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Abstract

We consider the conditional mean risk allocation for an insurance pool, as defined by Denuit and Dhaene (2012). Precisely, we study the asymptotic behavior of the respective relative contributions of the participants as the total loss of the pool tends to infinity. The numerical illustration in Denuit (2019) suggests that the application of the conditional mean risk sharing rule may ultimately produce a linear sharing of independent compound Poisson losses. This paper studies the validity of this empirical finding in the class of compound Panjer-Katz sums consisting of compound Binomial, compound Poisson, and compound Negative Binomial sums with either Gamma or Pareto severities. It is demonstrated that such a behavior does not hold in general since one term may ultimately dominate the other ones.

Keywords: conditional expectation, risk pooling, compound distributions, Panjer formula, size-biased transform.

1 Introduction

In this paper, we consider the conditional mean risk allocation of independent losses, as defined by Denuit and Dhaene (2012). According to this rule, each participant to an insurance pool contributes the conditional expectation of the loss brought to the pool, given the total loss experienced by the entire pool. The properties of the conditional mean risk allocation have been studied in Denuit (2019) and the present study originates from the empirical findings in the numerical illustration contained in that paper.

From a theoretical point of view, we investigate the relative behavior of the conditional expectations of random variables given their sum, when the realization of the sum tends to infinity. Conditions are given under which one of the following two cases occurs:

Case i) the conditional expectations are asymptotically in fixed, positive proportions to each other (the "linear" case);

Case ii) one of the random variables dominates, the conditional expectations of the others being asymptotically vanishingly small with respect to this one.

This research question is investigated for the class of compound Panjer-Katz sums consisting of compound Binomial, compound Poisson, and compound Negative Binomial sums. This class of distributions is central to actuarial mathematics so that the results derived in this paper are of wide applicability in insurance studies. As far as severities are concerned, we consider the heavy-tailed case with regularly varying tails. In particular, the Pareto law belongs to this class of distributions. We also discuss a light-tailed case where severities obey the Gamma distribution. Since the Gamma distribution is the prototype example of light-tailed distribution, widely used in theory as well as in actuarial applications (also in compound Poisson sums, giving rise to the Tweedie distribution), this choice appears to be relevant for our investigation. Independence is assumed in all cases.

Since all standard severity models correspond to absolutely continuous probability distributions, the present paper naturally concentrates on this situation. We nevertheless also consider the particular setting of the numerical illustration proposed by Denuit (2019), that is, compound Poisson sums with integer-valued claim severities. This is because these empirical findings motivated the present study.

The remainder of this paper is organized as follows. Section 2 provides the reader with economic motivation for the study conducted in the present paper by demonstrating its relevance for Peer-to-Peer (P2P) insurance. The problem under investigation is also properly positioned there, relative to the existing literature. In Section 3, we recall the connection of the conditional mean risk sharing rule with the size-biased transform. Section 4 derives the size-biased transforms and the conditional mean risk sharing of compound Panjer-Katz sums. In this paper, we favor direct reasoning specific to compound Panjer-Katz sums to recover their respective size-biased transforms. These results can be found in Denuit (2020) where they are derived by means of general results about size-biasing sums and mixtures. Sections 5-6 study the asymptotic linearity of the conditional mean risk allocation in the case of severities with regularly varying tails or obeying the Gamma distribution, respectively. Compared to other papers dealing with tails of sums of independent random variables, we must deal here first with non-identically distributed random variable and second with probability density functions and not tail probabilities. Section 7 goes back to the empirical findings in the numerical illustration proposed by Denuit (2019) that motivated the present paper. Specifically, we consider independent compound Poisson losses with integer-valued severities. We establish that the application of the conditional mean risk sharing principle ultimately produces a linear allocation when claim severities possess the same, finite upper endpoint to their support. Since a common finite upper endpoint to the support of claim severities can be obtained with the help of an excess-of-loss protection, this result appears to be particularly interesting for applications. But it is also shown there that the conditional mean risk sharing may fail to produce an asymptotically linear allocation. This is the case when the finite upper endpoints of the respective supports differ. For the case when the severities are heterogeneous with unbounded support, we provide an example where severities follow Logarithmic distribution showing that the asymptotic linearity does not hold in general.

All proofs are gathered in appendix. The following notation is adopted throughout the text. For two positive functions g_1 and g_2 defined in a neighborhood of infinity, we write $g_1 \sim g_2$ provided $\lim_{x\to\infty} g_1(x)/g_2(x) = 1$ and we write $g_1 = o(g_2)$ provided $\lim_{x\to\infty} g_1(x)/g_2(x) = 0$. We use $=_d$ to denote equality in distribution for two random variables. Independence is assumed throughout this text, among severities and frequencies involved in the sums, as well as between sums.

2 Motivation

The paper aims at contributing to the rich literature on risk sharing and risk allocation that are both core topics in actuarial science. After Karl Borch's seminal contribution, many papers have been devoted to risk sharing. We refer the interested reader to the reviews by Aase (1993, 2002). Within this vast topic, we concentrate on Peer-to-Peer (P2P) insurance schemes where participants share their respective losses, reviving the ancestral compensation mechanism consisting in using the contributions of the many to balance the misfortunes of the few. See, e.g., Abdikerimova and Feng (2019) and the references therein. The conditional mean risk sharing rule appears to be a very convenient way to distribute retained losses among participants, as shown by Denuit (2019).

Consider n participants to a P2P insurance pool, numbered $i = 1, 2, \ldots, n$. Each of them faces a risk X_i. By risk, we mean a non-negative random variable representing a monetary loss. In the remainder of this paper, we assume that $X_1, X_2, \ldots, X_n$ are independent and we adopt the notation $S = \sum_{i=1}^n X_i$ for the total risk of the pool. In a risk pooling scheme, each participant contributes ex-post an amount $h_i(s)$ where $s = \sum_{i=1}^n x_i$ is the sum of the realizations $x_1, x_2, \ldots, x_n$ of $X_1, X_2, \ldots, X_n$.

In the design of the scheme, it is important that the sharing rule represented by the functions h_i is both intuitively acceptable and transparent. In that respect, the conditional mean risk sharing (or allocation) h_i^* proposed by Denuit and Dhaene (2012) seems to be particularly attractive. Recall that this allocation is defined as

$$h_i^*(S) = E[X_i|S], \;\; i = 1, 2, \ldots, n. \tag{2.1}$$

In words, participant i must contribute the expected value of the risk X_i brought to the pool, given the total loss S. Clearly, the conditional mean risk sharing (2.1) allocates the full risk S as we obviously have

$$\sum_{i=1}^{n} h_i^*(S) = \sum_{i=1}^{n} E[X_i|S] = S$$

so that the sum of participants' contributions covers the entire loss S.

In the expected utility setting, every risk-averse decision-maker prefers $h_i^*(S)$ over the initial risk X_i so that the conditional mean risk sharing rule appears to be beneficial to all participants (as an application of Jensen's inequality). Conditions for Pareto-optimality have been provided by Denuit and Dhaene (2012).

The present paper investigates the question whether the respective relative contributions of the n participants tend to stabilize when the total loss of the pool increases, or equivalently if there exist constants δ_i, $i = 1, 2, \ldots, n$, such that

$$\delta_i > 0 \text{ for all } i \text{ and } \sum_{i=1}^{n} \delta_i = 1$$

and

$$h_i^*(s) = E[X_i|S = s] \sim \delta_i s \text{ for } i \in \{1, 2, \ldots, n\}. \tag{2.2}$$

When the total loss gets large, it can thus be shared among participants according to the proportions δ_i when (2.2) holds true. This is case i) as referred to in the introductory section of this paper. As it can be expected, certain symmetry relations must hold between the random variables under consideration for (2.2) to be valid.

It is worth to mention that Furman et al. (2018) investigated a related problem. Precisely, these authors studied conditions ensuring that the identity $h_i^*(s) = \delta_i s$ holds true for some δ_i depending on the means of the risks under consideration (see Theorem 3.2 in that paper). Compared to Furman et al. (2018), we only require asymptotic linearity in (2.2).

If one loss, X_1 say, dominates then the conditional expectations of the others may become asymptotically negligible with respect to this one, that is,

$$h_1^*(s) \sim s \text{ and } h_j^*(s) = o(s) \text{ for } j \in \{2, \ldots, n\} \tag{2.3}$$

holds true. Formula (2.3) corresponds to case ii) as referred to in the introductory section of this paper.

Let us now explain why (2.2)-(2.3) are relevant for applications to P2P insurance. Linear risk sharing rules have often been applied to allocate losses among members of a P2P community. Such rules are of the form

$$h_i^{\text{lin}}(S) = E[X_i] + a_i \left(S - E[S]\right), \quad i = 1, 2, \ldots, n,$$

where $\sum_{i=1}^{n} a_i = 1$. Clearly, a linear risk sharing scheme allocates the full risk S and satisfies the fairness constraint $E[h_i^{\text{lin}}(S)] = E[X_i]$ for $i = 1, \ldots, n$. With h_i^{lin}, participants agree to pay the pure premium $E[X_i]$ and to divide deviations of S from the total pure premium $E[S]$ (positive or negative) in proportion to the coefficients a_i.

As an example of linear rule, participants may agree to take a fixed percentage of the total loss S, in accordance with the expected values of the risks they bring to the pool compared to the total expected loss, that is,

$$h_i^{\text{prop}}(S) = E[X_i] + \frac{E[X_i]}{E[S]}(S - E[S]) = \frac{E[X_i]}{E[S]}S.$$

This rule, referred to as the proportional risk sharing rule has often been applied in the context of P2P insurance. However, volatility is not accounted for because participants i_1 and i_2 with $E[X_{i_1}] = E[X_{i_2}]$ contribute equally to the total loss even if the respective variances $V[X_{i_1}]$ and $V[X_{i_2}]$ strongly differ.

With linear sharing rules, the same proportion a_i of the total losses S is allocated to each participant, whatever the realization of S. In the case investigated by Furman et al. (2018) recalled above, such linear rules are perfectly appropriate. If (2.2) holds true then linearity remains relevant even for large realizations of S but the coefficients δ_i may differ from the assumed a_i. On the contrary, under (2.3) linear risk sharing rules depart from the conditional mean risk allocation. This means that when the pool experiences large losses, the application of the assumed proportions a_i leads to individual contributions that do not reflect the expected contribution of X_i given S. As an extreme situation, for a loss X_1 at most equal to b, say, we might end up with $h_1^{\text{lin}}(S) > b$ when S gets large whereas $h_1^*(S)$ always stays smaller than b. This questions the relevance of linear risk sharing rules in adverse scenarios. Of course, the upper layer of S is generally (re-)insured because of the limited risk-bearing capacity of the P2P community. The results derived in this paper then suggest that the price of the stop-loss protection for the upper layer should not be distributed among participants according to the same proportions a_i defining h_i^{lin}.

It is well known that

$$X_1, X_2, \ldots, X_n \text{ independent and identically distributed } \Rightarrow h_i^*(s) = \frac{s}{n}. \tag{2.4}$$

Thus, the conditional mean risk sharing is linear and $\delta_i = \frac{1}{n}$ in this case. The rule h_i^* extends this uniform allocation to heterogeneous losses. As a particular case of the results derived in this paper, we will discuss the homogeneous case (2.4) to recover this trivial situation (to some extent). It is interesting to point out that (2.4) is the key argument to derive Panjer recursive formula; see for instance the proof of Theorem 3.5.1 in Kaas et al. (2008).

3 Conditional mean risk sharing rule and size-biased transform

The size-biased transform appears to be useful to study the conditional mean risk sharing rule, as pointed out in Denuit (2019). Given a non-negative random variable X with distribution function F_X and strictly positive expected value $E[X]$, define $\widetilde{X}$ with distribution function

$$P[\widetilde{X} \le t] = \frac{E\big[XI[X \le t]\big]}{E[X]},$$

where $I[\cdot]$ denotes the indicator function (equal to 1 if the event appearing within the brackets is realized, and to 0 otherwise). Then, $\widetilde{X}$ is said to be a size-biased version of X, and the operator mapping the distribution function F_X of X to the distribution function $F_{\widetilde{X}}$ of $\widetilde{X}$ is called the size-biased transform. Henceforth, we assume that X and $\widetilde{X}$ are mutually independent.

The size-biased transform can be traced back to the late 1960s in the statistical literature. It has proven to be useful in the study of risk measures after the pioneering work by Furman and Landsman (2005, 2008) and Furman and Zitikis (2008a,b). The size-biased transform is an example of weighted distribution. Initially developed in order to unify various sampling distributions when the chance of being recorded by an observer varies, weighted distributions are closely related to weighted risk measures and weighted capital allocation rules. See Furman and Zitikis (2009) for an overview. Among these weighted distributions, the size-biased, or length-biased one corresponds to the identity weight function.

Compound sums with absolutely continuous severities have a probability mass at 0 and possess a probability density function over $(0,\infty)$. Such a distribution is said to be zero-augmented and it appears to be relevant to examine the effect of size-biasing in this case. Assume that X is a zero-augmented risk, i.e. it is equal to 0 with probability $P[X=0]>0$ or strictly positive with probability $P[X>0]$ and possesses the probability density function $f_{X|X>0}$ over $(0,\infty)$. Then, $\widetilde{X}$ is a strictly positive random variable with probability density function

$$f_{\widetilde{X}}(x) = \frac{x f_{X|X>0}(x)}{E[X|X>0]}. \tag{3.1}$$

See e.g. Property 2.1 in Denuit (2019).

Let us now give the reason why size-biasing appears to be useful in relation with the conditional mean risk sharing. Consider independent, zero-augmented risks $X_1, X_2, \ldots, X_n$ with positive expectations. Let $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$ be their corresponding size-biased versions, assumed to be independent and independent of $X_1, X_2, \ldots, X_n$. It is proved in Denuit (2019, Proposition 2.2 iii)) that, for any $s>0$,

$$E[X_i|S=s] = \frac{E[X_i] f_{S-X_i+\widetilde{X}_i}(s)}{\sum_{j=1}^n E[X_j] f_{S-X_j+\widetilde{X}_j}(s)} s. \tag{3.2}$$

If the random variables $X_1, X_2, \ldots, X_n$ are identically distributed then the ratio appearing in (3.2) is equal to $1/n$ and we recover (2.4).

Considering (3.2), we see that to study the asymptotic behavior of $h_i^*(s)$ as s is large, we must deal with probability density functions of sums of independent random variables for large values and not with survival functions. The literature on this topic is however limited in extreme value theory. The main contributions are found in Tauberian theory.

4 Size-biased transform and conditional mean risk sharing within the compound Panjer-Katz family

4.1 Compound Binomial sums

Assume that the loss X_i brought by participant i to the insurance pool can be represented as

$$X_i = \sum_{k=1}^{N_i} C_{i,k} \text{ with } N_i \sim \text{Binomial}(\nu_i, p_i),\ \ i = 1, 2, \ldots, \tag{4.1}$$

where ν_i is a positive integer, $p_i \in (0, 1)$, and where the claim severities $C_{i,k}$ are positive, absolutely continuous, distributed as C_i, all these random variables being independent.

The next result gives the size-biased transform of compound Binomial distributions; its proof can be found in Appendix A.

Proposition 4.1. *The size-biased version of the compound Binomial random variable $X = \sum_{k=1}^{N} C_k$ with N ∼Binomial(ν, p) and claim severities C_k that are positive, absolutely continuous, independent and identically distributed as C, all these random variables being independent, is given by $\widetilde{X} =_d \sum_{k=1}^{N'} C_k + \widetilde{C}$ where N' ∼Binomial$(\nu - 1, p)$, and where N', C_1, C_2, ..., $C_{\nu-1}$ and $\widetilde{C}$ are mutually independent.*

Proposition 4.1 allows us to deal with the situation where each participant to the insurance pool brings a loss of the form (4.1). To this end, let $I_{i,1}$, $I_{i,2}$,..., I_{i,ν_i} be independent Bernoulli distributed random variables with common mean p_i and independent of $C_{i,1}$, $C_{i,2}$,..., C_{i,ν_i}. Then,

$$X_i =_d \sum_{k=1}^{\nu_i} Y_{i,k} \text{ where } Y_{i,k} = I_{i,k} C_{i,k}.$$

Proceeding as in the proof of Proposition 4.1 (see Appendix A), we have that

$$\widetilde{X}_i =_d \sum_{k=1}^{\nu_i - 1} Y_{i,k} + \widetilde{Y}_{i,\nu_i}$$

where $Y_{i,1}$, $Y_{i,2}$, ..., Y_{i,ν_i-1} and $\widetilde{Y}_{i,\nu_i}$ are mutually independent. Using the fact that $\widetilde{Y}_{i,\nu_i} =_d \widetilde{C}_i$, where $\widetilde{C}_i$ is assumed to be independent of all $Y_{i,k}$, we get from (3.2) that

$$E\left[X_i | S = s\right] = \frac{E\left[X_i\right] f_{S - Y_{i,\nu_i} + \widetilde{C}_i}(s)}{\sum_{j=1}^{n} E\left[X_j\right] f_{S - Y_{j,\nu_j} + \widetilde{C}_j}(s)} s. \tag{4.2}$$

Now, assume that $C_1, C_2, \ldots, C_n$ are identically distributed. Then, $\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_n$ are also identically distributed. Assume also that $p_1 = ... = p_n$. Then (4.2) allows us to write

$$E[X_i | S = s] = \frac{\nu_i}{\nu_\bullet} s \tag{4.3}$$

where $\nu_\bullet = \nu_1 + \ldots + \nu_n$. The representation (4.3) shows that the conditional mean risk sharing rule is linear in this case with slopes $\delta_i = \nu_i / \nu_\bullet$. This result can be related to (2.4) by considering the independent and identically distributed random variables $Y_{i,k}$.

Remark 4.2. *In the limiting case $p_i = 1$, we recover sums with a deterministic numbers of terms, that is, losses X_i of the form $X_i = \sum_{k=1}^{\nu_i} C_{i,k}$. Identity (4.2) then becomes*

$$E[X_i|S=s] = \frac{E[X_i] f_{S-C_{i,\nu_i}+\widetilde{C}_i}(s)}{\sum_{j=1}^{n} E[X_j] f_{S-C_{j,\nu_i}+\widetilde{C}_j}(s)} s. \tag{4.4}$$

4.2 Compound Poisson sums

Assume that the loss X_i brought by participant i to the insurance pool can be represented as

$$X_i = \sum_{k=1}^{N_i} C_{i,k} \text{ with } N_i \sim \text{Poisson}(\lambda_i), \;\; i = 1, 2, \ldots, \tag{4.5}$$

where the claim severities C_{ik} are positive, absolutely continuous, distributed as C_i, all these random variables being independent.

Proposition 4.3. *The size-biased version of the compound Poisson random variable $X = \sum_{k=1}^{N} C_k$ with $N \sim$ Poisson(λ) and claim severities C_k that are positive, absolutely continuous, independent and identically distributed as C, all these random variables being independent, is given by $\widetilde{X} =_d X + \widetilde{C}$ where X and $\widetilde{C}$ are mutually independent.*

The proof of Proposition 4.3 is given in Appendix B. It can be seen there that Panjer formula can be invoked for compound Poisson sums with absolutely continuous severities, in order to recover the corresponding size-biased transform derived in Denuit (2020) from general results about size-biasing compound sums.

For each X_i in (4.5), using the fact that $\widetilde{X}_i$ is distributed as $X_i + \widetilde{C}_i$ where the size-biased version $\widetilde{C}_i$ of C_i is independent of X_i, we get from (3.2) that

$$E[X_i|S=s] = \frac{E[X_i] f_{S+\widetilde{C}_i}(s)}{\sum_{j=1}^{n} E[X_j] f_{S+\widetilde{C}_j}(s)} s. \tag{4.6}$$

Assume that $C_1, C_2, \ldots, C_n$ are identically distributed. Then, $\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_n$ are also identically distributed and (4.6) allows us to write

$$E[X_i|S=s] = \frac{\lambda_i}{\lambda_\bullet} s \tag{4.7}$$

where $\lambda_\bullet = \lambda_1 + \ldots + \lambda_n$. The representation (4.7) shows that the conditional mean risk sharing rule is linear in this case with slopes $\delta_i = \lambda_i/\lambda_\bullet$. It is interesting to compare (4.7) to (2.4) by considering $\lambda_1, \ldots, \lambda_n$ as proper volume measures.

4.3 Compound Negative Binomial sums

Assume that the loss X_i brought by participant i to the insurance pool can be represented as

$$X_i = \sum_{k=1}^{N_i} C_{i,k} \text{ with } N_i \sim \text{Negative Binomial}(\xi_i, \beta_i), \;\; i = 1, 2, \ldots, \tag{4.8}$$

with N_i obeying the Negative Binomial(ξ_i, β_i) distribution with positive parameters β_i and ξ_i, i.e.

$$P[N_i = k] = \frac{\beta_i^{\xi_i}}{(1+\beta_i)^{\xi_i+k}} \frac{\Gamma(\xi_i + k)}{k!\Gamma(\xi_i)}, \quad k = 0, 1, 2, \dots$$

and where the claim severities $C_{i,k}$ are positive, absolutely continuous, independent and distributed as C_i, all these random variables being independent.

Proposition 4.4. *The size-biased version of the compound Negative Binomial random variable* $X = \sum_{k=1}^{N} C_k$ *with* $N \sim$*Negative Binomial*(ξ, β) *and claim severities* C_k *that are positive, absolutely continuous, independent and identically distributed as* C*, all these random variables being independent, is given by* $\widetilde{X} =_d X + \widetilde{C} + Z$ *where* Z *is a compound Negative Binomial sum* $\sum_{k=1}^{M} C'_k$ *with* $M \sim$*Negative Binomial*$(1, \beta)$ *and* C'_k *distributed as* C_k*, all these random variables being independent.*

The proof of Proposition 4.4 is given in Appendix C. As for the compound Poisson case, it is based on Panjer recursive formula.

For each X_i of the form (4.8), using the fact that $\widetilde{X}_i$ is distributed as $X_i + \widetilde{C}_i + Z_i$ where the size-biased version $\widetilde{C}_i$ of C_i and Z_i are independent of X_i, we get from (3.2) that

$$E\left[X_i | S = s\right] = \frac{E\left[X_i\right] f_{S+\widetilde{C}_i+Z_i}\left(s\right)}{\sum_{j=1}^{n} E\left[X_j\right] f_{S+\widetilde{C}_j+Z_j}\left(s\right)} s \tag{4.9}$$

Assume that $C_1, C_2, \dots, C_n$ are identically distributed. Then, $\widetilde{C}_1, \widetilde{C}_2, \dots, \widetilde{C}_n$ are also identically distributed. If $\beta_1 = ... = \beta_n$ then (4.9) allows us to write

$$E[X_i | S = s] = \frac{\xi_i}{\xi_\bullet} s \tag{4.10}$$

where $\xi_\bullet = \xi_1 + \ldots + \xi_n$. The representation (4.10) shows that the conditional mean risk sharing rule is linear in this case with slopes $\delta_i = \xi_i / \xi_\bullet$. The comparison with (2.4) is again instructive, by introducing proper volume measures.

5 Severities with regularly varying tails

Let us now refine the results derived in Section 4 by adding some information about claim severities. In this section, we assume that claim severities have decreasing densities and regularly varying tails. This corresponds for instance to severities obeying the Pareto distribution. Precisely, we assume in this section that the loss X_i brought by participant i to the insurance pool can be represented as

$$X_i = \sum_{k=1}^{N_i} C_{i,k} \tag{5.1}$$

where N_i is a counting random variable, the claim severities $C_{i,k}$ are positive, absolutely continuous, distributed as C_i, all these random variables being independent. Moreover we assume that the tail functions $\bar{F}_{C_i}$ defined as $\bar{F}_{C_i}(t) = P[C_i > t]$ satisfy

$$\bar{F}_{C_i}\left(x\right) \sim x^{-\alpha_i} L_i\left(x\right) \tag{5.2}$$

where $L_i(\cdot)$ are slowly varying functions and $\alpha_i > 1$ for $i = 1, ..., n$. We refer the reader to Embrechts et al. (1997) for a description of this class of distributions and further bibliography on the topic. The following result establishes the ultimate behavior of the conditional mean risk sharing rule in that case.

Proposition 5.1. *Assume that C_i have decreasing densities f_{C_i} and that N_i are random variables such that there exist $\varepsilon_i > 0$ with $E\left[e^{\varepsilon_i N_i}\right] < \infty$, $i = 1, 2, \ldots, n$. The following results then hold true:*

(i) If $\alpha_1 = ... = \alpha_n = \alpha$ and $L_i(x) \sim c_i L(x)$ with $c_i > 0$ for $i = 1, ..., n$, then

$$E\left[X_i | S = s\right] \sim \frac{E[N_i]c_i}{\sum_{j=1}^{n} E[N_j]c_j} s \text{ for } i \in \{1, 2, \ldots, n\}.$$

(ii) If $\alpha_1 < \min\{\alpha_2, ..., \alpha_n\}$ then

$$E\left[X_1 | S = s\right] \sim s \text{ and } E[X_j | S = s] = o(s) \text{ for } j \in \{2, \ldots, n\}.$$

The proof of Proposition 5.1 is given in Appendix D. Proposition 5.1 applies in particular when N_i is a positive integer, or obeys the Binomial distribution, the Poisson distribution, or the Negative Binomial distribution. Proposition 5.1 thus covers compound Panjer-Katz sums when severities have decreasing densities and tails satisfying (5.2), as in the Pareto case for instance.

In addition to cases (i) and (ii) considered in Proposition 5.1, it is possible to encounter situations where several α_i parameters are equal and equal to the minimum of these parameters. For instance, we might have $\alpha_1 = \alpha_2 < \min\{\alpha_3, ..., \alpha_n\}$ and $L_i(x) \sim c_i L(x)$ with $c_i > 0$ for $i = 1, 2$. In this case, the respective relative contributions of the n participants satisfy

$$\begin{aligned} E\left[X_i | S = s\right] &\sim \frac{E[N_i]c_i}{\sum_{j=1}^{2} E[N_j]c_j} s \text{ for } i \in \{1, 2\}, \\ E\left[X_i | S = s\right] &= o(s) \text{ for } i \in \{3, \ldots, n\}. \end{aligned}$$

When claim severities are heavy-tailed, we see from Proposition 5.1 that only severities matter for (2.2) to hold as long as the number of terms N_i have a finite moment generating function in a neighborhood of the origin. This is generally the case when the tails of compound sums are studied. See for instance Robert and Segers (2008) and the references therein.

6 Gamma distributed severities

In Section 5, we have considered heavy-tailed severities. In this section, we consider a light-tailed case and we assume that C_i follows the Gamma(α_i, τ_i) distribution, with positive parameters α_i and τ_i, $i = 1, 2, \ldots, n$. Precisely, the probability density function of C_i is given by

$$f_{C_i}(x) = \frac{\tau_i^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i - 1} \exp(-x\tau_i), \quad x \geq 0.$$

Gamma distributions are prototype examples of light-tailed distributions. They have been widely applied in practice, because they belong to the exponential dispersion family to which the GLM machinery applies and also because, together with the Poisson distribution, they are the building blocks to the Tweedie distribution that is often used in insurance studies. This explains why results established in that particular setting remains relevant for applications.

The following results cover compound Panjer-Katz sums with severities obeying the Gamma distribution. Proposition 6.1 considers compound Binomial sums, Proposition 6.2 compound Poisson sums, and Proposition 6.4 compound Negative Binomial sums. Each time, item (ii) refers to the situation where (2.2) holds true whereas item (i) identifies the situation where (2.3) applies. Some comments are given to discuss all possible situations.

Proposition 6.1. *Assume that the loss X_i brought by participant i to the insurance pool is of the form* (4.1) *with $C_i \sim$Gamma(α_i, τ_i). The following results then hold true:*

(i) If $\tau_1 < \min\{\tau_2, ..., \tau_n\}$ then

$$E\left[X_1|S=s\right] \sim s \text{ and } E\left[X_j|S=s\right] = o\left(s\right) \text{ for } j \in \{2, \ldots, n\}.$$

(ii) If $\tau_1 = ... = \tau_n = \tau$ then

$$E\left[X_i|S=s\right] \sim \frac{\nu_i \alpha_i}{\sum_{j=1}^n \nu_j \alpha_j} s \text{ for } i \in \{1, 2, \ldots, n\}.$$

The proof of Proposition 6.1 is given in Appendix E. To get (2.2) in the compound Binomial case with Gamma-distributed severities, we thus see that all the parameters τ_i must be equal. If one of them differs from the others then we switch to (2.3) and the conditional expectation of the loss with the smallest τ_i dominates the others.

In addition to cases (i) and (ii) considered in Proposition 6.1, it is possible to encounter situations where several τ_i parameters are equal and equal to the minimum of these parameters. For instance, we might have $\tau_1 = \tau_2 < \min\{\tau_3, ..., \tau_n\}$. In this case, the respective relative contributions of the n participants satisfy

$$\begin{aligned} E\left[X_i|S=s\right] &\sim \frac{\nu_i \alpha_i}{\sum_{j=1}^2 \nu_j \alpha_j} s \text{ for } i \in \{1, 2\}, \\ E\left[X_i|S=s\right] &= o\left(s\right) \text{ for } i \in \{3, \ldots, n\}. \end{aligned}$$

Proposition 6.2. *Assume that the loss X_i brought by participant i to the insurance pool is of the form* (4.5) *with $C_i \sim$Gamma(α_i, τ_i). The following results then hold true:*

(i) Assume that $\tau_1 = ... = \tau_n$. If $\alpha_1 > \max\{\alpha_2, ..., \alpha_n\}$ then

$$E\left[X_1|S=s\right] \sim s \text{ and } E\left[X_j|S=s\right] = o\left(s\right) \text{ for } j \in \{2, \ldots, n\}.$$

(ii) Assume that $\tau_1 = ... = \tau_n$ and $\alpha_1 = ... = \alpha_n$. Then,

$$E\left[X_i|S=s\right] = \frac{\lambda_i}{\lambda_\bullet} s \text{ for } i \in \{1, 2, \ldots, n\}.$$

The proof of Proposition 6.2 is given in Appendix F. Compared to the compound Binomial case considered in Proposition 6.1, we see from Proposition 6.2 that we must impose stronger constraints on the Gamma parameters α_i and τ_i to get (2.2) in the compound Poisson case with Gamma-distributed severities as all the parameters α_i must be equal, not only the parameters τ_i. If not then we switch to (2.3) and the conditional expectation of the loss with the largest α_i dominates the others. Clearly, if the parameters τ_i are not equal then (2.2) cannot hold so that all cases are covered.

Remark 6.3. *Since C_i follows the Gamma(α_i, τ_i) distribution, the random variables X_i in Proposition 6.2 obey the Tweedie distribution. Precisely, the probability density function of each $X_i|X_i > 0$ is given by*

$$f_{X_i|X_i>0}\left(x\right) = \frac{e^{-\lambda_i}}{1-e^{-\lambda_i}} \exp\left(-x\tau_i\right) x^{-1} r_{\alpha_i}\left(\lambda_i \tau_i^{\alpha_i} x^{\alpha_i}\right) \text{ with } r_\alpha\left(x\right) = \sum_{j=1}^{\infty} \frac{1}{j!\Gamma\left(j\alpha\right)} x^j.$$

Proposition 6.4. *Assume that the loss X_i brought by participant i to the insurance pool is of the form (4.8) with $C_i \sim$Gamma(α_i, τ_i). The following results then hold true:*

(i) Assume that $\tau_1 = ... = \tau_n = \tau$, $\alpha_1 = ... = \alpha_n = \alpha$ and $\beta_1 > \max\{\beta_2, ..., \beta_n\}$. Then,

$$E\left[X_1|S=s\right] \sim s \text{ and } E\left[X_j|S=s\right] = o\left(s\right) \text{ for } j \in \{2,\ldots,n\}.$$

(ii) Assume that $\tau_1 = ... = \tau_n = \tau$, $\alpha_1 = ... = \alpha_n = \alpha$ and $\beta_1 = ... = \beta_n$. Then,

$$E\left[X_i|S=s\right] = \frac{\xi_i}{\xi_\bullet} s \text{ for } i \in \{1,2,\ldots,n\}.$$

The proof of Proposition 6.4 is given in Appendix G. Compared to the compound Binomial and compound Poisson cases, the conditions imposed on the parameters are even stronger in the compound Negative Binomial case. The Gamma parameters α_i and τ_i, as well the Negative Binomial parameters β_i must be equal for all participants to get (2.2) in the compound Negative Binomial case with Gamma-distributed severities. If not then we switch to (2.3) and the conditional expectation of the loss with the largest β_i dominates the others. Clearly, if the parameters α_i and τ_i are not equal then (2.2) cannot hold so that all cases are covered.

Compared to the heavy-tailed case considered in Section 5 where only claim severities mattered for (2.2) to hold, Proposition 6.4 also imposes conditions on the Negative Binomial parameters to get asymptotic linearity.

7 Discussion for Compound Poisson with discrete severities

In this section, we consider the situation investigated in the numerical illustration proposed by Denuit (2019) that motivated the present study. The aim is to explain the empirical findings in that paper and to provide preliminary results in the discrete case (that appear to be of independent interest).

In this section, we assume that the loss X_i brought by participant i to the insurance pool is of the form (4.5) where the claim severities $C_{i,k}$ are valued in $\{1, 2, 3, \ldots\}$. Based on the classical Panjer recursive formula, Denuit (2019) established that the conditional mean risk allocation for independent compound Poisson sums $X_1, \ldots, X_n$ in (4.5) is given by

$$E[X_i|S = s] = \frac{E[X_i]P[S + \widetilde{C}_i = s]}{\sum_{j=1}^{n} E[X_j]P[S + \widetilde{C}_j = s]} s. \tag{7.1}$$

In the particular case where $C_1, C_2, \ldots, C_n$ are identically distributed, so that $\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_n$ are also identically distributed, (7.1) allows us to see that (4.7) is still valid.

7.1 Heterogeneous claim severities with bounded support

Assume that the claim severities are bounded but not identically distributed among compound Poisson sums. The following result shows that the conditional mean risk sharing is asymptotically linear in the case where the severities C_i have a common finite upper endpoint to their support. The weights δ_i will be however different than those in the case (4.7) of homogeneous severities, i.e. $\lambda_i/\lambda_\bullet$.

Proposition 7.1. *Consider independent compound Poisson losses of the form* (4.5) *where the support of* C_i *is* $\{1, 2, \ldots, b_i\}$ *for some* $b_i < \infty$*, so that* $P[C_i = b_i] > 0$ *for all* $i = 1, 2, \ldots, n$*. The following results then hold true:*

(i) If $b_1 = b_2 = \ldots = b_n = b$ *then*

$$E[X_i|S = s] \sim \frac{\lambda_i P[C_i = b]}{\sum_{j=1}^{n} \lambda_j P[C_j = b]} s \text{ for } i \in \{1, 2, \ldots, n\}.$$

(ii) If $b_1 > \max\{b_2, \ldots, b_n\}$ *then*

$$E\left[X_1|S = s\right] \sim s \text{ and } E\left[X_j|S = s\right] = o\left(s\right) \text{ for } j \in \{2, \ldots, n\}.$$

The proof of Proposition 7.1 is given in Appendix H. We see from (i) that (2.2) holds true if all severities have the same, finite upper endpoint to their support. If not, (ii) indicates that (2.2) is no more valid and the share for participant with the largest finite upper endpoint to the support dominates.

Notice that, when claim severities are identically distributed, $b_i = b_j = b$ and $P[C_i = b] = P[C_j = b]$ for all i and j. The limiting result in Proposition 7.1(i) is in accordance with (4.7). The proportions $\lambda_i/\lambda_\bullet$ still apply asymptotically if $b_i = b_j = b$ and $P[C_i = b] = P[C_j = b]$ for all i and j, as in the numerical example described in Denuit (2019, Section 6.1) where the claim severities put the same probability mass 0.3 on $b = 4$. To be precise, there were 4 participants ($n = 4$) with respective claim severities C_1, C_2, C_3, and C_4 such that C_1 and C_3 are identically distributed, with probability masses 0.1, 0.2, 0.4, and 0.3 on 1, 2, 3, and 4, whereas C_2 and C_4 are identically distributed, with probability masses 0.15, 0.25, 0.3, and 0.3 on 1, 2, 3, and 4. Hence, we get from item (i) in Proposition 7.1 that

$$E[X_i|S=s] \sim \frac{\lambda_i}{\lambda_\bullet} s \text{ for } i \in \{1,2,3,4\} \text{ in this example}$$

and the asymptotic behavior thus coincides with (4.7) established for homogeneous severities. It is worth to stress that the respective shares $E[X_i|S=s]/s$ do not converge to $E[X_i]/E[S]$ as erroneously suggested in Denuit (2019) based on the numerical illustration contained in that paper.

7.2 Heterogeneous claim severities with Logarithmic distribution

It is tempting to deduce from Proposition 7.1(i) that the result remains valid letting b tend to infinity. The following example of heterogeneous claim severities with unbounded support shows that it is not necessarily the case. Specifically, assume that C_i obeys the Logarithmic distribution with parameter p_i, that is,

$$P[C_i = k] = \frac{-1}{\ln(1-p_i)} \frac{p_i^k}{k}, \quad k = 1, 2, \ldots.$$

The corresponding size-biased transform is given by

$$P[\widetilde{C}_i = k] = (1-p_i)p_i^{k-1}, \quad k = 1, 2, \ldots,$$

that is, $\widetilde{C}_i$ obeys the Geometric distribution with parameter $1-p_i$. The next result shows that the application of the conditional mean risk sharing rule fails to ultimately deliver a linear sharing if the severities C_i are not identically distributed.

Proposition 7.2. *Consider independent compound Poisson losses of the form* (4.5) *where* C_i *obeys the Logarithmic distribution with parameter* p_i*,* $i = 1, 2, \ldots, n$*. If* $p_1 > \max\{p_2, ..., p_n\}$*, then*

$$E[X_1|S=s] \sim s \text{ and } E[X_j|S=s] = o(s) \text{ for } j \in \{2, \ldots, n\}.$$

The proof of Proposition 7.2 is given in Appendix I.

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APPENDIX: PROOFS OF THE RESULTS

A Proof of Proposition 4.1

Define $Y_k = I_k C_k$ where $I_1, I_2, ..., I_\nu$ are independent Bernoulli distributed random variables with common mean p, that is, $P[I_k = 1] = 1 - P[I_k = 0] = p$ for $k = 1, 2, \ldots, \nu$. Clearly,

$$X =_d \sum_{k=1}^{\nu} Y_k.$$

We know (see, e.g., Corollary 3.2 in Denuit (2020)) that the size-biased version of a sum $Z = \sum_{k=1}^{\nu} D_k$ where ν is a positive integer and the random variables $D_1, D_2, \ldots, D_\nu$ are non-negative, independent and all distributed as D, is given by $\widetilde{Z} =_d \sum_{k=1}^{\nu-1} D_k + \widetilde{D}$ where $D_1, D_2, \ldots, D_{\nu-1}$ and $\widetilde{D}$ are mutually independent. Therefore,

$$\widetilde{X} =_d \sum_{k=1}^{\nu-1} Y_k + \widetilde{Y}_\nu$$

where Y_1, Y_2, ..., $Y_{\nu-1}$ and $\widetilde{Y}_\nu$ are mutually independent. The announced result then follows since $\widetilde{C} =_d \widetilde{Y}_\nu$ and $\sum_{k=1}^{N'} C_k =_d \sum_{k=1}^{\nu-1} Y_k$.

B Proof of Proposition 4.3

The random variable X obeys a mixture of a Dirac distribution at 0 with probability $e^{-\lambda}$ and a continuous distribution over $(0, \infty)$ with probability density function

$$f_{X|X>0}(x) = \frac{e^{-\lambda}}{1 - e^{-\lambda}} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} f_C^{*k}(x)$$

with probability $1 - e^{-\lambda}$, where f_C^{*k} is the probability density function of the sum $C_1 + \ldots + C_k$. Considering (3.1), the probability density function of the size-biased version $\widetilde{X}$ of X is given by

$$f_{\widetilde{X}}(x) = \frac{1 - e^{-\lambda}}{\lambda E[C]} x f_{X|X>0}(x).$$

Define

$$g(x) = \left(1 - e^{-\lambda}\right) f_{X|X>0}(x).$$

We know from Panjer (1981) that

$$g(x) = \lambda e^{-\lambda} f_C(x) + \frac{\lambda}{x} \int_0^x y f_C(y) g(x - y) \, \mathrm{d}y.$$

Therefore

$$x g(x) = \lambda E[C] e^{-\lambda} \frac{x f_C(x)}{E[C]} + \lambda E[C] \int_0^x \frac{y f_C(y)}{E[C]} g(x - y) \, \mathrm{d}y,$$

and it follows that

$$
\begin{aligned}
f_{\widetilde{X}}(x) &= e^{-\lambda} f_{\widetilde{C}}(x) + \left(1 - e^{-\lambda}\right) \int_0^x f_{\widetilde{C}}(y) f_{X|X>0}(x-y)\,\mathrm{d}y \\
&= e^{-\lambda} f_{\widetilde{C}}(x) + \left(1 - e^{-\lambda}\right) f_{X+\widetilde{C}|X>0}(x) \\
&= f_{X+\widetilde{C}}(x).
\end{aligned}
$$

This ends the proof.

C Proof of Proposition 4.4

The random variable X obeys a mixture distribution: a Dirac distribution at 0 with probability $P[N=0] = \beta^{\xi}/(1+\beta)^{\xi}$ and a continuous distribution over $(0,\infty)$ with probability density function

$$
f_{X|X>0}(x) = \frac{1}{P[N \geq 1]} \sum_{k=1}^{\infty} P[N=k] f_C^{*k}(x)
$$

with probability $P[N \geq 1]$. Considering (3.1), the probability density function of the size-biased version $\widetilde{X}$ of X is given by

$$
f_{\widetilde{X}}(x) = \frac{\beta P[N \geq 1]}{\xi E[C]} x f_{X|X>0}(x).
$$

Define

$$
g(x) = P[N \geq 1] f_{X|X>0}(x).
$$

We know from Panjer (1981) that

$$
g(x) = \frac{1}{1+\beta} \left(\xi \frac{\beta^{\xi}}{(1+\beta)^{\xi}} f_C(x) + \int_0^x \left(1 + (\xi - 1)\frac{y}{x}\right) f_C(y)\, g(x-y)\,\mathrm{d}y \right)
$$

which gives

$$
\begin{aligned}
xg(x) &= \frac{\xi}{1+\beta} E[C]\, \mathrm{P}[N=0] \frac{x f_C(x)}{E[C]} + \frac{\xi}{1+\beta} E[C] \int_0^x \frac{y f_C(y)}{E[C]} g(x-y)\,\mathrm{d}y \\
&\quad + \frac{1}{1+\beta} \int_0^x f_C(y)(x-y)\, g(x-y)\,\mathrm{d}y.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\xi E[C]}{\beta} f_{\widetilde{X}}(x) &= \frac{\xi E[C]}{1+\beta} P[N=0] f_{\widetilde{C}}(x) + \frac{\xi E[C]}{1+\beta} \int_0^x f_{\widetilde{C}}(y) g(x-y)\,\mathrm{d}y \\
&\quad + \frac{1}{1+\beta} \frac{\xi E[C]}{\beta} \int_0^x f_C(y)\, f_{\widetilde{X}}(x)\,(x-y)\,\mathrm{d}y
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
f_{\widetilde{X}}(x) &= \frac{\beta}{1+\beta} P[N=0] f_{\widetilde{C}}(x) + \frac{\beta}{1+\beta} P[N \geq 1] \int_0^x f_{\widetilde{C}}(y) f_{X|X>0}(x-y)\,\mathrm{d}y \\
&\quad + \frac{1}{1+\beta} \int_0^x f_C(y)\, f_{\widetilde{X}}(x)\,(x-y)\,\mathrm{d}y.
\end{aligned}
$$

This finally shows that

$$f_{\widetilde{X}}(x) = \frac{\beta}{1+\beta} f_{X+\widetilde{C}}\left(x\right) + \frac{1}{1+\beta} f_{\widetilde{X}+C}\left(x\right).$$

Denoting as $\mathcal{L}_X\left(t\right)$ the Laplace transform of X, we deduce from the previous equation that

$$\begin{aligned}
\mathcal{L}_{\widetilde{X}}\left(t\right) &= \frac{\beta}{1+\beta}\mathcal{L}_{X+\widetilde{C}}\left(t\right) + \frac{1}{1+\beta}\mathcal{L}_{\widetilde{X}+C}\left(t\right) \\
&= \frac{\beta}{1+\beta}\mathcal{L}_{X}\left(t\right)\mathcal{L}_{\widetilde{C}}\left(t\right) + \frac{1}{1+\beta}\mathcal{L}_{\widetilde{X}}\left(t\right)\mathcal{L}_{C}\left(t\right) \\
&= \mathcal{L}_{X}\left(t\right)\mathcal{L}_{\widetilde{C}}\left(t\right)\frac{\beta/(1+\beta)}{\left(1-\mathcal{L}_{C}\left(t\right)/(1+\beta)\right)} \\
&= \mathcal{L}_{X}\left(t\right)\mathcal{L}_{\widetilde{C}}\left(t\right)\mathcal{L}_{Z}\left(t\right)
\end{aligned}$$

which ends the proof.

D Proof of Proposition 5.1

We know from Embrechts et al. (1997, Theorem A3.20) that

$$\bar{F}_{X_i}\left(x\right) \sim E\left[N_i\right]\bar{F}_{C_i}\left(x\right).$$

Since the density function f_{C_i} is a decreasing function, the density function

$$f_{X_i|X_i>0}\left(x\right) = \sum_{k=1}^{\infty} P\left[N_i = k\right] f_{C_i}^{*k}\left(x\right)$$

is also a decreasing function. We then deduce from Embrechts et al. (1997, Theorem A3.7) that

$$f_{X_i|X_i>0}\left(x\right) \sim \frac{1}{P\left[N_i > 0\right]} E\left[N_i\right]\alpha_i x^{-\alpha_i - 1} L_i\left(x\right).$$

Moreover, from (3.1), we also have

$$f_{\widetilde{X}_i}(x) = \frac{x f_{X_i|X_i>0}(x)}{E[X_i|X_i>0]} \sim \frac{1}{E\left[C_i\right]}\alpha_i x^{-\alpha_i} L_i\left(x\right),$$

and by Karamata's theorem

$$\bar{F}_{\widetilde{X}_i}\left(x\right) \sim \frac{1}{E\left[C_i\right]}\frac{\alpha_i}{\alpha_i - 1} x^{-\alpha_i + 1} L_i\left(x\right).$$

We now consider separately the two cases (i)-(ii) in Proposition 5.1.

Considering (i), assume that $\alpha_1 = ... = \alpha_n = \alpha$ and $L_i(x) = c_i L(x)$ with $c_i > 0$ for $i = 1, ..., n$. We have that

$$\bar{F}_{S-X_i+\widetilde{X}_i}(x) \sim \bar{F}_{\widetilde{X}_i}(x) \sim \frac{c_i}{E[C_i]} \frac{\alpha}{(\alpha-1)} x^{-\alpha+1} L(x).$$

Since $f_{S-X_i+\widetilde{X}_i}$ is an ultimately decreasing function, we deduce that

$$f_{S-X_i+\widetilde{X}_i}(x) \sim \frac{c_i}{E[C_i]} \alpha x^{-\alpha} L(x)$$

The announced result then follows from (3.2).

Turning to (ii), assume that $\alpha_1 < \min\{\alpha_2, ..., \alpha_n\}$. Since the random variable $\widetilde{X}_1$ has a regularly varying tail with index $\alpha_1 - 1 < \alpha_1$, we have that

$$\bar{F}_{S-X_1+\widetilde{X}_1}(x) \sim \bar{F}_{\widetilde{X}_1}(x) \sim \frac{1}{E[C_1]} \frac{\alpha_1}{(\alpha_1-1)} x^{-\alpha_1+1} L_1(x).$$

For $j \in \{2, \ldots, n\}$, we have

$$\begin{aligned} \bar{F}_{S-X_j+\widetilde{X}_j}(x) &\sim \begin{cases} O\left(x^{-\alpha_1} L_1(x)\right) & \text{if } \alpha_1 < \alpha_j - 1 \\ \frac{\alpha_j}{E[C_j](\alpha_j-1)} x^{-\alpha_j+1} L_j(x) & \text{if } \alpha_1 > \alpha_j - 1 \end{cases} \\ &= o\left(\bar{F}_{S-X_1+\widetilde{X}_1}\right) \end{aligned}$$

and we deduce that

$$\bar{F}_{S-X_j+\widetilde{X}_j}(x) = o\left(\bar{F}_{S-X_1+\widetilde{X}_1}(x)\right).$$

Since $f_{S-X_j+\widetilde{X}_j}$, $j \in \{1, \ldots, n\}$, are ultimately decreasing functions, we deduce that

$$f_{S-X_j+\widetilde{X}_j}(x) = o\left(f_{S-X_1+\widetilde{X}_1}(x)\right) \quad \text{for } j \in \{2, \ldots, n\}$$

and the announced result follows.

E Proof of Proposition 6.1

E.1 Sums with deterministic numbers of terms

Before considering compound Binomial sums, we start with the limiting case considered in Remark 4.2, that is, with sums comprising a deterministic numbers of terms. Losses $X_1, X_2, \ldots, X_n$ are thus of the form $X_i = \sum_{k=1}^{\nu_i} C_{i,k}$ for some positive integers ν_i and $C_{i,k}$ ∼Gamma(α_i, τ_i), all the random variables being independent. We establish the validity of Proposition 6.1 in this limit case.

For (i), note that X_i ∼Gamma$(\nu_i\alpha_i, \tau_i)$ and $\widetilde{X}_i$ ∼Gamma$(\nu_i\alpha_i + 1, \tau_i)$ for all $i \in \{1, \ldots, n\}$. Since $\tau_1 < \min\{\tau_2, ..., \tau_n\}$, we deduce from Example 7.32 in Balkema et al. (1999) that

$$f_{S-X_1+\widetilde{X}_1}(x) \sim f_{\widetilde{X}_1}(x) \prod_{j=2,\ldots,n} M_{X_j}\left(\tau_j^{-1}\right)$$

and that, for $i > 1$,

$$f_{S-X_i+\widetilde{X}_i}(x) \sim f_{X_1}(x) M_{\widetilde{X}_i}\left(\tau_i^{-1}\right) \prod_{j=2,\ldots,n,j\neq i} M_{X_j}\left(\tau_j^{-1}\right).$$

The announced result follows immediately.

For (ii), since X_i ~Gamma$(\nu_i\alpha_i, \tau)$ and $\widetilde{X}_i$ ~Gamma$(\nu_i\alpha_i + 1, \tau)$ for all $i \in \{1, \ldots, n\}$ with $\tau = \tau_1 = ... = \tau_n$, we have

$$S - X_i + \widetilde{X}_i \sim \text{Gamma}\left(\sum_{j=1}^{n} \nu_j\alpha_j + 1, \tau\right).$$

The result then follows from (3.2) and by noting that $E[X_i] = \nu_i\alpha_i/\tau$.

E.2 Compound Binomial case

As in Section 4.1, we denote N_i' ~Binomial$(\nu_i - 1, p_i)$ and $\widetilde{X}_i = \sum_{k=1}^{N_i'} C_{i,k} + \widetilde{C}_i$ for $i = 1, \ldots, n$. We define

$$A_{i,m_1,\ldots,m_n} = \{N_1 = m_1, .., N_{i-1} = m_{i-1}, N_i' = m_i, N_{i+1} = m_{i+1}, ..., N_n = m_n\}.$$

For (i), given the event $A_{i,m_1,\ldots,m_n}$, we have

$$\widetilde{X}_i \sim \text{Gamma}\left((m_i + 1)\alpha_i, \tau_i\right)$$

and for $j \neq i$

$$\begin{aligned} X_j &\sim \text{Gamma}\left(m_j\alpha_j, \tau_j\right) \quad \text{if } m_j \geq 1, \\ X_j &= 0 \quad \text{if } m_j = 0. \end{aligned}$$

From the proof of the validity of Proposition 6.1 for sums with deterministic numbers of terms (see Subsection E.1 above), we deduce that there exist positive constants D_i, $i = 1, ..., n$ for which we must have

$$\begin{aligned} f_{S-X_1+\widetilde{X}_1}(x) &\sim D_1 x^{\nu_1\alpha_1} \exp\left(-x\tau_1\right) \\ f_{S-X_i+\widetilde{X}_i}(x) &\sim D_i x^{\nu_1\alpha_1 - 1} \exp\left(-x\tau_1\right) \text{ for } i = 2, \ldots, n, \end{aligned}$$

since

$$f_{S-X_i+\widetilde{X}_i}(x) = \sum_{m_1=0}^{\nu_1} \ldots \sum_{m_i=0}^{\nu_i - 1} \ldots \sum_{m_n=0}^{\nu_n} P[A_{i,m_1,\ldots,m_n}] f_{S-X_i+\widetilde{X}_i|A_{i,m_1,\ldots,m_n}}(x).$$

The announced result then follows.

For (ii), given the event $A_{i,m_1,\ldots,m_n}$, we have

$$S - X_i + \widetilde{X}_i \sim \text{Gamma}\left(\sum_{j=1}^{n} m_j\alpha_j + (\alpha_i + 1), \tau\right).$$

Therefore

$$\begin{aligned}
f_{S-X_i+\widetilde{X}_i}(x) &\sim P[N_1=\nu_1,..,N_{i-1}=\nu_{i-1},N_i'=\nu_i-1,N_{i+1}=\nu_{i+1},...,N_n=\nu_n] \\
&\quad \times \frac{\tau^{\sum_{j=1}^n \nu_j\alpha_j+1}}{\Gamma\left(\sum_{j=1}^n \nu_j\alpha_j+1\right)} x^{\sum_{j=1}^n \nu_j\alpha_j} \exp(-x\tau) \\
&= \frac{\prod_{j=1}^n p_j^{\nu_j}}{p_i} \frac{\tau^{\sum_{j=1}^n \nu_j\alpha_j+1}}{\Gamma\left(\sum_{j=1}^n \nu_j\alpha_j+1\right)} x^{\sum_{j=1}^n \nu_j\alpha_j} \exp(-x\tau).
\end{aligned}$$

The announced result then follows from (3.2).

F Proof of Proposition 6.2

To prove Proposition 6.2, we first discuss the asymptotic behavior of the conditional mean risk sharing rule for absolutely continuous risks with Gaussian tails.

F.1 Absolutely continuous risks with Gaussian tails

Suppose that the probability density functions of $X_1,...,X_n$ are positive on an interval I that is unbounded above. Assume further that these density functions are such that

$$f_{X_i}(x) \sim \gamma_i(x) e^{-\psi_i(x)} \text{ as } x\to\infty \text{ for } i=1,2,\ldots,n,$$

where the functions ψ_i and γ_i satisfy the following conditions:

(a) the function ψ_i is C^2 (i.e. twice differentiable);

(b) the function ψ_i is ultimately convex, that is, $\psi_i''(x)>0$ for large x;

(c) the function σ_i defined as $\sigma_i(x)=(\psi_i''(x))^{-1/2}$ is self-neglecting, that is,

$$\lim_{x\to\infty}\frac{\sigma_i(x+t\sigma_i(x))}{\sigma_i(x)}=1 \quad \text{locally uniformly in } t; \tag{F.1}$$

(d) the function γ_i satisfies the condition

$$\lim_{x\to\infty}\frac{\gamma_i(x+t\sigma_i(x))}{\gamma_i(x)}=1 \quad \text{locally uniformly in } t. \tag{F.2}$$

Finally, assume that $\tau_\infty=\lim_{x\to\infty}\psi_i'(x)$ is independent of i. The risks $X_1,X_2,\ldots,X_n$ are then said to have Gaussian tails after Barndorff-Nielsen and Klüppelberg (1992) and Balkema et al. (1995).

We then have the following result.

Proposition F.1. *Consider independent risks* $X_1, \ldots, X_n$ *with Gaussian tails. Assume that* $\lim_{s\to\infty} \sigma_i(s)/s = 0$ *for* $i = 1, ..., n$*. Let the function* q_i *be defined as*

$$q_i(s) = \left(\sum_{j=1}^{n} (\psi_j')^{(-1)} \circ \psi_i'\right)^{(-1)}(s).$$

The following results then hold true:

(i) If $q_i(s) \sim \beta_i q(s)$ *for a positive function* q *and some positive constants* β_i *for* $i \in \{1, \ldots, n\}$*, then*

$$E[X_i|S=s] \sim \frac{\beta_i}{\sum_{j=1}^{n} \beta_j} s \text{ for } i \in \{1, 2, \ldots, n\}.$$

(ii) If $q_j(s) = o(q_1(s))$ *for* $j \in \{2, \ldots, n\}$*, then*

$$E[X_1|S=s] \sim s \text{ and } E[X_j|S=s] = o(s) \text{ for } j \in \{2, \ldots, n\}.$$

Proof of Proposition F.1 The asymptotic behavior of the probability density function of the sum S of risks $X_1, X_2, \ldots, X_n$ with Gaussian tails is characterized in the following theorem, taken from Barndorff-Nielsen and Klüppelberg (1992).

Theorem F.2. *(Barndorff-Nielsen and Klüppelberg (1992)). The probability density function of* S *satisfies*

$$f_S(s) \sim \gamma_S(s) e^{-\psi_S(s)} \quad as \quad s \to \infty,$$

where ψ_S *is* C^2*,* $\psi_S''(s) > 0$ *for large* s*,* $\sigma_S(s) = (\psi_S''(s))^{-1/2}$ *is self-neglecting. Explicit formulas for* γ_S *and* ψ_S *can be given as follows: define the function* q_i *from* $\psi_i'(q_i) = \tau$ *where* $s = s(\tau) = q_1 + ... + q_n$*. Then* s *is a continuous strictly increasing function of* τ *and* $s(\tau) \uparrow \infty$ *as* $\tau \uparrow \tau_\infty$*. Now one may choose*

$$\begin{aligned} \psi_S(s) &= \psi_1(q_1) + ... + \psi_n(q_n) \\ \sigma_S^2(s) &= \sigma_1^2(q_1) + ... + \sigma_n^2(q_n) \end{aligned}$$

and

$$\sqrt{2\pi}\sigma_S(s)\gamma_S(s) = \prod_{i=1}^{n}\left(\sqrt{2\pi}\sigma_i(q_i)\gamma_i(q_i)\right).$$

Then $\sigma_S(s) = (\psi_S''(s))^{-1/2}$ *and* $\lim_{s\to\infty} \psi_S'(s) = \tau_\infty$*.*

The results stated under Proposition F.1 are then established as follows. For $i = 1, ..., n$, we have

$$f_{\widetilde{X}_i}(x) \sim \widetilde{\gamma}_i(x) e^{-\psi_i(x)} \quad \text{as} \quad x \to \infty,$$

where $\widetilde{\gamma}_i(x) = x\gamma_i(x)/E[X_i]$ and satisfies

$$\lim_{x\to\infty} \frac{\widetilde{\gamma}_i(x + t\sigma_i(x))}{\widetilde{\gamma}_i(x)} = 1 \quad \text{locally uniformly in } t.$$

Using Theorem F.2, we deduce that

$$\frac{f_{S - X_i + \widetilde{X}_i}(s)}{f_S(s)} \sim \frac{q_i(s)}{E[X_i]}.$$

The asymptotic behavior stated under (i) and (ii) are then easily deduced from (3.2).

F.2 Proof of Proposition 6.2

We are now ready to establish the validity of the results stated in Proposition 6.2. Considering item (i), it is proved in Withers and Nadarajah (2011) that the function r_α appearing in Remark 6.3 satisfies

$$r_\alpha(x) \sim \exp\left((1+\alpha)\,\xi_{x,\alpha}\right)\left(\frac{1}{2\pi(1+\alpha)}\alpha\xi_{x,\alpha}\right)^{1/2}\sum_{j=0}^{\infty} e_j \xi_{x,\alpha}^{-1}$$

where $\xi_{x,\alpha} = (x\alpha^{-\alpha})^{1/(1+\alpha)}$ and the coefficient e_j is given by formula (3.3) in Withers and Nadarajah (2013). It follows that

$$\begin{aligned} f_{X_i|X_i>0}(x) \quad &\sim \quad \frac{e^{-\lambda_i}}{1-e^{-\lambda_i}}\left(\frac{1}{2\pi(1+\alpha_i)}\alpha_i^{1-\alpha_i/(1+\alpha_i)}\right)^{1/2}\left(\lambda_i\tau_i^{\alpha_i}x^{\alpha_i}\right)^{1/2(1+\alpha_i)}x^{-1} \\ &\times \exp\left(-x\tau_i + (1+\alpha_i)\,\alpha_i^{-\alpha_i/(1+\alpha_i)}\left(\lambda_i\tau_i^{\alpha_i}x^{\alpha_i}\right)^{1/(1+\alpha_i)}\right) \end{aligned}$$

and

$$f_{X_i|X_i>0}(x) \sim \gamma_i(x)\,e^{-\psi_i(x)}$$

with

$$\begin{aligned} \gamma_i(x) \quad &= \quad \frac{e^{-\lambda_i}}{1-e^{-\lambda_i}}\left(\frac{1}{2\pi(1+\alpha_i)}\alpha_i^{1+\alpha_i/(1+\alpha_i)}\right)^{1/2}\left(\lambda_i\tau_i^{\alpha_i}x^{\alpha_i}\right)^{-1/2(1+\alpha_i)}x^{-1} \\ \psi_i(x) \quad &= \quad x\tau_i - (1+\alpha_i)\,\alpha_i^{-\alpha_i/(1+\alpha_i)}\left(\lambda_i\tau_i^{\alpha_i}x^{\alpha_i}\right)^{1/(1+\alpha_i)}. \end{aligned}$$

Note that ψ_i is C^2 and $\psi_i''(x) > 0$ for large x. Moreover, the function $\sigma_i(x) = (\psi_i''(x))^{-1/2}$ is self-neglecting, that is, (F.1) is valid. Also, the function γ_i satisfies condition (F.2). Furthermore, $\tau_\infty = \lim_{x\to\infty}\psi_i'(x)$ is independent of i if $\tau_1 = ... = \tau_n = \tau_\infty$.

Now, we have

$$f_{\widetilde{X}_i}(x) \sim \frac{x}{E[X_i]}\gamma_i(x)\,e^{-\psi_i(x)}.$$

Moreover $S - X_i + \widetilde{X}_i$ is absolutely continuous and

$$f_{S-X_i+\widetilde{X}_i}(s) = \sum_{\substack{k=0,\dots,n-1 \\ j_1,\dots,j_k\in\{1,2,\dots,n\}\setminus i}} P\left[A_{i,j_1,\dots,j_k}\right] f_{S-X_i+\widetilde{X}_i|A_{i,j_1,\dots,j_k}}(s) \tag{F.3}$$

with

$$A_{i,j_1,\dots,j_k} = \{X_{j_1} > 0, ..., X_{j_k} > 0, X_l = 0; l \neq j_1, ..., j_k, i\}.$$

Since

$$\begin{aligned} f_{S-X_i+\widetilde{X}_i}(s) \quad &\sim \quad P\left[X_1 > 0, ..., X_{i-1} > 0, X_{i+1} > 0, ..., X_n > 0\right] f_{S-X_i+\widetilde{X}_i|X_1>0,\dots,X_{i-1},X_{i+1},\dots,X_n>0}(s) \\ &= \quad \prod_{j\neq i}(1-e^{-\lambda_j}) f_{X_1|X_1>0} * ... * f_{X_{i-1}|X_{i-1}>0} * f_{\widetilde{X}_i} * f_{X_{i+1}|X_{i+1}>0} * ... * f_{X_n|X_n>0}(s) \end{aligned}$$

we can use Proposition F.1 ii) to conclude.

Turning to item (ii), we see that $C_1, C_1, , ..., C_n$ are identically distributed in this case and the result is given by (4.7).

G Proof of Proposition 6.4

(i) Note that $C_1, C_1, , ..., C_n$ are identically distributed with distribution Gamma(α, τ). Moreover

$$M_{C_i}(t) = M_C(t) = \frac{1}{(1+\tau^{-1}t)^{\alpha}}.$$

Let κ_i be such that $M_C(\kappa_i) = (1+\beta_i)$ and define

$$\nu_1 = \frac{1}{(1+\beta_1)} M'_C(\kappa_1).$$

Since $N_i \sim$Negative Binomial(ξ_i, β_i), we deduce that

$$\begin{aligned}
M_{N_i}(t) &= \left(\frac{\beta_i/(1+\beta_i)}{(1-e^t/(1+\beta_i))}\right)^{\xi_i} \text{ for } t < \ln(1+\beta_i), \\
M_{X_i}(t) &= \left(\frac{\beta_i/(1+\beta_i)}{(1-M_C(t)/(1+\beta_i))}\right)^{\xi_i} \text{ for } t < \tau\left((1+\beta_i)^{-1/\alpha} - 1\right), \\
M_S(t) &= \prod_{i=1}^{n} M_{X_i}(t) \text{ for } t < \tau\left((1+\beta_1)^{-1/\alpha} - 1\right).
\end{aligned}$$

Moreover we have

$$\begin{aligned}
M_{\widetilde{C}}(t) &= \frac{E\left[Ce^{tC}\right]}{E[C]} = \frac{1}{E[C]} M'_C(t) \\
M_{Z_i}(t) &= \left(\frac{\beta_i/(1+\beta_i)}{(1-M_C(t)/(1+\beta_i))}\right) \text{ for } t < \tau\left((1+\beta_i)^{-1/\alpha} - 1\right).
\end{aligned}$$

Let

$$U_i(s) = e^{\kappa_1 s} f_{S+\widetilde{C}_i+Z_i}(s)$$

and

$$\hat{U}_i(t) = t\int_0^{\infty} e^{-ts} U_i(s)\, ds = t\int_0^{\infty} e^{-(t-\kappa_1)s} f_{S+\widetilde{C}_i+Z_i}(s)\, ds = tM_{S+\widetilde{C}_i+Z_i}(\kappa_1 - t).$$

Note that

$$M_{S+\widetilde{C}_i+Z_i}(\kappa_1 - t) = M_{X_1}(\kappa_1 - t)\, M_{Z_i}(\kappa_1 - t)\, \frac{M'_C(\kappa_1 - t)}{E[C]} \left(\prod_{j=2}^{n} M_{X_j}(\kappa_1 - t)\right).$$

If $i = 1$ then

$$M_{S+\widetilde{C}_1+Z_1}(\kappa_1 - t) \underset{t\downarrow 0}{\sim} \left(\frac{\beta_1/(1+\beta_1)}{t\nu_1}\right)^{\xi_1} \frac{1}{t} \frac{\beta_1}{E[C]} \left(\prod_{j=2}^{n} M_{X_j}(\kappa_1)\right)$$

and for $i \in \{2, \ldots, n\}$

$$M_{S+\widetilde{C}_1+Z_1}(t) \underset{t\downarrow 0}{\sim} \left(\frac{\beta_1/(1+\beta_1)}{t\nu_1}\right)^{\xi_1} M_{Z_i}(\kappa_1)\, \frac{M'_C(\kappa_1)}{E[C]} \left(\prod_{j=2}^{n} M_{X_j}(\kappa_1)\right).$$

It follows that

$$\hat{U}_1(t) \underset{t\downarrow 0}{\sim} t^{-\xi_1} \left(\frac{\beta_1/(1+\beta_1)}{\nu_1}\right)^{\xi_1} \frac{\nu_1\beta_1}{E[C]} \left(\prod_{j=2}^{n} M_{X_j}(\kappa_1)\right)$$

and for $i \in \{2, \ldots, n\}$

$$\hat{U}_i(t) \underset{t\downarrow 0}{\sim} t^{-(\xi_1-1)} \left(\frac{\beta_1/(1+\beta_1)}{\nu_1}\right)^{\xi_1} M_{Z_i}(\kappa_1) \frac{M'_C(\kappa_1)}{E[C]} \left(\prod_{j=2}^{n} M_{X_j}(\kappa_1)\right).$$

By Theorem 1.7.6 in Bingham et al. (1987), we have

$$f_{S+\widetilde{C}_1+Z_1}(s) \sim s^{\xi_1} e^{-\kappa_1 s} \frac{1}{\Gamma(1+\xi_1)} \left(\frac{\beta_1/(1+\beta_1)}{\nu_1}\right)^{\xi_1} \frac{\nu_1\beta_1}{E[C]} \left(\prod_{j=2}^{n} M_{X_j}(\tau_1)\right)$$

and for $i \in \{2, \ldots, n\}$

$$f_{S+\widetilde{C}_i+Z_i}(s) \sim s^{\xi_1-1} e^{-\kappa_1 s} \frac{1}{\Gamma(\xi_1)} \left(\frac{\beta_1/(1+\beta_1)}{\nu_1}\right)^{\xi_1} M_{Z_i}(\kappa_1) \frac{M'_C(\kappa_1)}{E[C]} \left(\prod_{j=2}^{n} M_{X_j}(\kappa_1)\right).$$

This ends the proof for (i).

(ii) In this case, $C_1, C_1, , ..., C_n$ are identically distributed and $\beta_1 = ... = \beta_n$. So, the result is given by (4.10).

H Proof of Proposition 7.1

Let $b_* = \max\{b_1, b_2, \ldots, b_n\}$. We know that S obeys the compound Poisson distribution with Poisson parameter $\lambda_\bullet$ and claim severities distributed as Z, where for $k = 1, ..., b_*$,

$$P[Z = k] = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_\bullet} P[C_i = k]. \tag{H.1}$$

Panjer's formula ensures that for $s \geq b_*$, we have

$$P[S = s] = \lambda_\bullet \sum_{k=1}^{b_*} \frac{k}{s} P[Z = k] P[S = s-k]. \tag{H.2}$$

Note that $P[S = s] > 0$ for $s \geq 0$ since $P[Z = k] > 0$ for $k = 1, ..., b_*$. Let us define the sequence $(c_s)_{s \geq b_*}$ by

$$c_s = P[S = s] e^{s \ln s / b_*}.$$

Note that, for $k = 1, ..., b_*$,

$$P[S = s-k] = c_{s-k} e^{-(s-k)\ln(s-k)/b_*}$$

and that, for large s,

$$P[S = s-k] \sim c_{s-k} e^{-s \ln s / b_*} e^{k/b_*} s^{k/b_*}.$$

From (H.2), we have the following recurrence relation

$$c_s = \lambda_\bullet \sum_{k=1}^{b_*} s^{(k-b_*)/b_*} k P[Z=k] e^{-(s-k)\ln(1-k/s)/b_*} c_{s-k}$$

and, for large s, we see that

$$c_s \sim e\lambda_\bullet b_* P[Z=b_*] c_{s-b_*}.$$

Therefore we define

$$d_s = c_s \left(e\lambda_\bullet b_* P[Z=b_*]\right)^{s/b_*}$$

and derive the following recurrence relation

$$d_s = \lambda_\bullet \sum_{k=1}^{b_*} s^{(k-b_*)/b_*} e^{-(s-k)\ln(1-k/s)/b_*} \frac{kP[Z=k]}{\left(e\lambda_\bullet b_* P[Z=b_*]\right)^{k/b_*}} d_{s-k}.$$

We note that, for large s,

$$d_s \sim d_{s-b_*}.$$

It follows that

$$d_s \sim d_{s-1},$$

otherwise there would be a contradiction with the previous asymptotic relation.

We deduce that, for large s,

$$\frac{P[S=s]}{P[S=s-1]} \sim \left(e\lambda_\bullet b_* P[Z=b_*]\right)^{-1/b_*} s^{-1/b_*}$$

and then

$$\lim_{s\to\infty} \frac{P[S=s]}{P[S=s-1]} = 0.$$

For every $i = 1, ..., n$ and $s \geq b_*$, we obviously have

$$P[S+\widetilde{C}_i = s] = \sum_{k=1}^{b_i} P[\widetilde{C}_i = k] P[S=s-k].$$

The term dominating this sum corresponds to $k = b_i$ and we then deduce that

$$P[S+\widetilde{C}_i = s] \sim P[\widetilde{C}_i = b_i] P[S = s-b_i] \text{ for } i = 1, \ldots, n.$$

The announced result then follows from (7.1) which gives

$$E[X_i|S=s] \sim \frac{E\left[X_i\right] P[\widetilde{C}_i = b_i] P[S=s-b_i]}{\sum_{j=1}^n E\left[X_j\right] P[\widetilde{C}_j = b_j] P[S=s-b_j]} s$$

and ends the proof.

I Proof of Proposition 7.2

The probability generating functions of C_i and $\widetilde{C}_i$ are respectively given by

$$\begin{aligned} G_{C_i}(z) &= E\left[z^{C_i}\right] = \frac{\ln(1-p_i z)}{\ln(1-p_i)}, \\ G_{\widetilde{C}_i}(z) &= E\left[z^{\widetilde{C}_i}\right] = \frac{(1-p_i)z}{(1-p_i z)}, \end{aligned}$$

with $z \geq 0$. Moreover, with $\alpha_i = -\lambda_i / \ln(1-p_i)$, we have

$$G_{X_i}(z) = e^{\lambda_i\left(G_{C_i}(z)-1\right)} = \left(\frac{1-p_i}{1-p_i z}\right)^{\alpha_i}$$

showing that X_i is Negative Binomially distributed.

Define, for $0 \leq z < 1$,

$$A_S(z) = G_S(z/p_1) = \prod_{i=1}^{n} G_{X_i}(z/p_1).$$

We have, as $z \uparrow 1$,

$$A_S(z) \sim \left(\frac{1-p_1}{1-z}\right)^{\alpha_1} G_{S-X_1}\left(\frac{1}{p_1}\right).$$

In the same way, we get, as $z \uparrow 1$,

$$\begin{aligned} A_{S+\widetilde{C}_1}(z) &= G_{S+\widetilde{C}_1}(z/p_1) \sim \left(\frac{1-p_1}{1-z}\right)^{\alpha_1+1} G_{S-X_1}\left(\frac{1}{p_1}\right) \\ A_{S+\widetilde{C}_j}(z) &= G_{S+\widetilde{C}_j}(z/p_1) \sim \left(\frac{1-p_1}{1-z}\right)^{\alpha_1} G_{S-X_1}\left(\frac{1}{p_1}\right) G_{\widetilde{C}_j}\left(\frac{1}{p_1}\right), \quad j \in \{2,\ldots,n\}. \end{aligned}$$

Note that

$$G_{S+\widetilde{C}_1}(z) = \sum_{k=1}^{\infty} P[S+\widetilde{C}_1 = k] z^k$$

and

$$A_{S+\widetilde{C}_1}(z) = \sum_{k=1}^{\infty} a_{S+\widetilde{C}_1}(k) z^k \text{ with } a_{S+\widetilde{C}_1}(k) = P[S+\widetilde{C}_1 = k] p_1^{-k}.$$

By Corollary 1.7.3 in Bingham et al. (1987), we have

$$a_{S+\widetilde{C}_1}(k) \sim (1-p_1)^{\alpha_1+1} G_{S-X_1}\left(\frac{1}{p_1}\right) \frac{1}{\Gamma(1+\alpha_1)} k^{\alpha_1}$$

and

$$P[S+\widetilde{C}_1 = k] = a_{S+\widetilde{C}_1}(k) p_1^k \sim (1-p_1)^{\alpha_1+1} G_{S-X_1}\left(\frac{1}{p_1}\right) \frac{1}{\Gamma(1+\alpha_1)} k^{\alpha_1} p_1^k.$$

In the same way, we have for $j \in \{2, \ldots, n\}$

$$a_{S+\widetilde{C}_j}(k) \sim (1-p_1)^{\alpha_1} G_{S-X_1}\left(\frac{1}{p_1}\right) G_{\widetilde{C}_i}\left(\frac{1}{p_1}\right) \frac{1}{\Gamma(\alpha_1)} k^{\alpha_1 - 1}$$

and

$$P[S+\widetilde{C}_j = k] = a_{S+\widetilde{C}_j}(k) p_1^k \sim (1-p_1)^{\alpha_1} G_{S-X_1}\left(\frac{1}{p_1}\right) G_{\widetilde{C}_i}\left(\frac{1}{p_1}\right) \frac{1}{\Gamma(\alpha_1)} k^{\alpha_1 - 1} p_1^k.$$

The announced result then follows from identity (7.1).