DECOMPOSABILITY OF ORTHOGONAL INVOLUTIONS IN DEGREE 12

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Abstract. A theorem of Pfister asserts that every 12-dimensional quadratic form with trivial discriminant and trivial Clifford invariant over a field of characteristic different from 2 decomposes as a tensor product of a binary quadratic form and a 6-dimensional quadratic form with trivial discriminant. The main result of the paper extends Pfister's result to orthogonal involutions : every central simple algebra of degree 12 with orthogonal involution of trivial discriminant and trivial Clifford invariant decomposes into a tensor product of a quaternion algebra and a central simple algebra of degree 6 with orthogonal involutions. This decomposition is used to establish a criterion for the existence of orthogonal involutions with trivial invariants on algebras of degree 12, and to calculate the f_3-invariant of the involution if the algebra has index 2.

Every semi-simple algebraic group of classical type can be described in terms of a central simple algebra with involution, except for groups of type D in characteristic 2, where the involution should be replaced by a so-called quadratic pair [7, §26]. When the base field has characteristic 0, this was first observed by Weil [15] in the 60's for adjoint groups. In particular, over a field of characteristic different from 2, groups of type D_n are quotients of the Spin group of a degree $2n$ algebra with orthogonal involution. If the algebra is the endormophism ring of some $2n$-dimensional vector-space V, the involution is adjoint to a quadratic form q defined on V, unique up to a scalar factor, and the corresponding groups are quotients of the Spin group of this quadratic form.

Algebraic groups of low rank, and the corresponding algebras with involution, which have degree ≤ 14, play a special role in the theory. Indeed, these groups have specific properties, which in turn produce efficient tools to study and describe the underlying algebraic objects. In particular, we may mention the so-called exceptional isomorphisms, with consequences on algebras with involution explored in [7, § 15], triality, that is the action of the symmetric group in three letters on the Dynkin diagram D_4, see [7, Chapter X], and the existence of an open orbit for some representations of algebraic groups of low rank, allowing to view torsors under those groups as torsors under the stabilizer, see Garibaldi [3, Th. 9.3].

Even though they were first studied independently, these facts are related to the classification theorems describing quadratic forms of even dimension ≤ 12 with trivial discriminant and trivial Clifford invariant, which were proved by Pfister in 1966 [8], see also [6, Th. 8.1.1]. It appears that those forms always contain a nontrivial subform of even dimension and trivial discriminant, and admit a diagonalisation of a special shape, depending on the dimension of the form. In particular, the number of parameters required to describe such a form in general is less than what one may expect in view of the dimension. An analogous statement was obtained by Rost [12], more than thirty years later, for quadratic forms of dimension 14 (see also [3, Th. 21.3]), based on the representation argument mentioned above. From the point of view of algebraic groups, it is clear that those results do not extend to higher dimensional quadratic forms. This was formally proved by Merkurjev and Chernousov in [2], where they compute the essential dimension of a split spinor group Spin_n, for $n \geq 15$. Roughly speaking, since torsors under

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Spin_n are closely related to n-dimensional quadratic forms with trivial discriminant and trivial Clifford invariant, this essential dimension provides a measure of the number of parameters required to describe such a form in general. It follows from this computation that a general quadratic form of dimension ≥ 15 does not contain a subform of a given dimension and with trivial discriminant, with two possible exceptions (see [2, Th. 4.2] for a precise statement).

As opposed to this, Pfister's theorem does extend to algebras with orthogonal involutions. This was already known in dimension ≤ 10, and partial results in dimension 12 were discussed in [4] and [10]. The main result of this paper is Theorem 1.3, which is an improved version of these dimension 12 analogues, obtained by using the descent theorem for unitary involutions in degree 6 proven in [11, Th. 1.3]. This new statement is closer to Pfister's original result, which asserts that every 12-dimensional quadratic form with trivial discriminant and trivial Clifford invariant over a field of characteristic different from 2 decomposes as a tensor product of a binary quadratic form and a 6-dimensional quadratic form with trivial discriminant.

As a consequence, we characterize in Corollary 2.1 the biquaternion F-algebras D such that the matrix algebra $M_3(D)$ carries an orthogonal involution with trivial discriminant and trivial Clifford algebra. This property turns out to hold for every biquaternion F-algebra if the 2-cohomological dimension of F is at most 2; we show in Example 2.2 that it fails for certain totally ramified biquaternion F-algebras.

Another use of Theorem 1.3 is for the computation of a certain cohomological invariant. Recall from [10] that a cohomological invariant of degree 3 for orthogonal involutions with trivial discriminant and trivial Clifford invariant is defined on the model of the Arason invariant e_3 of quadratic forms. The generalized Arason invariant takes its values in a quotient of the third Galois cohomology group of $\mu_4^{\otimes 2}$; taking the square of a representative yields an invariant f_3 with values in the cohomology of μ_2. We show in Theorem 2.3 how this invariant can be calculated from a tensor product decomposition afforded by Theorem 1.3.

Throughout, F is a field of characteristic different from 2, and (A, σ) is a central simple F-algebra with orthogonal involution. A possible characterization of (A, σ) is the existence of a finite Galois extension L/F and a quadratic space (V, φ) over L such that

$$A \otimes_F L \simeq \mathrm{End}_L(V) \text{ and } \sigma \otimes \mathrm{Id} = \mathrm{ad}_\varphi$$

where ad_φ is the involution adjoint to φ (or, more precisely, to its polar bilinear form). We generally follow the notation used in [7], to which we refer for background information on involutions on central simple algebras. In particular, for any field K containing F, we write $(A, \sigma)_K$ for the K-algebra with involution $(A \otimes_F K, \sigma \otimes \mathrm{Id})$. If φ is a (nondegenerate) quadratic form on some F-vector space V, we write Ad_φ for $(\mathrm{End}_F(V), \mathrm{ad}_\varphi)$. The discriminant of a quadratic form and the even part of its Clifford algebra, which are invariant under similitudes, and may therefore be considered as invariants of the involution ad_φ, extend to non-split algebras with orthogonal involution [7, §7,8].

For $i \geq 1$, we let $H^i(F)$ denote the Galois cohomology group $H^i(F, \mu_2)$ and identify $H^1(F)$ with $F^\times/F^{\times 2}$ (written additively) and $H^2(F)$ with the 2-torsion subgroup of the Brauer group of F. For $a \in F^\times$ and A a central simple F-algebra of exponent 1 or 2, we write (a) for the square-class of a and $[A]$ for the Brauer class of A. For every orthogonal involution σ on a central simple F-algebra A of even degree, we let $e_1(\sigma) \in H^1(F)$ denote the discriminant of σ. If $e_1(\sigma) = 0$, the Clifford invariant $e_2(\sigma) \in H^2(F)/\{0, [A]\}$ is the coset represented by any of the two components of the Clifford algebra $C(A, \sigma)$.

1. Decomposability

Our first decomposition result does not require triviality of the Clifford invariant. It is premised instead on the existence of a quadratic extension making the involution hyperbolic, i.e., adjoint to a hyperbolic hermitian form.

Proposition 1.1. *Let (A,σ) be a central simple F-algebra with orthogonal involution of degree 12, and let $K = F(\sqrt{d})$ be a quadratic field extension of F. If A is split, assume additionally that σ is not adjoint to a quadratic form of odd Witt index.*

(i) *The algebra with involution $(A,\sigma)_K$ is hyperbolic if and only if (A,σ) decomposes as*

$$(A,\sigma) = (A_0,\sigma_0) \otimes (H,\rho)$$

where (A_0,σ_0) is a central simple algebra with orthogonal involution of degree 6 and (H,ρ) is a quaternion algebra with orthogonal involution of discriminant (d).

(ii) *The algebra with involution $(A,\sigma)_K$ is split and hyperbolic if and only if (A,σ) decomposes as*

$$(A,\sigma) = \mathrm{Ad}_\varphi \otimes (H,\rho)$$

where φ is a quadratic form of dimension 6 and (H,ρ) is a quaternion algebra with orthogonal involution of discriminant (d).

Proof. (i) The condition is obviously sufficient, since $(H,\rho)_K$ is hyperbolic. Assume conversely that $(A,\sigma)_K$ is hyperbolic. By [1, Th. 3.3], this means A contains a skew-symmetric element δ with square d. Writing ι for the nontrivial automorphism of K, we may then identify (K,ι) with a subalgebra of (A,σ). Let B be the centralizer of K in A. The involution σ induces an involution τ of B, which restricts to ι on K. Hence by the descent theorem of [11, Th. 1.3], $(B,\tau) = (A_0,\sigma_0)\otimes_F(K,\iota)$, for some algebra with orthogonal involution (A_0,σ_0). The centralizer of A_0 in A is a quaternion algebra H, which contains K, and by the double centralizer theorem, we have $A = A_0 \otimes H$. Moreover, since A_0 is σ-stable, H also is, and we get a decomposition

$$(A,\sigma) = (A_0,\sigma_0) \otimes (H,\rho),$$

with σ_0 and ρ of orthogonal type, and $(H,\rho) \supset (K,\iota)$. The latter inclusion shows that $e_1(\rho) = (d)$, and the proof of (i) is complete.

(ii) As in (i), the condition is sufficient because $(H,\rho)_K$ is hyperbolic. For the converse, we modify the argument in (i), taking into account the additional hypothesis that A_K is split. From this hypothesis, it follows that the algebra B is split, hence we may identify $B = \mathrm{End}_K(V)$ for some K-vector space V, and $\tau = \mathrm{ad}_h$ for some hermitian form h on V. Fix an orthogonal basis $(e_1,\dots,e_6)$ of V. The form h restricts to a symmetric bilinear form on the F-vector space V_0 spanned by $e_1, \dots, e_6$, and we may take $A_0 = \mathrm{End}_F(V_0)$ in the proof of (i). Thus, $(A_0,\sigma_0) = \mathrm{Ad}_\varphi$ where $\varphi(x) = h(x,x)$ on V_0. □

Remark 1.2. Let (A,σ) be a central simple F-algebra with orthogonal involution of degree $4m$ for some integer m (excluding the case where A is split and σ is adjoint to a quadratic form of odd Witt index). We compare the following statements:

(a) $(A,\sigma) = (A_0,\sigma_0)\otimes(H,\rho)$ for some quaternion algebra with orthogonal involution (H,ρ);
(b) there exists a quadratic field extension K of F such that $(A,\sigma)_K$ is hyperbolic;
(c) $e_1(\sigma) = 0$.

The implication (a) ⇒ (b) always holds, for we may take for K the subfield of H generated by a skew-symmetric element. (If the skew-symmetric elements in H do not generate a field, then (H,ρ) is hyperbolic and (b) clearly holds.) The implication (b) ⇒ (c) can be derived from the first step in the proof of Proposition 1.1 as follows: if $(A,\sigma)_K$ is hyperbolic, then (K,ι) embeds in (A,σ) by [1, Th. 3.3], hence A contains a skew-symmetric element α such that $\alpha^2 \in F^\times$. Let $\alpha^2 = a$. The reduced norm $\mathrm{Nrd}_A(\alpha)$ is $(-a)^{2m}$ and by definition $e_1(\sigma) = \big(\mathrm{Nrd}_A(\alpha)\big)$, so $e_1(\sigma) = 0$.

On the other hand, taking for A an indecomposable algebra of degree 8 yields examples where (b) holds but (a) does not (see [9, Ex. 3.6]), whereas Proposition 1.1 shows that (a) and (b) are equivalent when $\deg A = 12$. The implication (c) ⇒ (b) does not hold, even when A is split of degree 12: for instance, any quadratic form which is the orthogonal sum of a 3-fold and a 2-fold Pfister form is a 12-dimensional quadratic form with trivial discriminant, which need not be hyperbolic over a quadratic field extension of the base field. For an explicit example, consider for instance $\varphi = \pi_3 \oplus \langle\langle x, y\rangle\rangle$ over $F = k((x))((y))$, where π_3 is an arbitrary anisotropic 3-fold Pfister form over k.

Note also that Tao's computation in [13] shows that when (a) holds, then $e_2(\sigma)$ is represented by $[H] + (d, d_0)$ where $e_1(\rho) = (d)$ and $e_1(\sigma_0) = (d_0)$. It is therefore easy to see that (a) does not imply $e_2(\sigma) = 0$.

By contrast, the condition $e_1(\sigma) = e_2(\sigma) = 0$ turns out to be sufficient for the existence of a quadratic extension K such that $(A, \sigma)_K$ is hyperbolic (hence also for a decomposition as in Proposition 1.1(i)) when $\deg A = 12$. The following result may be regarded as a generalization of Pfister's theorem on 12-dimensional quadratic forms with trivial discriminant and trivial Clifford invariant.

Theorem 1.3. *Let (A, σ) be a central simple algebra with orthogonal involution of degree 12. The following conditions are equivalent:*

(a) $e_1(\sigma) = e_2(\sigma) = 0$*;*

(b) *there exists a central simple algebra with orthogonal involution (A_0, σ_0) of degree 6 and a quaternion algebra with orthogonal involution (H, ρ) such that, writing $e_1(\rho) = (d)$ and $e_1(\sigma_0) = (d_0)$,*

$$(A, \sigma) = (A_0, \sigma_0) \otimes (H, \rho) \quad \textit{and} \quad H = (d, d_0).$$

Proof. That (b) implies (a) follows from the computation of the discriminant and the Clifford algebra of decomposable algebras with involution, see [7, (7.3)] and [13].

The first part of the argument for the converse is borrowed from [4]. More precisely, assume condition (a) holds. Then one of the half-spin representations V of $\mathrm{Spin}(A, \sigma)$ is defined over F. By a classical result in representation theory, since the degree of A is 12, $\mathrm{Spin}(A, \sigma)$ has an open orbit in $\mathbb{P}(V)(F_{\mathrm{alg}})$, where F_{alg} is an algebraic closure of F. Using this open orbit, Garibaldi produced in (loc. cit., proof of Th. 3.1) a quadratic field extension $K = F(\sqrt{d})$ of F over which σ is hyperbolic. Therefore, Proposition 1.1 applies and yields a decomposition

$$(A, \sigma) = (A_0, \sigma_0) \otimes (H, \rho)$$

for some algebra with orthogonal involution (A_0, σ_0) of degree 6 and some quaternion algebra with orthogonal involution (H, ρ) such that $e_1(\rho) = (d)$. Let $e_1(\sigma_0) = (d_0)$. Tao's computation in [13] shows that the Clifford algebra of (A, σ) has two components, which are Brauer-equivalent to $[H] + (d, d_0)$ and $[A_0] + (d, d_0)$. Therefore, the triviality of $e_2(\sigma)$ implies that $(d, d_0) = [H]$ or $[A_0]$. The proof is complete if the first equation holds.

For the rest of the proof, assume $(d, d_0) = [A_0]$. Then K splits A_0 as well as H, hence it splits A. Therefore, by Proposition 1.1, we may assume $(A_0, \sigma_0) = \mathrm{Ad}_\varphi$ for some 6-dimensional quadratic form φ. Let $\langle \lambda_1, \dots, \lambda_6\rangle$ be a diagonalization of φ and let $q \in H$ be such that $\rho(x) = q\overline{x}q^{-1}$ for $x \in H$. Then $(d_0) = (-\lambda_1 \cdots \lambda_6)$, $F(q) \simeq K$, and $(A, \sigma) \simeq \mathrm{Ad}_h$ for the skew-hermitian form $h = \langle \lambda_1 q, \dots, \lambda_6 q\rangle$. Let $u \in H^\times$ be a quaternion that anticommutes with q, and let $c = u^2 \in F^\times$. Then $[H] = (c, d)$ and

$$\overline{ux} \cdot q \cdot ux = \overline{x} \cdot cq \cdot x \qquad \text{for } x \in H,$$

hence the skew-hermitian forms $\langle q\rangle$ and $\langle cq\rangle$ are isometric. Therefore,

$$h \simeq \langle \lambda_1 q, \dots, \lambda_5 q, c\lambda_6 q\rangle \simeq \varphi' \otimes \langle q\rangle \quad \text{for } \varphi' = \langle \lambda_1, \dots, \lambda_5, c\lambda_6\rangle,$$

and we have another decomposition

$$(A, \sigma) \simeq \mathrm{Ad}_{\varphi'} \otimes (H, \rho), \qquad \text{with } e_1(\varphi') = (cd_0).$$

Since $(d, d_0) = [A_0] = 0$ and $[H] = (c, d)$, it follows that $\big(e_1(\varphi'), e_1(\rho)\big) = [H]$, hence the latter decomposition satisfies the conditions in (b). □

To emphasize the analogy between Theorem 1.3 and Pfister's result in [8, pp. 123–124], we derive an additive decomposition of (A, σ) from the multiplicative decomposition in Theorem 1.3(b). Since $\deg A_0 = 6$ and $2[A_0] = 0$, there is a quaternion algebra H' such that

$A_0 \simeq M_3(H')$. The involution σ_0 is adjoint to some skew-hermitian form h over $(H', {}^{-})$. Pick a diagonalization $h = \langle q_1, q_2, q_3\rangle$, for some pure quaternions $q_i \in H'$. Denote $a_i = q_i^2$, and consider $b_i \in F^\times$ for $i = 1, 2, 3$ such that $H' = (a_1, b_1) = (a_2, b_2) = (a_3, b_3)$. Since $e_1(\sigma_0) = d_0$, we have $(a_1a_2a_3) = (d_0)$. The algebra with involution $(M_3(H'), \mathrm{ad}_h)$ is an orthogonal sum of the (H', ρ_i), where $\rho_i = \mathrm{Int}(q_i) \circ {}^{-}$ has discriminant a_i. This yields an additive decomposition of (A, σ), namely (in the notation of [10, §3.1])

$$(A, \sigma) \in \boxplus_{i=1}^{3} (H', \rho_i) \otimes (H, \rho). \tag{1}$$

Each term in this decomposition is a central simple algebra of degree 4 with orthogonal involution of trivial discriminant. It can be rewritten as a tensor product of two quaternion algebras with canonical involution

$$(H', \rho_i) \otimes (H, \rho) \simeq (H_i, {}^{-}) \otimes (Q_i, {}^{-}) \tag{2}$$

with $H_i = (a_i d_0, d)$ and $Q_i = (a_i, b_i d)$. (This follows from a calculation of Clifford algebras or, more elementarily, from a suitable choice of base change.) We thus recover the decomposition in Corollary 3.3 of [10].

If A is split, hence $(A, \sigma) = \mathrm{Ad}_\psi$ for some 12-dimensional form ψ of trivial discriminant and Clifford invariant, then $H \simeq H'$, hence each term on the right side of (1) can be written as Ad_{π_i} for some 2-fold Pfister form π_i, and (1) yields

$$\psi \simeq \langle \alpha_1\rangle \pi_1 \perp \langle \alpha_2\rangle \pi_2 \perp \langle \alpha_3\rangle \pi_3 \tag{3}$$

for some α_1, α_2, $\alpha_3 \in F^\times$. We thus get a decomposition of ψ as in [8, p. 124]. Note moreover that each summand $(H', \rho_i) \otimes (H, \rho)$ becomes hyperbolic over $K = F(\sqrt{d})$, hence $\pi_i \simeq \langle\langle \beta_i, d\rangle\rangle$ for some $\beta_i \in F^\times$. Since $e_2(\psi) = e_2(\pi_1) + e_2(\pi_2) + e_3(\pi_3) = 0$, we may assume $(\beta_1\beta_2\beta_3) = 0$. Equation (3) can be rewritten as

$$\psi \simeq (\langle \alpha_1\rangle\langle\langle \beta_1\rangle\rangle \perp \langle \alpha_2\rangle\langle\langle \beta_2\rangle\rangle \perp \langle \alpha_3\rangle\langle\langle \beta_3\rangle\rangle) \otimes \langle\langle d\rangle\rangle \qquad \text{with} \quad (\beta_1\beta_2\beta_3) = 0. \tag{4}$$

2. Applications

2.1. **Existence of orthogonal involutions with trivial invariants.** As a corollary of Theorem 1.3, we characterize the biquaternion algebras D such that $M_3(D)$ carries an orthogonal involution with trivial discriminant and Clifford invariant.

Corollary 2.1. *Let D be a biquaternion F-algebra. There exists an orthogonal involution on $M_3(D)$ having trivial discriminant and trivial Clifford invariant if and only if D admits a decomposition into quaternion algebras $D = H' \otimes H$ such that the reduced norm $n_{H'}$ and the pure subform n_H^0 of the reduced norm n_H (i.e., its restriction to the pure quaternions) have a common nonzero value.*

If $I^3F = 0$, this condition holds for every biquaternion F-algebra D.

Proof. Assume first there exists an orthogonal involution σ on $M_3(D)$ which has trivial discriminant and trivial Clifford invariant. The algebra with involution $(M_3(D), \sigma)$ admits a decomposition as in Theorem 1.3, with $A_0 = M_3(H')$ for some quaternion algebra H'. Consider the discriminant d_0 of the involution σ_0. We have $d_0 = -\mathrm{Nrd}_{M_3(H')}(s)$, where $s \in M_3(H')$ is any invertible skew-symmetric element, hence d_0 is a value of $n_{H'}$ by [5, Lemma 2.6.4]. In addition, $d_0 = j^2$ for some pure quaternion $j \in H = (d, d_0)$. Therefore $d_0 = -n_H^0(j)$, so the quadratic forms $n_{H'}$ and n_H^0 share $-d_0$ as a common nonzero value.

To prove the converse, assume $D = H' \otimes H$ for some quaternion algebras H and H', such that there exists a quaternion $q \in H'$ and a pure quaternion $j \in H$ satisfying $n_{H'}(q) = n_H(j) \neq 0$. Let $d_0 = j^2 = -n_{H'}(q)$, and let $H'_0 \subset H'$ be the vector subspace of pure quaternions. Pick an arbitrary invertible $q_3 \in H'_0$. The vector space $qq_3^{-1}H'_0 \subset H'$ has dimension 3, hence

$$\dim(qq_3^{-1}H'_0 \cap H'_0) \geq 2.$$

Since $\dim H'_0 = 3$, the Witt index of $n^0_{H'}$ is at most 1, hence $qq_3^{-1}H'_0 \cap H'_0$ contains anisotropic vectors. Therefore, there exist q_1, $q_2 \in H'_0$ invertible such that $qq_3^{-1}q_2^{-1} = q_1$, i.e., $q = q_1q_2q_3$. Then $d_0 = -n_{H'}(q)$ is the discriminant of the involution adjoint to the skew-hermitian form $h = \langle q_1, q_2, q_3\rangle$ over $(H', \bar{\ })$. Pick a pure quaternion i which anticommutes with j, and define $\rho = \mathrm{Int}(i) \circ \bar{\ }$. We get that $H = (d, d_0)$, where $d = i^2 = -n_H(i)$ is the discriminant of the orthogonal involution ρ on H. The involution $\sigma = \mathrm{ad}_h \otimes \rho$ on $M_3(H') \otimes H = M_3(D)$ satisfies

$$(M_3(D), \sigma) = (M_3(H'), \mathrm{ad}_h) \otimes (H, \rho) \qquad \text{with} \quad [H] = (d, d_0).$$

Therefore, Theorem 1.3 shows that $e_1(\sigma) = e_2(\sigma) = 0$.

If $I^3(F) = 0$, then the reduced norm form of every quaternion algebra represents every nonzero element in F, hence the condition holds for every biquaternion F-algebra D. □

Example 2.2. Let F_0 be an arbitrary field of characteristic different from 2, and let $F = F_0(\!(x_1)\!)(\!(y_1)\!)(\!(x_2)\!)(\!(y_2)\!)$ be the field of iterated Laurent series in four variables over F_0. The biquaternion algebra $D = (x_1, y_1) \otimes (x_2, y_2)$ carries a unique valuation v extending the $(x_1, \dots, y_2)$-adic valuation on F, and it is totally ramified over F. We claim that $M_3(D)$ does not carry any orthogonal involution with trivial discriminant and trivial Clifford invariant. To see this as a consequence of Corollary 2.1, consider a decomposition $D = H' \otimes H$ into quaternion subalgebras. Let Γ_D, $\Gamma_{H'}$, Γ_H, Γ_F be the value groups of D, H', H, F for the valuation v, so $\Gamma_F = \mathbb{Z}^4$ and $\Gamma_D = (\frac{1}{2}\mathbb{Z})^4$. By [14, Cor. 8.11] we have $\Gamma_D/\Gamma_F = (\Gamma_{H'}/\Gamma_F) \oplus (\Gamma_H/\Gamma_F)$, hence $\Gamma_{H'} \cap \Gamma_H = \Gamma_F$. For $x \in H'^{\times}$ we have $v(x) = \frac{1}{2}v\big(n_{H'}(x)\big)$ by [14, Th. 1.4], hence $v\big(n_{H'}(x)\big) \in 2\Gamma_{H'}$. Similarly, $v\big(n_H(y)\big) \in 2\Gamma_H$ for $y \in H^{\times}$. But the valuation on H is an "armature gauge" as defined on [14, p. 339], which means that for every standard quaternion basis 1, i, j, k of H and $\lambda_0, \dots, \lambda_3 \in F$

$$v(\lambda_0 + \lambda_1 i + \lambda_2 j + \lambda_3 k) = \min\{v(\lambda_0), v(\lambda_1 i), v(\lambda_2 j), v(\lambda_3 k)\}.$$

Since H is totally ramified over F, $v(1)$, $v(i)$, $v(j)$, and $v(k)$ are in different cosets of Γ_D modulo Γ_F. Therefore, if $y \in H^{\times}$ is a pure quaternion, then $v(y) \notin \Gamma_F$, hence $v\big(n_H(y)\big) \in 2\Gamma_H \setminus 2\Gamma_F$. In conclusion, it is impossible to find $x \in H'^{\times}$ and $y \in H_0$ such that $n_{H'}(x) = n_H(y)$, because $2\Gamma_{H'} \cap 2\Gamma_H = 2\Gamma_F$.

2.2. **A formula for the f_3-invariant.** In the situation of Theorem 1.3, the algebras H and A_0 occurring in the decomposition of (A, σ) with $e_1(\sigma) = e_2(\sigma) = 0$ are not uniquely determined, even up to Brauer-equivalence. Take for instance an arbitrary quaternion algebra $H = (d, d_0)$ with an orthogonal involution ρ of discriminant (d). As $-d_0$ is represented by the reduced norm form n_H, we may argue as in the proof of Corollary 2.1 to find pure quaternions q_1, q_2, $q_3 \in H$ such that $n_H(q_1q_2q_3) = -d_0$. On $A_0 = M_3(H)$, the orthogonal involution σ_0 adjoint to the skew-hermitian form $\langle q_1, q_2, q_3\rangle$ has discriminant (d_0). Therefore, $(A, \sigma) = (A_0, \sigma_0) \otimes (H, \rho)$ satisfies the conditions of Theorem 1.3. But A is split since A_0 and H_0 are Brauer-equivalent, hence $(A, \sigma) \simeq \mathrm{Ad}_\psi$ for some 12-dimensional quadratic form ψ with $e_1(\psi) = e_2(\psi) = 0$. By Pfister's result (see (4)), there is a decomposition $\psi \simeq \psi_0 \otimes \beta$ for some 6-dimensional form ψ_0 with $e_1(\psi_0) = 0$ and some 2-dimensional form β, hence another decomposition of (A, σ) as in Theorem 1.3:

$$(A_0, \sigma_0) \otimes (H, \rho) = (A, \sigma) \simeq \mathrm{Ad}_{\psi_0} \otimes \mathrm{Ad}_\beta.$$

Nevertheless, we show in this section that the invariant $f_3(\sigma)$ defined in [10, Def. 2.4] can be calculated from any decomposition as in Theorem 1.3, and can thus yield some information on the possible decompositions. The main ingredient of the proof is Theorem 5.4 in [10], which shows that $f_3(\sigma)$ is the Arason invariant of the sum of the norm forms of all quaternion algebras in a given decomposition group of (A, σ). Since the f_3 invariant is defined only when the underlying central simple algebra carries a hyperbolic involution, we need to assume in the following statement, which is the main result of this section, that the index of A is at most 2.

Theorem 2.3. *Let (A,σ) be a central simple algebra of degree 12 and index ≤ 2 with an orthogonal involution with trivial discriminant and trivial Clifford invariant. Pick a decomposition of (A,σ) as in Theorem 1.3,*

$$(A,\sigma) \simeq (A_0,\sigma_0) \otimes (H,\rho)$$

where (A_0,σ_0) is a central simple algebra with orthogonal involution of degree 6 and (H,ρ) is a quaternion algebra with orthogonal involution, and $H=(d,d_0)$ with $e_1(\rho)=d$ and $e_1(\sigma_0)=d_0$. Let Q and H' be the quaternion algebras that are Brauer-equivalent to A and A_0 respectively, and let n_Q, $n_{H'}$, n_H be the reduced norm forms of Q, H' and H respectively. With this notation,

$$f_3(\sigma) = e_3(n_Q - n_H - \langle d\rangle n_{H'}) \in H^3(F). \tag{5}$$

(Note that $n_Q - n_H - \langle d\rangle n_{H'} \in I^3F$ because $[Q]+[H]+[H']=0$.) Moreover, if $c\in F^\times$ is such that H, H' and Q are all split by $F(\sqrt{c})$, and $e\in F^\times$ is such that $H=(c,e)$, then

$$f_3(\sigma) = (de)\cdot[Q] = (de)\cdot[H']. \tag{6}$$

Proof. Consider the additive decomposition of (A,σ) in (1). Together with (2), it shows that

$$\{0,\ [Q],\ [Q_1],\ [H_1],\ [Q_2],\ [H_2],\ [Q_3],\ [H_3]\}$$

is a decomposition group of (A,σ) as defined in [10, Def. 3.6]. As a result, Theorem 5.4 in [10] yields

$$f_3(\sigma) = e_3\big(n_Q + \sum_{i=1}^{3} n_{H_i} + \sum_{i=1}^{3} n_{Q_i}\big).$$

In order to compute the Arason invariant of this quadratic form, we use the following identity in the Witt group of F:

$$\langle\langle \lambda,\mu\nu\rangle\rangle = \langle\langle \lambda,\mu\rangle\rangle + \langle\mu\rangle\langle\langle \lambda,\nu\rangle\rangle.$$

In particular, it shows that for $i=1$, 2, and 3, we have

$$n_{H_i} = \langle\langle a_i,d\rangle\rangle + \langle a_i\rangle n_H \quad\text{and}\quad n_{Q_i} = \langle\langle a_i,d\rangle\rangle + \langle d\rangle n_{H'}.$$

Therefore,

$$\sum_{i=1}^{3} n_{H_i} + \sum_{i=1}^{3} n_{Q_i} = \langle a_1,a_2,a_3\rangle n_H + \langle d,d,d\rangle n_{H'} + \sum_{i=1}^{3}\langle\langle -1,a_i,d\rangle\rangle. \tag{7}$$

Recall that $(d_0)=(a_1a_2a_3)$, hence

$$\langle a_1,a_2,a_3\rangle n_H \equiv \langle -d_0\rangle n_H \bmod I^4F.$$

Similarly,

$$\langle d,d,d\rangle n_{H'} \equiv \langle -d\rangle n_{H'} \bmod I^4F.$$

Therefore, (7) yields

$$\begin{aligned} e_3\big(n_Q + \sum_{i=1}^{3} n_{H_i} + \sum_{i=1}^{3} n_{Q_i}\big) &= e_3(n_Q - \langle d_0\rangle n_H - \langle d\rangle n_{H'}) + \sum_{i=1}^{3}(-1,a_i,d) \\ &= e_3(n_Q - n_H - \langle d\rangle n_{H'}) + (d_0)\cdot[H] + (-1,d_0,d). \end{aligned}$$

Now, since $H=(d,d_0)$ and $(d_0,d_0)=(-1,d_0)$, the last two terms on the right side of the last displayed equation cancel, and Formula (5) is proved.

To obtain Formula (6), choose $c\in F^\times$ such that $F(\sqrt{c})$ splits Q, H, and H', and let e, $e'\in F^\times$ be such that $H=(c,e)$ and $H'=(c,e')$, hence $Q=(c,ee')$. Then

$$\begin{aligned} n_Q - n_H - \langle d\rangle n_{H'} &= \langle\langle c,ee'\rangle\rangle - \langle\langle c,e\rangle\rangle - \langle d\rangle\langle\langle c,e'\rangle\rangle \\ &= \langle\langle c\rangle\rangle\langle e,-ee',-d,de'\rangle \\ &= \langle e\rangle\langle\langle c,e',de\rangle\rangle. \end{aligned}$$

Therefore, $f_3(\sigma) = (c, e', de) = (de) \cdot [H']$. As $H = (c, e) = (d, d_0)$, we have
$$(d) \cdot [H] = (-1) \cdot [H] = (e) \cdot [H],$$
hence $(de) \cdot [H] = 0$ and $(de) \cdot [H'] = (de) \cdot [Q]$. Formula (6) is thus proved. □

Corollary 2.4. *With the notation of Theorem 2.3, we have $f_3(\sigma) = 0$ if any of the following conditions holds:*

(i) *A is split;*
(ii) *A_0 is split;*
(iii) *A_0 is split by $F(\sqrt{d_0})$.*

Proof. Formula (6) readily shows that $f_3(\sigma) = 0$ when (i) or (ii) holds. In case (iii) we may take $c = d_0$ and $e = d$ in Formula (6) to obtain $f_3(\sigma) = 0$.

Alternatively, in case (i) we may argue that $(A, \sigma) = \mathrm{Ad}_\psi$ for some quadratic form $\psi \in I^3F$, hence $e_3(\sigma) = e_3(\psi) \in H^3(F, \mu_2)$ and therefore $f_3(\sigma) = 0$ by definition. Also, in case (ii) (A, σ) is split and hyperbolic over $F(\sqrt{d})$, hence $f_3(\sigma) = 0$ by [10, Prop. 5.6]. □

By contrast, $f_3(\sigma)$ does not necessarily vanish when H is split. In that case we may choose $e = 1$ in Formula (6) and derive the following: if $(A, \sigma) = (A_0, \sigma_0) \otimes \mathrm{Ad}_{\langle\langle d\rangle\rangle}$ and $e_1(\sigma_0) = (d_0)$ is such that (d, d_0) is split, then
$$f_3(\sigma) = (d) \cdot [A_0].$$
This also follows from [10, Cor. 2.18].

References

[1] E. Bayer-Fluckiger, D. Shapiro and J.-P. Tignol, Hyperbolic involutions, Math Z **214** (1993), no. 3, 461–476.
[2] V. Chernousov and A. Merkurjev, Essential dimension of spinor and Clifford groups. Algebra Number Theory **8** (2014), no. 2, 457–472.
[3] S. Garibaldi, *Cohomological invariants: exceptional groups and spin groups.* Mem. Amer. Math. Soc. 200 (2009), no. 937, xii+81 pp.
[4] S. Garibaldi and A. Quéguiner-Mathieu, Pfister's theorem for orthogonal involutions of degree 12, Proc. Amer. Math. Soc. **137** (2009), no. 4, 1215–1222.
[5] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, second edition, Cambridge Studies in Advanced Mathematics, 165, Cambridge University Press, Cambridge, 2017.
[6] B. Kahn, *Formes quadratiques sur un corps.* Cours Spécialisés, **15**. Société Mathématique de France, Paris, 2008. x+303 pp.
[7] M.-A. Knus et al., *The book of involutions*, Amer. Math. Soc., Providence, RI, 1998.
[8] A. Pfister, Quadratische Formen in beliebigen Körpern, Invent. Math. **1** (1966), 116–132.
[9] A. Quéguiner-Mathieu and J.-P. Tignol, Discriminant and Clifford algebras, Math. Z. **240** (2002), no. 2, 345–384.
[10] A. Quéguiner-Mathieu and J.-P. Tignol, The Arason invariant of orthogonal involutions of degree 12 and 8, and quaternionic subgroups of the Brauer group. Doc. Math. 2015, Extra vol.: Alexander S. Merkurjev's sixtieth birthday, 529–576.
[11] A. Quéguiner-Mathieu and J.-P. Tignol, Outer automorphisms of classical algebraic groups, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no. 1, 113–141.
[12] M. Rost, On 14-dimensional quadratic forms, their spinors, and the difference of two octonion algebras, March 1999, unpublished manuscript.
[13] D. Tao, The generalized even Clifford algebra, J. Algebra **172** (1995), no. 1, 184–204.
[14] J.-P. Tignol and A. R. Wadsworth, *Value functions on simple algebras, and associated graded rings*, Springer Monographs in Mathematics, Springer, Cham, 2015.
[15] A. Weil, Algebras with involutions and the classical groups, J. Indian Math. Soc. **24** (1961) 589–623.

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