FRACTIONAL HAWKES PROCESSES

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Abstract

Hawkes processes have a self-excitation mechanism used for modeling the clustering of events observed in natural or social phenomena. In the first part of this article, we find the forward differential equations ruling the probability density function and the Laplace's transform of the intensity of a Hawkes process, with an exponential decaying kernel. In the second part, we study the properties of the fractional version of this process. The fractional Hawkes process is obtained by subordinating the point process with the inverse of a α-stable Lévy process. This process is not Markov but the probability density function of its intensity is solution of a fractional Fokker-Planck equation. Finally, we present closed form expressions for moments and autocovariance of the fractional intensity.

1 Introduction

In many natural or social phenomena, shocks are rare events but their occurrence momentarily raises the risk of aftershocks. An endogenous way to model the clustering of events or shocks is provided by self-exciting point processes. In this category of processes, the instantaneous probability to observe a shock depends on the number of past events. Hawkes (1971a, b) and Hawkes and Oakes (1974) were among the firsts to propose a point process with this feature. In the most common and simplest specification, the intensity process is persistent and suddenly increases when a jump occurs. Moreover, the influence of an event on the intensity does not depend on its size and it decays over time more or less rapidly according to a kernel function. Hawkes processes have been successfully applied for modeling the clustering of shocks in seismology, finance, criminality and in many other fields. Without being exhaustive, we can cite e.g. Musmeci and Vere-Jones or Ogata (1998) who propose a space-time point-process for earthquake occurrences or Porter and White (2012) who use a self-exciting model for modeling terrorist activity. Hawkes processes are also used for modeling financial transactions (Bauwens and Hautsch, 2009, Bacry et al. , 2015, Hainaut 2017, Hainaut and Moraux 2018, Hainaut and Goutte 2019). Johnson (1996) develops a model for neuron activity based on self-exciting processes. We refer to Reinhart (2018) for a detailed review of other applications and properties of Hawkes processes.

In the common specification of Hawkes processes, the influence of past jumps on the probability of a new shock decays exponentially with time. Choosing this type of memory guarantees that the point process and its intensity form a bivariate Markov process. Properties of this process may be studied with standard tools from stochastic calculus, like the Itô's lemma for semi-martingales. In this framework, the autocovariance of the intensity decays exponentially with time. However this feature is not necessary adapted for modeling real phenomena that exhibit a long term memory of past events. A common method for modeling sub-exponential decreasing autocovariance consists to replace the exponential memory kernel by a power decaying function. In this case, the point process is not anymore Markov and the dynamics of the point process may not be described with backward stochastic differential equations. This article explores an alternative way for modeling a self-exciting intensity with sub-exponential covariance. This involves a non-Markov time change involving the inverse of an alpha stable subordinator.

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The time change approach is applied to diffusions in physics for describing the movement of heavy particles that can get immobilized (see e.g. Metzler and Klafter 2004 or Eliazar, Klafter 2004). This type of time-changed Brownian motions, called sub-diffusions, are also popular in econophysics (see Scalas 2006) and applied to financial derivatives by Magdziarz (2009, a) for illiquid markets. As shown e.g. in Magdziarz (2009, b), the probability density function of sub-diffusions is solution of a Fractional Fokker-Planck equation. This equation is proposed in Barkai et al. (2000) and Metzler et al. (1999). Articles of Leonenko et al. (2013, a) Leonenko et al. (2013, b) go a step further and explore fractional Pearson diffusions and their correlation structure.

The contributions of this article are multiple. We first establish the Fokker-Planck equations (FPE's) ruling the probability density function and the Laplace's transform of the intensity for a Hawkes process with an exponential decaying memory. To the best of our knowledge, these results are new and FPE's differ from these for jump-diffusions. We next provide the FPE for the joint statistical distribution of counting and intensity processes. We also propose a numerical method for solving the FPE of the Laplace's transform of intensity around zero. The second part of this work studies the same process, subordinated by the inverse of a stable Lévy process. We show that this process is ruled by similar FPE's but the derivative with respect to time is replaced by a Caputo's fractional derivative. We next present closed-form expressions for moments and for the autocovariance. Finally, we extend the non-fractional numerical framework for solving the fractional FPE of the Laplace's transform of intensity around zero.

2 The Hawkes process

The Hawkes process (1971 a,b) is a self-exciting point process for which the arrival of one event increases the probability of occurrence of new ones. Let us consider a probability space $(\Omega, \mathcal{F}, P)$ on which is defined a counting process $(N_t)_{t\geq 0}$ with random independent identically distributed marks denoted by $(\xi_k)_{k=1,N_t}$. The probability density function (pdf) of marks is a measure, $\nu(.)$, defined on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. The expectation and variance of the mark are respectively $\mathbb{E}(\xi) = \mu > 0$ and $\mathbb{V}(\xi) = \psi^2$. The sum of marks is a pure jump process, denoted by $(P_t)_{t\geq 0}$:

$$P_t = \sum_{k=1}^{N_t} \xi_k \,. \tag{1}$$

The rate of arrival of events is a stochastic process $(\lambda_t)_{t\geq 0}$ that depends upon the history of the point process $(P_t)_{t\geq 0}$ through the following auto-regressive relation:

$$\lambda_t = \theta + e^{-\kappa(t-s)} (\lambda_s - \theta) + \eta \int_s^t e^{-\kappa(t-u)} dP_u \quad t \geq s, \tag{2}$$

where $\theta, \eta, \kappa \in \mathbb{R}^+$. The natural filtration of the triplet (P_t, N_t, λ_t) is the collection of sigma-algebras: $(\mathcal{F}_t)_{t\geq 0} = \sigma(P_s, N_s, \lambda_s, s \leq t)$. Differentiating equation (2) allows us to reformulate the dynamics of the intensity as follows:

$$d\lambda_t \quad = \quad \kappa (\theta - \lambda_t) dt + \eta dP_t \,. \tag{3}$$

As illustrated in Figure 1, between two successive jumps, the process λ_t reverts towards θ at a speed κ. If an event occurs, the intensity increases by a random quantity $\eta\xi$. The point process P_t is not Markov since it depends upon λ_t. But the pair $X_t = (\lambda_t, P_t)_{t\in\mathbb{R}^+}$ is well Markov in the state space $D = \mathbb{R}_+ \times \mathbb{R}_+$. Therefore, the infinitesimal generator of X_t, denoted $\mathcal{A}$, is the operator acting on a sufficiently regular function $f : D \to \mathbb{R}$ such that

$$\mathcal{A}f(\boldsymbol{x}) \quad = \quad \lim_{h\to 0} \frac{\mathbb{E}_t \left[f(X_{t+h}) - f(\boldsymbol{x}) \right]}{h} \,,$$

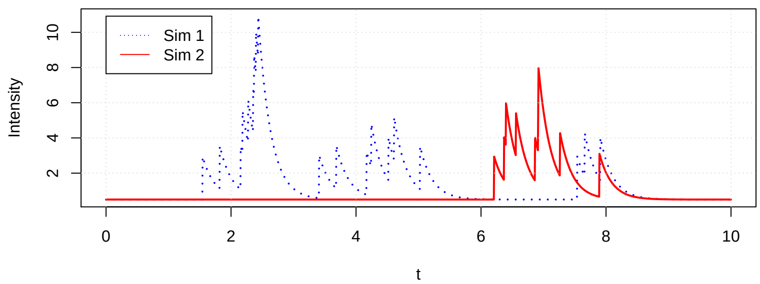

Figure 1: Simulated sample paths of the intensity of a Hawkes process.

where $\mathbb{E}_t = \mathbb{E}[.|\mathcal{F}_t]$ and $X_t = \boldsymbol{x} = (\lambda, p)$. Using standard arguments and the Itô's lemma, the generator is equal to:

$$\mathcal{A}f(\boldsymbol{x}) = \kappa(\theta - \lambda)\frac{\partial f}{\partial \lambda}(\boldsymbol{x}) + \lambda \mathbb{E}\left[f(\lambda + \eta\xi\,,\, p + \xi) - f(\boldsymbol{x})\right]. \tag{4}$$

On the other hand, the compensated process $(M_t)_{t\geq 0}$ defined as follows

$$M_t = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_u)du\,,$$

is a martingale relative to its natural filtration (see e.g. Proposition 1,6 of Chapter VII in Revuz and Yor (1999)). Thus, for $s > t$, we have that

$$\mathbb{E}_t\left[f(X_s) - \int_0^s \mathcal{A}f(X_u)du\right] = f(X_t) - \int_0^t \mathcal{A}f(X_u)du$$

by the martingale property. This leads to the Dynkin formula linking the expectation of a function of X_t to the infinitesimal generator:

$$\mathbb{E}_t\left[f(X_s)\right] = f(X_t) + \mathbb{E}_t\left[\int_t^s \mathcal{A}f(X_u)du\right]. \tag{5}$$

This last formula allows us to calculate the expectation and variance of the intensity. We remind these useful results and a proof may be found e.g. in Hainaut (2017).

Proposition 2.1. *The first moment of λ_t is given by*

$$\mathbb{E}_0[\lambda_t] = e^{(\eta\mu-\kappa)t}\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\left(e^{(\eta\mu-\kappa)t} - 1\right). \tag{6}$$

If we define $\rho_1 = \eta^2(\psi^2 + \mu^2)$ and $\rho_2 = \frac{\eta^2\kappa\theta(\psi^2+\mu^2)}{\eta\mu-\kappa}$ then the variance is equal to

$$\mathbb{V}_0[\lambda_t] = \frac{\rho_1\lambda_0 + \rho_2}{\eta\mu-\kappa}\left(e^{2(\eta\mu-\kappa)t} - e^{(\eta\mu-\kappa)t}\right) + \frac{\rho_2}{2(\eta\mu-\kappa)}\left(1 - e^{2(\eta\mu-\kappa)t}\right). \tag{7}$$

The expectation and variance of λ_t are well defined and meaningful only if the parameters η, μ and κ fulfills the following conditions:

$$\eta\mu - \kappa < 0\,. \tag{8}$$

If this condition is not satisfied, the speed of mean reversion κ is not sufficient to drive back the intensity to θ and λ_t tends to $+\infty$ when $t \to \infty$. If the equilibrium condition (8) is fulfilled, the expectation of the jump arrival intensity tends to a constant, $-\frac{\kappa\theta}{\eta\mu-\kappa}$, as t becomes large. Whereas the variance of λ_t converges to a constant, $\frac{\eta^2\kappa\theta(\psi^2+\mu^2)}{2(\eta\mu-\kappa)^2}$ when $t \to \infty$. The next proposition recalls the form of the intensity autocovariance.

Proposition 2.2. *The covariance between λ_s and λ_t for $t \leq s$ is proportional to the variance:*

$$\begin{aligned}\mathbb{C}_0\left[\lambda_t \lambda_s\right] =& \mathbb{E}_0\left[\lambda_t \lambda_s\right]-\mathbb{E}_0\left[\lambda_s\right] \mathbb{E}_0\left[\lambda_t\right] \\ =& e^{(\eta \mu-\kappa)(s-t)} \mathbb{V}_0\left[\lambda_t\right] .\end{aligned}$$

As mentioned in the introduction, the fractional Hawkes process is a self-exciting process that is time changed by the inverse of a α stable Lévy process. Since the subordinator is not Markov, the fractional process is not anymore a semi-martingale. Therefore, we cannot rely on tools from stochastic calculus, like the Itô's lemma to infer a backward differential equation satisfied by any smooth function of the fractional process. However, we will see the probability density function (pdf) of this process is solution of a forward differential equation, called fractional Fokker-Planck equation. In order to establish this result, we construct in the next section, the Fokker-Planck equation (FPE) satisfied by the pdf of the intensity λ_t.

3 Fokker-Planck equation for the pdf of $(\lambda_t)_{t\geq 0}$

To the best of our knowledge, this result is new and compared to a pure jump process, the Hawkes FPE has an additional term. From equation (2), we infer that for $t \geq s$, the pdf of λ_t, conditionally to $\mathcal{F}_s$ exclusively depends upon λ_s. Thus, we denote by $p(t,x|s,y)$ its probability density function that is defined as follows:

$$p(t,x|s,y)dx \quad = \quad P\left(\lambda_t \in [x, x+dx] \mid \lambda_s = y\right) ,$$

for $s \leq t$ and $x, y \in \mathbb{R}^+$. This pdf is solution of a forward differential equation as stated in the next proposition.

Proposition 3.1. *The pdf of λ_t is solution of the following Fokker-Planck equation, also called forward Kolmogorov equation:*

$$\begin{aligned}\frac{\partial p(t,x|s,y)}{\partial t} \quad = \quad & -\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)p(t,x|s,y)\right)-\eta \mathbb{E}\left[\xi p(t,x-\eta\xi|s,y)\right] \\ & +x\mathbb{E}\left[p(t,x-\eta\xi|s,y)-p(t,x|s,y)\right] ,\end{aligned} \tag{9}$$

with the initial condition: $p(s,x|s,y) = \delta_{\{x-y\}}$ where δ_z is the Dirac measure located at z.

Proof From the Kramers-Moyal forward expansion, we know that the pdf's, $p(t+\Delta, x|s,y)$ at time $t+\Delta$ and $p(t,x|s,y)$ at time t, are linked to moments of λ_t by the relation

$$p(t+\Delta,x|s,y)-p(t,x|s,y) \quad = \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n}\left[M_n(t,x,\Delta)p(t,x|s,y)\right] \tag{10}$$

where $M_n(t,x,\Delta)$ is the moment of order n of $\Delta\lambda_t = \lambda_{t+\Delta}-\lambda_t$:

$$\begin{aligned}M_n(t,x,\Delta) \quad &= \quad \mathbb{E}_t\left[\left(\lambda_{t+\Delta}-\lambda_t\right)^n \mid \lambda_t = x\right] \\ &= \quad \int_{-\infty}^{+\infty} \left(z-x\right)^n p(t+\Delta, z \mid t, x) dz .\end{aligned}$$

On the other hand, marks $(\xi_k)_{k=1,\dots,N_t}$ are independent from $X_t = (\lambda_t, P_t)$. Thus, from Equation (3) and for a small enough step of time, the centered moments of λ_t may be expanded as follows:

$$\begin{aligned}&\mathbb{E}_t\left[\lambda_{t+\Delta}-\lambda_t \middle| \lambda_t = x\right] = \left(\kappa\left(\theta-x\right)+\eta\mu x\right)\Delta + \mathcal{O}\left(\Delta_t^2\right) , \\ &\mathbb{E}_t\left[\left(\lambda_{t+\Delta}-\lambda_t\right)^2 \mid \lambda_t = x\right] = \eta^2 \mathbb{E}\left[\xi^2\right] x\Delta + \mathcal{O}\left(\Delta_t^2\right) , \\ &\vdots \\ &\mathbb{E}_t\left[\left(\lambda_{t+\Delta}-\lambda_t\right)^n \mid \lambda_t = x\right] = \eta^n \mathbb{E}\left[\xi^n\right] x\Delta + \mathcal{O}\left(\Delta_t^2\right) .\end{aligned}$$

Injecting these expansions in Equation (10) gives us:

$$\begin{aligned}\frac{p(t+\Delta,x|s,y)-p(t,x|s,y)}{\Delta} &= -\frac{\partial}{\partial x}\left[\kappa\left(\theta-x\right)p(t,x|s,y)\right] \\ &+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\eta^n\mathbb{E}\left[\xi^n\right]\frac{\partial^n\left(xp(t,x|s,y)\right)}{\partial^n x}+\mathcal{O}\left(\Delta\right).\end{aligned} \tag{11}$$

Since the n^{th} derivative of $x\,p(t,x|s,y)$ is equal to

$$\frac{\partial^n}{\partial x^n}\left(x\,p(t,x|s,y)\right) = n\frac{\partial^{n-1}p(t,x|s,y)}{\partial x^{n-1}}+x\frac{\partial^n p(t,x|s,y)}{\partial^n x},$$

Equation (11) may be rewritten as the following sum:

$$\begin{aligned}\frac{p(t+\Delta,x|s,y)-p(t,x|s,y)}{\Delta} &= -\frac{\partial}{\partial x}\left[\kappa\left(\theta-x\right)p(t,x|s,y)\right] \\ &+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!}\eta^n\mathbb{E}\left[\xi^n\right]\frac{\partial^{n-1}p(t,x|s,y)}{\partial x^{n-1}} \\ &+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\eta^n\mathbb{E}\left[\xi^n\right]x\frac{\partial^n p(t,x|s,y)}{\partial^n x}+\mathcal{O}\left(\Delta\right).\end{aligned} \tag{12}$$

the second term in this equation is equal to the Taylor's expansion of the following expectation:

$$\mathbb{E}\left[-\eta\xi\sum_{n=0}^{\infty}\frac{(-\eta\xi)^n}{n!}\frac{\partial^n p(t,x|s,y)}{\partial x^n}\right] = -\eta\mathbb{E}\left[\xi p(t,x-\eta\xi|s,y)\right].$$

Whereas the third term in Equation (12) is equal to the Taylor's expansion of :

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\eta^n\mathbb{E}\left[\xi^n\right]x\frac{\partial^n p(t,x|s,y)}{\partial^n x}=x\mathbb{E}\left[p(t,x-\eta\xi|s,y)-p(t,x|s,y)\right].$$

end

The probability density function may be rewritten as the conditional expectation of a Dirac function: $p(t,x|s,y)=\mathbb{E}_s\left(\delta_{\{\lambda_t-x\}}\right)$ and is therefore a martingale, From Equation (5), the pdf is then also solution of a backward Kolmogorov equation that is extensively studied in the literature on Hawkes processes:

$$\begin{aligned}-\frac{\partial p(t,x|s,y)}{\partial s} &= \kappa\left(\theta-y\right)\frac{\partial p(t,x|s,y)}{\partial y}+ \\ & y\mathbb{E}\left[p(t,x|s,y+\eta\xi)-p(t,x|s,y)\right].\end{aligned} \tag{13}$$

From Equation (2), the distribution of $\lambda_t|\lambda_s$ is time-homogeneous in the sense that

$$p(t,x|s,y)=p(t-s,x|0,y).$$

Therefore, the backward equation (13) is also equivalent to:

$$\begin{aligned}\frac{\partial p(t,x|s,y)}{\partial t} &= \kappa\left(\theta-y\right)\frac{\partial p(t,x|s,y)}{\partial y}+ \\ & y\mathbb{E}\left[p(t,x|s,y+\eta\xi)-p(t,x|s,y)\right].\end{aligned} \tag{14}$$

It is interesting to compare the FPE of a Hawkes process to the one of a process without any self-excitation mechanism. For this purpose, let us denote by $\left(N_t^{'}\right)_{t\geq 0}$, a pure Poisson process with a constant intensity,

$\rho \in \mathbb{R}^+$. The sum of marks is here noted $P_t^{'} = \sum_{k=1}^{N_t^{'}} \xi_k$. Let us define a process $\left(\lambda_t^{'}\right)_{t\geq 0}$ by the following SDE:

$$d\lambda_t^{'} \quad = \quad \kappa \left(\theta - \lambda_t^{'}\right) dt + \eta dP_t^{'} \,.$$

From the Chapter 7 of Hanson (2007), we know that the pdf $p(t,x|s,y)$ of $\lambda_t^{'}$, is solution of the following Fokker-Planck equation

$$\begin{aligned} \frac{\partial p(t,x|s,y)}{\partial t} &= -\frac{\partial}{\partial x}\left(\kappa\left(\theta - x\right) p(t,x|s,y)\right) \\ &\quad +\rho\mathbb{E}\left[p(t,x-\eta\xi|s,y) - p(t,x|s,y)\right] \end{aligned} \tag{15}$$

A comparison with Equation (9) reveals that the presence of self-excitation introduces one new term, $-\eta\mathbb{E}\left[\xi p(t,x-\eta\xi|s,y)\right]$, in the Fokker-Planck Equation (15).

Contrary to backward equations, solving the Fokker-Planck equation (9) is numerically more challenging because we have to approximate the Dirac measure in the initial condition. However, we will see in the next section that the Laplace's transform of the pdf is solution of a forward equation, easy to solve numerically.

4 Laplace's transform of $p(t,x|s,y)$

Let us denote by $\phi(.)$ the Laplace's transform of the pdf $p(t,x|s,y)$ of the intensity process. This transform is also equal to the following expectation

$$\phi(t,z|s,y) = \mathbb{E}\left(e^{-z\lambda_t}|\lambda_s = y\right)$$

for $z \in \mathbb{R}^+$. It is well known in the literature that this Laplace's transform is solution of a backward Kolmogorov equation. Details about this equation are reminded at the end of this section. However, this transform is also solution of a forward Kolmogorov equation, as stated in the next proposition.

Proposition 4.1. *The Laplace transform $\phi(t,z|0,y)$ satisfies the forward equation:*

$$\frac{\partial \phi(t,z|0,y)}{\partial t} \quad = \quad -\kappa\theta z\, \phi(t,z|0,y) + \left(1 - \kappa z - \mathbb{E}\left(e^{-z\eta\xi}\right)\right) \frac{\partial \phi(t,z|0,y)}{\partial z} \tag{16}$$

with the initial conditions $\phi(0,z|0,y) = e^{-zy}$ and $\phi(t,0|0,y) = 1$.

Proof Let $\Delta \in \mathbb{R}^+$ be a small step of time. The difference $\phi(t+\Delta,z|0,y) - \phi(t,z|0,y)$ when $\Delta \to 0$ is equal to

$$\lim_{\Delta\to 0} \frac{\phi(t+\Delta,z|0,y) - \phi(t,z|0,y)}{\Delta} = \lim_{\Delta\to 0} \int_0^\infty e^{-zx}\left(\frac{p(t+\Delta,x|0,y) - p(t,x|0,y)}{\Delta}\right) dx\,.$$

Using the FPE (9) for Hawkes processes, we infer that the derivative of $\phi(.)$ with respect to time is the sum of three terms:

$$\begin{aligned} \frac{\partial \phi(t,z|0,y)}{\partial t} &= \int_0^\infty e^{-zx}\left(-\frac{\partial}{\partial x}\left(\kappa\left(\theta - x\right) p(t,x|0,y)\right)\right) dx + \\ &\quad \int_0^\infty e^{-zx}\left(-\eta\mathbb{E}\left[\xi p(t,x-\eta\xi|0,y)\right]\right) dx + \\ &\quad \int_0^\infty e^{-zx}\left(x\mathbb{E}\left[p(t,x-\eta\xi|0,y) - p(t,x|0,y)\right]\right) dx\,. \end{aligned} \tag{17}$$

Using the relation

$$\int_0^\infty e^{-zx} x \frac{\partial}{\partial x} p(t,x|0,y) dx \quad = \quad -\phi(t,z|0,y) - z\frac{\partial \phi(t,z|0,y)}{\partial z}\,,$$

and properties of the Laplace's transform allows us to rewrite the first term of Equation (17) as

$$\begin{aligned}\int_0^\infty e^{-zx}\left(-\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)p(t,x|0,y)\right)\right)dx &= -\kappa\theta\left(z\phi(t,z|0,y)-\underbrace{p(t,0|s,y)}_{=0}\right)+\kappa\phi(t,z|0,y)\\ &\quad -\kappa\phi(t,z|0,y)-\kappa z\frac{\partial\phi(t,z|0,y)}{\partial z}\,.\end{aligned}$$

After a change of variable $x'=x-\eta\xi$, the second term of (17) becomes:

$$\begin{aligned}&-\eta\int_0^\infty \xi\left(\int_0^\infty e^{-zx}p(t,x-\eta\xi|0,y)dx\right)\nu(\xi)d\xi\\ &\quad=-\eta\mathbb{E}\left(\xi e^{-z\eta\xi}\right)\phi(t,z|0,y)\,.\end{aligned}$$

Given that $p(t,x|0,y)$ is null for $x\leq 0$, the third term of (17) is rewritten, after the same change of variable, as:

$$\begin{aligned}&\int_0^\infty\int_0^\infty e^{-zx}\left(xp(t,x-\eta\xi|0,y)\right)dx\nu(\xi)d\xi-\int_0^\infty e^{-zx}\left(xp(t,x|0,y)\right)dx\\ &=\int_0^\infty e^{-z\eta\xi}\,\nu(\xi)d\xi\int_0^\infty e^{-zx'}x'p(t,x'|0,y)dx'\\ &+\eta\int_0^\infty \xi e^{-z\eta\xi}\nu(\xi)d\xi\int_0^\infty e^{-zx'}p(t,x'|0,y)dx'+\frac{\partial\phi(t,z|0,y)}{\partial z}\\ &=-\mathbb{E}\left(e^{-z\eta\xi}\right)\frac{\partial\phi(t,z|0,y)}{\partial z}+\eta\mathbb{E}\left(\xi e^{-z\eta\xi}\right)\phi(t,z|0,y)+\frac{\partial\phi(t,z|0,y)}{\partial z}\,.\end{aligned}$$

Combining previous elements allows us to conclude. **end**.

We compare the forward Equation (16) with the backward one. From the Itô's lemma for semi-martingales, we know that $\phi(.)$ is solution of the backward partial differential equation:

$$0 \;=\; \frac{\partial\phi}{\partial s}+\kappa\left(\theta-y\right)\frac{\partial\phi}{\partial y}+y\mathbb{E}\left(\phi(t,z|s,y+\eta\xi)-\phi\right)\,. \tag{18}$$

If we do the assumption that $\phi(.)$ is a function of the form $\exp\left(A(s,t)+B(s,t)y\right)$, we can easily show that functions $A(.)$ and $B(.)$ are solutions of a system of ordinary differential equations (ODE) as stated in the next proposition:

Proposition 4.2. *$\phi(t,z|s,y)$ is equal to*

$$\phi(t,z|s,y)=\exp\left(A(s,t)+B(s,t)y\right)\,, \tag{19}$$

where $A(s,t)$ and $B(s,t)$ are functions that satisfy the following ODE's:

$$\begin{aligned}\frac{\partial A(s,t)}{\partial s} &= -\kappa\theta B(s,t)\,,\\ \frac{\partial B(s,t)}{\partial s} &= \kappa B(s,t)-\mathbb{E}\left(e^{B(s,t)\eta\xi}-1\right)\,,\end{aligned}$$

with the terminal conditions $A(t,t)=0$ and $B(t,t)=-z$.

This result is not new but we will use it to benchmark the efficiency of the numerical scheme proposed in the next section for solving Equation (16).

Solving the forward equation for the Laplace transform of λ_t

This section presents a numerical method based on differential transforms for solving the forward equation (16) ruling the Laplace's transform of the intensity, around zero. This method is widely used in physics and the interested reader may e.g. refer to Bildik et al. (2006) or Yang et al. (2001) for details. The method is based on the following observation: if a differentiable function $l(t,z)$ is the product of two $\mathcal{C}^{\infty}$ functions $f(t)$ and $g(z)$, then $l(t,z)$ is the product of their Taylor's expansions:

$$\begin{aligned} l(t,z) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!}\frac{1}{h!}\left[\frac{\partial^k f(t)}{\partial t^k}\right]_{t=t_0}\left[\frac{\partial^h g(z)}{\partial z^h}\right]_{z=z_0}(t-t_0)^k\,(z-z_0)^h \\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} w(k,h)(t-t_0)^k\,(z-z_0)^h \end{aligned} \tag{20}$$

where

$$(w(k,h))_{k,h>0} = \frac{1}{k!h!}\left[\frac{\partial l(t,z)}{\partial t^k \partial z^h}\right]_{t=t_0,z=z_0}$$

is called the spectrum of $l(t,z)$. The differential method for solving the Fokker-Planck equation consists to approach its solution by a series similar to (20). Here, we approximate the Laplace transform $\phi(t,z|0,y)$ by the following infinite sum:

$$\phi(t,z|0,y) \approx \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} w(k,h)\times z^h t^k\,,$$

where the differential weights in this sum are noted:

$$w(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h}\phi(t,z|.)\right]_{t=0,z=0}, \tag{21}$$

and may be computed by a forward iterative recursion. To establish this recursion, we will apply the differential operator $\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k\partial z^h}\right]_{t=0,z=0}$ to the forward equation (16). The next proposition provides some useful results.

Proposition 4.3. *The differential weights such as defined by Equation (21) satisfy the following relations:*

$$\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k\partial z^h}\frac{\partial\phi(.)}{\partial t}\right]_{t=0,z=0} = (k+1)w(k+1,h) \tag{22}$$

$$\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k\partial z^h}z\phi(.)\right]_{t=0,z=0} = w(k,h-1) \tag{23}$$

$$\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k\partial z^h}z\frac{\partial\phi(.)}{\partial z}\right]_{t=0,z=0} = hw(k,h) \tag{24}$$

$$\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k\partial z^h}e^{-z\eta\xi}\frac{\partial\phi(.)}{\partial z}\right]_{t=0,z=0} = \sum_{j=0}^{h}\frac{j+1}{(h-j)!}\left(-\eta\xi\right)^{h-j}w(k,j+1) \tag{25}$$

Proof. Equation (22) results from the definition of differential weights. The property (23) is a direct consequence of:

$$\frac{\partial^{k+h}}{\partial t^k\partial z^h}z\phi(.) = h\frac{\partial^{k+h-1}}{\partial t^k\partial z^{h-1}}\phi(.) + z\frac{\partial^{k+h}}{\partial t^k\partial z^h}\phi(.)\,,$$

whereas equation (24) comes from the relation:

$$\frac{\partial^{k+h}}{\partial t^k\partial z^h}z\frac{\partial\phi(.)}{\partial z} = h\frac{\partial^{k+h}}{\partial t^k\partial z^h}\phi(.) + z\frac{\partial^{k+h+1}}{\partial t^k\partial z^{h+1}}\phi(.)\,.$$

To show (25), we use the following Newton's formula:

$$
\begin{aligned}
&\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h} e^{-z\eta\xi} \frac{\partial \phi(t,z|.)}{\partial z}\right]_{t=0,z=0} \\
&= \left[\sum_{j=0}^{h} \frac{(j+1)}{(h-j)!k!(j+1)!} \frac{\partial^{k+j+1}\phi(t,z|.)}{\partial t^k \partial z^{j+1}} \left(-\eta\xi\right)^{h-j} e^{-z\eta\xi}\right]_{t=0,z=0} \\
&= \sum_{j=0}^{h} \frac{j+1}{(h-j)!} \left(-\eta\xi\right)^{h-j} w(k,j+1)\,.
\end{aligned}
$$

end.

The next proposition provides an approached expression for the Laplace's transform of the pdf $p(t,x|s,y)$.

Proposition 4.4. *The Laplace's transform $\phi(.)$ is approached by the following sum*

$$
\phi(t,z|0,y) \quad \approx \quad \sum_{k=0}^{K}\sum_{h=0}^{H} w(k,h) \times z^h t^k\,, \tag{26}
$$

with $K, H \in \mathbb{N}$ and differential weights satisfying the following recursion:

$$
\begin{aligned}
(k+1)w(k+1,h) \quad &= \quad -\kappa\theta w(k,h-1) - \kappa\, h\, w(k,h) + (h+1)w(k,h+1) \\
&\quad -\sum_{j=0}^{h} \frac{j+1}{(h-j)!}(-1)^{h-j}\mathbb{E}\left[\left(\eta\xi\right)^{h-j}\right] w(k,j+1)\,,
\end{aligned} \tag{27}
$$

determined by the initial conditions:

$$
\begin{aligned}
w(0,h) \quad &= \quad \frac{1}{h!}\left(-y\right)^h\,, \\
w(k,0) \quad &= \quad 0 \quad k>0\,.
\end{aligned}
$$

Proof To establish the recursion (27), we apply the differential operator $\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h}\right]_{t=0,z=0}$ to the forward equation (16). For $-\kappa\theta z\phi(t,z|0,y)$ we have

$$
-\frac{1}{k!h!}\kappa\theta\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h} z\phi(t,z|.)\right]_{t=0,z=0} = -\kappa\theta w(k,h-1)
$$

and

$$
\begin{aligned}
&\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h}\left(1-\kappa z - \mathbb{E}\left(e^{-z\eta\xi}\right)\right)\frac{\partial\phi(t,z|.)}{\partial z}\right]_{t=0,z=0} = (h+1)w(k,h+1) - \kappa\, h\, w(k,h) \\
&-\sum_{j=0}^{h}\frac{j+1}{(h-j)!}(-1)^{h-j}\mathbb{E}\left[\left(\eta\xi\right)^{h-j}\right] w(k,j+1)
\end{aligned}
$$

Since $e^{-yz} = \sum_{h=0}^{\infty} \frac{1}{h!}\left(-y\right)^h z^h$ therefore $w(0,h) = \frac{1}{h!}\left[\frac{\partial^h}{\partial z^h} e^{-zy}\right]_{z=0} = \frac{1}{h!}\left(-y\right)^h$. **end**

Figure 2 compares the Laplace's transform approached by the sum (26) to the numerical solution of the backward Kolmogorov's Equation (19). Since the solution of the forward equation is based on a Taylor's development of the Laplace's transform around $z=0$ and $t=0$, its accuracy deteriorates with the distance to zero. However, it is accurate enough to compute numerically the first four moments as shown in Table 1 by deriving the Laplace's transform in the neighborhood of $z=0$. This table also provides the theoretical expectation and variance computed with Equations (6) and (7).

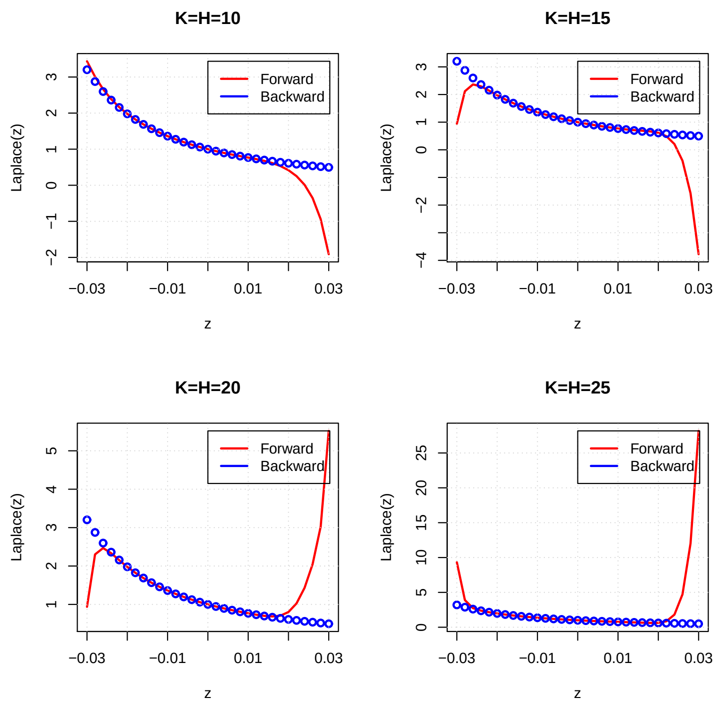

Figure 2: Laplace's transform of $\lambda_{t=1}$ computed with the forward and backward equations. $\eta = 7$, $\theta = 5$, $\kappa = 8.7$ and $\xi = 1$. The maturity is $t = 1$.

K, H	$\mathbb{E}_0(\lambda_t)$	$\mathbb{V}_0(\lambda_t)$	$\mathbb{S}_0(\lambda_t)$	$\mathbb{K}_0(\lambda_t)$
10	28.3333	21.1100	-0.2806	49.4070
15	28.3333	20.9698	1.6740	6.3504
20	28.3333	20.9698	1.6626	7.1083
25	28.3333	20.9698	1.6626	7.1318
Analytical	28.3333	20.9698		

Table 1: Comparison of numerical and theoretical moments, $\eta = 7$, $\theta = 5$, $\kappa = 8.7$ and $\xi = 1$. The maturity is $t = 1$. $y_0 = \lim_{t\to\infty} \mathbb{E}(\lambda_t) = 28.3333$

5 Fokker-Planck equation for the pdf of $(\lambda_t, P_t)_{t\geq 0}$

The compound jump process $(P_t)_{t\geq 0}$ is not Markov since its statistical distribution depends upon the arrival rate of events, λ_t. For this reason, we build the Fokker-Planck equation satisfied by the joint pdf of the pair (λ_t, P_t). This pdf is denoted by $p(t, x_1, x_2|s, y_1, y_2)$ and is defined as follows:

$$p(t, x_1, x_2|s, y_1, y_2) \quad = \quad P\left(\lambda_t \in [x_1, x_1 + dx], P_t \in [x_1, x_2 + dx] \,|\lambda_s = y_1, P_s = y_2\right) ,$$

for $s \leq t$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}^+$. The FPE of the joint pdf is obtained in a similar way to the one of λ_t and the proof relies on a 2 dimensions version of the Moyal's expansion.

Proposition 5.1. *The joint pdf $p(t, x_1, x_2|s, y_1, y_2)$ of (λ_t, P_t) is solution of the following Fokker-Planck equation:*

$$\begin{aligned} \frac{\partial p(t, x_1, x_2|.)}{\partial t} \quad &= \quad -\frac{\partial}{\partial x_1}\left(\kappa\left(\theta - x_1\right) p(t, x_1, x_2|.)\right) \\ &\quad -\eta\mathbb{E}\left[\xi p(t, x_1 - \eta\xi, x_2 - \xi|.)\right] \\ &\quad +x_1\mathbb{E}\left[p(t, x_1 - \eta\xi, x_2 - \xi|.) - p(t, x_1, x_2|.)\right] , \end{aligned} \tag{28}$$

with the initial condition: $p(s, x_1, x_2|s, y_1, y_2) = \delta_{\{x_1 - y_1, x_2 - y_2\}}$.

Proof The pdf's, $p(t + \Delta, x_1, x_2|.)$ at time $t + \Delta$ and $p(t, x_1, x_2|.)$ at time t, are related to moments by a bivariate version of the Moyal expansion.

$$\begin{aligned} &p(t + \Delta, x_1, x_2|.) - p(t, x_1, x_2|.) \\ &= \sum_{n=1}^{\infty}\sum_{j=0}^{n} \frac{(-1)^n}{j!n-j!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\left(M(j, n-j|x_1, x_2)p(t, x_1, x_2|.)\right) \end{aligned} \tag{29}$$

where $M(j, n-j|x_1, x_2)$ is the cross moment of variations:

$$M(j, n-j|x_1, x_2) \quad = \quad \mathbb{E}\left((\lambda_{t+\Delta} - \lambda_t)^j \left(P_{t+\Delta} - P_t\right)^{n-j} |\lambda_t = x_1, P_t = x_2\right) .$$

Since this expansion is not standard, we provide a proof in appendix A. For a small step of time Δ, most of terms in the Moyal expansion are of order $\mathcal{O}\left(\Delta^2\right)$, excepted the following cross moments:

$$\begin{aligned} M(1, 0|x_1, x_2) &= \mathbb{E}\left((\lambda_{t+\Delta} - \lambda_t)^1 \left(P_{t+\Delta} - P_t\right)^0 |\lambda_t = x_1, P_t = x_2\right) \\ &= \left(\kappa\left(\theta - x_1\right) + \eta\mu x_1\right)\Delta + \mathcal{O}\left(\Delta^2\right) , \end{aligned}$$

$$\begin{aligned} M(0, 1|x_1, x_2) \quad &= \quad \mathbb{E}\left((\lambda_{t+\Delta} - \lambda_t)^0 \left(P_{t+\Delta} - P_t\right)^1 |\lambda_t = x_1, P_t = x_2\right) \\ &= \quad \mu x_1\Delta + \mathcal{O}\left(\Delta^2\right) , \end{aligned}$$

and

$$\begin{aligned} M(j, n-j|x_1, x_2) &= \mathbb{E}\left((\lambda_{t+\Delta} - \lambda_t)^j \left(P_{t+\Delta} - P_t\right)^{n-j} |\lambda_t = x_1, P_t = x_2\right) \\ &= \eta^j\mathbb{E}\left[\xi^n\right] x_1\Delta + \mathcal{O}\left(\Delta^2\right) . \end{aligned}$$

Thus, Equation (29) becomes

$$\begin{aligned} &p(t + \Delta, x_1, x_2|.) - p(t, x_1, x_2|.) = \\ &\quad -\frac{\partial}{\partial x_1}\left(\left(\kappa\left(\theta - x_1\right) + \eta\mu x_1\right) p(t, x_1, x_2|.)\right)\Delta - \frac{\partial}{\partial x_2}\left(\mu x_1 p(t, x_1, x_2|.)\right)\Delta \\ &\quad +\sum_{n=2}^{\infty}\sum_{j=0}^{n} \frac{(-1)^n}{j!n-j!}\eta^j\mathbb{E}\left[\xi^n\right]\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\left(x_1 p(t, x_1, x_2|.)\right)\Delta + \mathcal{O}\left(\Delta^2\right) . \end{aligned} \tag{30}$$

Since the partial derivatives in the last term of this equation are also equal to

$$\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\left(x_1\,p(.)\right) \quad = \quad j\frac{\partial^{n-j}}{\partial x_2^{n-j}}\frac{\partial^{j-1}}{\partial x_1^{j-1}}p(.)+x_1\frac{\partial^{n-j}}{\partial x_2^{n-j}}\frac{\partial^j}{\partial^j x_1}p(.),$$

Equation (30) is equal to

$$\begin{aligned}
& p(t+\Delta,x_1,x_2|s,y_1,y_2)-p(t,x_1,x_2|s,y_1,y_2)=\mathcal{O}\left(\Delta^2\right)+ \\
& -\frac{\partial}{\partial x_1}\left(\kappa\left(\theta-x_1\right)p(t,x_1,x_2|.)\right)\Delta-\eta\mu p(t,x_1,x_2|.)\Delta \\
& -x_1\eta\frac{\partial}{\partial x_1}\left(\mu p(t,x_1,x_2|.)\right)\Delta-\frac{\partial}{\partial x_2}\left(\mu x_1 p(t,x_1,x_2|.)\right)\Delta \\
& +\sum_{n=2}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n j}{j!n-j!}\eta^j\mathbb{E}\left[\xi^n\right]\frac{\partial^{n-j}}{\partial x_2^{n-j}}\frac{\partial^{j-1}}{\partial x_1^{j-1}}\left(p(t,x_1,x_2|.)\right)\Delta \\
& +x_1\sum_{n=2}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!n-j!}\eta^j\mathbb{E}\left[\xi^n\right]\frac{\partial^{n-j}}{\partial x_2^{n-j}}\frac{\partial^j}{\partial^j x_1}p(t,x_1,x_2|.)\Delta\,.
\end{aligned} \tag{31}$$

The last term of this equation is related to the Taylor's expansion of $p(t,x_1-\eta\xi,x_2-\xi|.)-p(t,x_1,x_2|.)$:

$$\begin{aligned}
p(t,x_1-\eta\xi,x_2-\xi|.)-p(t,x_1,x_2|.) &= \sum_{n=1}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!(n-j)!}\eta^j\xi^n\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}p(t,x_1,x_2|.) \\
&= -\xi\frac{\partial}{\partial x_2}p(t,x_1,x_2|.)-\eta\xi\frac{\partial}{\partial x_1}p(t,x_1,x_2|.) \\
&\quad +\sum_{n=2}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!(n-j)!}\eta^j\xi^n\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}p(t,x_1,x_2|.)\,.
\end{aligned}$$

Whereas the second term of Equation (31) is also a Taylor's expansion:

$$\begin{aligned}
& \sum_{n=2}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n j}{j!n-j!}\eta^j\xi^n\frac{\partial^{n-j}}{\partial x_2^{n-j}}\frac{\partial^{j-1}}{\partial x_1^{j-1}}\left(p(t,x_1,x_2|.)\right) \\
& =-\eta\xi\sum_{n=1}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!n-j!}\left(\eta\xi\right)^j\xi^{n-j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\frac{\partial^j}{\partial x_1^j}\left(p(t,x_1,x_2|.)\right) \\
& =-\eta\xi\left(p(t,x_1-\eta\xi,x_2-\xi|.)-p(t,x_1,x_2|.)\right)
\end{aligned}$$

Dividing by Δ and considering the limit when $\Delta\to 0$ gives us after calculations,

$$\begin{aligned}
\frac{\partial p(t,x_1,x_2|.)}{\partial t} &= -\frac{\partial}{\partial x_1}\left(\kappa\left(\theta-x_1\right)p(t,x_1,x_2|.)\right)-\eta\mathbb{E}\left[\xi p(t,x_1-\eta\xi,x_2-\xi|.)\right] \\
&\quad +x_1\mathbb{E}\left[p(t,x_1-\eta\xi,x_2-\xi|.)-p(t,x_1,x_2|.)\right]
\end{aligned}$$

and we can conclude. **end**

We do not study the numerical method for solving the FPE ruling the pdf of (λ_t,P_t) because it is outside the scope of this article. However, in a similar manner to Section 4, we can construct the forward equation satisfied by the bivariate Laplace's transform of (λ_t,P_t) and solve it locally around the origin with the same approach as in Section 4.

6 The subordinator

As seen in Proposition 2.2, the autocovariance of the intensity of a Hawkes process with an exponential kernel decays exponentially with time. This model is therefore inappropriate for modeling intensity process with a longer memory of past events.

The solution that we explore in this article consists to introduce periods during which the intensity is motionless. A way for modeling these periods of intensity freeze consists to time-change the process by a subordinator that can be constant over relatively short periods of time. The subordinator is built as the inverse of an α stable process $(U_t)_{t\geq 0}$. This particular type of Lévy processes has a simple moment generating function given by:

$$\mathbb{E}_0\left(e^{-uU_t}\right) \;=\; e^{-t\,u^{\alpha}}\,.$$

The process U_t is a $\frac{1}{\alpha}$ self-similar process, meaning that:

$$U_{at} \;\stackrel{d}{=}\; (at)^{\frac{1}{\alpha}}\,U_1\,.$$

The α-stable processes are strictly increasing and may therefore be used as subordinator but they cannot duplicate motionless periods since they have an infinite activity. Therefore, we use U_t for defining a subordinator, noted $(S_t)_{t\geq 0}$ that is its inverse hitting time:

$$S_t = \inf\{\tau > 0 \,:\, U_\tau \geq t\}\,.$$

By definition and due to the self-similarity property, the cumulative distribution function (cdf) of S_t admits the following representation:

$$\begin{aligned}
P(S_t \leq \tau) &= P(U_\tau \geq t)\\
&= P(\tau^{\frac{1}{\alpha}}U_1 \geq t)\\
&= P\left(\left(\frac{t}{U_1}\right)^{\alpha} \leq \tau\right)
\end{aligned}$$

The distribution of S_t is then the same as the random variable $\left(\frac{t}{U_1}\right)^{\alpha}$. In the rest of the article, the pdf of S_t is denoted by $g(t,\tau) = \frac{d}{d\tau}P(\tau \leq S_t \leq \tau + d\tau)$ and $p_U(t,u)$ is the pdf of U_t. On the other-hand, the self-similarity leads to the relation:

$$\begin{aligned}
P(U_\tau \leq t) &= P(\tau^{\frac{1}{\alpha}}U_1 \leq t)\\
&= P(U_1 \leq t\tau^{-\frac{1}{\alpha}})\,.
\end{aligned}$$

If we derive this last expression with respect to t, we obtain that

$$p_U(\tau,t) \;=\; \tau^{-\frac{1}{\alpha}}p_U(1,t\tau^{-\frac{1}{\alpha}})\,,$$

and infer an important relation linking the pdf of S_t to the pdf of U_t:

$$\begin{aligned}
\tau\, g(t,\tau) &= \frac{t}{\alpha}\left(\tau^{-\frac{1}{\alpha}}p_U(1,t\tau^{-\frac{1}{\alpha}})\right)\\
&= \frac{t}{\alpha}p_U(\tau,t)\,.
\end{aligned} \tag{32}$$

On the other hand, the pdf of S_t is related to the pdf of U_t through the relation

$$\begin{aligned}
g(t,\tau) &= \frac{\partial}{\partial\tau}P(S_t \leq \tau) = -\frac{\partial}{\partial\tau}P(U_\tau \leq t)\\
&= -\frac{\partial}{\partial\tau}\int_0^t p_U(\tau,u)du\,.
\end{aligned}$$

Recalling that the Laplace transform of a function $\int_0^t f(u)du$ is equal to $\omega^{-1}\tilde{f}(\omega)$, the Laplace transform $\tilde{g}(\omega,\tau)$ of $g(t,\tau)$ with respect to time t is therefore equal to:

$$\begin{aligned}
\tilde{g}(\omega,\tau) &= -\frac{\partial}{\partial\tau}\widetilde{\int_0^t p_U(\tau,u)du}\\
&= -\frac{\partial}{\partial\tau}\left(\omega^{-1}e^{-\tau\,\omega^{\alpha}}\right)\\
&= \omega^{\alpha-1}e^{-\tau\,\omega^{\alpha}}\,.
\end{aligned} \tag{33}$$

The Laplace's transform of S_t conditionally to the information available at time zero is given by :

$$\mathbb{E}_0\left[e^{-\omega S_t}\right] = \int_0^\infty e^{-\omega\tau} g(t,\tau)d\tau \tag{34}$$
$$= E_\alpha(-\omega t^\alpha)$$

where E_α is the Mittag-Leffler function (for a proof see e.g. Piryatinska et al. 2005):

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)},$$

where $\Gamma(.)$ is the gamma function. This result clearly reveals that S_t is not a Lévy process since its Laplace's transform does not have an exponential form. However, we can easily compute the moments of S_t by deriving and cancelling its Laplace's transform:

$$\mathbb{E}_0\left(S_t^n\right) = \frac{n!t^{n\alpha}}{\Gamma\left(n\alpha+1\right)}.$$

The Mittag-Leffler function is closely related to the concept of fractional or Caputo's derivative that is also involved in the construction of the fractional Hawkes process. The Caputo's derivative of order $\alpha \in]0,1[$ for a function $h(t,x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $\mathcal{C}^1$ with respect to t is defined by

$$\frac{\partial^\alpha}{\partial t^\alpha} h(t,x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} h(s,x)ds - \frac{h(0,x)}{t^\alpha}. \tag{35}$$

An alternative writing is the following:

$$\frac{\partial^\alpha}{\partial t^\alpha} h(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial}{\partial s} h(s,x)ds \tag{36}$$

When $\alpha = 1$, this derivative corresponds to the derivative with respect to time. Let $\tilde{h}(\omega,x)$ be the usual Laplace's transform of a function $h(t,x)$ with respect to time t. A direct calculation shows that the Laplace's transform of the Caputo's derivative $\frac{\partial^\alpha}{\partial t^\alpha} h(t,x)$ is equal to:

$$\widetilde{\frac{\partial^\alpha h}{\partial t^\alpha}}(\omega,x) = \omega^\alpha \tilde{h}(\omega,x) - \omega^{\alpha-1} h(0,x),$$

which reduces to the familiar form when $\alpha = 1$. Notice that the Caputo's fractional derivative of a power function is given by

$$\frac{\partial^\alpha}{\partial t^\alpha} t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} & p \geq 1\,,\, p \in \mathbb{R} \\ 0 & p \leq 0\,,\, p \in \mathbb{N} \end{cases}.$$

On the other hand, the solution of the fractional differential equation:

$$\frac{\partial^\alpha}{\partial t^\alpha} y(t) = \lambda y(t) \qquad 0 < \alpha < 1$$

with the initial condition $y(0) = b_0$ is precisely the Mittag-Leffler function $y(t) = E_\alpha(\lambda t^\alpha)$.

7 The fractional Hawkes process

The rest of this article focuses on the properties of the time-changed Hawkes process (λ_{S_t}, P_{S_t}) where λ_t is ruled by the dynamics in Equation (3) and S_t is the inverse of an α-stable subordinator. We denote by $p_\alpha(t,x|s,y)$ its transition pdf that is defined as follows:

$$p_\alpha(t,x|s,y)dx = P\left(\lambda_{S_t} \in [x, x+dx] \mid \lambda_{S_s} = y\right).$$

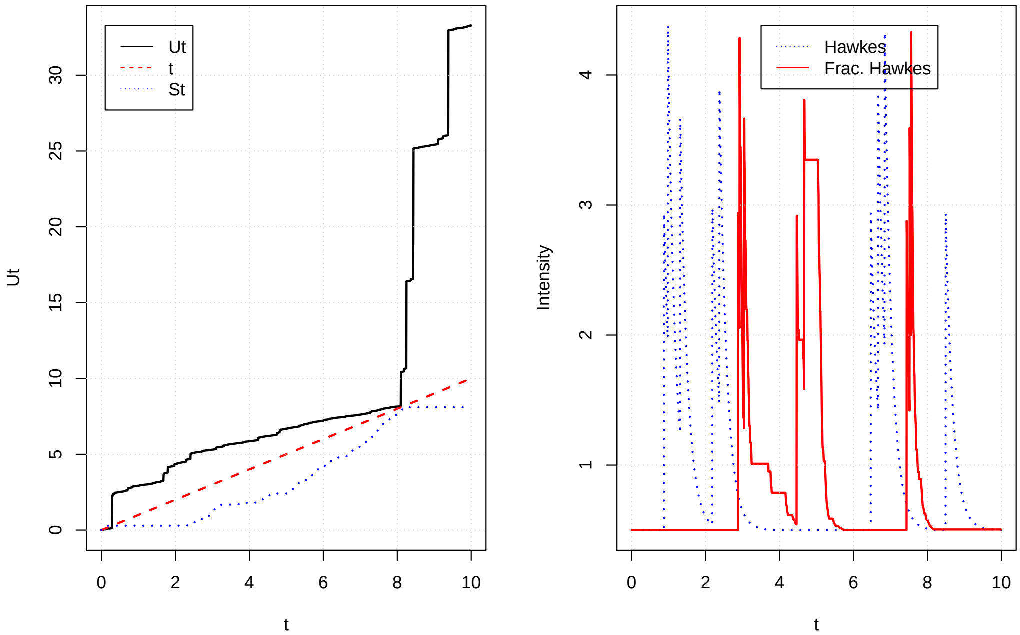

Figure 3: Left plot: simulated sample path of U_t and S_t. Right plot: simulated sample of λ_{S_t}.

Proposition 7.1. *The pdf $p_\alpha(t,x|0,y)$ is solution of a time-fractional Fokker-Planck equation (FFPE):*

$$\begin{aligned}\frac{\partial^\alpha p_\alpha(t,x|0,y)}{\partial t^\alpha} &= -\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)p_\alpha(t,x|0,y)\right)-\eta\mathbb{E}\left[\xi p_\alpha(t,x-\eta\xi|0,y)\right] \\ &\quad +x\mathbb{E}\left[p_\alpha(t,x-\eta\xi|0,y)-p_\alpha(t,x|0,y)\right].\end{aligned} \tag{37}$$

with the condition $p(0,x|0,y)=\delta_{\{x-y\}}$. This is also solution of the fractional backward Kolmogorov's equation:

$$\begin{aligned}\frac{\partial^\alpha p_\alpha(t,x|0,y)}{\partial t^\alpha} &= \kappa\left(\theta-y\right)\frac{\partial p_\alpha(t,x|0,y)}{\partial y} \\ &\quad +y\mathbb{E}\left[p_\alpha(t,x|0,y+\eta\xi)-p_\alpha(t,x|0,y)\right].\end{aligned}$$

Proof To lighten developments, we momentarily adopt the notations: $p(t,x):=p(t,x|0,y)$, $p_\alpha(t,x):=p_\alpha(t,x|0,y)$. As $g(t,\tau)$ is the pdf of $S_t|S_0=0$ and from the independence between P_t and S_t, we infer that

$$p_\alpha(t,x) = \int_0^\infty p(\tau,x)g(t,\tau)d\tau\,.$$

The Laplace's transform of $p_\alpha(t,x)$ with respect to time t is thus given by

$$\begin{aligned}\tilde{p}_\alpha(\omega,x) &= \int_0^\infty\int_0^\infty p(\tau,x)e^{-\omega t}g(t,\tau)d\tau dt \\ &= \int_0^\infty p(\tau,x)\,\tilde{g}(\omega,\tau)\,d\tau\,,\end{aligned}$$

where $\tilde{g}(\omega,\tau)=\int_0^\infty e^{-\omega t}g(t,\tau)dt$ is the Laplace's transform of $g(t,\tau)$ with respect to time. Since this transform is given by Equation (33), $\tilde{p}_\alpha(.)$ is equal to:

$$\begin{aligned}\tilde{p}_\alpha(\omega,x) &= \omega^{\alpha-1}\int_0^\infty p(\tau,x)\,e^{-\tau\omega^\alpha}\,d\tau \\ &= \omega^{\alpha-1}\tilde{p}(\omega^\alpha,x)\,.\end{aligned}$$

where $\tilde{p}(\omega,x)=\int_0^\infty e^{-\omega t}p(t,x)dt$ is the Laplace's transform of $p(t,x)$ with respect to time. From the FPE (9), we deduce that $\tilde{p}(\omega,x)$ is then solution of

$$\begin{aligned}\omega\tilde{p}(\omega,x)-p(0,x) &= -\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)\tilde{p}(\omega,x)\right)\\ &\quad -\eta\mathbb{E}\left[\xi\tilde{p}(\omega,x-\eta\xi)\right]+x\mathbb{E}\left[\tilde{p}(\omega,x-\eta\xi)-\tilde{p}(\omega,x)\right]\,.\end{aligned}$$

As $\tilde{p}_\alpha(\omega,x)=\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x)$, replacing ω by ω^α leads to

$$\begin{aligned}\omega^\alpha\tilde{p}(\omega^\alpha,x)-p(0,x) &= -\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)\tilde{p}(\omega^\alpha,x)\right)\\ &\quad +x\mathbb{E}\left[\tilde{p}(\omega^\alpha,x-\eta\xi)-\tilde{p}(\omega^\alpha,x)\right]-\eta\mathbb{E}\left[\xi\tilde{p}(\omega^\alpha,x-\eta\xi)\right]\,.\end{aligned}$$

If we multiply this last equation by $\omega^{\alpha-1}$, we obtain that

$$\begin{aligned}\omega^\alpha\left(\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x)\right)-\omega^{\alpha-1}p(0,x) &= -\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x)\right)\\ &\quad +x\mathbb{E}\left[\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x-\eta\xi)-\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x)\right]-\eta\mathbb{E}\left[\xi\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x-\eta\xi)\right]\,.\end{aligned}$$

Since $p_\alpha(0,x)=p(0,x)$, we have that

$$\begin{aligned}\omega^\alpha\tilde{p}_\alpha(\omega,x)-\omega^{\alpha-1}p_\alpha(0,x) &= -\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)\tilde{p}_\alpha(\omega,x)\right)\\ &\quad +x\mathbb{E}\left[\tilde{p}_\alpha(\omega,x-\theta\xi)-\tilde{p}_\alpha(\omega,\lambda)\right]-\eta\mathbb{E}\left[\xi\tilde{p}_\alpha(\omega,x-\theta\xi)\right]\,.\end{aligned}$$

The left-hand term is the Laplace's transform of the Caputo's derivative of $p_\alpha(t,x)$. Therefore this last equation is also the Laplace's transform of the FFPE (37). On the other hand, the density is solution of the backward Kolmogorov equation:

$$-\frac{\partial p(t,x|s,y)}{\partial s} \;=\; \kappa\left(\theta-y\right)\frac{\partial p(t,x|s,y)}{\partial y}+y\mathbb{E}\left[p(t,x|s,y+\eta\xi)-p(t,x|s,y)\right]\,. \tag{38}$$

As the distribution of $\lambda_t|\lambda_s$ is time-homogeneous in the sense that

$$p(t,x|s,y)=p(t-s,x|0,y)\,,$$

then the backward equation is rewritten as

$$\frac{\partial p(t,x|0,y)}{\partial t} \;=\; \kappa\left(\theta-y\right)\frac{\partial p(t,x|0,y)}{\partial y}+y\mathbb{E}\left[p(t,x|0,y+\eta\xi)-p(t,x|0,y)\right]\,. \tag{39}$$

Therefore, the Laplace's transform $\tilde{p}(t,x|0,y)$ with respect to time is solution of

$$\begin{aligned}&\omega\tilde{p}(\omega,x|0,y)-p(0,x|0,y)\\ &\quad =\kappa\left(\theta-y\right)\frac{\partial}{\partial y}\tilde{p}(\omega,x|0,y)+y\mathbb{E}\left[\tilde{p}(\omega,x|0,y+\theta\xi)-\tilde{p}(\omega,x|0,y)\right]\end{aligned}$$

Since $\tilde{p}_\alpha(\omega,x|0,y)=\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x|0,y)$, replacing ω by ω^α and multiplying by $\omega^{\alpha-1}$ leads to

$$\begin{aligned}&\omega^\alpha\left(\tilde{p}_\alpha(\omega,x|0,y)\right)-\omega^{\alpha-1}\tilde{p}_\alpha(0,x|0,y)=\kappa\left(\theta-y\right)\frac{\partial}{\partial y}\tilde{p}_\alpha(\omega,x|0,y)\\ &+y\mathbb{E}\left[\tilde{p}_\alpha(\omega,x|0,y+\theta\xi)-\tilde{p}_\alpha(\omega,x|0,y)\right]\,,\end{aligned}$$

where we have also used the relation $p_\alpha(0,x|0,y)=p(0,x|0,y)$. The left-hand term is the Caputo's derivative of $p_\alpha(t,\omega)$ and therefore we get the backward Kolmogorov Equation. **End**

We denote by $p_\alpha(t,x_1,x_2|s,y_1,y_2)$ the bivariate probability density function $(\lambda_{S_t},P_{S_t})_{t\geq 0}$ that is defined as follows:

$$p_\alpha(t,x_1,x_2|s,y_1,y_2) \;=\; P\left(\lambda_{S_t}\in[x_1,x_1+dx]\,,P_{S_t}\in[x_2,x_2+dx]\,|\lambda_s=y_1,P_s=y_2\right)$$

for $s\leq t$ and $x_1,x_2,y_1,y_2\in\mathbb{R}^+$. This joint pdf is obtained in a similar way to the one of the time-changed intensity. For this reason, we do not provide the proof.

Proposition 7.2. *The pdf $p_\alpha(t,x_1,x_2|s,y_1,y_2)$ of $(\lambda_{S_t},P_{S_t})_{t\geq 0}$ is solution of a time-fractional Fokker-Planck equation:*

$$\begin{aligned}\frac{\partial^\alpha p_\alpha(t,x_1,x_2|.)}{\partial t^\alpha} \;=\; & -\frac{\partial}{\partial x_1}\left(\kappa\left(\theta-x_1\right)p_\alpha(t,x_1,x_2|.)\right)-\eta\mathbb{E}\left[\xi p_\alpha(t,x_1-\eta\xi,x_2-\xi|.)\right] \\ & +x_1\mathbb{E}\left[p_\alpha(t,x_1-\eta\xi,x_2-\xi|.)-p_\alpha(t,x_1,x_2|.)\right]\,,\end{aligned} \tag{40}$$

with the initial condition: $p_\alpha(s,x_1,x_2|s,y_1,y_2)=\delta_{\{x_1-y_1,x_2-y_2\}}$

8 Moments of the fractional intensity

As the fractional Hawkes process is a time-changed model, then moments of λ_{S_t} are directly related to those of λ_t and S_t. While the expectation the fractional intensity is easy to establish, the variance and autocovariance require more attention.

Proposition 8.1. *The expectation of λ_{S_t} is equal to*

$$\mathbb{E}_0\left[\lambda_{S_t}\right]=E_\alpha\left(\left(\eta\mu-\kappa\right)t^\alpha\right)\left(\lambda_0+\frac{\kappa\theta}{\eta\mu-\kappa}\right)-\frac{\kappa\theta}{\eta\mu-\kappa}\,. \tag{41}$$

This result is found by combining Equations (6) and (34). The autocovariance of the fractional intensity is a function of the Laplace's transforms of S_t+S_s and S_t-S_s as stated in the proposition:

Proposition 8.2. *Let us denote $\beta:=-\left(\eta\mu-\kappa\right)>0$ then for $s\leq t$, the covariance between λ_{S_s} and λ_{S_t} is given by*

$$\begin{aligned}\mathbb{C}_0\left(\lambda_{S_s},\lambda_{S_t}\right)= & \frac{\rho_1\lambda_0+\rho_2}{\eta\mu-\kappa}\left(\mathbb{E}_0\left(e^{-\beta(S_t+S_s)}\right)-E_\alpha\left(-\beta t^\alpha\right)\right)+ \\ & \frac{\rho_2}{2\left(\eta\mu-\kappa\right)}\left(\mathbb{E}_0\left(e^{-\beta(S_t-S_s)}\right)-\mathbb{E}_0\left(e^{-\beta(S_t+S_s)}\right)\right)+ \\ & \left(\lambda_0+\frac{\kappa\theta}{\eta\mu-\kappa}\right)^2\left(\mathbb{E}_0\left(e^{-\beta(S_s+S_t)}\right)-E_\alpha\left(-\beta t^\alpha\right)E_\alpha\left(-\beta s^\alpha\right)\right)\,.\end{aligned} \tag{42}$$

Proof Let us denote by $(\mathcal{H}_t)_{t\geq 0}$, the filtration of $(S_t)_{t\geq 0}$. For $s\leq t$, the covariance between λ_{S_s} and λ_{S_t} is the sum of the expected $\mathcal{H}_t$-conditional covariance and of the covariance of $\mathcal{H}_t$-conditional expectations:

$$\begin{aligned}\mathbb{C}_0\left(\lambda_{S_s},\lambda_{S_t}\right) \;=\; & \mathbb{E}\left(\mathbb{C}_0\left(\lambda_{S_s},\lambda_{S_t}\right)|\mathcal{H}_t\vee\mathcal{F}_0\right)+ \\ & \mathbb{C}_0\left(\mathbb{E}\left(\lambda_{S_s}|\mathcal{H}_t\vee\mathcal{F}_0\right),\mathbb{E}\left(\lambda_{S_t}|\mathcal{H}_t\vee\mathcal{F}_0\right)\right)\,.\end{aligned} \tag{43}$$

From equation (7), we directly infer that

$$\begin{aligned}&\mathbb{E}\left(\mathbb{C}_0\left(\lambda_{S_s},\lambda_{S_t}\right)|\mathcal{H}_t\vee\mathcal{F}_0\right)= \\ &\quad\frac{\rho_1\lambda_0+\rho_2}{\eta\mu-\kappa}\left(\mathbb{E}_0\left(e^{-\beta(S_t+S_s)}\right)-\mathbb{E}_0\left(e^{-\beta S_t}\right)\right)+ \\ &\quad\frac{\rho_2}{2\left(\eta\mu-\kappa\right)}\left(\mathbb{E}_0\left(e^{-\beta(S_t-S_s)}\right)-\mathbb{E}_0\left(e^{-\beta(S_t+S_s)}\right)\right)\,.\end{aligned}$$

In order to calculate the covariance between conditional expectations, we introduce the following notations:

$$\begin{aligned}
Y_s := \mathbb{E}\left(\lambda_{S_s}|\mathcal{H}_t \vee \mathcal{F}_0\right) &= e^{(\eta\mu-\kappa)S_s}\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\left(e^{(\eta\mu-\kappa)S_s}-1\right) , \\
Y_t := \mathbb{E}\left(\lambda_{S_t}|\mathcal{H}_t \vee \mathcal{F}_0\right) &= e^{(\eta\mu-\kappa)S_t}\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\left(e^{(\eta\mu-\kappa)S_t}-1\right) .
\end{aligned}$$

and

$$\mathbb{C}_0\left(\mathbb{E}\left(\lambda_{S_s}|\mathcal{H}_t \vee \mathcal{F}_0\right), \mathbb{E}\left(\lambda_{S_t}|\mathcal{H}_t \vee \mathcal{F}_0\right)\right) = \mathbb{E}_0\left(Y_sY_t\right) - \mathbb{E}_0\left(Y_t\right)\mathbb{E}_0\left(Y_t\right) .$$

A direct calculation leads to

$$\begin{aligned}
\mathbb{E}_0\left(Y_sY_t\right) &= \mathbb{E}_0\left(e^{(\eta\mu-\kappa)(S_s+S_t)}\right)\left[\lambda_0^2 + \frac{2\kappa\theta}{\eta\mu-\kappa}\lambda_0 + \left(\frac{\kappa\theta}{\eta\mu-\kappa}\right)^2\right] \\
&\quad -\mathbb{E}_0\left(e^{(\eta\mu-\kappa)S_s}\right)\left(\left(\frac{\kappa\theta}{\eta\mu-\kappa}\right)^2 + \frac{\kappa\theta\lambda_0}{\eta\mu-\kappa}\right) \\
&\quad -\mathbb{E}_0\left(e^{(\eta\mu-\kappa)S_t}\right)\left(\left(\frac{\kappa\theta}{\eta\mu-\kappa}\right)^2 + \frac{\kappa\theta\lambda_0}{\eta\mu-\kappa}\right) + \left(\frac{\kappa\theta}{\eta\mu-\kappa}\right)^2 .
\end{aligned}$$

Given that

$$\begin{aligned}
\mathbb{E}_0\left(Y_s\right) &= \mathbb{E}_0\left(e^{(\eta\mu-\kappa)S_s}\right)\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\left(\mathbb{E}_0\left(e^{(\eta\mu-\kappa)S_s}\right)-1\right) , \\
\mathbb{E}_0\left(Y_t\right) &= \mathbb{E}_0\left(e^{(\eta\mu-\kappa)S_t}\right)\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\left(\mathbb{E}_0\left(e^{(\eta\mu-\kappa)S_t}\right)-1\right) ,
\end{aligned}$$

we infer the result. **end**

Before providing analytical expressions for the Laplace's transforms of $S_t + S_s$ and $S_t - S_s$, we calculate the variance of the fractional intensity:

Corollary 8.3. *If we denote by* $\rho_1 = \eta^2(\psi^2+\mu^2)$ *and* $\rho_2 = \frac{\eta^2\kappa\theta(\psi^2+\mu^2)}{\eta\mu-\kappa}$ *the variance of* λ_{S_t} *is given by*

$$\begin{aligned}
\mathbb{V}_0\left[\lambda_{S_t}\right] &= \frac{\rho_1\lambda_0+\rho_2}{\eta\mu-\kappa}\left(E_\alpha\left(2\left(\eta\mu-\kappa\right)t^\alpha\right) - E_\alpha\left(\left(\eta\mu-\kappa\right)t^\alpha\right)\right) \\
&\quad + \frac{\rho_2}{2\left(\eta\mu-\kappa\right)}\left(1 - E_\alpha\left(2\left(\eta\mu-\kappa\right)t^\alpha\right)\right) + \\
&\quad \left(\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\right)^2\left(E_\alpha\left(2\left(\eta\mu-\kappa\right)t^\alpha\right) - \left(E_\alpha\left(\left(\eta\mu-\kappa\right)t^\alpha\right)\right)^2\right) .
\end{aligned} \tag{44}$$

This corollary is a direct consequence of Proposition 2.2. The Laplace's transforms of expectations involved in the autocovariance, in Equation (42), admit integral representations presented in the next two propositions.

Proposition 8.4. *For* $t \geq s$ *and* $\beta < 0$*, The Laplace's transform of* $e^{-\beta(S_t-S_s)}$ *is equal to*

$$\mathbb{E}_0\left(e^{-\beta(S_t-S_s)}\right) = \frac{\alpha\beta}{\Gamma\left(1+\alpha\right)}\int_{y=0}^{s} y^{\alpha-1}E_\alpha\left(-\beta(t-y)^\alpha\right)dy + E_\alpha\left(-\beta t^\alpha\right) . \tag{45}$$

A proof of this result may be found in Leonenko et al. (2013, b), Pproposition 3.1. At the best of our knowledge, we haven't found in the literature any expression for the Laplace's transform of $S_t + S_s$, detailed in the next proposition.

Proposition 8.5. *For $t \geq s$ and $\beta < 0$, The Laplace's transform of $e^{-\beta(S_t+S_s)}$ is equal to*

$$\mathbb{E}_0\left(e^{-\beta(S_t+S_s)}\right) = \frac{1}{2}\int_{y=0}^{s}\left(\frac{d}{dy}E_\alpha\left(-2\beta y^\alpha\right)\right)E_\alpha\left(-\beta(t-y)^\alpha\right)dy + E_\alpha\left(-\beta t^\alpha\right) \tag{46}$$

where

$$\frac{d}{dy}E_\alpha\left(-2\beta y^\alpha\right) = \sum_{n=0}^{\infty}\frac{(-2\beta)^n\, y^{n\alpha-1}}{\Gamma(n\alpha)}.$$

Proof We respectively denote by $h(u,v)$ and $H(u,v)$, the bivariate pdf and cdf of the pair (S_s, S_t). By definition $H(u,\infty) = P\left(S_s \leq u\right)$, $H(\infty, v) = P\left(S_t \leq v\right)$ and $H(\infty,\infty) = 1$. The Laplace's transform is hence equal to

$$\mathbb{E}_0\left(e^{-\beta(S_t+S_s)}\right) = \int_0^\infty\int_0^\infty \underbrace{e^{-\beta(v+u)}}_{f(u,v)}H(du,dv).$$

Using a bivariate by part integration leads to

$$\begin{aligned}\int_0^\infty\int_0^\infty f(u,v)H(du,dv) &= \int_0^\infty\int_0^\infty H([u,\infty]\times[v,\infty])f(du,dv) \\ &+\int_0^\infty H([u,\infty]\times[0,\infty])f(du,0) \\ &+\int_0^\infty H([0,\infty]\times[v,\infty])f(0,dv) \\ &+f(0,0)H([0,\infty]\times[0,\infty])W,\end{aligned} \tag{47}$$

where $f(0,0)H([0,\infty]\times[0,\infty]) = 1$. On the other hand,

$$\begin{aligned}H([u,\infty]\times[0,\infty]) &= P\left(S_s \geq u\right) \\ &= 1 - P\left(S_s \leq u\right),\end{aligned}$$

therefore, the second term in Equation (47) becomes

$$\begin{aligned}\int_0^\infty H([u,\infty]\times[0,\infty])f(du,0) &= -\beta\int_0^\infty e^{-\beta u}\left(1-P\left(S_s\leq u\right)\right)du \\ &= \left[e^{-\beta u}\left(1-P\left(S_s\leq u\right)\right)\right]_0^\infty + \mathbb{E}\left(e^{-\beta S_s}\right) \\ &= E_\alpha\left(-\beta s^\alpha\right) - 1.\end{aligned}$$

In a similar way, we find that the third term in Equation (47) is:

$$\begin{aligned}\int_0^\infty H([0,\infty]\times[v,\infty])f(0,dv) &= -\beta\int_0^\infty e^{-\beta v}\left(1-P\left(S_t\leq v\right)\right)dv \\ &= E_\alpha\left(-\beta t^\alpha\right) - 1.\end{aligned}$$

The first term of Equation (47) is a double integral

$$\int_0^\infty\int_0^\infty H([u,\infty]\times[v,\infty])f(du,dv) = \int_0^\infty\int_0^\infty P\left(u\leq S_s, v\leq S_t\right)\beta^2 e^{-\beta(v+u)}dudv$$

Since S_t is increasing and discontinuous and that $P\left(u\leq S_s, v\leq S_t\right) = P\left(u\leq S_s\right)$ for $u > v$, this integral may be split:

$$\begin{aligned}\int_0^\infty\int_0^\infty P\left(u\leq S_s, v\leq S_t\right)\beta^2 e^{-\beta(v+u)}dudv &= \int_0^\infty\int_0^v P\left(u\leq S_s, v\leq S_t\right)\beta^2 e^{-\beta(v+u)}dudv \\ &+\int_0^\infty\int_v^\infty P\left(u\leq S_s\right)\beta^2 e^{-\beta(v+u)}dudv.\end{aligned} \tag{48}$$

Integrating by parts allows us to rewrite the second term of this last equation:

$$\begin{aligned}
&\int_0^\infty \int_v^\infty P\left(u \leq S_s\right) \beta^2 e^{-\beta(v+u)} du dv = \int_0^\infty \int_0^u P\left(u \leq S_s\right) \beta^2 e^{-\beta(v+u)} dv du \\
&= -\beta \int_0^\infty P\left(u \leq S_s\right) \left(e^{-\beta(2u)} - e^{-\beta u}\right) du \\
&= -\beta \int_0^\infty P\left(u \leq S_s\right) e^{-2\beta u} du + \beta \int_0^\infty P\left(u \leq S_s\right) e^{-\beta u} du \,.
\end{aligned}$$

The first and second integrals are respectively equal to

$$\begin{aligned}
&-\beta \int_0^\infty P\left(u \leq S_s\right) e^{-2\beta u} du = \int_0^\infty \left(1 - P\left(S_s \leq u\right)\right) \left(-\beta e^{-2\beta u}\right) du \\
&= \left[P\left(u \leq S_s\right) \frac{1}{2} e^{-2\beta u}\right]_{u=0}^{u=\infty} + \frac{1}{2} \int_0^\infty g\left(s, u\right) \left(e^{-2\beta u}\right) du \\
&= \frac{1}{2}\left(E_\alpha\left(-2\beta s^\alpha\right) - 1\right) \,,
\end{aligned}$$

and

$$\beta \int_0^\infty P\left(u \leq S_s\right) e^{-\beta u} du = 1 - E_\alpha\left(-\beta s^\alpha\right) \,.$$

The first term of Equation (48) is hence given by:

$$\int_0^\infty \int_v^\infty P\left(u \leq S_s\right) \beta^2 e^{-\beta(v+u)} du dv = \frac{1}{2} + \frac{1}{2} E_\alpha\left(-2\beta s^\alpha\right) - E_\alpha\left(-\beta s^\alpha\right) \,.$$

On the other hand , for $s \leq t$ and $u \leq v$, given that U_t has independent self-similar increments, the cdf of (S_s, S_t) is such that

$$\begin{aligned}
P\left(u \leq S_s, v \leq S_t\right) &= P\left(U_u \leq s, U_v \leq t\right) \\
&= P\left(U_u \leq s, U_u + \left(U_v - U_u\right) \leq t\right) \\
&= \int_{y=0}^s p_U(u,y) dy \int_0^{t-y} p_U(v-u,x) dx dy
\end{aligned}$$

From equation (32), we know that the pdf of U_t is related to the one of S_t as follows:

$$\begin{aligned}
\frac{y}{\alpha} p_U(u,y) &= u g(y,u) \,, \\
\frac{x}{\alpha} p_U(v-u,x) &= (v-u) g(x, v-u) \,.
\end{aligned}$$

The first term of Equation (48) is developped as follows:

$$\begin{aligned}
&\beta^2 \int_{u=0}^\infty \int_{v=u}^\infty P\left(u \leq S_s, v \leq S_t\right) e^{-\beta(v+u)} dv du \qquad (49) \\
&= \beta^2 \int_{u=0}^\infty \int_{v=u}^\infty \int_{y=0}^s p_U(u,y) dy \int_{x=0}^{t-y} p_U(v-u,x) dx dy e^{-\beta(v+u)} dv du \\
&= \beta^2 \int_{y=0}^s \int_{x=0}^{t-y} \int_{u=0}^\infty p_U(u,y) \int_{v=u}^\infty p_U(v-u,x) e^{-\beta(v+u)} dv du dx dy \\
&= \beta^2 \int_{y=0}^s \frac{\alpha}{y} \int_{x=0}^{t-y} \frac{\alpha}{x} \int_{u=0}^\infty u g(y,u) \int_{v=u}^\infty \left(v-u\right) g(x, v-u) e^{-\beta(v-u+2u)} dv du dx dy \\
&= \beta^2 \int_{y=0}^s \frac{\alpha}{y} \int_{x=0}^{t-y} \frac{\alpha}{x} \int_{u=0}^\infty u g(y,u) e^{-2\beta u} du \int_{z=0}^\infty z g(x,z) e^{-\beta z} dz dx dy
\end{aligned}$$

Leonenko et al. (2013, b) in the proof of proposition 3.1, have shown that

$$\int_{z=0}^{\infty} zg(x,z)e^{-\beta z}dz = -\frac{x}{\alpha\beta}\frac{d}{dx}E_\alpha\left(-\beta x^\alpha\right),$$

and that

$$\int_{u=0}^{\infty} ug(y,u)e^{-2\beta u}du = -\frac{y}{2\alpha\beta}\frac{d}{dy}E_\alpha\left(-2\beta y^\alpha\right).$$

Equation (49) may be simplified as follows:

$$\begin{aligned}
&\int_0^\infty \int_0^v P\left(u \leq S_s, v \leq S_t\right)\beta^2 e^{-\beta(v+u)}dudv \\
&= -\frac{1}{2}\int_{y=0}^{s}\frac{d}{dy}E_\alpha\left(-2\beta y^\alpha\right)\int_{x=0}^{t-y}\frac{d}{dx}E_\alpha\left(-\beta x^\alpha\right)dxdy \\
&= -\frac{1}{2}\int_{y=0}^{s}\frac{d}{dy}E_\alpha\left(-2\beta y^\alpha\right)\left(E_\alpha\left(-\beta(t-y)^\alpha\right)-1\right)dy \\
&= -\frac{1}{2}\int_{y=0}^{s}\left(\frac{d}{dy}E_\alpha\left(-2\beta y^\alpha\right)\right)E_\alpha\left(-\beta(t-y)^\alpha\right)dy + \frac{1}{2}\left(E_\alpha\left(-2\beta s^\alpha\right)-1\right).
\end{aligned}$$

Collecting all terms allows us to deduce Equation (46). **end**

Figure 4 shows the autocorrelogram and the expected intensity for a fractional Hawkes process with constant marks ($\eta = 5$, $\theta = 5$, $\kappa = 8.5$, $\xi = 2$, $s = 0.1$ and $t = 0.1$ to 2.0). The upper graph clearly emphasizes that the autocorrelation decays at a sub-exponential rate for values of α lower than one. The long term mean of the intensity is equal to $\lim_{t\to\infty}\mathbb{E}\left(\lambda_t\right) = 28.33$ and λ_0 is set to $\theta = 5$. The second graph reveals that the expected intensity converges to 28.33 but a lower pace than a Hawkes process when α is lower than one.

9 Laplace's transform of $p_\alpha(t,x|0,y)$ and numerical solution

Let us denote by $\phi_\alpha(.)$ the Laplace's transform of the transition pdf $p_\alpha(t,x|0,y)$ of the time changed intensity:

$$\begin{aligned}
\phi_\alpha(t,z|0,y) &= \mathbb{E}\left(e^{-z\lambda_{S_t}}|\lambda_0 = y\right) \\
&= \int_0^\infty \phi(\tau,z|0,y)g(t,\tau)d\tau
\end{aligned}$$

for $z \in \mathbb{R}^+$. This Laplace's transform is solution of a forward equation.

Proposition 9.1. *The Laplace's transform $\phi_\alpha(t,z|0,y)$ is solution of the following forward equation:*

$$\frac{\partial^\alpha \phi_\alpha(t,z|0,y)}{\partial t^\alpha} = -\kappa\theta z\,\phi_\alpha(t,z|0,y) + \left(1-\kappa z - \mathbb{E}\left(e^{-z\eta\xi}\right)\right)\frac{\partial\phi_\alpha(t,z|0,y)}{\partial z} \tag{50}$$

with the initial condition $\phi_\alpha(0,z|0,y) = e^{-zy}$.

We do not provide the proof of this results since it is obtained in the same manner as Proposition 7.1.

We now propose a numerical method for solving the forward equation (50) for z and t in the neighbourhood of zero. As in Section 4, we consider a expansion of the Laplace's transform $\phi_\alpha(t,z|0,y)$. From e.g. Usero (2008), we know that for any continuous differentiable function, $u(t,x)$ with respect to time and x may be rewritten as an infinite sum

$$u(t,x) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^{k\alpha},$$

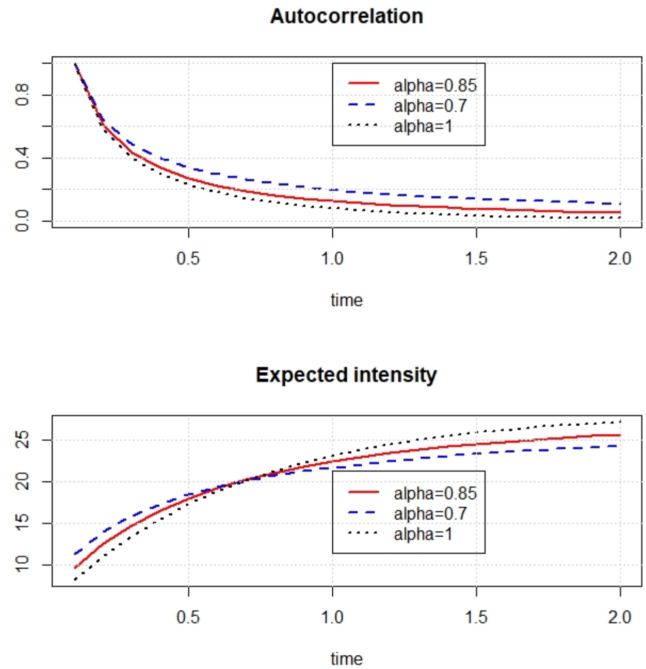

Figure 4: Upper graph: autocorrelogram of fractional Hawkes processes. Lower graph: expected intensity for different α.

where

$$U_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\left(\frac{\partial}{\partial t^{\alpha}}\right)^k u(t,x)\right]_{t=t_0}.$$

Therefore we assume that this Laplace's transform can be approached by the following sum:

$$\phi_\alpha(t,z|0,y) \approx \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} w_\alpha(k,h)\times z^h t^{\alpha k}, \tag{51}$$

The differential weights in this sum are defined by:

$$w_\alpha(k,h) = \frac{1}{\Gamma(k\alpha+1)\,h!}\left[\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^h}\phi_\alpha(t,z|.)\right]_{t=0,z=0}. \tag{52}$$

Notice that we use the notation

$$\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^h} = \left(\frac{\partial}{\partial t^{\alpha}}\right)^k \frac{\partial^h}{\partial z^h},$$

but the reader must be aware that $\left(\frac{\partial}{\partial t^{\alpha}}\right)^k \neq \frac{\partial^k}{\partial t^{k\alpha}}$ for the Caputo's derivative. The equation (51) is exact if $\phi_\alpha(t,z|0,y)$ is the product of one function of t and of one function of z. Differential weights are computed recursively.

Proposition 9.2. *The Laplace transform is approximated by the sum*

$$\phi_\alpha(t,z|0,y) = \sum_{k=0}^{K}\sum_{h=0}^{H} w_\alpha(k,h)\times z^h t^{\alpha k}, \tag{53}$$

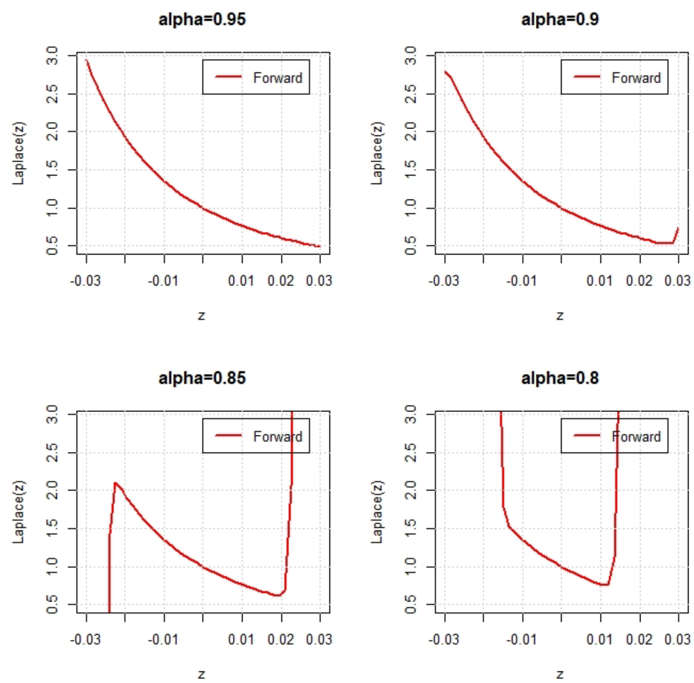

Figure 5: Laplace's transform of λ_{S_t} computed with the forward and backward equations. $\eta = 7$, $\theta = 5$, $\kappa = 8.5$ and $\xi = 1$. The maturity is $t = 0.5$. $K = 50$ and $H = 80$.

with differential weights satisfying the following recursive equation:

$$\begin{aligned}\frac{\Gamma\left((k+1)\alpha+1\right)}{\Gamma\left(k\alpha+1\right)}w(k+1,h) &= -\kappa\theta w(k,h-1)-\kappa\, h\, w(k,h)+(h+1)w(k,h+1) \\ &\quad -\sum_{j=0}^{h}\frac{j+1}{(h-j)!}(-1)^{h-j}\mathbb{E}\left[(\eta\xi)^{h-j}\right]w(k,j+1)\,. \end{aligned} \tag{54}$$

The initial conditions that are used for initializing the recursion are:

$$\begin{aligned} w(0,h) &= \frac{1}{h!}\left(-y\right)^{h}\,, \\ w(k,0) &= 0 \quad k>0\,. \end{aligned}$$

Proof To establish the recursion (54), we apply the differential operator $\frac{1}{\Gamma(k\alpha+1)h!}\left[\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^{h}}\right]_{t=0,z=0}$ to the forward equation (16). Since $e^{-yz} = \sum_{h=0}^{\infty}\frac{1}{h!}\left(-y\right)^{h}z^{h}$ therefore $w(0,h) = \frac{1}{h!}\left(-y\right)^{h}$. **end**

Figure 5 shows the Laplace's transform approached by the sum (53). As in the non-fractional case, the accuracy deteriorates with the distance to zero. However, it is accurate enough to compute numerically the first four moments as shown in Table 2 by deriving the Laplace's transform in the neighbourhood of $z = 0$. This table also provides the theoretical expectation and variance computed with Equations (41) and (44).

10 Conclusions

The first part of this article presents the forward or Fokker-Planck Equation (FPE) for a Hawkes process with an exponential decaying memory. Due to self-exciting jumps, the FPE differs from the one of a pure jump process. We also provide an numerical method for solving the forward equation satisfied by the Laplace's transform of the intensity process. This method is enough accurate in the neighbourhood of zero for computing the first four moments. Next, we provide the FPE ruling the joint pdf of point and intensity process.

	Numerical				Theoretical	
α	$\mathbb{E}_0(\lambda_t)$	$\mathbb{V}_0(\lambda_t)$	$\mathbb{S}_0(\lambda_t)$	$\mathbb{K}_0(\lambda_t)$	$\mathbb{E}_0(\lambda_t)$	$\mathbb{V}_0(\lambda_t)$
0.95	28.3333	18.9750	1.5213	6.4740	28.3333	18.9744
0.90	28.3333	18.9709	1.5532	6.7081	28.3333	18.9695
0.85	28.3333	18.9516	1.5814	6.9127	28.3333	18.9510
0.80	28.3333	18.9523	1.6055	7.0893	28.3333	18.9231

Table 2: Comparison of numerical and theoretical moments, $\eta = 7$, $\theta = 8.5$, and $\xi = 1$. The maturity is $t = 0.5$. $y_0 = \lim_{t\to\infty} \mathbb{E}(\lambda_t) = 28.3333$. $K = 50$ and $H = 80$.

The second part of the article studies the features of the fractional Hawkes process. We build this process with a non-Markov time-change. We show that the pdf of its intensity satisfies a forward equation in which the derivative with respect to time is replaced by the Caputo's derivative. Next, we find closed form expressions for expectation, variance and autocovariance of the intensity. Finally, we develop a numerical method for computing the Laplace's transform of the fractional intensity. This numerical scheme allows us to estimate the first moments.

This work opens the way to further research. As for Hawkes processes, we have very few information about the properties of N_{S_t}. This deserves further investigation. Secondly, It would be interesting to develop a multivariate model with self-excitation and contagion between jumps of different point processes. Finally, the statistical estimation of parameters is an open question.

11 Appendix A, the bivariate Kramers-Moyal expansion

In this appendix, we sketch the proof of the bivariate Kramers-Moyal expansion.

Proposition 11.1. *The bivariate pdf of (λ_t, P_t) is related to cross moments through the relation:*

$$\begin{aligned}&p(t+\Delta, x_1, x_2|s, y_1, y_2) - p(t, x_1, x_2|s, y_1, y_2)\\&\quad = \sum_{n=1}^{\infty}\sum_{j=0}^{n} \frac{(-1)^n}{j!n-j!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\left(M(j, n-j|x_1, x_2)p(t, x_1, x_2|s, y_1, y_2)\right),\end{aligned}$$

where $M(j, n-j|x_1, x_2)$ is defined as:

$$M(j, n-j|x_1, x_2) \quad = \quad \mathbb{E}\Big((\lambda_{t+\Delta} - \lambda_t)^j (P_{t+\Delta} - P_t)^{n-j} |\lambda_t = x_1, P_t = x_2\Big).$$

Proof. We start from the joint characteristic function of $(\lambda_{t+\Delta} - \lambda_t)$ and $(P_{t+\Delta} - P_t)$, conditionally up to the information up to time t. This function is also the Fourier transform of the joint pdf $p(t+\Delta, x_1, x_2|t, y_1, y_2)$ that we denote by

$$\begin{aligned}&\hat{p}(t+\Delta, u_1, u_2|t, y_1, y_2)\\&\quad = \mathbb{E}\left(\exp\left(iu_1(\lambda_{t+\Delta} - \lambda_t) + iu_2(P_{t+\Delta} - P_t)\right) | \lambda_t = y_1, I_t = y_2\right)\\&\quad = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{iu_1(x_1-y_1)+iu_2(x_2-y_2)} p(t+\Delta, x_1, x_2|t, y_1, y_2)dx_1dx_2\,.\end{aligned}$$

Since the Taylor expansion of $f(x, y) = e^{x+y}$ around $(0, 0)$ is equal to the sum

$$\begin{aligned}e^{x+y} &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{1}{j!(n-j)!}(x)^j(y)^{n-j}\left[\frac{\partial^j}{\partial x^j}\frac{\partial^{n-j}}{\partial y^{n-j}}e^{a+b}\right]_{a=b=0}\\&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{1}{j!(n-j)!}(x)^j(y)^{n-j}\,,\end{aligned}$$

the characteristic function may be expanded explicitly as follows:

$$
\begin{aligned}
&\hat{p}(t+\Delta, u_1, u_2|t, y_1, y_2) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\left(iu_1\left(x_1-y_1\right)\right)^j\left(iu_2\left(x_2-y_2\right)\right)^{n-j}}{j!(n-j)!}p(t+\Delta, x_1, x_2|.)dx_1dx_2 \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(iu_1)^j(iu_2)^{n-j}}{j!(n-j)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x_1-y_1)^j(x_2-y_2)^{n-j}p(t+\Delta, x_1, x_2|.)dx_1dx_2 \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(iu_1)^j(iu_2)^{n-j}}{j!(n-j)!}\mathbb{E}\left(\left(\lambda_{t+\Delta}-\lambda_t\right)^j\left(P_{t+\Delta}-P\right)^{n-j}|\lambda_t=y_1, P_t=y_2\right) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(iu_1)^j(iu_2)^{n-j}}{j!(n-j)!}M(j, n-j|y_1, y_2)\,.
\end{aligned}
$$

Therefore, inverting the Fourier transform leads to the following expansion for the transition probability

$$
\begin{aligned}
&p(t+\Delta, x_1, x_2|t, y_1, y_2) \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-iu_1(x_1-y_1)-iu_2(x_2-y_2)}\hat{p}(t+\Delta, u_1, u_2|t, y_1, y_2)du_1du_2 \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{M(j, n-j|y_1, y_2)}{j!(n-j)!}\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sum_{u_2\in\mathbb{N}}(iu_1)^j(iu_2)^{n-j}e^{-iu_1(x_1-y_1)-iu(x_2-y_2)}du_1du_2\,.
\end{aligned}
$$

Using the ansatz that

$$
\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(iu_1)^j(iu_2)^{n-j}e^{-iu_1(x_1-y_1)-iu_2(x_2-y_2)}du_1du_2 \\
&\quad= \left(-\frac{\partial}{\partial x_1}\right)^j\left(-\frac{\partial}{\partial x_2}\right)^{n-j}\delta\left(x_1-y_1\right)\delta\left(x_2-y_2\right)\,,
\end{aligned}
$$

and as for any function bivariate function $f(y_1, y_2)$, we have that

$$
\begin{aligned}
&\delta\left(x_1-y_1\right)\delta\left(x_2-y_2\right)f(y_1, y_2) \\
&\quad= \delta\left(x_1-y_1\right)\delta\left(x_2-y_2\right)f(x_1, x_2)\,,
\end{aligned}
$$

we obtain the following expansion for $p(t+\Delta, x_1, x_2|t, y_1, y_2)$:

$$
\begin{aligned}
&p(t+\Delta, x_1, x_2|t, y_1, y_2) = \\
&\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!(n-j)!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}M(j, n-j|y_1, y_2)\delta\left(x_1-y_1\right)\delta\left(x_2-y_2\right)
\end{aligned}
$$

Finally, using the Chapman-Kolmogorov equation allows us to rewrite $p(t+\Delta, x_1, x_2|s, y_1, y_2)$:

$$
\begin{aligned}
&p(t+\Delta, x_1, x_2|s, y_1, y_2) \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}p(t+\Delta, x_1, x_2|t, z_1, z_2)p(t, z_1, z_2|s, y_1, y_2)dz_1dz_2 \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!(n-j)!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}M(j, n-j|z_1, z_2) \\
&\times p(t, z_1, z_2|s, y_1, y_2)\delta\left(x_1-z_1\right)\delta\left(x_2-z_2\right)dz_1dz_2\,.
\end{aligned}
$$

end

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Fractional Hawkes processes

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Abstract

Hawkes processes have a self-excitation mechanism used for modeling the clustering of events observed in natural or social phenomena. In the first part of this article, we find the forward differential equations ruling the probability density function and the Laplace's transform of the intensity of a Hawkes process, with an exponential decaying kernel. In the second part, we study the properties of the fractional version of this process. The fractional Hawkes process is obtained by subordinating the point process with the inverse of a α-stable Lévy process. This process is not Markov but the probability density function of its intensity is solution of a fractional Fokker-Planck equation. Finally, we present closed form expressions for moments and autocovariance of the fractional intensity.

1 Introduction

In many natural or social phenomena, shocks are rare events but their occurrence momentarily raises the risk of aftershocks. An endogenous way to model the clustering of events or shocks is provided by self-exciting point processes. In this category of processes, the instantaneous probability to observe a shock depends on the number of past events. Hawkes (1971a, b) and Hawkes and Oakes (1974) were among the firsts to propose a point process with this feature. In the most common and simplest specification, the intensity process is persistent and suddenly increases when a jump occurs. Moreover, the influence of an event on the intensity does not depend on its size and it decays over time more or less rapidly according to a kernel function. Hawkes processes have been successfully applied for modeling the clustering of shocks in seismology, finance, criminality and in many other fields. Without being exhaustive, we can cite e.g. Musmeci and Vere-Jones or Ogata (1998) who propose a space-time point-process for earthquake occurrences or Porter and White (2012) who use a self-exciting model for modeling terrorist activity. Hawkes processes are also used for modeling financial transactions (Bauwens and Hautsch, 2009, Bacry et al. , 2015, Hainaut 2017, Hainaut and Moraux 2018, Hainaut and Goutte 2019). Johnson (1996) develops a model for neuron activity based on self-exciting processes. We refer to Reinhart (2018) for a detailed review of other applications and properties of Hawkes processes.

In the common specification of Hawkes processes, the influence of past jumps on the probability of a new shock decays exponentially with time. Choosing this type of memory guarantees that the point process and its intensity form a bivariate Markov process. Properties of this process may be studied with standard tools from stochastic calculus, like the Itô's lemma for semi-martingales. In this framework, the autocovariance of the intensity decays exponentially with time. However this feature is not necessary adapted for modeling real phenomena that exhibit a long term memory of past events. A common method for modeling sub-exponential decreasing autocovariance consists to replace the exponential memory kernel by a power decaying function. In this case, the point process is not anymore Markov and the dynamics of the point process may not be described with backward stochastic differential equations. This article explores an alternative way for modeling a self-exciting intensity with sub-exponential covariance. This involves a non-Markov time change involving the inverse of an alpha stable subordinator.

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The time change approach is applied to diffusions in physics for describing the movement of heavy particles that can get immobilized (see e.g. Metzler and Klafter 2004 or Eliazar, Klafter 2004). This type of time-changed Brownian motions, called sub-diffusions, are also popular in econophysics (see Scalas 2006) and applied to financial derivatives by Magdziarz (2009, a) for illiquid markets. As shown e.g. in Magdziarz (2009, b), the probability density function of sub-diffusions is solution of a Fractional Fokker-Planck equation. This equation is proposed in Barkai et al. (2000) and Metzler et al. (1999). Articles of Leonenko et al. (2013, a) Leonenko et al. (2013, b) go a step further and explore fractional Pearson diffusions and their correlation structure.

The contributions of this article are multiple. We first establish the Fokker-Planck equations (FPE's) ruling the probability density function and the Laplace's transform of the intensity for a Hawkes process with an exponential decaying memory. To the best of our knowledge, these results are new and FPE's differ from these for jump-diffusions. We next provide the FPE for the joint statistical distribution of counting and intensity processes. We also propose a numerical method for solving the FPE of the Laplace's transform of intensity around zero. The second part of this work studies the same process, subordinated by the inverse of a stable Lévy process. We show that this process is ruled by similar FPE's but the derivative with respect to time is replaced by a Caputo's fractional derivative. We next present closed-form expressions for moments and for the autocovariance. Finally, we extend the non-fractional numerical framework for solving the fractional FPE of the Laplace's transform of intensity around zero.

2 The Hawkes process

The Hawkes process (1971 a,b) is a self-exciting point process for which the arrival of one event increases the probability of occurrence of new ones. Let us consider a probability space $(\Omega, \mathcal{F}, P)$ on which is defined a counting process $(N_t)_{t\geq 0}$ with random independent identically distributed marks denoted by $(\xi_k)_{k=1,N_t}$. The probability density function (pdf) of marks is a measure, $\nu(.)$, defined on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. The expectation and variance of the mark are respectively $\mathbb{E}(\xi) = \mu > 0$ and $\mathbb{V}(\xi) = \psi^2$. The sum of marks is a pure jump process, denoted by $(P_t)_{t\geq 0}$:

$$P_t = \sum_{k=1}^{N_t} \xi_k \,. \tag{1}$$

The rate of arrival of events is a stochastic process $(\lambda_t)_{t\geq 0}$ that depends upon the history of the point process $(P_t)_{t\geq 0}$ through the following auto-regressive relation:

$$\lambda_t = \theta + e^{-\kappa(t-s)} (\lambda_s - \theta) + \eta \int_s^t e^{-\kappa(t-u)} dP_u \quad t \geq s, \tag{2}$$

where $\theta, \eta, \kappa \in \mathbb{R}^+$. The natural filtration of the triplet (P_t, N_t, λ_t) is the collection of sigma-algebras: $(\mathcal{F}_t)_{t\geq 0} = \sigma(P_s, N_s, \lambda_s, \, s \leq t)$. Differentiating equation (2) allows us to reformulate the dynamics of the intensity as follows:

$$d\lambda_t \;\; = \;\; \kappa(\theta - \lambda_t)\, dt + \eta dP_t \,. \tag{3}$$

As illustrated in Figure 1, between two successive jumps, the process λ_t reverts towards θ at a speed κ. If an event occurs, the intensity increases by a random quantity $\eta\xi$. The point process P_t is not Markov since it depends upon λ_t. But the pair $X_t = (\lambda_t, P_t)_{t\in\mathbb{R}^+}$ is well Markov in the state space $D = \mathbb{R}_+ \times \mathbb{R}_+$. Therefore, the infinitesimal generator of X_t, denoted $\mathcal{A}$, is the operator acting on a sufficiently regular function $f : D \to \mathbb{R}$ such that

$$\mathcal{A}f(\boldsymbol{x}) \;\; = \;\; \lim_{h\to 0} \frac{\mathbb{E}_t \left[f(X_{t+h}) - f(\boldsymbol{x}) \right]}{h} \,,$$

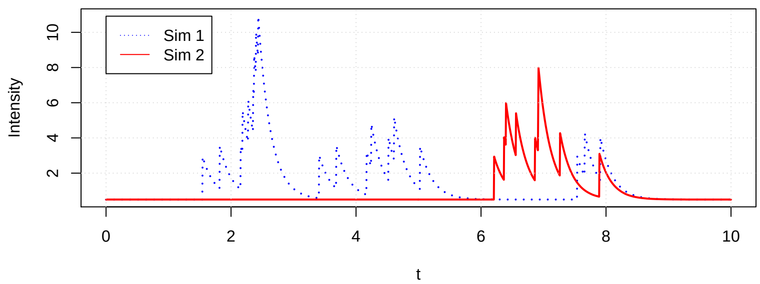

Figure 1: Simulated sample paths of the intensity of a Hawkes process.

where $\mathbb{E}_t = \mathbb{E}\left[.|\mathcal{F}_t\right]$ and $X_t = \boldsymbol{x} = (\lambda, p)$. Using standard arguments and the Itô's lemma, the generator is equal to:

$$\mathcal{A}f(\boldsymbol{x}) \quad = \quad \kappa\left(\theta-\lambda\right)\frac{\partial f}{\partial\lambda}(\boldsymbol{x})+\lambda\mathbb{E}\left[f(\lambda+\eta\xi\,,\,p+\xi)-f(\boldsymbol{x})\right]. \tag{4}$$

On the other hand, the compensated process $(M_t)_{t\geq 0}$ defined as follows

$$M_t \quad = \quad f(X_t)-f(X_0)-\int_0^t \mathcal{A}f(X_u)du\,,$$

is a martingale relative to its natural filtration (see e.g. Proposition 1,6 of Chapter VII in Revuz and Yor (1999)). Thus, for $s>t$, we have that

$$\mathbb{E}_t\left[f(X_s)-\int_0^s \mathcal{A}f(X_u)du\right] \quad = \quad f(X_t)-\int_0^t \mathcal{A}f(X_u)du$$

by the martingale property. This leads to the Dynkin formula linking the expectation of a function of X_t to the infinitesimal generator:

$$\mathbb{E}_t\left[f\left(X_s\right)\right] \quad = \quad f(X_t)+\mathbb{E}_t\left[\int_t^s \mathcal{A}f(X_u)du\right]. \tag{5}$$

This last formula allows us to calculate the expectation and variance of the intensity. We remind these useful results and a proof may be found e.g. in Hainaut (2017).

Proposition 2.1. *The first moment of λ_t is given by*

$$\mathbb{E}_0\left[\lambda_t\right] = e^{(\eta\mu-\kappa)t}\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\left(e^{(\eta\mu-\kappa)t}-1\right). \tag{6}$$

If we define $\rho_1 = \eta^2(\psi^2+\mu^2)$ and $\rho_2 = \frac{\eta^2\kappa\theta(\psi^2+\mu^2)}{\eta\mu-\kappa}$ then the variance is equal to

$$\mathbb{V}_0\left[\lambda_t\right] \quad = \quad \frac{\rho_1\lambda_0+\rho_2}{\eta\mu-\kappa}\left(e^{2(\eta\mu-\kappa)t}-e^{(\eta\mu-\kappa)t}\right)+\frac{\rho_2}{2\left(\eta\mu-\kappa\right)}\left(1-e^{2(\eta\mu-\kappa)t}\right). \tag{7}$$

The expectation and variance of λ_t are well defined and meaningful only if the parameters η, μ and κ fulfills the following conditions:

$$\eta\mu-\kappa<0\,. \tag{8}$$

If this condition is not satisfied, the speed of mean reversion κ is not sufficient to drive back the intensity to θ and λ_t tends to $+\infty$ when $t\to\infty$. If the equilibrium condition (8) is fulfilled, the expectation of the jump arrival intensity tends to a constant, $-\frac{\kappa\theta}{\eta\mu-\kappa}$, as t becomes large. Whereas the variance of λ_t converges to a constant, $\frac{\eta^2\kappa\theta(\psi^2+\mu^2)}{2(\eta\mu-\kappa)^2}$ when $t\to\infty$. The next proposition recalls the form of the intensity autocovariance.

Proposition 2.2. *The covariance between λ_s and λ_t for $t \leq s$ is proportional to the variance:*

$$\begin{aligned}\mathbb{C}_0\left[\lambda_t\lambda_s\right] =& \mathbb{E}_0\left[\lambda_t\lambda_s\right] - \mathbb{E}_0\left[\lambda_s\right]\mathbb{E}_0\left[\lambda_t\right]\\ =& e^{(\eta\mu-\kappa)(s-t)}\mathbb{V}_0\left[\lambda_t\right] .\end{aligned}$$

As mentioned in the introduction, the fractional Hawkes process is a self-exciting process that is time changed by the inverse of a α stable Lévy process. Since the subordinator is not Markov, the fractional process is not anymore a semi-martingale. Therefore, we cannot rely on tools from stochastic calculus, like the Itô's lemma to infer a backward differential equation satisfied by any smooth function of the fractional process. However, we will see the probability density function (pdf) of this process is solution of a forward differential equation, called fractional Fokker-Planck equation. In order to establish this result, we construct in the next section, the Fokker-Planck equation (FPE) satisfied by the pdf of the intensity λ_t.

3 Fokker-Planck equation for the pdf of $(\lambda_t)_{t\geq 0}$

To the best of our knowledge, this result is new and compared to a pure jump process, the Hawkes FPE has an additional term. From equation (2), we infer that for $t \geq s$, the pdf of λ_t, conditionally to $\mathcal{F}_s$ exclusively depends upon λ_s. Thus, we denote by $p(t,x|s,y)$ its probability density function that is defined as follows:

$$p(t,x|s,y)dx \quad = \quad P\left(\lambda_t \in [x, x+dx] \mid \lambda_s = y\right) ,$$

for $s \leq t$ and $x, y \in \mathbb{R}^+$. This pdf is solution of a forward differential equation as stated in the next proposition.

Proposition 3.1. *The pdf of λ_t is solution of the following Fokker-Planck equation, also called forward Kolmogorov equation:*

$$\begin{aligned}\frac{\partial p(t,x|s,y)}{\partial t} \quad = \quad & -\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)p(t,x|s,y)\right) - \eta\mathbb{E}\left[\xi p(t,x-\eta\xi|s,y)\right] \\ & +x\mathbb{E}\left[p(t,x-\eta\xi|s,y) - p(t,x|s,y)\right] ,\end{aligned} \tag{9}$$

with the initial condition: $p(s,x|s,y) = \delta_{\{x-y\}}$ where δ_z is the Dirac measure located at z.

Proof From the Kramers-Moyal forward expansion, we know that the pdf's, $p(t+\Delta, x|s,y)$ at time $t+\Delta$ and $p(t,x|s,y)$ at time t, are linked to moments of λ_t by the relation

$$p(t+\Delta,x|s,y) - p(t,x|s,y) \quad = \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial x^n}\left[M_n(t,x,\Delta)p(t,x|s,y)\right] \tag{10}$$

where $M_n(t,x,\Delta)$ is the moment of order n of $\Delta\lambda_t = \lambda_{t+\Delta} - \lambda_t$:

$$\begin{aligned}M_n(t,x,\Delta) \quad &= \quad \mathbb{E}_t\left[\left(\lambda_{t+\Delta}-\lambda_t\right)^n \mid \lambda_t = x\right] \\ &= \quad \int_{-\infty}^{+\infty}\left(z-x\right)^n p(t+\Delta, z\,|\,t,x)dz\,.\end{aligned}$$

On the other hand, marks $(\xi_k)_{k=1,\dots,N_t}$ are independent from $X_t = (\lambda_t, P_t)$. Thus, from Equation (3) and for a small enough step of time, the centered moments of λ_t may be expanded as follows:

$$\begin{aligned}&\mathbb{E}_t\left[\lambda_{t+\Delta}-\lambda_t \middle|\, \lambda_t = x\right] = \left(\kappa\left(\theta-x\right)+\eta\mu x\right)\Delta + \mathcal{O}\left(\Delta_t^2\right) ,\\ &\mathbb{E}_t\left[\left(\lambda_{t+\Delta}-\lambda_t\right)^2 \mid \lambda_t = x\right] = \eta^2\mathbb{E}\left[\xi^2\right]x\Delta + \mathcal{O}\left(\Delta_t^2\right) ,\\ &\vdots\\ &\mathbb{E}_t\left[\left(\lambda_{t+\Delta}-\lambda_t\right)^n \mid \lambda_t = x\right] = \eta^n\mathbb{E}\left[\xi^n\right]x\Delta + \mathcal{O}\left(\Delta_t^2\right) .\end{aligned}$$

Injecting these expansions in Equation (10) gives us:

$$\frac{p(t+\Delta,x|s,y)-p(t,x|s,y)}{\Delta} = -\frac{\partial}{\partial x}\left[\kappa\left(\theta-x\right)p(t,x|s,y)\right] \tag{11}$$
$$+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\eta^n\mathbb{E}\left[\xi^n\right]\frac{\partial^n\left(xp(t,x|s,y)\right)}{\partial^n x}+\mathcal{O}\left(\Delta\right)\,.$$

Since the n^{th} derivative of $x\,p(t,x|s,y)$ is equal to

$$\frac{\partial^n}{\partial x^n}\left(x\,p(t,x|s,y)\right) \;=\; n\frac{\partial^{n-1}p(t,x|s,y)}{\partial x^{n-1}}+x\frac{\partial^n p(t,x|s,y)}{\partial^n x}\,,$$

Equation (11) may be rewritten as the following sum:

$$\frac{p(t+\Delta,x|s,y)-p(t,x|s,y)}{\Delta} = -\frac{\partial}{\partial x}\left[\kappa\left(\theta-x\right)p(t,x|s,y)\right] \tag{12}$$
$$+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!}\eta^n\mathbb{E}\left[\xi^n\right]\frac{\partial^{n-1}p(t,x|s,y)}{\partial x^{n-1}}$$
$$+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\eta^n\mathbb{E}\left[\xi^n\right]x\frac{\partial^n p(t,x|s,y)}{\partial^n x}+\mathcal{O}\left(\Delta\right)\,.$$

the second term in this equation is equal to the Taylor's expansion of the following expectation:

$$\mathbb{E}\left[-\eta\xi\sum_{n=0}^{\infty}\frac{(-\eta\xi)^n}{n!}\frac{\partial^n p(t,x|s,y)}{\partial x^n}\right] \;=\; -\eta\mathbb{E}\left[\xi p(t,x-\eta\xi|s,y)\right]\,.$$

Whereas the third term in Equation (12) is equal to the Taylor's expansion of :

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\eta^n\mathbb{E}\left[\xi^n\right]x\frac{\partial^n p(t,x|s,y)}{\partial^n x}=x\mathbb{E}\left[p(t,x-\eta\xi|s,y)-p(t,x|s,y)\right]\,.$$

end

The probability density function may be rewritten as the conditional expectation of a Dirac function: $p(t,x|s,y)=\mathbb{E}_s\left(\delta_{\{\lambda_t-x\}}\right)$ and is therefore a martingale, From Equation (5), the pdf is then also solution of a backward Kolmogorov equation that is extensively studied in the literature on Hawkes processes:

$$-\frac{\partial p(t,x|s,y)}{\partial s} \;=\; \kappa\left(\theta-y\right)\frac{\partial p(t,x|s,y)}{\partial y}+ \tag{13}$$
$$y\mathbb{E}\left[p(t,x|s,y+\eta\xi)-p(t,x|s,y)\right]\,.$$

From Equation (2), the distribution of $\lambda_t|\lambda_s$ is time-homogeneous in the sense that

$$p(t,x|s,y)=p(t-s,x|0,y).$$

Therefore, the backward equation (13) is also equivalent to:

$$\frac{\partial p(t,x|s,y)}{\partial t} \;=\; \kappa\left(\theta-y\right)\frac{\partial p(t,x|s,y)}{\partial y}+ \tag{14}$$
$$y\mathbb{E}\left[p(t,x|s,y+\eta\xi)-p(t,x|s,y)\right]\,.$$

It is interesting to compare the FPE of a Hawkes process to the one of a process without any self-excitation mechanism. For this purpose, let us denote by $\left(N_t^{'}\right)_{t\geq 0}$, a pure Poisson process with a constant intensity,

$\rho \in \mathbb{R}^+$. The sum of marks is here noted $P_t^{'} = \sum_{k=1}^{N_t^{'}} \xi_k$. Let us define a process $\left(\lambda_t^{'}\right)_{t\geq 0}$ by the following SDE:

$$d\lambda_t^{'} = \kappa\left(\theta - \lambda_t^{'}\right)dt + \eta dP_t^{'} .$$

From the Chapter 7 of Hanson (2007), we know that the pdf $p(t,x|s,y)$ of $\lambda_t^{'}$, is solution of the following Fokker-Planck equation

$$\begin{aligned}\frac{\partial p(t,x|s,y)}{\partial t} &= -\frac{\partial}{\partial x}\left(\kappa\left(\theta - x\right)p(t,x|s,y)\right) \\ &\quad +\rho\mathbb{E}\left[p(t,x-\eta\xi|s,y) - p(t,x|s,y)\right]\end{aligned} \tag{15}$$

A comparison with Equation (9) reveals that the presence of self-excitation introduces one new term, $-\eta\mathbb{E}\left[\xi p(t,x-\eta\xi|s,y)\right]$, in the Fokker-Planck Equation (15).

Contrary to backward equations, solving the Fokker-Planck equation (9) is numerically more challenging because we have to approximate the Dirac measure in the initial condition. However, we will see in the next section that the Laplace's transform of the pdf is solution of a forward equation, easy to solve numerically.

4 Laplace's transform of $p(t,x|s,y)$

Let us denote by $\phi(.)$ the Laplace's transform of the pdf $p(t,x|s,y)$ of the intensity process. This transform is also equal to the following expectation

$$\phi(t,z|s,y) = \mathbb{E}\left(e^{-z\lambda_t}|\lambda_s = y\right)$$

for $z \in \mathbb{R}^+$. It is well known in the literature that this Laplace's transform is solution of a backward Kolmogorov equation. Details about this equation are reminded at the end of this section. However, this transform is also solution of a forward Kolmogorov equation, as stated in the next proposition.

Proposition 4.1. *The Laplace transform $\phi(t,z|0,y)$ satisfies the forward equation:*

$$\frac{\partial \phi(t,z|0,y)}{\partial t} = -\kappa\theta z\,\phi(t,z|0,y) + \left(1 - \kappa z - \mathbb{E}\left(e^{-z\eta\xi}\right)\right)\frac{\partial \phi(t,z|0,y)}{\partial z} \tag{16}$$

with the initial conditions $\phi(0,z|0,y) = e^{-zy}$ and $\phi(t,0|0,y) = 1$.

Proof Let $\Delta \in \mathbb{R}^+$ be a small step of time. The difference $\phi(t+\Delta,z|0,y) - \phi(t,z|0,y)$ when $\Delta \to 0$ is equal to

$$\lim_{\Delta\to 0}\frac{\phi(t+\Delta,z|0,y) - \phi(t,z|0,y)}{\Delta} = \lim_{\Delta\to 0}\int_0^\infty e^{-zx}\left(\frac{p(t+\Delta,x|0,y) - p(t,x|0,y)}{\Delta}\right)dx\,.$$

Using the FPE (9) for Hawkes processes, we infer that the derivative of $\phi(.)$ with respect to time is the sum of three terms:

$$\begin{aligned}\frac{\partial \phi(t,z|0,y)}{\partial t} &= \int_0^\infty e^{-zx}\left(-\frac{\partial}{\partial x}\left(\kappa\left(\theta - x\right)p(t,x|0,y)\right)\right)dx + \\ &\quad \int_0^\infty e^{-zx}\left(-\eta\mathbb{E}\left[\xi p(t,x-\eta\xi|0,y)\right]\right)dx + \\ &\quad \int_0^\infty e^{-zx}\left(x\mathbb{E}\left[p(t,x-\eta\xi|0,y) - p(t,x|0,y)\right]\right)dx\,.\end{aligned} \tag{17}$$

Using the relation

$$\int_0^\infty e^{-zx}x\frac{\partial}{\partial x}p(t,x|0,y)dx = -\phi(t,z|0,y) - z\frac{\partial \phi(t,z|0,y)}{\partial z}\,,$$

and properties of the Laplace's transform allows us to rewrite the first term of Equation (17) as

$$\begin{aligned}\int_0^\infty e^{-zx}\left(-\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)p(t,x|0,y)\right)\right)dx &= -\kappa\theta\left(z\phi(t,z|0,y)-\underbrace{p(t,0|s,y)}_{=0}\right)+\kappa\phi(t,z|0,y)\\ &\quad -\kappa\phi(t,z|0,y)-\kappa z\frac{\partial\phi(t,z|0,y)}{\partial z}\,.\end{aligned}$$

After a change of variable $x' = x - \eta\xi$, the second term of (17) becomes:

$$\begin{aligned}&-\eta\int_0^\infty \xi\left(\int_0^\infty e^{-zx}p(t,x-\eta\xi|0,y)dx\right)\nu(\xi)d\xi\\ &\quad = -\eta\mathbb{E}\left(\xi e^{-z\eta\xi}\right)\phi(t,z|0,y)\,.\end{aligned}$$

Given that $p(t,x|0,y)$ is null for $x \leq 0$, the third term of (17) is rewritten, after the same change of variable, as:

$$\begin{aligned}&\int_0^\infty\int_0^\infty e^{-zx}\left(xp(t,x-\eta\xi|0,y)\right)dx\nu(\xi)d\xi-\int_0^\infty e^{-zx}\left(xp(t,x|0,y)\right)dx\\ &=\int_0^\infty e^{-z\eta\xi}\,\nu(\xi)d\xi\int_0^\infty e^{-zx'}x'p(t,x'|0,y)dx'\\ &+\eta\int_0^\infty \xi e^{-z\eta\xi}\nu(\xi)d\xi\int_0^\infty e^{-zx'}p(t,x'|0,y)dx'+\frac{\partial\phi(t,z|0,y)}{\partial z}\\ &=-\mathbb{E}\left(e^{-z\eta\xi}\right)\frac{\partial\phi(t,z|0,y)}{\partial z}+\eta\mathbb{E}\left(\xi e^{-z\eta\xi}\right)\phi(t,z|0,y)+\frac{\partial\phi(t,z|0,y)}{\partial z}\,.\end{aligned}$$

Combining previous elements allows us to conclude. **end**.

We compare the forward Equation (16) with the backward one. From the Itô's lemma for semi-martingales, we know that $\phi(.)$ is solution of the backward partial differential equation:

$$0 = \frac{\partial\phi}{\partial s}+\kappa\left(\theta-y\right)\frac{\partial\phi}{\partial y}+y\mathbb{E}\left(\phi(t,z|s,y+\eta\xi)-\phi\right)\,. \tag{18}$$

If we do the assumption that $\phi(.)$ is a function of the form $\exp\left(A(s,t)+B(s,t)y\right)$, we can easily show that functions $A(.)$ and $B(.)$ are solutions of a system of ordinary differential equations (ODE) as stated in the next proposition:

Proposition 4.2. *$\phi(t,z|s,y)$ is equal to*

$$\phi(t,z|s,y)=\exp\left(A(s,t)+B(s,t)y\right)\,, \tag{19}$$

where $A(s,t)$ and $B(s,t)$ are functions that satisfy the following ODE's:

$$\begin{aligned}\frac{\partial A(s,t)}{\partial s} &= -\kappa\theta B(s,t)\,,\\ \frac{\partial B(s,t)}{\partial s} &= \kappa B(s,t)-\mathbb{E}\left(e^{B(s,t)\eta\xi}-1\right)\,,\end{aligned}$$

with the terminal conditions $A(t,t)=0$ and $B(t,t)=-z$.

This result is not new but we will use it to benchmark the efficiency of the numerical scheme proposed in the next section for solving Equation (16).

Solving the forward equation for the Laplace transform of λ_t

This section presents a numerical method based on differential transforms for solving the forward equation (16) ruling the Laplace's transform of the intensity, around zero. This method is widely used in physics and the interested reader may e.g. refer to Bildik et al. (2006) or Yang et al. (2001) for details. The method is based on the following observation: if a differentiable function $l(t,z)$ is the product of two $\mathcal{C}^{\infty}$ functions $f(t)$ and $g(z)$, then $l(t,z)$ is the product of their Taylor's expansions:

$$\begin{aligned} l(t,z) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!}\frac{1}{h!}\left[\frac{\partial^k f(t)}{\partial t^k}\right]_{t=t_0}\left[\frac{\partial^h g(z)}{\partial z^h}\right]_{z=z_0}(t-t_0)^k\,(z-z_0)^h \\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} w(k,h)(t-t_0)^k\,(z-z_0)^h \end{aligned} \tag{20}$$

where

$$(w(k,h))_{k,h>0} = \frac{1}{k!h!}\left[\frac{\partial l(t,z)}{\partial t^k \partial z^h}\right]_{t=t_0,z=z_0}$$

is called the spectrum of $l(t,z)$. The differential method for solving the Fokker-Planck equation consists to approach its solution by a series similar to (20). Here, we approximate the Laplace transform $\phi(t,z|0,y)$ by the following infinite sum:

$$\phi(t,z|0,y) \quad\approx\quad \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} w(k,h)\times z^h t^k\,,$$

where the differential weights in this sum are noted:

$$w(k,h) \quad=\quad \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h}\phi(t,z|.)\right]_{t=0,z=0}\,, \tag{21}$$

and may be computed by a forward iterative recursion. To establish this recursion, we will apply the differential operator $\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h}\right]_{t=0,z=0}$ to the forward equation (16). The next proposition provides some useful results.

Proposition 4.3. *The differential weights such as defined by Equation (21) satisfy the following relations:*

$$\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h}\frac{\partial\phi(.)}{\partial t}\right]_{t=0,z=0} = (k+1)w(k+1,h) \tag{22}$$

$$\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h}z\phi(.)\right]_{t=0,z=0} = w(k,h-1) \tag{23}$$

$$\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h}z\frac{\partial\phi(.)}{\partial z}\right]_{t=0,z=0} = hw(k,h) \tag{24}$$

$$\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h}e^{-z\eta\xi}\frac{\partial\phi(.)}{\partial z}\right]_{t=0,z=0} = \sum_{j=0}^{h}\frac{j+1}{(h-j)!}\left(-\eta\xi\right)^{h-j} w(k,j+1) \tag{25}$$

Proof. Equation (22) results from the definition of differential weights. The property (23) is a direct consequence of:

$$\frac{\partial^{k+h}}{\partial t^k \partial z^h}z\phi(.) = h\frac{\partial^{k+h-1}}{\partial t^k \partial z^{h-1}}\phi(.) + z\frac{\partial^{k+h}}{\partial t^k \partial z^h}\phi(.)\,,$$

whereas equation (24) comes from the relation:

$$\frac{\partial^{k+h}}{\partial t^k \partial z^h}z\frac{\partial\phi(.)}{\partial z} = h\frac{\partial^{k+h}}{\partial t^k \partial z^h}\phi(.) + z\frac{\partial^{k+h+1}}{\partial t^k \partial z^{h+1}}\phi(.)\,.$$

To show (25), we use the following Newton's formula:

$$\begin{aligned}
&\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h} e^{-z\eta\xi}\frac{\partial \phi(t,z|.)}{\partial z}\right]_{t=0,z=0} \\
&= \left[\sum_{j=0}^{h}\frac{(j+1)}{(h-j)!k!(j+1)!}\frac{\partial^{k+j+1}\phi(t,z|.)}{\partial t^k \partial z^{j+1}}\left(-\eta\xi\right)^{h-j} e^{-z\eta\xi}\right]_{t=0,z=0} \\
&= \sum_{j=0}^{h}\frac{j+1}{(h-j)!}\left(-\eta\xi\right)^{h-j} w(k,j+1)\,.
\end{aligned}$$

end.

The next proposition provides an approached expression for the Laplace's transform of the pdf $p(t,x|s,y)$.

Proposition 4.4. *The Laplace's transform $\phi(.)$ is approached by the following sum*

$$\phi(t,z|0,y) \quad \approx \quad \sum_{k=0}^{K}\sum_{h=0}^{H} w(k,h)\times z^h t^k\,, \tag{26}$$

with $K,H \in \mathbb{N}$ and differential weights satisfying the following recursion:

$$\begin{aligned}
(k+1)w(k+1,h) \quad &= \quad -\kappa\theta w(k,h-1) - \kappa\, h\, w(k,h) + (h+1)w(k,h+1) \\
&\quad -\sum_{j=0}^{h}\frac{j+1}{(h-j)!}(-1)^{h-j}\mathbb{E}\left[\left(\eta\xi\right)^{h-j}\right] w(k,j+1)\,,
\end{aligned} \tag{27}$$

determined by the initial conditions:

$$\begin{aligned}
w(0,h) \quad &= \quad \frac{1}{h!}\left(-y\right)^h\,, \\
w(k,0) \quad &= \quad 0 \quad k>0\,.
\end{aligned}$$

Proof To establish the recursion (27), we apply the differential operator $\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h}\right]_{t=0,z=0}$ to the forward equation (16). For $-\kappa\theta z\phi(t,z|0,y)$ we have

$$-\frac{1}{k!h!}\kappa\theta\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h} z\phi(t,z|.)\right]_{t=0,z=0} = -\kappa\theta w(k,h-1)$$

and

$$\begin{aligned}
&\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial z^h}\left(1-\kappa z-\mathbb{E}\left(e^{-z\eta\xi}\right)\right)\frac{\partial \phi(t,z|.)}{\partial z}\right]_{t=0,z=0} = (h+1)w(k,h+1) - \kappa\, h\, w(k,h) \\
&-\sum_{j=0}^{h}\frac{j+1}{(h-j)!}(-1)^{h-j}\mathbb{E}\left[\left(\eta\xi\right)^{h-j}\right] w(k,j+1)
\end{aligned}$$

Since $e^{-yz} = \sum_{h=0}^{\infty}\frac{1}{h!}\left(-y\right)^h z^h$ therefore $w(0,h) = \frac{1}{h!}\left[\frac{\partial^h}{\partial z^h}e^{-zy}\right]_{z=0} = \frac{1}{h!}\left(-y\right)^h$. **end**

Figure 2 compares the Laplace's transform approached by the sum (26) to the numerical solution of the backward Kolmogorov's Equation (19). Since the solution of the forward equation is based on a Taylor's development of the Laplace's transform around $z=0$ and $t=0$, its accuracy deteriorates with the distance to zero. However, it is accurate enough to compute numerically the first four moments as shown in Table 1 by deriving the Laplace's transform in the neighborhood of $z=0$. This table also provides the theoretical expectation and variance computed with Equations (6) and (7).

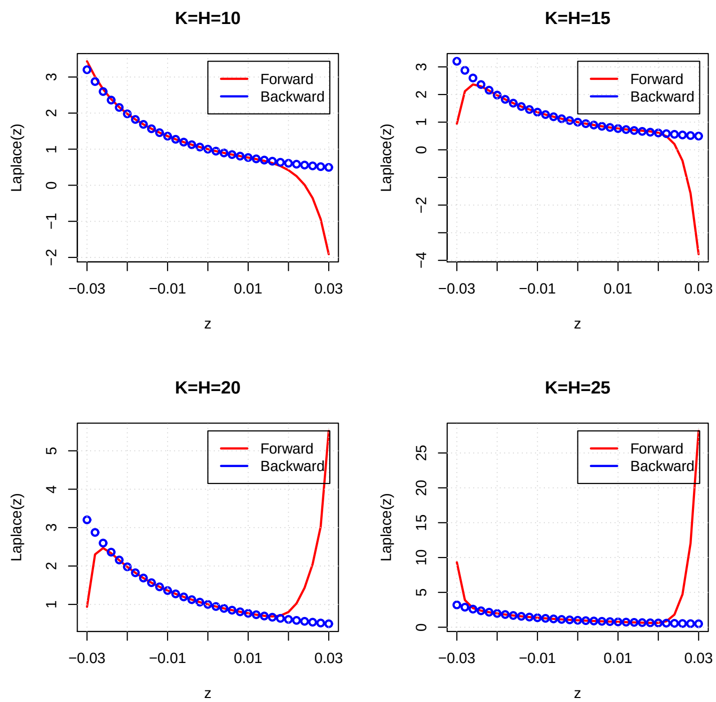

Figure 2: Laplace's transform of $\lambda_{t=1}$ computed with the forward and backward equations. $\eta = 7$, $\theta = 5$, $\kappa = 8.7$ and $\xi = 1$. The maturity is $t = 1$.

K, H	$\mathbb{E}_0(\lambda_t)$	$\mathbb{V}_0(\lambda_t)$	$\mathbb{S}_0(\lambda_t)$	$\mathbb{K}_0(\lambda_t)$
10	28.3333	21.1100	-0.2806	49.4070
15	28.3333	20.9698	1.6740	6.3504
20	28.3333	20.9698	1.6626	7.1083
25	28.3333	20.9698	1.6626	7.1318
Analytical	28.3333	20.9698		

Table 1: Comparison of numerical and theoretical moments, $\eta = 7$, $\theta = 5$, $\kappa = 8.7$ and $\xi = 1$. The maturity is $t = 1$. $y_0 = \lim_{t\to\infty} \mathbb{E}(\lambda_t) = 28.3333$

5 Fokker-Planck equation for the pdf of $(\lambda_t, P_t)_{t\geq 0}$

The compound jump process $(P_t)_{t\geq 0}$ is not Markov since its statistical distribution depends upon the arrival rate of events, λ_t. For this reason, we build the Fokker-Planck equation satisfied by the joint pdf of the pair (λ_t, P_t). This pdf is denoted by $p(t, x_1, x_2|s, y_1, y_2)$ and is defined as follows:

$$p(t,x_1,x_2|s,y_1,y_2) \quad = \quad P\left(\lambda_t \in [x_1, x_1+dx], P_t \in [x_1, x_2+dx] \,|\lambda_s = y_1, P_s = y_2\right) ,$$

for $s \leq t$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}^+$. The FPE of the joint pdf is obtained in a similar way to the one of λ_t and the proof relies on a 2 dimensions version of the Moyal's expansion.

Proposition 5.1. *The joint pdf $p(t, x_1, x_2|s, y_1, y_2)$ of (λ_t, P_t) is solution of the following Fokker-Planck equation:*

$$\begin{aligned}\frac{\partial p(t,x_1,x_2|.)}{\partial t} &= -\frac{\partial}{\partial x_1}\left(\kappa\left(\theta - x_1\right) p(t,x_1,x_2|.)\right) \\ &\quad -\eta\mathbb{E}\left[\xi p(t,x_1-\eta\xi,x_2-\xi|.)\right] \\ &\quad +x_1\mathbb{E}\left[p(t,x_1-\eta\xi,x_2-\xi|.) - p(t,x_1,x_2|.)\right] ,\end{aligned} \tag{28}$$

with the initial condition: $p(s, x_1, x_2|s, y_1, y_2) = \delta_{\{x_1-y_1, x_2-y_2\}}$.

Proof The pdf's, $p(t + \Delta, x_1, x_2|.)$ at time $t + \Delta$ and $p(t, x_1, x_2|.)$ at time t, are related to moments by a bivariate version of the Moyal expansion.

$$\begin{aligned}&p(t+\Delta,x_1,x_2|.) - p(t,x_1,x_2|.) \\ &= \sum_{n=1}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!n-j!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\left(M(j,n-j|x_1,x_2)p(t,x_1,x_2|.)\right)\end{aligned} \tag{29}$$

where $M(j, n - j|x_1, x_2)$ is the cross moment of variations:

$$M(j,n-j|x_1,x_2) \quad = \quad \mathbb{E}\Big(\left(\lambda_{t+\Delta}-\lambda_t\right)^j\left(P_{t+\Delta}-P_t\right)^{n-j}|\lambda_t = x_1, P_t = x_2\Big) .$$

Since this expansion is not standard, we provide a proof in appendix A. For a small step of time Δ, most of terms in the Moyal expansion are of order $\mathcal{O}\left(\Delta^2\right)$, excepted the following cross moments:

$$\begin{aligned}M(1,0|x_1,x_2) &= \mathbb{E}\Big(\left(\lambda_{t+\Delta}-\lambda_t\right)^1\left(P_{t+\Delta}-P_t\right)^0|\lambda_t = x_1, P_t = x_2\Big) \\ &= \left(\kappa\left(\theta-x_1\right)+\eta\mu x_1\right)\Delta + \mathcal{O}\left(\Delta^2\right) ,\end{aligned}$$

$$\begin{aligned}M(0,1|x_1,x_2) &= \mathbb{E}\Big(\left(\lambda_{t+\Delta}-\lambda_t\right)^0\left(P_{t+\Delta}-P_t\right)^1|\lambda_t = x_1, P_t = x_2\Big) \\ &= \mu x_1\Delta + \mathcal{O}\left(\Delta^2\right) ,\end{aligned}$$

and

$$\begin{aligned}M(j,n-j|x_1,x_2) &= \mathbb{E}\Big(\left(\lambda_{t+\Delta}-\lambda_t\right)^j\left(P_{t+\Delta}-P_t\right)^{n-j}|\lambda_t = x_1, P_t = x_2\Big) \\ &= \eta^j\mathbb{E}\left[\xi^n\right]x_1\Delta + \mathcal{O}\left(\Delta^2\right) .\end{aligned}$$

Thus, Equation (29) becomes

$$\begin{aligned}&p(t+\Delta,x_1,x_2|.) - p(t,x_1,x_2|.) = \\ &-\frac{\partial}{\partial x_1}\left(\left(\kappa\left(\theta-x_1\right)+\eta\mu x_1\right)p(t,x_1,x_2|.)\right)\Delta - \frac{\partial}{\partial x_2}\left(\mu x_1 p(t,x_1,x_2|.)\right)\Delta \\ &+\sum_{n=2}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!n-j!}\eta^j\mathbb{E}\left[\xi^n\right]\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\left(x_1 p(t,x_1,x_2|.)\right)\Delta + \mathcal{O}\left(\Delta^2\right) .\end{aligned} \tag{30}$$

Since the partial derivatives in the last term of this equation are also equal to

$$\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\left(x_1\, p(.)\right) \quad = \quad j\frac{\partial^{n-j}}{\partial x_2^{n-j}}\frac{\partial^{j-1}}{\partial x_1^{j-1}}p(.)+x_1\frac{\partial^{n-j}}{\partial x_2^{n-j}}\frac{\partial^j}{\partial^j x_1}p(.),$$

Equation (30) is equal to

$$\begin{aligned}
&p(t+\Delta,x_1,x_2|s,y_1,y_2)-p(t,x_1,x_2|s,y_1,y_2)=\mathcal{O}\left(\Delta^2\right)+ \\
&-\frac{\partial}{\partial x_1}\left(\kappa\left(\theta-x_1\right)p(t,x_1,x_2|.)\right)\Delta-\eta\mu p(t,x_1,x_2|.)\Delta \\
&-x_1\eta\frac{\partial}{\partial x_1}\left(\mu p(t,x_1,x_2|.)\right)\Delta-\frac{\partial}{\partial x_2}\left(\mu x_1 p(t,x_1,x_2|.)\right)\Delta \\
&+\sum_{n=2}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n j}{j!n-j!}\eta^j\mathbb{E}\left[\xi^n\right]\frac{\partial^{n-j}}{\partial x_2^{n-j}}\frac{\partial^{j-1}}{\partial x_1^{j-1}}\left(p(t,x_1,x_2|.)\right)\Delta \\
&+x_1\sum_{n=2}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!n-j!}\eta^j\mathbb{E}\left[\xi^n\right]\frac{\partial^{n-j}}{\partial x_2^{n-j}}\frac{\partial^j}{\partial^j x_1}p(t,x_1,x_2|.)\Delta\,.
\end{aligned} \tag{31}$$

The last term of this equation is related to the Taylor's expansion of $p(t,x_1-\eta\xi,x_2-\xi|.)-p(t,x_1,x_2|.)$:

$$\begin{aligned}
p(t,x_1-\eta\xi,x_2-\xi|.)-p(t,x_1,x_2|.) &= \sum_{n=1}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!(n-j)!}\eta^j\xi^n\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}p(t,x_1,x_2|.) \\
&= -\xi\frac{\partial}{\partial x_2}p(t,x_1,x_2|.)-\eta\xi\frac{\partial}{\partial x_1}p(t,x_1,x_2|.) \\
&\quad +\sum_{n=2}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!(n-j)!}\eta^j\xi^n\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}p(t,x_1,x_2|.)\,.
\end{aligned}$$

Whereas the second term of Equation (31) is also a Taylor's expansion:

$$\begin{aligned}
&\sum_{n=2}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n j}{j!n-j!}\eta^j\xi^n\frac{\partial^{n-j}}{\partial x_2^{n-j}}\frac{\partial^{j-1}}{\partial x_1^{j-1}}\left(p(t,x_1,x_2|.)\right) \\
&=-\eta\xi\sum_{n=1}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!n-j!}\left(\eta\xi\right)^j\xi^{n-j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\frac{\partial^j}{\partial x_1^j}\left(p(t,x_1,x_2|.)\right) \\
&=-\eta\xi\left(p(t,x_1-\eta\xi,x_2-\xi|.)-p(t,x_1,x_2|.)\right)
\end{aligned}$$

Dividing by Δ and considering the limit when $\Delta\to 0$ gives us after calculations,

$$\begin{aligned}
\frac{\partial p(t,x_1,x_2|.)}{\partial t} &= -\frac{\partial}{\partial x_1}\left(\kappa\left(\theta-x_1\right)p(t,x_1,x_2|.)\right)-\eta\mathbb{E}\left[\xi p(t,x_1-\eta\xi,x_2-\xi|.)\right] \\
&\quad +x_1\mathbb{E}\left[p(t,x_1-\eta\xi,x_2-\xi|.)-p(t,x_1,x_2|.)\right]
\end{aligned}$$

and we can conclude. **end**

We do not study the numerical method for solving the FPE ruling the pdf of (λ_t, P_t) because it is outside the scope of this article. However, in a similar manner to Section 4, we can construct the forward equation satisfied by the bivariate Laplace's transform of (λ_t, P_t) and solve it locally around the origin with the same approach as in Section 4.

6 The subordinator

As seen in Proposition 2.2, the autocovariance of the intensity of a Hawkes process with an exponential kernel decays exponentially with time. This model is therefore inappropriate for modeling intensity process with a longer memory of past events.

The solution that we explore in this article consists to introduce periods during which the intensity is motionless. A way for modeling these periods of intensity freeze consists to time-change the process by a subordinator that can be constant over relatively short periods of time. The subordinator is built as the inverse of an α stable process $(U_t)_{t\geq 0}$. This particular type of Lévy processes has a simple moment generating function given by:

$$\mathbb{E}_0\left(e^{-uU_t}\right) \;=\; e^{-t\,u^{\alpha}}\,.$$

The process U_t is a $\frac{1}{\alpha}$ self-similar process, meaning that:

$$U_{at} \;\stackrel{d}{=}\; (at)^{\frac{1}{\alpha}}\,U_1\,.$$

The α-stable processes are strictly increasing and may therefore be used as subordinator but they cannot duplicate motionless periods since they have an infinite activity. Therefore, we use U_t for defining a subordinator, noted $(S_t)_{t\geq 0}$ that is its inverse hitting time:

$$S_t = \inf\{\tau > 0 \,:\, U_\tau \geq t\}\,.$$

By definition and due to the self-similarity property, the cumulative distribution function (cdf) of S_t admits the following representation:

$$\begin{aligned}
P(S_t \leq \tau) &= P(U_\tau \geq t) \\
&= P(\tau^{\frac{1}{\alpha}} U_1 \geq t) \\
&= P\left(\left(\frac{t}{U_1}\right)^{\alpha} \leq \tau\right)
\end{aligned}$$

The distribution of S_t is then the same as the random variable $\left(\frac{t}{U_1}\right)^{\alpha}$. In the rest of the article, the pdf of S_t is denoted by $g(t,\tau) = \frac{d}{d\tau}P(\tau \leq S_t \leq \tau + d\tau)$ and $p_U(t,u)$ is the pdf of U_t. On the other-hand, the self-similarity leads to the relation:

$$\begin{aligned}
P(U_\tau \leq t) &= P(\tau^{\frac{1}{\alpha}} U_1 \leq t) \\
&= P(U_1 \leq t\tau^{-\frac{1}{\alpha}})\,.
\end{aligned}$$

If we derive this last expression with respect to t, we obtain that

$$p_U(\tau,t) \;=\; \tau^{-\frac{1}{\alpha}} p_U(1, t\tau^{-\frac{1}{\alpha}})\,,$$

and infer an important relation linking the pdf of S_t to the pdf of U_t:

$$\begin{aligned}
\tau\, g(t,\tau) &= \frac{t}{\alpha}\left(\tau^{-\frac{1}{\alpha}} p_U(1, t\tau^{-\frac{1}{\alpha}})\right) \\
&= \frac{t}{\alpha} p_U(\tau,t)\,.
\end{aligned} \tag{32}$$

On the other hand, the pdf of S_t is related to the pdf of U_t through the relation

$$\begin{aligned}
g(t,\tau) &= \frac{\partial}{\partial\tau} P(S_t \leq \tau) = -\frac{\partial}{\partial\tau} P(U_\tau \leq t) \\
&= -\frac{\partial}{\partial\tau}\int_0^t p_U(\tau,u)du\,.
\end{aligned}$$

Recalling that the Laplace transform of a function $\int_0^t f(u)du$ is equal to $\omega^{-1}\tilde{f}(\omega)$, the Laplace transform $\tilde{g}(\omega,\tau)$ of $g(t,\tau)$ with respect to time t is therefore equal to:

$$\begin{aligned}
\tilde{g}(\omega,\tau) &= -\frac{\partial}{\partial\tau}\widetilde{\int_0^t p_U(\tau,u)du} \\
&= -\frac{\partial}{\partial\tau}\left(\omega^{-1} e^{-\tau\,\omega^{\alpha}}\right) \\
&= \omega^{\alpha-1} e^{-\tau\,\omega^{\alpha}}\,.
\end{aligned} \tag{33}$$

The Laplace's transform of S_t conditionally to the information available at time zero is given by :

$$\begin{aligned}\mathbb{E}_0\left[e^{-\omega S_t}\right] &= \int_0^\infty e^{-\omega\tau} g(t,\tau)d\tau \\ &= E_\alpha(-\omega t^\alpha)\end{aligned} \tag{34}$$

where E_α is the Mittag-Leffler function (for a proof see e.g. Piryatinska et al. 2005):

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)},$$

where $\Gamma(.)$ is the gamma function. This result clearly reveals that S_t is not a Lévy process since its Laplace's transform does not have an exponential form. However, we can easily compute the moments of S_t by deriving and cancelling its Laplace's transform:

$$\mathbb{E}_0\left(S_t^n\right) = \frac{n!t^{n\alpha}}{\Gamma\left(n\alpha+1\right)}.$$

The Mittag-Leffler function is closely related to the concept of fractional or Caputo's derivative that is also involved in the construction of the fractional Hawkes process. The Caputo's derivative of order $\alpha \in]0,1[$ for a function $h(t,x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $\mathcal{C}^1$ with respect to t is defined by

$$\frac{\partial^\alpha}{\partial t^\alpha} h(t,x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} h(s,x)ds - \frac{h(0,x)}{t^\alpha}. \tag{35}$$

An alternative writing is the following:

$$\frac{\partial^\alpha}{\partial t^\alpha} h(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial}{\partial s} h(s,x)ds \tag{36}$$

When $\alpha = 1$, this derivative corresponds to the derivative with respect to time. Let $\tilde{h}(\omega,x)$ be the usual Laplace's transform of a function $h(t,x)$ with respect to time t. A direct calculation shows that the Laplace's transform of the Caputo's derivative $\frac{\partial^\alpha}{\partial t^\alpha} h(t,x)$ is equal to:

$$\frac{\widetilde{\partial^\alpha h}}{\partial t^\alpha}(\omega,x) = \omega^\alpha \tilde{h}(\omega,x) - \omega^{\alpha-1} h(0,x),$$

which reduces to the familiar form when $\alpha = 1$. Notice that the Caputo's fractional derivative of a power function is given by

$$\frac{\partial^\alpha}{\partial t^\alpha} t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} & p \geq 1\,,\, p \in \mathbb{R} \\ 0 & p \leq 0\,,\, p \in \mathbb{N} \end{cases}.$$

On the other hand, the solution of the fractional differential equation:

$$\frac{\partial^\alpha}{\partial t^\alpha} y(t) = \lambda y(t) \qquad 0 < \alpha < 1$$

with the initial condition $y(0) = b_0$ is precisely the Mittag-Leffler function $y(t) = E_\alpha(\lambda t^\alpha)$.

7 The fractional Hawkes process

The rest of this article focuses on the properties of the time-changed Hawkes process (λ_{S_t}, P_{S_t}) where λ_t is ruled by the dynamics in Equation (3) and S_t is the inverse of an α-stable subordinator. We denote by $p_\alpha(t,x|s,y)$ its transition pdf that is defined as follows:

$$p_\alpha(t,x|s,y)dx = P\left(\lambda_{S_t} \in [x, x+dx] \mid \lambda_{S_s} = y\right).$$

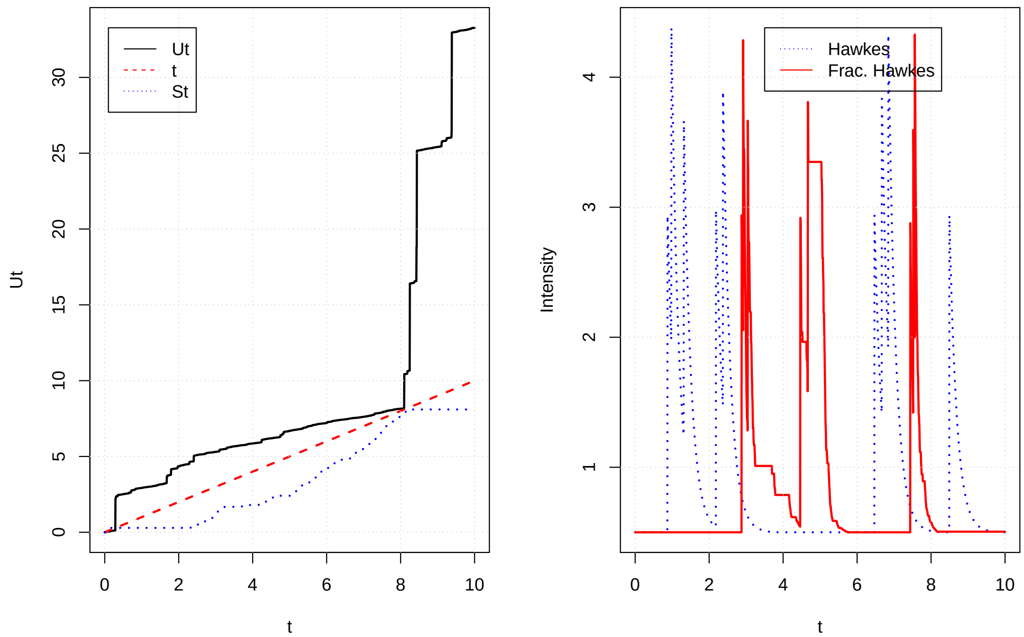

Figure 3: Left plot: simulated sample path of U_t and S_t. Right plot: simulated sample of λ_{S_t}.

Proposition 7.1. *The pdf $p_\alpha(t,x|0,y)$ is solution of a time-fractional Fokker-Planck equation (FFPE):*

$$\begin{aligned}\frac{\partial^\alpha p_\alpha(t,x|0,y)}{\partial t^\alpha} &= -\frac{\partial}{\partial x}\left(\kappa\left(\theta-x\right)p_\alpha(t,x|0,y)\right)-\eta\mathbb{E}\left[\xi p_\alpha(t,x-\eta\xi|0,y)\right] \\ &\quad +x\mathbb{E}\left[p_\alpha(t,x-\eta\xi|0,y)-p_\alpha(t,x|0,y)\right].\end{aligned} \tag{37}$$

with the condition $p(0,x|0,y)=\delta_{\{x-y\}}$. This is also solution of the fractional backward Kolmogorov's equation:

$$\begin{aligned}\frac{\partial^\alpha p_\alpha(t,x|0,y)}{\partial t^\alpha} &= \kappa\left(\theta-y\right)\frac{\partial p_\alpha(t,x|0,y)}{\partial y} \\ &\quad +y\mathbb{E}\left[p_\alpha(t,x|0,y+\eta\xi)-p_\alpha(t,x|0,y)\right].\end{aligned}$$

Proof To lighten developments, we momentarily adopt the notations: $p(t,x):=p(t,x|0,y)$, $p_\alpha(t,x):=p_\alpha(t,x|0,y)$. As $g(t,\tau)$ is the pdf of $S_t|S_0=0$ and from the independence between P_t and S_t, we infer that

$$p_\alpha(t,x) = \int_0^\infty p(\tau,x)g(t,\tau)d\tau\,.$$

The Laplace's transform of $p_\alpha(t,x)$ with respect to time t is thus given by

$$\begin{aligned}\tilde{p}_\alpha(\omega,x) &= \int_0^\infty\int_0^\infty p(\tau,x)e^{-\omega t}g(t,\tau)d\tau dt \\ &= \int_0^\infty p(\tau,x)\,\tilde{g}(\omega,\tau)\,d\tau\,,\end{aligned}$$

where $\tilde{g}(\omega,\tau)=\int_0^\infty e^{-\omega t}g(t,\tau)dt$ is the Laplace's transform of $g(t,\tau)$ with respect to time. Since this transform is given by Equation (33), $\tilde{p}_\alpha(.)$ is equal to:

$$\begin{aligned}\tilde{p}_\alpha(\omega,x) &= \omega^{\alpha-1}\int_0^\infty p(\tau,x)\,e^{-\tau\omega^\alpha}\,d\tau \\ &= \omega^{\alpha-1}\tilde{p}(\omega^\alpha,x)\,.\end{aligned}$$

where $\tilde{p}(\omega, x) = \int_0^\infty e^{-\omega t} p(t,x) dt$ is the Laplace's transform of $p(t,x)$ with respect to time. From the FPE (9), we deduce that $\tilde{p}(\omega, x)$ is then solution of

$$\omega\tilde{p}(\omega, x) - p(0,x) = -\frac{\partial}{\partial x}\left(\kappa\left(\theta - x\right)\tilde{p}(\omega, x)\right)$$
$$-\eta\mathbb{E}\left[\xi\tilde{p}(\omega, x-\eta\xi)\right] + x\mathbb{E}\left[\tilde{p}(\omega, x-\eta\xi) - \tilde{p}(\omega, x)\right] .$$

As $\tilde{p}_\alpha(\omega, x) = \omega^{\alpha-1}\tilde{p}(\omega^\alpha, x)$, replacing ω by ω^α leads to

$$\omega^\alpha\tilde{p}(\omega^\alpha, x) - p(0,x) = -\frac{\partial}{\partial x}\left(\kappa\left(\theta - x\right)\tilde{p}(\omega^\alpha, x)\right)$$
$$+x\mathbb{E}\left[\tilde{p}(\omega^\alpha, x-\eta\xi) - \tilde{p}(\omega^\alpha, x)\right] - \eta\mathbb{E}\left[\xi\tilde{p}(\omega^\alpha, x-\eta\xi)\right] .$$

If we multiply this last equation by $\omega^{\alpha-1}$, we obtain that

$$\omega^\alpha\left(\omega^{\alpha-1}\tilde{p}(\omega^\alpha, x)\right) - \omega^{\alpha-1}p(0,x) = -\frac{\partial}{\partial x}\left(\kappa\left(\theta - x\right)\omega^{\alpha-1}\tilde{p}(\omega^\alpha, x)\right)$$
$$+x\mathbb{E}\left[\omega^{\alpha-1}\tilde{p}(\omega^\alpha, x-\eta\xi) - \omega^{\alpha-1}\tilde{p}(\omega^\alpha, x)\right] - \eta\mathbb{E}\left[\xi\omega^{\alpha-1}\tilde{p}(\omega^\alpha, x-\eta\xi)\right] .$$

Since $p_\alpha(0,x) = p(0,x)$, we have that

$$\omega^\alpha\tilde{p}_\alpha(\omega, x) - \omega^{\alpha-1}p_\alpha(0,x) = -\frac{\partial}{\partial x}\left(\kappa\left(\theta - x\right)\tilde{p}_\alpha(\omega, x)\right)$$
$$+x\mathbb{E}\left[\tilde{p}_\alpha(\omega, x-\theta\xi) - \tilde{p}_\alpha(\omega, \lambda)\right] - \eta\mathbb{E}\left[\xi\tilde{p}_\alpha(\omega, x-\theta\xi)\right] .$$

The left-hand term is the Laplace's transform of the Caputo's derivative of $p_\alpha(t,x)$. Therefore this last equation is also the Laplace's transform of the FFPE (37). On the other hand, the density is solution of the backward Kolmogorov equation:

$$-\frac{\partial p(t,x|s,y)}{\partial s} = \kappa\left(\theta - y\right)\frac{\partial p(t,x|s,y)}{\partial y} + y\mathbb{E}\left[p(t,x|s,y+\eta\xi) - p(t,x|s,y)\right] . \tag{38}$$

As the distribution of $\lambda_t|\lambda_s$ is time-homogeneous in the sense that

$$p(t,x|s,y) = p(t-s,x|0,y) ,$$

then the backward equation is rewritten as

$$\frac{\partial p(t,x|0,y)}{\partial t} = \kappa\left(\theta - y\right)\frac{\partial p(t,x|0,y)}{\partial y} + y\mathbb{E}\left[p(t,x|0,y+\eta\xi) - p(t,x|0,y)\right] . \tag{39}$$

Therefore, the Laplace's transform $\tilde{p}(t,x|0,y)$ with respect to time is solution of

$$\omega\tilde{p}(\omega,x|0,y) - p(0,x|0,y)$$
$$= \kappa\left(\theta - y\right)\frac{\partial}{\partial y}\tilde{p}(\omega,x|0,y) + y\mathbb{E}\left[\tilde{p}(\omega,x|0,y+\theta\xi) - \tilde{p}(\omega,x|0,y)\right]$$

Since $\tilde{p}_\alpha(\omega,x|0,y) = \omega^{\alpha-1}\tilde{p}(\omega^\alpha,x|0,y)$, replacing ω by ω^α and multiplying by $\omega^{\alpha-1}$ leads to

$$\omega^\alpha\left(\tilde{p}_\alpha(\omega,x|0,y)\right) - \omega^{\alpha-1}\tilde{p}_\alpha(0,x|0,y) = \kappa\left(\theta - y\right)\frac{\partial}{\partial y}\tilde{p}_\alpha(\omega,x|0,y)$$
$$+y\mathbb{E}\left[\tilde{p}_\alpha(\omega,x|0,y+\theta\xi) - \tilde{p}_\alpha(\omega,x|0,y)\right] ,$$

where we have also used the relation $p_\alpha(0,x|0,y) = p(0,x|0,y)$. The left-hand term is the Caputo's derivative of $p_\alpha(t,\omega)$ and therefore we get the backward Kolmogorov Equation. **End**

We denote by $p_\alpha(t,x_1,x_2|s,y_1,y_2)$ the bivariate probability density function $(\lambda_{S_t},P_{S_t})_{t\geq 0}$ that is defined as follows:

$$p_\alpha(t,x_1,x_2|s,y_1,y_2) = P\left(\lambda_{S_t}\in[x_1,x_1+dx],P_{S_t}\in[x_2,x_2+dx]\,|\lambda_s=y_1,P_s=y_2\right)$$

for $s\leq t$ and $x_1,x_2,y_1,y_2\in\mathbb{R}^+$. This joint pdf is obtained in a similar way to the one of the time-changed intensity. For this reason, we do not provide the proof.

Proposition 7.2. *The pdf $p_\alpha(t,x_1,x_2|s,y_1,y_2)$ of $(\lambda_{S_t},P_{S_t})_{t\geq 0}$ is solution of a time-fractional Fokker-Planck equation:*

$$\begin{aligned}\frac{\partial^\alpha p_\alpha(t,x_1,x_2|.)}{\partial t^\alpha} &= -\frac{\partial}{\partial x_1}\left(\kappa\left(\theta-x_1\right)p_\alpha(t,x_1,x_2|.)\right)-\eta\mathbb{E}\left[\xi p_\alpha(t,x_1-\eta\xi,x_2-\xi|.)\right] \\ &\quad +x_1\mathbb{E}\left[p_\alpha(t,x_1-\eta\xi,x_2-\xi|.)-p_\alpha(t,x_1,x_2|.)\right]\,,\end{aligned} \tag{40}$$

with the initial condition: $p_\alpha(s,x_1,x_2|s,y_1,y_2)=\delta_{\{x_1-y_1,x_2-y_2\}}$

8 Moments of the fractional intensity

As the fractional Hawkes process is a time-changed model, then moments of λ_{S_t} are directly related to those of λ_t and S_t. While the expectation the fractional intensity is easy to establish, the variance and autocovariance require more attention.

Proposition 8.1. *The expectation of λ_{S_t} is equal to*

$$\mathbb{E}_0\left[\lambda_{S_t}\right]=E_\alpha\left(\left(\eta\mu-\kappa\right)t^\alpha\right)\left(\lambda_0+\frac{\kappa\theta}{\eta\mu-\kappa}\right)-\frac{\kappa\theta}{\eta\mu-\kappa}\,. \tag{41}$$

This result is found by combining Equations (6) and (34). The autocovariance of the fractional intensity is a function of the Laplace's transforms of S_t+S_s and S_t-S_s as stated in the proposition:

Proposition 8.2. *Let us denote $\beta:=-\left(\eta\mu-\kappa\right)>0$ then for $s\leq t$, the covariance between λ_{S_s} and λ_{S_t} is given by*

$$\begin{aligned}\mathbb{C}_0\left(\lambda_{S_s},\lambda_{S_t}\right) &= \frac{\rho_1\lambda_0+\rho_2}{\eta\mu-\kappa}\left(\mathbb{E}_0\left(e^{-\beta(S_t+S_s)}\right)-E_\alpha\left(-\beta t^\alpha\right)\right)+ \\ &\quad \frac{\rho_2}{2\left(\eta\mu-\kappa\right)}\left(\mathbb{E}_0\left(e^{-\beta(S_t-S_s)}\right)-\mathbb{E}_0\left(e^{-\beta(S_t+S_s)}\right)\right)+ \\ &\quad \left(\lambda_0+\frac{\kappa\theta}{\eta\mu-\kappa}\right)^2\left(\mathbb{E}_0\left(e^{-\beta(S_s+S_t)}\right)-E_\alpha\left(-\beta t^\alpha\right)E_\alpha\left(-\beta s^\alpha\right)\right)\,.\end{aligned} \tag{42}$$

Proof Let us denote by $(\mathcal{H}_t)_{t\geq 0}$, the filtration of $(S_t)_{t\geq 0}$. For $s\leq t$, the covariance between λ_{S_s} and λ_{S_t} is the sum of the expected $\mathcal{H}_t$-conditional covariance and of the covariance of $\mathcal{H}_t$-conditional expectations:

$$\begin{aligned}\mathbb{C}_0\left(\lambda_{S_s},\lambda_{S_t}\right) &= \mathbb{E}\left(\mathbb{C}_0\left(\lambda_{S_s},\lambda_{S_t}\right)|\mathcal{H}_t\vee\mathcal{F}_0\right)+ \\ &\quad \mathbb{C}_0\left(\mathbb{E}\left(\lambda_{S_s}|\mathcal{H}_t\vee\mathcal{F}_0\right),\mathbb{E}\left(\lambda_{S_t}|\mathcal{H}_t\vee\mathcal{F}_0\right)\right)\,.\end{aligned} \tag{43}$$

From equation (7), we directly infer that

$$\begin{aligned}&\mathbb{E}\left(\mathbb{C}_0\left(\lambda_{S_s},\lambda_{S_t}\right)|\mathcal{H}_t\vee\mathcal{F}_0\right)= \\ &\quad \frac{\rho_1\lambda_0+\rho_2}{\eta\mu-\kappa}\left(\mathbb{E}_0\left(e^{-\beta(S_t+S_s)}\right)-\mathbb{E}_0\left(e^{-\beta S_t}\right)\right)+ \\ &\quad \frac{\rho_2}{2\left(\eta\mu-\kappa\right)}\left(\mathbb{E}_0\left(e^{-\beta(S_t-S_s)}\right)-\mathbb{E}_0\left(e^{-\beta(S_t+S_s)}\right)\right)\,.\end{aligned}$$

In order to calculate the covariance between conditional expectations, we introduce the following notations:

$$\begin{aligned} Y_s := \mathbb{E}\left(\lambda_{S_s} | \mathcal{H}_t \vee \mathcal{F}_0\right) &= e^{(\eta\mu-\kappa)S_s}\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\left(e^{(\eta\mu-\kappa)S_s}-1\right) , \\ Y_t := \mathbb{E}\left(\lambda_{S_t} | \mathcal{H}_t \vee \mathcal{F}_0\right) &= e^{(\eta\mu-\kappa)S_t}\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\left(e^{(\eta\mu-\kappa)S_t}-1\right) . \end{aligned}$$

and

$$\mathbb{C}_0\left(\mathbb{E}\left(\lambda_{S_s} | \mathcal{H}_t \vee \mathcal{F}_0\right), \mathbb{E}\left(\lambda_{S_t} | \mathcal{H}_t \vee \mathcal{F}_0\right)\right) = \mathbb{E}_0\left(Y_s Y_t\right) - \mathbb{E}_0\left(Y_t\right)\mathbb{E}_0\left(Y_t\right) .$$

A direct calculation leads to

$$\begin{aligned} \mathbb{E}_0\left(Y_s Y_t\right) &= \mathbb{E}_0\left(e^{(\eta\mu-\kappa)(S_s+S_t)}\right)\left[\lambda_0^2 + \frac{2\kappa\theta}{\eta\mu-\kappa}\lambda_0 + \left(\frac{\kappa\theta}{\eta\mu-\kappa}\right)^2\right] \\ &\quad -\mathbb{E}_0\left(e^{(\eta\mu-\kappa)S_s}\right)\left(\left(\frac{\kappa\theta}{\eta\mu-\kappa}\right)^2 + \frac{\kappa\theta\lambda_0}{\eta\mu-\kappa}\right) \\ &\quad -\mathbb{E}_0\left(e^{(\eta\mu-\kappa)S_t}\right)\left(\left(\frac{\kappa\theta}{\eta\mu-\kappa}\right)^2 + \frac{\kappa\theta\lambda_0}{\eta\mu-\kappa}\right) + \left(\frac{\kappa\theta}{\eta\mu-\kappa}\right)^2 . \end{aligned}$$

Given that

$$\begin{aligned} \mathbb{E}_0\left(Y_s\right) &= \mathbb{E}_0\left(e^{(\eta\mu-\kappa)S_s}\right)\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\left(\mathbb{E}_0\left(e^{(\eta\mu-\kappa)S_s}\right)-1\right) , \\ \mathbb{E}_0\left(Y_t\right) &= \mathbb{E}_0\left(e^{(\eta\mu-\kappa)S_t}\right)\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\left(\mathbb{E}_0\left(e^{(\eta\mu-\kappa)S_t}\right)-1\right) , \end{aligned}$$

we infer the result. **end**

Before providing analytical expressions for the Laplace's transforms of $S_t + S_s$ and $S_t - S_s$, we calculate the variance of the fractional intensity:

Corollary 8.3. *If we denote by* $\rho_1 = \eta^2(\psi^2+\mu^2)$ *and* $\rho_2 = \frac{\eta^2\kappa\theta(\psi^2+\mu^2)}{\eta\mu-\kappa}$ *the variance of* λ_{S_t} *is given by*

$$\begin{aligned} \mathbb{V}_0\left[\lambda_{S_t}\right] &= \frac{\rho_1\lambda_0+\rho_2}{\eta\mu-\kappa}\left(E_\alpha\left(2\left(\eta\mu-\kappa\right)t^\alpha\right) - E_\alpha\left(\left(\eta\mu-\kappa\right)t^\alpha\right)\right) \\ &\quad + \frac{\rho_2}{2\left(\eta\mu-\kappa\right)}\left(1 - E_\alpha\left(2\left(\eta\mu-\kappa\right)t^\alpha\right)\right) + \\ &\quad \left(\lambda_0 + \frac{\kappa\theta}{\eta\mu-\kappa}\right)^2\left(E_\alpha\left(2\left(\eta\mu-\kappa\right)t^\alpha\right) - \left(E_\alpha\left(\left(\eta\mu-\kappa\right)t^\alpha\right)\right)^2\right) . \end{aligned} \tag{44}$$

This corollary is a direct consequence of Proposition 2.2. The Laplace's transforms of expectations involved in the autocovariance, in Equation (42), admit integral representations presented in the next two propositions.

Proposition 8.4. *For* $t \geq s$ *and* $\beta < 0$*, The Laplace's transform of* $e^{-\beta(S_t-S_s)}$ *is equal to*

$$\mathbb{E}_0\left(e^{-\beta(S_t-S_s)}\right) = \frac{\alpha\beta}{\Gamma\left(1+\alpha\right)}\int_{y=0}^{s} y^{\alpha-1}E_\alpha\left(-\beta(t-y)^\alpha\right)dy + E_\alpha\left(-\beta t^\alpha\right) . \tag{45}$$

A proof of this result may be found in Leonenko et al. (2013, b), Pproposition 3.1. At the best of our knowledge, we haven't found in the literature any expression for the Laplace's transform of $S_t + S_s$, detailed in the next proposition.

Proposition 8.5. *For $t \geq s$ and $\beta < 0$, The Laplace's transform of $e^{-\beta(S_t+S_s)}$ is equal to*

$$\mathbb{E}_0\left(e^{-\beta(S_t+S_s)}\right) = \frac{1}{2}\int_{y=0}^{s}\left(\frac{d}{dy}E_\alpha\left(-2\beta y^\alpha\right)\right)E_\alpha\left(-\beta(t-y)^\alpha\right)dy + E_\alpha\left(-\beta t^\alpha\right) \tag{46}$$

where

$$\frac{d}{dy}E_\alpha\left(-2\beta y^\alpha\right) = \sum_{n=0}^{\infty}\frac{(-2\beta)^n\, y^{n\alpha-1}}{\Gamma(n\alpha)}.$$

Proof We respectively denote by $h(u,v)$ and $H(u,v)$, the bivariate pdf and cdf of the pair (S_s, S_t). By definition $H(u,\infty) = P(S_s \leq u)$, $H(\infty, v) = P(S_t \leq v)$ and $H(\infty,\infty) = 1$. The Laplace's transform is hence equal to

$$\mathbb{E}_0\left(e^{-\beta(S_t+S_s)}\right) = \int_0^\infty\int_0^\infty \underbrace{e^{-\beta(v+u)}}_{f(u,v)}H(du,dv).$$

Using a bivariate by part integration leads to

$$\begin{aligned}\int_0^\infty\int_0^\infty f(u,v)H(du,dv) &= \int_0^\infty\int_0^\infty H([u,\infty]\times[v,\infty])f(du,dv) \\ &\quad+\int_0^\infty H([u,\infty]\times[0,\infty])f(du,0) \\ &\quad+\int_0^\infty H([0,\infty]\times[v,\infty])f(0,dv) \\ &\quad+f(0,0)H([0,\infty]\times[0,\infty])W,\end{aligned} \tag{47}$$

where $f(0,0)H([0,\infty]\times[0,\infty]) = 1$. On the other hand,

$$\begin{aligned}H([u,\infty]\times[0,\infty]) &= P(S_s \geq u) \\ &= 1 - P(S_s \leq u),\end{aligned}$$

therefore, the second term in Equation (47) becomes

$$\begin{aligned}\int_0^\infty H([u,\infty]\times[0,\infty])f(du,0) &= -\beta\int_0^\infty e^{-\beta u}\left(1-P(S_s\leq u)\right)du \\ &= \left[e^{-\beta u}\left(1-P(S_s\leq u)\right)\right]_0^\infty + \mathbb{E}\left(e^{-\beta S_s}\right) \\ &= E_\alpha\left(-\beta s^\alpha\right) - 1.\end{aligned}$$

In a similar way, we find that the third term in Equation (47) is:

$$\begin{aligned}\int_0^\infty H([0,\infty]\times[v,\infty])f(0,dv) &= -\beta\int_0^\infty e^{-\beta v}\left(1-P(S_t\leq v)\right)dv \\ &= E_\alpha\left(-\beta t^\alpha\right) - 1.\end{aligned}$$

The first term of Equation (47) is a double integral

$$\int_0^\infty\int_0^\infty H([u,\infty]\times[v,\infty])f(du,dv) = \int_0^\infty\int_0^\infty P\left(u\leq S_s, v\leq S_t\right)\beta^2 e^{-\beta(v+u)}dudv$$

Since S_t is increasing and discontinuous and that $P(u \leq S_s, v \leq S_t) = P(u \leq S_s)$ for $u > v$, this integral may be split:

$$\begin{aligned}\int_0^\infty\int_0^\infty P\left(u\leq S_s, v\leq S_t\right)\beta^2 e^{-\beta(v+u)}dudv &= \int_0^\infty\int_0^v P\left(u\leq S_s, v\leq S_t\right)\beta^2 e^{-\beta(v+u)}dudv \\ &\quad+\int_0^\infty\int_v^\infty P\left(u\leq S_s\right)\beta^2 e^{-\beta(v+u)}dudv.\end{aligned} \tag{48}$$

Integrating by parts allows us to rewrite the second term of this last equation:

$$\int_0^\infty \int_v^\infty P\left(u \leq S_s\right) \beta^2 e^{-\beta(v+u)} du dv = \int_0^\infty \int_0^u P\left(u \leq S_s\right) \beta^2 e^{-\beta(v+u)} dv du$$
$$= -\beta \int_0^\infty P\left(u \leq S_s\right) \left(e^{-\beta(2u)} - e^{-\beta u}\right) du$$
$$= -\beta \int_0^\infty P\left(u \leq S_s\right) e^{-2\beta u} du + \beta \int_0^\infty P\left(u \leq S_s\right) e^{-\beta u} du \,.$$

The first and second integrals are respectively equal to

$$-\beta \int_0^\infty P\left(u \leq S_s\right) e^{-2\beta u} du = \int_0^\infty \left(1 - P\left(S_s \leq u\right)\right) \left(-\beta e^{-2\beta u}\right) du$$
$$= \left[P\left(u \leq S_s\right) \frac{1}{2} e^{-2\beta u}\right]_{u=0}^{u=\infty} + \frac{1}{2} \int_0^\infty g\left(s, u\right) \left(e^{-2\beta u}\right) du$$
$$= \frac{1}{2} \left(E_\alpha\left(-2\beta s^\alpha\right) - 1\right) \,,$$

and

$$\beta \int_0^\infty P\left(u \leq S_s\right) e^{-\beta u} du = 1 - E_\alpha\left(-\beta s^\alpha\right) \,.$$

The first term of Equation (48) is hence given by:

$$\int_0^\infty \int_v^\infty P\left(u \leq S_s\right) \beta^2 e^{-\beta(v+u)} du dv = \frac{1}{2} + \frac{1}{2} E_\alpha\left(-2\beta s^\alpha\right) - E_\alpha\left(-\beta s^\alpha\right) \,.$$

On the other hand , for $s \leq t$ and $u \leq v$, given that U_t has independent self-similar increments, the cdf of (S_s, S_t) is such that

$$\begin{aligned} P\left(u \leq S_s, v \leq S_t\right) &= P\left(U_u \leq s, U_v \leq t\right) \\ &= P\left(U_u \leq s, U_u + \left(U_v - U_u\right) \leq t\right) \\ &= \int_{y=0}^s p_U(u,y) dy \int_0^{t-y} p_U(v-u,x) dx dy \end{aligned}$$

From equation (32), we know that the pdf of U_t is related to the one of S_t as follows:

$$\frac{y}{\alpha} p_U(u,y) = u g(y,u) \,,$$
$$\frac{x}{\alpha} p_U(v-u,x) = (v-u) g(x, v-u) \,.$$

The first term of Equation (48) is developped as follows:

$$\beta^2 \int_{u=0}^\infty \int_{v=u}^\infty P\left(u \leq S_s, v \leq S_t\right) e^{-\beta(v+u)} dv du \tag{49}$$
$$= \beta^2 \int_{u=0}^\infty \int_{v=u}^\infty \int_{y=0}^s p_U(u,y) dy \int_{x=0}^{t-y} p_U(v-u,x) dx dy e^{-\beta(v+u)} dv du$$
$$= \beta^2 \int_{y=0}^s \int_{x=0}^{t-y} \int_{u=0}^\infty p_U(u,y) \int_{v=u}^\infty p_U(v-u,x) e^{-\beta(v+u)} dv du dx dy$$
$$= \beta^2 \int_{y=0}^s \frac{\alpha}{y} \int_{x=0}^{t-y} \frac{\alpha}{x} \int_{u=0}^\infty u g(y,u) \int_{v=u}^\infty \left(v-u\right) g(x, v-u) e^{-\beta(v-u+2u)} dv du dx dy$$
$$= \beta^2 \int_{y=0}^s \frac{\alpha}{y} \int_{x=0}^{t-y} \frac{\alpha}{x} \int_{u=0}^\infty u g(y,u) e^{-2\beta u} du \int_{z=0}^\infty z g(x,z) e^{-\beta z} dz dx dy$$

Leonenko et al. (2013, b) in the proof of proposition 3.1, have shown that

$$\int_{z=0}^{\infty} zg(x,z)e^{-\beta z}dz = -\frac{x}{\alpha\beta}\frac{d}{dx}E_{\alpha}\left(-\beta x^{\alpha}\right) ,$$

and that

$$\int_{u=0}^{\infty} ug(y,u)e^{-2\beta u}du = -\frac{y}{2\alpha\beta}\frac{d}{dy}E_{\alpha}\left(-2\beta y^{\alpha}\right) .$$

Equation (49) may be simplified as follows:

$$\begin{aligned}
&\int_0^{\infty}\int_0^{v} P\left(u \leq S_s, v \leq S_t\right)\beta^2 e^{-\beta(v+u)}dudv \\
&= -\frac{1}{2}\int_{y=0}^{s}\frac{d}{dy}E_{\alpha}\left(-2\beta y^{\alpha}\right)\int_{x=0}^{t-y}\frac{d}{dx}E_{\alpha}\left(-\beta x^{\alpha}\right)dxdy \\
&= -\frac{1}{2}\int_{y=0}^{s}\frac{d}{dy}E_{\alpha}\left(-2\beta y^{\alpha}\right)\left(E_{\alpha}\left(-\beta(t-y)^{\alpha}\right)-1\right)dy \\
&= -\frac{1}{2}\int_{y=0}^{s}\left(\frac{d}{dy}E_{\alpha}\left(-2\beta y^{\alpha}\right)\right)E_{\alpha}\left(-\beta(t-y)^{\alpha}\right)dy + \frac{1}{2}\left(E_{\alpha}\left(-2\beta s^{\alpha}\right)-1\right) .
\end{aligned}$$

Collecting all terms allows us to deduce Equation (46). **end**

Figure 4 shows the autocorrelogram and the expected intensity for a fractional Hawkes process with constant marks ($\eta = 5$, $\theta = 5$, $\kappa = 8.5$, $\xi = 2$, $s = 0.1$ and $t = 0.1$ to 2.0). The upper graph clearly emphasizes that the autocorrelation decays at a sub-exponential rate for values of α lower than one. The long term mean of the intensity is equal to $\lim_{t\to\infty}\mathbb{E}\left(\lambda_t\right) = 28.33$ and λ_0 is set to $\theta = 5$. The second graph reveals that the expected intensity converges to 28.33 but a lower pace than a Hawkes process when α is lower than one.

9 Laplace's transform of $p_{\alpha}(t, x|0, y)$ and numerical solution

Let us denote by $\phi_{\alpha}(.)$ the Laplace's transform of the transition pdf $p_{\alpha}(t, x|0, y)$ of the time changed intensity:

$$\begin{aligned}
\phi_{\alpha}(t,z|0,y) &= \mathbb{E}\left(e^{-z\lambda_{S_t}}|\lambda_0 = y\right) \\
&= \int_0^{\infty}\phi(\tau,z|0,y)g(t,\tau)d\tau
\end{aligned}$$

for $z \in \mathbb{R}^+$. This Laplace's transform is solution of a forward equation.

Proposition 9.1. *The Laplace's transform $\phi_{\alpha}(t, z|0, y)$ is solution of the following forward equation:*

$$\frac{\partial^{\alpha}\phi_{\alpha}(t,z|0,y)}{\partial t^{\alpha}} = -\kappa\theta z\,\phi_{\alpha}(t,z|0,y) + \left(1-\kappa z-\mathbb{E}\left(e^{-z\eta\xi}\right)\right)\frac{\partial\phi_{\alpha}(t,z|0,y)}{\partial z} \tag{50}$$

with the initial condition $\phi_{\alpha}(0, z|0, y) = e^{-zy}$.

We do not provide the proof of this results since it is obtained in the same manner as Proposition 7.1.

We now propose a numerical method for solving the forward equation (50) for z and t in the neighbourhood of zero. As in Section 4, we consider a expansion of the Laplace's transform $\phi_{\alpha}(t, z|0, y)$. From e.g. Usero (2008), we know that for any continuous differentiable function, $u(t, x)$ with respect to time and x may be rewritten as an infinite sum

$$u(t,x) = \sum_{k=0}^{\infty}U_k(x)(t-t_0)^{k\alpha} ,$$

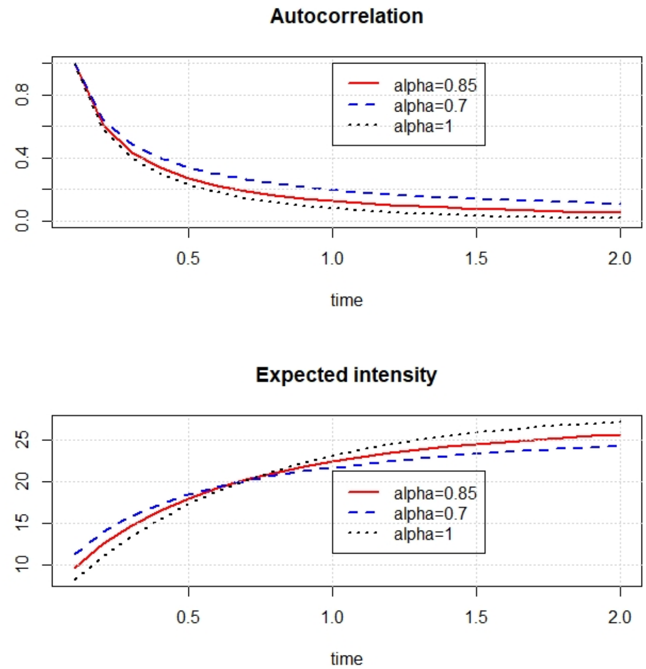

Figure 4: Upper graph: autocorrelogram of fractional Hawkes processes. Lower graph: expected intensity for different α.

where

$$U_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\left(\frac{\partial}{\partial t^{\alpha}}\right)^k u(t,x)\right]_{t=t_0}.$$

Therefore we assume that this Laplace's transform can be approached by the following sum:

$$\phi_\alpha(t,z|0,y) \approx \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} w_\alpha(k,h)\times z^h t^{\alpha k}, \tag{51}$$

The differential weights in this sum are defined by:

$$w_\alpha(k,h) = \frac{1}{\Gamma(k\alpha+1)\,h!}\left[\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^h}\phi_\alpha(t,z|.)\right]_{t=0,z=0}. \tag{52}$$

Notice that we use the notation

$$\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^h} = \left(\frac{\partial}{\partial t^{\alpha}}\right)^k \frac{\partial^h}{\partial z^h},$$

but the reader must be aware that $\left(\frac{\partial}{\partial t^{\alpha}}\right)^k \neq \frac{\partial^k}{\partial t^{k\alpha}}$ for the Caputo's derivative. The equation (51) is exact if $\phi_\alpha(t,z|0,y)$ is the product of one function of t and of one function of z. Differential weights are computed recursively.

Proposition 9.2. *The Laplace transform is approximated by the sum*

$$\phi_\alpha(t,z|0,y) = \sum_{k=0}^{K}\sum_{h=0}^{H} w_\alpha(k,h)\times z^h t^{\alpha k}, \tag{53}$$

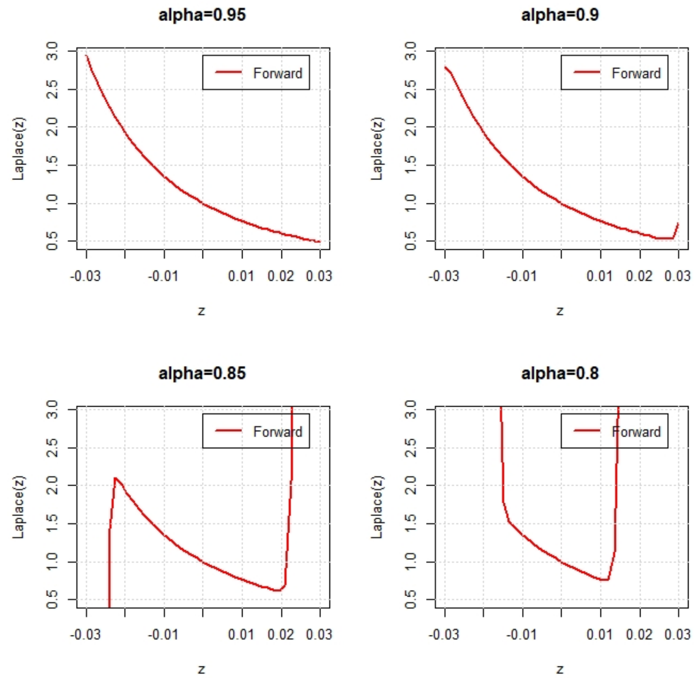

Figure 5: Laplace's transform of λ_{S_t} computed with the forward and backward equations. $\eta = 7$, $\theta = 5$, $\kappa = 8.5$ and $\xi = 1$. The maturity is $t = 0.5$. $K = 50$ and $H = 80$.

with differential weights satisfying the following recursive equation:

$$\begin{aligned}\frac{\Gamma\left((k+1)\alpha+1\right)}{\Gamma\left(k\alpha+1\right)}w(k+1,h) &= -\kappa\theta w(k,h-1) - \kappa\, h\, w(k,h) + (h+1)w(k,h+1) \\ &\quad - \sum_{j=0}^{h}\frac{j+1}{(h-j)!}(-1)^{h-j}\mathbb{E}\left[(\eta\xi)^{h-j}\right]w(k,j+1)\,. \end{aligned} \tag{54}$$

The initial conditions that are used for initializing the recursion are:

$$\begin{aligned} w(0,h) &= \frac{1}{h!}\left(-y\right)^{h}\,, \\ w(k,0) &= 0 \quad k>0\,. \end{aligned}$$

Proof To establish the recursion (54), we apply the differential operator $\frac{1}{\Gamma(k\alpha+1)h!}\left[\frac{\partial^{k\alpha+h}}{\partial t^{k\alpha}\partial z^{h}}\right]_{t=0,z=0}$ to the forward equation (16). Since $e^{-yz} = \sum_{h=0}^{\infty}\frac{1}{h!}\left(-y\right)^{h}z^{h}$ therefore $w(0,h) = \frac{1}{h!}\left(-y\right)^{h}$. **end**

Figure 5 shows the Laplace's transform approached by the sum (53). As in the non-fractional case, the accuracy deteriorates with the distance to zero. However, it is accurate enough to compute numerically the first four moments as shown in Table 2 by deriving the Laplace's transform in the neighbourhood of $z = 0$. This table also provides the theoretical expectation and variance computed with Equations (41) and (44).

10 Conclusions

The first part of this article presents the forward or Fokker-Planck Equation (FPE) for a Hawkes process with an exponential decaying memory. Due to self-exciting jumps, the FPE differs from the one of a pure jump process. We also provide an numerical method for solving the forward equation satisfied by the Laplace's transform of the intensity process. This method is enough accurate in the neighbourhood of zero for computing the first four moments. Next, we provide the FPE ruling the joint pdf of point and intensity process.

	Numerical				Theoretical	
α	$\mathbb{E}_0(\lambda_t)$	$\mathbb{V}_0(\lambda_t)$	$\mathbb{S}_0(\lambda_t)$	$\mathbb{K}_0(\lambda_t)$	$\mathbb{E}_0(\lambda_t)$	$\mathbb{V}_0(\lambda_t)$
0.95	28.3333	18.9750	1.5213	6.4740	28.3333	18.9744
0.90	28.3333	18.9709	1.5532	6.7081	28.3333	18.9695
0.85	28.3333	18.9516	1.5814	6.9127	28.3333	18.9510
0.80	28.3333	18.9523	1.6055	7.0893	28.3333	18.9231

Table 2: Comparison of numerical and theoretical moments, $\eta = 7$, $\theta = 8.5$, and $\xi = 1$. The maturity is $t = 0.5$. $y_0 = \lim_{t\to\infty} \mathbb{E}(\lambda_t) = 28.3333$. $K = 50$ and $H = 80$.

The second part of the article studies the features of the fractional Hawkes process. We build this process with a non-Markov time-change. We show that the pdf of its intensity satisfies a forward equation in which the derivative with respect to time is replaced by the Caputo's derivative. Next, we find closed form expressions for expectation, variance and autocovariance of the intensity. Finally, we develop a numerical method for computing the Laplace's transform of the fractional intensity. This numerical scheme allows us to estimate the first moments.

This work opens the way to further research. As for Hawkes processes, we have very few information about the properties of N_{S_t}. This deserves further investigation. Secondly, It would be interesting to develop a multivariate model with self-excitation and contagion between jumps of different point processes. Finally, the statistical estimation of parameters is an open question.

11 Appendix A, the bivariate Kramers-Moyal expansion

In this appendix, we sketch the proof of the bivariate Kramers-Moyal expansion.

Proposition 11.1. *The bivariate pdf of (λ_t, P_t) is related to cross moments through the relation:*

$$\begin{aligned}
&p(t+\Delta, x_1, x_2|s, y_1, y_2) - p(t, x_1, x_2|s, y_1, y_2) \\
&\quad = \sum_{n=1}^{\infty}\sum_{j=0}^{n} \frac{(-1)^n}{j!n-j!} \frac{\partial^j}{\partial x_1^j} \frac{\partial^{n-j}}{\partial x_2^{n-j}} \left(M(j, n-j|x_1, x_2) p(t, x_1, x_2|s, y_1, y_2) \right) ,
\end{aligned}$$

where $M(j, n-j|x_1, x_2)$ is defined as:

$$M(j, n-j|x_1, x_2) \quad = \quad \mathbb{E}\Big((\lambda_{t+\Delta} - \lambda_t)^j (P_{t+\Delta} - P_t)^{n-j} |\lambda_t = x_1, P_t = x_2 \Big) .$$

Proof. We start from the joint characteristic function of $(\lambda_{t+\Delta} - \lambda_t)$ and $(P_{t+\Delta} - P_t)$, conditionally up to the information up to time t. This function is also the Fourier transform of the joint pdf $p(t+\Delta, x_1, x_2|t, y_1, y_2)$ that we denote by

$$\begin{aligned}
&\hat{p}(t+\Delta, u_1, u_2|t, y_1, y_2) \\
&\quad = \mathbb{E}\left(\exp\left(iu_1 (\lambda_{t+\Delta} - \lambda_t) + iu_2 (P_{t+\Delta} - P_t)\right) | \lambda_t = y_1, I_t = y_2\right) \\
&\quad = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{iu_1(x_1-y_1)+iu_2(x_2-y_2)} p(t+\Delta, x_1, x_2|t, y_1, y_2) dx_1 dx_2 .
\end{aligned}$$

Since the Taylor expansion of $f(x, y) = e^{x+y}$ around $(0, 0)$ is equal to the sum

$$\begin{aligned}
e^{x+y} &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{1}{j!(n-j)!} (x)^j (y)^{n-j} \left[\frac{\partial^j}{\partial x^j}\frac{\partial^{n-j}}{\partial y^{n-j}} e^{a+b}\right]_{a=b=0} \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{1}{j!(n-j)!} (x)^j (y)^{n-j} ,
\end{aligned}$$

the characteristic function may be expanded explicitly as follows:

$$
\begin{aligned}
&\hat{p}(t+\Delta, u_1, u_2|t, y_1, y_2) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\left(iu_1\left(x_1-y_1\right)\right)^j\left(iu_2\left(x_2-y_2\right)\right)^{n-j}}{j!(n-j)!}p(t+\Delta, x_1, x_2|.)dx_1dx_2 \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(iu_1)^j(iu_2)^{n-j}}{j!(n-j)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x_1-y_1)^j(x_2-y_2)^{n-j}p(t+\Delta, x_1, x_2|.)dx_1dx_2 \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(iu_1)^j(iu_2)^{n-j}}{j!(n-j)!}\mathbb{E}\left(\left(\lambda_{t+\Delta}-\lambda_t\right)^j\left(P_{t+\Delta}-P\right)^{n-j}|\lambda_t=y_1, P_t=y_2\right) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(iu_1)^j(iu_2)^{n-j}}{j!(n-j)!}M(j, n-j|y_1, y_2)\,.
\end{aligned}
$$

Therefore, inverting the Fourier transform leads to the following expansion for the transition probability

$$
\begin{aligned}
&p(t+\Delta, x_1, x_2|t, y_1, y_2) \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-iu_1(x_1-y_1)-iu_2(x_2-y_2)}\hat{p}(t+\Delta, u_1, u_2|t, y_1, y_2)du_1du_2 \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{M(j, n-j|y_1, y_2)}{j!(n-j)!}\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sum_{u_2\in\mathbb{N}}(iu_1)^j(iu_2)^{n-j}e^{-iu_1(x_1-y_1)-iu(x_2-y_2)}du_1du_2\,.
\end{aligned}
$$

Using the ansatz that

$$
\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(iu_1)^j(iu_2)^{n-j}e^{-iu_1(x_1-y_1)-iu_2(x_2-y_2)}du_1du_2 \\
&\quad = \left(-\frac{\partial}{\partial x_1}\right)^j\left(-\frac{\partial}{\partial x_2}\right)^{n-j}\delta\left(x_1-y_1\right)\delta\left(x_2-y_2\right)\,,
\end{aligned}
$$

and as for any function bivariate function $f(y_1, y_2)$, we have that

$$
\begin{aligned}
&\delta\left(x_1-y_1\right)\delta\left(x_2-y_2\right)f(y_1, y_2) \\
&\quad = \delta\left(x_1-y_1\right)\delta\left(x_2-y_2\right)f(x_1, x_2)\,,
\end{aligned}
$$

we obtain the following expansion for $p(t+\Delta, x_1, x_2|t, y_1, y_2)$:

$$
\begin{aligned}
&p(t+\Delta, x_1, x_2|t, y_1, y_2) = \\
&\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!(n-j)!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}M(j, n-j|y_1, y_2)\delta\left(x_1-y_1\right)\delta\left(x_2-y_2\right)
\end{aligned}
$$

Finally, using the Chapman-Kolmogorov equation allows us to rewrite $p(t+\Delta, x_1, x_2|s, y_1, y_2)$:

$$
\begin{aligned}
&p(t+\Delta, x_1, x_2|s, y_1, y_2) \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}p(t+\Delta, x_1, x_2|t, z_1, z_2)p(t, z_1, z_2|s, y_1, y_2)dz_1dz_2 \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^n}{j!(n-j)!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}M(j, n-j|z_1, z_2) \\
&\times p(t, z_1, z_2|s, y_1, y_2)\delta\left(x_1-z_1\right)\delta\left(x_2-z_2\right)dz_1dz_2\,.
\end{aligned}
$$

end

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