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Exploiting separability in a multisectoral model of oligopolistic competition

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ABSTRACT

The paper uses the most general version of a Dixit–Stiglitz economy and the concept of oligopolistic equilibrium, defined in previous work, with firms maximizing profits in prices and quantities under a market share and a market size constraint. The purpose here is to take even more advantage of separability so as to partition the oligopolistic sector into groups. Weak separability simplifies quantity conjectures and homothetic separability simplifies price conjectures. Oligopolistic equilibria can in addition be approximated by introducing group expenditure conjectures. Finally, the way different groups interact within the same industry is illustrated within the same framework.

1. Introduction

The main obstacles in developing a theory of General Oligopolistic Equilibrium[1] have been well understood. They are of two types. First, from the modeler's point of view, combining the difficulties inherent to oligopoly theory, already present in a partial equilibrium context, with those of general equilibrium theory leads easily to intractability and even to non-existence of equilibrium. Second, from the players' point of view, it may be unrealistic to suppose that they are able or willing to take into account all the conceivable interactions, however weak, that might concern them. Fortunately, Dixit and Stiglitz (1977) contribution afforded a popular and successful way to bypass these obstacles, by reducing the economy to two sectors, one competitive, the other oligopolistic with identical firms supplying final products to a representative consumer, and then by considering the limiting monopolistic competition case of insignificant firms so as to eliminate strategic interactions (each firm behaving as a monopoly in its own niche). Because of its tractability, this type of modeling has become dominant in trade and macroeconomic applications, first on the basis of the simple CES sub-utility case and, more recently, beyond the CES, by relaxing either homotheticity or additivity.[2]

We want to argue that going beyond the CES case is not the only way of gaining in model flexibility and applicability. Even allowing for strategic interactions between large heterogeneous firms in the oligopolistic sector, alternative simplifications may keep the model tractable and give a reasonable account of firms' conjectures and calculations. In previous work (d' Aspremont and Dos Santos Ferreira, 2016, 2017) we have defined a concept of oligopolistic equilibrium by referring to a Dixit–Stiglitz economy, with two sectors corresponding to the two arguments of the representative consumer's separable utility function[3] $U(X(\mathbf{x}), z)$, where X is a function aggregating n differentiated goods into a single composite good and z is the quantity of a numeraire good, the composition of which is left implicit.[4] Here, we want to go a step further, by assuming separability of X itself:[5] $X(\mathbf{x}) \equiv \widetilde{X}\left(X^1\left(\mathbf{x}^1\right), \ldots, X^K\left(\mathbf{x}^K\right)\right)$, meaning that the set of the n differentiated goods can be partitioned into K groups, aggregated each into a composite good through the corresponding aggregator function X^k ($k = 1, \ldots, K$).

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1 For a survey, see for instance Hart (1985), Bonanno (1990) or d' Aspremont et al. (1999).

2 Examples of homotheticity relaxations are to be found in Krugman (1979), Behrens and Murata (2007) and Zhelobodko et al. (2012). Models allowing for non-additive preferences appear in Feenstra (2003) and Melitz and Ottaviano (2008) or Bertoletti and Epifani (2014).

3 Vectors are denoted in bold.

4 As well emphasized by Neary (2000), "previous writers had debated the appropriate definition of an "industry", or, in Chamberlin's preferred term, a "group". Typically, definitions were given in terms of cross-elasticities of demand, sometimes of *both* direct and inverse demand functions. [...] DS cut through all this fog: instead of restricting the demand functions by imposing arbitrary limits on inter- and intra-industry substitutability, they made a single restriction on the utility function, which implies that (in symmetric equilibria) all products within an industry should have the *same* degree of substitutability with other goods" (p.4).

5 We denote $\mathbf{x}^k \equiv \left(x_i^k\right)_{i=1}^{n_k} \in \mathbb{R}_+^{n_k}$ and $X^k : \mathbf{x}^k \mapsto X^k\left(\mathbf{x}^k\right) \in \mathbb{R}_+$.

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We could of course continue to use the same equilibrium concept, ignoring the fact that the consumer's preferences have more structure now. However, if we suppose that the goods in each group are traded in the same *relevant market*, we can exploit the additional separability by redefining the oligopolistic equilibrium concept in a way that reduces the range of firms' conjectures and simplifies their calculations. The partition of the oligopolistic sector into relevant markets is indeed the basis for the formation of conjectures by the actors about their competitors' actions and their general environment. These conjectures are detailed when the competitors are close rivals or partners, but tend to become coarser and coarser as they cover more and more distant activities. Hence, the actors' conjectures are naturally structured by this partition, which determines the level of aggregation of the required information. In the terminology we shall use, the partition of the oligopolistic sector into relevant markets is identified with the separability of sub-utility X into *groups* of goods.[6] We keep Dixit–Stiglitz two-*sector* distinction when referring to the separability between the numeraire good z and the differentiated goods $\mathbf{x}$.

Our basic concept of oligopolistic equilibrium supposes that firms behave strategically in price-quantity pairs, maximizing profits under two constraints, on market share and on market size. Taking advantage of the partition of the oligopolistic sector into relevant markets, the market share constraint of each firm will concern the conjectured actions of the sole group of competitors acting in the corresponding relevant market. Also, the market size constraint will involve less informational requirements: conjectures about quantities in other groups are not part of them and in addition, under homothetic separability, only conjectures about group-specific price indices are required.

Different degrees and types of separability will be exploited, with two different orientations. One is to make more apparent the general equilibrium structure of the model by considering groups of goods that are linked by close relations of substitutability or complementarity, with simplified interactions between groups. The other is to focus on the partial equilibrium dimension, looking more closely, within an industry, at the interaction of groups of firms characterized by similar degrees of competitive toughness.

The main output of our approach to oligopolistic behavior is to derive an equilibrium markup formula to be used to measure market conduct. The formula derived in the previous (above-mentioned) work has already been used in an empirical application. Sakamoto and Stiegert (2018) study sales data of ground coffee in the US in order to evaluate, using this formula, the market conduct of the main brands, clustered into two groups, identified to the dominant group and the competitive fringe but distinguished on the basis of preferences separability. They find that "the methodology is not burdensome to implement empirically because its primary requirements are estimates of elasticities of substitution". The objective of the present paper is to make the methodology even less burdensome, by restricting the relevant market of the competitors. As compared to the New Empirical Industrial Organization (NEIO) approach when extended to differentiated products (*e.g.* Nevo, 1998), our approach is more parsimonious in the parameter space: the number of conduct parameters increases linearly with the number of goods and not with the square of the number of goods. But it keeps the flexibility of the NEIO approach, since the conduct parameters to be estimated are continuous, in contrast to the so-called "menu approach" where a menu of models to be tested (say Bertrand vs. collusion) is fixed in advance. As Schmalensee (2012) points out, "the best way forward may be to attempt to develop and employ parsimonious parameterizations in the spirit of the 'conjectural variations' approach that can provide reliable reduced-form estimates of the location of conduct along 'the in-between range of incomplete collusion'" (p.172).

The consequences of weak separability in simplifying quantity conjectures are analyzed in Section 2. Those of stronger – homothetic – separability in further simplifying price conjectures are examined in Section 3. Section 4 introduces the idea of approximating oligopolistic equilibria by supposing that each firm forms a conjecture about the income to be spent in its group as if this group were independent. Section 5 considers the way different groups interact within the same industry. We formulate some concluding remarks in Section 6.

2. Oligopolistic competition under weak separability

We consider two sectors, one oligopolistic, producing a composite good, the other perfectly competitive, producing the numeraire good. Each firm in the oligopolistic sector produces a single good. The oligopolistic sector can be highly heterogeneous, containing groups of firms differentiated either in terms of the degree of substitutability between their products or in terms of the degree of competitive toughness they display. We are interested in exploiting such differences by identifying groups of firms which produce goods that are closely linked by relations either of substitutability or of complementarity, or else compete among themselves in a more or less uniform way. In addition, we want to diminish by aggregation the information used by each firm on the strategies of competitors belonging to other groups. As mentioned in the introduction, in order to pursue these objectives, we assume *weak separability* of the representative consumer's utility into two sectors, the competitive sector producing the numeraire good z and the oligopolistic sector itself divided into K groups of goods,

$$\begin{aligned} U\left(X\left(\mathbf{x}\right),z\right) &= U\left(\widetilde{X}\left(X^{1}\left(\mathbf{x}^{1}\right),\ldots,X^{K}\left(\mathbf{x}^{K}\right)\right),z\right) \\ &\equiv \widetilde{U}\left(X^{1}\left(\mathbf{x}^{1}\right),\ldots,X^{K}\left(\mathbf{x}^{K}\right),z\right), \end{aligned} \tag{1}$$

where X^1, …, X^K are increasing functions.

2.1. Demand

Thanks to separability, we may consider two stages when solving the utility maximization program: (i) minimizing the expenditure on each composite good $X^k\left(\mathbf{x}^k\right)$, $k = 1,\ldots,K$, by choosing the appropriate quantity x_i^k of each differentiated good $i = 1,\ldots,n_k$ in group k, while ensuring at least some level X_k of the aggregate; (ii) maximizing $U\left(X_1,\ldots,X_K,z\right)$ under the budget constraint. We denote $n \equiv \sum_k n_k$.

We obtain at the first stage, for each k, the expenditure function:

$$\min_{\mathbf{x}^k\in\mathbb{R}_+^{n_k}}\left\{\mathbf{p}^k\mathbf{x}^k\left|X^k\left(\mathbf{x}^k\right)\geq X_k\right.\right\}\equiv e^k\left(\mathbf{p}^k,X_k\right). \tag{2}$$

The solution to the preceding expenditure minimization problem satisfies:[7]

$$p_i^k = \partial_X e^k\left(\mathbf{p}^k,X_k\right)\partial_i X^k\left(\mathbf{x}^k\right) \qquad \text{(first order condition)} \tag{3}$$

$$x_i^k = \partial_i e^k\left(\mathbf{p}^k,X_k\right)\equiv H_i^k\left(\mathbf{p}^k,X_k\right) \qquad \text{(Shephard's lemma)}, \tag{4}$$

where H_i^k is the *Hicksian demand function* for good i in group k.

6 The term "industry" (as used for instance in the S&P 500 index) is more ambiguous and may or may not correspond to a group in our sense.

7 We denote $\partial_x f\left(x,y\right)\equiv\partial f\left(x,y\right)/\partial x$ and, when there is no ambiguity, $\partial_i f\left(\mathbf{x},y\right)\equiv\partial f\left(\mathbf{x},y\right)/\partial x_i$.

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Referring to the representative consumer's income Y, the second stage program can then be written as

$$\max_{(X_1,\ldots,X_K,z)\in\mathbb{R}_+^{K+1}} \left\{ \widetilde{U}\left(X_1,\ldots,X_K,z\right) \middle| \sum_{k=1}^{K} e^k\left(\mathbf{p}^k, X_k\right) + z \leq Y \right\}, \quad (5)$$

leading to the solution $X_k = D^k\left(\mathbf{p}^1,\ldots,\mathbf{p}^K,Y\right)$ for each k, and $z = Y - \sum_{k=1}^{K} e^k\left(\mathbf{p}^k, D^k\left(\mathbf{p}^1,\ldots,\mathbf{p}^K,Y\right)\right)$, where D^k is the *Marshallian demand function* for the composite good k.[8]

2.2. Profit maximization and equilibrium

A strict application of the oligopolistic equilibrium concept defined in d' Aspremont and Dos Santos Ferreira (2016, 2017) would lead us to consider the maximization in $\left(p_i^k, x_i^k\right)$ by the firm producing good i in group k of its profit $\left(p_i^k - c_i^k\right)x_i^k$ (where c_i^k is the constant marginal cost) under two constraints, a constraint on *market share*, expressing competition against other firms in the oligopolistic sector, and a constraint on *market size*, expressing competition against the competitive firms in the other sector. Such an application can of course be done in the present context, but it does not take into account the decomposition of the oligopolistic sector, which leads to the natural association of relevant markets with groups, rather than the reference to an integrated market associated with the whole oligopolistic sector. If we identify a market k with the group k, the constraint on market share of any firm in this group will refer to competition against other firms in the same group. By the same token, the constraint on market size will refer to competition against oligopolistic firms in the other groups, as well as against competitive firms in the other sector.

We thus adapt the concept defined in d' Aspremont and Dos Santos Ferreira (2016) along these lines.[9] In order to define an equilibrium concept at the economy level, we will further assume an inelastic supply of L units of labor and a wage equal to 1, the constant unit cost in the competitive sector. Income is accordingly equal to the sum of wages and profits, namely $Y = L + \Pi$, where Π is the profit of the imperfectly competitive sector (the profit of the other sector being necessarily zero). We shall also use the standard simplifying notations $\mathbf{p} = \left(\mathbf{p}^1,\ldots,\mathbf{p}^K\right) = \left(\mathbf{p}^k, \mathbf{p}^{-k}\right) = \left(p_i^k, \mathbf{p}_{-i}^k, \mathbf{p}^{-k}\right)$ in order to point to the (price) strategy of firm i in group k.

Definition 1. An *oligopolistic equilibrium* is a K-tuple of $2n_k$-tuples $\left(\mathbf{p}^{k*}, \mathbf{x}^{k*}\right)_k$ in $\mathbb{R}_+^{2n}$ such that, for any i in group k,

$$\left(p_i^{k*}, x_i^{k*}\right) \in \arg\max_{\left(p_i^k, x_i^k\right)\in\mathbb{R}_+^2} \left(p_i^k - c_i^k\right)x_i^k \quad (6)$$

$$\text{s.t.} \quad x_i^k \leq H_i^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right)\right)$$

$$\text{and } X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right) \leq D^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, \mathbf{p}^{-k*}, Y^*\right),$$

with $Y^* = L + \sum_{k=1}^{K}\sum_{i=1}^{n_k}\left(p_i^{k*} - c_i^k\right)x_i^{k*}$ and no rationing of the consumer.

Naturally, we retrieve the oligopolistic equilibrium concept of d' Aspremont and Dos Santos Ferreira (2016) when there is a single group ($K = 1$). In addition to the standard profit maximization requirement for an equilibrium, this definition requires that the individual firms' income conjectures are correct at equilibrium and excludes consumer rationing. This last condition means that both constraints are saturated for every firm at equilibrium, which is not implied by profit maximization by all firms. Without fulfillment of this condition, the analytical apparatus we have applied to the representative consumer's decision would not be valid.

As an example of oligopolistic equilibrium, we may refer to the strategy pair formed by a price equilibrium[10] vector and the corresponding vector of quantities, as we are going to show. An implication of this example is that conditions ensuring existence of a price equilibrium also ensure existence of an oligopolistic equilibrium. Denoting Walrasian demand for good i in group k by $Q_i^k(\mathbf{p},Y) \equiv H_i^k\left(\mathbf{p}^k, D^k(\mathbf{p},Y)\right)$, a price equilibrium $\mathbf{p}^*$ is solution to the following program for each firm i in each group k:

$$\max_{p_i^k\in\left[c_i^k,\infty\right)} \left(p_i^k - c_i^k\right)Q_i^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, \mathbf{p}^{-k*}, Y^*\right), \quad (7)$$

together with the condition $Y^* = L + \sum_{k=1}^{K}\sum_{i=1}^{n_k}\left(p_i^{k*} - c_i^k\right)Q_i^k\left(\mathbf{p}^*,Y^*\right)$.

Now, take a price equilibrium $\mathbf{p}^*$ and suppose that, for some i in some group k and some $\left(p_i^k, x_i^k\right)$ satisfying the two constraints in (6), we have: $\left(p_i^k - c_i^k\right)x_i^k > \left(p_i^{k*} - c_i^k\right)Q_i^k\left(p_i^{k*}, \mathbf{p}_{-i}^{k*}, \mathbf{p}^{-k*}, Y^*\right)$, so that $\left(\mathbf{p}^{k*}, \mathbf{Q}^k\left(\mathbf{p}^*,Y^*\right)\right)_{k=1,\ldots,K}$ is not an oligopolistic equilibrium. Then, $X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right) \leq D^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, \mathbf{p}^{-k*}, Y^*\right)$ by the market size constraint and, by the market share constraint,

$$\begin{aligned} x_i^k &\leq H_i^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right)\right) \leq H_i^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, D^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, \mathbf{p}^{-k*}, Y^*\right)\right) \\ &= Q_i^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, \mathbf{p}^{-k*}, Y^*\right). \end{aligned}$$

We thus obtain a contradiction, since

$$\begin{aligned} \left(p_i^k - c_i^k\right)Q_i^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, \mathbf{p}^{-k*}, Y^*\right) &\geq \left(p_i^k - c_i^k\right)x_i^k \\ &> \left(p_i^{k*} - c_i^k\right)Q_i^k\left(\mathbf{p}^*,Y^*\right), \end{aligned}$$

meaning that $\mathbf{p}^*$ is not a price equilibrium.

2.3. The markup formula

Necessary first order conditions for profit maximization at an oligopolistic equilibrium allow us to establish a markup formula, such that the Lerner index of any firm is a weighted harmonic mean of the reciprocals of two demand elasticities of x_i^k with respect to p_i^k. The first elasticity is computed along the market share frontier and refers to the Hicksian demand. By totally differentiating the market share constraint equation, we obtain the *elasticity of the market share frontier* at the equilibrium point:[11]

$$\begin{aligned} &-\left.\frac{dx_i^k}{dp_i^k}\frac{p_i^k}{x_i^k}\right|_{x_i^k = H_i^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right)\right)} \quad (8) \\ &= \frac{-\epsilon_i H_i^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right)\right)}{1 - \epsilon_X H_i^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right)\right)\epsilon_i X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right)} \equiv s_i^k. \end{aligned}$$

[8] This two-stage procedure requires *decentralisability* (entailed by weak separability) but is less demanding than two-stage budgeting, which requires in addition *price aggregation* with respect to the partition into groups, hence homothetic separability, as assumed in Section 3 (see Blackorby and Russell, 1997).

[9] We are restricting our analysis to price and quantity feedback effects, and neglecting income feedback effects (the so-called Ford effects), introduced in d' Aspremont and Dos Santos Ferreira (2017).

[10] Price equilibrium fits Chamberlin (1933) monopolistic competition equilibrium when firms size is not negligible with respect to market size (the *small group* case). It is more usual nowadays to restrict the term *monopolistic competition* to competition between insignificant firms whose market power derives from product differentiation only, that is, when the *large group* assumption applies. See for example Thisse and Ushchev (2018).

[11] We denote by ϵ_{p_i} or ϵ_{x_i} (ϵ_i when there is no ambiguity) the elasticity operator with respect to p_i or x_i, respectively, applied to some function of which these variables are an argument.

Table 1
Elasticities appearing in the markup formula.

Impact of x_i^k on X^k	$\alpha_i^k \equiv \epsilon_i X^k\left(\mathbf{x}^k\right)$
Impact of X^k on x_i^k via H_i^k	$\beta_i^k \equiv \epsilon_X H_i^k\left(\mathbf{p}^k, X^k\left(\mathbf{x}^k\right)\right)$
Elasticity of the market share frontier	$s_i^k \equiv \frac{-\epsilon_i H_i^k\left(\mathbf{p}^k, X^k\left(\mathbf{x}^k\right)\right)}{1-\alpha_i^k\beta_i^k}$
Elasticity of the market size frontier	$\sigma_i^k \equiv \frac{-\epsilon_i D^k(\mathbf{p},Y)}{\alpha_i^k}$

This elasticity can be viewed as the *intra-group* elasticity of substitution of good i in group k (for the composite good produced by the group).[12] The second elasticity is computed along the market size frontier and refers to the Marshallian demand for the composite good k. By totally differentiating the market size constraint equation, we obtain the *elasticity of the market size frontier* at the equilibrium point:

$$-\left.\frac{dx_i^k}{dp_i^k}\frac{p_i^k}{x_i^k}\right|_{X^k\left(x_i^k,\mathbf{x}_{-i}^{k*}\right)=D^k\left(p_i^k,\mathbf{p}_{-i}^{k*},\mathbf{p}^{-k*},Y^*\right)} = \frac{-\epsilon_{p_i^k} D^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, \mathbf{p}^{-k*}, Y^*\right)}{\epsilon_i X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right)} \equiv \sigma_i^k. \tag{9}$$

It measures the intensity of the response of the consumption x_i^k to a change in the price p_i^k taking into account the variation of the Marshallian demand D^k and so expresses an *inter-group* elasticity of substitution of good i.

The expressions in Eqs. (8) and (9) involve other elasticities, for which it will be convenient to introduce concise notations in Table 1 (together with the expressions for s_i^k and σ_i^k).

The following proposition gives the oligopolistic equilibrium markup formula. The elasticity of the isoprofit curve through the intersection of the two frontiers, the equilibrium point, is equal to the reciprocal of the Lerner index, which must indeed take an intermediate value between the elasticities of those frontiers.

Proposition 1. *Assume weak separability of the representative consumer's utility function U into K groups of goods produced in the oligopolistic sector. Let $\left(\mathbf{p}^{k*}, \mathbf{x}^{k*}\right)_{k=1,\ldots,K}$ be an oligopolistic equilibrium. Then the relative markup of each firm i in each group k is given by*

$$\frac{p_i^{k*}-c_i^k}{p_i^{k*}} = \frac{\theta_i^{k*}\left(1-\alpha_i^{k*}\beta_i^{k*}\right)+\left(1-\theta_i^{k*}\right)\alpha_i^{k*}}{\theta_i^{k*}\left(1-\alpha_i^{k*}\beta_i^{k*}\right)s_i^{k*}+\left(1-\theta_i^{k*}\right)\alpha_i^{k*}\sigma_i^{k*}} \equiv \mu_i^{k*}, \tag{10}$$

for some $\theta_i^{k*} \in [0,1]$.

Proof. We start by making dimensionally homogeneous the two constraints in program (6) of firm i in group k, rewriting them in terms of two ratios:

$$\frac{x_i^k}{H_i^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right)\right)} \leq 1 \text{ and } \frac{X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right)}{D^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, \mathbf{p}^{-k*}, Y^*\right)} \leq 1. \tag{11}$$

The conditions for profit maximization at $\left(p_i^{k*}, x_i^{k*}\right)$ under constraints (11) (holding as equalities at equilibrium because of the no-rationing condition) can then be expressed, for non-negative Lagrange multipliers λ_i^{k*} and ν_i^{k*}, as[13]

$$x_i^{k*} = \lambda_i^{k*}\frac{-\partial_{p_i^k}H_i^{k*}}{H_i^{k*}} + \nu_i^{k*}\frac{-\partial_{p_i^k}D^{k*}}{D^{k*}} \tag{12}$$

and

$$p_i^{k*} - c_i^k = \lambda_i^{k*}\frac{1-\partial_X H_i^{k*}\partial_i X^{k*}}{H_i^{k*}} + \nu_i^{k*}\frac{\partial_i X^{k*}}{X^{k*}}. \tag{13}$$

By dividing both hand sides of the second equality by the corresponding hand sides of the first, and then multiplying them by x_i^{k*}/p_i^{k*}, we obtain the following formula, in terms of elasticities, for the markup of firm i at the equilibrium $\left(\mathbf{p}^{k*}, \mathbf{x}^{k*}\right)$:

$$\frac{p_i^{k*}-c_i^k}{p_i^{k*}} = \frac{\lambda_i^{k*}\left(1-\epsilon_X H_i^{k*}\epsilon_i X^{k*}\right)+\nu_i^{k*}\epsilon_i X^{k*}}{\lambda_i^{k*}\left(-\epsilon_{p_i^k}H_i^{k*}\right)+\nu_i^{k*}\left(-\epsilon_{p_i^k}D^{k*}\right)}. \tag{14}$$

Denoting $\theta_i^k \equiv \lambda_i^k/\left(\lambda_i^k+\nu_i^k\right) \in [0,1]$ and using Table 1, we can rewrite this expression as formula (10). ■

For each firm i in group k, the parameter θ_i^{k*} measures the relative weight put, at equilibrium $\left(\mathbf{p}^{k*}, \mathbf{x}^{k*}\right)_{k=1,\ldots,K}$, on the market share constraint. This constraint reflects the intra-group rivalry, whereas the market size constraint reflects the convergent interests of group k firms in their competition against the rest of the economy. We accordingly call θ_i^{k*} the *degree of competitive toughness* displayed by firm i in group k on its rivals of the same group, as evaluated at the reference equilibrium.

Notice that when firm i in group k has a negligible size relative to the size of the group (when $\alpha_i^{k*} \simeq 0$), there are neither inter-group nor intersectoral feedback effects, so that its equilibrium markup coincides with the reciprocal of the elasticity of the market share frontier: $\mu_i^{k*} \simeq 1/s_i^{k*}$. Of course, this result applies to all firms in group k if we are in the presence of a *large group* in the sense of Chamberlin, which is the usual case of *monopolistic competition*. The same outcome results alternatively from tough conduct of any firm ($\theta_i^{k*} \simeq 1$) even in a *small group*. Although we do not require goods to be perfect substitutes, this effect is reminiscent of the Bertrand paradox which does not require a large number of competitors: two is enough to obtain the perfectly competitive outcome.

3. Assuming homothetic separability in order to aggregate information

Weak separability of consumer's utility into K groups of goods produced in the oligopolistic sector has allowed us, by associating those groups with relevant markets, to construct a concept of oligopolistic equilibrium such that it would be pointless for firms in some group to form conjectures about quantities decided in other groups. This does not apply to price conjectures. However, by assuming a stronger form of separability, homothetic separability, price conjectures concerning other groups can be simplified through aggregation into price index values. Further simplifications on the hypothesized conjectures naturally result from stronger assumptions either (a) on the degree of substitutability within each group or (b) on the interrelation between groups. These simplifications aim at improving the model tractability while keeping its general equilibrium nature in case (a) or on contrary by focusing on partial equilibrium features in case (b).

[12] This is the elasticity (in absolute value) of $x_i^k/X_k = x_i^k/H_i^k\left(\mathbf{p}^k, X_k\right)$ with respect to $p_i^k/P_k = p_i^k/\partial_X e^k\left(\mathbf{p}^k, X_k\right)$, where P_k denotes the shadow price $\partial_X e^k\left(\mathbf{p}^k, X_k\right)$ of the composite good k (see d' Aspremont and Dos Santos Ferreira, 2016, Appendix).

[13] For shortness, we use for equilibrium values the notations $F^* \equiv F\left(\mathbf{x}^*\right)$ and $\partial_i F^* \equiv \partial_i F\left(\mathbf{x}^*\right)$.

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3.1. Homothetic separability

All the aggregator functions X^k ($k = 1, \ldots, K$) are assumed to be homogeneous of degree one. Homothetic separability entails separability of the indirect utility function and the possibility of defining K price indices for the K groups. Indeed, the expenditure function defined by the consumer's first stage program (2) is now linear in the utility level: $e^k(\mathbf{p}^k, X_k) = P^k(\mathbf{p}^k) X_k$, with $P^k(\mathbf{p}^k)$ viewed as a price index for group k. As a consequence, the Marshallian demand derived from the solution to the consumer's second stage program (5) is separable in prices: $D^k(\mathbf{p}, Y) = \widetilde{D}^k(P^1(\mathbf{p}^1), \ldots, P^K(\mathbf{p}^K), Y)$. Thus, producers in each group may be assumed to conjecture price index values for the other groups, rather than having to form dispensable conjectures on the corresponding price vectors.

Another significant difference introduced by homothetic separability is the linearity in X_k of the Hicksian demand function for the group k: $H_i^k(\mathbf{p}^k, X_k) = \partial_i P^k(\mathbf{p}^k) X_k$ (by Shephard's lemma). We consequently have $\beta_i^k = 1$ in this case. Also, by referring in addition to the first order condition of the consumer's first stage program, we obtain:

$$\epsilon_i P^k(\mathbf{p}^k) = \frac{p_i^k x_i^k}{P^k(\mathbf{p}^k) X^k(\mathbf{x}^k)} = \epsilon_i X^k(\mathbf{x}^k) \equiv \alpha_i^k, \tag{15}$$

which can now be identified with the budget share of good i in group k. Finally, because of separability of Marshallian demand, we obtain

$$\begin{aligned}\sigma_i^k &\equiv -\epsilon_i D^k(\mathbf{p}, Y)/\alpha_i^k = -\epsilon_k \widetilde{D}^k(P^k(\mathbf{p}^k), P^{-k}(\mathbf{p}^{-k}), Y) \\ &\quad \times \epsilon_i P^k(\mathbf{p}^k)/\alpha_i^k \\ &= -\epsilon_k \widetilde{D}^k(P^k(\mathbf{p}^k), P^{-k}(\mathbf{p}^{-k}), Y) \equiv \sigma^k,\end{aligned}$$

the same elasticity of the market size frontier for any firm in group k.

The oligopolistic equilibrium will accordingly be defined as follows.

Definition 2. Under homothetic separability, an *oligopolistic equilibrium* can be defined as a K-tuple of $(2n_k + 1)$-tuples $(\mathbf{p}^{k*}, \mathbf{x}^{k*}, P_k^*)_k$ in $\mathbb{R}_+^{2n+K}$ such that $P_k^* = P^k(\mathbf{p}^{k*})$ for $k = 1, \ldots, K$, and such that, for any i in group k,

$$\begin{aligned}&(p_i^{k*}, x_i^{k*}) \in \arg \max_{(p_i^k, x_i^k) \in \mathbb{R}_+^2} (p_i^k - c_i^k) x_i^k \\ &\text{s.t.} \quad x_i^k \leq \partial_i P^k(p_i^k, \mathbf{p}_{-i}^{k*}) X^k(x_i^k, \mathbf{x}_{-i}^{k*}) \\ &\text{and } X^k(x_i^k, \mathbf{x}_{-i}^{k*}) \leq \widetilde{D}^k(P^k(p_i^k, \mathbf{p}_{-i}^{k*}), \mathbf{P}_{-k}^*, Y^*),\end{aligned} \tag{16}$$

with $Y^* = L + \sum_{k=1}^K \sum_{i=1}^{n_k} (p_i^{k*} - c_i^k) x_i^{k*}$ and no rationing of the consumer.

The markup formula can then be written as

$$\frac{p_i^{k*} - c_i^k}{p_i^{k*}} = \frac{\theta_i^{k*}(1 - \alpha_i^{k*}) + (1 - \theta_i^{k*})\alpha_i^{k*}}{\theta_i^{k*}(1 - \alpha_i^{k*}) s_i^{k*} + (1 - \theta_i^{k*})\alpha_i^{k*}\sigma^{k*}} \equiv \mu_i^{k*}. \tag{17}$$

3.2. Limit cases of substitutability within a group

We now consider the dual limit cases of (i) perfect substitutability and (ii) perfect complementarity within some group k (with $n_k > 1$): $X^k(\mathbf{x}^k) = \sum_i x_i^k$ and $P^k(\mathbf{p}^k) = \min(p_1^k, \ldots, p_{n^k}^k)$ in case (i) or $X^k(\mathbf{x}^k) = \min(x_1^k, \ldots, x_{n^k}^k)$ and $P^k(\mathbf{p}^k) = \sum_i p_i^k$ in case (ii). The market share frontier in space $x_i^k \times p_i^k$ degenerates in these cases into a horizontal line (in case (i)) or into a vertical line (in case (ii)). Notice also that differentiability is lost for P^k in case (i) and for X^k in case (ii), so that we must then argue in terms of left- and right-hand derivatives.[14]

Take first case (i) of perfect substitutability. Referring to a profile $(\mathbf{p}^k, \mathbf{x}^k) \in \mathbb{R}_{++}^{2n^k}$ with equal prices, any tentative upward price deviation would result in a single binding constraint, the market share one, leading to the Bertrand zero markup, since $s_i^k = \infty$. By contrast, any tentative downward price deviation would result in a single binding constraint, the market size one, with x_i^k bounded by the residual demand $\widetilde{D}^k(p_i^k, \mathbf{P}_{-k}, Y) - \sum_{j \neq i} x_j^k$. The associated markup would consequently be the Cournot one, namely $1/\sigma_i^k = \alpha_i^k / (-\epsilon_k \widetilde{D}^k)$, the reciprocal of the absolute value of the elasticity of the residual demand. The markup of an oligopolistic equilibrium may thus take any intermediate value between the Bertrand and Cournot markups, corresponding to upward and downward price deviations, respectively. It may be written as

$$\mu_i^k = (1 - \theta_i^k) \frac{\alpha_i^k}{-\epsilon_k \widetilde{D}^k} = \frac{1 - \theta_i^k}{\sigma_i^k}. \tag{18}$$

Case (ii) of perfect complementarity is symmetric with respect to the preceding one. Referring to a profile $(\mathbf{p}^k, \mathbf{x}^k) \in \mathbb{R}_{++}^{2n_k}$ with equal quantities, the market share constraint takes the form $x_i^k = \min(\mathbf{x}_{-i}^k)$ whatever the price set by firm i. In particular, a tentative downward price deviation would let firm i facing this sole binding constraint (corresponding to $\theta_i^k = 1$), with $s_i^k = 0$, resulting in an infinite markup. By contrast, a tentative upward price deviation would make the market size constraint the sole binding constraint (corresponding to $\theta_i^k = 0$), leading to markup $1/\sigma_i^k = 1/(-\epsilon_k \widetilde{D}^k)(p_i^k / \sum_j p_j^k)$, which is the Cournot markup in the case of complementary monopolies (the regime designated by Cournot as "producers' concurrence"). Of course, the equilibrium markup can take any value larger than its Cournot value (between this value and infinity).

An interesting situation is characterized by perfect substitutability and $\theta_i^k = 0$ prevailing in all K groups and for any firm i in any group k. We then obtain the outcome of a *Cournotian monopolistic competition equilibrium*, a concept introduced in d' Aspremont et al. (1991, 1997)[15]: producers play Cournot in the markets for their own products, taking other goods prices as given. Formally,

Definition 3. Under perfect substitutability within each group, a *Cournotian monopolistic competition equilibrium* is a K-tuple of $(n_k + 1)$-tuples $(\mathbf{x}^{k*}, P_k^*)_k$ in $\mathbb{R}_+^{n+K}$ such that, for any i in group k,

$$\begin{aligned}&(P_k, x_i^{k*}) \in \arg \max_{(P_k, x_i^k) \in \mathbb{R}_+^2} (P_k - c_i^k) x_i^k \\ &\text{s.t. } x_i^k + \sum_{j \neq i} \mathbf{x}_{-j}^{k*} \leq \widetilde{D}^k(P_k, \mathbf{P}_{-k}^*, Y^*),\end{aligned} \tag{19}$$

with $Y^* = L + \sum_{k=1}^K \sum_{i=1}^{n_k} (P_k^* - c_i^k) x_i^{k*}$ and no rationing of the consumer.

This concept is generalized by the concept of oligopolistic equilibrium, which may lead to any markup values between zero and the Cournot one, as already mentioned. Notice also that the Cournotian monopolistic competition equilibrium becomes a regular price equilibrium when $n_k = 1$ for any k. Conditions ensuring the existence of a Cournotian monopolistic competition equilibrium are given in d' Aspremont et al. (1991, p.980).

14 So, $\partial_i^- P^k(p, \ldots, p) = 1$ and $\partial_i^+ P^k(p, \ldots, p) = 0$ in case (i) and $\partial_i^- X^k(x, \ldots, x) = 1$ and $\partial_i^+ X^k(x, \ldots, x) = 0$ in case (ii). As a consequence, $\sigma_i^k \equiv -\epsilon_k \widetilde{D}^k(P^k(p, \ldots, p), P^{-k}(\mathbf{p}^{-k}), Y)/\alpha_i^k$ for a downward price deviation in case (i) and $\sigma_i^k \equiv -\epsilon_k \widetilde{D}^k(P^k(p, \ldots, p), P^{-k}(\mathbf{p}^{-k}), Y)\epsilon_i P^k(p, \ldots, p)$ for a downward quantity deviation in case (ii).

15 See also Costa (2004) and Brito et al. (2013).

4. Approximating oligopolistic equilibria as if groups were independent

In this section, we introduce another kind of simplified conjectures, not requiring homothetic separability. The idea is to approximate oligopolistic equilibria by supposing that each firm forms a conjecture about the income to be spent in its group as if this group were independent.

Let us start by introducing independence between groups, thus reinforcing the partial equilibrium dimension, by assuming that the consumer's utility function is Cobb–Douglas:

$$\widetilde{U}(\mathbf{X}, z) = \prod_k X_k^{\alpha_k} z^{1-\alpha},$$

with $\alpha = \sum_k \alpha_k$. The income that the representative consumer wishes to spend in goods of group k is then $Y^k = \alpha_k Y$ independently of the prices chosen by the different firms. Groups of oligopolistic firms become independent from each other and firms in group k do not have to make conjectures on prices and quantities prevailing outside group k. Conjecturing $Y^k = \alpha_k Y$ is sufficient in a context which is essentially one of partial equilibrium. The market size constraint impending upon firm i in group k can then be simply written as

$$e^k\left(p_i^k, \mathbf{p}_{-i}^k, X^k\left(x_i^k, \mathbf{x}_{-i}^k\right)\right) \leq \alpha_k Y. \tag{20}$$

Such a partial equilibrium approach, consisting in attributing to firm i, in a given group k, a conjecture Y^k which fixes directly the market size of the group without taking price and quantity feedback effects into account results here from the assumption of a Cobb–Douglas utility function of the representative consumer. It can however be exploited under more general preferences, as a way to *approximate* oligopolistic equilibria. More explicitly, we directly assume that the firms in group k conjecture the income Y^k left to be spent in their group and require this conjecture to be verified at equilibrium:

Definition 4. An *oligopolistic equilibrium with conjectured incomes* is a K-tuple of triples $\left(\mathbf{p}^{k*}, \mathbf{x}^{k*}, Y^{k*}\right)_{k=1,\ldots,K} \in \mathbb{R}_+^{2n+K}$ such that, for any $i = 1, \ldots, n_k$ and any $k = 1, \ldots, K$,

$$\begin{aligned}\left(p_i^{k*}, x_i^{k*}\right) \in \arg &\max_{\left(p_i^k, x_i^k\right) \in \mathbb{R}_+^2} \left(p_i^k - c_i^k\right) x_i^k \\ \text{s.t.} \quad & x_i^k \leq H_i^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right)\right)\end{aligned} \tag{21}$$

and $e^k\left(p_i^k, \mathbf{p}_{-i}^{k*}, X^k\left(x_i^k, \mathbf{x}_{-i}^{k*}\right)\right) \leq Y^{k*}$,

with $X^k\left(\mathbf{x}^{k*}\right) = D^k\left(\mathbf{p}^*, Y^*\right)$, $Y^* = L + \sum_{k=1}^K \sum_{i=1}^{n^k} \left(p_i^{k*} - c_i^k\right) x_i^{k*}$ and no rationing of the consumer.

Following the procedure used in the proof of Proposition 1, it is straightforward to derive the equilibrium markup formula for this case:

$$\frac{p_i^{k*} - c_i^k}{p_i^{k*}} = \frac{\theta_i^{k*}\left(1 - \alpha_i^{k*}\beta_i^{k*}\right) + \left(1 - \theta_i^{k*}\right) p_i^{k*} x_i^{k*}/Y^{k*}}{\theta_i^{k*}\left(1 - \alpha_i^{k*}\beta_i^{k*}\right) s_i^{k*} + \left(1 - \theta_i^{k*}\right) p_i^{k*} x_i^{k*}/Y^{k*}} \equiv \mu_i^{k*}. \tag{22}$$

First, notice that the Lerner index is still a weighted harmonic mean of the reciprocals of the two demand elasticities of x_i^k with respect to p_i^k at $\left(p_i^{k*}, x_i^{k*}\right)$, namely s_i^{k*} for the market share frontier and 1 for the market size frontier, which freezes the expenditure in group k. Second, notice that we find in the weight put on market size the budget share $p_i^{k*} x_i^{k*}/Y^{k*}$ of good i in group k at equilibrium, instead of the elasticity $\alpha_i^{k*} \equiv \epsilon_i X^k\left(\mathbf{x}^{k*}\right)$. Of course, under homothetic separability, the equilibrium budget share coincides with α_i^{k*}, so that formula (22) is a particular case of the general formula (10). More generally, by the consumer's first order condition (3),

$$\frac{p_i^{k*} x_i^{k*}}{Y^{k*}} = \alpha_i^{k*} \epsilon_X e^k\left(\mathbf{p}^{k*}, X_k\left(\mathbf{x}^{k*}\right)\right), \tag{23}$$

where $\epsilon_X e^k\left(\mathbf{p}^{k*}, X_k\left(\mathbf{x}^{k*}\right)\right)$ is not necessarily equal to one.

Outside the case previously considered of a Cobb–Douglas function $\widetilde{U}$ where the income that the representative consumer wishes to spend in goods of group k is independent, treating as given the incomes $\left(Y^1, \ldots, Y^K\right)$ amounts to suppose that producers are not able to exploit all the relevant information on the consumer's utility and on the derived demand structure. This is the price to be paid, in the absence of homothetic separability, to avoid the assumption that they have to conjecture all the individual prices in other groups.

5. Group-specific competitive conduct: an industry with a collusive and a competitive group

We have focused on the relations between goods as determined by the structure of the representative consumer's preferences, in order to consider competition within and between groups. In so doing, we did not emphasize the impact of differences in competitive conduct, for instance between collusive and competitive firms coexisting within the same group, a situation allowed by our model, where no symmetry is imposed on demand, cost or conduct. In such a situation, rather than viewing the whole set of competitive firms as just constraining, through their prices, the market size, each collusive firm contemplates any competitive firm as a direct rival, constraining its market share. This formulation may reflect an industrial situation where the products of all firms in the group are perceived similarly by the consumers.

In this section, we want to focus on the case where the goods produced by the collusive and the competitive firms are on the contrary perceived differently by the consumers. In other words, the initial group may be advantageously split into two groups with opposite conducts, according to an instance of preference separability which reflects the difference in conducts.

What we have in mind is the case where one of the groups is "dominant" (firms having large market shares), the other group being a "competitive fringe" where firms are small. The perception by the consumers of the products offered in the two groups is different and this difference is reflected in their preferences. In their study of sales data of ground coffee in the US, Sakamoto and Stiegert (2018) use our model to evaluate the market conduct of the main brands. But before, they test for weak separability and determine that Folgers and Maxwell House (which have more than 50% of revenue share) are the two dominant firms. The partition into two groups, identified as the dominant group and the competitive fringe, is made on the basis of preference separability.

5.1. An example

For explicitness, we will use an example building upon the following representative consumer's utility function:

$$U(X, z) = \frac{b^{1/\sigma}}{1 - 1/\sigma} X^{1-1/\sigma} + z, \tag{24}$$

with $b > 0$ and $\sigma > 1$, where

$$X = \left(X_1^{(s-1)/s} + X_2^{(s-1)/s}\right)^{s/(s-1)} \text{ and } X_k = \left(\sum_{i=1}^{n_k} \left(x_i^k\right)^{\left(s^k-1\right)/s^k}\right)^{s^k/\left(s^k-1\right)} \tag{25}$$

for $k = 1, 2$, with $s > 0$ and $s^k > 0$. Because of homothetic separability of X, indirect utility is also separable, with the corresponding CES price indices as arguments:

$$P = \left(P_1^{1-s} + P_2^{1-s}\right)^{1/(1-s)} \text{ and } P_k = \left(\sum_{i=1}^{n_k} \left(p_i^k\right)^{1-s^k}\right)^{1/\left(1-s^k\right)}. \quad (26)$$

For firm i in group k, the Hicksian demand is then

$$H_i^k\left(\mathbf{p}_i^k, X_k\right) = \left(p_i^k/P_k\right)^{-s^k} X_k, \quad (27)$$

and the Marshallian demand for the composite good produced in this group is

$$\widetilde{D}^k\left(P_k, P_{-k}, Y\right) = \left(P_k/P\right)^{-s} \min\left(Y/P, bP^{-\sigma}\right). \quad (28)$$

If $Y/P < bP^{-\sigma}$, all the income is spent in the oligopolistic sector, because the marginal utility of the composite good X deflated by its price P is larger than the corresponding marginal utility of the numeraire good, namely 1. In the computations that follow, we will focus on the case of a high aggregate income $Y > bP^{1-\sigma}$, the one leading to a positive activity of the numeraire sector, which allows to take into account the interaction between the oligopolistic and the competitive sectors, reinforcing the general equilibrium dimension of the analysis.

At equilibrium, the elasticities of the market share and market size frontiers at their intersection are equal in absolute value to s^k and σ^{k*}, respectively, with

$$\sigma^{k*} = \left(1-\alpha_k^*\right)s + \alpha_k^*, \text{ if } Y^* < bP^{*1-\sigma} \text{ or}$$
$$\sigma^{k*} = \left(1-\alpha_k^*\right)s + \alpha_k^*\sigma, \text{ if } Y^* > bP^{*1-\sigma}, \quad (29)$$

where $\alpha_k = \epsilon_k P = P_k^{1-s}/\left(P_1^{1-s} + P_2^{1-s}\right)$, so that σ^k is a function of P_1 and P_2.[16]

The competitive group will be characterized by a high degree of competitive toughness eroding the firms' market power (and possibly by a large number n_2 of firms, accounting for relatively small market shares). In the limit case where $\theta_i^2 = 1$ for any firm i in group 2 (or if $n_2 \to \infty$), the equilibrium markup reduces to $\mu_i^2 = 1/s^2$, leading to the equilibrium price $p_i^2 = c_i^2/\left(1 - 1/s^2\right)$. This markup is not necessarily close to zero, because the goods produced by competitive firms may be sufficiently differentiated among themselves to keep each producer so to say in its own dedicated niche. Also, even if the constraints on market share are the only binding constraints, the constraints on market size are still present, as they must be satisfied at equilibrium, so that price and quantity decisions taken by any competitive firm cannot be taken independently from the prices set by the collusive firms. However, as far as the *equilibrium* prices in the competitive group reflect the sole market share constraints, they are in fact independent from the collusive firms' decisions. The interesting case, from a general equilbrium point of view, is consequently the one where competitive firms' toughness is high but not maximal and where their number n_2 is not arbitrarily large.

By contrast with the competitive group, we expect the collusive firms to be in small number n_1, which allows for relatively high market shares, and to display a low degree of competitive toughness, although full collusion ($\theta_i^1 = 0$ for any i in group 1) may be neither feasible nor optimal. We shall however consider in the following the limit case of equilibria with a two-firm fully collusive group ($n_1 = 2$, $\theta_1^{1*} = \theta_2^{1*} = 0$ and $\mu_1^{1*} = \mu_2^{1*} = 1/\sigma^{1*}$). In such equilibria, both collusive firms maximize their respective profits under the same market size constraint.

16 We are applying formulas (8) and (9), and using the equilibrium property $\epsilon_i X^1\left(\mathbf{x}^{1*}\right) = \epsilon_i P^1\left(\mathbf{p}^{1*}\right) = \alpha_i^{1*}$ entailed by homothetic separability.

5.2. Equilibria with a collusive group: enforceability

As just mentioned, full collusion may not be feasible. To illustrate this issue we shall assume symmetry ($c_1^1 = c_2^1 = c^1$). The collusive price $\widehat{p}^1$ (the same, by symmetry, for both collusive firms) verifies the condition of tangency of the market size frontier and of the isoprofit curve through the potential equilibrium point:

$$\sigma^{1*} = \alpha_1^*\sigma + \left(1-\alpha_1^*\right)s = \frac{\left(2^{1/\left(1-s^1\right)}\widehat{p}^1\right)^{1-s}\sigma + P_2^{1-s}s}{\left(2^{1/\left(1-s^1\right)}\widehat{p}^1\right)^{1-s} + P_2^{1-s}}$$
$$= \frac{\widehat{p}^1}{\widehat{p}^1 - c^1}, \quad (30)$$

a condition which determines the collusive price as a function of the price P_2, of marginal cost c^1 and of the elasticities of substitution s^1, s and σ. For the collusive price to be an equilibrium price, it must be compatible with the simultaneous satisfaction of the market share and the market size equations in the case of a symmetric profile. Symmetry is enough as regards the market share equation. As to the market size equation, it determines the collusive quantity $\widehat{x}^1$:

$$\left(2^{s^1/\left(s^1-1\right)}\widehat{x}^1\right) = \left(2^{1/\left(1-s^1\right)}\widehat{p}^1\right)^{-s}$$
$$\times\, b\left(\left(2^{1/\left(1-s^1\right)}\widehat{p}^1\right)^{1-s} + P_2^{1-s}\right)^{(s-\sigma)/(1-s)}, \quad (31)$$

taking the case of a high aggregate income ($Y^* > bP^{*1-\sigma}$). Now, is the symmetric collusive profile with $\left(\widehat{p}^1, \widehat{x}^1\right)$ for each dominant firm enforceable as an oligopolistic equilibrium, conditional upon the price index value P_2 for the competitive group?

To answer this question, we resort to a graphical illustration. We take $\widehat{p}^1 = \widehat{x}^1 = 1$ by an appropriate choice of units of the goods produced in the two sectors.[17] We represent in Fig. 1, by thick curves, the market share frontier (for $p^1 > 1$) and the market size frontier (for $p^1 < 1$), as well as, by a thin curve, the isoprofit curve through their point of intersection, for the following parameter values: $s^1 = 2$, $s = 0.5$, $\sigma = 2$, $P_2 = 0.6$. The parameter values $c^1 \simeq 0.18$ and $b \simeq 9.2$ are chosen so as to ensure that $\widehat{p}^1 = \widehat{x}^1 = 1$ (see footnote 17).

The collusive profile is clearly enforceable. Each collusive firm i maximizes its profit under the two constraints at $\left(p_i^1, x_i^1\right) = (1, 1)$, when the other collusive firm chooses $\left(p_j^1, x_j^1\right) = (1, 1)$ and when the price index value of the competitive group is $P_2 = 0.6$. Enforceability is however eventually lost if we start to increase the elasticity of substitution within the collusive group. A high elasticity of substitution, determining a strong demand response to a downward price deviation, makes such deviation attractive in terms of a larger profit. We represent such a situation in Fig. 2, drawn with the same parameter values, except $s^1 = 5$, where we see that somewhat higher isoprofit curves would not violate the market size constraint.

17 This amounts to choose the unit of the goods produced by the collusive firms so as to obtain the marginal cost

$$c^1 = \frac{2^{(1-s)/\left(1-s^1\right)}(\sigma-1) + P_2^{1-s}(s-1)}{2^{(1-s)/\left(1-s^1\right)}\sigma + P_2^{1-s}s},$$

and to choose the unit of the numeraire good so as to obtain the parameter value

$$b = 2^{\left(s^1-s\right)/\left(s^1-1\right)}\left(2^{(1-s)/\left(1-s^1\right)} + P_2^{1-s}\right)^{(\sigma-s)/(1-s)}.$$

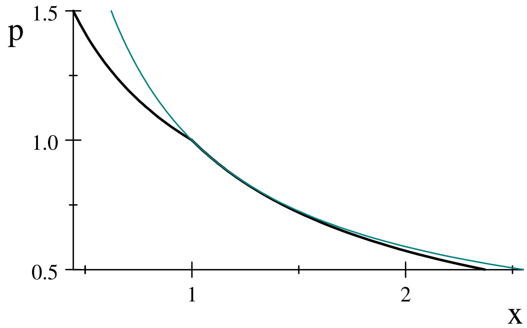

Fig. 1. Collusive equilibrium (for $s^1 = 2$).

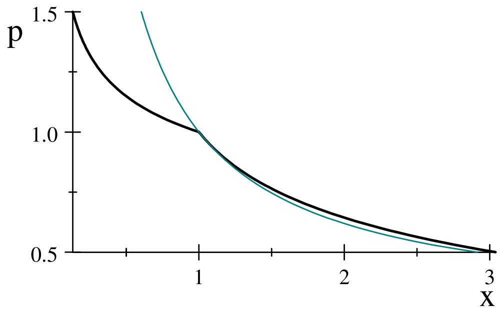

Fig. 2. Unenforceable collusive profile (for $s^1 = 5$).

This example shows that the collusive solution for group 1 is not always enforceable as an oligopolistic equilibrium.

5.3. Equilibria with a collusive group: efficiency

Symmetry is a natural condition for the collusive outcome associated with zero degrees of competitive toughness to be efficient from the viewpoint of the collusive group: without cost uniformity, it would be possible to increase joint profits through a redistribution of X_1 from the least productive to the most productive firm. Symmetry is however not enough. The profits of the collusive group are conditional on the price index value P_2 of the competitive group, which might depend upon the decisions of the collusive firms, a dependence which is not taken into account by Nash conjectures. As already emphasized, we should avoid the limit case in which $\theta_i^2 = 1$ for any firm i in group 2, and concentrate on the case where competitive toughness is high but not maximal. Indeed, equilibrium prices in group 2 are then dependent upon those in group 1, so that the non-cooperative conduct of the collusive firms with respect to the competitive group may be source of inefficiency.

In the example we have been using, the equilibrium joint profits of the collusive group are (for $Y^* > bP^{*1-\sigma}$ and given P_2^*):

$$\Pi^1\left(P_1^*, P_2^*\right) = \left(2^{1/\left(s^1-1\right)}P_1^* - c^1\right) bP_1^{*-s}\left(P_1^{*1-s} + P_2^{*1-s}\right)^{\frac{s-\sigma}{1-s}}. \tag{32}$$

The collusive solution results from maximizing $\Pi^1\left(\cdot, P_2^*\right)$ while taking P_2^* as a constant, leading to the first order condition $\partial\Pi^1\left(P_1^*, P_2^*\right)/\partial P_1 = 0$. However, if P_2 is a function of P_1, the first order condition for the maximization of equilibrium joint profits of the collusive group is

$$\frac{\partial\Pi^1\left(P_1, P_2\right)}{\partial P_1} + \frac{\partial\Pi^1\left(P_1, P_2\right)}{\partial P_2}\frac{\partial P_2}{\partial P_1} = 0. \tag{33}$$

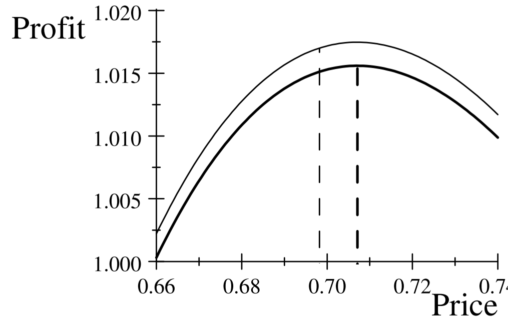

Fig. 3. Inefficient collusive equilibrium.

Hence, if the second term on the LHS of this condition is negative, $\partial\Pi^1\left(P_1, P_2\right)/\partial P_1$ should be positive and the optimal value of P_1 smaller than the collusive value. In our example, with $\sigma > s$, $\Pi^1\left(P_1, \cdot\right)$ is decreasing, so that $\partial\Pi^1\left(P_1, P_2\right)/\partial P_2 < 0$. Also, P_2 is decreasing (through μ^2) in σ^2, which is itself decreasing in α_1 (hence in P_1), leading to $\partial P_2/\partial P_1 > 0$. The condition for having an optimal value of P_1 smaller than the collusive value P_1^* is thus fulfilled. Such a smaller value of P_1 (hence of μ^1) should in addition be attainable through a positive θ^1, which is possible only if $s^1 > \sigma^1$.

Fig. 3 illustrates such a situation, with the thick curve representing the graph of the joint profit function $\Pi^1(\cdot, 0.6)$ of the collusive group.[18] This function is maximized at the collusive price index value $\widehat{P}_1 = 2^{-0.5} \simeq 0.707$ (which corresponds to the normalized price $\widehat{p}^1 = 1$), leading to $\Pi^1(0.707, 0.6) \simeq 1.015\,6$. The thin curve represents the graph of $\Pi^1(\cdot, 0.5995)$, with a lower value of the index price of the competitive group. We see that it locally dominates the preceding curve, maximized at the collusive price index value. A decrease of the price index P_2 from 0.6 to 0.5995 may be obtained at equilibrium by decreasing the price index P_1 from 0.707 to 0.698, a variation associated with an increase of the degree θ^1 of competitive toughness of the collusive firms from 0 to 0.33.[19] Deviating from the fully collusive (zero) degree of competitive toughness to a higher degree, say 0.33,[20] would allow collusive firms to have access to a higher profit $\Pi^1(0.698, 0.5995) = 1.017$.

This example shows that the collusive solution for group 1 may be inefficient from the viewpoint of this group.

6. Concluding remarks

The present paper elaborates on our previously introduced concept of oligopolistic equilibrium, with firms maximizing profits under two constraints, one on market share, which expresses the competitors' conflicting interests, the other on market size, which takes their convergent interests into account. This combination allows to recover under the same concept a plurality of competition regimes from the toughest to the softest feasible.

By introducing a partition of the oligopolistic sector into groups, we have now allowed for a significant simplification of

18 The figure is computed on the basis of the following parameter values: $s^1 = 3$, $s^2 = 2$, $s = 0.3$, $\sigma = 5$, $n_2 = 5$, $\theta^2 = 0.5$. According to footnote 17, the normalization $\widehat{p} = \widehat{x} = 1$ implies in addition $b \simeq 36.087$ and $c^1 \simeq 0.64093$.

19 For computation of P_1, we use the ratio $P_2/\widehat{P}_2 = \left(1 - \widehat{\mu}^2\right)/\left(1 - \mu^2\right)$ with μ^2 (which depends upon P_1) given by (17). The value of θ^1 can then be established from the equation $P_1 = 2^{0.5}c^1/\left(1 - \mu^1\right)$, where μ^1 depends upon θ^1.

20 Notice that the present computations, while establishing the inefficiency of the collusive equilibrium, do not allow to determine so far the optimal degree of competitive toughness for the collusive group.

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the conjectured constraints supposed to be considered by the competitors. The first constraint refers exclusively to the share of the market for products in the same group, making irrelevant any conjecture on quantities outside the group. The other groups appear only, through their prices, in the second constraint, and these prices can be aggregated into group indices by reinforcing separability, or else replaced through approximation by the conjectured expenditure within the own group.

These simplifications result in a stronger ''partial equilibrium flavor'' (Hart, 1985), realistically assuming that the competitors can take rational decisions without holding detailed representations of distant actions. However, the general equilibrium dimension is maintained and given more structure by the partition into groups. Even when focusing on a specific industry, the suggested approach confers to the analysis – of a collusive group and a competitive group in our example – a ''general equilibrium flavor''.

More generally, we have tried to illustrate the flexibility of the concept in its applications to different kinds of problems, by allowing for a large diversity of types of firm conjectures and the resulting competitive behaviors.

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