Computation of the maximal invariant set of linear systems with quasi-smooth nonlinear constraints

Zheming Wang, Raphaël M. Jungers and Chong-Jin Ong

Abstract— **In this paper, we consider the problem of computing the maximal invariant set of linear systems with a class of nonlinear constraints that admit quadratic relaxations. With these quadratic relaxations, we are able to determine a sufficient condition on the maximal invariant set. Using the sufficient condition, a new algorithm is presented by solving a set of linear matrix inequalities. Under mild assumptions, the proposed algorithm will terminate in finite time. The performance of this algorithm is demonstrated on several numerical examples.**

I. INTRODUCTION

Invariant set theory is an important tool for stability analysis and control design of constrained dynamical systems and it has been successfully used to solve various problems in system and control; see, for instance, [1]–[3] and the references therein. An invariant set is a region such that all trajectories generated by the dynamical system remain in the set if their initial states lie within it. One well-known application is in Model Predictive Control (MPC) [4], where invariant sets are often used to ensure recursive feasibility and stability.

Considerable research has been devoted to the characterization and computation of invariant sets of constrained systems. Recursive algorithms have been provided in [5]–[7] to compute polyhedral invariant sets of linear systems. For disturbed linear systems, robust invariant sets are introduced and different algorithms have been proposed for computing these sets in [8]–[13]. Methods to characterize and compute invariant sets of nonlinear systems are also available in the literature [14]–[20]. The study of invariant sets can also be extended to hybrid systems. For instance, the works [21]–[26] have investigated the computation of invariant sets of switching systems. Among various invariant sets, the maximal invariant set is of particular interest. The standard algorithm for computing the maximal invariant set of linear systems subject to polyhedral constraints is presented in [5], [8] with sufficient conditions for finite determinability. For switching linear systems, algorithms to compute the maximal invariant set are also provided in the cases of polyhedral/convex constraints [21], [25], [26] and semialgebraic constraints [23], [24]. In [19], an infinite-dimensional convex characterization of the maximal invariant is derived for polynomial systems with semialgebraic constraints. By solving finite-dimensional relaxations, outer approximations of the maximal invariant can be obtained. However, in the case of general nonlinear constraints, computing the maximal invariant set is still a challenging problem, even for linear systems. In this paper, we aim to compute the exact maximal invariant set of linear systems with a class of nonlinear constraints. A new approach will be proposed by solving a set of Linear Matrix Inequalities (LMI), which are constructed by the use of the S-procedure [27]. Based on the solution of these LMIs, a sufficient condition for the maximal invariant set can be established. The tightness of the sufficient condition largely depends on the conservatism of the S-procedure [28]. Under mild assumptions, finite determinability can be guaranteed with the proposed sufficient condition.

The rest of the paper is organized as follows. This section ends with the notations, followed by the next section on the review of preliminary results on the invariant sets of linear systems. Section III presents the proposed approach for computing the maximal invariant set of linear systems with nonlinear constraints. Several numerical examples are provided Section IV. The last section concludes the work. Some of the proofs are not given due to the page limitation.

The notations used in this paper are as follows. Nonnegative and positive integer sets are indicated respectively by $\mathbb{Z}_0^+$ and $\mathbb{Z}^+$ with $\mathbb{Z}^M := \{1, 2, \cdots, M\}$ and $\mathbb{Z}_L^M := \{L, L+1, \cdots, M\}, M \geq L,\ M, L \in \mathbb{Z}_0^+$. $\mathbb{S}^n$ denotes the set of symmetric matrices in $\mathbb{R}^{n\times n}$. I_n (the subscription is omitted when the dimension is clear from the context) is the $n \times n$ identity matrix and $\mathbf{1}_n$ denote the vector of n ones. For a square matrix Q, $Q \succ (\succeq) 0$ means Q is positive definite (semi-definite). The p-norm of $x \in \mathbb{R}^n$ is $\|x\|_p$ while $\|x\|_Q^2 = x^T Q x$ for $Q \succeq 0$. Additional notations are introduced as required in the text.

II. PRELIMINARIES

This section reviews some known results on the invariant sets of constrained discrete-time linear systems. We consider the linear system

$$x(t+1) = Ax(t), \quad \forall t \in \mathbb{Z}_0^+, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector. The system is subject to state constraints: $x(t) \in \Omega$, where $\Omega \subseteq \mathbb{R}^n$ is a quadratic

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set in the form of

$$\Omega = \{x \in \mathbb{R}^n : x^T Q_i x + 2q_i^T x \leq 1, i \in \mathbb{Z}^p\} \tag{2}$$

where $Q_i \in \mathbb{S}^n$, $q_i \in \mathbb{R}^n$ and p is the number of constraints. When $Q_i = 0$, for all $i \in \mathbb{Z}^p$, Ω becomes a polyhedron. More generally, other nonlinear constraints may also be imposed on the system:

$$x(t) \in \Theta := \{x \in \mathbb{R}^n : H_i(x) \leq 0, i \in \mathbb{Z}^m\}, \forall t \in \mathbb{Z}_0^+ \tag{3}$$

where $H_i : \mathbb{R}^n \to \mathbb{R}$ is a continuous nonlinear function and $m \in \mathbb{Z}^+$ is the number of other nonlinear constraints. The actual state constraint set is the intersection of Ω and Θ:

$$x(t) \in X := \Omega \bigcap \Theta, \forall t \in \mathbb{Z}_0^+ \tag{4}$$

For computational reasons, we treat quadratic constraints and general nonlinear constraints differently. The following assumptions are made.

Assumption 1: The matrix A is Schur stable, i.e., for any eigenvalue λ of A, $|\lambda|$ is smaller than one

Assumption 2: The set Ω is compact and contains the origin in its interior. There exists an open ball $\mathcal{B}$ around the origin and $\epsilon > 0$ such that $H_i(x) \leq -\epsilon$ for all $x \in \mathcal{B}$ and $i \in \mathbb{Z}^m$.

Assumption 3: For all $i \in \mathbb{Z}^m$, $H_i : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and there exist a vector $H_i^\nabla \in \mathbb{R}^n$ and a scalar $L_i \geq 0$ such that

$$|H_i(x) - H_i(0) - (H_i^\nabla)^T x| \leq \frac{L_i}{2}\|x\|^2 \tag{5}$$

for all $x \in \Omega$.

Assumptions 1 and 2 are standard and necessary for the problem to be well-defined, see [5], [8]. We will refer to a function satisfying (5) as a quasi-smooth function. Clearly, for functions with Lipschitz continuous gradient, the condition in Assumption 3 will be satisfied. Suppose that, for any $i \in \mathbb{Z}^m$, H_i is a continuously differentiable function with Lipschitz gradient:

$$\|\nabla H_i(x) - \nabla H_i(y)\| \leq L_i\|x - y\|, \forall x, y \in \Omega, \tag{6}$$

Assumption 3 is satisfied with $H_i^\nabla = \nabla H_i(0)$, see, e.g., Lemma 6.9.1 in [29]. For notational simplicity, let

$$q := [q_1 \ q_2 \ \cdots \ q_p] \tag{7}$$

$$H(x) := (H_1(x), H_2(x), \cdots, H_m(x)), \tag{8}$$

where $q \in \mathbb{R}^{n \times p}$ and $H(x) \in \mathbb{R}^m$.

We will define the central topic of this paper.

Definition 1: [2], [4] The nonempty set $Z \subseteq X$ is a CA-invariant (constraint admissible invariant) set for system (1) if and only if for any $x \in Z$ one has that $Ax \in Z$.

With Assumptions 1 and 2, there often exist multiple CA-invariant sets. In many applications, it is desirable to compute the maximal CA-invariant set, which is defined below.

Definition 2: [5] The nonempty set O_∞ is the maximal CA-invariant set for system (1) if and only if O_∞ is a CA-invariant set and contains all CA-invariant sets in X.

It is a standard result that the maximal CA-invariant set exists (see [5] for general conditions guaranteeing its existence), and that it can be computed recursively by the following iteration:

$$O_0 := X \tag{9}$$

$$O_{k+1} := O_k \bigcap \{x \in \mathbb{R}^n : Ax \in O_k\}, k \in \mathbb{Z}_0^+. \tag{10}$$

With these iterates, it is easy to verify that

$$O_k = \{x \in \mathbb{R}^n : A^\ell x \in X, \ell \in \mathbb{Z}_0^k\}. \tag{11}$$

Thus, the maximal CA-invariant set can be expressed as

$$O_\infty := \bigcap_{k \in \mathbb{Z}_0^+} O_k = \{x \in \mathbb{R}^n : A^k x \in X, k \in \mathbb{Z}_0^+\}. \tag{12}$$

From Assumptions 1 and 2, the set O_∞ defined in (12) has the following properties [5]: (i) if $Z \subseteq \mathbb{R}^n$ is a CA-invariant set of system (1), $Z \subseteq O_\infty$; (ii) there exists a finite k^* such that $O_{k^*+1} = O_{k^*}$; (iii) for any k^* satisfying (ii), it can be shown that $O_k = O_{k^*}$ for all $k \geq k^*$ and $O_\infty = O_{k^*}$. From the these properties, the problem of computing O_∞ boils down to the search for a k^* such that $O_{k^*+1} = O_{k^*}$. The standard procedure is to increase k from 0 until $O_{k+1} = O_k$, which is equivalent to

$$O_k \subseteq \{x \in \mathbb{R}^n : A^{k+1} x \in X\}, \tag{13}$$

see [5] for details. This condition can be treated as a stopping criterion for the algorithm in (9)-(10). Observe that $\{x \in \mathbb{R}^n : A^{k+1}x \in X\}$ can be rewritten as $\{x \in \mathbb{R}^n : (A^{k+1}x)^T Q_i A^{k+1} x + 2q_i^T A^{k+1} x \leq 1 i \in \mathbb{Z}^p, H(A^{k+1}x) \leq 0\}, \forall k \in \mathbb{Z}_0^+$. During the computation procedure, we aim to find the minimal k satisfies (13). Let $k_{\min} := \arg\min_{k \in \mathbb{Z}_0^+}\{k : (13)\}$. O_∞ can be determined for any upper bound on $k_{\min}$. To evaluate (13), we basically need to solve a set of nonlinear optimization problems. For general nonlinear constraints in (4), these problems are nonconvex and it is difficult to reach the global optimality. For this reason, we will aim to develop a sufficient condition for (13) without solving nonconvex problems.

III. The Proposed Approach

This section discusses the computation of the exact maximal CA-invariant set with nonlinear constraints. An algorithm will be presented to compute an upper bound on $k_{\min}$ and its finite determinability can be ensured under mild assumptions.

For the quadratic (or linear) constraints, the following nonlinear optimization problem is defined at the k^{th} iteration of (10):

$$g_i^k := \max_x (A^{k+1}x)^T Q_i A^{k+1} x + 2q_i^T A^{k+1} x - 1 \tag{14a}$$

$$\text{s.t.} \quad x \in O_k \tag{14b}$$

for $i \in \mathbb{Z}^p$. Let $g_{\max}^k := \max_{i \in \mathbb{Z}^p} g_i^k$ for all $k \in \mathbb{Z}_0^+$. If $g_{\max}^k \leq 0$ for some $k \in \mathbb{Z}_0^+$, $O_k \subseteq \{x \in \mathbb{R}^n : (A^{k+1}x)^T Q_i A^{k+1} x + 2q_i^T x \leq 1, i \in \mathbb{Z}^p\}$. Similarly, for other nonlinear constraints, the following nonlinear optimization problem is defined at the k^{th} iteration of (10):

$$h_i^k := \max_x H_i(A^{k+1}x) \tag{15a}$$
$$\text{s.t.} \quad x \in O_k \tag{15b}$$

for $i \in \mathbb{Z}^m$. Let $h_{\max}^k := \max_{i\in\mathbb{Z}^m} h_i^k$ for all $k \in \mathbb{Z}_0^+$. If $h_{\max}^k \le 0$ for some $k \in \mathbb{Z}_0^+$, $O_k \subseteq \{x \in \mathbb{R}^n : H(A^{k+1}x) \le 0\}$. Using (14) and (15), $k_{\min}$ can be determined via $\min_{k\in\mathbb{Z}_0^+}\{k : g_{\max}^k \le 0, h_{\max}^k \le 0\}$. To do so, we need in principle to solve (14) and (15) and get their global optimal solutions. However, for general nonlinear constraints, both (14) and (15) are nonlinear nonconvex problems. Even if Ω and Θ are convex sets, (14) and (15) are may not be convex problems. Therefore, we do not attempt to solve (14) and (15) directly. Instead, we will solve LMI problems that provide upper bounds on their optimal values and obtain an upper bound on $k_{\min}$.

A. Quadratic constraints

Before we discuss general nonlinear functions, let us first focus on quadratic functions. In this case, we assume that $\Theta = \mathbb{R}^n$ and $X = \Omega$. For notational convenience, let

$$\bar{Q}_i^\ell := \begin{pmatrix} (A^\ell)^T Q_i A^\ell & (A^\ell)^T q_i \\ q_i^T A^\ell & -1 \end{pmatrix} \tag{16}$$

for all $i \in \mathbb{Z}^p$ and $\ell \in \mathbb{Z}_0^+$. Following the definition above, we can see that

$$\bar{Q}_i^{\ell+1} := \begin{pmatrix} A^T & 0 \\ 0 & 1 \end{pmatrix} \bar{Q}_i^\ell \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \tag{17}$$

for all $i \in \mathbb{Z}^p$ and $\ell \in \mathbb{Z}_0^+$. Using the notations above, O_k at the k^{th} iteration can be rewritten as

$$\{x \in \mathbb{R}^n : \begin{pmatrix} x \\ 1 \end{pmatrix}^T \bar{Q}_i^\ell \begin{pmatrix} x \\ 1 \end{pmatrix} \le 0, i \in \mathbb{Z}^p, \ell \in \mathbb{Z}_0^k\} \tag{18}$$

From the S-procedure, see Section 2.6.3 in [27], the following lemma can be obtained.

Lemma 1: Suppose $\Theta = \mathbb{R}^n$ and $X = \Omega$. Let $\bar{Q}_i^k$ be defined in (16) and the set O_k be defined by the procedure in (9)-(10) for all $k \in \mathbb{Z}_0^+$. For any $i \in \mathbb{Z}^p$, if there exists a non-negative sequence $\{\tau_{(j,\ell+1)}^i \ge 0, j \in \mathbb{Z}^p, \ell \in \mathbb{Z}_0^k\}$ for some $k \in \mathbb{Z}_0^+$ such that

$$\bar{Q}_i^{k+1} \preceq \sum_{j=1}^{p}\sum_{\ell=0}^{k} \tau_{(j,\ell+1)}^i \bar{Q}_j^\ell, \tag{19}$$

then, $(A^{k+1}x)^T Q_i A^{k+1}x + 2q_i^T A^{k+1}x - 1 \le 0$ for any $x \in O_k$.

As we have seen, under Assumptions 1 and 2, the formal algorithm described in (9)-(10) always terminate in finite time. This algorithm is easily implementable when X is a polyhedron, see [2], [5]. In many cases, it is not directly implementable because of the nonlinearity in X. Even if X is convex, the optimization problem (14) is still non-convex and it is difficult to find the global optimum. However, the same algorithm with (19) would be implementable, since these inequalities are LMIs, which can be efficiently solved using interior point methods [27]. To recover the nice finiteness property of the former algorithm, an additional assumption is needed.

Assumption 4: There exists $D_x > 0$ such that $\|x\|^2 \le D_x$ for all $x \in \Omega$.

This assumption is made completely without loss of generality for the compact set Ω as we can always add the redundant ball constraint of the form $\|x\|^2 \le D_x$ to Ω. With this additional assumption, we can let $Q_1 = \frac{1}{D_x}I$ and $q_1 = 0$ in (2). We now show the finiteness property of the former algorithm still holds for the LMI version.

Lemma 2: Suppose Assumptions 1, 2 and 4 hold, $\Theta = \mathbb{R}^n$, and $X = \Omega$. Let $Q_1 = \frac{1}{D_x}I$ and $q_1 = 0$ in (2). For any $i \in \mathbb{Z}^p$, there always exists some finite k such that the LMI (19) holds for some non-negative sequence $\{\tau_{(j,\ell+1)}^i \ge 0, j \in \mathbb{Z}^p, \ell \in \mathbb{Z}_0^k\}$.

Proof of Lemma 2: Since $Q_1 = \frac{1}{D_x}I$ and $q_1 = 0$, for any $k \in \mathbb{Z}_0^+$, one choice of the sequence $\{\tau_{(j,\ell+1)}^i \ge 0, j \in \mathbb{Z}^p, \ell \in \mathbb{Z}_0^k\}$ in (19) can be given as:

$$\tau_{(1,1)}^i = \beta, \tau_{(j,\ell+1)}^i = 0, \forall (j,\ell) \ne (1,0) \tag{20}$$

for some $0 < \beta < 1$. With this choice, the LMI (19) reduces to

$$\begin{pmatrix} (A^{k+1})^T Q_i A^{k+1} & (A^{k+1})^T q_i \\ q_i^T A^{k+1} & -1 \end{pmatrix} \preceq \beta \begin{pmatrix} \frac{1}{D_x}I & 0 \\ 0 & -1 \end{pmatrix} \tag{21}$$

From Assumption 1, A^{k+1} goes to 0 as k increases. Hence, there always exists a k such that (21) holds. $\square$

Based on Lemma 2, the following LMI optimization problem is defined for all $i \in \mathbb{Z}^p$ and $k \in \mathbb{Z}_0^+$:

$$r_i^k := \min_{r,\tau^i\in\mathbb{R}^{p\times(k+1)}} r \tag{22a}$$
$$s.t. \quad \tau^i \ge 0, \tag{22b}$$
$$\bar{Q}_i^{k+1} \preceq \sum_{j=1}^{p}\sum_{\ell=0}^{k} \tau_{(j,\ell+1)}^i \bar{Q}_j^\ell + rI \tag{22c}$$

where τ^i denote the matrix expression of the sequence $\{\tau_{(j,\ell+1)}^i \ge 0, j \in \mathbb{Z}^p, \ell \in \mathbb{Z}_0^k\}$, i.e., the $(j,\ell+1)^{th}$ entry of τ^i is $\tau_{(j,\ell+1)}^i$. The properties of this LMI problem are stated in the following lemma.

Lemma 3: Suppose Assumptions 1, 2 and 4 hold, $\Theta = \mathbb{R}^n$, and $X = \Omega$. Let r_i^k be defined in (22) for all $i \in \mathbb{Z}^p$ and $k \in \mathbb{Z}_0^+$. Then, for all $i \in \mathbb{Z}^p$, there exists a finite $\bar{k}_i$ such that $r_i^{\bar{k}_i} \le 0$ and $r_i^k \le 0$ for all $k \ge \bar{k}_i$.

Proof of Lemma 3: Only a sketch of the proof is given. From Lemma 2, there always exist a finite $\bar{k}_i$ and a non-negative sequence $\{\bar{\tau}_{(j,\ell+1)}^i \ge 0, j \in \mathbb{Z}^p, \ell \in \mathbb{Z}_0^{\bar{k}_i}\}$ such that (19) holds. With this sequence, it is easy to verify that $(0, \bar{\tau}^i)$ is a feasible solution to (22). From the optimality, $r_i^{\bar{k}_i} \le 0$. To show that $r_i^k \le 0$ for all $k \ge \bar{k}_i$, we only need to show that $r_i^{\bar{k}_i} \le 0$ implies $r_i^{\bar{k}_i+1} \le 0$. Let $\hat{\tau}_{(j,\ell+1)}^i := \bar{\tau}_{(j,\ell)}^i$ for all $\ell \in \mathbb{Z}^{\bar{k}_i+1}$ and $\hat{\tau}_{(j,1)}^i := 0$ for all $j \in \mathbb{Z}^p$. It can be shown that $(0, \hat{\tau}^i)$ is a feasible solution to (22). From the optimality, we can get $r_i^{\bar{k}_i+1} \leq 0$ all $i \in \mathbb{Z}^p$. By induction, $r_i^k \leq 0$ for all $k \geq \bar{k}_i$ all $i \in \mathbb{Z}^p$. □

In the following theorem, we show that the LMI problem (22) can be used to establish a stopping criterion for the algorithm summarized in (9)-(10).

Theorem 1: Suppose Assumptions 1, 2 and 4 hold, $\Theta = \mathbb{R}^n$, and $X = \Omega$. Let the set O_k be defined by the procedure in (9)-(10) for all $k \in \mathbb{Z}_0^+$. For all $i \in \mathbb{Z}^p$ and $k \in \mathbb{Z}_0^+$, define r_i^k in (22) and let $r_{\max}^k := \max_{i \in \mathbb{Z}^p} r_i^k$. Then, there exists some finite k^* such that $r_{\max}^{k^*} \leq 0$ and $O_\infty = O_{k^*}$.

Based on the discussion above, the algorithm to compute the maximal CA-invariant set with quadratic constraints is summarized in the Algorithm 1.

Algorithm 1 Computation of the maximal CA-invariant set with quadratic constraints

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Input: A and $\{Q_i, q_i\}_{i=1}^p$
Output: O_{k^*}

1: *Initialization*: let $X := \{x \in \mathbb{R}^n : x^T Q_i x + 2q_i^T x \leq 1, i \in \mathbb{Z}^p\}$, and set $k = 0$ and $O_0 = X$;
2: Obtain r_i^k from (22) for all $i \in \mathbb{Z}^p$;
3: Let $r_{\max}^k := \max_{i \in \mathbb{Z}^p} r_i^k$. If $r_{\max}^k \leq 0$, let $k^* = k$ and terminate; otherwise, let $O_{k+1} := O_k \bigcap \{x \in \mathbb{R}^n : A^{k+1}x \in X\}$, set $k \leftarrow k+1$ and go to Step 2.

As (14) is not directly solved, the k^* obtained from Algorithm 1 is an upper bound on $k_{\min}$. For a loose upper bound k^*, the description of O_{k^*} may not be tight enough though it is still true that $O_{k^*} = O_\infty$. However, in some cases, k^* is not necessarily a loose upper bound. It can be close or equal to $k_{\min}$. One example is the case with only linear constraints, i.e., $\Theta = \mathbb{R}^n$ and $Q_i = 0$ for all $i \in \mathbb{Z}^p$. In the absence of quadratic and nonlinear constraints, the k^* obtained from Algorithm 1 is exactly equal to $k_{\min}$, as stated in the proposition below. In this case, Assumption 4 is not needed as the constraints are all linear.

Proposition 1: Suppose Assumption 1 holds, $\Theta = \mathbb{R}^n$ and $Q_i = 0$ for all $i \in \mathbb{Z}^p$. The constraint set X can be expressed as $\{x \in \mathbb{R}^n : 2q^T x \leq \mathbf{1}_p\}$ without any nonlinear constraint. Let $\{r_{\max}^k, O_k\}$ be generated by Algorithm 1. For any $k \in \mathbb{Z}_0^+$, $r_{\max}^k \leq 0$ if and only if $O_k \subseteq \{x \in \mathbb{R}^n : A^{k+1}x \in X\}$.

Proposition 1 suggests that the conservatism of k^* obtained from Algorithm 1 depends on the loss of the S-procedure in Lemma 1. If the LMI (19) is a necessary and sufficient condition of the set inclusion in (13), the S-procedure is lossless and k^* is exactly equal to $k_{\min}$. However, for general quadratic constraints, this is not true, see, e.g., [28]. More precisely, k^* can be larger than $k_{\min}$ in most of the cases. However, the size of the resulting O_∞ is not affected although there are redundant constraints in the description of the set.

B. Quasi-smooth nonlinear constraints

In the rest of this section, the proposed approach will be employed to handle general nonlinear constraints that satisfy Assumption 3. This is possible by making use of the quadratic upper and lower bounds in (5). With these quadratic bounds, we are able to establish the quadratic relaxations of (15) and apply the idea above. For notational simplicity, let

$$\begin{aligned} H_i^u(x) :&= H_i(0) + (H_i^\nabla)^T x + \frac{L_i}{2}\|x\|^2 \\ &= \begin{pmatrix} x \\ 1 \end{pmatrix}^T \begin{pmatrix} \frac{L_i}{2}I & \frac{1}{2}H_i^\nabla \\ \frac{1}{2}(H_i^\nabla)^T & H_i(0) \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \end{aligned} \tag{23}$$

$$\begin{aligned} H_i^l(x) :&= H_i(0) + (H_i^\nabla)^T x - \frac{L_i}{2}\|x\|^2 \\ &= \begin{pmatrix} x \\ 1 \end{pmatrix}^T \begin{pmatrix} -\frac{L_i}{2}I & \frac{1}{2}H_i^\nabla \\ \frac{1}{2}(H_i^\nabla)^T & H_i(0) \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \end{aligned} \tag{24}$$

for all $i \in \mathbb{Z}^m$. With the quadratic lower bounds above, a relaxed quadratic constraint set of O_k can be obtained for all $k \in \mathbb{Z}_0^+$:

$$\begin{aligned} \tilde{O}_k := \{x \in \mathbb{R}^n : \begin{pmatrix} x \\ 1 \end{pmatrix}^T \bar{Q}_i^\ell \begin{pmatrix} x \\ 1 \end{pmatrix} \leq 0, i \in \mathbb{Z}^p, \\ H^l(A^\ell x) \leq 0, \ell \in \mathbb{Z}_0^k\} \end{aligned} \tag{25}$$

where $H^l(A^\ell x) := (H_1^l(A^\ell x), H_2^l(A^\ell x), \cdots, H_m^l(A^\ell x))$. Based on this relaxed constraint set, a modification of (14) can be given by

$$\bar{g}_i^k := \max_x (A^{k+1}x)^T Q_i A^{k+1}x + 2q_i^T A^{k+1}x - 1 \tag{26a}$$

$$\text{s.t.} \quad x \in \tilde{O}_k \tag{26b}$$

for any $i \in \mathbb{Z}^p$ and $k \in \mathbb{Z}_0^+$. As $O_k \subseteq \tilde{O}_k$, $\bar{g}_i^k \geq g_i^k$ for all $i \in \mathbb{Z}^p$ and $k \in \mathbb{Z}_0^+$. Similarly, we can also modify (15) using the relaxed set. Since the cost function of (15) is also nonlinear, we will replace it by its quadratic upper bound (23). With the relaxed set and the quadratic upper bound of the cost function, the corresponding modification of (15) is given by

$$\bar{h}_i^k := \max_x H_i^u(A^{k+1}x) \tag{27a}$$

$$\text{s.t.} \quad x \in \tilde{O}_k \tag{27b}$$

for all $i \in \mathbb{Z}^m$. Again, we can see that $\bar{h}_i^k \geq h_i^k$ for all $i \in \mathbb{Z}^m$ and $k \in \mathbb{Z}_0^+$. Using the S-procedure, the following lemma can be obtained immediately.

Lemma 4: Suppose Assumption 3 holds. Let the set O_k be defined by the procedure in (9)-(10) and the relaxed quadratic set $\tilde{O}_k$ be defined in (25) using the quadratic lower bounds (24) for all $k \in \mathbb{Z}_0^+$. The following results hold.
(i) For any $i \in \mathbb{Z}^p$, if there exist some non-negative sequences $\{\tau_{(j,\ell+1)}^i \geq 0, j \in \mathbb{Z}^p, \ell \in \mathbb{Z}_0^k\}$ and $\{\pi_{(j,\ell+1)}^i \geq 0, j \in \mathbb{Z}^m, \ell \in \mathbb{Z}_0^k\}$ for some $k \in \mathbb{Z}_0^+$ such that

$$\begin{aligned} \bar{Q}_i^{k+1} \preceq &\sum_{\ell=0}^k \sum_{j=1}^p \tau_{(j,\ell+1)}^i \bar{Q}_j^\ell \\ &+ \sum_{\ell=0}^k \sum_{j=1}^m \pi_{(j,\ell+1)}^i \begin{pmatrix} -\frac{L_j}{2}(A^\ell)^T A^\ell & \frac{1}{2}(A^\ell)^T H_j^\nabla \\ \frac{1}{2}(H_j^\nabla)^T A^\ell & H_j(0) \end{pmatrix} \end{aligned} \tag{28}$$

holds, then, $(A^{k+1}x)^T Q_i A^{k+1}x + 2q_i^T A^{k+1}x - 1 \leq 0$ for any $x \in O_k$.
(ii) For any $i \in \mathbb{Z}^m$, if there exist some non-negative

sequences $\{\tau^i_{(j,\ell+1)} \geq 0, j \in \mathbb{Z}^p, \ell \in \mathbb{Z}^k_0\}$ and $\{\pi^i_{(j,\ell+1)} \geq 0, j \in \mathbb{Z}^m, \ell \in \mathbb{Z}^k_0\}$ for some $k \in \mathbb{Z}^+_0$ such that

$$\begin{pmatrix} \frac{L_i}{2}(A^{k+1})^T A^{k+1} & \frac{1}{2}(A^{k+1})^T H_i^\nabla \\ \frac{1}{2}(H_i^\nabla)^T A^{k+1} & H_i(0) \end{pmatrix} \preceq \sum_{\ell=0}^{k}\sum_{j=1}^{p} \tau^i_{(j,\ell+1)} \bar{Q}^\ell_j$$
$$+ \sum_{\ell=0}^{k}\sum_{j=1}^{m} \pi^i_{(j,\ell+1)} \begin{pmatrix} -\frac{L_j}{2}(A^\ell)^T A^\ell & \frac{1}{2}(A^\ell)^T H_j^\nabla \\ \frac{1}{2}(H_j^\nabla)^T A^\ell & H_j(0) \end{pmatrix} \tag{29}$$

holds, then, $H_i(A^{k+1}x) \leq 0$ for any $x \in O_k$.

From the lemma above, we can see that it is also possible to implement the formal algorithm in (9)-(10) using the LMIs in (28)-(29) for general nonlinear constraints that satisfy Assumption 3. The finiteness of the algorithm is discussed in the next lemma.

Lemma 5: Suppose Assumptions 1-4 hold, the relaxed quadratic set $\tilde{O}_k$ is defined in (25) using the quadratic lower bounds (24) for all $k \in \mathbb{Z}^+_0$. Then, the following results hold.
(i) For any $i \in \mathbb{Z}^p$, there always exists some finite k such that (28) holds for some non-negative sequences $\{\tau^i_{(j,\ell+1)} \geq 0, j \in \mathbb{Z}^p, \ell \in \mathbb{Z}^k_0\}$ and $\{\pi^i_{(j,\ell+1)} \geq 0, j \in \mathbb{Z}^m, \ell \in \mathbb{Z}^k_0\}$.
(ii) For any $i \in \mathbb{Z}^m$, there always exists some finite k such that (29) holds for some non-negative sequences $\{\tau^i_{(j,\ell+1)} \geq 0, j \in \mathbb{Z}^p, \ell \in \mathbb{Z}^k_0\}$ and $\{\pi^i_{(j,\ell+1)} \geq 0, j \in \mathbb{Z}^m, \ell \in \mathbb{Z}^k_0\}$.

Based on Lemma 5, we can define LMI problems for both quadratic and nonlinear constraints. For the quadratic constraints, let us define:

$$r^k_i := \min_{r,\tau^i \in \mathbb{R}^{p\times(k+1)}, \pi^i \in \mathbb{R}^{m\times(k+1)}} r \tag{30a}$$
$$s.t. \quad \tau^i \geq 0, \pi^i \geq 0, \tag{30b}$$
$$\bar{Q}^{k+1}_i \preceq rI + \sum_{\ell=0}^{k}\sum_{j=1}^{p} \tau^i_{(j,\ell+1)} \bar{Q}^\ell_j \tag{30c}$$
$$+ \sum_{\ell=0}^{k}\sum_{j=1}^{m} \pi^i_{(j,\ell+1)} \begin{pmatrix} -\frac{L_j}{2}(A^\ell)^T A^\ell & \frac{1}{2}(A^\ell)^T H_j^\nabla \\ \frac{1}{2}(H_j^\nabla)^T A^\ell & H_j(0) \end{pmatrix}$$

for all $i \in \mathbb{Z}^p$ and $k \in \mathbb{Z}^+_0$ with τ^i and π^i being the reshaping matrix of the sequences $\{\tau^i_{(j,\ell+1)} \geq 0, j \in \mathbb{Z}^p, \ell \in \mathbb{Z}^k_0\}$ and $\{\pi^i_{(j,\ell+1)} \geq 0, j \in \mathbb{Z}^m, \ell \in \mathbb{Z}^k_0\}$. For the nonlinear constraints, let us define:

$$\tilde{r}^k_i := \min_{\tilde{r},\tau^i \in \mathbb{R}^{p\times(k+1)}, \pi^i \in \mathbb{R}^{m\times(k+1)}} \tilde{r} \tag{31a}$$
$$s.t. \quad \tau^i \geq 0, \pi^i \geq 0, \tag{31b}$$
$$\begin{pmatrix} \frac{L_i}{2}(A^{k+1})^T A^{k+1} & \frac{1}{2}(A^{k+1})^T H_i^\nabla \\ \frac{1}{2}(H_i^\nabla)^T A^{k+1} & H_i(0) \end{pmatrix} \tag{31c}$$
$$\preceq \tilde{r}I + \sum_{\ell=0}^{k}\sum_{j=1}^{p} \tau^i_{(j,\ell+1)} \bar{Q}^\ell_j$$
$$+ \sum_{\ell=0}^{k}\sum_{j=1}^{m} \pi^i_{(j,\ell+1)} \begin{pmatrix} -\frac{L_j}{2}(A^\ell)^T A^\ell & \frac{1}{2}(A^\ell)^T H_j^\nabla \\ \frac{1}{2}(H_j^\nabla)^T A^\ell & H_j(0) \end{pmatrix}$$

for all $i \in \mathbb{Z}^m$ and $k \in \mathbb{Z}^+_0$. The properties of these LMI problems are given in the following lemma.

Lemma 6: Suppose Assumptions 1-4 hold. The LMI problems defined in (30) and (31) have the following properties.
(i) For all $i \in \mathbb{Z}^p$, let r^k_i be defined in (30), then there exists a finite k^*_i such that $r^{k^*_i}_i \leq 0$ and $r^k_i \leq 0$ for all $k \geq k^*_i$.
(ii) For all $i \in \mathbb{Z}^m$, let $\tilde{r}^k_i$ be defined in (31), then there exists a finite $\tilde{k}^*_i$ such that $\tilde{r}^{\tilde{k}^*_i}_i \leq 0$ and $\tilde{r}^k_i \leq 0$ for all $k \geq \tilde{k}^*_i$.

Based on Lemmas 4 - 6, the algorithm for computing the maximal CA-invariant set with nonlinear constraints is summarized in Algorithm 2.

Algorithm 2 Computation of the maximal constraint admissible invariant set with nonlinear constraints

Input: A, $\{Q_i, q_i\}^p_{i=1}$, and $\{H_i(x)\}^m_{i=1}$
Output: O_{k^*}

1: *Initialization*: let $X := \{x \in \mathbb{R}^n : (x)^T Q_i x + 2q_i^T x \leq 1, i \in \mathbb{Z}^p, H(x) \leq 0\}$, and set $k = 0$ and $O_0 = X$;
2: Obtain r^k_i from (30) for all $i \in \mathbb{Z}^p$;
3: Obtain $\tilde{r}^k_i$ from (31) for all $i \in \mathbb{Z}^m$;
4: Let $r^k_{\max} := \max_{i\in\mathbb{Z}^p} r^k_i$ and $\tilde{r}^k_{\max} := \max_{i\in\mathbb{Z}^m} \tilde{r}^k_i$. If $r^k_{\max} \leq 0$ and $\tilde{r}^k_{\max} \leq 0$, let $k^* = k$ and terminate; otherwise, let $O_{k+1} := O_k \bigcap \{x \in \mathbb{R}^n : A^{k+1}x \in X\}$, set $k \leftarrow k+1$ and go to Step 2.

Similar to Algorithm 1, Algorithm 2 will also terminate after a finite time as stated in the next theorem.

Theorem 2: Suppose Assumptions 1-4 hold, let $\{r^k_{\max}, \tilde{r}^k_{\max}, O_k\}$ be generated from Algorithm 2. Then, there exists some finite k^* such that $r^{k^*}_{\max} \leq 0$ and $\tilde{r}^{k^*}_{\max} \leq 0$ and $O_\infty = O_{k^*}$.

IV. Numerical Examples

We consider the linear system studied in [23, Example 1]: $A = [1.0216\ 0.3234; -0.6597\ 0.5226]$. In the first example, we consider the case when the constraint set is the unit circle given by $\Omega := \{x \in \mathbb{R}^2 : x^T x \leq 1\}$ and $\Theta = \mathbb{R}^n$. Algorithm 1 is used to the maximal CA-invariant set and the result is given in Figure 1. It can been seen from Figure 1 that Algorithm 1 takes 3 iterations to obtain this set. For the same setting, the algorithm in [23] takes 6 iterations.

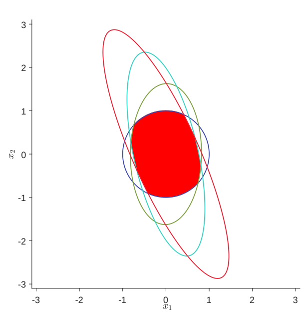


Fig. 1: The maximal CA-invariant set of Example 1.

In the second example, we consider more quadratic constraints. Let the quadratic constraint set be $\Omega := \{x \in \mathbb{R}^2 : x^T x \leq 1, 2x_1^2 - x_2^2 + 0.4x_1x_2 \leq 1, (x_1 + 0.5)^2 + x_2^2 \geq \frac{1}{16}, (x_1 - 0.5)^2 + x_2^2 \geq \frac{1}{16}\}$. Note that there are 4 quadratic constraints and that this set is nonconvex. Using Algorithm

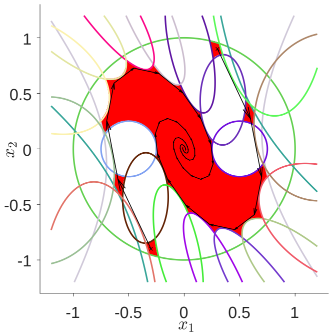
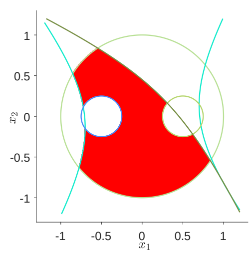


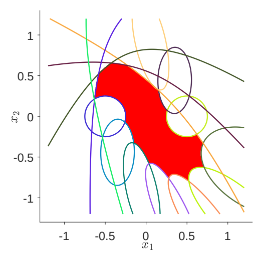
Fig. 2: The maximal CA-invariant set of Example 2

1, the maximal CA-invariant set can be obtained within 8 iterations and it is shown in Figure 2.

In the third example, we also consider a nonlinear constraint in addition to the quadratic constraints in the second example. Let $\Theta = \{x \in \mathbb{R}^2 : H_1(x) := \sqrt{x_1^2 + x_2^2 + 1} + 2x_1 + 2x_2 - 2 \leq 0\}$. It is easy to very that Assumption 3 is satisfied with $H_1^{\nabla} = [2\ 2]^T$ and $L_1 = 1$. Using Algorithm 2, the maximal CA-invariant set can be obtained within 8 iterations and it is shown in Figure 3.



(a)



(b)

Fig. 3: The maximal CA-invariant set of Example 3: (a) shows $\Omega \bigcap \Theta$ and (b) O_∞.

V. Conclusions

We have studied the computation of the maximal invariant set of linear systems with a class of nonlinear constraints, where the nonlinear functions have quadratic lower and upper bounds. By the use of these quadratic bounds, a sufficient condition is developed for computing the maximal invariant set. Based on this sufficient condition, a new algorithm is presented by solving a set of LMIs. Under mild assumptions, finite determinability can be guaranteed. Finally, we have illustrated the proposed algorithm by several numerical examples.

References

[1] J.P. Aubin. *Viability theory.* Basel: Birkhauser, 1991.

[2] F. Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999.

[3] F. Blanchini and S. Miani. *Set-Theoretic Methods in Control*. Birkhauser, 2008.

[4] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[5] E.G. Gilbert and K. T. Tan. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control*, 36:1008–1020, 1991.

[6] C. E. T. Dorea and J.C. Hennet. (A, B)-invariant polyhedral sets of linear discrete-time systems. *Journal of Optimization Theory and Applications*, 103(3):521–542, 1999.

[7] B. Pluymers, J.A. Rossiter, J. A. K. Suykens, and B. De Moor. The efficient computation of polyhedral invariant sets for linear systems with polytopic uncertainty. In *Proceedings of the American Control Conference*, pages 804–809, 2005.

[8] I. Kolmanovsky and E. G. Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4:317–367, 1998.

[9] S. V. Rakovic, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne. Invariant approximations of the minimal robust positively invariant set. *IEEE Transactions on Automatic Control*, 50(3):406–410, 2005.

[10] C. J. Ong and E. G. Gilbert. The minimal disturbance invariant set: Outer approximations via its partial sums. *Automatica*, 42(9):1563–1568, 2006.

[11] S. V. Raković, E. C. Kerrigan, D. Q. Mayne, and K. I. Kouramas. Optimized robust control invariance for linear discrete-time systems: Theoretical foundations. *Automatica*, 43(5):831–841, 2007.

[12] Z. Artstein and S. V. Raković. Feedback and invariance under uncertainty via set-iterates. *Automatica*, 44(2):520–525, 2008.

[13] P. Trodden. A one-step approach to computing a polytopic robust positively invariant set. *IEEE Transactions on Automatic Control*, 61(12):4100–4105, 2016.

[14] J. M. Bravo, D.l Limón, T. Alamo, and E. F. Camacho. On the computation of invariant sets for constrained nonlinear systems: An interval arithmetic approach. *Automatica*, 41(9):1583–1589, 2005.

[15] T. Alamo, A. Cepeda, M. Fiacchini, and E. F. Camacho. Convex invariant sets for discrete-time lure systems. *Automatica*, 45(4):1066–1071, 2009.

[16] M. Fiacchini, T. Alamo, and E. F. Camacho. On the computation of convex robust control invariant sets for nonlinear systems. *Automatica*, 46(8):1334–1338, 2010.

[17] M. A. B. Sassi and A. Girard. Computation of polytopic invariants for polynomial dynamical systems using linear programming. *Automatica*, 48(12):3114–3121, 2012.

[18] D. Henrion and M. Korda. Convex computation of the region of attraction of polynomial control systems. *IEEE Transactions on Automatic Control*, 59(2):297–312, 2014.

[19] M. Korda, D. Henrion, and C. N. Jones. Convex computation of the maximum controlled invariant set for polynomial control systems. *SIAM Journal on Control and Optimization*, 52(5):2944–2969, 2014.

[20] S. V. Raković and M. E. Villanueva. The maximal positively invariant set: Polynomial setting. *arXiv preprint arXiv:1712.01150*, 2017.

[21] M. Dehghan and C.J. Ong. Discrete-time switching linear system with constraints: Characterization and computation of invariant sets under dwell-time consideration. *Automatica*, 5(48):964–969, 2012.

[22] M. A Hernández-Mejías, A. Sala, C. Ariño, and A. Querol. Reliable controllable sets for constrained markov-jump linear systems. *International Journal of Robust and Nonlinear Control*, 26(10):2075–2089, 2016.

[23] N. Athanasopoulos and R. M. Jungers. Computing the domain of attraction of switching systems subject to non-convex constraints. In *Proceedings of the 19th International Conference on Hybrid Systems: Computation and Control*, pages 41–50. ACM, 2016.

[24] N. Athanasopoulos and R. M. Jungers. Invariant sets for switching affine systems subject to semi-algebraic constraints. *IFAC-PapersOnLine*, 49(18):158–163, 2016.

[25] N. Athanasopoulos, K. Smpoukis, and R. M. Jungers. Invariant sets analysis for constrained switching systems. *IEEE Control Systems Letters*, 1(2):256–261, 2017.

[26] N. Athanasopoulos and R. M. Jungers. Combinatorial methods for invariance and safety of hybrid systems. *Automatica*, 98:130–140, 2018.

[27] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics (SIAM), 1994.

[28] K. Derinkuyu and M. Pınar. On the S-procedure and some variants. *Mathematical Methods of Operations Research*, 64(1):55–77, 2006.

[29] D. P. Bertsekas. *Convex optimization theory*. Athena Scientific, 2009.