplex B of $C(W)$ such that:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & C(W) \\
& B \arrow[ur, hook] &
\end{tikzcd}$$

According to Remark 4.9, the cells of B consist of pairs $(\alpha_1, \varepsilon_1), ..., (\alpha_n, \varepsilon_n)$, where α_i 's are cells of C and $\varepsilon_i : V_{\alpha_i} \otimes \triangle^{k_i} \longrightarrow W$ are non-degenerate simplices in $Hom_{Ch}(V_{\alpha_i} \otimes \triangle^\bullet, W)$.

Since the domain of each ε_i is a finite chain complex, there is a natural factorization

$$\begin{tikzcd}
V_{\alpha_i} \otimes \triangle^{k_i} \arrow[rr] \arrow[dr] & & W \\
& L_i \arrow[ur, hook] &
\end{tikzcd}$$

where L_i is a finite subcomplex of W. Let L be a finite subcomplex of W which contains the complexes $L_1, ..., L_n$. We have then built the factorization

$$\begin{tikzcd}
A \arrow[rrr] \arrow[dr] & & & W \\
& B \arrow[r, hook] & C(L) \arrow[ur] &
\end{tikzcd}$$

□

Proof of Proposition 4.13. We give the proof by induction on the cells. We consider the following single cell attachment

$$\begin{tikzcd}
V_0 \otimes \widetilde{Ch}(V,-) \arrow[r] \arrow[d] & E' \arrow[d] \\
V_1 \otimes \widetilde{Ch}(V,-) \arrow[r] & E
\end{tikzcd}$$

where α restricts to $\alpha_{|E'} : E' \longrightarrow C'D'$ with C' (resp. D') a finite subcomplex of F (resp. G). The Yoneda lemma applied to the diagram

$$\begin{tikzcd}
V_0 \otimes \widetilde{Ch}(V,-) \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & C'D' \arrow[d] \\
V_1 \otimes \widetilde{Ch}(V,-) \arrow[r] & E \arrow[r] & FG
\end{tikzcd}$$

is equivalent to diagram

$$\begin{tikzcd}
V_0 \arrow[r] \arrow[d] & E'(V) \arrow[r] \arrow[d] & C'D'(V) \arrow[d] \\
V_1 \arrow[r] & E(V) \arrow[r] & FG(V)
\end{tikzcd}$$

Since $D'(V)$ and $G(D)$ are cell complexes and that V_0 and V_1 are finite, we obtain from Lemma 4.15 the factorization

$$\begin{tikzcd} V_0 \arrow[r] \arrow[d] & C'(L_0) \arrow[r] \arrow[d, dashed] & C'D'(V) \arrow[d] \\ V_1 \arrow[r] & F(L_1) \arrow[r] & FG(V) \end{tikzcd}$$

where L_0 and L_1 are finite subcomplexes of $D'(V)$ and $G(V)$ respectively. Since $D'(V)$ is a subcomplex of $G(V)$, we will assume that $L_0 \subset L_1$.

Applying the Yoneda lemma on the first square, we get:

$$\begin{tikzcd} V_0 \otimes \widetilde{Ch}(L_1, -) \arrow[r] \arrow[d] & C' \arrow[d] \\ V_1 \otimes \widetilde{Ch}(L_1, -) \arrow[r] & F \end{tikzcd}$$

Using Lemma 4.14, the map $V_1 \otimes \widetilde{Ch}(L_1, -) \longrightarrow F$ factors through a finite subcomplex C of F which we assume contains C'. We then have the factorization

$$V_1 \otimes \widetilde{Ch}(L_1, -) \longrightarrow C \longrightarrow F.$$

We also have a diagram

$$\begin{tikzcd} L_0 \arrow[r] \arrow[d] & D'(V) \arrow[d] \\ L_1 \arrow[r] & G(V) \end{tikzcd}$$

which is equivalent by the Yoneda lemma equivalent to the diagram

$$\begin{tikzcd} L_0 \otimes \widetilde{Ch}(V, -) \arrow[r] \arrow[d] & D' \arrow[d] \\ L_1 \otimes \widetilde{Ch}(V, -) \arrow[r] & G \end{tikzcd}$$

Using the same trick as in the case of F, let D be a finite subcomplex of G which contains D' and fitting into a factorization

$$L_1 \otimes \widetilde{Ch}(V, -) \longrightarrow D \longrightarrow G.$$

The above factorizations gives the morphism

$$V_1 \longrightarrow CD(V) \longleftarrow FG(V).$$

Finally, using the universal property of the pushout, we have the restriction map

$$\alpha_{|E} : E \longrightarrow CD$$

□

The second and independent result of this section is the decomposition of cellular functor in term of their finite subcomplexes. This is an algebraic version of [AC11, Cor 5.6, Cor 5.7].

Proposition 4.16.

1. *If $\widetilde{F} : \widetilde{\mathcal{C}}^{fin} \longrightarrow Ch$ is a presented cell functor, then there is an isomorphism*

$$\widetilde{F} \cong \underset{C \in Sub(\widetilde{F})}{colim} C$$

2. *If $\widetilde{F} : \widetilde{\mathcal{C}}^{fin} \longrightarrow Ch$ is a simplicial functor, then there is a zig zag of weak equivalences*

$$\widetilde{F} \simeq \underset{C \in Sub(Q\widetilde{F})}{hocolim} C$$

Proof. 1. We assume that $\widetilde{F} : \widetilde{\mathcal{C}}^{fin} \longrightarrow Ch$ is a presented cell functor. We will make an inductive approach. We consider the cellular decomposition $\widetilde{F} = \underset{i}{colim}\widetilde{F}_i$ and we assume that the inclusion $\widetilde{F}_i \longrightarrow \widetilde{F}$ factors as:

$$\widetilde{F}_i \longrightarrow \underset{C \in Sub(\widetilde{F})}{\operatorname{colim}} C \longrightarrow \widetilde{F}.$$

Using Lemma 4.14, a $i+1$ cell $V_1 \otimes \widetilde{\mathcal{C}}(X, -) \longrightarrow \widetilde{F}_{i+1} \longrightarrow \widetilde{F}$ factors via a finite subcomplex C of F :

$$\begin{tikzcd}
V_1 \otimes \widetilde{\mathcal{C}}(X,-) \arrow[r] \arrow[dr] & \widetilde{F}_{i+1} \arrow[r] & \widetilde{F} \\
& C \arrow[ur, hook] &
\end{tikzcd}$$

We then have the following $i+1$-cell attachment pushout

$$\begin{tikzcd}
\bigoplus_{\alpha \in A_i} V_0^\alpha \otimes \widetilde{\mathcal{C}}(X_\alpha,-) \arrow[r] \arrow[d] & \widetilde{F}_i \arrow[d] \arrow[rr, hook, dashed] \arrow[drr, hook, dashed] & & \widetilde{F} \\
\bigoplus_{\alpha \in A_i} V_1^\alpha \otimes \widetilde{\mathcal{C}}(X_\alpha,-) \arrow[r] \arrow[drr, dashed] & \widetilde{F}_{i+1} \arrow[urr, dashed] & & \\
& & & \underset{C \in Sub(\widetilde{F})}{\operatorname{colim}} C \arrow[uu, hook, dashed]
\end{tikzcd}$$

where all the doted arrows fit into commutative diagrams. Using the universal property of the pushout $\widetilde{F}_{i+1}$, we have the factorization $\widetilde{F}_{i+1} \longrightarrow \underset{C \in Sub(\widetilde{F})}{\operatorname{colim}} C \longrightarrow \widetilde{F}$ of the map $\widetilde{F}_{i+1} \longrightarrow \widetilde{F}$.

We then deduce that the composition $\widetilde{F} \longrightarrow \underset{C \in Sub(\widetilde{F})}{\operatorname{colim}} C \longrightarrow \widetilde{F}$ is the identity. The other composition $\underset{C \in Sub(\widetilde{F})}{\operatorname{colim}} C \longrightarrow \widetilde{F} \longrightarrow \underset{C \in Sub(\widetilde{F})}{\operatorname{colim}} C$ is obviously the identity.

2. If $\widetilde{F} : \widetilde{\mathcal{C}}^{fin} \longrightarrow Ch$ is an arbitrary simplicial functor, then using 1. and the fact that the cofibrant replacement $Q\widetilde{F}$ is a presented cell functor, we have

$$\widetilde{F} \xleftarrow{\simeq} Q\widetilde{F} \cong \underset{C \in Sub(Q\widetilde{F})}{\operatorname{colim}} C$$

It remains now to prove that this colimit is equivalent to the hocolim of the same diagram. In fact this is a filtered colimit and each C is a chain complexes valued functor. Since homology of chain complexes commutes with filtered colimits, we deduce that

$$\underset{C \in Sub(Q\widetilde{F})}{\operatorname{colim}} C \simeq \underset{C \in Sub(Q\widetilde{F})}{\operatorname{hocolim}} C.$$

□

4.4 Homotopy property of simplicial functors

When we do homotopy theory, we are very sensitive to weak equivalences in the same way as we care about homotopy functors (i.e. preserve weak equivalences) when we do functor calculus. We have previously defined simplicial functors and it is natural to ask whether theses functors, more precisely their associated functor described in Definition 3.8 are homotopy. We prove in this section in two lemmas that:

- Any simplicial functor $F : Ch_+ \longrightarrow Ch$ is a homotopy functor.
- Any simplicial functor $F : \mathrm{Alg}_{\mathcal{O}}^{fin} \longrightarrow Ch$ is a homotopy functor.

Lemma 4.17. *If $\Bbbk$ is a field of any characteristic and $F : Ch_+ \longrightarrow Ch$ is a simplicial functor, then F preserves homotopy equivalences.*

Proof. Let

$$V \underset{g}{\overset{f}{\rightleftarrows}} W$$

be a homotopy equivalence pair between two chain complexes. Having a homotopy of maps $gf \sim 1_V$ is equivalent to have a homotopy $H : V \otimes \triangle^1 \longrightarrow Y$ such that

$$H_{|0} : V \otimes \Bbbk(0) \longrightarrow V \text{ is } gf, \text{ and } H_{|1} : V \otimes \Bbbk(1) \longrightarrow V \text{ is } 1_V.$$

We want to construct a homotopy equivalence pair

$$F(V) \rightleftarrows F(W)$$

We consider the simplicial structure map of chain complexes associated to V :

$$F_{V,V} : Ch_+(V, V) \longrightarrow \underline{hom}(F(V), F(V));$$

We then make the following computations:

$$\begin{aligned} F_{V,V}(dH) =& F_{V,V}(H_{|0} - H_{|1}) \\ =& F_{V,V}(gf) - 1_{F(V)} \\ =& F_{W,V}(g) \circ F_{V,W}(f) - 1_{F(V)}; \end{aligned}$$

On the other hand, $d(F_{V,V})(dH) = d_V F_{V,V}(H) + F_{V,V}(H) d_V$. Since $F_{V,V}$ is a morphism of chain complexes, we have

$$F_{W,V}(g) \circ F_{V,W}(f) \sim 1_{F(V)} \text{ with the homotopy } F_{V,V}(H).$$

A similar argument gives

$$F_{V,W}(f) \circ F_{W,V}(g) \sim 1_{F(W)}.$$

□

Remark 4.18. *If $\Bbbk$ is a field of characteristic 0, then any quasi-isomorphism in Ch is a homotopy equivalence. In this case, Lemma 4.17 says that any simplicial functor $F : Ch_+ \longrightarrow Ch$ preserves quasi-isomorphisms.*

Lemma 4.19. *Any simplicial functor $F : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ preserves weak equivalences.*

Proof. 1. When $F = V \otimes \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, -)$, it is known from the previous chapter that F preserves weak equivalences.

2. We now suppose that F is a presented cell functor with the presentation

$$0 = F_0 \longrightarrow F_1 \longrightarrow ... \longrightarrow F_i \longrightarrow ... \longrightarrow F$$

and suppose that F_i preserves weak equivalences.

Since the pushout diagram

$$\begin{tikzcd} \bigoplus\limits_{\alpha} V_0^{\alpha} \otimes \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X_\alpha, -) \arrow[r] \arrow[d] & F_i \arrow[d] \\ \bigoplus\limits_{\alpha} V_1^{\alpha} \otimes \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X_\alpha, -) \arrow[r] & F_{i+1} \end{tikzcd}$$

is a homotopy pushout, thus it follows that F_{i+1} preserves weak equivalences. We then conclude that F preserves weak equivalences.

3. Now we take an arbitrary simplicial functor F and consider QF its cofibrant replacement in $[\mathrm{Alg}_{\mathcal{O}}^{fin}, Ch]$ which has a presented cell structure. If $f : Y \xrightarrow{\simeq} Z$ be a weak equivalence in $\mathrm{Alg}_{\mathcal{O}}$ then $QF(f)$ is a quasi isomorphism from (2). We form the following commutative diagram

$$\begin{tikzcd} QF(Y) \arrow[r, "\simeq"] \arrow[d, "QF(f)"', "\simeq"] & F(Y) \arrow[d, "F(f)"] \\ QF(Z) \arrow[r, "\simeq"] & F(Z) \end{tikzcd}$$

We deduce that $F(f)$ is a weak equivalence.

□

4.5 Computation of the chain complexes $Nat(Q(I^{\otimes n}), I^{\otimes m})$

In this section, $I : Ch_+^{fin} \longrightarrow Ch$ denotes the inclusion functor viewed as a simplicial functor in Example 3.11. We also denote by $QI \xrightarrow{\simeq} I$ the cofibrant replacement of I in the model category $[Ch_+^{fin}, Ch]_0$.

The goal of this section is to prove the the following:

Proposition 4.20. *We assume that the ground field is of characteristic* 0. *If* $n, m \in \mathbb{N}$, *then*

$$Nat(Q(I^{\otimes n}), I^{\otimes m}) \simeq \begin{cases} \Bbbk[\Sigma_n] & \text{if } n = m; \\ 0 & \text{if } n \neq m. \end{cases}$$

where $Q(I^{\otimes n})$ denotes the cofibrant replacement of the functor $Ch_+^{fin} \ni V \longmapsto V^{\otimes n}$ in the model category $[Ch_+^{fin}, Ch]_0$.

This result was inspired by the work of Kuhn in [Kuh94, Lemma 6.12]. Before we give the proof of this result, we will need some homotopical properties of the bi-functor $Nat(-,-)$.

Proposition 4.21. *Let $\mathcal{C} = Alg_{\mathcal{O}}$, Ch_+ or Ch. The objects $Nat(\widetilde{F}, \widetilde{G})$ make the functor category $[\mathcal{C}^{fin}, Ch]_0$ into an enriched model category over the symmetric monoidal category Ch.*

Proof. We need to show that if $\widetilde{F} \rightarrowtail \widetilde{F'}$ is a cofibration and $\widetilde{G'} \twoheadrightarrow \widetilde{G}$ is a fibration in $[\mathcal{C}^{fin}, Ch]_0$ then the map

$$Nat(\widetilde{F'}, \widetilde{G'}) \longrightarrow Nat(\widetilde{F}, \widetilde{G'}) \underset{Nat(\widetilde{F}, \widetilde{G})}{\times} Nat(\widetilde{F'}, \widetilde{G}) \tag{4.5}$$

is a fibration in Ch, and that it is a quasi-isomorphism if either $\widetilde{F} \longrightarrow \widetilde{F'}$ or $\widetilde{G'} \longrightarrow \widetilde{G}$ is. Since the category $[\mathcal{C}^{fin}, Ch]_0$ is cofibrantly generated, it is sufficient to consider the case $\widetilde{F} \longrightarrow \widetilde{F'}$ is either a generating cofibration or a generating trivial cofibration $V_0 \otimes \widetilde{\mathcal{C}}(X, -) \xrightarrow{r \otimes 1} V_1 \otimes \widetilde{\mathcal{C}}(X, -)$ in $[\mathcal{C}^{fin}, Ch]_0$, where $r : V_0 \longrightarrow V_1$ is either a cofibration or a generating cofibration in Ch.

Using the strong Yoneda Lemma to Equation (4.5), we get equivalently the morphism of chain complexes

$$\underline{hom}(V_1, \widetilde{G'}(X)) \longrightarrow \underline{hom}(V_0, \widetilde{G'}(X)) \underset{\underline{hom}(V_0, \widetilde{G}(X))}{\times} \underline{hom}(V_1, \widetilde{G}(X)) \tag{4.6}$$

And this a is fibration since $r : V_0 \longrightarrow V_1$ is a cofibration and the map $\widetilde{G'}(X) \longrightarrow \widetilde{G}(X)$ is a fibration in Ch.

In addition, If r or $\widetilde{G'}(X) \longrightarrow \widetilde{G}(X)$, $(\forall X)$ is a quasi-isomorphism, then so is the map (4.6). ☐

A straight consequence of Lemma 4.21 is the next result.

Corollary 4.22. *If $\widetilde{F} \rightarrowtail \widetilde{F'}$ is a trivial cofibration and $\widetilde{G'}$ (which is always fibrant) in $[\mathcal{C}^{fin}, Ch]_0$, then the natural map of chain complexes*

$$Nat(\widetilde{F'}, \widetilde{G'}) \longrightarrow Nat(\widetilde{F}, \widetilde{G'})$$

is a quasi-isomorphism.

Moreover, if $\widetilde{F} \longrightarrow \widetilde{F'}$ is simply a weak equivalence with $\widetilde{F}$ and $\widetilde{F'}$ cofibrant functors in $[\mathcal{C}^{fin}, Ch]_0$, using the Ken Brown's Lemma, we deduce from Corollary 4.22 that $Nat(\widetilde{F'}, \widetilde{G'}) \longrightarrow Nat(\widetilde{F}, \widetilde{G'})$ is a quasi-isomorphism.

Now we start the analysis for the proof of Proposition 4.20 which uses the following fundamental lemmas in the particular case $n = m = 1$.

Lemma 4.23. *We assume that the ground field is of characteristic* 0. *There is a quasi isomorphism of chain complexes*

$$Nat(QI, I) \simeq \Bbbk$$

The proof of this lemma will use the following key result:

Lemma 4.24. *We assume that the ground field is of characteristic* 0. *There is a weak equivalence in $[Ch_+^{fin}, Ch]_0$*

$$\rho : \underset{p}{hocolim}\ s^{-p} N\widetilde{\Bbbk Hom}_{Ch_+}(N\Bbbk\triangle^\bullet, s^p-) \xrightarrow{\simeq} I$$

Proof. Consider the maps l_p $(\forall p \geq 0)$ defined as follows:

$$N\widetilde{\Bbbk Hom}_{Ch_+}(N\Bbbk\triangle^\bullet, s^p-) \xrightarrow{l_p} NHom_{Ch_+}(N\Bbbk\triangle^\bullet, s^p-) \cong s^p I$$
$$\bigoplus_i f_i \longmapsto \sum_i f_i$$

The map ρ in the lemma is defined by first taking the adjoint of the maps l_p's:

$$\widetilde{l_p} : s^{-p} N\widetilde{\Bbbk Hom}_{Ch_+}(N\Bbbk\triangle^\bullet, s^p-) \xrightarrow{\simeq} I$$

and then applying $\underset{p}{hocolim}$ on the domain.

We will prove in the first two items that the domain and the co-domain of the natural projection l_p are simplicial functors and that l_p is itself a simplicial natural transformation. We use these maps l_p ,$\forall p$, to prove in 3. that ρ is a simplicial natural transformation between simplicial functors. In the last item we show that ρ is a weak equivalence, that is a level-wise quasi-isomorphism.

1. The simplicial structure on the functor

$$N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^p-) : \widetilde{Ch}_+ \longrightarrow Ch$$

is almost defined as the simplicial structure of the representable functor $N\Bbbk Hom_{Ch_+}(N\Bbbk\triangle^\bullet, -)$ of Example 3.10. More precisely the simplicial structure map

$$\widetilde{Ch}_+(V, W) \xrightarrow{\Gamma_p} \underline{hom}(N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pV), N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pW))$$

is adjoint to a map (in Ch)

$$\widetilde{Ch}_+(V, W) \otimes N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pV) \xrightarrow{\Gamma_p^{'}} N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pW)$$

To define this map $\Gamma_p^{'}$, we consider the morphism of simplicial sets

$$\begin{aligned} Hom_{Ch_+}(N\Bbbk\triangle^\bullet \otimes V, W) \times Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pV) &\xrightarrow{\gamma} Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pW) \\ f \otimes g &\longmapsto \gamma(f \otimes g) \end{aligned}$$

where $\gamma(f \otimes g)$ is the composite

$$\begin{tikzcd} N\Bbbk\triangle^\bullet \arrow[r, "\triangle"] & N\Bbbk\triangle^\bullet \otimes N\Bbbk\triangle^\bullet \arrow[d, "g\otimes N\Bbbk\triangle^\bullet"] & \\ & s^pV \otimes N\Bbbk\triangle^\bullet \cong s^p\Bbbk \otimes (V \otimes N\Bbbk\triangle^\bullet) \arrow[r, "s^p\Bbbk\otimes f"] & s^p\Bbbk \otimes W \end{tikzcd}$$

In addition since the functors N and $\Bbbk(-)$ are monoidal, we apply $N\Bbbk(-)$ to γ and form the commutative diagram

$$\begin{tikzcd} N\Bbbk Hom_{Ch_+}(N\Bbbk\triangle^\bullet \otimes V, W) \otimes N\Bbbk* \arrow[r, "N\Bbbk(\gamma)"] \arrow[d, "1\otimes i"] & N\Bbbk* \arrow[d, "i"] \\ N\Bbbk Hom_{Ch_+}(N\Bbbk\triangle^\bullet \otimes V, W) \otimes N\Bbbk Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pV) \arrow[r, "N\Bbbk(\gamma)"] & N\Bbbk Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pW) \end{tikzcd}$$

Taking the fiber of the vertical maps $1 \otimes i$ and i, we get the map $\Gamma_p^{'}$. A similar argument with item (a) in the proof of Proposition 3.4 permits to claim that γ is associative and satisfies the unit axiom. Therefore $\Gamma_p^{'}$ and thus Γ_p satisfy these properties since N and $\Bbbk(-)$ are monoidal.

2. The functor $NHom_{Ch_+}(N\Bbbk\triangle^\bullet, s^p-) : Ch_+ \longrightarrow Ch$ which is isomorphic using the Dold-Kan correspondence to the functor $s^p\Bbbk \otimes I : Ch_+ \longrightarrow Ch$ induces clearly a simplicial functor as I does so (see Example 3.11).

3. We prove now that ρ is a simplicial natural transformation. We consider the natural projection

$$N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^p-) \xrightarrow{l_p} NHom_{Ch_+}(N\Bbbk\triangle^\bullet, s^p-) \cong s^pI$$

which is a simplicial natural transformation in $[\widetilde{Ch}_+, Ch]_0$. In fact we have the commutative diagram:

$$\begin{tikzcd}
\widetilde{Ch}_+(V,W) \arrow[r, "\Gamma_p"] \arrow[d, "s^p\Bbbk\otimes\widetilde{I}_{V,W}"] & \underline{hom}(N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pV), N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pW)) \arrow[d, "\underline{hom}(1,l_{pW})"] \\
\underline{hom}(s^pV, s^pW) \arrow[r, "\underline{hom}(l_{pV},1)"'] & \underline{hom}(N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pV), s^pW)
\end{tikzcd}$$

which is equivalent by adjunction to the commutative diagram

$$\begin{tikzcd}
N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pV)\otimes\widetilde{Ch}_+(V,W) \arrow[r, "\Gamma_p"] \arrow[d] & N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pW) \arrow[d] \\
N\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pV)\otimes\underline{hom}(s^pV, s^pW) \arrow[r] & s^pW
\end{tikzcd}$$

Note that $\underline{hom}(s^pV, s^pW) \cong \underline{hom}(V,W)$. By applying $\operatorname{hocolim}_p s^{-p}-$ to the above diagram, we deduce that ρ is a simplicial natural transformation.

4. We finally need to show that ρ is a weak equivalence. According to the Hurewicz theorem (see [GJ99, Chap III, Thm 3.7] and [KK04]), we have the $2p$-connected morphism of simplicial sets

$$h_p : Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pW) \longrightarrow \widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pW)$$

which is deduced from the unit of the adjoint pair $\Bbbk(-) : sVect_\Bbbk \rightleftarrows sSet : U$. On the other hand, considering the projection l_p defined above in item 3. we form the composite

$$Hom_{Ch_+}(N\Bbbk\triangle^\bullet s^pW) \xrightarrow{h_p} \widetilde{\Bbbk}Hom_{Ch_+}((N\Bbbk\triangle^\bullet, s^pW) \xrightarrow{l_p} Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pW)$$

which is the identity on $Hom_{Ch_+}(\triangle^\bullet, s^pW) - \{0\}$. Thus we deduce that l_p is also $2p$-connected. We then have the p-connected map of simplicial vector spaces:

$$\Omega^p\widetilde{\Bbbk}Hom_{Ch_+}((N\Bbbk\triangle^\bullet, s^pW) \xrightarrow{\Omega^p(l_p)} \Omega^pHom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pW).$$

By applying the functor $\operatorname{hocolim}_p-$ on this map, we get the a weak equivalence of simplicial vector spaces

$$\operatorname{hocolim}_p \Omega^p\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pW) \longrightarrow \operatorname{hocolim}_p \Omega^pHom_{Ch_+}(N\Bbbk\triangle^\bullet, s^pW).$$

Finally we apply the normalization functor N(which is a left and thus commutes with colimits) to get the quasi-isomorphism of chain complexes

$$\operatorname*{hocolim}_p s^{-p} N\widetilde{\Bbbk Hom}_{Ch_+}(N\Bbbk\triangle^\bullet, s^p W) \longrightarrow \operatorname*{hocolim}_p s^{-p} N Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^p W).$$

The computation on the co-domain of this later map using the Dold Kan correspondence gives:

$$\operatorname*{hocolim}_p s^{-p} N Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^p W) \simeq \operatorname*{hocolim}_p s^{-p} s^p W \simeq W$$

□

Proof of Lemma 4.23. Using Lemma 4.24 and the fact that $Nat(-, I)$ preserves weak equivalences between cofibrant functors, we make the following computation

$$\begin{aligned} Nat(QI, I) &\simeq Nat(\operatorname*{hocolim}_p s^{-p} QN\widetilde{\Bbbk Hom}_{Ch_+}(N\Bbbk\triangle^\bullet, s^p -), I) && (4.7)\\ &\simeq \operatorname*{holim}_p s^p Nat(QN\widetilde{\Bbbk Hom}_{Ch_+}(N\Bbbk\triangle^\bullet, s^p -), I) && (4.8)\end{aligned}$$

At this point, we will compute the chain complex $Nat(QN\widetilde{\Bbbk Hom}_{Ch_+}(N\Bbbk\triangle^\bullet, s^p -), I)$. We consider the (homotopy) pushout diagram of simplicial vector spaces

$$\begin{tikzcd} \Bbbk * \cong Hom_{Ch_+}(0 \otimes N\Bbbk\triangle^\bullet, W) \arrow[r] \arrow[d] & 0 \arrow[d] \\ \Bbbk Hom_{Ch_+}(\Bbbk \otimes N\Bbbk\triangle^\bullet, W) \arrow[r] & \widetilde{\Bbbk Hom}_{Ch_+}(\Bbbk \otimes N\Bbbk\triangle^\bullet, W) \end{tikzcd}$$

which induces a level-wise homotopy pushout diagram in chain complexes

$$\begin{tikzcd} N\Bbbk Hom_{Ch_+}(0 \otimes N\Bbbk\triangle^\bullet, s^p -) \arrow[r] \arrow[d] & 0 \arrow[d] \\ N\Bbbk Hom_{Ch_+}(\Bbbk \otimes N\Bbbk\triangle^\bullet, s^p -) \arrow[r] & N\widetilde{\Bbbk Hom}_{Ch_+}(\Bbbk \otimes N\Bbbk\triangle^\bullet, s^p -) \end{tikzcd}$$

When we apply $Nat(Q-, I)$ to this diagram, we get the homotopy pullback diagram

$$\begin{tikzcd} Nat(QN\Bbbk Hom_{Ch_+}(0 \otimes N\Bbbk\triangle^\bullet, s^p -), I) & 0 \arrow[l] \\ Nat(QN\Bbbk Hom_{Ch_+}(\Bbbk \otimes N\Bbbk\triangle^\bullet, s^p -), I) \arrow[u] & Nat(QN\widetilde{\Bbbk Hom}_{Ch_+}(\Bbbk \otimes N\Bbbk\triangle^\bullet, s^p -), I) \arrow[l] \arrow[u] \end{tikzcd}$$

$(*)$

We observe that the most right vertical map of the diagram $(*)$ is a weak equivalence. In fact the functor $N\Bbbk Hom_{Ch_+}(0 \otimes N\Bbbk\triangle^\bullet, s^p-)$ is cofibrant (see Appendix (6)). We then make the computation:

$$Nat(QN\Bbbk Hom_{Ch_+}(0 \otimes N\Bbbk\triangle^\bullet, s^p-), I) \simeq Nat(N\Bbbk Hom_{Ch_+}(0 \otimes N\Bbbk\triangle^\bullet, s^p-), I) \tag{4.9}$$

$$\simeq s^{-p}I(0) = 0 \tag{4.10}$$

where the equivalence (4.10) is roughly given by the Yoneda lemma and explained in Appendix (6)−(5.51).

It follows that the most left vertical map of the pullback diagram $(*)$ is also a weak equivalence. Therefore we have

$$Nat(QN\widetilde{\Bbbk}Hom_{Ch_+}(\Bbbk \otimes N\Bbbk\triangle^\bullet, s^p-), I) \simeq Nat(QN\Bbbk Hom_{Ch_+}(\Bbbk \otimes N\Bbbk\triangle^\bullet, s^p-), I) \tag{4.11}$$

$$\simeq Nat(N\Bbbk Hom_{Ch_+}(\Bbbk \otimes N\Bbbk\triangle^\bullet, s^p-), I) \tag{4.12}$$

$$\simeq Nat(N\Bbbk Hom_{Ch_{\geq p}}(\Bbbk \otimes N\Bbbk\triangle^\bullet, -), s^{-p}I) \tag{4.13}$$

$$= s^{-p}\Bbbk. \tag{4.14}$$

where

- Equation (4.12) comes from the fact that the functor $N\Bbbk Hom_{Ch_+}(\Bbbk \otimes N\Bbbk\triangle^\bullet, s^p-)$ is cofibrant (see Appendix (6)).
- Equation (4.14) comes from the Yoneda lemma (see Appendix-(6)-(5.51)).

Using the equations (4.8) and (4.14) we deduce the following computation

$$\begin{aligned} Nat(QI, I) &\simeq \operatorname*{holim}_p\ s^p Nat(QN\widetilde{\Bbbk}Hom_{Ch_+}(N\Bbbk\triangle^\bullet, s^p-), I) \\ &\simeq \operatorname*{holim}_p\ s^p s^{-p}\Bbbk \simeq \Bbbk. \end{aligned}$$

□

We now consider the two functors:

$$\begin{aligned} \triangle : Ch_+^{fin} &\longrightarrow Ch_+^{fin} \times Ch_+^{fin} \\ V &\longmapsto (V, V) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \Pi : Ch_+^{fin} \times Ch_+^{fin} &\longrightarrow Ch_+^{fin} \\ (V, W) &\longmapsto V \oplus W \end{aligned}$$

which extend to simplicial functors $\widetilde{\triangle}$ and $\widetilde{\Pi}$ in Example 3.15 between the Ch-enriched categories $\widetilde{Ch}_+^{fin} \times \widetilde{Ch}_+^{fin}$ and $\widetilde{Ch} = Ch$.

We denote by $Nat_{bi}(-,-)$ the functor whose input are the simplicial bifunctors $\widetilde{Ch}_+^{fin} \times \widetilde{Ch}_+^{fin} \longrightarrow \widetilde{Ch}$ and returns a chain complex. This is obtained explicitly by replacing $\widetilde{\mathcal{C}}^{fin}$ in Definition 4.1 by $\widetilde{Ch}_+^{fin} \times \widetilde{Ch}_+^{fin}$.

The pairs $(\triangle, \Pi)$ and $(\Pi, \triangle)$ are adjoint; therefore we deduce the following lemma which is an enriched case of a general argument (see [Emi17, Prop 4.4.6]).

Lemma 4.25. *Given any two simplicial functors* $\widetilde{F} : Ch_+^{fin} \longrightarrow Ch$ *and* $\widetilde{G} : Ch_+^{fin} \times Ch_+^{fin} \longrightarrow Ch$, *there are isomorphisms*

$$Nat(\widetilde{F}, \widetilde{G} \circ \widetilde{\triangle}) \cong Nat_{bi}(\widetilde{F} \circ \widetilde{\Pi}, \widetilde{G}) \tag{4.15}$$

and

$$Nat(\widetilde{G} \circ \widetilde{\triangle}, \widetilde{F}) \cong Nat_{bi}(\widetilde{G}, \widetilde{F} \circ \widetilde{\Pi}) \tag{4.16}$$

Proof. We will only prove Equation (4.15) as the proof for Equation (4.16) follows an analogous argument.

Let $f = \{f_{V,W} : \widetilde{F}(V \oplus W) \longrightarrow \widetilde{G}(V,W)\}_{V,W} \in Nat_{bi}(\widetilde{F} \circ \widetilde{\Pi}, \widetilde{G})$ and $g = \{g_V : \widetilde{F}(V) \longrightarrow \widetilde{G}(V,V)\}_V \in Nat(\widetilde{F}, \widetilde{G} \circ \widetilde{\triangle})$;

If we denote by $\eta : 1 \longrightarrow \triangle\Pi$ and $\varepsilon : \Pi\triangle \longrightarrow 1$ the unit and the co-unit of the adjunction $(\Pi, \triangle)$ respectively. We define the morphisms γ and γ' as follows:

$$\gamma : Nat_{bi}(\widetilde{F} \circ \widetilde{\Pi}, \widetilde{G}) \longrightarrow \prod_V \underline{hom}(\widetilde{F}(V), \widetilde{G}(V,V))$$

$$f \longmapsto \{\widetilde{F}(V) \stackrel{\widetilde{F}(\eta)}{\longrightarrow} \widetilde{F}(V \oplus V) \stackrel{f_{V,V}}{\longrightarrow} \widetilde{G}(V,V)\}_V$$

$$\gamma' : Nat(\widetilde{F}, \widetilde{G} \circ \widetilde{\triangle}) \longrightarrow \prod_{V,W} \underline{hom}(\widetilde{F}(V \oplus W), \widetilde{G}(V,W))$$

$$g \longmapsto \{\widetilde{F}(V \oplus W) \stackrel{g_{V \oplus W}}{\longrightarrow} \widetilde{G}(V \oplus W, V \oplus W) \stackrel{\widetilde{G}(\varepsilon)}{\longrightarrow} \widetilde{G}(V,W)\}_{V,W}$$

One can check that $\gamma(f)$ and $\gamma'(g)$ satisfy the naturality in the underlying diagrams to $Nat(\widetilde{F}, \widetilde{G} \circ \widetilde{\triangle})$ and $Nat_{bi}(\widetilde{F} \circ \widetilde{\Pi}, \widetilde{G})$ respectively. Therefore these maps factors uniquely to $Nat(\widetilde{F}, \widetilde{G} \circ \widetilde{\triangle})$ and $Nat_{bi}(\widetilde{F} \circ \widetilde{\Pi}, \widetilde{G})$ respectively.

In addition we have the identity $\gamma'\gamma(f) = f$ explained by the following commutative diagram

$$\begin{tikzcd}
\widetilde{F}(V \oplus W) \arrow[r, "\widetilde{F}(\Pi\eta)"] \arrow[dr, "="'] & \widetilde{F}(V \oplus W \oplus V \oplus W) \arrow[r, "f_{V\oplus W, V\oplus W}"] \arrow[d, "\widetilde{F}(\varepsilon\Pi)"] & \widetilde{G}(V \oplus W, V \oplus W) \arrow[r, "\widetilde{G}(\varepsilon)"] & \widetilde{G}(V,W) \\
 & \widetilde{F}(V \oplus W) \arrow[urr, "f_{V,W}"'] & &
\end{tikzcd}$$

The first triangle commutes by the triangle identity property that satisfy the adjunction; The second triangle comes from the naturality in the diagram underlying $Nat_{bi}(\widetilde{F} \circ \widetilde{\Pi}, \widetilde{G})$ that satisfy f.

A similar argument gives the identity $\gamma\gamma'(g) = g$. □

Given two simplicial functors $\widetilde{F}, \widetilde{H} : Ch_+^{fin} \longrightarrow Ch$, we define the bi-functor

$$\widetilde{F} \boxtimes \widetilde{H}(V,W) := \widetilde{F}(V) \otimes \widetilde{H}(W)$$

Using this definition, the tensor product of two simplicial functors introduced in Example 3.12 can be re-written as $\widetilde{F} \otimes \widetilde{H} := (\widetilde{F} \boxtimes \widetilde{H}) \circ \triangle$.

Lemma 4.26. *If $\widetilde{F}, \widetilde{G} : \widetilde{Ch}_{+}^{fin} \longrightarrow Ch$ are two presented cell functors in $[Ch_{+}^{fin}, Ch]_0$, then the tensor product $\widetilde{F} \otimes \widetilde{G}$ is cofibrant.*

Proof. Let $\widetilde{H} \overset{\beta}{\underset{\simeq}{\twoheadrightarrow}} \widetilde{K}$ be a fibration in $[Ch_{+}^{fin}, Ch]_0$. We want to prove that the natural map

$$Nat(\widetilde{F} \otimes \widetilde{G}, \widetilde{H}) \xrightarrow{\beta_*} Nat(\widetilde{F} \otimes \widetilde{G}, \widetilde{K})$$

induced by β is a surjection. Here $Nat(-,-)$ denotes the set of enriched natural transformations of simplicial sets.

1. We assume that $\widetilde{F} = V \otimes \widetilde{Ch}_{+}(V', -)$ and $\widetilde{G} = W \otimes \widetilde{Ch}_{+}(W', -)$. We have

$$\begin{aligned} Nat(\widetilde{F} \otimes \widetilde{G}, \widetilde{H}) &\cong Nat(\widetilde{F}, \underline{hom}(W, \widetilde{H}(W' \oplus -))) && (4.17)\\ &\cong \underline{hom}(V, \underline{hom}(W, \widetilde{H}(W' \oplus V'))) && (4.18)\\ &\cong \underline{hom}(V \otimes W, \widetilde{H}(W' \oplus V')) && (4.19) \end{aligned}$$

   where the equations (4.17) and (4.18) are given by the enriched Yoneda lemma and Lemma 4.25.

   We deduce similarly that $Nat(\widetilde{F} \otimes \widetilde{G}, \widetilde{K}) \cong \underline{hom}(V \otimes W, \widetilde{K}(W' \oplus V'))$, and that the map β_* is equivalent to the surjection:

$$\underline{hom}(V \otimes W, \widetilde{H}(W' \oplus V')) \longrightarrow \underline{hom}(V \otimes W, \widetilde{K}(W' \oplus V'))$$

2. We now consider the following presentation of $\widetilde{F}$:

$$\widetilde{F}_0 = * \longrightarrow \widetilde{F}_1 \longrightarrow ... \longrightarrow \widetilde{F}_k \longrightarrow ... \longrightarrow \widetilde{F}$$

   and the following attachment

$$\begin{tikzcd} \bigoplus\limits_{\alpha} V_0^{\alpha} \otimes \widetilde{Ch}_{+}(V'_{\alpha}, -) \arrow[r] \arrow[d] & \widetilde{F}_i \arrow[d] \\ \bigoplus\limits_{\alpha} V_1^{\alpha} \otimes \widetilde{Ch}_{+}(V'_{\alpha}, -) \arrow[r] & \widetilde{F}_{i+1} \end{tikzcd}$$

   The tensor product of this diagram with an arbitrary presented cell $\widetilde{G}$ gives the pushout diagram

$$\begin{tikzcd}
\bigoplus_{\alpha} V_0^{\alpha} \otimes \widetilde{Ch}_+(V'_{\alpha}, -) \otimes \widetilde{G} \arrow[r] \arrow[d, "(3)"'] & \widetilde{F}_i \otimes \widetilde{G} \arrow[d, "(4)"] \\
\bigoplus_{\alpha} V_1^{\alpha} \otimes \widetilde{Ch}_+(V'_{\alpha}, -) \otimes \widetilde{G} \arrow[r] & \widetilde{F}_{i+1} \otimes \widetilde{G}
\end{tikzcd}$$

Now since $\widetilde{G}$ is cofibrant, all the maps

$$V_0^{\alpha} \otimes \widetilde{Ch}_+(V'_{\alpha}, -) \otimes \widetilde{G} \longrightarrow V_1^{\alpha} \otimes \widetilde{Ch}_+(V'_{\alpha}, -) \otimes \widetilde{G}$$

are cofibrations. One then deduce that the map (3) is a cofibration and therefore that (4) is a cofibration.

The result then follows inductively using 1. and 2.

$\square$

Proof of Proposition 4.20 . We first remark that $Q(I)^{\otimes n}$ is also a cofibrant replacement of $I^{\otimes n}$ (see Lemma 4.26) , therefore

$$Nat(Q(I^{\otimes n}), I^{\otimes m}) \simeq Nat(Q(I)^{\otimes n}, I^{\otimes m}).$$

On the other hand we make the following computation (using Lemma 4.25)

$$\begin{aligned} Nat(Q(I)^{\otimes n}, I^{\otimes m}) &\cong Nat(Q(I)^{\otimes n}, I^{\boxtimes m} \circ \triangle_m) \\ &\cong Nat(Q(I)^{\otimes n} \circ \Pi_m, I^{\boxtimes m}) \end{aligned}$$

where $\triangle_m : Ch_+^{fin} \longrightarrow (Ch_+^{fin})^{\times m}$ and $\Pi_m : (Ch_+^{fin})^{\times m} \longrightarrow Ch_+^{fin}$ are the $(m-1)^{th}$-iteration of $\triangle$ and Π respectively.

Thus until now we have the equivalence

$$Nat(Q(I^{\otimes n}), I^{\otimes m}) \simeq Nat(Q(I)^{\otimes n} \circ \Pi_m, I^{\boxtimes m}).$$

It remains to develop the term $Q(I)^{\otimes n} \circ \Pi_m$.

Let $V_1, ..., V_m \in Ch_+^{fin}$; then

$$\begin{aligned} Q(I)^{\otimes n} \circ \Pi_m(V_1, ..., V_m) &= (Q(I)(V_1 \oplus ... \oplus V_m))^{\otimes n} \\ &= \bigoplus_{i_1,...,i_n \in \{1,...,m\}} Q(I)(V_{i_1}) \otimes ... \otimes Q(I)(V_{i_n}) \\ &= \bigoplus_{i_1,...,i_n \in \{1,...,m\}} Q(I) \boxtimes ... \boxtimes Q(I)(V_{i_1}, ..., V_{i_n}) \end{aligned}$$

1. If $n < m$ then

$$Nat(Q(I)^{\boxtimes n}, I^{\boxtimes m}) \cong Nat(Q(I), I)^{\otimes n} \otimes Nat(\Bbbk, I)^{\otimes m-n}$$

where we denote abusively $\Bbbk$ to mean the constant functor which is $\Bbbk$ for any given input.

Since $Nat(\Bbbk, I) = 0$, we have that $Nat(Q(I)^{\boxtimes n}, I^{\boxtimes m}) \cong 0$.

We then deduce that $Nat(Q(I^{\otimes n}), I^{\otimes m}) \simeq 0$

2. When $n > m$ this is the dual case and analogue of the previous case.
3. If $n = m$ we start with the easy case $n = 2$ and we generalize the constructions for any arbitrary n.

$$Q(I) \otimes Q(I) \circ \Pi \cong (Q(I) \otimes Q(I)) \circ pr_1 \oplus (Q(I) \otimes Q(I)) \circ pr_2 \\ \oplus \Bbbk\Sigma_2 \otimes (Q(I) \boxtimes Q(I))$$

Since

$$Nat((Q(I) \otimes Q(I)) \circ pr_1, I^{\boxtimes 2}) \cong Nat(Q(I) \otimes Q(I), I \otimes (I(0))) \cong 0,$$

and $Nat((Q(I) \otimes Q(I)) \circ pr_2, I^{\boxtimes 2}) \cong Nat(Q(I) \otimes Q(I), (I(0)) \otimes I) \cong 0$, we deduce that

$$Nat(Q(I) \otimes Q(I) \circ \Pi, I^{\boxtimes 2}) \cong Nat(\Bbbk\Sigma_2 \otimes (Q(I) \boxtimes Q(I)), I^{\boxtimes 2}) \\ \cong \Bbbk\Sigma_2 \otimes Nat(Q(I), I)^{\otimes 2}$$

We know from Lemma 4.23 that $Nat(Q(I), I) \simeq \Bbbk$. Therefore, we deduce that $Nat(Q(I) \otimes Q(I), I \otimes I) \simeq \Bbbk\Sigma_2$.

For a general integer n, we remark that

$$Q(I)^{\otimes n} \circ \Pi_n = \bigoplus_{\underline{n} \xrightarrow{f} \underline{k}, k \geq n} Q(I)^{\boxtimes n} \circ Pr_f$$

where $Pr_f(V_1, ..., V_n) := (V_{f(1)}, ..., V_{f(n)})$.

If $f \notin \Sigma_n$, then as the the case $n = 2$, we have

$$Nat(Q(I)^{\boxtimes n} \circ Pr_f, I^{\boxtimes n}) \simeq 0.$$

Therefore we have

$$\begin{aligned} Nat(Q(I)^{\otimes n} \circ \Pi_n, I^{\boxtimes n}) &\simeq \bigoplus_{\Sigma_n} Nat(Q(I)^{\boxtimes n}, I^{\boxtimes n}) \\ &\simeq \bigoplus_{\Sigma_n} Nat(Q(I), I)^{\otimes n} \\ &\simeq \bigoplus_{\Sigma_n} \Bbbk = \Bbbk\Sigma_n \end{aligned}$$

□

Remark 4.27. *An analogous and equivalent statement of Lemma 4.20 is the following: If A and B are finite sets, then*

$$Nat(Q(I^{\otimes A}), I^{\otimes B}) \simeq \Bbbk[FinSet(A, B)]$$

where $FinSet(A, B)$ denotes the set of bijections from A to B.

CHAPTER 5

Taylor tower of simplicial functors

In this chapter, we will give an explicit and fairly computable formula for the Taylor tower $\{P_nF\}$ of a functor $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$. This is the context of Theorem 5.35 which roughly states that when F is simplicial and finitary, then

$$P_nF(X) \simeq B(\partial_{*\leq n}F, B^cB(\mathcal{O}), X) \tag{5.1}$$

Let us explain this formula in more detail. The external B is the bar-construction with coefficients and the cobar-bar $B^B(\mathcal{O})$ is the cofibrant replacement of $\mathcal{O}$. The left $B^cB(\mathcal{O})$-module structure on X is induced from the obvious left $\mathcal{O}$-module structure on the $\mathcal{O}$-algebra X.

For formula (5.1) to make, sense we also need to give a right $B^cB(\mathcal{O})$-module structure on the truncated derivatives

$$\partial_{*\leq n}F = (\partial_0F, ..., \partial_nF, 0, 0, ...)$$

In order to do so, we will prove in Theorem 5.10 the formula of the form

$$\partial_*F \simeq B(Nat(F\Omega^\infty I, I^{\otimes *}), B(\mathcal{O})^\vee, \mathbb{I})^\vee \tag{5.2}$$

which induces for general reason a strict $B^cB(\mathcal{O})$-module structure on the right hand side. Here $Nat(F\Omega^\infty I, I^{\otimes *})$ is the chain complex of natural transformation between the functors :

$$F\Omega^\infty I : Ch_+ \longrightarrow Ch, V \longmapsto F(\Omega^\infty I(V));$$

$$I^{\otimes n} : Ch_+ \longrightarrow Ch, V \longrightarrow V^{\otimes n},$$

To give sense to formula (5.2), we need a left $B(\mathcal{O})^\vee$-module structure on $Nat(F\Omega^\infty I, I^{\otimes *})$. This comes from the natural morphism of symmetric sequences

$$Nat(F\Omega^\infty I, I^{\otimes *}) \circ Nat(\Sigma^\infty\Omega^\infty I, I^{\otimes *}) \longrightarrow Nat(F\Omega^\infty I, I^{\otimes *}) \tag{5.3}$$

(see the map ϱ of Equation (5.18) in the proof of Proposition 5.15) which turns out to be associative because $\Sigma^\infty\Omega^\infty$ is a comonad, and a natural map

$$\lambda_* : B(\mathcal{O})^\vee \longrightarrow Nat(TI, I^{\otimes *}) \tag{5.4}$$

explained at Equation (5.13) (which ought to be an equivalence). Combining (5.3) and (5.4) gives the desired $B(\mathcal{O})^\vee$-module structure on $Nat(F\Omega^\infty I, I^{\otimes *})$.

Note that in this introduction we have limited our formulas in the special case where F if a finite cellular functor. The general formulas will be stated for general simplicial functors by taking a suitable filtered homotopy functors or if we can prove that the derivative is of finite type.

Let us now explain the strategy of our proofs. The proof of Theorems 5.10 and 5.35 will pass through a first study of functors of chain complexes

$$F : Ch_+ \longrightarrow Ch$$

Indeed in §5.1, for such functor we will associate the functor

$$\Psi_n F : Ch_+ \longrightarrow Ch, W \longmapsto \underline{hom}(Nat(F, I^{\otimes n}), W^{\otimes n})^{\Sigma_n}$$

and we will prove that the natural evaluation map

$$F \longrightarrow \Psi_n F$$

induces an equivalence on the n-th layers, hence we will get the formula

$$\partial_* F \simeq Nat(F, I^{\otimes *})^\vee$$

We will prove this formula by proving this first on representable functors and our computation of derivatives from Chapter 2. We will then infer this for any cellular functors by standard arguments. With this new formula for the derivatives, we will deduce our formula for the derivatives of functors $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$ by taking the standard cosimplicial resolution of F through $F\Omega^\infty$ and the comonad $\Sigma^\infty\Omega^\infty$ as explained in §5.3.3

For the proof of Theorem 5.35, we will compare the right hand side of Equation (5.1) with the "fake Taylor tower" inspired by Arone-Ching and defined as (see Lemma 5.27)

$$\Phi_n F(X) \simeq Map^{right}_{B(\mathcal{O})^\vee}(B(\partial^* F\Omega^\infty, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee), (\Sigma^\infty X)^{\otimes * \leq n}).$$

In this chapter, our based operad $\mathcal{O}$ is again a reduced operad on Ch_+. Since the dual of the bar construction $B(\mathcal{O})$, that we denote $B(\mathcal{O})^\vee$, is not always a cooperad, we will assume that our operad $\mathcal{O}$ is aritywise finite dimensional. That is $\forall n, dim(\mathcal{O}(n)) < \infty$. In many of our construction, we make a use of Proposition 4.20 thus we will consider that the ground field $\Bbbk$ is of characteristic 0. The functors $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$ that we consider here are all homotopy functors (preserve weak equivalences). Note that simplicial functors $F : Ch_+ \longrightarrow Ch$ are automatically homotopy functors (see Remark 4.18), and the simplicial functors $F : \mathrm{Alg}^{fin}_{\mathcal{O}} \longrightarrow Ch$ are also homotopy functors (see Lemma 4.19).

The chapter has the following guidelines:

- In §5.1, we give a new model for the Goodwillie derivatives of simplicial functors $F : Ch_+ \longrightarrow Ch$. This new model is expressed in term of the enriched natural transformations described in Definition 4.4.

- In §5.2, we prove that the new model for the derivatives described in §5.1, has a chain rule property.

- In §5.3, we give a new model for the derivatives of functors $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$ and we show that this model has a natural module structure over the operad $B^c(B(\mathcal{O}))$.

- We describe in §5.4, the Taylor tower of simplicial functors. This is the main goal of the chapter, and we use in this description the main results that we have developed throughout the other sections of this chapter.

- Finally in §5.5, we compute as example the Taylor tower of two functors: the representable functor and the forgetful functor

$$\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, -), IU(-) : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch.$$

5.1 D_n-approximation of functors $F : Ch_+ \longrightarrow Ch$

In this section, we give a D_n-approximation of presented cell functors $F : Ch_+ \longrightarrow Ch$. More precisely, we define $\forall n$ a functor $\Psi_n F$ such that:

- The derivatives of $\Psi_n F$ are easily computable;

- There is a natural transformation $\psi_F : F \longrightarrow \Psi_n F$ which is a D_n-equivalence. Namely, $D_n\psi_F : D_n F \longrightarrow D_n \Psi_n F$ is a weak equivalence.

Definition 5.1. *Let $F : Ch_+^{fin} \longrightarrow Ch$ be a presented cell functor. Then we define the functor $\Psi_n F$ by*

$$\Psi_n F(W) := \underset{C \in Sub(F)}{hocolim}\, \underline{hom}(Nat(C, I^{\otimes n}), W^{\otimes n})^{\Sigma_n} \tag{5.5}$$

where $Sub(F)$ is the set of finite subcomplexes of F.

let $F : Ch_+ \longrightarrow Ch$ be a presented cell functor and $C \in Sub(F)$. We define the morphism

$$\psi_C : C(W) \longrightarrow \underline{hom}(Nat(C, I^{\otimes n}), W^{\otimes n})^{\Sigma_n}$$

to be the adjoint of the evaluation(equivariant) map :

$$C(W) \otimes Nat(C, I^{\otimes n}) \longrightarrow W^{\otimes n}$$

We have proved in Proposition 4.16 that $F \simeq \underset{C \in Sub(F)}{\mathrm{hocolim}}\, C$, thus we can define the morphism

$$\psi_F : F(W) \simeq \underset{C \in Sub(F)}{\mathrm{hocolim}}\, C(W) \longrightarrow \Psi_n F(W),$$

which is induced by the morphisms ψ_C, and that we simply denote by

$$\psi_F : F \longrightarrow \Psi_n F.$$

The main result of this section is the following proposition.

Proposition 5.2. *We assume char($\Bbbk$)=0. If $F : Ch_+^{fin} \longrightarrow Ch$ is a presented cell functor, then the morphism*

$$\psi_F : F \longrightarrow \Psi_n F$$

is a D_n-equivalence.

A consequence of Proposition 5.2 is that we get an expression of derivatives of simplicial functors in term of natural transformations. Given a simplicial functor $F : Ch_+ \longrightarrow Ch$, we denote by QF the cofibrant replacement of its restriction in the category $[Ch_+^{fin}, Ch]$.

Proposition 5.3 (Model for derivatives)**.** *We assume char($\Bbbk$)=0. If F : $Ch_+ \longrightarrow Ch$ is a simplicial functor, then a model for the Goodwillie derivatives of F is given by:*

$$\partial_* F \simeq \underset{C \in Sub(QF)}{hocolim} \underline{hom}(Nat(C, I^{\otimes *}), \Bbbk)$$

where $Sub(QF)$ denotes the category of finite subcomplexes of QF.

Proof of Proposition 5.3. If $F : Ch_+ \longrightarrow Ch$ is a simplicial functor, then its restriction to Ch_+^{fin} that we abusively denote F has a cofibrant replacement $F \xleftarrow{\simeq} QF$, where QF has a presented cell structure. Using Proposition 5.2, we have

$$\partial_n F \xleftarrow{\simeq} \partial_n QF \xrightarrow{\simeq} \partial_n \Psi_n QF.$$

On the other hand, we proved in Proposition 2.43 that

$$\begin{aligned}\partial_n \Psi_n QF &\simeq \underset{C \in Sub(QF)}{\operatorname{hocolim}} \partial_n \Psi_n C \\ &\simeq \underset{C \in Sub(QF)}{\operatorname{hocolim}} \underline{hom}(Nat(C, I^{\otimes *}), \Bbbk)\end{aligned}$$

Therefore we get the result. □

Proof of Proposition 5.2. The proof is done in 3 steps. We address the cases where F is a representable functor, a finite presented cell functor and an arbitrary presented cell functor respectively.

1. We consider $C = W_0 \otimes \widetilde{Ch}_+(V, -) : Ch_+^{fin} \longrightarrow Ch$, where $V, \in Ch_+^{fin}$, $W_0 \in Ch^{fin}$ and we want to prove that the morphism

$$\psi_C : C(W) \longrightarrow \underline{hom}(Nat(C, I^{\otimes n}), W^{\otimes n})^{\Sigma_n}$$

is a D_n-equivalence. Note that C is a homotopy functor from §4.4. Using the characterization of the layers $D_n C$ in Corollary 2.22, it will be sufficient to prove that the morphism

$$\widehat{\triangle}_n(\psi_C) : \widehat{\triangle}_n C(\Bbbk) \longrightarrow \widehat{\triangle}_n(\underline{hom}(Nat(C, I^{\otimes n}), I^{\otimes n})^{\Sigma_n})(\Bbbk)$$

is a quasi-isomorphism, where $\widehat{\triangle}_n C$ is given in Definition 2.30 as the stabilization of the cross effect $cr_n C$.

We start by computing the cross effect of the source and the target of ψ_C.

(a) For the source of ψ_C, since C takes values in Ch, by Lemma 2.40 we can replace the cross-effect by the co-cross-effect of Definition 2.39. Moreover, since $W_0 \otimes - : Ch \longrightarrow Ch$ is a Quillen left adjoint it commutes with hocolims, and since the co-cross-effect is an iterated homotopy colimit, we make the computation:

$$\begin{aligned} cr_n(W_0 \otimes \widetilde{Ch}_+(V, -))(W_1, ..., W_n) &\simeq cr^n(W_0 \otimes \widetilde{Ch}_+(V, -))(W_1, ..., W_n) \\ &\simeq W_0 \otimes cr^n(\widetilde{Ch}_+(V, -))(W_1, ..., W_n) \end{aligned}$$

Using Equation (2.32) in the proof of Proposition 2.44, we deduce that:

$$\begin{gathered} cr_n(W_0 \otimes \widetilde{Ch}_+(V, -))(W_1, ..., W_n) \simeq \\ W_0 \otimes N\widetilde{\Bbbk Hom}_{Ch_+}(V \otimes \triangle^\bullet, W_1) \otimes ... \otimes N\widetilde{\Bbbk Hom}_{Ch_+}(V \otimes \triangle^\bullet, W_n). \end{gathered}$$

(b) For the target of ψ_C, we write for short $E = Nat(C, I^{\otimes n})$ which is a chain complex with a Σ_n-action. Note that E is finite since we have by the Yoneda Lemma the isomorphism

$$\begin{aligned} E &\cong Nat(W_0 \otimes \widetilde{Ch}_+(V, -), I^{\otimes n}) \\ &\cong \underline{hom}(W_0, V^{\otimes n}). \end{aligned}$$

We then use again Lemma 2.40 as in (a) to make the following computation:

$$\begin{aligned} cr_n(\underline{hom}(E, I^{\otimes n})^{\Sigma_n})(W_1, ..., W_n) &\simeq cr^n(\underline{hom}(E, I^{\otimes n})^{\Sigma_n})(W_1, ..., W_n) \\ &\simeq cr^n(E^\vee \underset{\Sigma_n}{\otimes} I^{\otimes n})(W_1, ..., W_n) \\ &\simeq E^\vee \underset{\Sigma_n}{\otimes} cr^n(I^{\otimes n})(W_1, ..., W_n) \\ &= E^\vee \underset{\Sigma_n}{\otimes} [\bigoplus_{\sigma \in \Sigma_n} W_{\sigma(1)} \otimes ... \otimes W_{\sigma(n)}] \\ &\simeq E^\vee \otimes W_1 \otimes ... \otimes W_n \\ &\simeq \underline{hom}(E, W_1 \otimes ... \otimes W_n) \\ &\simeq \underline{hom}(\underline{hom}(W_0, V^{\otimes n}), W_1 \otimes ... \otimes W_n). \end{aligned}$$

(c) The map $cr_n \psi_C$ is equivalent to $cr^n \psi_C$ which is, by the above computations, equivalent to the composite:

$$\begin{array}{c}
W_0 \otimes N\widetilde{\Bbbk}Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, W_1) \otimes ... \otimes N\widetilde{\Bbbk}Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, W_n) \\
\Big\downarrow \\
W_0 \otimes \underline{hom}(V, W_1) \otimes ... \otimes \underline{hom}(V, W_n) \\
\Big\downarrow \cong \\
\underline{hom}(\underline{hom}(W_0, V^{\otimes n}), W_1 \otimes ... \otimes W_n)
\end{array} \tag{5.6}$$

where the first vertical map is given by the projection

$$N\widetilde{\Bbbk}Hom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, -) \longrightarrow NHom_{Ch_+}(V \otimes N\Bbbk\triangle^\bullet, -) \longrightarrow \underline{hom}(V, -)$$

and the second map is the composite of the adjoint of the obvious evaluation maps. Note that this later map is an isomorphism since $W_i, i = 0, ..., n$ and V are finite dimensional chain complexes.
In addition, we showed at the end of the proof of Proposition 2.44 that, when we replace $W_i = s^{p_i}\Bbbk$ and apply the functor $\operatorname{hocolim}_{p\to\infty}$ to the projection map

$$\begin{aligned}
s^{-p}N\widetilde{\Bbbk}Hom_{Ch_+}(V \otimes \triangle^\bullet, s^p\Bbbk) &\longrightarrow s^{-p}NHom_{Ch_+}(V \otimes \triangle^\bullet, s^p\Bbbk) \\
&\longrightarrow s^{-p}NHom_{Ch_+}(\triangle^\bullet, \underline{hom}(V, s^p\Bbbk)) \\
&\longrightarrow s^{-p}\underline{hom}(V, s^p\Bbbk)
\end{aligned}$$

we get a quasi-isomorphism when $char(\Bbbk) = 0$:

$$\operatorname*{hocolim}_{p\to\infty} s^{-p}N\Bbbk Hom_{Ch_+}(V \otimes \triangle^\bullet, s^p\Bbbk) \xrightarrow{\simeq} \underline{hom}(V, \Bbbk).$$

Applying this to each factor of the source of the map in (5.6), we get a quasi-isomorphism

$$\widehat{\triangle}_n C(\Bbbk) \simeq W_0 \otimes \underline{hom}(V, \Bbbk)^{\otimes n}$$

This completes the proof that $\widehat{\triangle}_n(\psi_C)$ is a quasi-isomorphism.

2. Now consider the following single cell attachment:

$$\begin{tikzcd}
W_0 \otimes \widetilde{Ch}_+(V, -) \arrow[r] \arrow[d] & C' \arrow[d] \\
W_1 \otimes \widetilde{Ch}_+(V, -) \arrow[r] & C
\end{tikzcd} \tag{5.7}$$

This is also an objectwise (homotopy) pushout diagram. We apply the functor $Nat(-, I^{\otimes n})$ to this diagram, we get the (homotopy) pullback diagram in Ch :

$$\begin{tikzcd} Nat(C, I^{\otimes n}) \arrow[r] \arrow[d] & \underline{hom}(W_1, V^{\otimes n}) \arrow[d] \\ Nat(C', I^{\otimes n}) \arrow[r] & \underline{hom}(W_0, V^{\otimes n}) \end{tikzcd}$$

The above diagram is again a (homotopy) pushout since the category Ch is stable. Therefore we apply $\underline{hom}(-, W^{\otimes n})$ to this diagram and get the (homotopy) pullback diagram:

$$\begin{tikzcd} \underline{hom}(\underline{hom}(W_0, V^{\otimes n}), W^{\otimes n}) \arrow[r] \arrow[d] & \underline{hom}(Nat(C', I^{\otimes n}), W^{\otimes n}) \arrow[d] \\ \underline{hom}(\underline{hom}(W_1, V^{\otimes n}), W^{\otimes n}) \arrow[r] & \underline{hom}(Nat(C, I^{\otimes n}), W^{\otimes n}) \end{tikzcd} \tag{5.8}$$

We have a map between the two (homotopy) pushout diagrams (5.7) and (5.8) induced by the morphisms ψ_F.

3. Since the functor D_n commutes with homotopy pushouts (for chain complex valued functors), we deduce that if $\psi_{C'}$ is a D_n-equivalence then so is ψ_C. In conclusion, the proof that ψ_C is a D_n-equivalence, for any arbitrary finite presented cell functor C, follows by induction.

□

Corollary 5.4. *If $F : Ch_+ \longrightarrow Ch$ is a cellular functor, then there is a quasi-isomorphism*

$$(\partial_* F)^\vee \simeq Nat(F, I^{\otimes *})$$

Proof. We know from Proposition 4.16-(1) that $QF \cong \operatorname{colim}_{C \in Sub(QF)} C$ so we deduce

$$Nat(QF, I^{\otimes *}) \cong \lim_{C \in Sub(QF)} Nat(C, I^{\otimes *})$$

On the other hand, given any finite cell functor C, the chain complex $Nat(C, I^{\otimes *})$ is finite and we write

$$Nat(C, I^{\otimes *}) \cong \underline{hom}(\underline{hom}(Nat(C, I^{\otimes *})$$

We then deduce

$$\begin{aligned} Nat(QF, I^{\otimes *}) &\cong \lim_{C \in Sub(QF)} \underline{hom}(\underline{hom}(Nat(C, I^{\otimes *}), \Bbbk), \Bbbk) \\ &\cong \underline{hom}(\operatorname{colim}_{C \in Sub(QF)} \underline{hom}(Nat(C, I^{\otimes *}), \Bbbk), \Bbbk) \\ &\simeq \underline{hom}(\partial_* F, \Bbbk) \end{aligned}$$

□

Corollary 5.5. *Let $F : Ch_+ \longrightarrow Ch$ is a cellular functor. If $\partial_* F$ or $Nat(F, I^{\otimes *})$ is of finite type then, there is a quasi-isomorphism*

$$\partial_* F \simeq Nat(F, I^{\otimes *})^\vee$$

Proof. This result is a consequence of Corollary 5.4 and the fact that any chain complex of finite type is quasi-isomorphic to its bi-dual. □

We will often use Corollary 5.5 to compute the derivatives of the tensor powers of the inclusion $I : Ch_+ \longrightarrow Ch$.

Remark 5.6.

1. *The result of Proposition 4.20 says that the chain complex $Nat(QI^{\otimes n}, I^{\otimes m})$ is of finite type $\forall n, m \geq 0$. Then using Corollary 5.5 , we have the quasi-isomorphism*

$$\partial_* I^{\otimes n} \simeq \underline{hom}(Nat(QI^{\otimes n}, I^{\otimes *}), \Bbbk).$$

2. *We have an analogous result for the comonad $T := \Sigma^\infty \Omega^\infty : Ch \longrightarrow Ch$ or more precisely the composite*

$$TI = \bigoplus_n B(\mathcal{O})(n) \underset{\Sigma_n}{\otimes} I^{\otimes n} : Ch_+ \longrightarrow Ch.$$

We have shown in Proposition 2.42 that $\partial_* T \simeq B(\mathcal{O})$ which is of finite type. On the other hand, it is straightforward that $\partial_* T \simeq \partial_* TI$. Hence we deduce from Corollary 5.5 that there is a quasi-isomorphism

$$\partial_* TI \simeq \underline{hom}(Nat(QTI, I^{\otimes *}), \Bbbk) \tag{5.9}$$

5.2 Chain rule property on derivatives

This section is dedicated to give the chain rule property for the new model of derivatives established in Proposition 5.3. The main result is the following:

Proposition 5.7 (Chain rule for simplicial functors in Ch)**.** *We assume char($\Bbbk$)=0. Let $F, G : Ch \longrightarrow Ch$ be two simplicial functors, with F finitary. Then there is a zig-zag of weak equivalences of symmetric sequences*

$$\partial_*(F.G) \simeq \partial_* F \circ \partial_* G$$

For the proof of this result, we define a pro-object $\partial^* F$ (associated to each functor F), and we remind the next elementary lemma which appears in [Chi10, §6] in the context of spectra and in [AC11, Prop. 3.1], in a little more general setting.

Lemma 5.8. *If $Ch \xrightarrow{G} Ch \xrightarrow{F} Ch$ are two homotopy functors, then $\forall n \geq 0$, the natural map*

$$P_n(FG) \xrightarrow{P_n(p_n F)} P_n((P_n F)G)$$

which is induced by $p_n F : F \longrightarrow P_n F$, is an equivalence.

Proof. Let $X \in Ch$ and U be a finite subset of $\underline{n}$. The composite

$$G(X) \longrightarrow \coprod_U G(X) \longrightarrow G(\coprod_U X)$$

induces

$$G(X) \longrightarrow G(X) * U \longrightarrow G(X * U)$$

and we deduce $t_n(FG)$: $FG \xrightarrow{t_nF} (T_nF)G \longrightarrow T_n(FG)$, and in general ,$\forall k$, we have the map(natural in k):

$$t_n^k(FG) : \; T_n^k(FG) \xrightarrow{T_n^k(t_nF)} T_n^k((T_nF)G) \longrightarrow T_n^{k+1}(FG)$$

When we apply $\underset{k}{hocolim}(-)$ to the $t_n^k(FG)$, we get:

$$P_n(FG) \xrightarrow{P_n(t_nF)} P_n((T_nF)G) \xrightarrow{u(F,G)} P_n(FG)$$

Note that by construction $u(F,G) \circ P_n(t_nF)$ is equivalent to the identity $1 : P_n(FG) \longrightarrow P_n(FG)$ and we denote it by $u(F,G) \circ P_n(t_nF) \sim 1$. An iterated version of this construction gives the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & P_n(T_n^{k+1}FG) \arrow[r, "u(T_n^kF,G)"] & P_n((T_n^kF)G) \arrow[r] & \ldots \arrow[r, "u(F,G)"] & P_n(FG) \\
& P_n(T_n^kFG) \arrow[u, "P_n(t_nT_n^kF)"] & P_n(T_n^{k-1}FG) \arrow[u, "P_n(t_nT_n^{k-1}F)"] & & \\
& P_n(T_n^{k-1}FG) \arrow[u, "P_n(t_nT_n^{k-1}F)"] & \ldots \arrow[u, "P_n(t_nT_n^{k-2}F)"] & & \\
& \ldots \arrow[u, "P_n(t_nT_n^{k-2}F)"] & & &
\end{tikzcd}$$

where, $\forall k$, we have $u(T_n^kF,G) \circ P_n(t_nT_n^kF) \sim 1$. Therefore the maps $u(T_n^kF,G)$ induce a map $v_n(F,G) : P_n(P_n(F)G) \longrightarrow P_n(FG)$ so that

$$v_n(F,G) \circ P_n(p_nF) \sim 1.$$

Now we want to prove the inverse equivalence $P_n(p_nF) \circ v_n(F,G) \sim 1$. For this we consider the following commutative diagram

$$\begin{tikzcd}
P_n(P_n(F)G) \arrow[r, "v_n(F,G)"] \arrow[d, "P_n(p_nP_nF)"] & P_n(FG) \arrow[d, "P_n(p_nF)"] \\
P_n(P_n^2(F)G) \arrow[r, "v_n(P_nF,G)"'] & P_n(P_n(F)G)
\end{tikzcd}$$

Since $v_n(P_n(F), G) \circ P_n(p_n P_n F) \sim 1$, we deduce that

$$P_n(p_n F) \circ v_n(F, G) \sim 1.$$

$\square$

Definition 5.9 (Pro object $\partial^* F$)**.** *If $F : Ch_+^{fin} \longrightarrow Ch$ is a presented cell functor, then there is a pro-symmetric sequence of chain complexes, denoted $\partial^* F$, and given by*

$$\partial^* F := \{Nat(C, I^{\otimes *})\}_{C \in Sub(F)}$$

Proof of Proposition 5.7. For the proof of this result, we will firstly define the two maps involved in the zig-zag, and secondly prove that each of these maps is a weak equivalence.

Let $F, G : Ch \longrightarrow Ch$ be two simplicial functors. If E is a finite subcomplex of $Q(QF.QG)$, then using Proposition 4.13, the cofibrant resolution $\alpha : Q(QF.QG) \xrightarrow{\simeq} QF.QG$ restricts to a natural transformation $E \longrightarrow CD$, where $C \in Sub(QF)$ and $D \in Sub(QG)$. We then deduce the composite

$$\begin{aligned} Nat(C, I^{\otimes *}) \circ Nat(D, I^{\otimes *}) &\longrightarrow Nat(CD, D^{\otimes *}) \circ Nat(D, I^{\otimes *}) \\ &\longrightarrow Nat(CD, I^{\otimes *}) \\ &\longrightarrow Nat(E, I^{\otimes *}) \end{aligned}$$

which produces the morphism of pro- symmetric sequences:

$$\mu^* : \partial^* QF \circ \partial^* QG \longrightarrow \partial^* Q(QF.QG)$$

The continuous dual of this morphism gives the morphism of chain complexes

$$\mu_* : \partial_*(QF.QG) \longrightarrow \partial_* F \circ \partial_* G$$

Remark that this construction of μ_* is natural in F and G. On the other hand, If F is finitary then the cofibrant resolution $QF \longrightarrow F$ in $[Ch^{fin}, Ch]_0$ produces the weak equivalences $QF(X) \xrightarrow{\simeq} F(X)$, for any $X \in Ch$. In particular we have the weak equivalence $QF.QG \xrightarrow{\simeq} F.QG$. In addition, we also get the weak equivalence $F.QG \xrightarrow{\simeq} FG$ since F is finitary and preserves weak equivalences (see § 4.4). In summary we form the composite

$$QF.QG \xrightarrow{\simeq} F.QG \xrightarrow{\simeq} FG,$$

and the zig-zag

$$\partial_*(F.G) \xleftarrow{\simeq} \partial_*(QF.QG) \xrightarrow{\mu_*} \partial_* F \circ \partial_* G$$

At this point, it remains to prove that μ_* is a weak equivalence.

Let us consider the diagram

$$\begin{tikzcd} \partial_n(F.G) \arrow[r, "\mu_*"] \arrow[d, "\simeq"] & (\partial_* F \circ \partial_* G)(n) \arrow[d, "\simeq"] \\ \partial_n(P_n F.G) \arrow[r, "\mu_*"] & (\partial_* P_n F \circ \partial_* G)(n) \end{tikzcd}$$

where the vertical maps are induced by the natural transformation $p_nF : F \longrightarrow P_nF$; The most left vertical map is an equivalence using Lemma 5.8 and the fact that , for a given functor $F, \partial_n(P_nF) \simeq \partial_nF$. The right vertical quasi-isomorphism is justified by the fact that: $\forall k \leq n, \partial_kP_nF \simeq \partial_kF$.

One deduce from this diagram that our proof reduces in proving that the bottom horizontal map is a quasi-isomorphism. At this point we remark that the functor $\partial_*(-)$ respects fiber sequences and in particular the fiber sequence

$$D_nF \longrightarrow P_nF \longrightarrow P_{n-1}F$$

Therefore we only need to prove that we have the quasi-isomorphism

$$\partial_n(D_nF.G) \xrightarrow{\mu_*} (\partial_*D_nF \circ \partial_*G)(n)$$

or more generally to prove , that given a k-homogeneous functor $F(V) = E \underset{\Sigma_k}{\otimes} V^{\otimes k}$, we have the quasi-isomorphism

$$E \underset{\Sigma_k}{\otimes} \partial_n G^{\otimes k} \xrightarrow{\mu_*} \underset{\underline{n}\twoheadrightarrow\underline{k}}{\oplus} E \underset{\Sigma_k}{\otimes} \partial_{\underline{n_1}}G \otimes ... \otimes \partial_{\underline{n_k}}G$$

where

- $\partial_{\underline{n_i}}(-) := \partial_{n_i}(-), \forall i$;
- The direct sum on the right hand side is under the set of all surjections $\underline{n} \twoheadrightarrow \underline{k}$, with k fixed;
- Each sequence $\underline{n_1}, ..., \underline{n_k}$ is the partition of $\underline{n}$ obtained by a surjection $\underline{n} \twoheadrightarrow \underline{k}$.

In more general case, given k functors $G_1, ..., G_k : Ch \longrightarrow Ch$, we want to prove that the following map is a quasi-isomorphism

$$\partial_n(G_1 \otimes ... \otimes G_k) \xrightarrow{\mu_*} \underset{\underline{n}\twoheadrightarrow\underline{k}}{\oplus} \partial_{\underline{n_1}}G_1 \otimes ... \otimes \partial_{\underline{n_k}}G_k$$

Again, since the functor $\partial_*(-)$ commutes with fibration sequence, we can reduce to the case $G_i(V) := C_i \underset{\Sigma_{m_i}}{\otimes} V^{\otimes \underline{m_i}}$, and the map μ_* is equivalent in this case to

$$C \underset{\Sigma_{m_1}\times...\times\Sigma_{m_k}}{\otimes} \partial_n I^{\otimes \underline{m}} \xrightarrow{\mu_*} \underset{\underline{n}\twoheadrightarrow\underline{k}}{\oplus} C \underset{\Sigma_{m_1}\times...\times\Sigma_{m_k}}{\otimes} \partial_{\underline{n_1}} I^{\otimes \underline{m_1}} \otimes ... \otimes \partial_{\underline{n_k}} I^{\otimes \underline{m_k}}$$

where $C = C_1 \otimes ... \otimes C_k$.

By Remark 5.9, the dual of the map

$$\partial_n I^{\otimes \underline{m}} \xrightarrow{\mu_*} \underset{\underline{n}\twoheadrightarrow\underline{k}}{\oplus} \partial_{\underline{n_1}} I^{\otimes \underline{m_1}} \otimes ... \otimes \partial_{\underline{n_k}} I^{\otimes \underline{m_k}}$$

is equivalent to the morphism

$$\underset{\underline{n}\twoheadrightarrow\underline{k}}{\oplus} Nat(QI^{\otimes \underline{m_1}}, I^{\otimes \underline{n_1}}) \otimes ... \otimes Nat(QI^{\otimes \underline{m_k}}, I^{\otimes \underline{n_k}}) \longrightarrow Nat(QI^{\otimes \underline{m}}, I^{\otimes \underline{n}}) \quad (5.10)$$

Now using Remark 4.27, we re-write Equation (5.10) as the composition map

$$\bigoplus_{\underline{n}\twoheadrightarrow\underline{k}} \Bbbk[FinSet(\underline{m_1},\underline{n_1})\times ... \times FinSet(\underline{m_k},\underline{n_k})] \longrightarrow \Bbbk[FinSet(\underline{m},\underline{n})] \tag{5.11}$$

and since this is an isomorphism, we are done.

□

5.3 New model for derivatives of functors $F:\mathbf{Alg}_{\mathcal{O}}\longrightarrow Ch$

In this section, we give a new model for the Goodwillie derivatives of simplicial functors $F:\mathrm{Alg}_{\mathcal{O}}\longrightarrow Ch$. The goal is to establish the following theorem.

Theorem 5.10. *We assume char($\Bbbk$)=0. If $F: Alg_{\mathcal{O}}\longrightarrow Ch$ is a simplicial finitary functor, then there is a quasi-isomorphism*

$$\partial_* F \simeq \underset{C\in Sub(QF)}{hocolim}\, \underline{hom}(B(Nat(C\Omega^\infty I, I^{\otimes *}), B(\mathcal{O})^\vee, \mathbb{I}), \Bbbk) \tag{5.12}$$

where $\mathbb{I}(J)=\Bbbk$ *if* $|J|=1$ *and* $\mathbb{I}(J)=0$ *otherwise.*

We remind that $I: Ch_+\longrightarrow Ch$ denotes the embedding functor.

Note that filtered colimits and filtered homotopy colimits of chain complexes are equivalent since homology of chain complexes commutes with filtered colimits. Therefore the hocolim of Theorem 5.10 can be replaced by a strict colimit.

The straight consequence of this result is the module structure that it endows on the derivatives $\partial_* F$.

Corollary 5.11. *Equation (5.12) of Theorem 5.10 endows the symmetric sequence $\partial_* F$ with a structure of $B^cB(\mathcal{O})$-module.*

Proof. The proof is straightforward from Theorem 5.10. We use in fact the usual decomposition of the bar-construction. This shows in fact that

$$B(Nat(C\Omega^\infty I, I^{\otimes *}), B(\mathcal{O})^\vee, \mathbb{I})$$

is a right $BB^c(\mathcal{O}^\vee)$-comodule. Then we take the linear dual of this comodule structure map

$$B(Nat(C\Omega^\infty I, I^{\otimes *}), B(\mathcal{O})^\vee, \mathbb{I}) \longrightarrow B(Nat(C\Omega^\infty I, I^{\otimes *}), B(\mathcal{O})^\vee, \mathbb{I})\circ BB^c(\mathcal{O}^\vee)$$

to get the module structure map

$$B(Nat(C\Omega^\infty I, I^{\otimes *}), B(\mathcal{O})^\vee, \mathbb{I})^\vee \circ B^cB(\mathcal{O}) \longrightarrow B(Nat(C\Omega^\infty I, I^{\otimes *}), B(\mathcal{O})^\vee, \mathbb{I})^\vee.$$

□

The rest of this section is based on the proof of Theorem 5.10. We have divided the proof into three major steps:

1. We first prove that, if $F:\mathrm{Alg}_{\mathcal{O}}^{fin}\longrightarrow Ch$ is a presented cell functor then $\partial^*(F\Omega^\infty) := \{Nat(C\Omega^\infty I, I^{\otimes *})\}_{C\in Sub(F)}$ is a pro-right-module over $B(\mathcal{O})^\vee$.

2. Given a simplicial functor $F : \mathrm{Alg}_{\mathcal{O}}^{fin} \longrightarrow Ch$, we build an associated cosimplicial functor $Res^{\bullet}(F)$ whose the totalization is equivalent to F. The terms of this resolution are built with functors in $[Ch_{+}^{fin}, Ch]_0$.

3. Finally, we will build a so-called D_n-approximation of F. This follows an analogous idea of § 5.1. In fact, we will construct a functor $\Phi_n F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$ along with a natural transformation $\psi_F : F \longrightarrow \Phi_n F$ associated to F such that:

   - The derivatives of $\Phi_n F$ are easily computable;
   - The natural transformation $\psi_F : F \longrightarrow \Phi_n F$ is a D_n-equivalence. Namely $D_n \psi_F : D_n F \longrightarrow D_n \Phi_n F$ is a weak equivalence.

   This last assertion will use the cosimplicial resolution $Res^{\bullet}(F)$ of F and the D_n-approximation developed in Section 5.1 for functors $Ch_{+} \longrightarrow Ch$. Finally we will obtain Theorem 5.10 by computing the derivative of $\Phi_n F$.

5.3.1 Module structure on $\partial^* F\Omega^\infty$

Definition 5.12. *If $F : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ is a presented cell functor, then we denote by $\partial^*(F\Omega^\infty)$ the pro-symmetric sequence of chain complexes*

$$\partial^*(F\Omega^\infty) := \{Nat(C\Omega^\infty I, I^{\otimes *})\}_{C \in Sub(F)}.$$

We denote by T the comonad $T = (\Sigma^\infty\Omega^\infty, m_T, \varepsilon_T)$, where the coproduct

$$m_T : T \longrightarrow TT \text{ (resp. the co-unit } \varepsilon_T : T \longrightarrow 1 \text{)}$$

is induced naturally by the cooperad coproduct

$$m^c : B(\mathcal{O}, -) \longrightarrow B(\mathcal{O}) \circ B(\mathcal{O}, -) \cong B(\mathcal{O}, -)\Omega^\infty\Sigma^\infty$$

$$\text{(resp. co-unit } \varepsilon : B(\mathcal{O}) \longrightarrow 1).$$

Given $F : \mathrm{Alg}_{\mathcal{O}}^{fin} \longrightarrow Ch$ a presented cell functor, the right module structure that we want to construct on $\partial^* F\Omega^\infty$ arises from a right T-comodule structure on $F\Omega^\infty$. This later structure is detailed in the next lemma.

Proposition 5.13. *Let $F : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ be a presented cell functor and T be the comonad $T = (\Sigma^\infty\Omega^\infty, m_T, \varepsilon_T)$. Then the functor $F\Omega^\infty$ is a right T-comodule, with the structure map denoted $\eta : F\Omega^\infty \longrightarrow F\Omega^\infty T$.*

To prove this theorem, we will need the next lemma which uses the natural $B(\mathcal{O})$-coproduct m^c.

Lemma 5.14. *Given $X \in Alg_{\mathcal{O}}$, there exists a natural map of chain complexes*

$$\widetilde{Alg}_{\mathcal{O}}(X, -) \xrightarrow{\delta} \widetilde{Alg}_{\mathcal{O}}(X, -)\Omega^\infty\Sigma^\infty$$

such that the following diagram commutes

$$\begin{tikzcd}
\widetilde{Alg}_{\mathcal{O}}(X,-) \arrow[r, "\delta"] \arrow[d, "\delta"'] & \widetilde{Alg}_{\mathcal{O}}(X,-)\Omega^\infty\Sigma^\infty \arrow[d, "Alg_{\mathcal{O}}(X,-)\Omega^\infty m^c"] \\
\widetilde{Alg}_{\mathcal{O}}(X,-)\Omega^\infty\Sigma^\infty \arrow[r, "\delta\Omega^\infty\Sigma^\infty"'] & \widetilde{Alg}_{\mathcal{O}}(X,-)\Omega^\infty\Sigma^\infty\Omega^\infty\Sigma^\infty
\end{tikzcd}$$

Proof. Let $Y \in \mathrm{Alg}_{\mathcal{O}}$. We have the isomorphism

$$\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,Y) \cong N\Bbbk Hom_{\mathrm{coAlg}_{B(\mathcal{O})}}(B(\mathcal{O},X), B(\mathcal{O}, red_0(Y \otimes Apl_\bullet)))$$

The multiplication of the element of the algebra $Apl_\bullet$ on the leaves of trees gives the map (defined in §3.1.2 – (I))

$$\sigma : UB(\mathcal{O}, red_0(Y \otimes Apl_\bullet)) \longrightarrow red_0(IUB(\mathcal{O},Y) \otimes Apl_\bullet).$$

Using the map σ and the coalgebra comultiplication m^c, we make the following computation

$$\begin{aligned}
B(\mathcal{O}, red_0(Y \otimes Apl_\bullet)) &\xrightarrow{m^c} B(\mathcal{O}) \circ B(\mathcal{O}, red_0(Y \otimes Apl_\bullet)) \\
&\xrightarrow{\cong} B(\mathcal{O})(UB(\mathcal{O}, red_0(Y \otimes Apl_\bullet))) \\
&\xrightarrow{\sigma} B(\mathcal{O})(red_0(IUB(\mathcal{O},Y) \otimes Apl_\bullet)) \\
&\xrightarrow{\cong} B(\mathcal{O}, (red_0(IUB(\mathcal{O},Y) \otimes Apl_\bullet))_{triv})
\end{aligned}$$

On the other hand, we have the equivalences of $\mathcal{O}$-algebras

$$\begin{aligned}
red_0(IUB(\mathcal{O},\Omega^\infty V) \otimes Apl_\bullet))_{triv} &\cong [red_0((Ired_0)IUB(\mathcal{O},\Omega^\infty V) \otimes Apl_\bullet)]_{triv} \\
&\cong red_0(I(red_0 IUB(\mathcal{O},Y))_{triv} \otimes Apl_\bullet) \\
&\cong red_0(I\Omega^\infty\Sigma^\infty(Y) \otimes Apl_\bullet)
\end{aligned}$$

Therefore the map δ follows.

Finally, since the multiplication $m : Apl_\bullet \otimes Apl_\bullet \longrightarrow Apl_\bullet$ is associative and that $m^c : B(\mathcal{O},-) \longrightarrow B(\mathcal{O}) \circ B(\mathcal{O},-) \cong B(\mathcal{O},-)\Omega^\infty\Sigma^\infty$ is co-associative, we deduce that the diagram given in the statement is commutative. ☐

Proof of Proposition 5.13. We make the proof by induction on the cells of F.

1. We first assume that F is the representable functor $F = \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,-)$ (with $X \in \mathrm{Alg}_{\mathcal{O}}$), and we want to prove that $F\Omega^\infty$ is a right T-comodule. Let $V \in Ch$. According to Lemma 5.14. We have a morphism of chain complexes
$$\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,-) \xrightarrow{\delta} \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,-)\Omega^\infty\Sigma^\infty$$
Composing this map with Ω^∞, we get $\eta = \delta\Omega^\infty$
$$\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,\Omega^\infty -) \xrightarrow{\eta} \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,-)\Omega^\infty\Sigma^\infty\Omega^\infty = \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,\Omega^\infty -)T.$$
The coassociativity and the counit properties of m^c lead to that η is a right T-comodule structure map.

2. Let consider that $F = C$ is a finite presented cell functor. Let consider in particular the single cell attachment

$$\begin{tikzcd} W_0 \otimes \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, \Omega^\infty -) \arrow[r] \arrow[d] & C'\Omega^\infty \arrow[d] \\ W_1 \otimes \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, \Omega^\infty -) \arrow[r] & C\Omega^\infty \end{tikzcd} \qquad (*)$$

This is objectwise a pushout diagram. Therefore the diagram

$$\begin{tikzcd} W_0 \otimes \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, \Omega^\infty T -) \arrow[r] \arrow[d] & C'\Omega^\infty T \arrow[d] \\ W_1 \otimes \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, \Omega^\infty T -) \arrow[r] & C\Omega^\infty T \end{tikzcd} \qquad (**)$$

is also objectwise a pushout diagram. Therefore if there is a map $\eta' : C'\Omega^\infty \longrightarrow C'\Omega^\infty T$ so that the appropriate diagrams formed from $(*)$ to $(**)$ commute, then the universal property of pushouts induces a morphism $\eta : C\Omega^\infty \longrightarrow C\Omega^\infty T$.

We assume in addition that the map $\eta' : C'\Omega^\infty \longrightarrow C'\Omega^\infty T$ is coassociative, which means that the following diagram commutes:

$$\begin{tikzcd} C'\Omega^\infty \arrow[r, "\eta'"] \arrow[d, "\eta'"] & C'\Omega^\infty T \arrow[d, "\eta' T"] \\ C'\Omega^\infty T \arrow[r, "C' m_T"'] & C'\Omega^\infty TT \end{tikzcd}$$

Since the diagram

$$\begin{tikzcd} W_0 \otimes \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, \Omega^\infty TT -) \arrow[r] \arrow[d] & C'\Omega^\infty TT \arrow[d] \\ W_1 \otimes \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, \Omega^\infty TT -) \arrow[r] & C\Omega^\infty TT \end{tikzcd} \qquad (***)$$

is objectwise a pushout diagram, and looking at the different maps between the diagrams $(*)$ and $(***)$, we deduce from the universal property of pushouts that the two composites

$$C\Omega^\infty \xrightarrow{\eta} C\Omega^\infty T \xrightarrow{\eta T} C\Omega^\infty TT$$
and
$$C\Omega^\infty \xrightarrow{\eta} C\Omega^\infty T \xrightarrow{C\Omega^\infty m_T} C\Omega^\infty TT$$

are equal. It follows that if η' is a right T-comodule map, then so is η. It then follows by induction that for any finite presented cell functor C, there is a right T comodule map $\eta : C\Omega^\infty \longrightarrow C\Omega^\infty T$.

A similar argument works for the counit axiom.

3. For any arbitrary presented cell functor F, since we have the isomorphism (see Proposition 4.16)

$$F \cong \operatorname*{colim}_{C \in Sub(F)} C,$$

we deduce that $F\Omega^\infty$ is a right T-comodule with the structure map $\eta : F\Omega^\infty \longrightarrow F\Omega^\infty T$ induced by the maps $\eta : C\Omega^\infty \longrightarrow C\Omega^\infty T$ ($\forall C \in Sub(F)$).

□

We use the notations of Proposition 5.13 to deduce the module structure:

Proposition 5.15 (module structure on $\partial^*(F\Omega^\infty)$)**.** *Let $F : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ be a presented cell functor. Then the right T-comodule structure map η of Lemma 5.13 induces a morphism of pro-symmetric sequence*

$$\eta^* : \partial^*(F\Omega^\infty) \circ B(\mathcal{O})^\vee \longrightarrow \partial^*(F\Omega^\infty)$$

which makes $\partial^*(F\Omega^\infty)$ into a pro-right-module over $B(\mathcal{O})^\vee$.

It remains now to prove this result. In the proof we will use the map λ_* :

$$\lambda_* : B(\mathcal{O})^\vee \longrightarrow Nat(TI, I^{\otimes *}) \tag{5.13}$$

defined level-wise by: $\forall n, \lambda_n$ as the composite

$$\begin{aligned} B(\mathcal{O})^\vee(n) \cong B(\mathcal{O})^\vee(n) \otimes \Bbbk &\longrightarrow B(\mathcal{O})^\vee(n) \underset{\Sigma_n}{\otimes} Nat(I^{\otimes n}, I^{\otimes n}) &(5.14)\\ &\underset{\cong}{\longrightarrow} Nat(B(\mathcal{O})(n) \underset{\Sigma_n}{\otimes} I^{\otimes n}, I^{\otimes n}) &(5.15)\\ &\longrightarrow Nat(TI, I^{\otimes n}) &(5.16)\end{aligned}$$

where,

- The morphism (5.14) is induced by the map $\Bbbk \longrightarrow Nat(I^{\otimes n}, I^{\otimes n})$ which sends the unit $1 \in \Bbbk$ to the class of the identity $W^{\otimes n} \xrightarrow{Id} W^{\otimes n}$;

  The diagonal Σ_n-action on the right hand of (5.14) is defined as follows: Σ_n acts on $B(\mathcal{O})^\vee(n)$ by its induced action on $B(\mathcal{O})(n)$; and Σ_n acts on $Nat(F^{\otimes n}, G^{\otimes n})$ by permuting the components in the domain $F^{\otimes n}$.

- The morphism (5.15) is the isomorphism

$$\begin{aligned} B(\mathcal{O})^\vee(n) \underset{\Sigma_n}{\otimes} Nat(I^{\otimes n}, I^{\otimes n}) &\cong \underline{hom}(B(\mathcal{O})(n), Nat(I^{\otimes n}, I^{\otimes n}))^{\Sigma_n}\\ &\cong Nat(B(\mathcal{O})(n) \underset{\Sigma_n}{\otimes} I^{\otimes n}, I^{\otimes n})\end{aligned}$$

- The morphism (5.16) is induced by the projection

$$TW \longrightarrow B(\mathcal{O})(n) \underset{\Sigma_n}{\otimes} W^{\otimes n}.$$

We also define the morphism

$$Q\lambda_* : B(\mathcal{O})^{\vee} \longrightarrow Nat(QTI, I^{\otimes *}) \tag{5.17}$$

as the composite:

$$B(\mathcal{O})^{\vee} \xrightarrow{\lambda_*} Nat(TI, I^{\otimes *}) \overset{Nat(t,I^{\otimes *})}{\longrightarrow} Nat(QTI, I^{\otimes *})$$

where $t : QT \xrightarrow{\simeq} T$ is a cofibrant resolution of T in $[Ch^{fin}, Ch]_0$.

Proof of Proposition 5.15. To define η^*, we need to define the intermediate morphism ϱ of pro-objects

$$\varrho : \partial^* F\Omega^{\infty} \circ Nat(T, I^{\otimes *}) \longrightarrow \partial^* F\Omega^{\infty} \tag{5.18}$$

as the collection of the morphisms: $\forall C \in Sub(F)$,

$$\begin{aligned} Nat(C\Omega^{\infty}I, I^{\otimes *}) \circ Nat(TI, I^{\otimes *}) &\longrightarrow Nat(C\Omega^{\infty}TI, T^{\otimes *}) \circ Nat(TI, I^{\otimes *}) \\ &\longrightarrow Nat(C\Omega^{\infty}TI, I^{\otimes *}) \\ &\overset{Nat(\eta,I^{\otimes *})}{\longrightarrow} Nat(C\Omega^{\infty}I, I^{\otimes *}) \end{aligned}$$

We then define the morphism η^* as the composite

$$\eta^* : \partial^*(F\Omega^{\infty}) \circ B(\mathcal{O})^{\vee} \overset{\partial^*(F\Omega^{\infty})\circ\lambda_*}{\longrightarrow} \partial^*(F\Omega^{\infty}) \circ Nat(TI, I^{\otimes *}) \xrightarrow{\varrho} \partial^*(F\Omega^{\infty}),$$

where λ_* is given in Equation (5.13). To make the notation easier, we will replace the product $\partial^*(F\Omega^{\infty})\circ\lambda_*$ by $1\circ\lambda_*$, to mean that this morphism applies λ_* on the second term in the circle product, and the first term in unchanged. We will also adopt the same convention in general to define maps on the circle products.

1. To prove that η^* is a module structure map, we first prove the associativity. Let $C \in Sub(F)$, we sometimes use the notation $\partial^*(C\Omega^{\infty}) = Nat(C\Omega^{\infty}I, I^{\otimes *})$ to simplify the expressions. We consider the following diagram :

$$\begin{tikzcd}
\partial^*(C\Omega^{\infty}) \circ (B(\mathcal{O})^{\vee})^{\circ 2} \arrow[r, "="] \arrow[d, "="] & \partial^*(C\Omega^{\infty}) \circ (B(\mathcal{O})^{\vee})^{\circ 2} \arrow[r, "\eta^*\circ 1"'] \arrow[d, "1\circ\lambda_*\circ\lambda_*"] & \partial^*(C\Omega^{\infty}) \circ B(\mathcal{O})^{\vee} \arrow[d, "1\circ\lambda_*"] \\
\partial^*(C\Omega^{\infty}) \circ (B(\mathcal{O})^{\vee})^{\circ 2} \arrow[r, "1\circ\lambda_*\circ\lambda_*"'] \arrow[d, "1\circ(m^c)^{\vee}"] & \partial^*(C\Omega^{\infty}) \circ Nat(TI, I^{\otimes *})^{\circ 2} \arrow[r, "\varrho\circ 1"'] \arrow[d, "1\circ m_T^*"] & \partial^*(C\Omega^{\infty}) \circ Nat(TI, I^{\otimes *}) \arrow[d, "\varrho"] \\
\partial^*(C\Omega^{\infty}) \circ (B(\mathcal{O})^{\vee}) \arrow[r, "1\circ\lambda_*"'] & \partial^*(C\Omega^{\infty}) \circ Nat(TI, I^{\otimes *}) \arrow[r, "\varrho"'] & \partial^*(C\Omega^{\infty})
\end{tikzcd}$$

where,

- the multiplication $(m^c)^\vee : B(\mathcal{O})^\vee \circ B(\mathcal{O})^\vee \longrightarrow B(\mathcal{O})^\vee$ is the dual of the cooperad coproduct m^c;
- m_T^* is the composite

$$\begin{aligned} Nat(TI, I^{\otimes *}) \circ Nat(TI, I^{\otimes *}) &\longrightarrow Nat(TTI, TI^{\otimes *}) \circ Nat(TI, I^{\otimes *}) \\ &\longrightarrow Nat(TTI, I^{\otimes *}) \\ &\overset{Nat(m_T, I^{\otimes *})}{\longrightarrow} Nat(TI, I^{\otimes *}) \end{aligned}$$

One can easily check that the two top squares are commutative. The bottom most left square commutes since one can check that the following diagram is commutative

$$\begin{tikzcd} B(\mathcal{O})^\vee \circ B(\mathcal{O})^\vee \arrow[r, "\lambda_* \circ \lambda_*"'] \arrow[d, "(m^c)^\vee"] & Nat(TI, I^{\otimes *}) \circ Nat(TI, I^{\otimes *}) \arrow[d, "m_T^*"] \\ B(\mathcal{O})^\vee \arrow[r, "\lambda_*"'] & Nat(TI, I^{\otimes *}) \end{tikzcd}$$

The most right bottom square commutes due to the coassociativity of the comodule structure map $\eta : C\Omega^\infty \longrightarrow C\Omega^\infty T$ of Proposition 5.13 :

$$\begin{tikzcd} C\Omega^\infty \arrow[r, "\eta"] \arrow[d, "\eta"] & C\Omega^\infty T \arrow[d, "C\Omega^\infty m_T"] \\ C\Omega^\infty T \arrow[r, "\eta T"'] & C\Omega^\infty TT \end{tikzcd}$$

2. If $\varepsilon^\vee$ denotes the dual of the cooperad co-unit $\varepsilon : B(\mathcal{O}) \longrightarrow 1$, then the unit diagram,

$$\begin{tikzcd} Nat(C\Omega^\infty I, I^{\otimes *}) \cong Nat(C\Omega^\infty I, I^{\otimes *}) \circ 1 \arrow[r, "1 \circ \varepsilon^\vee"] \arrow[dr, "="'] & Nat(C\Omega^\infty I, I^{\otimes *}) \circ B(\mathcal{O})^\vee \arrow[d, "\eta^*"] \\ & Nat(C\Omega^\infty I, I^{\otimes *}) \end{tikzcd}$$

is commutative using the counit property of $\eta : C\Omega^\infty \longrightarrow C\Omega^\infty T$

$$\begin{tikzcd} C\Omega^\infty(W) \arrow[r, "\eta"] \arrow[dr, "="'] & C\Omega^\infty T(W) \arrow[d, "C\Omega^\infty(\varepsilon)"] \\ & C\Omega^\infty 1(W) \end{tikzcd}$$

$\square$

5.3.2 Cosimplicial resolution of functors $F : \mathbf{Alg}_{\mathcal{O}}^{fin} \longrightarrow Ch$

Let $F : \mathrm{Alg}_{\mathcal{O}}^{fin} \longrightarrow Ch$ be a presented cell functor. We define the cosimplicial object $Res^{\bullet}(F)$ associated to F and the comonad $T = \Sigma^{\infty}\Omega^{\infty}$ as follows:

$$\text{for any integer } r,\ Res^{r}(F) := F\Omega^{\infty}T^{r}\Sigma^{\infty}$$

The right T-comodule structure map $\eta : F\Omega^{\infty} \longrightarrow F\Omega^{\infty}T$ (see Lemma 5.13) and the comonad coproduct $m_T : T \longrightarrow TT$ are used in the classical way to construct the cofaces $d^0 : Res^r(F) \longrightarrow Res^{r+1}(F)$ and ,for $0 < i \leq r$, $d^i : Res^r(F) \longrightarrow Res^{r+1}(F)$ respectively. The codegeneracies $s^j : Res^{r+1}(F) \longrightarrow Res^r(F)$, for $0 \leq j < r$, are induced by the comonad unit $T \longrightarrow 1$.

We define the totalization of cosimplicial chain complexes as it appears in [Fre17, § 3.3.13.].

Definition 5.16 (Totalization functor)**.** *We define the totalization of a cosimplicial chain complex $X^{\bullet}$ as the end (in the category of chain complexes):*

$$Tot(X^{\bullet}) = \int_{\underline{n}\in\triangle} \underline{hom}(N\Bbbk\triangle^n, X^n)$$

Remark 5.17. *In our reference [Fre17, § 3.3.13.], it is explicitly stated that the object in the argument of the end $\int_{\underline{n}\in\triangle}$ is a simplicial frame([Fre17, § 3.2.2]). In our case (chain complex) one can see that , given $V \in Ch$, $\underline{hom}(N\Bbbk\triangle^n, V)$ is a simplicial frame associated to V.*

Now we are ready to state the main result of this part.

Proposition 5.18. *If $F : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ is a presented cell functor, then there is a weak equivalence*

$$F \xrightarrow{\simeq} \widetilde{Tot}(Res^{\bullet}(F))$$

where $\widetilde{Tot}(Res^{\bullet}(F))$ is the totalization functor applied to the (Reedy) fibrant replacement of $Res^{\bullet}(F)$.

Remark 5.19. *Since totalization Tot of chain complexes is equivalent to homotopy totalization $\widetilde{Tot}$ (see [ALTV08, Remark, p6], [BK72]), Proposition 5.18 is equivalent to saying that we have a weak equivalence*

$$F \xrightarrow{\simeq} Tot(Res^{\bullet}(F))$$

The rest of this part is dedicated to the proof of Proposition 5.18. We remind the following definition of "extra codegeneracies" as it is reported in [MV15, Definition 9.1.24].

Definition 5.20 (Extra codegeneracies)**.**

1. *An augmentation of a cosimplicial chain complex $X^{\bullet}$ is a space X^{-1} and a map $d^0 : X^{-1} \longrightarrow X^0$ such that*
$$d^1 d^0 = d^0 d^0 : X^{-1} \longrightarrow X^1 \tag{5.19}$$
2. *We call extra codegeneracies of an augmented cosimplicial space $X^{-1} \longrightarrow X^{\bullet}$, a collection of maps*

$$s^{-1} : X^{n+1} \longrightarrow X^n, n \geq -1$$

satisfying

$$s^{-1}d^0 = Id,$$
$$s^{-1}d^i = d^{i-1}s^{-1}, i \geq 1,$$
$$s^{-1}s^i = s^{i-1}s^{-1}, i \geq 0.$$

Given an augmented cosimplicial chain complex $X^\bullet$ with the augmentation $d^0 : X^{-1} \longrightarrow X^0$, The Equation (5.19) ensures that there is a unique map $X^{-1} \longrightarrow X^n, n > 0$ given by composing $d^0 : X^{-1} \longrightarrow X^0$ with the cofaces in $X^\bullet$. We consider the map

$$(d^0)^{n+1} : X^{-1} \longrightarrow X^n, n > 0$$

by composing $d^0 : X^{-1} \longrightarrow X^0$ with the cofaces $d^0 : X^i \longrightarrow X^{i+1}, i < n$. This defines a morphism of cosimplicial chain complexes

$$p : X^{-1} \longrightarrow X^\bullet \tag{5.20}$$

where X^{-1} is taken as a constant cosimplicial chain complexes.

We state the following result which says literally that an augmented cosimplicial chain complex with extra codegeneracies has a totalization equivalent to its augmentation. The proof imitates [MV15, Proposition 9.1.25].

Proposition 5.21. *If an augmented cosimplicial chain complex $X^{-1} \longrightarrow X^\bullet$ has extra codegeneracies, then the map $Tot(p) : X^{-1} \longrightarrow Tot(X^\bullet)$, induced by the map p from (5.20), is a homotopy equivalence of chain complexes.*

Proof. Our candidate to be the homotopy inverse of $Tot(p)$ is the map

$$Tot(q) : Tot(X^\bullet) \longrightarrow X^{-1}$$

induced by the map $q : X^\bullet \longrightarrow X^{-1}$ which is itself obtained by the collection of maps

$$(s^{-1})^{n+1} : X^n \longrightarrow X^{-1}$$

defined by composing the extra codegeneracies $n+1$-times.

One can remark that the composite

$$X^{-1} \overset{(d^0)^n}{\longrightarrow} X^n \overset{(s^{-1})^n}{\longrightarrow} X^{-1}$$

is the identity since $s^{-1}d_0 = Id$. It follows that $Tot(q)Tot(p) = Id$ on X^{-1}.

It remains now to show that $Tot(p)Tot(q) \sim Id$ on $Tot(X^\bullet)$.

Let $(f_0, f_1, ..., f_n, ...)$ be an object in $Tot(X^\bullet) \subset \prod_n \underline{hom}(N\Bbbk\triangle^n, X^n)$. We will prove inductively that $(d^0)^{k+1}(s^{-1})^{k+1}f_n \sim f_n$, for $k \leq n$

1. We prove here that $d^0s^{-1}f_n \sim f_n$. The morphism $f_1 : N\Bbbk\triangle^1 \longrightarrow X^1$ gives a chain homotopy between d^0f_0 and d^1f_0. Therefore $s^{-1}f_1$ gives a chain homotopy between $s^{-1}d^0f_0 = f_0$ and $s^{-1}d^1f_0 = d^0s^{-1}f_0$.

   For f_n, in general we consider the restriction of f_{n+1} to

$$N\Bbbk\triangle^1 \hookrightarrow N\Bbbk\triangle^{n+1} \longrightarrow X^{n+1}$$

which gives a chain homotopy between $s^{-1}d^0 f_n = f_n$ and $s^{-1}d^1 f_n = d^0 s^{-1} f_n$.

2. We make the following computation

$$\begin{aligned}
(d^0)^{k+1}(s^{-1})^{k+1} f_n &= (d^0)^k (s^{-1})^{k+1} d^{k+1} f_n && (5.21)\\
&\sim (d^0)^k (s^{-1})^{k+1} d^0 f_n && (5.22)\\
&= (d^0)^k (s^{-1})^k f_n && (5.23)\\
&\sim d^0 s^{-1} f_n && (5.24)\\
&\sim f_n && (5.25)
\end{aligned}$$

where

- In (5.22), the chain homotopy $d^{k+1} f_n \sim d^0 f_n$ is given by the restriction of f_{n+1} to the edge $N\Bbbk\triangle^1 \hookrightarrow N\Bbbk\triangle^{n+1} \longrightarrow X^{n+1}$ whose vertices are $[0]$ and $[k+1]$;
- The homotopy (5.24) is obtained by repeating a similar process as (5.21) and (5.22) to reduce to $k-1$ and so on.
- The homotopy (5.25) is deduced from 1.

In conclusion we have $(d^0)^{n+1}(s^{-1})^{n+1} f_n \sim f_n$ and this proves the $Tot(p)Tot(q) \sim Id$ on $Tot(X^\bullet)$.

□

Proof of Proposition 5.18. Using Remark 5.19, we have to prove that

$$F \xrightarrow{\simeq} Tot(Res^\bullet(F)).$$

Moreover, based on Proposition 5.21, we only have to prove that there is a morphism $F \longrightarrow Res^0(F)$ such that $F \longrightarrow Res^\bullet(F)$ is an augmented cosimplicial chain complex with extra codegeneracies.

(1) The cosimplicial object $Res^\bullet(F)$ has a natural augmentation

$$F(X) \longrightarrow Res^0(F)(X) = F\Omega^\infty\Sigma^\infty(X)$$

defined inductively from the representable functors. Namely,

(a) $\forall X \in \mathrm{Alg}_{\mathcal{O}}$, there is a natural map

$$\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, -) \xrightarrow{\delta} \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, -)\Omega^\infty\Sigma^\infty$$

given by Lemma 5.14. Roughly speaking, this is given by the $B(\mathcal{O})$-coalgebra coproduct

$$m^c : B(\mathcal{O}, -) \longrightarrow B(\mathcal{O}) \circ B(\mathcal{O}, -).$$

(b) The morphism in (a) extends inductively to a morphism

$$\delta : C \longrightarrow C\Omega^\infty\Sigma^\infty$$

for any finite presented cell functor $C : \mathrm{Alg}_{\mathcal{O}}^{fin} \longrightarrow Ch$. In addition, since the multiplication $m : Apl_\bullet \otimes Apl_\bullet \longrightarrow Apl_\bullet$ is associative and that $m^c : B(\mathcal{O}, -) \longrightarrow B(\mathcal{O}) \circ B(\mathcal{O}, -) \cong B(\mathcal{O}, -)\Omega^\infty\Sigma^\infty$ is co-associative, we form the following commutative diagram

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d, "\delta"'] & C\Omega^\infty\Sigma^\infty \arrow[d, "C\Omega^\infty m^c"] \\ C\Omega^\infty\Sigma^\infty \arrow[r, "\eta\Sigma^\infty"'] & C\Omega^\infty\Sigma^\infty\Omega^\infty\Sigma^\infty \end{tikzcd}$$

Finally, since $F \cong \underset{C \in Sub(F)}{colim} C$, this construction generalizes to a morphism $\delta : F \longrightarrow F\Omega^\infty\Sigma^\infty$ and this defines an augmentation of the cosimplicial object $Res^\bullet(F)$.

(2) The cosimplicial object $Res^\bullet(F)$ has extra codegeneracies

$$s^{-1} : Res^{k+1}(F) \longrightarrow Res^k(F), (\forall k \geq -1)$$

constructed again inductively from representable functors. Namely $\forall Z, X \in \mathrm{Alg}_{\mathcal{O}}$, there is a sequence:

$$\begin{tikzcd} ... \arrow[r] & Res^2(\widetilde{\mathrm{Alg}}_{\mathcal{O}}(Z,-))(X) \arrow[r] & Res^1(\widetilde{\mathrm{Alg}}_{\mathcal{O}}(Z,-))(X) \arrow[d] \\ & & Res^0(\widetilde{\mathrm{Alg}}_{\mathcal{O}}(Z,-))(X) = \widetilde{\mathrm{Alg}}_{\mathcal{O}}(Z,X) \end{tikzcd}$$

induced by the sequence:

$$... \longrightarrow B(\mathcal{O}) \circ B(\mathcal{O}) \circ B(\mathcal{O}, X) \xrightarrow{\varepsilon \circ B(\mathcal{O}) \circ B(\mathcal{O},X)} B(\mathcal{O}) \circ B(\mathcal{O}, X) \xrightarrow{\varepsilon \circ B(\mathcal{O},X)} B(\mathcal{O}, X)$$

where $\varepsilon : B(\mathcal{O}) \longrightarrow \mathbb{k}$ is the cooperad co-unit.

□

5.3.3 D_n-approximation of functors $F : \mathbf{Alg}_{\mathcal{O}} \longrightarrow Ch$

In this section, we define a D_n-approximation of simplicial functors $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$. The development in this part follows the following road-map:

1. We remind the construction of the map of modules;
2. We make the construction of $\Phi_n F$;
3. Then we make the construction of the map $\psi_F : F \longrightarrow \Phi_n F$;
4. Finally, we prove that ψ_F is a D_n-approximation.

(1) Map of modules

Definition 5.22 (Mapping object)**.**

1. *Let M, N be two symmetric sequences of chain complexes. We define the chain complex*

$$Map_{\Sigma}(M,N) := \bigoplus_{r=1}^{\infty} \underline{hom}(M(r), N(r))^{\Sigma_r}$$

2. *Let M, N, P be two symmetric sequences in Ch. Then there is a natural map*

$$Map_{\Sigma}(M,N) \longrightarrow Map_{\Sigma}(M \circ P, N \circ P)$$

constructed from the maps:

$$\underline{hom}(M(r), N(r)) \longrightarrow \underline{hom}(M(r) \otimes P(n_1) \otimes ... \otimes P(n_r), N(r) \otimes P(n_1) \otimes ... \otimes P(n_r))$$

which are themselves induced by the evaluation maps of the form

$$M(r) \otimes \underline{hom}(M(r), N(r)) \longrightarrow N(r)$$

Definition 5.23 (Mapping objects for modules)**.** *Let $\mathcal{O}$ be a reduced operad on Ch, and M, N be right $\mathcal{O}$-modules. We define*

$$Map_{\mathcal{O}}^{right}(M,N) = lim\ (Map_{\Sigma}(M,N) \rightrightarrows Map_{\Sigma}(M \circ \mathcal{O}, N))$$

where one of the arrows on the right hand side of this equation is induced by the module structure map $M \circ P \longrightarrow M$, and the other map is the composite

$$Map_{\Sigma}(M,N) \longrightarrow Map_{\Sigma}(M \circ \mathcal{O}, N \circ \mathcal{O}) \longrightarrow Map_{\Sigma}(M \circ \mathcal{O}, N)$$

where the first map is the map constructed in Definition 5.22-(2)*, and the second morphism is produced by the module structure map $N \circ \mathcal{O} \longrightarrow N$.*

Definition 5.24 (Mapping objects for pro-symmetric sequences)**.** *Let $M : \mathcal{I} \longrightarrow Ch$ and $N : \mathcal{J} \longrightarrow Ch$ be two pro-symmetric sequences on chain complexes. The map between M and N, denoted $Map_{\Sigma}(M,N)^{pro}$, the chain complex*

$$Map_{\Sigma}^{pro}(M,N) := \underset{j \in \mathcal{J}}{lim}\ \underset{i \in \mathcal{I}}{colim}\ Map_{\Sigma}(M(i), N(j))$$

(2) Construction of $\Phi_n F$

Definition 5.25 (Construction of $\Phi_n F$)**.** *Let $F : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ be a simplicial functor. We define by $\Phi_n F : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ the functor which assigns to any $\mathcal{O}$-algebra X, the chain complex*

$$\Phi_n F(X) := \underset{C \in Sub(QF)}{hocolim}\ \widetilde{Tot}(Map_{B(\mathcal{O})^{\vee}}^{right}(\partial^* C\Omega^{\infty} \circ (B(\mathcal{O})^{\vee})^{\bullet+1}, (\Sigma^{\infty} X)^{\otimes * \leq n}))$$

This definition uses the fact that the symmetric sequence $(\Sigma^\infty X)^{\otimes *}$ is a right $B(\mathcal{O})^\vee$-module as it is explained here below.

Lemma 5.26. *If X is an $\mathcal{O}$-algebra, then the symmetric sequence $(\Sigma^\infty X)^{\otimes *}$ is a right module over $B(\mathcal{O})^\vee$*

Proof. The structure map

$$(\Sigma^\infty X)^{\otimes *} \circ B(\mathcal{O})^\vee \longrightarrow (\Sigma^\infty X)^{\otimes *}$$

is given by the composite

$$\begin{aligned}(\Sigma^\infty X)^{\otimes *} \circ B(\mathcal{O})^\vee &\xrightarrow{(m^c)^*} (T\Sigma^\infty X)^{\otimes *} \circ B(\mathcal{O})^\vee \\ &\longrightarrow (T\Sigma^\infty X)^{\otimes *} \circ Nat(TI, I^{\otimes *}) \\ &\longrightarrow (\Sigma^\infty X)^{\otimes *}\end{aligned}$$

where,

- the first map $(m^c)^*$ is induced by the $B(\mathcal{O})$-coalgebra structure map

$$m^c : B(\mathcal{O}, -) \longrightarrow B(\mathcal{O}) \circ B(\mathcal{O}, -) \cong T\Sigma^\infty;$$

- the second map is induced by the map $\lambda_* : B(\mathcal{O})^\vee \longrightarrow Nat(TI, I^{\otimes *})$;
- the third map is induced by the evaluation maps of the form

$$T(\Sigma^\infty X) \circ Nat(TI, I^{\otimes n}) \longrightarrow (\Sigma^\infty X)^{\otimes n}.$$

□

There is an equivalent description of the functor $\Phi_n F$. Though we will essentially use the version provided in Definition 5.25 in this section, it is also important to consider this other description that we will use in the next section to characterize $\{P_n F\}_n$.

Lemma 5.27. *Let $F : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ be a finite presented cell functor. Then there is an equivalence*

$$\Phi_n F(X) \simeq Map^{right}_{B(\mathcal{O})^\vee}(B(\partial^* F\Omega^\infty, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee), (\Sigma^\infty X)^{\otimes * \leq n})$$

Proof. By definition, we have

$$\begin{aligned}\Phi_n F(X) &:= \widetilde{Tot}(Map^{right}_{B(\mathcal{O})^\vee}(\partial^* F\Omega^\infty \circ (B(\mathcal{O})^\vee)^{\bullet+1}, (\Sigma^\infty X)^{\otimes * \leq n})) \\ &\simeq Tot(Map^{right}_{B(\mathcal{O})^\vee}(\partial^* F\Omega^\infty \circ (B(\mathcal{O})^\vee)^{\bullet+1}, (\Sigma^\infty X)^{\otimes * \leq n}))\end{aligned}$$

where the last equivalence is deduced from Remark 5.19 where we have explained that totalization and homotopy totalization coincide on chain complexes. On the other hand, Tot which is technically a limit commutes with $Map^{right}_{B(\mathcal{O})^\vee}(-, (\Sigma^\infty X)^{\otimes * \leq n})$. We then have

$$\Phi_n F(X) \simeq Map^{right}_{B(\mathcal{O})^\vee}(N(\partial^* F\Omega^\infty \circ (B(\mathcal{O})^\vee)^{\bullet+1}), (\Sigma^\infty X)^{\otimes * \leq n})$$

where $N(\partial^* F\Omega^\infty \circ (B(\mathcal{O})^\vee)^{\bullet+1})$ denotes the realization of the simplicial right $B(\mathcal{O})^\vee$-module $\partial^* C\Omega^\infty \circ (B(\mathcal{O})^\vee)^{\bullet+1}$.

Fresse showed in [Fre04, $Thm 4.1.8$] that, there is a quasi-isomorphism

$$N(\partial^* F\Omega^\infty \circ (B(\mathcal{O})^\vee)^{\bullet+1}) \simeq B(\partial^* F\Omega^\infty, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee),$$

and this is actually a morphism of right $B(\mathcal{O})^\vee$-module using the fact that

$$N(\partial^* F\Omega^\infty \circ (B(\mathcal{O})^\vee)^{\bullet+1}) \simeq N(\partial^* F\Omega^\infty \circ (B(\mathcal{O})^\vee)^{\bullet}) \circ B(\mathcal{O})^\vee.$$

Therefore, we deduce

$$\Phi_n F(X) \simeq Map^{right}_{B(\mathcal{O})^\vee}(B(\partial^* F\Omega^\infty, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee), (\Sigma^\infty X)^{\otimes * \leq n}).$$

□

Remark 5.28. *Given an arbitrary simplicial functor $F : Alg^{fin}_{\mathcal{O}} \longrightarrow Ch$, then there is an equivalence*

$$\Phi_n F(X) \simeq \underset{C\in Sub(QF)}{hocolim}\, Map^{right}_{B(\mathcal{O})^\vee}(B(\partial^* C\Omega^\infty, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee), (\Sigma^\infty X)^{\otimes * \leq n}) \tag{5.26}$$

(3) Construction of the map $\psi_F : F \longrightarrow \Phi_n F$.

Let $F : \mathrm{Alg}^{fin}_{\mathcal{O}} \longrightarrow Ch$ be a finite presented cell functor. To describe the map ψ_F which appears in this proposition, we will need the next three lemmas.

We consider the composite: $\forall X \in \mathrm{Alg}_{\mathcal{O}}$,

$$\psi'_F : F(X) \xrightarrow{\delta} F\Omega^\infty\Sigma^\infty(X) \longrightarrow Map_\Sigma(Nat(F\Omega^\infty I, I^{\otimes *}), (\Sigma^\infty X)^{\otimes *})$$

where δ is the map defined in Lemma 5.14 and the second map is induced by the evaluation map: $\forall W \in Ch_+$,

$$F\Omega^\infty I(W) \longrightarrow Map_\Sigma(Nat(F\Omega^\infty I, I^{\otimes *}), W^{\otimes *}).$$

Using the right $B(\mathcal{O})^\vee$-module structure on $(\Sigma^\infty X)^{\otimes *}$ defined in Lemma 5.26, we prove in the next lemma that ψ'_F factors through the module maps.

Lemma 5.29. *Let $F : Alg^{fin}_{\mathcal{O}} \longrightarrow Ch$ be a finite presented cell functor.*

The above defined morphism ψ'_F factors via the corresponding mapping object for right $B(\mathcal{O})^\vee$-modules . Namely, we have the commutative diagram

$$\begin{tikzcd}
F(X) \arrow[rr, "\psi'_F"] \arrow[dr, "\psi''_F"'] & & Map_\Sigma(Nat(F\Omega^\infty I, I^{\otimes *}), (\Sigma^\infty X)^{\otimes *}) \\
& Map^{right}_{B(\mathcal{O})^\vee}(Nat(F\Omega^\infty I, I^{\otimes *}), (\Sigma^\infty X)^{\otimes *}) \arrow[ur] &
\end{tikzcd}$$

Proof. It is sufficient for us to prove that the following diagram commutes: for every partition $r = r_1 + ... + r_k$,

$$\begin{tikzcd} F(X)\otimes Nat(F\Omega^\infty I, I^{\otimes k})\otimes B(\mathcal{O})^\vee(r_1)\otimes ...\otimes B(\mathcal{O})^\vee(r_k) \arrow[r] \arrow[d, "\psi'_F"] & F(X)\otimes Nat(F\Omega^\infty I, I^{\otimes r}) \arrow[d, "\psi'_F"] \\ (\Sigma^\infty X)^{\otimes k}\otimes B(\mathcal{O})^\vee(r_1)\otimes ...\otimes B(\mathcal{O})^\vee(r_k) \arrow[r] & (\Sigma^\infty X)^{\otimes r} \end{tikzcd}$$

where the horizontal morphisms are module structure maps and the vertical ones are, roughly speaking, induced in the natural way by ψ'_F.

Let $\alpha \in Nat(F\Omega^\infty, I^{\otimes k}) \subseteq \prod_{V\in Ch^{fin}} \underline{hom}(F\Omega^\infty(V), V^{\otimes k})$ and , given a partition $r = r_1 + ... + r_k$, we take $\alpha_i \in B(\mathcal{O})^\vee(r_i)$ and $\lambda_*(\alpha_i)$ the corresponding natural transformation in $Nat(T, I^{\otimes r_i})$ (via the map $\lambda_{r_i} : B(\mathcal{O})^\vee(r_i) \longrightarrow Nat(T, I^{\otimes r_i})$), for $i = 1, ..., k$.

We have the following commutative diagram which is induced by the naturality of α

$$\begin{tikzcd} F(X) \arrow[r, "\delta"] & F\Omega^\infty\Sigma^\infty X \arrow[r, "F\Omega^\infty m^c_X"] \arrow[d, "\alpha"] & F\Omega^\infty T\Sigma^\infty X \arrow[d, "\alpha"] & \\ & (\Sigma^\infty X)^{\otimes k} \arrow[r, "(m^c_X)^{\otimes k}"'] & (T\Sigma^\infty X)^{\otimes k} \arrow[r, "\lambda_*(\alpha_1)\otimes...\otimes\lambda_*(\alpha_k)"'] & (\Sigma^\infty X)^{\otimes r} \end{tikzcd},$$

where the map δ (defined in the proof of Proposition 5.18) is induced naturally by the $B(\mathcal{O})$-coalgebra coproduct

$$m^c_X : B(\mathcal{O}, X) \longrightarrow B(\mathcal{O}) \circ B(\mathcal{O}, X) \cong B(\mathcal{O}, -)\Omega^\infty\Sigma^\infty X$$

Now since the structure m^c_X is co-associative, we have $(F\Omega^\infty m^c_X) \circ \delta = \eta \circ \delta$ (see the above proof of Proposition 5.18) and therefore we deduce the commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "\delta"] \arrow[d, "\delta"] & F\Omega^\infty\Sigma^\infty X \arrow[r, "\eta"] & F\Omega^\infty T\Sigma^\infty X \arrow[d, "\alpha"] & \\ F\Omega^\infty\Sigma^\infty X \arrow[r, "\alpha"] & (\Sigma^\infty X)^{\otimes k} \arrow[r, "(m^c_X)^{\otimes k}"'] & (T\Sigma^\infty X)^{\otimes k} \arrow[r, "\lambda_*(\alpha_1)\otimes...\otimes\lambda_*(\alpha_k)"'] & (\Sigma^\infty X)^{\otimes r} \end{tikzcd}$$

□

On the other hand, we define the map

$$\eta^{*,r} : Nat(F\Omega^\infty I, I^{\otimes *}) \longrightarrow Nat(F\Omega^\infty I, I^{\otimes *}) \circ (B(\mathcal{O})^\vee)^{r+1} \tag{5.27}$$

which correspond to the r-th iteration of the $B(\mathcal{O})^\vee$-module structure morphism

$$\eta^* : Nat(F\Omega^\infty I, I^{\otimes *}) \circ B(\mathcal{O})^\vee \longrightarrow Nat(F\Omega^\infty I, I^{\otimes *})$$

defined in Proposition 5.15.

Definition 5.30. *We define the map*

$$\psi_F : F(X) \longrightarrow \Phi_n F(X) \tag{5.28}$$

which is induced by the composite of the map $\psi_F^{''}$ of Lemma 5.29 and the maps $Map^{right}_{B(\mathcal{O})^\vee}(\eta^{*,r}, (\Sigma^\infty X)^{\otimes *})$ $(\forall r)$:

$$Map^{right}_{B(\mathcal{O})^\vee}(\partial^* F\Omega^\infty, (\Sigma^\infty X)^{\otimes *}) \longrightarrow Map^{right}_{B(\mathcal{O})^\vee}(\partial^* F\Omega^\infty \circ (B(\mathcal{O})^\vee)^{r+1}, (\Sigma^\infty X)^{\otimes *})$$

(4) Statement and proof of the D_n-approximation

Proposition 5.31. *We assume char($\Bbbk$)=0. If $C : Alg^{fin}_{\mathcal{O}} \longrightarrow Ch$ is a finite presented cell functor, then the morphism*

$$\psi_C : C \longrightarrow \Phi_n C$$

given in Definition 5.30 is a D_n-equivalence.

The straight consequence of this proposition is the next result which follows from the fact that the functor D_n- commutes with the filtered colimit functor.

Corollary 5.32. *We assume char($\Bbbk$)=0. If $F : Alg^{fin}_{\mathcal{O}} \longrightarrow Ch$ is a simplicial functor, then the morphism*

$$\psi_F : F \simeq \underset{C \in Sub(QF)}{hocolim}\ C \longrightarrow \Phi_n F$$

which is induced by the maps $\psi_C : C \longrightarrow \Phi_n C$ of Proposition 5.31, is a D_n-equivalence.

To prove Proposition 5.31, we will use the cosimplicial approximation $Res^\bullet(C)$ associated to C developed in the previous section. Namely, we define the morphism of chain complexes: $\forall r \in \mathbb{N}$,

$$\begin{aligned}\psi_{C,r} : C\Omega^\infty T^r(\Sigma^\infty X) \longrightarrow & Map_\Sigma(Nat(C\Omega^\infty I, I^{\otimes *}) \circ (B(\mathcal{O})^\vee)^r, (\Sigma^\infty X)^{\otimes \leq n}) \\ & \cong Map^{right}_{B(\mathcal{O})^\vee}(Nat(C\Omega^\infty I, I^{\otimes *}) \circ (B(\mathcal{O})^\vee)^{r+1}, (\Sigma^\infty X)^{\otimes \leq n})\end{aligned}$$

as follows: If β is the composite

$$\begin{aligned}\beta : Nat(C\Omega^\infty I, I^{\otimes *}) \circ (B(\mathcal{O})^\vee)^r & \xrightarrow{1\circ\lambda^r_*} Nat(C\Omega^\infty I, I^{\otimes *}) \circ Nat(TI, I^{\otimes *})^r \\ & \longrightarrow Nat(C\Omega^\infty TI^r, (TI^r)^{\otimes *}) \circ Nat(TI^r, I^{\otimes *}) \\ & \longrightarrow Nat(C\Omega^\infty TI^r, I^{\otimes *})\end{aligned}$$

and $ev : Nat(C\Omega^\infty TI^r, I^{\otimes *}) \otimes C\Omega^\infty T^r(\Sigma^\infty X) \longrightarrow (\Sigma^\infty X)^{\otimes *}$ is the evaluation map, we then set $\psi_{C,r}$ as the adjoint of the map $ev.(\beta \otimes 1)$:

$$(Nat(C\Omega^\infty I, I^{\otimes *}) \circ (B(\mathcal{O})^\vee)^r) \otimes C\Omega^\infty T^r(\Sigma^\infty X) \xrightarrow{ev.(\beta\otimes 1)} (\Sigma^\infty X)^{\otimes *}$$

Lemma 5.33. *We assume char($\mathbb{k}$)=0. Let $C : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ be a finite presented cell functor, and $r \geq 0$. Then the morphism*

$$\psi_{C,r} : C\Omega^\infty T^r(\Sigma^\infty X) \longrightarrow Map^{right}_{B(\mathcal{O})^\vee}(Nat(C\Omega^\infty I, I^{\otimes *})\circ(B(\mathcal{O})^\vee)^{r+1}, (\Sigma^\infty X)^{\otimes \leq n})$$

is a D_n-equivalence.

Proof. In this proof, we will use the map of pro-symmetric sequences described in Definition 5.24. We will sometimes use the notation $\partial^* C\Omega^\infty = Nat(C\Omega^\infty I, I^{\otimes *})$ and $\mathbb{X}_n = (\Sigma^\infty X)^{\otimes \leq n}$ to reduce the length of expressions.

When $r = 1$, we consider the following commutative diagram

$$\begin{tikzcd}
\operatorname*{colim}_{E\in Sub(Q(C\Omega^\infty QT))} Map_\Sigma(Nat(E, I^{\otimes *}), \mathbb{X}_n) \arrow[d, "(a)"] & \\
\operatorname*{colim}_{D\in Sub(QT)} Map_\Sigma(Nat(C\Omega^\infty D, I^{\otimes *}), \mathbb{X}_n) \arrow[r, "(e)"] \arrow[d, "(b)"] & Map_\Sigma(Nat(C\Omega^\infty QT, I^{\otimes *}), \mathbb{X}_n) \arrow[d, "(d)"] \\
\operatorname*{colim}_{D\in Sub(QT)} Map_\Sigma(\partial^* C\Omega^\infty \circ Nat(D, I^{\otimes *}), \mathbb{X}_n) \arrow[r, "(c)"] & Map_\Sigma(\partial^* C\Omega^\infty \circ Nat(QT, I^{\otimes *}), \mathbb{X}_n)
\end{tikzcd}$$

where

- The map (a) is induced by the cofibrant resolution $\alpha : Q(C\Omega^\infty QT) \xrightarrow{\simeq} C\Omega^\infty QT$. Namely given any finite sub complex $E \in Q(C\Omega^\infty QT)$, using Proposition 4.13, the natural transformation α restricts to $E \longrightarrow C\Omega^\infty D$, for some finite subcomplex $D \in Sub(QT)$. We then deduce the composite $Nat(C\Omega^\infty D, I^{\otimes *}) \longrightarrow Nat(E, I^{\otimes *})$.
- The two vertical maps (b) and (d) in the square are induced by the compositions $Nat(C\Omega^\infty I, I^{\otimes *}) \circ Nat(D, I^{\otimes *}) \longrightarrow Nat(C\Omega^\infty D, I^{\otimes *})$ and $Nat(C\Omega^\infty I, I^{\otimes *}) \circ Nat(QTI, I^{\otimes *}) \longrightarrow Nat(C\Omega^\infty QTI, I^{\otimes *})$ respectively; The two horizontal maps are induced naturally by the inclusions $D \hookrightarrow QT$.

This diagram induces the commutative diagram

$$\begin{tikzcd}
Map^{pro}_\Sigma(\partial^* Q(C\Omega^\infty QT), \mathbb{X}_n) \arrow[r, "(e)\circ(a)"] \arrow[d, "(b)\circ(a)"] & Map^{pro}_\Sigma(Nat(C\Omega^\infty QTI, I^{\otimes *}), \mathbb{X}_n) \arrow[d, "(d)"] \\
Map^{pro}_\Sigma(Nat(C\Omega^\infty I, I^{\otimes *}) \circ \partial^* QT, \mathbb{X}_n) \arrow[r, "(c)"] & Map_\Sigma(Nat(C\Omega^\infty I, I^{\otimes *}) \circ Nat(QTI, I^{\otimes *}), \mathbb{X}_n)
\end{tikzcd}$$

where

- The map $(b) \circ (a)$ is induced by the map

$$\mu^* : Nat(C\Omega^\infty I, I^{\otimes *}) \circ \partial^* QT \longrightarrow \partial^* Q(C\Omega^\infty QT)$$

constructed in the proof of Proposition 5.7. We also proved there that its continuous dual μ_* is a weak equivalence, and here this means that $(b) \circ (a)$ is a D_n-equivalence.

- The map (c) is a D_n-equivalence using Remark 5.6-(2).

The above diagram generalizes by iteration , $\forall r$, to produce the following commutative diagram

$$\begin{tikzcd}
Map^{pro}_{\Sigma}(\partial^* Q^r(C\Omega^\infty QT)), \mathbb{X}_n) \arrow[r, "(4)"] \arrow[d, "(1)"] & Map_{\Sigma}(Nat(C\Omega^\infty (QT)^r, I^{\otimes *}), \mathbb{X}_n) \arrow[d, "(3)"] \\
Map^{pro}_{\Sigma}(Nat(C\Omega^\infty I, I^{\otimes *}) \circ (\partial^* QT)^r, \mathbb{X}_n) \arrow[r, "(2)"] & Map_{\Sigma}(Nat(C\Omega^\infty I, I^{\otimes *}) \circ Nat(QTI, I^{\otimes *})^r, \mathbb{X}_n)
\end{tikzcd}$$

where

- $Q^r(C\Omega^\infty QT) := Q(...Q(Q(C\Omega^\infty QT)QT)QT...)$. This is an iterated construction which consists of taking the cofibrant replacement of the functor obtained when we pre-compose with QT.

- The map (1) is an iterated version of the map μ^*.

- The map (4) is constructed as previously but iteratively on the cofibrant resolution sequence α :

$$Q(Q(...Q(C\Omega^\infty QT)...)QT) \xrightarrow{\simeq} ... \xrightarrow{\simeq} Q(C\Omega^\infty (QT)^r) \xrightarrow{\simeq} C\Omega^\infty (QT)^r$$

For the same reasons as previously, the maps (1) and (2) are D_n-equivalences. On the other hand, we have the following commutative diagram

$$\begin{tikzcd}
Q^r(C\Omega^\infty QT)(\Sigma^\infty X) \arrow[r, "(5)"] \arrow[d, "\simeq"', "\alpha"] & Map^{pro}_{\Sigma}(\partial^* Q^r(C\Omega^\infty QT), (\Sigma^\infty X)^{\otimes \leq n}) \arrow[d, "(4)"] \\
C\Omega^\infty (QT)^r(\Sigma^\infty X) \arrow[r, "(6)"] & Map_{\Sigma}(Nat(C\Omega^\infty (QTI)^r, I^{\otimes *}), (\Sigma^\infty X)^{\otimes \leq n})
\end{tikzcd}$$

where (5) is given by the evaluation maps: $\forall C \in Sub(Q^r(C\Omega^\infty QT))$,

$$C(\Sigma^\infty X) \longrightarrow Map_{\Sigma}(Nat(CI, I^{\otimes *}), (\Sigma^\infty X)^{\otimes \leq n})$$

and (6) is the evaluation map.

Using Proposition 5.2, we deduce that (5) is a D_n-equivalence after the following remark:

$$\begin{aligned}
\partial_n Map^{pro}_{\Sigma}(\partial^* Q^r(C\Omega^\infty QT), (\Sigma^\infty X)^{\otimes \leq n}) &\simeq \underset{C' \in Sub(Q^r(C\Omega^\infty QT))}{\mathrm{hocolim}} \partial_n Map_{\Sigma}(Nat(C'I, I^{\otimes *}), (\Sigma^\infty X)^{\otimes \leq n}) \\
&\simeq \underset{C' \in Sub(Q^r(C\Omega^\infty QT))}{\mathrm{hocolim}} \partial_n \underline{hom}(Nat(C'I, I^{\otimes n}), (\Sigma^\infty X)^{\otimes n}) \\
&\simeq \partial_n \Psi_n Q^r(C\Omega^\infty QT)(\Sigma^\infty X)
\end{aligned}$$

We now consider the following commutative diagram

$$\begin{tikzcd}
Q^r(C\Omega^\infty QT)(\Sigma^\infty X) \arrow[r, "(1)\circ(5)"] \arrow[d, "\alpha"', "\simeq"] & Map^{pro}_{\Sigma}(Nat(C\Omega^\infty I, I^{\otimes *}) \circ (\partial^* QT)^r, (\Sigma^\infty X)^{\otimes \leq n}) \arrow[d, "(2)"] \\
C\Omega^\infty (QT)^r(\Sigma^\infty X) \arrow[r, "(3)\circ(6)"] \arrow[d, "(7)"', "\simeq"] & Map_{\Sigma}(Nat(C\Omega^\infty I, I^{\otimes *}) \circ Nat(QTI, I^{\otimes *})^r, (\Sigma^\infty X)^{\otimes \leq n}) \arrow[d, "(8)"] \\
C\Omega^\infty T^r(\Sigma^\infty X) \arrow[r, "\psi_{C,r}"'] & Map^{right}_{B(\mathcal{O})^\vee}(Nat(C\Omega^\infty I, I^{\otimes *}) \circ (B(\mathcal{O})^\vee)^{r+1}, (\Sigma^\infty X)^{\otimes \leq n})
\end{tikzcd}$$

where,

- The map (7) is given by $(QT)^r \longrightarrow T^r$, where $t : QT \xrightarrow{\simeq} T$ is the cofibrant resolution of T;
- The morphism (8) is induced by

$$Q\lambda_* : B(\mathcal{O})^\vee \longrightarrow Nat(QTI, I^{\otimes *})$$

  (in Equation (5.17)) which is a weak equivalence by the computations done in Remark 5.6. This means in particular that the map (8) is a D_n-equivalence (by computation using Proposition 5.2).

Finally, since the composite $(1) \circ (5)$ is a D_n-equivalence, we deduce that $(3) \circ (6)$ is a D_n-equivalence and therefore that $\psi_{C,r}$ is also a D_n- equivalence. $\square$

Lemma 5.34. *Let $C : Alg^{fin}_{\mathcal{O}} \longrightarrow Ch$ be a finite presented cell functor. There is a commutative diagram of the form*

$$\begin{tikzcd}
C\Omega^\infty(\Sigma^\infty X) \arrow[r, "\psi''_C"] \arrow[d, "\eta^r"'] & Map^{right}_{B(\mathcal{O})^\vee}(Nat(C\Omega^\infty I, I^{\otimes *}), (\Sigma^\infty X)^{\otimes *}) \arrow[d, "Map^{right}_{B(\mathcal{O})^\vee}(\eta^{*,r}, (\Sigma^\infty X)^{\otimes *})"] \\
C\Omega^\infty T^r(\Sigma^\infty X) \arrow[r, "\psi_{C,r}"'] & Map^{right}_{B(\mathcal{O})^\vee}(Nat(C\Omega^\infty I, I^{\otimes *}) \circ (B(\mathcal{O})^\vee)^{r+1}, (\Sigma^\infty X)^{\otimes *})
\end{tikzcd}$$

where $\eta^{*,r}$ (resp. η^r) is the $r-th$ iteration of the $B(\mathcal{O})^\vee$-module (resp.T-comodule) structure morphism $\eta^* : Nat(C\Omega^\infty I, I^{\otimes *}) \circ B(\mathcal{O})^\vee \longrightarrow Nat(C\Omega^\infty I, I^{\otimes *})$ (resp. $\eta : C\Omega^\infty \longrightarrow C\Omega^\infty T$) defined in Proposition 5.15 (resp. Proposition 5.13).

Proof. The proof is only based on computation using the fact that the following diagram is commutative

$$\begin{tikzcd}
C\Omega^\infty(\Sigma^\infty X) \arrow[r] \arrow[d, "\eta^r"'] & Map_{\Sigma}(Nat(C\Omega^\infty I, I^{\otimes *}), (\Sigma^\infty X)^{\otimes *}) \arrow[d, "Map_{\Sigma}(Nat(\eta^r, I^{\otimes *}), (\Sigma^\infty X)^{\otimes *})"] \\
C\Omega^\infty T^r(\Sigma^\infty X) \arrow[r] & Map_{\Sigma}(Nat(C\Omega^\infty T I^r, I^{\otimes *}), (\Sigma^\infty X)^{\otimes *})
\end{tikzcd}$$

where the horizontal morphisms are evaluation maps. □

We can now prove Proposition 5.31.

Proof of Proposition 5.31. The proof is based on the bellow diagram

$$\begin{tikzcd}
C(X) \arrow[rr, "\delta"] \arrow[d, "\delta"] & & C\Omega^{\infty}\Sigma^{\infty}(X) \arrow[d, "\psi''_C"] \\
C\Omega^{\infty}\Sigma^{\infty}(X) \arrow[d, "\eta^{\bullet}"] & & Map^{right}_{B(\mathcal{O})^{\vee}}(Nat(C\Omega^{\infty}I, I^{\otimes *}), (\Sigma^{\infty}X)^{\otimes * \leq n}) \arrow[d, "Map^{right}_{B(\mathcal{O})^{\vee}}(\eta^{*,r}, (\Sigma^{\infty}X)^{\otimes *})"] \\
\widetilde{Tot}(C\Omega^{\infty}T^{\bullet}\Sigma^{\infty})(X) \arrow[r, "\psi_{C,r}"'] & \widetilde{Tot}(Map^{right}_{B(\mathcal{O})^{\vee}}(Nat(C\Omega^{\infty}I, I^{\otimes *}) \circ (B(\mathcal{O})^{\vee})^{\bullet+1}, (\Sigma^{\infty}X)^{\otimes * \leq n}) &
\end{tikzcd}$$

This diagram is commutative using Lemma 5.34. In addition, the most left vertical arrow is an equivalence (using §5.3.2), and hence is a D_n-equivalence. In addition Lemma 5.33 shows that the bottom horizontal map is also a D_n-equivalence. This completes this proof. □

Now we have all the ingredient to prove Theorem 5.10. In fact, since we have constructed the D_n-equivalence

$$\psi_F : F \longrightarrow \Phi_n F,$$

to compute the derivatives of F, we will simply compute the derivatives of $\Phi_n F$.

Proof of Theorem 5.10. Since the functor D_n- commutes with the totalization Tot, we make the following computation:

$$\begin{aligned}
\partial_n F \simeq \partial_n QF \simeq \partial_n \Phi_n QF & \qquad (5.29)\\
\simeq \underset{C \in Sub(QF)}{hocolim}\ \widetilde{\mathrm{Tot}}\ \partial_n Map^{right}_{B(\mathcal{O})^{\vee}}(Nat(C\Omega^{\infty}I, I^{\otimes *}) \circ (B(\mathcal{O})^{\vee})^{\bullet+1}, (\Sigma^{\infty}-)^{\otimes * \leq n}) & \qquad (5.30)\\
\simeq \underset{C \in Sub(QF)}{hocolim}\ \widetilde{\mathrm{Tot}}\ \partial_n Map_{\Sigma}(Nat(C\Omega^{\infty}I, I^{\otimes *}) \circ (B(\mathcal{O})^{\vee})^{\bullet}, (\Sigma^{\infty}-)^{\otimes * \leq n}) & \qquad (5.31)\\
\simeq \underset{C \in Sub(QF)}{hocolim}\ \widetilde{\mathrm{Tot}}\ \partial_n \underline{hom}(Nat(C\Omega^{\infty}I, I^{\otimes *}) \circ (B(\mathcal{O})^{\vee})^{\bullet}(n), (\Sigma^{\infty}-)^{\otimes n})^{\Sigma_n} & \qquad (5.32)\\
\simeq \underset{C \in Sub(QF)}{hocolim}\ \widetilde{Tot}\ \underline{\mathrm{hom}}(Nat(C\Omega^{\infty}I, I^{\otimes *}) \circ (B(\mathcal{O})^{\vee})^{\bullet}(n), \mathbb{k}) & \qquad (5.33)\\
\simeq \underset{C \in Sub(QF)}{hocolim}\ \underline{hom}(N(Nat(C\Omega^{\infty}I, I^{\otimes *}) \circ (B(\mathcal{O})^{\vee})^{\bullet}(n)), \mathbb{k}) & \qquad (5.34)\\
\simeq \underset{C \in Sub(QF)}{hocolim}\ \underline{hom}(B(Nat(C\Omega^{\infty}I, I^{\otimes *}), B(\mathcal{O})^{\vee}, \mathbb{I})(n), \mathbb{k}) & \qquad (5.35)
\end{aligned}$$

where

- The quasi-isomorphism (5.29) comes from Corollary 5.32;
- The quasi-isomorphism (5.33) comes analogously using the cross effect as in the proof of Proposition 5.2 in 1.(b).
- The functor $N : sAb \longrightarrow Ch$ in (5.34) denotes the normalization functor;
- The quasi-isomorphism (5.35) is induced by the Fresse's result ([Fre04, Thm 4.1.8]):

$$N(Nat(C\Omega^{\infty}I, I^{\otimes *}) \circ (B(\mathcal{O})^{\vee})^{\bullet}) \simeq B(Nat(C\Omega^{\infty}I, I^{\otimes *}), B(\mathcal{O})^{\vee}, \mathbb{I}).$$

□

5.4 The Taylor tower of simplicial functors

The aim of this section is to characterize the Taylor tower of simplicial functors $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$ out of the characterization of homogeneous functors developed in Chapter 2. Indeed, we will use the additional structure we have found on the derivatives $\partial_* F$ along the way in this chapter.

This characterization will be given by functors of the form

$$\begin{aligned} B(M, \mathcal{O}, -) : \mathrm{Alg}_{\mathcal{O}} &\longrightarrow Ch \\ X &\longmapsto B(M, \mathcal{O}, X) = \bigoplus_{\underline{n}} (B(M, \mathcal{O}, \widehat{X})(\underline{n}), \partial_0 + \partial) \end{aligned}$$

where

- M is a right $\mathcal{O}$-module;
- $\widehat{X} = (X, 0, ..., 0, ...)$ is the left $\mathcal{O}$-module associated to X.

An interest to this functor in Functor Calculus is not new. In fact the Goodwillie tower of this functor has been studied in [KP17, § 2.6] with a different notation: F^R_M, where R is the ground ring. These authors proved that the Taylor tower $B(M, P, -)$ identifies with

$$B(M^{* \leq 1}, P, X) \longleftarrow B(M^{* \leq 2}, P, X) \longleftarrow B(M^{* \leq 3}, P, X) \longleftarrow ...$$

where $M^{\leq *}$ is the truncation (above) of M.

In our case we will prove that, for a specific value for M and $R = \Bbbk$, this tower describes the whole category of polynomial simplicial (and finitary) functors $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$.

The main result of this section is given in the next theorem.

Theorem 5.35. *If $F : Alg_{\mathcal{O}} \longrightarrow Ch$ is a simplicial finitary functor and $P := B^c(B(\mathcal{O}))$, then the Taylor tower of F is given by*

$$P_n F(X) \simeq B(\partial_{* \leq n} F, P, X),$$

To prove this result, we will use the following road-map:

1. We first show that the Taylor tower P_*F is equivalent to the Fake tower (that we will define);
2. We will next show that the fake tower is equivalent to a tower of functors $B(\partial_{*\leq n}F, P, -), \forall n$.

After these two development, we will then prove the theorem at the end of the section.

(1) The Taylor tower is equivalent to the fake tower

Definition 5.36. *(Fake tower) Let $F : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ be a simplicial functor. The fake tower $\{\Phi_n F\}$ of F is given by*

$$... \xrightarrow{f_{n+1}} \Phi_{n+1}F \xrightarrow{f_n} \Phi_n F \xrightarrow{f_{n-1}} ... \xrightarrow{f_1} \Phi_1 F \tag{5.36}$$

where

- $\Phi_n F(X) = \underset{C \in Sub(QF)}{hocolim} Map^{right}_{B(\mathcal{O})^\vee}(B(\partial^* C\Omega^\infty, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee), (\Sigma^\infty X)^{\otimes * \leq n})$ *as defined in Lemma 5.27;*
- *The maps $f_n : \Phi_{n+1}F \longrightarrow \Phi_n F$ are naturally induced by the projection of symmetric sequences $(\Sigma^\infty X)^{*\leq n+1} \longrightarrow (\Sigma^\infty X)^{*\leq n}$.*

We have constructed in Definition 5.30 the natural transformation

$$\psi_F : F(X) \longrightarrow \Phi_n F(X)$$

and we showed in Corollary 5.32 that this is a D_n- equivalence. Now we will show that the fake tower is equivalent to the Taylor tower in characteristic 0.

Theorem 5.37. *If $F : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ is a simplicial functor, then the fake tower of Definition 5.36 is equivalent to the Taylor tower of F.*

In other to prove Theorem 5.37, we will first prove that the fake tower is really made by excisive functors.

Lemma 5.38. *Let $F : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ be a simplicial functor. The functor $\Phi_n F : Alg_{\mathcal{O}}^{fin} \longrightarrow Ch$ described in Definition 5.25 is n-excisive.*

Proof. For an arbitrary simplicial functor F, since there is an equivalence

$$F \simeq \underset{C \in Sub(QF)}{hocolim} C$$

and since the functor P_n- commutes with the filtered colimit $\underset{C \in Sub(QF)}{hocolim}$ (as always), the proof resumes to the finite cellular functor case. We now assume that F is a finite presented functor and we prove this result inductively.

1. We consider the first term
$$\Phi_1 F(X) = Map^{right}_{B(\mathcal{O})^\vee}(B(\partial^* F\Omega^\infty, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee), \Sigma^\infty X)$$
Since $\Sigma^\infty X$ viewed as a symmetric sequence in a single degree has a trivial $B(\mathcal{O})^\vee$-module structure, it implies the computation
$$\begin{aligned}\Phi_1 F(X) \cong & Map_\Sigma(B(\partial^* F\Omega^\infty, B(\mathcal{O})^\vee, \mathbb{I}), \Sigma^\infty X) && (5.37)\\ \cong & \underline{hom}(B(\partial^* F\Omega^\infty, B(\mathcal{O})^\vee, \mathbb{I})(1), \Sigma^\infty X)^{\Sigma_1} && (5.38)\\ \cong & B(\partial^* F\Omega^\infty, B(\mathcal{O})^\vee, \mathbb{I})^\vee(1) \underset{\Sigma_1}{\otimes} \Sigma^\infty X. && (5.39)\end{aligned}$$
The last isomorphism is due to the fact that the chain complex $B(\partial^* F\Omega^\infty, B(\mathcal{O})^\vee, \mathbb{I})(1)$ is of finite type as F is finite.
We conclude from this step that the functor $\Phi_1 F$ is 1-homogeneous and then in particular is 1-excisive.
2. To generate the inductive construction, we consider the fiber $\triangle_n F$ of the map $\Phi_n \overset{f_{n-1}}{\longrightarrow} \Phi_{n-1} F$ which is given by the formula
$$\triangle_n F(X) = Map^{right}_{B(\mathcal{O})^\vee}(B(\partial^* F\Omega^\infty, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee), (\Sigma^\infty X)^{\otimes n})$$
As in 1. since $(\Sigma^\infty X)^{\otimes n}$ viewed as a symmetric sequence in a single degree has a trivial module structure, we have the isomorphism
$$\triangle_n F(X) \cong Map_\Sigma(B(\partial^* F\Omega^\infty, B(\mathcal{O})^\vee, \mathbb{I}), (\Sigma^\infty X)^{\otimes n})$$
and computations as previously gives
$$\triangle_n F(X) \cong B(\partial^* F\Omega^\infty, B(\mathcal{O})^\vee, \mathbb{I})^\vee(n) \underset{\Sigma_n}{\otimes} (\Sigma^\infty X)^{\otimes n}.$$
This last equation shows that the functor $\triangle_n F$ is n-homogeneous and thus in particular is n-excisive. On the other hand, we showed in Lemma 2.9 that given a fiber sequence of homotopy functors, if two of the functors are excisive, then so is the third one. We apply this result here to claim inductively that the functor $\Phi_n F$ is n-excisive.

□

Proof of Theorem 5.37. Since $\Phi_n F$ is n-excisive (see Lemma 5.38), the map $F \longrightarrow \Phi_n F$ of Proposition 5.31 factors via the morphism $P_n F \longrightarrow \Phi_n F$. We then form the following commutative diagram:

$$\begin{tikzcd} D_n F \arrow[r, "D_n\psi_F"] \arrow[d] & D_n \Phi_n F \arrow[d] \\ P_n F \arrow[r, "P_n\psi_F"] \arrow[d] & P_n \Phi_n F \simeq \Phi_n F \arrow[d] \\ P_{n-1} F \arrow[r, "P_{n-1}\psi_F"] & P_{n-1}\Phi_{n-1} F \simeq \Phi_{n-1} F \end{tikzcd}$$

$D_n\psi_F$ is a weak equivalence using Corollary 5.32. Therefore we deduce inductively that $P_n\psi_F$ is a weak equivalence. □

(2) The fake tower is equivalent to the tower provided in Theorem 5.35

Definition 5.39. *Let $F : Alg_{\mathcal{O}} \longrightarrow Ch$ be a simplicial functor, $n \geq 0$ and $P := B^c(B(\mathcal{O}))$. We define the functor $G_n : Alg_{\mathcal{O}} \longrightarrow Ch$ by*

$$G_n(X) := B(\partial_{*\leq n}F, P, X).$$

There is a tower of fibrations

$$... \xrightarrow{g_n} G_n \xrightarrow{g_{n-1}} G_{n-1} \longrightarrow ... \xrightarrow{g_1} G_1$$

where the morphism $g_{n-1} : G_n \longrightarrow G_{n-1}$ is induced by the projection of symmetric sequences $\partial_{*\leq n}F \longrightarrow \partial_{*\leq n-1}F$.

The main result of this part is the following:

Theorem 5.40. *If $F : Alg_{\mathcal{O}} \longrightarrow Ch$ is a simplicial functor and $G_n : Alg_{\mathcal{O}} \longrightarrow Ch$ is the functor described in Definition 5.39, then there is a weak equivalence*

$$G_n \simeq \Phi_n F$$

Before proving this theorem, we will first show in the next lemma the properties of the functor G_n.

Lemma 5.41. *Let $F : Alg_{\mathcal{O}} \longrightarrow Ch$ be a simplicial functor and $n \geq 0$. The functor $G_n : Alg_{\mathcal{O}} \longrightarrow Ch$ described in Definition 5.39 is simplicial, finitary and n-excisive.*

Proof. We showed in Example 3.14 that G_n is simplicial. On the other hand G_n is finitary since colimits distribute over the graded tensor product $- \otimes -($ which is a left adjoint). It remains now to prove that G_n is n-excisive. For this we consider the following tower

$$... \xrightarrow{g_n} G_n \xrightarrow{g_{n-1}} G_{n-1} \longrightarrow ... \xrightarrow{g_1} G_1$$

We have the fiber sequence

$$B(\partial_n F, P, X) \longrightarrow G_n(X) \xrightarrow{g_{n-1}} G_{n-1}(X) \tag{5.40}$$

We make the computation

$$\begin{aligned} B(\partial_n F, P, X) &\simeq \partial_n F \underset{\Sigma_n}{\otimes} (UB(P, X))^{\otimes n} &(5.41)\\ &\simeq \partial_n F \underset{\Sigma_n}{\otimes} (UB(\mathcal{O}, X))^{\otimes n} &(5.42)\\ &\simeq \partial_n F \underset{\Sigma_n}{\otimes} (\Sigma^\infty X)^{\otimes n} &(5.43)\end{aligned}$$

where the map (5.42) is given by the quasi-isomorphism $UB(P, X) \xrightarrow{\simeq} UB(\mathcal{O}, X)$ proved in Lemma 1.11. We deduce that the functors G_1 and $B(\partial_n F, P, -)$, $\forall n$, are n-homogeneous and in particular n-excisive.

At this point, the result follows inductively using Lemma 2.9 applied each time, $\forall n$, on the fiber sequence

$$B(\partial_n F, P, X) \longrightarrow G_n(X) \overset{g_{n-1}}{\longrightarrow} G_{n-1}(X)$$

□

Now, to prove Theorem 5.40, we will need to build the map (of vector spaces)

$$\Gamma_n : \Phi_n F \longrightarrow G_n$$

We will then prove that Γ_n is a map of chain complexes and finally prove that this is a weak equivalence. This construction follows a general argument which is illustrated in the next two lemmas.

Lemma 5.42. *Let R be a finite right $B(\mathcal{O})^\vee$-module and X be an $\mathcal{O}$-algebra. Then there is an isomorphism of chain complexes*

$$B^c(R^\vee, B(\mathcal{O}), B(\mathbb{I}, \mathcal{O}, \widehat{X})) \underset{\cong}{\overset{\phi}{\longrightarrow}} Map_{B(\mathcal{O})^\vee}(B(R, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee), B(\mathcal{O}, X)^{\otimes *})$$

Lemma 5.43. *Let R be a finite right $B(\mathcal{O})^\vee$-module and X be an $\mathcal{O}$-algebra. Then there is a quasi-isomorphism*

$$B^c(R^\vee, B(\mathcal{O}), B(\mathbb{I}, \mathcal{O}, \widehat{X})) \underset{\simeq}{\overset{\psi}{\longrightarrow}} B(B^c(R^\vee, B(\mathcal{O}), \mathbb{I}), B^c B(\mathcal{O}), X)$$

Before proving the above two lemmas, we need to fix some notations.

Notation 3. *1. A tree T in $B^c(R^\vee, B(\mathcal{O}), B(\mathbb{I}, \mathcal{O}, \widehat{X}))$ has three levels:*

- *the first level consists of a root $r \in R^\vee(k)$, for some integer k;*
- *the second level or the middle consists of trees $\beta_1, ..., \beta_k$, where $\beta_i \in B^c B(\mathcal{O})(n_i)$, for some integer n_i;*
- *the last level or the top one consists of trees $T_1, ..., T_u \in B(\mathcal{O}, X)$.*

The tree T will be denoted $T = [r(\beta_1, ..., \beta_k)](T_1, ..., T_u)$.

2. Similarly, a tree $T' \in B(R, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee)$ has tree level:

- *the first level consists of a root $r' \in R(k')$, for some integer k';*
- *the second level or the middle consists of trees $\alpha_1, ..., \alpha_k$, where $\alpha_j \in BB^c(\mathcal{O}^\vee)(m_j)$, for some integer m_j;*
- *the last level or the top one consists of trees $q_1, ..., q_v$ in $B(\mathcal{O})^\vee$.*

The tree T' will be denoted $T' = [r'(\alpha_1, ..., \alpha_{k'})](q_1, ..., q_v)$.

Proof of Lemma 5.42. Let $T = [r(\beta_1, ..., \beta_k)](T_1, ..., T_u) \in B^c(R^\vee, B(\mathcal{O}), B(\mathbb{I}, \mathcal{O}, \widehat{X}))$. The morphism

$$\begin{aligned} \phi(T) : B(R, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee) &\longrightarrow B(\mathcal{O}, X)^{\otimes *} \\ T' = [r'(\alpha_1, ..., \alpha_{k'})](q_1, ..., q_v) &\longmapsto \phi(T)(T') \end{aligned}$$

is defined by:

- if $k = k', n_i = m_i, \forall i$ and $u = v$, then

  $$\phi(T)(T') =< r, r' >< \beta_1, \alpha_1 > ... < \beta_k, \alpha_k > m(T_1 \otimes ... \otimes T_u \otimes q_1 ... \otimes q_u)$$

  where $m : B(\mathcal{O}, X)^{\otimes *} \circ B(\mathcal{O})^\vee \longrightarrow B(\mathcal{O}, X)^{\otimes *}$ is the structure map of the right $B(\mathcal{O})^\vee$-module $B(\mathcal{O}, X)^{\otimes *}$.

- If any of the above condition is not satisfied, then $\phi(T)(T') = 0$.

By construction $\phi(T)$ is a well defined morphism of right $B(\mathcal{O})^\vee$-modules. In addition, ϕ is a well defined map of chain complexes as it is basically defined by the means of evaluation maps.

On the other hand, let $U : Ch \longrightarrow gVect_{\mathbb{k}}$ be the forgetful functor from the chain complex category Ch to the category $gVect_{\mathbb{k}}$ of graded vector spaces. The functor essentially forget the differential of chain complexes. We make the following computation:

$$\begin{aligned} UMap_{B(\mathcal{O})^\vee}(B(R, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee), B(\mathcal{O}, X)^{\otimes *}) &\cong UMap_\Sigma(B(R, B(\mathcal{O})^\vee, \mathbb{I}), B(\mathcal{O}, X)^{\otimes *}) \\ &\cong UB(R, B(\mathcal{O})^\vee, \mathbb{I})^\vee \circ B(\mathcal{O}, X) \\ &\cong UR^\vee \circ B^c B\mathcal{O} \circ B(\mathbb{I}, \mathcal{O}, \widehat{X}) \\ &\cong UB^c(R^\vee, B^c B\mathcal{O}, B(\mathbb{I}, \mathcal{O}, \widehat{X})) \end{aligned}$$

Note that $U\phi$ is one of the map in this isomorphism and let ϕ' being its inverse. It remains now to show that ϕ' commutes with the differentials from its domain and codomain.

Since $\phi d = d\phi$, we have equivalently $d = \phi' d\phi$. When we pre-compose this last equation with ϕ', we get $d\phi' = (\phi' d\phi)\phi' = \phi' d$. Thus ϕ' is a morphism of chain complexes. This completes the proof.

□

Proof of Lemma 5.43. We define the morphism of vector spaces

$$\psi : \underbrace{B^c(R^\vee, B(\mathcal{O}), B(\mathbb{I}, \mathcal{O}, \widehat{X}))}_{= R^\vee \circ B^c B(\mathcal{O}) \circ B(\mathcal{O}) \circ \widehat{X}} \longrightarrow \underbrace{B(B^c(R^\vee, B(\mathcal{O}), \mathbb{I}), B^c B(\mathcal{O}), X)}_{= R^\vee \circ B^c B(\mathcal{O}) \circ B B^c B(\mathcal{O}) \circ \widehat{X}}$$

as

$$\psi = R^\vee \circ B^c B(\mathcal{O}) \circ \eta \circ \widehat{X}$$

where $\eta : B(\mathcal{O}) \longrightarrow B B^c B(\mathcal{O})$ is the unit of the cobar-bar adjunction $B^c \vdash B$ applied to the cooperad $B(\mathcal{O})$.

By definition, the morphism ψ preserves almost all the differentials. We only have to show that the twisting differential δ_0 induced by the left $B(\mathcal{O})$-comodule

$$B(\mathcal{O}, X) \longrightarrow B(\mathcal{O}) \circ B(\mathcal{O}, X)$$

in the source $B^c(R^\vee, B(\mathcal{O}), B(\mathbb{I}, \mathcal{O}, \widehat{X}))$ is converted (through ψ) into the twisting differential δ_1 induced by the right $B^cB(\mathcal{O})$-module map

$$\underbrace{B^c(R^\vee, B(\mathcal{O}), \mathbb{I}) \circ B^cB\mathcal{O}}_{=R^\vee \circ B^cB\mathcal{O} \circ B^cB\mathcal{O}} \longrightarrow \underbrace{B^c(R^\vee, B(\mathcal{O}), \mathbb{I})}_{=R^\vee \circ B^cB\mathcal{O}}$$

in the target $B(B^c(R^\vee, B(\mathcal{O}), \mathbb{I}), B^cB(\mathcal{O}), X)$.

We have in fact the commutative diagram:

$$\begin{tikzcd}
B^cB(\mathcal{O}) \circ B(\mathcal{O}) \circ \widehat{X} \arrow[r, "B^cB(\mathcal{O})\circ\eta\circ\widehat{X}"] \arrow[d, "B^cB(\mathcal{O})\circ m^c\circ\widehat{X}"] & B^cB(\mathcal{O}) \circ BB^cB(\mathcal{O}) \circ \widehat{X} \arrow[d, "B^cB(\mathcal{O})\circ\gamma'\circ\widehat{X}"] \\
B^cB(\mathcal{O}) \circ B(\mathcal{O}) \circ B(\mathcal{O}) \circ \widehat{X} \arrow[d, "\gamma\circ B(\mathcal{O})\circ\widehat{X}"] & B^cB(\mathcal{O}) \circ B^cB(\mathcal{O}) \circ BB^cB(\mathcal{O}) \circ \widehat{X} \arrow[d, "\overline{m}\circ BB^cB(\mathcal{O})\circ\widehat{X}"] \\
B^cB(\mathcal{O}) \circ B(\mathcal{O}) \circ \widehat{X} \arrow[r, "B^cB(\mathcal{O})\circ\eta\circ\widehat{X}"] & B^cB(\mathcal{O}) \circ BB^cB(\mathcal{O}) \circ \widehat{X}
\end{tikzcd}$$

where

- The map $m^c : B(\mathcal{O}) \longrightarrow B(\mathcal{O}) \circ B(\mathcal{O})$ is the cooperad co-multiplication;
- The map $\gamma : B^cB(\mathcal{O}) \circ B(\mathcal{O}) \longrightarrow B^cB(\mathcal{O})$ is the composite

$$B^cB(\mathcal{O}) \circ B(\mathcal{O}) \longrightarrow B^cB(\mathcal{O}) \circ B^cB(\mathcal{O}) \longrightarrow B^cB(\mathcal{O})$$

  where the first map is induced by the -1 inclusion map $B(\mathcal{O}) \longrightarrow B^cB(\mathcal{O})$ and the second map is the operad multiplication. Note that this multiplication is basically due by grafting trees;
- The map $\gamma' : BB^cB(\mathcal{O}) \longrightarrow B^cB(\mathcal{O}) \circ BB^cB(\mathcal{O})$ is the composite

$$BB^cB(\mathcal{O}) \longrightarrow BB^cB(\mathcal{O}) \circ BB^cB(\mathcal{O}) \longrightarrow B^cB(\mathcal{O}) \circ BB^cB(\mathcal{O})$$

  where the first map is the cooperad coproduct and the second map is induced by the -1 projection $BB^cB(\mathcal{O}) \longrightarrow B^cB(\mathcal{O})$;
- The map $\overline{m} : B^cB(\mathcal{O})\circ B^cB(\mathcal{O}) \longrightarrow B^cB(\mathcal{O})$ is the operad multiplication. Note that this multiplication is basically due by grafting trees.

The most left vertical composite of this diagram is the twisting differential δ_0 while the most right vertical composite is the twisting differential δ_1. Therefore we can conclude that ψ is a map of chain complexes. In addition, it is a quasi-isomorphism as η is a quasi-isomorphism. $\square$

Proof of Theorem 5.40. Our goal in this proof is to establish that there is a natural map $\Gamma_n : \Phi_n F \longrightarrow G_n$ which is a weak equivalence.

Note that

$$G_n(X) = \operatorname*{colim}_{C \in Sub(QF)} B(B(\partial^* C\Omega^\infty, B(\mathcal{O})^\vee, \mathbb{I})^\vee_{\leq n}, P, X) \tag{5.44}$$

Since the colimit of Equation (5.44) is a filtered colimit, this reduces to simply define a map

$$\Gamma_{n,C} : \Phi_n C \longrightarrow B(B(\partial^* C\Omega^\infty, B(\mathcal{O})^\vee, \mathbb{I})^\vee_{\leq n}, P, -)$$

and prove that it is a weak equivalence, $\forall C \in Sub(QF)$.

In the case that $R = \partial^* C\Omega^\infty$, the combination of Lemma 5.42 and Lemma 5.43 gives the diagram

$$\begin{tikzcd} B^c(R^\vee, B(\mathcal{O}), B(\mathbb{I}, \mathcal{O}, \widehat{X})) \arrow[r, "\phi", "\cong"'] \arrow[d, "\psi", "\simeq"'] & Map_{B(\mathcal{O})^\vee}(B(R, B(\mathcal{O})^\vee, B(\mathcal{O})^\vee), B(\mathcal{O}, X)^{\otimes *}) \\ B(B^c(R^\vee, B(\mathcal{O}), \mathbb{I}), B^c B(\mathcal{O}), X) & \end{tikzcd}$$

Therefore $\Gamma_{n,C}$ is simply the restriction of the composite $\psi \circ \phi^{-1}$ to the domain $\Phi_n C(X)$. Thus we are done.

$\square$

Proof of Theorem 5.35. We observe the following fact: Let X be a $\mathcal{O}$-algebra. We have the cofibrant resolution $\Omega_{\mathcal{O}}(B(\mathcal{O}, X)) \xrightarrow{\simeq} X$ which provides X with a functorial quasi-free $\mathcal{O}$-algebra resolution. On the other hand, any quasi-free $\mathcal{O}$-algebra is a filtered colimit of finitely generated $\mathcal{O}$-algebras.

Using this observation, and the fact that the two functors F and $B(\partial_{*\leq n} F, O, -)$ are both finitary (and thus $P_n F$ also), our argument reduces to proving that

$$P_n F(X) \simeq B(\partial_{*\leq n} F, P, X),\ \forall X \in \mathrm{Alg}^{fin}_{\mathcal{O}}$$

On the other hand we proved in Theorem 5.40 that $G_n \simeq \Phi_n F$. In addition, we proved in Theorem 5.37 that $P_n F \simeq \Phi_n F$. The result then follows. $\square$

5.5 Examples

In this thesis, we have characterized the Taylor tower of simplicial functors $F : \mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$ which is given by the formula

$$P_n F(X) \simeq B(\partial_{*\leq n} F, B^c B(\mathcal{O}), X) \tag{5.45}$$

We will now use this formula to describe the approximation of two functors, which means concretely computing their derivatives (as a module) and plug them respectively into Equation(5.45).

5.5.1 Example 1: Taylor tower of $IU : \mathbf{Alg}_{\mathcal{O}} \longrightarrow Ch$

We consider the composite

$$\mathrm{Alg}_{\mathcal{O}} \xrightarrow{U} Ch_+ \xrightarrow{I} Ch$$

where U is the forgetful functor and I is the embedding functor. The computation of the derivatives of the functor IU using the formula cross-effect is straightforward as in Proposition 2.41 and gives

$$\partial_* IU \simeq \mathcal{O}$$

This proves also that $\partial_* IU$ is in particular of finite type. Therefore using a similar argument as in Corollary 5.5 and using Theorem 5.10, we can deduce that

$$\partial_* IU \simeq \underline{hom}(B(Nat(QIU\Omega^\infty I, I^{\otimes *}), B(\mathcal{O})^\vee, \mathbb{I}), \Bbbk)$$

On the other hand, $IU\Omega^\infty I = I$ and since

$$Nat(Q(I^{\otimes n}), I^{\otimes m}) \simeq \begin{cases} \Bbbk[\Sigma_n] & \text{if } n = m; \\ 0 & \text{if } n \neq m. \end{cases}$$

we get the equivalence

$$\partial_* IU \simeq \underline{hom}(B(\mathbb{I}, B(\mathcal{O})^\vee, \mathbb{I}), \Bbbk) \cong B^c B(\mathcal{O}).$$

In conclusion, we get the Taylor tower of IU by plugging this last computation in Equation(5.45):

$$P_n IU(X) \simeq B(B^c B(\mathcal{O})_{*\leq n}, B^c B(\mathcal{O}), X)$$

which is also equivalent to

$$P_n IU(X) \simeq B(\mathcal{O}_{*\leq n}, \mathcal{O}, X)$$

5.5.2 Example 2: Taylor tower of the representable functor $\widetilde{\mathbf{Alg}}_{\mathcal{O}}(X, -)$

We consider in this example the representable functor $\mathrm{Alg}_{\mathcal{O}} \longrightarrow Ch$ defined as

$$\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, -) := N\Bbbk Hom_{\mathrm{Alg}_{\mathcal{O}}}(\Omega_{\mathcal{O}} B(\mathcal{O}, X), red_0(Apl_\bullet \otimes -))$$

and we want to construct its Taylor tower. As previously we first compute its derivatives using Theorem 5.10. This is given in the next proposition and its proof is provided at the end of the section.

Proposition 5.44. *Let $X \in Alg_{\mathcal{O}}$. The derivative of the representable functor $\widetilde{Alg}_{\mathcal{O}}(X, -) : Alg_{\mathcal{O}} \longrightarrow Ch$ is given by the equivalence:*

$$\partial_* \widetilde{Alg}_{\mathcal{O}}(X, -) \simeq B((\Sigma^\infty X)^{\otimes *}, B(\mathcal{O})^\vee, \mathbb{I})^\vee$$

Now we use this result to get the Taylor tower of $\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, -)$ by plugging the computation of its derivatives in Equation(5.45):

$$P_n \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, -)(Y) \simeq B(B((\Sigma^\infty X)^{\otimes *}, B(\mathcal{O})^\vee, \mathbb{I})^\vee_{*\leq n}, B^c B(\mathcal{O}), Y)$$

Equivalently, we can use in addition Lemma 5.43, and give the approximation on the form

$$P_n\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,-)(Y) \simeq B^c(R^{\vee}_{*\leq n}, B(\mathcal{O}), B(\mathbb{I},\mathcal{O},\widehat{Y}))$$

where

- $R = (\Sigma^\infty X)^{\otimes *}$;
- $\widehat{Y}$ is the left $\mathcal{O}$-module associated to the $\mathcal{O}$-algebra Y.

Proof of Proposition 5.44. Using Theorem 5.10, we have the equivalence

$$\partial_*\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,-) \simeq B(Nat(\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,\Omega^\infty-), I^{\otimes *}), B(\mathcal{O})^\vee, \mathbb{I})^\vee \tag{5.46}$$

Thus it remains to prove the equivalence : $\forall n$,

$$Nat(\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,\Omega^\infty-), I^{\otimes n}) \simeq (\Sigma^\infty X)^n. \tag{5.47}$$

For this computation, we will need two arguments:

1. The functor $\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,\Omega^\infty-)$ is cofibrant in $[Ch^{fin}, Ch]_0$. In fact consider the diagram

$$\begin{tikzcd} & G \arrow[d, two heads, "\simeq"] \\ \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,\Omega^\infty-) \arrow[r] & F \end{tikzcd}$$

When we compose this diagram with $\Sigma^\infty : \widetilde{\mathrm{Alg}}_{\mathcal{O}} \longrightarrow Ch$, we form the following diagram in $[\mathrm{Alg}^{fin}_{\mathcal{O}}, Ch]_0$:

$$\begin{tikzcd} & & G\Sigma^\infty \arrow[d, two heads, "\simeq"] \\ \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,-) \arrow[r, "\delta"] & \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,\Omega^\infty\Sigma^\infty-) \arrow[r] & F\Sigma^\infty \end{tikzcd}$$

where δ is the natural transformation defined in Lemma 5.14.
Now since $\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,-)$ is cofibrant in $[\mathrm{Alg}^{fin}_{\mathcal{O}}, Ch]_0$, there is a map

$$\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,-) \longrightarrow G\Sigma^\infty$$

such that we get the following diagram

$$\begin{tikzcd} & G\Sigma^\infty \arrow[d, two heads, "\simeq"] \\ \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X,-) \arrow[r] \arrow[ur] & F\Sigma^\infty \end{tikzcd}$$

When we compose the above diagram with Ω^∞, we get the diagram

$$\begin{tikzcd} & G\Sigma^\infty\Omega^\infty \arrow[r] \arrow[d, "\simeq"] & G \arrow[d, "\simeq"] \\ \widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, \Omega^\infty -) \arrow[r] \arrow[ur] & F\Sigma^\infty\Omega^\infty \arrow[r] & F \end{tikzcd}$$

where the most right square is induced by the comonad counit

$$T = \Sigma^\infty\Omega^\infty \longrightarrow 1.$$

This prove our claim.

2. We showed in the proof of Lemma 4.11 the weak equivalence

$$\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, \Omega^\infty I-) \simeq \widetilde{Ch}_+(UB(\mathcal{O}, X), -), \tag{5.48}$$

where $U : \mathrm{coAlg}_{B(\mathcal{O})} \longrightarrow Ch_+$ is the forgetful functor.

Since the left and right hand side of Equation (5.48) is cofibrant in $[Ch^{fin}, Ch]_0$, then we use the fact that $Nat(-, I^{\otimes n})$ preserves weak equivalences between cofibrant functors (see Corollary 4.22) to deduce that

$$\begin{aligned} Nat(\widetilde{\mathrm{Alg}}_{\mathcal{O}}(X, \Omega^\infty -), I^{\otimes n}) &\simeq Nat(\widetilde{Ch}_+(UB(\mathcal{O}, X), -), I^{\otimes n}) & (5.49) \\ &\simeq (\Sigma^\infty X)^n, & (5.50) \end{aligned}$$

where the last equation is deduced from the Yoneda Lemma.

□

Appendix

In this section, we generalize some of our constructions in Ch_+ to the category $Ch_{\geq p}$ of p-bounded bellow chain complexes, for an arbitrary integer p. Note that these two categories are equivalent. The main objective of the section is to compute the chain complex $Nat(\widetilde{Ch}_+(V, s^p-), I)$.

(1) The Ch-enriched category $\widetilde{Ch}_+$ defined in §3.2 can be extended to the Ch-category $\widetilde{Ch}_{\geq p}$. That is the enriched category whose

- Objects are chain complexes in $Ch_{\geq p}$;
- The enriched hom functor denoted $\widetilde{Ch}_{\geq p}(-, -)$ is given by: $\forall V, W \in Ch_{\geq p}$,

$$\widetilde{Ch}_{\geq p}(V, W) := N\Bbbk Hom_{Ch_{\geq p}}(N\Bbbk\triangle^\bullet \otimes V, W).$$

(2) The relation between $\widetilde{Ch}_+$ and $\widetilde{Ch}_{\geq p}$ is given through the following computation:

$$\begin{aligned}\widetilde{Ch}_+(V,W) &= N\Bbbk Hom_{Ch_+}(N\Bbbk\triangle^\bullet \otimes V, W)\\ &\cong N\Bbbk Hom_{Ch}(N\Bbbk\triangle^\bullet, \underline{hom}(V,W))\\ &\cong N\Bbbk Hom_{Ch}(N\Bbbk\triangle^\bullet, \underline{hom}(s^pV, s^pW))\\ &\cong N\Bbbk Hom_{Ch_+}(N\Bbbk\triangle^\bullet \otimes s^pV, s^pW)\\ &\cong N\Bbbk Hom_{Ch_{\geq p}}(N\Bbbk\triangle^\bullet \otimes s^pV, s^pW) = \widetilde{Ch}_{\geq p}(s^pV, s^pW).\end{aligned}$$

(3) The notion of simplicial functors $\widetilde{F} : \widetilde{Ch}_{\geq p} \longrightarrow \widetilde{Ch}$ can also be defined as in §3.3.

(4) We define the chain complex $Nat_{Ch_{\geq p}}(\widetilde{F}, \widetilde{G})$ of natural transformation between two simplicial functors $\widetilde{F}, \widetilde{G} : \widetilde{Ch}_{\geq p} \longrightarrow \widetilde{Ch}$ similarly to Definition 4.1. More precisely,

$Nat_{Ch_{\geq p}}(\widetilde{F}, \widetilde{G}) := lim(\prod_{V\in Ch^{fin}_{\geq p}} \underline{hom}(\widetilde{F}(V), \widetilde{G}(V)) \rightrightarrows \prod_{V,W\in Ch^{fin}_{\geq p}} \underline{hom}(\widetilde{Ch}_{\geq p}(V,W)\otimes$
$\widetilde{F}(V), \widetilde{G}(W)))$

(5) There is a general version of the strong Yoneda lemma (see Lemma 4.6):

$$Nat_{Ch_{\geq p}}(V_0 \otimes \widetilde{Ch}_{\geq p}(V,-), \widetilde{G}) \cong \underline{hom}(V_0, \widetilde{G}(V)).$$

The proof is analogous to the proof of Lemma 4.6.

(6) Using the above items, we are ready to show that

$$Nat(\widetilde{Ch}_+(V, s^p-), I) \cong s^{-p}V \tag{5.51}$$

We make the following computation

$$Nat(\widetilde{Ch}_+(V, s^p-), I) = \tag{5.52}$$

$$lim(\prod_{V' \in Ch_+^{fin}} \underline{hom}(\widetilde{Ch}_+(V, s^pV'), V') \rightrightarrows \prod_{V', W \in Ch_+^{fin}} \underline{hom}(\widetilde{Ch}_+(V', W) \otimes \widetilde{Ch}_+(V, s^pV'), W)) = \tag{5.53}$$

$$lim(\prod_{V' \in Ch_+^{fin}} \underline{hom}(\widetilde{Ch}_+(V, s^pV'), V') \rightrightarrows \prod_{V', W \in Ch_+^{fin}} \underline{hom}(\widetilde{Ch}_{\geq p}(s^pV', s^pW) \otimes \widetilde{Ch}_+(V, s^pV'), W)) = \tag{5.54}$$

$$lim(\prod_{V' \in Ch_{\geq p}^{fin}} \underline{hom}(\widetilde{Ch}_{\geq p}(V, V'), s^{-p}V) \rightrightarrows \prod_{V', W \in Ch_{\geq p}^{fin}} \underline{hom}(\widetilde{Ch}_{\geq p}(V', W) \otimes \widetilde{Ch}_{\geq p}(V, V'), s^{-p}W)) = \tag{5.55}$$

$$= Nat_{Ch_{\geq p}}(\widetilde{Ch}_{\geq p}(V, -), s^{-p}I) \cong s^{-p}V \tag{5.56}$$

More generally, there is an isomorphism (obtained in the similar way)

$$Nat_{Ch_+}(V_0 \otimes \widetilde{Ch}_+(V, s^p-), \widetilde{F}) \cong Nat_{Ch_{\geq p}}(V_0 \otimes \widetilde{Ch}_{\geq p}(V, -), \widetilde{F}s^{-p}-) \tag{5.57}$$
$$\cong \underline{hom}(V_0, \widetilde{F}(s^{-p}V)) \tag{5.58}$$

There is a model structure on $[Ch_{\geq p}^{fin}, Ch]_0$ analogous to §4.2 and Equation (5.57) permits to claim that the functor $V_0 \otimes \widetilde{Ch}_+(V, s^p-) : Ch_+ \longrightarrow Ch$ is cofibrant in $[Ch_+^{fin}, Ch]_0$.

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