A Linear Bound on the K-Rendezvous Time for Primitive Sets of NZ Matrices

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Abstract. A set of nonnegative matrices is called primitive if there exists a product of these matrices that is entrywise positive. Motivated by recent results relating synchronizing automata and primitive sets, we study the length of the shortest product of a primitive set having a column or a row with k positive entries (the k-RT). We prove that this value is at most linear w.r.t. the matrix size n for small k, while the problem is still open for synchronizing automata. We then report numerical results comparing our upper bound on the k-RT with heuristic approximation methods.

Keywords: Primitive set of matrices · Synchronizing automaton · Černý conjecture

1 Introduction

Primitive Sets of Matrices. The notion of *primitive* matrix[1], introduced by Perron and Frobenius at the beginning of the 20th century in the theory that carries their names, can be extended to *sets* of matrices: a set of nonnegative matrices $\mathcal{M} = \{M_1, \ldots, M_m\}$ is called *primitive* if there exists some indices $i_1, \ldots, i_r \in \{1, \ldots, m\}$ such that the product $M_{i_1} \cdots M_{i_r}$ is entrywise positive. A product of this kind is called a *positive* product and the length of the shortest positive product of a primitive set $\mathcal{M}$ is called its *exponent* and it is denoted by $exp(\mathcal{M})$. The concept of primitive set was just recently formalized by Protasov and Voynov [31], but has been appearing before in different fields as in stochastic switching systems [20,30] and time-inhomogeneous Markov chains [19,34]. It has lately gained more importance due to its applications in consensus of discrete-time multi-agent systems [9], cryptography [12] and automata theory [4,6,15].

[1] A nonnegative matrix M is *primitive* if there exists $s \in \mathbb{N}$ such that $M^s > 0$ entrywise.

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Deciding whether a set is primitive is a PSPACE-complete problem for sets of two matrices [15], while it is an NP-hard problem for sets of at least three matrices [4]. Computing the exponent of a primitive set is usually hard, namely it is an $\mathrm{FP}^{\mathrm{NP[log]}}$-complete problem [15]; for the complexity of other problems related to primitivity and the computation of the exponent, we refer the reader to [15]. For sets of matrices having at least one positive entry in every row and every column (called *NZ* [15] or *allowable* matrices [18,20]), the primitivity problem becomes decidable in polynomial-time [31], although computing the exponent remains NP-hard [15]. Methods for approximating the exponent have been proposed [7] as well as upper bounds that depend just on the matrix size; in particular, if we denote with $exp_{NZ}(n)$ the maximal exponent among all the primitive sets of $n \times n$ NZ matrices, it is known that $exp_{NZ}(n) \leq (15617n^3 + 7500n^2 + 56250n - 78125)/46875$ [4,36]. Better upper bounds have been found for some classes of primitive sets (see e.g. [15] and [19], Theorem 4.1). The NZ condition is often met in applications and in particular in the connection with synchronizing automata.

Synchronizing Automata. A *(complete deterministic finite state) automaton* is a 3-tuple $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$ where $Q = \{q_1, \ldots, q_n\}$ is a finite set of states, $\Sigma = \{a_1, \ldots, a_m\}$ is a finite set of input symbols (the *letters* of the automaton) and $\delta : Q \times \Sigma \to Q$ is the *transition function*. Let $i_1, i_2, \ldots, i_l \in \{1, \ldots, m\}$ be indices. Then $w = a_{i_1} a_{i_2} \ldots a_{i_l}$ is called a word and we define $\delta(q, w) = \delta(\delta(q, a_{i_1} a_{i_2} \ldots a_{i_{l-1}}), a_{i_l})$. An automaton is *synchronizing* if it admits a word w, called a *synchronizing* or a *reset* word, and a state q such that $\delta(q', w) = q$ for any state $q' \in Q$. In other words, the reset word w brings the automaton from every state to the same fixed state.

Remark 1. The automaton $\mathcal{A}$ can be equivalently represented by the set of matrices $\{A_1, \ldots, A_m\}$ where, for all $i = 1, \ldots, m$ and $l, k = 1, \ldots, n$, $(A_i)_{lk} = 1$ if $\delta(q_l, a_i) = q_k$, $(A_i)_{lk} = 0$ otherwise. The action of a letter a_i on a state q_j is represented by the product $e_j^T A_i$, where e_j is the j-th element of the canonical basis. Notice that the matrices $A_1, \ldots, A_m$ are binary[2] and row-stochastic, i.e. each of them has exactly one entry equal to 1 in every row and zero everywhere else. In this representation, the automaton $\mathcal{A}$ is synchronizing if and only if there exists a product of its matrices with a column whose entries are all equal to 1 (also called an *all-ones* column).

The idea of synchronization is quite simple: we want to restore control over a device whose current state is unknown. For this reason, synchronizing automata are often used as models of error-resistant systems [8,11], but they also find application in other fields such as in symbolic dynamics [25], in robotics [26] or in resilience of data compression [33,37]. For a recent survey on synchronizing automata we refer the reader to [42]. We are usually interested in the length of the shortest reset word of a synchronizing automaton $\mathcal{A}$, called its *reset threshold* and denoted by $rt(\mathcal{A})$. Despite the fact that determining whether an automaton is synchronizing can be done in polynomial time (see e.g. [42]), computing

[2] A *binary* matrix is a matrix having entries in $\{0, 1\}$.

its reset threshold is an NP-hard problem [11][3]. One of the most longstanding open questions in automata theory concerns the maximal reset threshold of a synchronizing automaton, presented by Černý in 1964 in his pioneering paper:

Conjecture 1. (The Černý conjecture [39]*).* Any synchronizing automaton on n states has a synchronizing word of length at most $(n-1)^2$.

Černý also presented in [39] a family of automata having reset threshold of exactly $(n-1)^2$, thus demonstrating that the bound in his conjecture (if true) cannot be improved. Exhaustive search confirmed the Černý conjecture for small values of n [2,5,24,38] and within certain classes of automata (see e.g. [22,35,41]), but despite a great effort has been made to prove (or disprove) it in the last decades, its validity still remains unclear. Indeed on the one hand, the best upper bound known on the reset threshold of any synchronizing n-state automaton is cubic in n [13,28,36], while on the other hand automata having quadratic reset threshold, called *extremal* automata, are very difficult to find and few of them are known (see e.g. [10,16,23,32]). Some of these families have been found by Ananichev et al. [3] by coloring the digraph of primitive matrices having large exponent; this has been probably the first time where primitivity has been successfully used to shed light on synchronization.

Connecting Primitivity and Synchronization. The following definition and theorem establish the connection between primitive sets of binary NZ matrices and synchronizing automata. From here on, we will use the matrix representation of deterministic finite automata as reported in Remark 1.

Definition 1. *Let $\mathcal{M}$ be a set of binary NZ matrices. The* automaton associated to *the set $\mathcal{M}$ is the automaton $Aut(\mathcal{M})$ such that $A \in Aut(\mathcal{M})$ if and only if A is a binary and row-stochastic matrix and there exists $M \in \mathcal{M}$ such that $A \leq M$ (entrywise). We denote with $Aut(\mathcal{M}^T)$ the automaton associated to the set $\mathcal{M}^T = \{M_1^T, \ldots, M_m^T\}$.*

Theorem 1. ([4] Theorems 16–17, [15] Theorem 2). *Let $\mathcal{M} = \{M_1, \ldots, M_m\}$ be a primitive set of binary NZ matrices. Then $Aut(\mathcal{M})$ and $Aut(\mathcal{M}^T)$ are synchronizing and it holds that:*

$$rt\big(Aut(\mathcal{M})\big) \leq exp(\mathcal{M}) \leq rt\big(Aut(\mathcal{M})\big) + rt\big(Aut(\mathcal{M}^T)\big) + n - 1. \qquad (1)$$

Notice that the requirement in Theorem 1 that the set $\mathcal{M}$ has to be made of *binary* matrices is not restrictive, as the primitivity property does not depend on the magnitude of the positive entries of the matrices of the set. We can thus restrict ourselves to the set of binary matrices by using the Boolean product between them[4], that is setting for any A and B binary matrices, $(AB)_{ij} = 1$ any time that $\sum_s A_{is}B_{sj} > 0$. In this framework, primitivity can be also rephrased

[3] Moreover, even approximating the reset threshold of an n-state synchronizing automaton within a factor of $n^{1-\epsilon}$ is known to be NP-hard for any $\epsilon > 0$, see [14].

[4] In other words, we work with matrices over the Boolean semiring.

as a *membership problem* (see e.g. [27,29]), where we ask whether the all-ones matrix belongs to the semigroup generated by the matrix set. The following example reports a primitive set $\mathcal{M}$ of NZ matrices and the synchronizing automata $Aut(\mathcal{M})$ and $Aut(\mathcal{M}^T)$.

Example 1. Here we present a primitive set and its associated automata:
$\mathcal{M}=\left\{\begin{pmatrix} 0&1&0\\0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix} 0&1&0\\1&0&1\\0&0&1\end{pmatrix}\right\}, Aut(\mathcal{M})=\left\{a=\begin{pmatrix} 0&1&0\\0&0&1\\1&0&0\end{pmatrix}, b_1=\begin{pmatrix} 0&1&0\\1&0&0\\0&0&1\end{pmatrix}, b_2=\begin{pmatrix} 0&1&0\\0&0&1\\0&0&1\end{pmatrix}\right\}$,
$Aut(\mathcal{M}^T)=\left\{a'=\begin{pmatrix} 0&0&1\\1&0&0\\0&1&0\end{pmatrix}, b_1=\begin{pmatrix} 0&1&0\\1&0&0\\0&0&1\end{pmatrix}, b_2'=\begin{pmatrix} 0&1&0\\1&0&0\\0&1&0\end{pmatrix}\right\}$
It holds that $exp(\mathcal{M})=7$, $rt\big(Aut(\mathcal{M})\big)=2$ and $rt\big(Aut(\mathcal{M}^T)\big)=3$. See also Fig. 1.

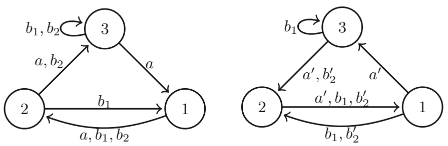


Fig. 1. The automata $Aut(\mathcal{M})$ (left) and $Aut(\mathcal{M}^T)$ (right) of Example 1.

Equation (1) shows that the behavior of the exponent of a primitive set of NZ matrices is tightly connected to the behavior of the reset threshold of its associated automaton. A primitive set $\mathcal{M}$ with quadratic exponent implies that one of the automata $Aut(\mathcal{M})$ or $Aut(\mathcal{M}^T)$ has quadratic reset threshold; in particular, a primitive set with exponent greater than $2(n-1)^2+n-1$ would disprove the Černý conjecture. This property has been used by the authors in [6] to construct a randomized procedure for finding extremal synchronizing automata.

The synchronization problem for automata is about finding the length of the shortest word mapping the whole set of states onto one single state. We can weaken this request by asking what is the length of the shortest word w such that there exists a set of $k \geq 2$ states mapped by w onto one single state. In the matrix framework, we are asking what is the length of the shortest product having a column with k positive entries. The case $k = 2$ is trivial, as any synchronizing automaton has a letter mapping two states onto one; for $k = 3$ Gonze and Jungers [17] presented a quadratic upper bound in the number of the states of the automaton while, to the best of our knowledge, the cases $k \geq 4$ are still open. Clearly, the case $k = n$ is the problem of computing the reset threshold.

In view of the connection between synchronizing automata and primitive sets, we extend the above described problem to primitive sets by introducing the *k-rendezvous time* (k-RT): the k-RT of a primitive set $\mathcal{M}$ is the length of the shortest product having a row or a column with k positive entries. The following proposition shows how the k-RT of a primitive set $\mathcal{M}$ of NZ matrices (denoted by $rt_k(\mathcal{M})$) is linked to the length of the shortest word for which it exists a set of k states mapped by it onto a single state in the automata $Aut(\mathcal{M})$ and $Aut(\mathcal{M}^T)$ (lengths denoted respectively by $rt_k(Aut(\mathcal{M}))$ and $rt_k(Aut(\mathcal{M}^T))$).

Proposition 1. *Let $\mathcal{M}$ be a primitive set of $n \times n$ binary NZ matrices and let $Aut(\mathcal{M})$ and $Aut(\mathcal{M}^T)$ be the automata defined in Definition 1. Then for every $2 \le k \le n$, it holds that $rt_k(\mathcal{M}) = \min\{rt_k(Aut(\mathcal{M})), rt_k(Aut(\mathcal{M}^T))\}$.*

Proof. Omitted due to length restrictions.

Our Contribution. In this paper we prove that for any primitive set $\mathcal{M}$ of $n \times n$ NZ matrices, the k-rendezvous time $rt_k(\mathcal{M})$ is upper bounded by a linear function in n for any fixed $k \le \sqrt{n}$, problem that is still open for synchronizing automata. Our result also implies that $\min\{rt_k(Aut(\mathcal{M})), rt_k(Aut(\mathcal{M}^T))\}$ is upper bounded by a linear function in n for any fixed $k \le \sqrt{n}$, in view of Proposition 1. We then show that our technique for upper bounding $rt_k(\mathcal{M})$ cannot be much improved as it is, and so new strategies have to be implemented in order to possibly achieve better upper bounds. Finally, we report some numerical experiments comparing our theoretical upper bound on the k-RT with the real k-RT (or an approximation of it when it becomes too hard to compute it) for some examples of primitive sets.

2 Notation and Preliminaries

The set $\{1, \ldots, n\}$ is represented by $[n]$. The *support* of a nonnegative vector v is the set $supp(v) = \{i : v_i > 0\}$ and the *weight* of a nonnegative vector v is the cardinality of its support.

Given a matrix A, we denote by A_{*j} its j-th column and by A_{i*} its i-th row. A *permutation* matrix is a binary matrix having exactly one positive entry in every row and every column. We remind that an $n \times n$ matrix A is called *irreducible* if for any $i, j \in [n]$, there exists a natural number k such that $A^k_{ij} > 0$. A matrix A is called *reducible* if it is not irreducible.

Given $\mathcal{M}$ a set of matrices, we denote with $\mathcal{M}^d$ the set of all the products of at most d matrices from $\mathcal{M}$. A set of matrices $\mathcal{M} = \{M_1, \ldots, M_m\}$ is *reducible* if the matrix $\sum_i M_i$ is reducible, otherwise it is called *irreducible*. Irreducibility is a necessary but not sufficient condition for a matrix set to be primitive (see [31], Sect. 1). Given a directed graph $D = (V, E)$, we denote by $v \to w$ the directed edge leaving v and entering in w and with $v \to w \in E$ the fact that the edge $v \to w$ belongs to the digraph D.

Lemma 1. *Let $\mathcal{M}$ be an irreducible set of $n \times n$ NZ matrices, $A \in \mathcal{M}$ and $i, j \in [n]$. Then there exists a matrix $B \in \mathcal{M}^{n-1}$ such that $supp(A_{*i}) \subseteq supp((AB)_{*j})$.*

Proof. We consider the labeled directed graph $\mathscr{D}_{\mathcal{M}} = (V, E)$ where $V = [n]$ and $i \to j \in E$ iff there exists a matrix $A \in \mathcal{M}$ such that $A_{ij} > 0$. We label the edge $i \to j \in E$ by all the matrices $A \in \mathcal{M}$ such that $A_{ij} > 0$. We remind that a directed graph is *strongly connected* if there exists a directed path from any vertex to any other vertex. Notice that a path in $\mathscr{D}_{\mathcal{M}}$ from vertex k to vertex l having the edges sequentially labeled by the matrices $A_{s_1}, \ldots, A_{s_r}$ from $\mathcal{M}$ means that $(A_{s_1} \cdots A_{s_r})_{kl} > 0$. Since $\mathcal{M}$ is irreducible, it follows that $\mathscr{D}_{\mathcal{M}}$

is strongly connected and so, since V has cardinality n, any pair of vertices in $\mathscr{D}_{\mathcal{M}}$ are connected by a path of length at most $n-1$. Consider a path connecting vertex i to vertex j whose edges are sequentially labeled by the matrices $A_{s_1}, \ldots, A_{s_t}$ from $\mathcal{M}$ and let $B = A_{s_1} \cdots A_{s_t}$. Clearly $B \in \mathcal{M}^{n-1}$; furthermore it holds that $B_{ij} > 0$ and so $supp(A_{*i}) \subseteq supp\big((AB)_{*j}\big)$. $\square$

Definition 2. *Let $\mathcal{M}$ be a finite set of $n \times n$ NZ matrices. We define the* pair digraph *of the set $\mathcal{M}$ as the labeled directed graph $\mathcal{PD}(\mathcal{M}) = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{(i,j) : 1 \le i \le j \le n\}$ is the vertex set and $(i,j) \to (i',j') \in \mathcal{E}$ if and only if there exists $A \in \mathcal{M}$ such that*

$$A_{ii'} > 0 \text{ and } A_{jj'} > 0, \text{ or } A_{ij'} > 0 \text{ and } A_{ji'} > 0. \tag{2}$$

An edge $(i,j) \to (i',j') \in \mathcal{E}$ is labeled by any matrix $A \in \mathcal{M}$ for which Eq. (2) holds. A vertex of the form (s,s) is called a singleton.

Lemma 2. *Let $\mathcal{M}$ be a finite set of $n \times n$ NZ matrices and let $\mathcal{PD}(\mathcal{M}) = (\mathcal{V}, \mathcal{E})$ be its pair digraph. Let $i, j, k \in [n]$ and suppose that there exists a path in $\mathcal{PD}(\mathcal{M})$ from the vertex (i,j) to the singleton (k,k) having the edges sequentially labeled by the matrices $A_{s_1}, \ldots, A_{s_l}$ from $\mathcal{M}$. Then it holds that for every $A \in \mathcal{M}$, $supp(A_{*i}) \cup supp(A_{*j}) \subseteq supp((AA_{s_1} \cdots A_{s_l})_{*k})$. Furthermore if $\mathcal{M}$ is irreducible, then $\mathcal{M}$ is primitive if and only if for any $(i,j) \in \mathcal{V}$ there exists a path in $\mathcal{PD}(\mathcal{M})$ from (i,j) to some singleton.*

Proof. By the definition of the pair digraph $\mathcal{PD}(\mathcal{M})$ (Definition 2), the existence of a path in $\mathcal{PD}(\mathcal{M})$ from vertex (i,j) to vertex (k,k) labeled by the matrices $A_{s_1}, \ldots, A_{s_l}$ implies that $(A_{s_1} \cdots A_{s_l})_{ik} > 0$ and $(A_{s_1} \cdots A_{s_l})_{jk} > 0$. By Lemma 1, it follows that $supp(A_{*i}) \cup supp(A_{*j}) \subseteq supp\big((AA_{s_1} \cdots A_{s_l})_{*k}\big)$.

Suppose now that $\mathcal{M}$ is irreducible. If $\mathcal{M}$ is primitive, then there exists a product M of matrices from $\mathcal{M}$ such that for all i, j, $M_{ij} > 0$. By the definition of $\mathcal{PD}(\mathcal{M})$, this implies that any vertex in $\mathcal{PD}(\mathcal{M})$ is connected to any other vertex. On the other hand, if every vertex in $\mathcal{PD}(\mathcal{M})$ is connected to some singleton, then for every $i, j, k \in [n]$ there exists a product $A_{s_1} \cdots A_{s_l}$ of matrices from $\mathcal{M}$ such that $(A_{s_1} \cdots A_{s_l})_{ik} > 0$ and $(A_{s_1} \cdots A_{s_l})_{jk} > 0$. This suffices to establish the primitivity of $\mathcal{M}$ by Theorem 1 in [1][5]. $\square$

3 The K-Rendezvous Time and a Recurrence Relation for Its Upper Bound

In this section, we define the k-rendezvous time of a primitive set of $n \times n$ NZ matrices and we prove a recurrence relation for a function $B_k(n)$ that upper bounds it.

[5] The theorem states that the following condition is sufficient for an irreducible matrix set $\mathcal{M}$ to be primitive: for all indices i, j, there exists an index k and a product M of matrices from $\mathcal{M}$ such that $M_{ik} > 0$ and $M_{jk} > 0$.

Definition 3. *Let $\mathcal{M}$ be a primitive set of $n \times n$ NZ matrices and $2 \le k \le n$. We define the k-*rendezvous time *(k-RT) to be the length of the shortest product of matrices from $\mathcal{M}$ having a column or a row with k positive entries and we denote it by $rt_k(\mathcal{M})$. We indicate with $rt_k(n)$ the maximal value of $rt_k(\mathcal{M})$ among all the primitive sets $\mathcal{M}$ of $n \times n$ NZ matrices.*

Our goal is to find, for any $n \ge 2$ and $2 \le k \le n$, a function $B_k(n)$ such that $rt_k(n) \le B_k(n)$.

Definition 4. *Let n, k integers such that $n \ge 2$ and $2 \le k \le n-1$. We denote by $\mathcal{S}_k^n$ the set of all the $n \times n$ NZ matrices having every row and column of weight at most k and at least one column of weight exactly k. For any $A \in \mathcal{S}_n^k$, let $\mathcal{C}_A$ be the set of the indices of the columns of A having weight equal to k. We define $a_k^n(A) = \min_{c \in \mathcal{C}_A} |\{i : supp(A_{*i}) \nsubseteq supp(A_{*c})\}|$ and $a_k^n = \min_{A \in \mathcal{S}_n^k} a_k^n(A)$.*

In other words, $a_k^n(A)$ is the minimum over all the indices $c \in \mathcal{C}_A$ of the number of columns of A whose support is not contained in the support of the c-th column of A. Since the matrices are NZ, it clearly holds that for any $A \in \mathcal{S}_n^k$, $1 \le a_k^n \le a_k^n(A)$. The following theorem shows that for every $n \ge 2$, we can recursively define a function $B_k(n) \ge rt_k(n)$ on k by using the term a_k^n.

Theorem 2. *Let $n \ge 2$ integer. The following recursive function $B_k(n)$ is such that for all $2 \le k \le n$, $rt_k(n) \le B_k(n)$.*

$$\begin{cases} B_2(n) = 1 \\ B_{k+1}(n) = B_k(n) + n(1 + n - a_k^n)/2 \quad \text{for } 2 \le k \le n-1. \end{cases} \tag{3}$$

Proof. We prove the theorem by induction.

Let $k = 2$. Any primitive set of NZ matrices must have a matrix with a row or a column with two positive entries, as otherwise it would be made of just permutation matrices and hence it would not be primitive. This trivially implies that $rt_2(n) = 1 \le B_2(n)$.

Suppose now that $rt_k(n) \le B_k(n)$, we show that $rt_{k+1}(n) \le B_{k+1}(n)$. We remind that we denote with $\mathcal{M}^d$ the set of all the products of matrices from $\mathcal{M}$ having length smaller than or equal to d. If in $\mathcal{M}^{rt_k(\mathcal{M})+n-1}$ there exists a product having a column or a row with $k+1$ positive entries then $rt_{k+1}(\mathcal{M}) \le rt_k(\mathcal{M}) + n - 1 \le B_{k+1}(n)$. Suppose now that this is not the case. This means that in $\mathcal{M}^{rt_k(\mathcal{M})+n-1}$ every matrix has all the rows and columns of weight at most k. Let $A \in \mathcal{M}^{rt_k(\mathcal{M})}$ be a matrix having a row or a column of weight k, and suppose it is a column. The case when A has a row of weight k will be studied later. By Lemma 1 applied on the matrix A, for every $i \in [n]$ there exists a matrix $W_i \in \mathcal{M}^{rt_k(\mathcal{M})+n-1}$ having the i-th column of weight k (and all the other columns and rows of weight $\le k$). Every W_i has at least a_k^n (see Definition 4) columns whose support is not contained in the support of the i-th column of W_i: let $c_i^1, c_i^2, \ldots, c_i^{a_k^n}$ be the indices of these columns. Notice that any product B of matrices from $\mathcal{M}$ of length l such that $B_{is} > 0$ and $B_{c_i^j s} > 0$ for some $s \in [n]$ and $j \in [a_k^n]$ would imply that $W_i B$ has the s-th column of weight at least $k+1$

and so $rt_{k+1}(\mathcal{M}) \leq rt_k(\mathcal{M}) + n - 1 + l$. We now want to minimize this length l over all $i, s \in [n]$ and $j \in [a_k^n]$: we will prove that there exists $i, s \in [n]$ and $j \in [a_k^n]$ such that $l \leq n(n-1-a_k^n)/2 + 1$. To do this, we consider the pair digraph $\mathcal{PD}(\mathcal{M}) = (\mathcal{V}, \mathcal{E})$ (see Definition 2) and the vertices

$$(1, c_1^1), (1, c_1^2), \ldots, (1, c_1^{a_k^n}), (2, c_2^1), \ldots, (2, c_2^{a_k^n}), \ldots, (n, c_n^1), \ldots, (n, c_n^{a_k^n}). \quad (4)$$

By Lemma 2, for each vertex in Eq. (4) there exists a path in $\mathcal{PD}(\mathcal{M})$ connecting it to a singleton. By the same lemma, a path of length l from (i, c_i^j) to a singleton (s, s) would result in a product B_j of matrices from $\mathcal{M}$ of length l such that $W_i B_j$ has the s-th column of weight at least $k+1$. We hence want to estimate the minimal length among the paths connecting the vertices in Eq. (4) to a singleton. Notice that Eq. (4) contains at least $\lceil n a_k^n/2 \rceil$ different elements, since each element occurs at most twice. It is clear that the shortest path from a vertex in the list (4) to a singleton does not contain any other element from that list. The vertex set $\mathcal{V}$ of $\mathcal{PD}(\mathcal{M})$ has cardinality $n(n+1)/2$ and it contains n vertices of type (s, s). It follows that the length of the shortest path connecting some vertex from the list (4) to some singleton is at most of $n(n+1)/2 - n - \lceil n a_k^n/2 \rceil + 1 \leq n(n-1-a_k^n)/2 + 1$. In view of what said before, we have that there exists a product B of matrices from $\mathcal{M}$ of length $\leq n(n-1-a_k^n)/2 + 1$ and $i \in [n]$ such that $W_i B_j$ has a column of weight at least $k+1$. Since $W_i B_j$ belongs to $\mathcal{M}^{rt_k(\mathcal{M}) + n - 1 + n(n-1-a_k^n)/2 + 1}$, it follows that $rt_{k+1}(\mathcal{M}) \leq rt_k(\mathcal{M}) + n(n+1-a_k^n)/2 \leq B_{k+1}(n)$.

Suppose now $A \in \mathcal{M}^{rt_k(\mathcal{M})}$ has a row of weight k. We can use the same argument as above on the matrix set $\mathcal{M}^T$ made of the transpose of all the matrices in $\mathcal{M}$. $\square$

Notice that the above argument stays true if we replace a_k^n by a function $b(n, k)$ such that for all $n \geq 2$ and $2 \leq k \leq n-1$, $1 \leq b(n, k) \leq a_k^n$. It follows that Eq. (3) still holds true if we replace a_k^n by $b(n, k)$.

4 Solving the Recurrence

We now find an analytic expression for a lower bound on a_k^n and we then solve the recurrence (3) in Theorem 2 by using this lower bound. We then show that this is the best estimate on a_k^n we can hope for.

Lemma 3. *Let n, k integers such that $n \geq 2$ and $2 \leq k \leq n-1$, and let a_k^n as in Definition 4. It holds that $a_k^n \geq \max\{n - k(k-1) - 1, \lceil (n-k)/k \rceil, 1\}$.*

Proof. We have that $a_k^n \geq 1$ since $k \leq n-1$ and the matrices are NZ.
Let now $A \in S_n^k$ (see Definition 4) and let a be one of its columns of weight k. Let $S = supp(a)$; by assumption, the rows of A have at most k positive entries, so there can be at most $(k-1)k$ columns of A different from a whose support is contained in S. Therefore, since A is NZ, there must exist at least $n - k(k-1) - 1$ columns of A whose support is not contained in $supp(a)$ and so $a_k^n \geq n - k(k-1) - 1$.

Let again $A \in S_n^k$ and let a be one of its columns of weight k. Let $S = [n]\backslash supp(a)$; S has cardinality $n-k$ and since A is NZ, for every $s \in S$ there exists $s' \in [n]$ such that $A_{ss'} > 0$. By assumption each column of A has weight of at most k, so there must exist at least $\lceil (n-k)/k \rceil$ columns of A different from a whose support is not contained in $supp(a)$. It follows that $a_k^n \geq \lceil (n-k)/k \rceil$. □

Since $\lceil (n-k)/k \rceil \geq (n-k)/k$, $n-k(k-1)-1 \geq (n-k)/k$ for $k \leq \lfloor \sqrt{n} \rfloor$ and $(n-k)/k \geq 1$ for $k \leq \lfloor n/2 \rfloor$, the recursion (3) with a_k^n replaced by $\max\{n-k(k-1)-1, (n-k)/k, 1\}$ now reads as:

$$\tilde{B}_{k+1}(n) = \begin{cases} 1 & \text{if } k = 1 \\ \tilde{B}_k(n) + n(1 + k(k-1)/2) & \text{if } 2 \leq k \leq \lfloor \sqrt{n} \rfloor \\ \tilde{B}_k(n) + n(1 + n(k-1)/2k) & \text{if } \lfloor \sqrt{n} \rfloor + 1 \leq k \leq \lfloor n/2 \rfloor \\ \tilde{B}_k(n) + n^2/2 & \text{if } \lfloor n/2 \rfloor + 1 \leq k \leq n-1 \end{cases}. \tag{5}$$

The following proposition shows the solution of the recursion (5):

Proposition 2. *Equation (5) is fulfilled by the following function:*

$$\tilde{B}_k(n) = \begin{cases} \dfrac{n(k^3 - 3k^2 + 8k - 12)}{6} + 1 & \textit{if } 2 \leq k \leq \lfloor \sqrt{n} \rfloor \\ \tilde{B}_{\lfloor \sqrt{n} \rfloor}(n) + \dfrac{n(n+2)(k - \lfloor \sqrt{n} \rfloor)}{2} - \dfrac{n^2}{2} \displaystyle\sum_{i=\lfloor \sqrt{n} \rfloor}^{k-1} \frac{1}{i} & \textit{if } \lfloor \sqrt{n} \rfloor + 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \\ \tilde{B}_{\lfloor \frac{n}{2} \rfloor}(n) + \dfrac{(k - \lfloor \frac{n}{2} \rfloor)n^2}{2} & \textit{if } \lfloor \frac{n}{2} \rfloor + 1 \leq k \leq n \end{cases}. \tag{6}$$

Therefore, for any constant k such that $k \leq \sqrt{n}$, the k-rendezvous time $rt_k(n)$ is at most linear in n.

Proof. If $2 \leq k \leq \lfloor \sqrt{n} \rfloor$, let $C_k(n) = \tilde{B}_k(n)/n$. By Eq. (5), it holds that $C_{k+1}(n) - C_k(n) = 1 + k(k-1)/2$. By setting $C_k(n) = \alpha k^3 + \beta k^2 + \gamma k + \delta$, it follows that $3\alpha k^2 + (3\alpha + 2\beta)k + \alpha + \beta + \gamma = k^2/2 - k/2 + 1$. Since this must be true for all k, by equating the coefficients we have that $C_k(n) = k^3/6 - k^2/2 + 4k/3 + \delta$. Imposing the initial condition $\tilde{B}_2(n) = 1$ gives finally the desired result $\tilde{B}_k(n) = n(k^3 - 3k^2 + 8k - 12)/6 + 1$.

If $\lfloor \sqrt{n} \rfloor + 1 \leq k \leq \lfloor n/2 \rfloor$, let again $C_k(n) = \tilde{B}_k(n)/n$. By Eq. (5), it holds that $C_{k+1}(n) - C_k(n) = 1 + n(k-1)/2k$ and so $C_k(n) = C_{\lfloor \sqrt{n} \rfloor}(n) + (k-2)(1 + n/2) - (n/2)\sum_{i=\lfloor \sqrt{n} \rfloor}^{k-1} i^{-1}$. Since $C_{\lfloor \sqrt{n} \rfloor}(n) = \tilde{B}_{\lfloor \sqrt{n} \rfloor}(n)/n$, it follows that $\tilde{B}_k(n) = \tilde{B}_{\lfloor \sqrt{n} \rfloor}(n) + (k - \lfloor \sqrt{n} \rfloor)n(n+2)/2 - (n^2/2)\sum_{i=\lfloor \sqrt{n} \rfloor}^{k-1} i^{-1}$.

If $\lfloor n/2 \rfloor + 1 \leq k \leq n-1$, by Eq. (5) it is easy to see that $\tilde{B}_k(n) = \tilde{B}_{\lfloor n/2 \rfloor}(n) + (k - \lfloor n/2 \rfloor)n^2/2$, which concludes the proof. □

We now show that $a_k^n = \max\{n - k(k-1) - 1, \lceil (n-k)/k \rceil, 1\}$, and so we cannot improve the upper bound $\tilde{B}_k(n)$ on $rt_k(n)$ by improving our estimate of a_k^n.

Lemma 4. *Let n, k integers such that $n \geq 2$ and $2 \leq k \leq n-1$. It holds that:*

$$1 \leq a_k^n \leq u(n,k) := \begin{cases} n-k(k-1)-1 & \text{if } n-k(k-1)-1 \geq \lceil (n-k)/k \rceil \\ \lceil (n-k)/k \rceil & \text{otherwise} \end{cases}.$$

Proof. We need to show that for every $n \geq 2$ and $2 \leq k \leq n-1$, there exists a matrix $A \in \mathcal{S}_n^k$ such that $a_k^n(A) = u(n,k)$ (see Definition 4). We define the matrix $C_i^{m_1 \times m_2}$ as the $m_1 \times m_2$ matrix having all the entries of the i-th column equal to 1 and all the other entries equal to 0, and the matrix $R_i^{m_1 \times m_2}$ as the $m_1 \times m_2$ matrix having all the entries of the i-th row equal to 1 and all the other entries equal to 0. We indicate with $\mathbf{0}^{m_1 \times m_2}$ the $m_1 \times m_2$ matrix having all its entries equal to zero and with $\mathbf{I}^{m \times m}$ the $m \times m$ identity matrix. Let $v_k^n = \lceil (n-k)/k \rceil + 1$ and $q = n \mod k$.
Suppose that $n-k(k-1)-1 \geq \lceil (n-k)/k \rceil$ and set $\alpha = n-k(k-1)-1-\lceil (n-k)/k \rceil$. Then the following matrix $\hat{A}$ is such that $a_k^n(\hat{A}) = n-k(k-1)-1 = u(n,k)$:

$$\hat{A} = \left[\begin{array}{c|cccc|c} C_1^{k \times v_k^n} & R_1^{k \times (k-1)} & R_2^{k \times (k-1)} & \cdots & R_k^{k \times (k-1)} & \\ C_2^{k \times v_k^n} & & & & & \\ \vdots & & & \mathbf{0}^{(n-k) \times [k(k-1)]} & & D \\ C_{v_k^n - 1}^{k \times v_k^n} & & & & & \\ C_{v_k^n}^{q \times v_k^n} & & & & & \end{array}\right], D = \begin{bmatrix} \mathbf{0}^{k \times \alpha} \\ \mathbf{I}^{\alpha \times \alpha} \\ \mathbf{0}^{(n-k-\alpha) \times \alpha} \end{bmatrix}.$$

Indeed by construction, the first column of $\hat{A}$ has exactly k positive entries. The columns of $\hat{A}$ whose support is not contained in $\hat{A}_{*1}$ are the columns $\hat{A}_{*i}$ for $i = 2, \ldots, v_k^n$ and all the columns of D. In total we have $\lceil (n-k)/k \rceil + \alpha = n-k(k-1)-1$ columns, so it holds that $a_k^n(\hat{A}) = n-k(k-1)-1$.

Suppose that $n-k(k-1)-1 \leq \lceil (n-k)/k \rceil$. Then the following matrix $\tilde{A}$ is such that $a_k^n(\tilde{A}) = \lceil (n-k)/k \rceil = u(n,k)$:

$$\tilde{A} = \left[\begin{array}{c|ccccc} C_1^{k \times v_k^n} & R_1^{k \times (k-1)} & R_2^{k \times (k-1)} & \cdots & R_{k-1}^{k \times (k-1)} & R_k^{k \times (n - v_k^n - (k-1)^2)} \\ \hline C_2^{k \times v_k^n} & & & & & \\ \vdots & & & & & \\ C_{v_k^n - 1}^{k \times v_k^n} & & & \mathbf{0}^{(n-k) \times (n - v_k^n)} & & \\ C_{v_k^n}^{q \times v_k^n} & & & & & \end{array}\right]$$

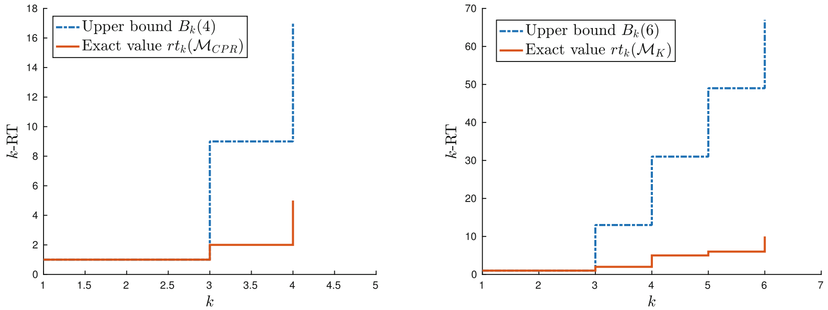
Indeed by construction, the first column of $\tilde{A}$ has exactly k positive entries and the columns of $\tilde{A}$ whose support is not contained in $\tilde{A}_{*1}$ are the columns $\tilde{A}_{*i}$ for $i = 2, \ldots, v_k^n$. Therefore it holds that $a_k^n(\tilde{A}) = v_k^n - 1 = \lceil (n-k)/k \rceil$. $\square$

5 Numerical Results

We report here some numerical results that compare the theoretical bound $\tilde{B}_k(n)$ on $rt_k(n)$ of Eq. (6) with either the exact k-RT or with an heuristic approximation of the k-RT when the computation of the exact value is not computationally feasible. In Fig. 2 we compare our bound with the real k-RT of the primitive sets $\mathcal{M}_{CPR}$ and $\mathcal{M}_K$ reported here below:

$$\mathcal{M}_{CPR} = \left\{ \begin{pmatrix} 0&0&1&0\\ 1&1&0&0\\ 1&0&0&0\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&0&0 \end{pmatrix} \right\}, \mathcal{M}_K = \left\{ \begin{pmatrix} 1&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0 \end{pmatrix} \right\}.$$

The sets $\mathcal{M}_K$ and $\mathcal{M}_{CPR}$ are primitive sets of matrices that are based on the Kari automaton [21] and the Černý-Piricka-Rozenaurova automaton [40] respectively. We can see that for small values of k, the upper bound is fairly close to the actual value of $rt_k(\mathcal{M})$.

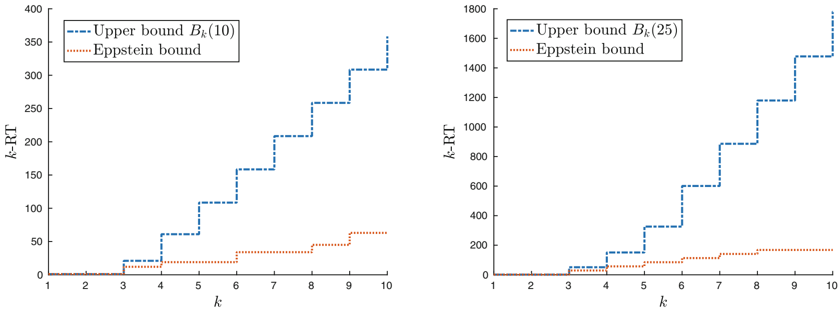


(a) $n = 4$, $\mathcal{M} = \mathcal{M}_{CPR}$ (b) $n = 6$, $\mathcal{M} = \mathcal{M}_K$

Fig. 2. Comparison between the bound $\tilde{B}_k(n)$, valid for all primitive NZ sets, and $rt_k(\mathcal{M})$ for $\mathcal{M} = \mathcal{M}_{CPR}$ (left) and $\mathcal{M} = \mathcal{M}_K$ (right).

When n is large, computing the k-RT for every $2 \leq k \leq n$ becomes hard, so we compare our upper bound on the k-RT with a method for approximating it. The *Eppstein heuristic* is a greedy algorithm developed by Eppstein in [11] for approximating the reset threshold of a synchronizing automaton. Given a primitive set $\mathcal{M}$ of binary NZ matrices, we can apply a slightly modified Eppstein heuristic to obtain, for any k, an upper bound on $rt_k(\mathcal{M})$. The description of this modified heuristic is not reported here due to length restrictions.

In Fig. 3 we compare our upper bound with the results of the Eppstein heuristic on the k-RT of the primitive sets with quadratic exponent presented by Catalano and Jungers in [6], Sect. 4; here we denote these sets by $\mathcal{M}_{C_n}$ where n is the matrix dimension. Finally, Fig. 4 compares the evolution of our bound with the results of the Eppstein heuristic on the k-RT of the family $\mathcal{M}_{C_n}$ for fixed $k = 4$ and as n varies. It can be noticed that the bound $\tilde{B}_k(n)$ does not increase very rapidly as compared to the Eppstein approximation.



(a) $n = 10$, $\mathcal{M} = \mathcal{M}_{C_{10}}$ (b) $n = 25$, $\mathcal{M} = \mathcal{M}_{C_{25}}$

Fig. 3. Comparison between $\tilde{B}_k(n)$ and the Eppstein approx. of $rt_k(\mathcal{M})$, for $\mathcal{M} = \mathcal{M}_{C_{10}}$ (left) and $\mathcal{M} = \mathcal{M}_{C_{25}}$ (right). We recall that $\tilde{B}_k(n)$ is a generic bound valid for all primitive NZ sets, while the Eppstein bound is computed on each particular set.

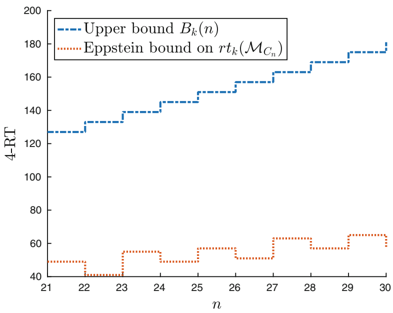


Fig. 4. Comparison between $\tilde{B}_k(n)$ and the Eppstein approx. of $rt_k(\mathcal{M}_{C_n})$ for $k = 4$. We recall that $\tilde{B}_k(n)$ is a generic bound valid for all primitive NZ sets, while the Eppstein bound is computed on each particular set.

6 Conclusions

In this paper we have shown that we can upper bound the length of the shortest product of a primitive NZ set $\mathcal{M}$ having a column or a row with k positive entries by a linear function of the matrix size n, for any constant $k \leq \sqrt{n}$. We have called this length the *k-rendezvous time* (k-RT) of the set $\mathcal{M}$, and we have shown that the same linear upper bound holds for $\min\{rt_k(Aut(\mathcal{M})), rt_k(Aut(\mathcal{M}^T))\}$, where $Aut(\mathcal{M})$ and $Aut(\mathcal{M}^T)$ are the synchronizing automata defined in Definition 1. We have also showed that our technique cannot be improved as it already takes into account the worst cases, so new strategies have to be implemented in order to possibly obtain a better upper bound on $rt_k(n)$. The notion of k-RT for primitive sets comes as an extension to primitive sets of the one introduced for synchronizing automata. For automata, the problem whether there exists a linear upper bound on the k-RT for small k is still open, as the only nontrivial result on the k-RT that appears in the literature, to the best of our knowledge, proves a quadratic upper bound on the 3-RT [17]. We believe that our result could help in shedding light to this problem and possibly to the Černý conjecture, in view of the connection between synchronizing automata and primitive NZ sets established by Theorem 1.

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