Canad. J. Math. Vol. **70** (2), 2018 pp. 265–293
http://dx.doi.org/10.4153/CJM-2017-021-3

Rational Models of the Complement of a Subpolyhedron in a Manifold with Boundary

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Abstract. Let W be a compact simply connected triangulated manifold with boundary and let $K \subset W$ be a subpolyhedron. We construct an algebraic model of the rational homotopy type of $W\backslash K$ out of a model of the map of pairs $(K, K \cap \partial W) \hookrightarrow (W, \partial W)$ under some high codimension hypothesis.

We deduce the rational homotopy invariance of the configuration space of two points in a compact manifold with boundary under 2-connectedness hypotheses. Also, we exhibit nice explicit models of these configuration spaces for a large class of compact manifolds.

1 Introduction

Let W be a compact and simply-connected manifold with boundary (in this paper all manifolds are triangulated). Let $f: K \hookrightarrow W$ be the inclusion of a subpolyhedron. The first goal of this paper is to determine the rational homotopy type of the complement $W\backslash K$. We will then apply this to deduce the rational homotopy type of the configuration space of two points in a manifold with boundary under 2-connectedness hypotheses. Hence, this paper extends the results of [7, 8] to the case of manifolds with boundary.

The main result of [8] is an explicit description of the rational homotopy type of $W\backslash K$ when W is a closed manifold and K is a subpolyhedron of codimension at least $(\dim W)/2 + 2$. This rational homotopy type depends only on the rational homotopy class of the inclusion $K \hookrightarrow W$ ([8, Theorem 1.2]).

The situation for manifolds with boundary is different. For example, let W be an n-dimensional disk D^n and let K be a point. If K is embedded in the interior of D^n, then $W\backslash K \simeq S^{n-1}$. On the contrary, if K is embedded in the boundary of D^n, then $W\backslash K \simeq *$. Hence the complements $W\backslash K$ have different rational homotopy types, although the two inclusions $K \hookrightarrow W$ are homotopic. These examples show that we need more information to determine the rational homotopy type of $W\backslash K$. Our main result is that the only extra information needed is related to the inclusion of $\partial W \cap K$ in ∂W. More precisely, we have the following result.

Received by the editors March 30, 2016; revised May 8, 2017.
Published electronically November 7, 2017.
H. C. B. and P. L. are supported by the Belgian Federal Fund PAI Dygest. D. S. is supported by NSERC of Canada.
AMS subject classification: **55P62**, 55R80.
Keywords: Lefschetz duality, Sullivan model, configuration space.

Theorem 1.1 (Corollary 4.6 and Theorem 4.5) *Let W be a compact simply connected triangulated manifold with boundary and let K be a subpolyhedron in W.*

Assume that

$$\dim W \geq 2 \dim K + 3. \tag{1.1}$$

Then the rational homotopy type of $W \backslash K$ depends only on the rational homotopy type of the square of inclusions

$$\begin{array}{ccc} (K \cap \partial W) & \hookrightarrow & \partial W \\ \downarrow & & \downarrow \\ K & \hookrightarrow & W. \end{array} \tag{1.2}$$

Moreover, a CDGA model of $W \backslash K$ (that is, an algebraic model in the sense of Sullivan of this rational homotopy type, see Section 2.1) can be explicitly constructed out of any CDGA model of Diagram (1.2).

Actually, we will see that the high codimension hypothesis (1.1) can be weakened. Indeed, we will establish a sharp *unknotting condition*, which is an inequality relating the connectivity of the inclusion maps and the dimensions of the manifold and the subpolyhedron (see (4.5) in Corollary 4.6), under which we still get a CDGA model of the complement.

There is an interesting application of this theorem to the study of configuration spaces of 2 points in W;

$$\mathrm{Conf}(W, 2) := \left\{ (x_1, x_2) \in W \times W : x_1 \neq x_2 \right\}.$$

Indeed, this configuration space is the complement

$$\mathrm{Conf}(W, 2) = W \times W \setminus \Delta(W),$$

where $\Delta : W \hookrightarrow W \times W$ is the diagonal embedding. We will deduce from Theorem 1.1 the following result.

Theorem 1.2 (Corollary 5.5) *Let W be a 2-connected compact manifold with a 2-connected or empty boundary. The rational homotopy type of the configuration space* $\mathrm{Conf}(W, 2)$ *depends only on the rational homotopy type of the pair $(W, \partial W)$.*

In [3] we prove that a large class of compact manifolds with boundary admit CDGA models of a special form that we call *surjective pretty models*. This class contains, in particular, even-dimensional disk bundles over a closed manifold and complements of high codimensional polyhedra in closed manifolds. As a consequence, such manifolds admit a CDGA model of the form P/I, where P is a Poincaré duality CDGA and I is some differential ideal. Poincaré duality CDGAs come with a natural diagonal class $\Delta \in (P \otimes P)^n$. We then get the following elegant model for the configuration space (see Section 5.3 for more details).

Theorem 1.3 (Theorem 5.8) *Let W be a compact manifold of dimension n with boundary and assume that W and ∂W are 2-connected. If $(W, \partial W)$ admits a surjective pretty model in the sense of* [3], *then a CDGA model of* $\mathrm{Conf}(W, 2)$ *is given* *by*

$$(P/I \otimes P/I) \oplus_{\overline{\Delta^!}} ss^{-n} P/I,$$

where P is the Poincaré duality CDGA and I the ideal associated with the pretty model, and $\overline{\Delta^!}$ is a map induced by multiplication by the diagonal class $\Delta \in (P \otimes P)^n$.

When W is a closed manifold, we have $I = 0$ and the model of Theorem 1.3 is exactly that of [7].

In a paper in preparation we will show how to build a model (of dgmodules) of $\mathrm{Conf}(W, k)$, $k \geq 2$, which enables us to compute effectively the homology of the space of configurations of any number of points in a manifold with boundary. This model will be of the form

$$\left(\frac{(P/I)^{\otimes k} \otimes \Lambda(g_{ij} : 1 \leq i < j \leq k)}{(\text{Arnold and symmetry relations})}, d(g_{ij}) = \pi_{ij}^*(\overline{\Delta}) \right),$$

mimicking the model in [10].

Here is the plan of the paper. Section 2 contains a very short review of rational homotopy theory, the notion of truncation of a CDGA, a discussion on CDGA structures on mapping cones, and the notion of homotopy kernel. Section 3 is a first step to the understanding of a dgmodule model of the complement $W \setminus K$, and in Section 4 we establish a CDGA model of that complement. In Section 5 we apply the previous results to the model of the configuration space of 2 points in compact manifolds, with some developments of the examples of configuration spaces on a disk bundle or in the complement of a polyhedron in a closed manifold.

2 Truncation of Dgmodules and CDGA's, and CDGA Structures on Mapping Cones

This section contains a quick review of some classical topics that we will need with some special development. In particular, in Section 2.3 we explain some notion of truncation of a CDGA, and in Section 2.4 we show how to endow a mapping cone (or its truncation) with the structure of a CDGA.

2.1 Rational Homotopy Theory

In this paper we will use the standard tools and results of rational homotopy theory, following [5]. Recall that A_{PL} is the Sullivan–de Rham functor and that for a 1-connected space of finite type, X, $A_{PL}(X)$ is a commutative differential graded algebra (CDGA for short), which completely encodes the rational homotopy type of X. Any CDGA weakly equivalent to $A_{PL}(X)$ is called a *CDGA model of* X. All our dgmodules and CDGAs are over the field $\mathbb{Q}$.

2.2 Truncation of a Dgmodule

The classical truncation of a cochain complex, *i.e.*, $\mathbb{Q}$-dgmodule, C, is classically defined by (see [12, Section 1.2.7])

$$(\widehat{\tau}^{\leq N} C)^i = \begin{cases} C^i & \text{if } i < N, \\ C^N \cap \ker d & \text{if } i = N, \\ 0 & \text{if } i > N. \end{cases} \tag{2.1}$$

This comes with an inclusion $\iota\colon \widehat{\tau}^{\leq N} C \hookrightarrow C$, which induces isomorphisms $H^i(\iota)$, for $i \leq N$, and such that $H^{>N}(\widehat{\tau}^{\leq N} C) = 0$.

When R is an A-dgmodule, the truncation $\widehat{\tau}^{\leq N} R$ is not necessarily an A-dgmodule. In that case a better replacement would be to take for the truncation a quotient R/I where I is a suitable A-dgsubmodule such that $I^i = R^i$ for $i > N$. In this paper we will use the following definition.

Definition 2.1 Let R be an A-dgmodule and let N be a positive integer. A *truncation below degree N of R* is an A-dgmodule, $\tau^{\leq N} R$, and a morphism $\pi\colon R \to \tau^{\leq N} R$ of A-dgmodules verifying the two following conditions:

(i) $(\tau^{\leq N} R)^{>N} = 0$ and $(\tau^{\leq N} R)^{<N} \cong R^{<N}$,

(ii) the morphism π is a surjection of A-dgmodules such that $H^i(\pi)$ is an isomorphism for $0 \leq i \leq N$.

Contrary to $\widehat{\tau}^{\leq N}$ from (2.1), our truncation $\tau^{\leq N} R$ is not unique and is not a functorial construction.

2.3 Truncation of a CDGA

Definition 2.2 Let A be a connected CDGA. *A CDGA truncation below degree N of A* is a truncation of A-dgmodule $(\tau^{\leq N} A, \pi)$ such that $\tau^{\leq N} A$ is a CDGA and $\pi\colon A \to \widehat{\tau}^{\leq N} A$ is a CDGA morphism.

Equivalently, a CDGA truncation can be seen as a projection $\pi\colon A \to A/I$ where I is an ideal of A such that $I^{<N} = 0$, $I^{>N} = A^{>N}$, and $I^N \oplus (\ker d \cap A^N) = A^N$.

Proposition 2.3 *Any two CDGA truncations below degree N of a given connected CDGA are weakly equivalent.*

Proof Let A be a connected CDGA and $N \in \mathbb{N}$. It is easy to construct a relative Sullivan model

$$\iota\colon A \rightarrowtail (A \otimes \Lambda V, D)$$

such that $H^{\leq N}(\iota)$ is an isomorphism, $V = V^{\geq N}$ and $H^{>N}(A \otimes \Lambda V, D) = 0$. Indeed, one builds inductively $V = V^{\geq N}$ by adding generators to eliminate all the homology in degrees $> N$. It is straightforward to check that any CDGA truncation $\pi\colon A \to \tau^{\leq N} A$

factors as follows

$$\begin{array}{ccc} A & \xrightarrow{\pi} & \tau^{\leq N} A \\ & {\scriptstyle \iota}\searrow \quad \nearrow{\scriptstyle m} & \\ & (A \otimes \Lambda V, D) & \end{array}$$

where $m(V) = 0$. Since $H^{\leq N}(\iota)$ and $H^{\leq N}(\pi)$ are isomorphisms and

$$H^{>N}(A \otimes \Lambda V, D) = H^{>N}(\tau^{\leq N} A) = 0,$$

we deduce that m is a quasi-isomorphism. Therefore, any two truncation of A are quasi-isomorphic to $(A \otimes \Lambda V, D)$, and hence are weakly equivalent ■

2.4 Semi-trivial C(D)GA Structures on Mapping Cones

Let A be a CDGA and let R be an A-dgmodule. We will denote by $s^k R$ the k-th suspension of R, *i.e.*, $(s^k R)^p = R^{k+p}$, and for a map of A-dgmodules, $f: R \to Q$, we denote by $s^k f$ the k-th suspension of f. Furthermore, we will use # to denote the linear dual of a vector space, $\#V = \hom(V, \mathbb{Q})$, and $\#f$ to denote the linear dual of a map f.

If $f: Q \to R$ is an A-dgmodule morphism, *the mapping cone* of f is the A-dgmodule

$$C(f) := (R \oplus_f sQ, \delta)$$

defined by $R \oplus sQ$ as an A-module and with a differential δ such that $\delta(r, sq) = (d_R(r) + f(q), -sd_Q(q))$.

When $R = A$, the mapping cone $C(f: Q \to A)$ can be equipped with a unique commutative graded algebra (CGA) structure that extends the algebra structure on A, respects the A-dgmodule structure, and such that $(sq) \cdot (sq') = 0$, for $q, q' \in Q$. We will call this structure the *semi-trivial CGA structure* on the mapping cone $A \oplus_f sQ$ (see [8, Section 4]). The following result is very useful to detect when this CGA structure is, in fact, a CDGA structure.

Definition 2.4 Let A be a CDGA. An A-dgmodule morphism $f: Q \to A$ is *balanced* if $f(x)y = xf(y)$ for all $x, y \in Q$.

The importance of this notion comes from the following proposition.

Proposition 2.5 *Let Q be an A-dgmodule and let $f: Q \to A$ be an A-dgmodule morphism. The mapping cone $C(f) = A \oplus_f sQ$ endowed with the semi-trivial CGA structure is a CDGA if and only if f is balanced.*

Proof In one direction, assume that f is balanced. The only non-trivially verified condition for $C(f)$ being a CDGA is the Leibniz rule for the differential. Let $a, a' \in A$ and $q, q' \in Q$. For products of the form $(a, 0)(a', 0)$ and of the form $(a, 0)(0, sq)$ the Leibniz rule is verified because A is a CDGA and Q is an A-dgmodule. For products of the form $(0, sq)(0, sq')$, by semi-triviality of the CDGA structure of the mapping cone, we have to verify that

$$\big(\delta(0, sq)\big)(0, sq') + (-1)^{|q|+1}(0, sq)\big(\delta(0, sq')\big) = 0, \tag{2.2}$$

which is a direct consequence of the hypothesis that f is balanced and the formula for the differential on a mapping cone.

In the other direction, if the Leibniz rule is satisfied for the semi-trivial multiplication, then (2.2) holds for any $q, q' \in Q$, and, again by the formula for the differential on a mapping cone and the semi-trivial multiplication, this implies that $f(q)q' = qf(q')$, hence f is balanced. ∎

Proposition 2.6 *Let A be a connected CDGA and let $f\colon Q \to A$ be an A-dgmodule morphism. Let p and N be natural integers such that $Q^{<p} = 0$ and $N \leq 2p - 3$. Then the semi-trivial CGA structure on the mapping cone $C(f)$ induces a CDGA structure on $\tau^{\leq N}(C(f))$, and*

$$A \longrightarrow \tau^{\leq N}(C(f))$$

is a CDGA morphism.

Proof First we verify that the truncation $\tau^{\leq N}(C(f))$ is a CDGA. As in Proposition 2.5 we have to verify the Leibniz rule. For multiplications between an element of A and an element of Q the Leibniz rule is verified, because of the A-dgmodule structure on Q. Furthermore, for $sq \in sQ$ and $sq' \in sQ$ we have that the non-zero elements in $C(f)$ of the form $(0, sq)(0, sq')$ are, by the hypothesis on Q, of degree $\geq 2p - 2$. Therefore, these products vanish in the truncation $\tau^{\leq N}(C(f))$, and the Leibniz rule is trivially verified.

The fact that the morphism $A \to \tau^{\leq N}(C(f))$ is itself a CDGA comes directly from the CDGA structure of A. ∎

Remark 2.7 In the rest of this paper, when a mapping cone is equipped with a CDGA structure it will be understood that it comes from the semi-trivial structure.

2.5 Homotopy Kernel

In this section we recall the notion of homotopy kernel and some of its properties.

Definition 2.8 Let $f\colon M \to N$ be a morphism of A-dgmodules. The *homotopy kernel* of f is the A-dgmodule mapping cone

$$\operatorname{hoker} f := s^{-1}N \oplus_{s^{-1}f} M,$$

which comes with an obvious map

$$\operatorname{hoker} f \longrightarrow M; (s^{-1}n, m) \longmapsto m.$$

The following result is a consequence of the five lemma and justifies the terminology “homotopy kernel”.

Proposition 2.9 *Let $f\colon M \to N$ be a surjective morphism of A-dgmodules. Then the morphism*

$$\begin{array}{rccc} \varphi\colon & \ker f & \xrightarrow{\simeq} & \operatorname{hoker} f \\ & m & \longmapsto & (0, m) \end{array}$$

is an A-dgmodule quasi-isomorphism.

3 Lefschetz Duality for Manifolds with Boundary

The aim of this section is to prove Proposition 3.1, which is a first step towards the description of the rational homotopy type of the complement of a subpolyhedron in a manifold with boundary.

Let W be a compact connected oriented triangulated manifold of dimension n with boundary and let $f\colon K \hookrightarrow W$ be the inclusion of a connected subpolyhedron of dimension k in W. Denote by ∂W the boundary of W and set

$$\partial_W K := K \cap \partial W. \tag{3.1}$$

In this section we will construct a dgmodule model of $W \setminus K$, extending [10, Theorem 6.3] to manifolds with boundary. Consider the diagram

$$\begin{array}{ccc} W & \xleftarrow{\ f\ } & K \\ \uparrow & & \uparrow \\ \partial W & \xleftarrow[\partial f]{} & \partial_W K, \end{array} \tag{3.2}$$

which after applying the A_{PL} functor gives

$$\begin{array}{ccc} A_{PL}(W) & \xrightarrow{A_{PL}(f)} & A_{PL}(K) \\ \downarrow & & \downarrow \\ A_{PL}(\partial W) & \longrightarrow & A_{PL}(\partial_W K). \end{array} \tag{3.3}$$

Recall that for a map of spaces $Y \to X$, we set

$$A_{PL}(X, Y) = \ker(A_{PL}(X) \longrightarrow A_{PL}(Y)).$$

The inclusion of pairs

$$i\colon (K, \partial_W K) \hookrightarrow (W, \partial W)$$

induces an $A_{PL}(W)$-dgmodule morphism

$$A_{PL}(i)\colon A_{PL}(W, \partial W) \longrightarrow A_{PL}(K, \partial_W K).$$

Using our notation for mapping cones, suspension and linear duals from Section 2.4, consider the map

$$s^{-n}\#A_{PL}(i)\colon s^{-n}\#A_{PL}(K, \partial_W K) \longrightarrow s^{-n}\#A_{PL}(W, \partial W)$$

and its mapping cone

$$C\big(s^{-n}\#A_{PL}(i)\big) = s^{-n}\#A_{PL}(W, \partial W) \oplus_{s^{-n}\#A_{PL}(i)} ss^{-n}\#A_{PL}(K, \partial_W K) \tag{3.4}$$

with the inclusion

$$\iota\colon s^{-n}\#A_{PL}(W, \partial W) \hookrightarrow C\big(s^{-n}\#A_{PL}(i)\big). \tag{3.5}$$

Since $(W, \partial W)$ is an oriented compact manifold of dimension n, Poincaré duality induces a quasi-isomorphism of $A_{PL}(W)$-dgmodules

$$\Phi_W\colon A_{PL}(W) \xrightarrow{\simeq} s^{-n}\#A_{PL}(W, \partial W) \tag{3.6}$$

(see (3.7) in the proof of Proposition 3.1 for an explicit description of Φ_W.)

Proposition 3.1 *The map*

$$A_{PL}(W) \longrightarrow A_{PL}(W\backslash K)$$

is weakly equivalent in the category of $A_{PL}(W)$-dgmodules to the map

$$\iota \circ \Phi_W : A_{PL}(W) \longrightarrow C\big(s^{-n}\# A_{PL}(i)\big),$$

where $C(s^{-n}\# A_{PL}(i))$ is the mapping cone (3.4)*, ι is from* (3.5)*, and Φ_W is from* (3.6).

Proof First we review from [10, Section 4] a variation of the functor A_{PL} defined on ordered simplicial complex and having an improved excision property. Recall from [5, Chapter 10] that A_{PL} is actually defined first on simplicial sets. Consider the category, $\mathcal{K}$, of ordered simplicial complexes. With any ordered simplicial complex, K, we can associate naturally a simplicial set, $K_\bullet$, whose non-degenerate simplices are exactly the simplices of K (see [4, p. 108]). Define the functor

$$\widehat{A_{PL}} : \mathcal{K} \longrightarrow ADGC; K \longrightarrow A_{PL}(K_\bullet).$$

This functor verifies the two following properties (see [10, Section 4]):

(a) $A_{PL}(|K|) \simeq \widehat{A_{PL}}(K)$ naturally for every ordered simplicial complex (where $|K|$ is the geometric realization).

(b) *Strong excision property*: Let (K, L) be a pair of ordered simplicial complexes. Let $K' \subset K$ a sub-complex and $L' = K' \cap L$. If $K' \cup L = K$, then the inclusion $j: (K', L') \hookrightarrow (K, L)$ induces an *isomorphism*

$$\widehat{A_{PL}}(j) : \widehat{A_{PL}}(K, L) \xrightarrow{\cong} \widehat{A_{PL}}(K', L').$$

(Note that $A_{PL}(j)$ is a quasi-isomorphism by the classical excision property.)

Consider now the triangulated compact manifold W and its subpolyhedron K. Replace those polyhedra W and K by their second barycentric subdivision. Denote by T the star of K in W, which is a regular neighborhood (see [6, chapters 1 and 2]), hence T is a codimension 0 submanifold with boundary and it retracts by deformation onto K. It is clear that the topological closure $\overline{W \smallsetminus T}$ of $W \smallsetminus T$ is homotopy equivalent to $W\backslash K$. Set

$$\partial_+ T = \partial T \cap \partial W,$$
$$\partial_- T = \overline{(\partial T \cap (W \smallsetminus \partial W))} = T \cap \overline{W \smallsetminus T},$$
$$\partial_0 T = \partial_+ T \cap \partial_- T,$$

which gives a decomposition of the boundary of T, $\partial T = \partial_+ T \cup_{\partial_0 T} \partial_- T$.

Our next goal is to set up Diagram (3.8). Let us fix an arbitrary order on the vertices of the simplicial complex W such that W and the subpolyhedron T, ∂T, $\partial_+ T$, $\partial_- T$, $\partial_0 T$, K, and $\partial_W K$ turn into ordered simplicial complexes. We can apply to them the functor $\widehat{A_{PL}}$ which is naturally quasi-isomorphic to A_{PL}. To prove the result, it suffices to show that the mapping cone $C(s^{-n}\# \widehat{A_{PL}}(i))$ is a model of $\widehat{A_{PL}}(W)$-dgmodule of $\widehat{A_{PL}}(\overline{W\backslash T})$. To ease notations, in the rest of this proof we will write A_{PL} instead of $\widehat{A_{PL}}$.

By the strong excision property above, the inclusion of the pair

$$(T, \partial T) \hookrightarrow (W, \overline{W\backslash T} \cup \partial W)$$

induces an isomorphism

$$A_{PL}(W, \overline{W\backslash T} \cup \partial W) \xrightarrow{\cong} A_{PL}(T, \partial T).$$

Denote by n the dimension of W. By Poincaré duality of the pair $(W, \partial W)$, there exists an orientation

$$\epsilon_W : A_{PL}(W, \partial W) \longrightarrow s^{-n}\mathbb{Q},$$

i.e., a morphism of cochain complexes that induces an isomorphism in cohomology in degree n. Using this morphism we can define a morphism of $A_{PL}(W)$-dgmodules

$$\begin{array}{rccc} \Phi_W: & A_{PL}(W) & \longrightarrow & s^{-n}\#A_{PL}(W, \partial W) \\ & \alpha & \longmapsto & (\Phi_W(\alpha): \beta \mapsto \epsilon_W(\alpha\beta)), \end{array} \tag{3.7}$$

which is a quasi-isomorphism by Poincaré duality of the pair $(W, \partial W)$. The composition

$$\epsilon_T : A_{PL}(T, \partial T) \cong A_{PL}(W, \overline{W\backslash T} \cup \partial W) \xrightarrow{A_{PL}(\text{incl})} A_{PL}(W, \partial W) \xrightarrow{\epsilon_W} s^n\mathbb{Q}$$

induces an isomorphism in cohomology in degree n. Define

$$\begin{array}{rccc} \Phi_T: & A_{PL}(T) & \longrightarrow & s^{-n}\#A_{PL}(T, \partial T) \\ & \alpha & \longmapsto & \big(\Phi_T(\alpha): \beta \mapsto \epsilon_T(\alpha\beta)\big) \end{array},$$

which is a quasi-isomorphism of $A_{PL}(W)$-dgmodules by Poincaré duality of the pair $(T, \partial T)$. Also, using the quasi-isomorphism above and the five lemma, it is not difficult to see that the morphism

$$\begin{array}{rccc} \widetilde{\Phi}_T: & A_{PL}(T, \partial_- T) & \longrightarrow & s^{-n}\#A_{PL}(T, \partial_+ T) \\ & \alpha & \longmapsto & \big(\widetilde{\Phi}_T(\alpha): \beta \mapsto \epsilon_T(\alpha\beta)\big) \end{array}$$

is a quasi-isomorphism of $A_{PL}(T)$-dgmodules, hence of $A_{PL}(W)$-dgmodules.

The inclusion $(K, \partial_W K) \hookrightarrow (T, \partial_+ T)$ is a homotopy equivalence and induces a weak equivalence of $A_{PL}(W)$-dgmodules

$$A_{PL}(T, \partial_+ T) \xrightarrow{\simeq} A_{PL}(K, \partial_W K).$$

By the strong excision property, the inclusion

$$(T, \partial_- T) \hookrightarrow (W, \overline{W\backslash T})$$

induces an isomorphism

$$A_{PL}(W, \overline{W\backslash T}) \xrightarrow{\cong} A_{PL}(T, \partial_- T).$$

Combining all of these morphisms, we get the following commutative diagram of $A_{PL}(W)$-dgmodules

(3.8)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & A_{PL}(W) & = & A_{PL}(W) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle 0} & (*) & \Vert & & {\scriptstyle A_{PL}(j)}\big\downarrow & & \\
0 & \longrightarrow & A_{PL}(W,\overline{W\backslash T}) & \longrightarrow & A_{PL}(W) & \xrightarrow[A_{PL}(j)]{} & A_{PL}(\overline{W\backslash T}) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \cong\ \mathrm{exc}} & & & & & & \\
 & & A_{PL}(T,\partial_- T) & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle \Phi_W} & & & & \\
 & & \big\downarrow{\scriptstyle \simeq\widetilde{\Phi}_T} & & & & & & \\
 & & s^{-n}\#A_{PL}(T,\partial_+ T) & \longrightarrow & s^{-n}\#A_{PL}(W,\partial W) & & & & \\
 & & \big\uparrow{\scriptstyle \simeq} & & \Vert & & & & \\
 & & s^{-n}\#A_{PL}(K,\partial_W K) & \xrightarrow[s^{-n}\#A_{PL}(i)]{} & s^{-n}\#A_{PL}(W,\partial W), & & & &
\end{array}$$

and the two top lines are short exact sequences.

Properties of mapping cones and of short exact sequences imply that, in the category of $A_{PL}(W)$-dgmodules, the morphism

$$A_{PL}(j): A_{PL}(W) \longrightarrow A_{PL}(\overline{W\backslash T}) \tag{3.9}$$

on the top right of (3.8) is equivalent to the map induced between the mapping cones of the horizontal maps of the square $(*)$ in Diagram (3.8),

$$\mathrm{id}_{A_{PL}(W)} \oplus s0: A_{PL}(W) \oplus s0 \longrightarrow A_{PL}(W) \oplus sA_{PL}(W,\overline{W\backslash T}). \tag{3.10}$$

Since the vertical maps below the second line of (3.8) are quasi-isomorphisms, the morphism $\mathrm{id}_{A_{PL}(W)} \oplus s0$ in (3.10) is equivalent to

$$\iota \circ \Phi_W: A_{PL}(W) \longrightarrow C(s^{-n}\#A_{PL}(i)).$$

The morphism $A_{PL}(j)$ of (3.9) is clearly equivalent to

$$A_{PL}(W) \longrightarrow A_{PL}(W\backslash K).$$

This finishes the proof. ■

4 Rational Model of the Complement of a Subpolyhedron in a Manifold with Boundary

In this section we establish the CDGA model of the complement $W \smallsetminus K$ under some unknotting condition, in particular when the codimension of the subpolyhedron is high (Theorem 4.5). We also state a partial CDGA model without unknotting condition (Proposition 4.7.) We end the section with a few examples that illustrate our results.

Consider the same setting as at the beginning of Section 3, in particular, Diagram (3.2). Suppose given a commutative diagram of CDGAs

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \partial A & \xrightarrow[\partial\varphi]{} & \partial B \end{array} \tag{4.1}$$

that is a CDGA model of

$$\begin{array}{ccc} W & \xleftarrow{f} & K \\ \uparrow & & \downarrow \\ \partial W & \xleftarrow[\partial f]{} & \partial_W K, \end{array} \tag{4.2}$$

in other words, Diagram (4.1) is quasi-isomorphic to Diagram (3.3). Note that in Diagram (4.1), ∂A and ∂B are just the names of some CDGAs.

The goal of this section is to construct from Diagram (4.1) a CDGA model of $A_{PL}(W\backslash K)$.

4.1 Dgmodule Model of the Complement $W\backslash K$

Let $\widehat{A}$ be a CDGA such that we have the following zig-zag of quasi-isomorphisms

$$A \xleftarrow[\simeq]{\rho} \widehat{A} \xrightarrow[\simeq]{\rho'} A_{PL}(W).$$

The morphism ρ' induces a structure of $\widehat{A}$-dgmodule on Diagram (3.3), and the morphism ρ induces a structure of $\widehat{A}$-dgmodule on Diagram (4.1). From Diagram (4.1) we deduce an $\widehat{A}$-dgmodules morphism between the homotopy kernels of α and β (see Section 2.5)

$$\overline{\varphi}\colon \mathrm{hoker}\,\alpha \longrightarrow \mathrm{hoker}\,\beta.$$

Note also that by Poincaré duality of the pair $(W, \partial W)$, we have a quasi-isomorphism of A-dgmodules

$$\theta_A\colon A \xrightarrow{\simeq} s^{-n}\#\,\mathrm{hoker}\,\alpha.$$

Proposition 4.1 *An $\widehat{A}$-dgmodule model of $A_{PL}(W) \to A_{PL}(W\backslash K)$ is given by the composite*

$$A \xrightarrow[\theta_A]{\simeq} s^{-n}\#\,\mathrm{hoker}\,\alpha \xhookrightarrow[\iota]{} C(s^{-n}\#\overline{\varphi}),$$

where $C(s^{-n}\#\overline{\varphi})$ is the mapping cone of the $\widehat{A}$-dgmodules morphism

$$s^{-n}\#\overline{\varphi}\colon s^{-n}\#\,\mathrm{hoker}\,\beta \longrightarrow^{-n}\#\,\mathrm{hoker}\,\alpha.$$

Proof Since (4.1) is a CDGA model of (3.3), $\mathrm{hoker}\,\alpha$ is weakly equivalent as an $\widehat{A}$-dgmodule to $A_{PL}(W, \partial W)$ and $\mathrm{hoker}\,\beta$ is weakly equivalent as an $\widehat{A}$-dgmodule to $A_{PL}(K, \partial_W K)$. Hence, the result is a direct consequence of Proposition 3.1. ∎

Remark 4.2 If the morphisms α and β are surjective, then we can work with the genuine kernel instead of the homotopy kernel.

The major flaw of the dgmodule model of $W \smallsetminus K$ of Proposition 4.1 is that there is no natural CDGA structure on it. The next proposition is a first step to endow this dgmodule model of $W \smallsetminus K$ with the structure of a CDGA.

Proposition 4.3 *Assume we have an $\widehat{A}$-dgmodule morphism $\varphi^!: Q \to A$ weakly equivalent to*

$$s^{-n}\#\overline{\varphi}: s^{-n}\#\,\mathrm{hoker}\,\beta \to s^{-n}\#\,\mathrm{hoker}\,\alpha.$$

Then an $\widehat{A}$-dgmodule model of $A_{PL}(W) \to A_{PL}(W\backslash K)$ is given by $A \hookrightarrow C(\varphi^!)$, where $C(\varphi^!)$ is the mapping cone $A \oplus_{\varphi^!} sQ$.

Proof This is a direct consequence of Proposition 4.1. ■

Remark 4.4 The existence of such a morphism $\varphi^!$ is guaranteed if we take for Q a cofibrant $\widehat{A}$-dgmodule model of $s^{-n}\#$ hoker β.

This new dgmodule model $C(\varphi^!) = A \oplus_{\varphi^!} sQ$ of $W\backslash K$ has the advantage that A is a CDGA, and therefore, under some dimension hypotheses, the semi-trivial CGA structure on the mapping cone described in Section 2.4 makes it into a CDGA. We develop this in the next section.

4.2 CDGA Model of the Complement $W\backslash K$

We work in the set-up of diagrams (4.1) and (4.2) described at the beginning of the section. Remember also the notion of semi-trivial CDGA structure on a mapping cone from Section 2.4 and the notion of CDGA truncation from Section 2.3. Under some codimension and connectedness hypothesis for the inclusion $f: K \hookrightarrow W$ we can construct a CDGA model of $W\backslash K$. More precisely, we have the following theorem.

Theorem 4.5 *Let W be a compact connected oriented triangulated manifold of dimension n with boundary, and let $K \subset W$ be a subpolyhedron of dimension k. Consider Diagram (4.2) and its CDGA model (4.1). Let r be an integer such that the induced morphisms on homology $H_*(f;\mathbb{Q})$ and $H_*(\partial f;\mathbb{Q})$ are r-connected, that is,*

$$H_{\leq r}(W, K;\mathbb{Q}) = 0 \quad \text{and} \quad H_{\leq r}(\partial W, \partial_W K;\mathbb{Q}) = 0.$$

Suppose we have an A-dgmodule Q weakly equivalent to $s^{-n}\#\,\mathrm{hoker}\,\beta$ such that $Q^{<n-k} = 0$ and an A-dgmodules morphism

$$\varphi^!: Q \longrightarrow A \tag{4.3}$$

weakly equivalent to

$$s^{-n}\#\overline{\varphi}: s^{-n}\#\,\mathrm{hoker}\,\beta \to s^{-n}\#\,\mathrm{hoker}\,\alpha.$$

If

$$r \geq 2k - n + 2, \tag{4.4}$$

then every truncation $\tau^{\leq n-r-1}(C(\varphi^!))$ of the mapping cone $C(\varphi^!) = A \oplus_{\varphi^!} sQ$ equipped with the semi trivial structure is a CDGA, and the morphism

$$A \longrightarrow \tau^{\leq n-r-1}(C(\varphi^!))$$

is a CDGA model of the inclusion $W\backslash K \hookrightarrow W$.

Moreover, it is always possible to construct an A-dgmodule Q and a morphism $\varphi^!$ as in (4.3).

This generalizes the main result of [8, Theorem 1.2] to manifolds with boundary. A first direct consequence of this theorem is the following corollary on the rational homotopy invariance of the complement under some connectedness-codimension hypotheses.

Corollary 4.6 *Let W be a compact triangulated manifold with boundary and $K \subset W$ be a subpolyhedron. Assume that W and ∂W are 1-connected and that the inclusions*

$$K \hookrightarrow W \quad \text{and} \quad K \cap \partial W \hookrightarrow \partial W$$

are r-connected with

$$r \geq 2(\dim K) - \dim W + 2. \tag{4.5}$$

Then the rational homotopy type of $W\backslash K$ depends only on the rational homotopy type of the diagram

$$\begin{array}{ccc} \partial_W K & \overset{\partial f}{\hookrightarrow} & \partial W \\ \downarrow & & \downarrow \\ K & \underset{f}{\hookrightarrow} & W. \end{array}$$

The hypothesis (4.4) (or equivalently (4.5)) is called *the unknotting condition*.

In Example 4.9 we will show that the unknotting hypothesis cannot be removed from the statement of the theorem, because without this condition there exists homotopic embeddings with non-rationally equivalent complements (this cannot happen under the unknotting condition, essentially because under this hypothesis the homotopy class of the embedding determines its isotopy class). At the end of the section we will give other examples illustrating the main theorems of this section (Examples 4.10–4.12).

Proof of Theorem 4.5 Let $\widehat{A}$ be a CDGA such that we have a zig-zag of CDGA quasi-isomorphisms

$$A_{PL}(W) \underset{\rho'}{\overset{\simeq}{\longleftarrow}} \widehat{A} \underset{\rho}{\overset{\simeq}{\longrightarrow}} A.$$

Set $N := 2(n-k) - 3$. By Proposition 2.6 (with $p = n - k$), $\tau^{\leq N} C(\varphi^!)$ admits the structure of a CDGA induced by the semi-trivial CGA structure on the mapping cone, and the composite

$$A \overset{\iota}{\longrightarrow} C(\varphi^!) \longrightarrow \tau^{\leq N} C(\varphi^!)$$

is a CDGA morphism.

We now prove that $H^{>N}(W \smallsetminus K) = 0$, where (co)homology of spaces is understood with coefficients in $\mathbb{Q}$. By excision and the connectedness hypotheses on $H(\partial f)$ and $H(f)$,

$$\begin{aligned} H_{\leq r}(K \cup_{\partial_W K} \partial W, K) &\cong H_{\leq r}(\partial W, \partial_W K) = 0, \\ H_{\leq r}(W, K) &= 0. \end{aligned}$$

Lefschetz duality and the long exact sequence of the triple $(W, K \cup_{\partial_W K} \partial W, K)$ give

$$H^{\geq n-r}(W \smallsetminus K) \cong H_{\leq r}(W, K \cup_{\partial_W K} \partial W) = 0.$$

The unknotting hypothesis (4.4) implies that $N \geq n-r-1$; therefore, $H^{>N}(W \smallsetminus K) = 0$.

By Proposition 4.3, $A \to C(\varphi^!)$ is an $\widehat{A}$-dgmodule model of $A_{PL}(W) \to A_{PL}(W \backslash K)$. This implies that

$$H^{>N}(C(\varphi^!)) \cong H^{>N}(W \backslash K) = 0;$$

therefore,

$$\text{proj}\colon C(\varphi^!) \longrightarrow \tau^{\leq N}(C(\varphi^!))$$

is a quasi-isomorphism.

Thus, the CDGA morphism $A \to \tau^{\leq N}(C(\varphi^!))$ is a model of $\widehat{A}$-dgmodules of $A_{PL}(W) \to A_{PL}(W \backslash K)$. We will prove that it is actually a CDGA model.

Take a minimal relative Sullivan model (in the sense of [5, Chapter 14])

$$(4.6) \qquad \begin{array}{ccccc} \widehat{A} & \xrightarrow[\simeq]{\rho'} & A_{PL}(W) & \longrightarrow & A_{PL}(W \backslash K) \\ & \searrow & & \nearrow_{\lambda', \simeq} & \\ & & (\widehat{A} \otimes \Lambda V, D). & & \end{array}$$

By Proposition 4.3, $\widehat{A} \rightarrowtail \widehat{A} \otimes \Lambda V$ is an $\widehat{A}$-dgmodule model of $A \to C(\varphi^!)$. Since $(\widehat{A} \otimes \Lambda V, D)$ is a cofibrant $\widehat{A}$-dgmodule, we can construct a weak equivalence of $\widehat{A}$-dgmodules

$$\lambda\colon \widehat{A} \otimes \Lambda V \longrightarrow C(\varphi^!),$$

making the following diagram commute, where the upper part is of CDGA and the lower part is of $\widehat{A}$-dgmodules,

$$(4.7) \qquad \begin{array}{ccccc} A_{PL}(W) & \longrightarrow & A_{PL}(W \backslash K) & & \\ \uparrow{\scriptstyle \rho', \simeq} & & \uparrow{\scriptstyle \lambda', \simeq} & & \\ \widehat{A} & \rightarrowtail & \widehat{A} \otimes \Lambda V & & \\ \downarrow{\scriptstyle \rho, \simeq} & & \downarrow{\scriptstyle \lambda, \simeq} & \searrow^{\bar{\lambda} = \text{proj} \circ \lambda}_{\simeq} & \\ A & \longrightarrow & C(\varphi^!) & \xrightarrow[\text{proj}]{\simeq} & \tau^{\leq N} C(\varphi^!). \end{array}$$

By Lefschetz duality and the hypothesis on the dimension of K,

$$H^{<n-k}(W, W \backslash K) \cong H_{>k}(K, \partial_W K) = 0.$$

By minimality of the Sullivan relative model (4.6), this implies that $V^{<n-k-1} = 0$. Therefore, $(\Lambda^{\geq 2} V)^{\leq N} = 0$ and, since $(\tau^{\leq N} C(\varphi^!))^{>N} = 0$, this implies that the composition

$$\overline{\lambda}\colon (\widehat{A} \otimes \Lambda V, D) \xrightarrow{\lambda} C(\varphi^!) \xrightarrow{\text{proj}} \tau^{\leq N}(C(\varphi^!))$$

is a morphism of CDGA. Thus, all the solid arrows in Diagram (4.7) are of CDGAs. This proves that $A \to \tau^{\leq N}(C(\varphi^!))$ is a CDGA model of $W\backslash K \hookrightarrow W$, as claimed.

It remains to prove the existence of an A-dgmodule Q and a morphism $\varphi^!$. Since $H^{>k}(\text{hoker } \beta) \cong H^{>k}(K, \partial_W K) = 0$, we have $H^{<n-k}(s^{-n}\#\text{ hoker } \beta) = 0$. Therefore, there exists a cofibrant A-dgmodule model Q of $s^{-n}\#$ hoker β such that $Q^{<n-k} = 0$. Since, by Poincaré duality, $s^{-n}\#$ hoker $\alpha \simeq A$, there exists an A-dgmodule morphism

$$\varphi^!\colon Q \longrightarrow A$$

weakly equivalent to $s^{-n}\#\overline{\varphi}$. ■

Actually, even when the unknotting condition (4.4) of Theorem 4.5 is not satisfied, we still get a partial model of $W\backslash K$. More precisely, we get a CDGA model of $W\backslash K$ up to some degree, *i.e.*, a model of the truncation of $A_{PL}(W\backslash K)$. This is the content of the next proposition.

Proposition 4.7 *Consider the same hypotheses as in Theorem* 4.5 *except that we do not assume the unknotting condition* (4.4)

Let $l\colon A(W) \to A(W\backslash K)$ be a CDGA model of $A_{PL}(W) \to A_{PL}(W\backslash K)$ such that $A(W)$ and $A(W\backslash K)$ are connected. Set $N = 2(n-k) - 3$. Then the CDGA morphism

$$A \longrightarrow \tau^{\leq N} C(\varphi^!)$$

is a CDGA model of the composite

$$\pi \circ l\colon A(W) \hookrightarrow A(W\backslash K) \longrightarrow \tau^{\leq N} A(W\backslash K).$$

Proof of Proposition 4.7 The proof is very similar to that of Theorem 4.5. The details to change are left to the reader. ■

Remark 4.8 We would have preferred in Proposition 4.7 to state that $A \to \tau^{\leq N}(C(\varphi^!))$ is a CDGA model of $A_{PL}(W) \to \tau^{\leq N}(A_{PL}(W\backslash K))$, but the latter is not well defined because $A_{PL}(W\backslash K)$ is not connected, and hence we cannot take its truncation. This is the reason for considering instead a model $l\colon A(W) \to A(W\backslash K)$ between connected CDGAs.

Note that $N = n - (2k - n + 2) - 1$, and therefore, under the unknotting condition $r \geq 2k - n + 2$, we have that $N \geq n - r - 1$. But, Poincaré duality and the r-connectedness imply that $H^{\geq n-r}(W\backslash K) = 0$. Hence, Theorem 4.5 is actually a corollary of Proposition 4.7

We finish this section by illustrating our main results with a few examples.

Example 4.9 We sketch first an example showing that the unknotting condition (4.4) cannot be removed in Theorem 4.5 and Corollary 4.6. This example is fully detailed and generalized in [8, Section 9]. Take $W = S^{15}$ the 15-dimensional sphere, with empty boundary, and $K = S^3 \times S^7$. We will construct two embeddings $f_i\colon S^3 \times S^7 \hookrightarrow S^{15}$,

$i = 1, 2$, which are homotopic but with non rationally equivalent complements. Therefore, a CDGA model of their complements cannot be uniquely deduced from a CDGA model of the embeddings.

The first embedding is obtained as the composite

$$f_1 \colon S^3 \times S^7 \hookrightarrow \mathbb{R}^4 \times \mathbb{R}^8 = \mathbb{R}^{12} \subset \mathbb{R}^{15} \subset \mathbb{R}^{15} \cup \{\infty\} = S^{15}.$$

Since this embedding factors through an equator S^{14}, it is easy to check that the complement $W \setminus f_1(K)$ is homotopy equivalent to the suspension of the complement in S^{14}. Therefore, all products in the rational cohomology of $W \setminus f_1(K)$ are trivial. Actually, one can show that $W \setminus f_1(K) \simeq S^4 \vee S^7 \vee S^{11}$.

To define the second embedding, we use the Hopf fibration $\pi \colon S^{15} \to S^8$ with fibre S^7. Consider a subequator $S^3 \subset S^8$. Then $\pi^{-1}(S^3)$ is homeomorphic to $S^3 \times S^7$, and this defines an embedding

$$f_2 \colon S^3 \times S^7 \cong \pi^{-1}(S^3) \hookrightarrow S^{15},$$

whose complement is $W \setminus f_2(K) = \pi^{-1}(S^8 \setminus S^3) \simeq S^4 \times S^7$. Therefore, there are non-trivial products in the rational cohomology algebra of $W \setminus f_2(K)$.

In this example, with the notation of Theorem 4.5,

$$n = \dim(W) = 15, \quad k = \dim(K) = 10, \quad \text{and} \quad H_2(W, K; \mathbb{Q}) \neq 0,$$

so we need to take $r < 2$, but $2k - n + 2 = 7$; thus, inequality (4.4) is not satisfied.

Example 4.10 We now illustrate Theorem 4.5 with two elementary examples of embeddings of the point in a disk. Let $W = D^n$ be the n-disk (with $n \geq 3$) with boundary $\partial W = S^{n-1}$, and let $K = \{*\}$ be a single point.

(a) Consider a first embedding $f \colon K \hookrightarrow W$ as a point in the interior of the disk. Then Diagram (4.2) becomes

$$\begin{array}{ccc} W = D^n & \xleftarrow{\ f\ } & K = \{*\} \\ \uparrow & & \uparrow \\ \partial W = S^{n-1} & \xleftarrow[\partial f]{} & \partial_W K = \varnothing, \end{array}$$

and a CDGA model of this diagram is

$$\begin{array}{ccc} A = \mathbb{Q} & \xrightarrow{\varphi = \mathrm{id}} & B = \mathbb{Q} \\ {\scriptstyle \alpha = \mathrm{incl}} \downarrow & & \downarrow {\scriptstyle \beta = 0} \\ \partial A = (\mathbb{Q} \oplus \mathbb{Q} z_{n-1}, 0) & \xrightarrow[\partial\varphi = 0]{} & \partial B = 0. \end{array}$$

Then $\operatorname{hoker} \alpha \simeq \mathbb{Q}\overline{z_n}$ and $\operatorname{hoker} \beta = \mathbb{Q}$. Therefore, the zero-map

$$\varphi^! = 0 \colon \mathbb{Q}\overline{u_n} \longrightarrow A = \mathbb{Q}$$

is a A-dgmodule model of

$$s^{-n}\#\overline{\varphi} \colon s^{-n}\# \operatorname{hoker} \beta \longrightarrow s^{-n}\# \operatorname{hoker} \alpha.$$

Thus,

$$C(\varphi^!) = \mathbb{Q} \oplus_{\varphi^!} s(\mathbb{Q}\overline{u_n}) = (\mathbb{Q} \oplus \mathbb{Q}u_{n-1}, 0)$$

with the obvious algebra structure is a CDGA model of the complement, which is expected, since $W \setminus f(K) \simeq S^{n-1}$.

(b) Now consider an embedding $f: K \hookrightarrow S^{n-1} \subset D^n$ of the point in the boundary of the disk. Then $\partial_W K = \{*\}$ and a CDGA model of Diagram (4.2) is now

$$\begin{array}{ccc} A = \mathbb{Q} & \xrightarrow{\varphi=\mathrm{id}} & B = \mathbb{Q} \\ \downarrow{\scriptstyle \alpha=\mathrm{incl}} & & \downarrow{\scriptstyle \beta=\mathrm{id}} \\ \partial A = (\mathbb{Q} \oplus \mathbb{Q}z_{n-1}, 0) & \xrightarrow[\partial\varphi=\mathrm{proj}]{} & \partial B = \mathbb{Q}. \end{array}$$

Then hoker $\beta \simeq 0$ and a model of $s^{-n}\#\overline{\varphi}$ is again a zero-map $\varphi^!: 0 \to A = \mathbb{Q}$ but with a different domain. Then $C(\varphi^!) = \mathbb{Q} \oplus 0 = \mathbb{Q}$ is a CDGA model of the complement, as expected, since $W \setminus f(K)$ is contractible.

Example 4.11 Here is a more interesting example illustrating Theorem 4.5. Consider a real vector bundle ξ of rank 10 over the base $S^5 \times S^5$ with non-zero Euler class. Let $W = D\xi$ be the associated disk bundle with boundary $\partial W = S\xi$, the sphere bundle. Denote the projection of the bundle by π and set $K = S^5 \times S^5$. One can construct an embedding $f: K \hookrightarrow W$ such that πf is homotopic to the identity map and $\partial_W K = S^5 \times \{x_0\}$, where $x_0 \in S^5$ is the base point. A CDGA model of Diagram (4.2) for such an embedding is given by

$$\begin{array}{ccc} A = (\wedge(x,y),0) & \xrightarrow{\varphi=\mathrm{id}} & B = (\wedge(x,y),0) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \partial A = (\wedge(x,y,z), dz = xy) & \xrightarrow[\partial\varphi]{} & \partial B = (\wedge(x),0) \end{array}$$

with $\deg(x) = \deg(y) = 5$ and $\deg(z) = 9$, where each map in the diagram sends each generator to itself or to 0 when it disappears.

One computes that a model of the A-dgmodule morphism $s^{-20}\#\overline{\varphi}$ between the suspensions of the linear dual of the homotopy kernels of the vertical maps is given by

$$\varphi^!: Q = (\mathbb{Q}\overline{u} \oplus \mathbb{Q}\overline{v}, 0) \longrightarrow A = (\wedge(x,y),0)$$

with $\deg(\overline{u}) = 10$, $\deg(\overline{v}) = 15$, the A-dgmodule structure on Q determined by $x\overline{u} = \overline{v}$ and $y\overline{u} = 0$, and $\varphi^!(\overline{u}) = xy$. Thus,

$$\begin{aligned} C(\varphi^!) &\cong \big(A \oplus \mathbb{Q}\langle u, xu\rangle, Du = xy\big) \\ &= \big(\mathbb{Q}\langle 1, x, y, xy, u, xu\rangle, Du = xy\big), \end{aligned}$$

with $\deg(u) = 9$, is a CDGA model of the complement $W \setminus K$. Therefore, this complement has the same rational cohomology algebra as $S^5 \vee S^5 \vee S^{14}$ but is not a formal space, because there is a non-trivial Massey product.

Example 4.12 Our last example illustrates the partial model of Proposition 4.7. Recall the two non-isotopic embeddings $f_i\colon S^3 \times S^7 \hookrightarrow S^{15}$, $i = 1, 2$, from Example 4.9. Although the two complements do not have the same rational homotopy type, we will show that Proposition 4.7 implies that their 7-skeletons are rationally equivalent. Indeed, for both embeddings a CDGA model of Diagram (4.2) is given by

$$\begin{array}{ccc} A = (\wedge(z), 0) & \xrightarrow{\varphi} & B = (\wedge(x, y), 0) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \partial A = 0 & \xrightarrow[\partial\varphi]{} & \partial B = 0 \end{array}$$

with $\deg(z) = 15$, $\deg(x) = 3$, and $\deg(y) = 7$. The map between the homotopy kernels of the vertical maps is equivalent to φ, and a model of its suspended dual $s^{-15}\#\overline{\varphi}$ is given by

$$\varphi^!\colon Q = \big(\mathbb{Q}\langle \overline{\varepsilon}_5, \overline{\eta}_8, \overline{\xi}_{12}, \overline{\omega}_{15}\rangle, 0\big) \longrightarrow A = \wedge(z_{15}, 0),$$

where the A-module structure on Q is trivial and $\varphi^!$ is determined by $\varphi^!(\overline{\omega}) = z$, and its mapping cone is

$$C(\varphi^!) \cong \big(\mathbb{Q}\langle 1, \varepsilon_4, \eta_7, \xi_{11}, \omega_{14}, z\rangle, D(\omega) = z\big).$$

Using the notation of Proposition 4.7, $n = \dim(W) = 15$, $k = \dim(K) = 10$, and $N = 2(n-k) - 3 = 7$. Thus, the N-th truncation of $C(\varphi^!)$ is

$$\tau^{\leq 7} C(\varphi^!) = \big(\mathbb{Q}\langle 1, \varepsilon_4, \eta_7\rangle, 0\big),$$

which is a CDGA model of $S^4 \vee S^7$. Proposition 4.7 states that it is also a model of both seventh truncations of connected CDGA models of the complements $W \smallsetminus f_i(K)$. This was expected, since these complements are homotopy equivalent to $S^4 \vee S^7 \vee S^{11}$ and $S^4 \times S^7$, which have 7-skeletons equivalent to $S^4 \vee S^7$.

5 Rational Model of the Configuration Space of Two Points in a Manifold with Boundary

In this section we use the results of Section 4 to describe the rational homotopy type of the configuration space of two points in a compact manifold with boundary under the 2-connectedness hypothesis. In particular, we prove in Corollary 5.5 that the rational homotopy type of $\mathrm{Conf}(W, 2)$ depends only on the rational homotopy type of the pair $(W, \partial W)$ when W and ∂W are 2-connected. We also construct in Theorem 5.4 an explicit CDGA model of $\mathrm{Conf}(W, 2)$. Moreover, in Theorem 5.8 we describe an elegant CDGA model for $\mathrm{Conf}(W, 2)$ when the pair $(W, \partial W)$ admits a pretty surjective model in the sense of [3].

Fix a compact connected orientable manifold of dimension n, W, with boundary ∂W. Let

$$\Delta\colon W \hookrightarrow W \times W; \quad x \longmapsto (x, x)$$

be the diagonal embedding. The configuration space of two points in W is the complementary space

$$\mathrm{Conf}(W,2) := (W \times W)\backslash\Delta(W) = \{(x,y) \in W \times W | x \neq y\}.$$

Notice that the diagonal embedding Δ is such that $\Delta(\partial W) \cong \partial W$ and $\Delta^{-1}(\partial W \times \partial W) = \partial W$. In other words, with the notation of (3.1),

$$\partial_{W\times W}\big(\Delta(W)\big) = \Delta(\partial W) \cong \partial W.$$

Therefore, according to Corollary 4.6, if W and ∂W are connected enough, then the rational homotopy type of $\mathrm{Conf}(W,2) = W \times W\backslash\Delta(W)$ is determined by the square (5.1) of Proposition 5.1. The goal of the next section is to compute a CDGA model of that square.

5.1 CDGA Model of the Diagonal Embedding of the Pair $(W, \partial W)$ into $(W \times W, \partial(W \times W))$

The goal of this section is to prove the following proposition.

Proposition 5.1 *Let W be a compact connected orientable manifold with boundary ∂W. Suppose given a CDGA surjective model $\beta\colon B \twoheadrightarrow \partial B$ of the inclusion $\partial W \hookrightarrow W$. Then a CDGA model of the square*

(5.1)
$$\begin{array}{ccc}
W \times W & \xleftarrow{\ \Delta\ } & W \\
\uparrow & & \uparrow \\
\partial(W \times W) & \xleftarrow[\ \partial\Delta\ ]{} & \partial W,
\end{array}$$

where Δ is the diagonal map and $\partial\Delta$ is the composition $\partial W \overset{\Delta}{\hookrightarrow} \partial W \times \partial W \hookrightarrow \partial(W \times W)$ is given by the CDGA square

(5.2)
$$\begin{array}{ccc}
B \otimes B & \xrightarrow{\ \mu\ } & B \\
\alpha \downarrow & & \downarrow \beta \\
\frac{B\otimes B}{(\ker\beta \otimes \ker\beta)} & \xrightarrow[\ \widetilde{\mu}\ ]{} & \partial B,
\end{array}$$

where μ is the multiplication, α is the projection on the quotient, and $\widetilde{\mu}$ the map induced by $\beta \circ \mu$.

The rest of this section is devoted to the proof of this result and for the rest of it we will use the notations introduced in the proposition. First, notice that, since W is a manifold with boundary, $W \times W$ is also a manifold with boundary

$$\partial(W \times W) = W \times \partial W \cup_{\partial W \times \partial W} \partial W \times W.$$

In other words we have a pushout (and homotopy pushout)

(5.3)
$$\begin{array}{ccc} \partial(W\times W) & \hookleftarrow & W\times(\partial W) \\ \uparrow & \text{pushout} & \uparrow \\ (\partial W)\times W & \hookleftarrow & \partial W\times\partial W. \end{array}$$

The key argument to prove Proposition 5.1 is that Diagram (5.1) is the right upper half of the following diagram

(5.4)
$$\begin{array}{ccccc} W\times W & & \overset{\Delta}{\hookleftarrow} & & W \\ \uparrow & & & & \\ \partial(W\times W) & \hookleftarrow & W\times(\partial W) & & \\ \uparrow & \text{pushout} & \uparrow & & \uparrow \\ (\partial W)\times W & \hookleftarrow & \partial W\times\partial W & \hookleftarrow & \partial W, \end{array}$$

where the maps are the obvious inclusions and diagonals, and the small left lower square in (5.4) is the homotopy pushout (5.3).

Lemma 5.2 *The following diagram is a CDGA model of diagram (5.4):*

(5.5)
$$\begin{array}{ccccc} B\otimes B & & \xrightarrow{\mu} & & B \\ \downarrow{\scriptstyle\alpha} & & & & \\ P & \longrightarrow & B\otimes\partial B & & \downarrow{\scriptstyle\beta} \\ \downarrow & \textit{pullback} & \downarrow{\scriptstyle\beta\otimes\mathrm{id}} & & \\ \partial B\otimes B & \xrightarrow[\mathrm{id}\otimes\beta]{} & \partial B\otimes\partial B & \xrightarrow{\mu} & \partial B, \end{array}$$

where P is the pullback of the small square, α is the morphism given by the universal property, and μ are the multiplication morphisms.

Proof of Lemma 5.2 Using the classical CDGA models for products and diagonal maps on spaces, the fact that A_{PL} turns homotopy pushout of topological spaces into homotopy pullbacks of CDGAs, that a pullback of CDGA surjections is a homotopy pullback, and standard techniques in rational homotopy theory, we get that a CDGA model of Diagram (5.4) is given by the following diagram, where P' denotes the pullback of the left bottom corner of the square

$$\begin{array}{ccccc}
A_{PL}(W)\otimes A_{PL}(W) & & \xrightarrow{\text{mult}} & & A_{PL}(W) \\
\downarrow & & & & \\
P' & \longrightarrow & A_{PL}(W)\otimes A_{PL}(\partial W) & & \downarrow \\
\downarrow & \text{pullback} & \downarrow & & \\
A_{PL}(\partial W)\otimes A_{PL}(W) & \twoheadrightarrow & A_{PL}(\partial W)\otimes A_{PL}(\partial W) & \longrightarrow & A_{PL}(\partial W).
\end{array}$$

This diagram is easily seen to be equivalent to Diagram (5.5).

■

The following lemma computes the small lower left pullback square in Diagram (5.5).

Lemma 5.3 *We have a pullback in CDGA:*

$$\begin{array}{ccc}
\frac{B\otimes B}{(\ker\beta\otimes\ker\beta)} & \xrightarrow{\overline{id_B\otimes\beta}} & B\otimes\partial B \\
{\scriptstyle \overline{\beta\otimes id_B}}\downarrow & & \downarrow{\scriptstyle \beta\otimes id_{\partial B}} \\
\partial B\otimes B & \xrightarrow[id_{\partial B}\otimes\beta]{} & \partial B\otimes\partial B.
\end{array}$$

Proof Consider the following diagram of CDGA's where the internal square is a pullback and α is the map induced by the universal property:

$$\begin{array}{ccccc}
B\otimes B & & & & \\
 & \overset{\alpha}{\searrow} & & \overset{\beta\otimes id_B}{\searrow} & \\
 & & P & \longrightarrow & B\otimes\partial B \\
 & {\scriptstyle id_\beta\otimes\beta}\searrow & \downarrow & & \downarrow{\scriptstyle \beta\otimes id_{\partial B}} \\
 & & \partial B\otimes B & \xrightarrow[id_{\partial B}\otimes\beta]{} & \partial B\otimes\partial B.
\end{array}$$

It is straightforward to check that α is surjective and that $\ker\alpha = \ker\beta\otimes\ker\beta$. Therefore, we have an induced isomorphism

$$\overline{\alpha}\colon \frac{B\otimes B}{\ker\beta\otimes\ker\beta} \xrightarrow{\cong} P.$$

■

Proof of Proposition 5.1 Diagram (5.1) is the upper right part of Diagram (5.4); therefore, by Lemma 5.2, a CDGA model of (5.1) is given by the upper right part of (5.5). Using Lemma 5.3, which computes the pullback P, we deduce that this CDGA model is (5.2). ■

5.2 A First CDGA Model of $\mathrm{Conf}(W,2)$

Let $\beta\colon B \twoheadrightarrow \partial B$ be a surjective CDGA model of $i\colon \partial W \hookrightarrow W$. Using the results of Section 4, a CDGA model of $\mathrm{Conf}(W,2) = W \times W \backslash \Delta(W)$ can be obtained from a CDGA model of

$$\begin{array}{ccc} W \times W & \xleftarrow{\ \Delta\ } & W \\ \uparrow & & \uparrow \\ \partial(W \times W) & \xleftarrow[\partial\Delta]{} & \partial W = \partial_{W\times W} W, \end{array}$$

which, by Proposition 5.1, is given by

$$\begin{array}{ccc} B \otimes B & \xrightarrow{\ \mu\ } & B \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ \frac{B\otimes B}{(\ker\beta\otimes\ker\beta)} & \xrightarrow[\tilde{\mu}]{} & \partial B. \end{array}$$

Theorem 5.4 *Let W be a compact triangulated manifold with boundary such that W and ∂W are 2-connected. Let $\beta\colon B \twoheadrightarrow \partial B$ be a surjective CDGA model of $\partial W \hookrightarrow W$ and consider the map*

$$\overline{\mu}\colon \ker\beta \otimes \ker\beta \longrightarrow \ker\beta$$

induced by the multiplication $\mu\colon B\otimes B \to B$. Suppose given a $B\otimes B$-dgmodule morphism

$$\delta^{!}\colon D \longrightarrow B \otimes B$$

weakly equivalent to

$$s^{-2n}\#\overline{\mu}\colon s^{-2n}\#\ker\beta \longrightarrow s^{-2n}\#(\ker\beta \otimes \ker\beta)$$

and such that $D^{<n} = 0$.

Then every truncation $\tau^{\leq 2n-3}C(\delta^{!})$ of the mapping cone of $\delta^{!}$ admits a semi-trivial CDGA structure and

$$B \otimes B \longrightarrow \tau^{\leq 2n-3}C(\delta^{!})$$

is a CDGA model of $A_{PL}(W \times W) \to A_{PL}(\mathrm{Conf}(W,2))$.

Proof Since W and ∂W are 2-connected, the morphisms $\Delta\colon W \hookrightarrow W \times W$ and $\partial\Delta\colon \partial W \hookrightarrow \partial(W \times W)$ are 2-connected. So we are under the hypothesis of Theorem 4.5 with $r = 2$, and the result is a direct consequence of it. ∎

We deduce the rational homotopy invariance of $\mathrm{Conf}(W,2)$.

Corollary 5.5 *Let W be a compact manifold with boundary. If W and ∂W are 2-connected, then the rational homotopy type of $\mathrm{Conf}(W,2)$ depends only of the rational homotopy type of the pair $(W,\partial W)$.*

The rational homotopy invariance of Conf $(W, 2)$ when W is closed and 2-connected was established in [7], and [2] gives partial results in the 1-connected case. When W is not simply-connected, [11] shows that there is no rational homotopy invariance.

Remark 5.6 If we have a CDGA quasi-isomorphism $B \xrightarrow{\simeq} B'$ and a $B' \otimes B'$-dgmodule morphism $\delta'^{!}: D' \to B' \otimes B'$ that is weakly equivalent as a $B \otimes B$-dgmodule morphism to $s^{-2n}\#\overline{\mu}$, then it follows immediately from Theorem 5.4 that

$$B' \otimes B' \longrightarrow \tau^{\leq 2n-3} C(\delta'^{!})$$

is also a CDGA model of $A_{PL}(W \times W) \to A_{PL}(\text{Conf}(W, 2))$.

5.3 A CDGA Model of $\text{Conf}(W, 2)$ when $(W, \partial W)$ Admits a Surjective Pretty Model

Let W be a compact manifold of dimension n with boundary ∂W such that both W and ∂W are 2-connected. In this section we will construct an elegant CDGA model of $\text{Conf}(W, 2)$ when the pair $(W, \partial W)$ admits a surjective pretty model in the sense of [3, Definition 4.2]. Let us recall what this means. Suppose we have

(a) a connected Poincaré duality CDGA, P, in dimension n,
(b) a connected CDGA, Q,
(c) a CDGA morphism, $\varphi: P \to Q$.

Since P is a Poincaré Duality CDGA, there exists an isomorphism of P-dgmodules

$$\theta_P: P \xrightarrow{\cong} s^{-n}\#P.$$

Consider the composite

$$\varphi^{!}: s^{-n}\#Q \xrightarrow{s^{-n}\#\varphi} s^{-n}\#P \xrightarrow{\theta_P^{-1}} P, \tag{5.6}$$

which is a morphism of P-dgmodules. Assume that the morphism

$$\varphi\varphi^{!}: s^{-n}\#Q \longrightarrow Q$$

is balanced (see Definition 2.4) and consider the CDGA morphism

$$\varphi \oplus \text{id}: P \oplus_{\varphi^{!}} ss^{-n}\#Q \longrightarrow Q \oplus_{\varphi\varphi^{!}} ss^{-n}\#Q. \tag{5.7}$$

When (5.7) is a CDGA model of the inclusion $\partial W \hookrightarrow W$ we say that it is a *pretty model* of the pair $(W, \partial W)$. If, moreover, φ is surjective (and hence also (5.7)), we say that is is a surjective pretty model. Then if we consider the differential ideal

$$I = \varphi^{!}(s^{-n}\#Q) \subset P, \tag{5.8}$$

[3, Corollary 4.4] states that the CDGA P/I is a CDGA model of W. In [3] we proved that many compact manifolds admit surjective pretty models as examples even-dimensional disk bundles over closed manifolds, complements of high codimensional polyhedra in a closed manifold, as well as any compact manifold whose boundary retracts rationally on its half-skeleton (see [3, Definition 6.1].)

The objective in this section is to use this model, P/I, of W to construct an elegant model for $\mathrm{Conf}(W, 2)$, analogous to the one constructed in [7] for configuration spaces in closed manifolds.

Since P is a Poincaré duality CDGA, for any homogeneous basis $\{a_i\}_{0\leq i\leq N}$ of P, there exists a Poincaré dual basis $\{a_i^*\}_{0\leq i\leq N}$ characterized by $\epsilon(a_i a_j^*) = \delta_{ij}$ where $\epsilon : P^n \to \mathbb{Q}$ is an orientation of P and δ_{ij} is the Kronecker symbol. Let $\Delta \in (P\otimes P)^n$ be the diagonal class of $P\otimes P$ defined as

$$\Delta = \sum_{i=0}^{N}(-1)^{|a_i|}a_i \otimes a_i^*.$$

Denote the projection by $\pi\colon P \to P/I$. Taking the image of the diagonal Δ by the projection $\pi\otimes\pi\colon P\otimes P \to P/I\otimes P/I$ we get a truncated diagonal class

$$\overline{\Delta} = (\pi\otimes\pi)(\Delta) \in (P/I\otimes P/I)^n.$$

Define the map

$$\overline{\Delta}^! \colon s^{-n}P/I \to P/I\otimes P/I\,;\; s^{-n}x \mapsto \overline{\Delta}\cdot(1\otimes x). \tag{5.9}$$

Lemma 5.7 *The map $\overline{\Delta}^!\colon s^{-n}P/I \to P/I\otimes P/I$ defined in (5.9) is a $P/I\otimes P/I$-dgmodules morphism.*

Proof In [7, Lemma 5.1] it is shown that for P a connected Poincaré duality CDGA, the morphism $\Delta^!\colon s^{-n}P \to P\otimes P; s^{-n}x \mapsto \Delta(1\otimes x)$ is a $P\otimes P$-dgmodules morphism.

We have the commutative diagram

$$\begin{array}{ccc} s^{-n}P & \xrightarrow{\Delta^!} & P\otimes P \\ \downarrow{\scriptstyle s^{-n}\pi} & & \downarrow{\scriptstyle \pi\otimes\pi} \\ s^{-n}P/I & \xrightarrow[\overline{\Delta}^!]{} & P/I\otimes P/I. \end{array}$$

Since P/I is a P-dgmodule generated by $1 \in P/I$, this implies that $\overline{\Delta}^!$ is a $P\otimes P$-dgmodules morphism, and the surjectivity of the morphism $\pi\colon P\to P/I$ implies that $\overline{\Delta}^!$ is a $P/I\otimes P/I$-dgmodules morphism. ∎

The main result of this section is the following theorem.

Theorem 5.8 *Let W be a 2 connected compact manifold of dimension n whose boundary is 2-connected. Suppose that $(W, \partial W)$ admits a surjective pretty model of the form (5.7), and let $\overline{\Delta}^!$ be the $P/I\otimes P/I$-dgmodules morphism defined in (5.9). Then the mapping cone*

$$C(\overline{\Delta}^!) = (P/I\otimes P/I)\oplus_{\overline{\Delta}^!} ss^{-n}P/I$$

equipped with the semi-trivial structure is a CDGA model of $\mathrm{Conf}(W, 2)$.

Before proving the theorem, let us fix some notation and prove a lemma. Set

$$B = P\oplus_{\varphi^!} ss^{-n}\#Q \quad \text{and} \quad \partial B = Q\oplus ss^{-n}\#Q.$$

By hypothesis

$$\beta := \varphi \oplus \mathrm{id}\colon B \twoheadrightarrow \partial B$$

is a surjective CDGA model for the inclusion $\partial W \hookrightarrow W$. Also, let $B' := P/I$ and notice that the obvious projection $\pi \oplus 0\colon B \xrightarrow{\simeq} B'$ is a quasi-isomorphism of CDGA. According to Theorem 5.4 and Remark 5.6, we only need to show that $\overline{\Delta}^!$ is equivalent to $s^{-2n}\#\overline{\mu}$, which is the content of the following lemma.

Lemma 5.9 *There exists a $B \otimes B$-dgmodules commutative square:*

$$\begin{array}{ccc} s^{-n}P/I & \xrightarrow{\overline{\Delta}^!} & P/I \otimes P/I \\ {\scriptstyle \overline{\theta}_P}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \overline{\theta}_{P\otimes P}} \\ s^{-2n}\#\ker\beta & \xrightarrow[s^{-n}\#\overline{\mu}]{} & s^{-2n}\#\ker\beta \otimes \ker\beta. \end{array}$$

Proof By Poincaré duality of the CDGA P, we have a P-dgmodules isomorphism $\theta_P : P \xrightarrow{\cong} s^{-n}\#P$. This morphism induces, by construction of the differential ideal $I \subset P$ (see (5.6) and (5.8)), a P-dgmodules isomorphism

$$\overline{\theta}_P\colon P/I \xrightarrow{\cong} s^{-n}\#\ker\varphi.$$

The morphism

$$\beta := \varphi \oplus \mathrm{id}\colon B \twoheadrightarrow \partial B$$

is a surjective CDGA model of $\partial W \hookrightarrow W$. We have an obvious isomorphism $\ker\beta \cong \ker\varphi$ as P-dgmodules. So, we have a P-dgmodules isomorphism (that we will also denote $\overline{\theta}_P$)

$$\overline{\theta}_P\colon P/I \xrightarrow{\cong} s^{-n}\#\ker\beta.$$

An easy computation shows that for $(p,u) \in B = P \oplus ss^{-n}\#Q$ and $x \in P/I$,

$$\overline{\theta}_P\big((p,u)\cdot x\big) = (p,u)\overline{\theta}(x).$$

Thus, $\overline{\theta}_P$ is a morphism of B-dgmodules and, via the multiplication $\mu\colon B\otimes B \to B$, it is a $B\otimes B$-dgmodules morphism. As a direct consequence, we have the $B\otimes B$-dgmodules isomorphism

$$\overline{\theta}_{P\otimes P}\colon P/I \otimes P/I \xrightarrow{\cong} s^{-n}\#\ker\beta \otimes s^{-n}\#\ker\beta \cong s^{-2n}\#(\ker\beta\otimes\ker\beta).$$

By Lemma 5.7, the morphism $\overline{\Delta}^!$ is a $P/I \otimes P/I$-dgmodules morphism, and hence it is also a morphism of $B \otimes B$-dgmodules.

Consider the diagram of $B \otimes B$-dgmodules

$$\begin{array}{ccc} s^{-n}P/I & \xrightarrow{\overline{\Delta}^!} & P/I \otimes P/I \\ {\scriptstyle \overline{\theta}_P}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \overline{\theta}_P\otimes\overline{\theta}_P} \\ s^{-2n}\#\ker\beta & \xrightarrow[s^{-n}\#\overline{\mu}]{} & s^{-2n}\#\ker\beta \otimes \ker\beta, \end{array}$$

and let us show that it commutes. Since P/I is a $B \otimes B$-dgmodule generated by the element $1 \in P/I$, it suffices to prove that

$$\overline{\theta}_P \otimes \overline{\theta}_P(\overline{\Delta}^!(s^{-n}1)) = s^{-n} \# \overline{\mu}(\overline{\theta}_P(s^{-n}1)).$$

A straightforward computation shows that this is the case. ■

Proof of Theorem 5.8 Since W and ∂W are 2-connected, Lemma 5.9, Remark 5.6, and Theorem 5.4 imply that

$$P/I \otimes P/I \longrightarrow \tau^{\leq 2n-3} C(\overline{\Delta}^!)$$

is a CDGA model of $\mathrm{Conf}(W,2) \hookrightarrow W \times W$. Moreover, we can verify that the morphism $\overline{\Delta^!}$ is balanced, therefore $C(\overline{\Delta}^!)$ is also a CDGA when equipped with the semi-trivial structure. By the 2-connectedness of the manifold W and for degree reasons we have that

$$C(\overline{\Delta}^!) \xrightarrow{\simeq} \tau^{\leq 2n-3} C(\overline{\Delta}^!)$$

is a CDGA quasi-isomorphism. ■

5.4 A CDGA Model for $\mathrm{Conf}(W,2)$ when W is a Disk Bundle of Even Rank Over a Closed Manifold

We apply the model constructed in Section 5.3 to disk bundles.

Let ξ be a vector bundle of even rank, $2k$, for some $k \geq 2$, over some 2-connected closed manifold, M, of dimension m. Then the disk bundle $D\xi$ is a compact manifold of dimension $m + 2k$ with boundary the sphere bundle $S\xi$.

Let Q be a Poincaré duality CDGA model of M, let $\Delta_Q \in (Q \otimes Q)^m$ be a diagonal class for Q, and let $e \in Q^{2k} \cap \ker(d_Q)$ be a representative of the Euler class of ξ. Denote by $(\Delta_Q \cdot (e \otimes 1))^!$ the $Q \otimes Q$-dgmodule morphism

$$\big(\Delta_Q \cdot (e \otimes 1)\big)^! : s^{-(m+2k)} Q \longrightarrow Q \otimes Q, s^{-(m+2k)} q \longmapsto \Delta_Q \cdot (e \otimes q),$$

which is balanced. Consider the mapping cone

$$Q \otimes Q \bigoplus_{(\Delta_Q \cdot (1 \otimes e))^!} s s^{-(m+2k)} Q,$$

which is a CDGA.

Theorem 5.10 *With the notation above, assume that the vector bundle ξ is of even rank $2k \geq 4$, and that the base, M, is a 2-connected closed manifold. Then*

$$Q \otimes Q \bigoplus_{(\Delta_Q \cdot (1 \otimes e))^!} s s^{-(m+2k)} Q$$

is a CDGA model of $\mathrm{Conf}(D\xi, 2)$.

Before proving this theorem, let us first deduce the rational homotopy invariance of that configuration space.

Corollary 5.11 *The rational homotopy type of the configuration space of 2 points in a disk bundle of even rank ≥ 4 over a 2-connected closed manifold depends only on the rational homotopy type of the base and on the Euler class.*

Proof of Corollary 5.11 By the main result of [9], the base of the bundle admits a Poincaré duality CDGA model, Q. Let $e \in Q \cap \ker(d_Q)$ be a representative of the Euler class. By Theorem 5.10, a CDGA model of the configuration space, and hence its rational homotopy type, since it is simply connected, depends only on those data. ■

Proof of Theorem 5.10 Denote by $\bar{z}$ a generator of degree $2k$ and define the CDGA

$$P := \Big(\frac{Q \otimes \wedge \bar{z}}{(\bar{z}^2 - e\bar{z})}, D\bar{z} = 0 \Big),$$

which is a Poincaré CDGA in dimension $n = m + 2k$. Define the CDGA morphism

$$\varphi: P \longrightarrow Q$$

by $\varphi(q_1 + q_2\bar{z}) = q_1 + q_2 \cdot e$, for $q_1, q_2 \in Q$. Then [3, Theorem 5.1] and its proof establish that the pair $(D\xi, S\xi)$ admits a surjective pretty model associated with φ. We will then use Theorem 5.8 to establish the model of $\mathrm{Conf}(D\xi, 2)$.

Following the notation of [3, proof of Theorem 5.1], one computes that

$$I = \varphi^!(s^{-n}\#Q) = \Phi^!(s^{-2k}Q) = \bar{z} \cdot Q.$$

We need to compute the truncated diagonal class $\overline{\Delta} \in P/I \otimes P/I$. Let $\{q_i\}$ be a homogeneous basis of Q and let $\{q_i^*\}$ be its Poincaré dual basis. Denote by $\omega \in Q^m$ the fundamental class of Q, so that we have

$$q_i \cdot q_j^* = \delta_{ij} \cdot \omega \mod Q^{<m}.$$

Then

$$\{q_i\} \cup \{q_i \cdot \bar{z}\} \tag{5.10}$$

is a homogeneous basis of P and $-\omega\bar{z}$ is a fundamental class of P. Then the Poincaré dual basis of (5.10) is given by,

$$\{q_i^* \cdot (e - \bar{z})\} \cup \{-q_i^*\}$$

because of the four equations

$$\begin{aligned} q_i \cdot q_j^*(e - \bar{z}) &= -\delta_{ij}\omega\bar{z} && \mod P^{<n}, \\ q_i \cdot (-q_j^*) &= 0 && \mod P^{<n}, \\ (q_i\bar{z}) \cdot (q_j^*(e - \bar{z})) &= q_i q_j^*(\bar{z}e - \bar{z}^2) = 0 && \mod P^{<n}, \\ (q_i\bar{z}) \cdot (-q_j^*) &= -q_i q_j^* \bar{z} = -\delta_{ij}\omega\bar{z} && \mod P^{<n}. \end{aligned}$$

Therefore, the diagonal class in P is given by

$$\Delta_P = \sum_i (-1)^{|q_i|} \big(q_i \otimes q_i^*(e - \bar{z}) - q_i\bar{z} \otimes q_i^* \big) \in P \otimes P,$$

and, since $I = Q\bar{z}$, the truncated diagonal class is

$$\overline{\Delta}_P = \sum_i (-1)^{|q_i|} q_i \otimes q_i^* e \in P/I \otimes P/I.$$

The diagonal class of Q is

$$\Delta_Q = \sum_i (-1)^{|q_i|} q_i \otimes q_i^* \in Q \otimes Q;$$

therefore, using the canonical isomorphism $P/I \cong Q$, we have

$$\overline{\Delta}_P = \Delta_Q \cdot (1 \otimes e).$$

The theorem is then a direct consequence of Theorem 5.8. ∎

Note that the total space, $E\xi$, of the vector bundle ξ is homeomorphic to the interior of $D\xi$, and therefore $\mathrm{Conf}(E\xi, 2) \simeq \mathrm{Conf}(D\xi, 2)$. In particular, when the bundle is trivial, the above gives a model of $\mathrm{Conf}(M \times \mathbb{R}^{2k}, 2)$. Hence, we recover partially the result [1, Theorem1].

Interestingly enough, we get different models when the bundle is not trivial. Consider for example the quaternionic Hopf line bundle, η, over S^4, of rank 4. In that case we can take $Q = (\mathbb{Q}[x]/(x^2), d_Q = 0)$, with $\deg(x) = 4$, as a model for S^4 and the Euler class is represented by $e = x$. Using the model of Theorem 5.10, one easily computes that the rational cohomology algebra of $\mathrm{Conf}(D\eta, 2)$ is the same as $H^*(S^4 \vee S^4 \vee S^{11}; \mathbb{Q})$, but $\mathrm{Conf}(D\eta, 2)$ is not formal, because it admits a non-trivial Massey product in degree 11.

By contrast, for the trivial bundle of rank 4 over S^4, $\epsilon = S^4 \times \mathbb{R}^4$, one computes that $\mathrm{Conf}(D\epsilon, 2)$ is formal and its rational cohomology algebra is given by

$$H^*\big(\mathrm{Conf}(S^4 \times \mathbb{R}^4, 2); \mathbb{Q}\big) \cong \frac{\wedge(x, x', u)}{(x^2, x'^2, ux - ux')},$$

with $\deg(x) = \deg(x') = 4$ and $\deg(u) = 7$.

Thus, the two compact manifolds $D\eta$ and $D\epsilon$ of dimension 8 are homotopy equivalent but their configuration spaces have different Poincaré series. This is because their boundaries, $\partial D\eta = S^7$ and $\partial D\epsilon = S^4 \times S^3$, are not homotopy equivalent.

5.5 A CDGA Model for $\mathrm{Conf}(W, 2)$ when W is the Complement of a Subpolyhedron in a Closed Manifold

Let V be a 2-connected closed manifold of dimension n. Let $K \subset V$ be a 2-connected subpolyhedron such that $\dim V \geq 2\dim(K) + 3$. In this section we explain how to build a CDGA model of $\mathrm{Conf}(V \setminus K, 2)$.

Let T be a regular neighborhood of K in V; in other words, T is a compact codimension 0 submanifold of V that retracts by deformation on K. Then let W be the closure of $V \setminus T$ in M, which is a compact manifold with boundary $\partial W = \partial T$. The interior of W is homeomorphic to $V \setminus K$. Therefore, $\mathrm{Conf}(V \setminus K, 2)$ is homotopy equivalent to $\mathrm{Conf}(W, 2)$.

Let us recall how to build a pretty surjective model of $(W, \partial W)$. By [3, Proposition 4.5] one can construct a surjective CDGA model $\varphi \colon P \twoheadrightarrow Q$ of $K \hookrightarrow V$, where P is a Poincaré duality CDGA and $Q^{\geq n/2-1} = 0$, such that the pretty model associated with φ,

$$\varphi \oplus \mathrm{id} \colon P \bigoplus_{\varphi^!} ss^{-m}\#Q \longrightarrow Q \bigoplus_{\varphi\varphi^!} ss^{-n}\#Q,$$

is a CDGA model of $(W, \partial W)$. Therefore, by Theorem 5.8, a CDGA model of $\mathrm{Conf}(V \smallsetminus K, 2)$ is given by $P/I \otimes P/I \bigoplus_{\overline{\Delta}^!} ss^{-n}(P/I)$.

Let us illustrate this for the configuration space of a punctured manifold. Let V be a closed 2-connected manifold and set $W = V \smallsetminus \{x_0\}$, with $x_0 \in V$. Let P be a Poincaré duality CDGA model of V with fundamental class $\omega \in P^n$. Pick a homogeneous basis $\{a_i\}_{0 \le i \le N}$ of P with $a_0 = 1$ and $a_N = \omega$. Let $\{a_i^*\}$ be the Poincaré dual basis. Then the diagonal class is

$$\Delta = 1 \otimes \omega + (-1)^n \omega \otimes 1 + \sum_{i=1}^{N-1} (-1)^{|a_i|} a_i \otimes a_i^*$$

with $1 \le \deg(a_i) \le n-1$, for $1 \le i \le N-1$. In that case we can take $I = \mathbb{Q} \cdot \omega$ and we get that $\overline{P} = P/(\mathbb{Q} \cdot \omega)$ is a CDGA model of $V \smallsetminus \{x_0\}$ and the truncated diagonal is

$$\overline{\Delta} = \sum_{i=1}^{N-1} (-1)^{|a_i|} a_i \otimes a_i^*.$$

Thus,

$$\overline{P} \otimes \overline{P} \bigoplus_{\overline{\Delta}^!} ss^{-n} \overline{P}$$

is a CDGA model of $\mathrm{Conf}(V \smallsetminus \{x_0\}, 2)$.

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