A kernel-based method for dense-mesh problems

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Abstract — **Fast methods for the evaluation of the EFIE operator at a given current distribution undergo some difficulties at low frequency. In particular, it is well known that the MLFMA at low frequency undergoes a breakdown due to the translation operator that becomes unstable. In this paper, an innovative method that provides a fast way to calculate the evaluation of the EFIE operator at a given current distribution and for low-frequency problem is presented. It is based on local Taylor approximation of the Green's function coupled with one-time (once and for all geometry) SVD compression technique.**

1 INTRODUCTION

One of the best known kernel-dependent methods is the Fast Multipole Method (FMM) [6] with a complexity of $O(N^{1.5})$. An updated version of the FMM, the Multi-Level Fast Multipole Algorithm (MLFMA) [14], is a direct improvement of the FMM, which consists of subdividing the domain into an octree of parents and children domains. Then, the complexity of the aggregation operations, translation operations and the disaggregation operations can be reduced to a complexity of $O(N\log(N))$. However, the basis and test functions must be placed at least a half wavelength apart, otherwise the FFM-MLFMA undergoes a breakdown. At low frequency, the translation operator computation becomes unstable. Therefore, for close basis and testing functions, a classical computation of the impedance matrix is generally performed. This means that, when the size of the basis and testing functions becomes relatively small (smaller than a hundredth of a wavelength), the computation of the impedance matrix in the near-field zone becomes cumbersome because of $O(N^2)$ complexity. Several authors have been trying to stabilize the FMM-MLFMA breakdown in the past decade [7], [3], [4]. However, all these methods suffer from uncontrolled approximations and sometimes even from somewhat contradictory hypotheses [8], which then requires a tedious empirical tuning.

For low-frequency problems, several authors have focused on kernel-free methods [10]. Basically, they focus more on the algebraic property of the rank deficiency of the off-diagonal blocks of the impedance matrix itself than on the kernel used to calculate the matrix. These methods are said to be kernel-free because they can be applied to any matrix that contains rank-deficient off-diagonal blocks. However, although the iterative solver converges faster, the full impedance matrix is still required. Several methods not requiring the knowledge of the full matrix have been developed, such as the Adaptive Cross Approximation [2] and its multi-level version [15]. The ACA-SVD [15] method consists of applying a low-cost SVD on an ACA decomposition to further reduce the two low rank matrices obtained with the ACA. However, although this method provides a matrix-vector multiplication in $O(Nlog(N))$ the computational complexity of the matrix compression (pre-processing) has a $O(N^2log(N))$ complexity. The kernel-free methods suffer from an important default: the control over the accuracy of the solution. The process that consists in stopping the columns and lines generation is somewhat empirical. In particular, it is relatively easy to imagine a PEC shape that breaks down the ACA when applied to a MoM problem.

In this work, an alternative kernel-dependant method is proposed in order to reduce the computation time due to the direct computation of the EFIE impedance matrix and the matrix-vector products in the iterative solver. The relative error in comparison to a MoM direct method can be potentially controlled which is an important difference with present fast methods. The idea resides in the observation that for basis functions much smaller than the wavelength, the local variations of the Green's function are relatively small. An alternative approach consists of using a Taylor expansion of the Green's function for source and test points given on a regular grid and of combining this approximation with a SVD compression technique. Using a regular grid of only few different sizes allows one to precalculate the SVD compressions once and for all, independently from the scatterers shapes. It will be shown that the method allows the separability of the EFIE integral. The method has been validated on a sphere and on a Radio Frequency Quadrupole (RFQ) which is a cavity placed at the very beginning of a linear accelerator [16], [5]. A comparison

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between a Methods of Moments approach solution of the RFQ and our proposed solver will be presented. The proposed method is called the Fast Near Field Method of Moments Solver (FNFMMS).

2 Mathematical formulation

The Taylor expansion is used to initiate the separability between source and observation coordinates. Indeed, the Taylor expansion expresses a given function as the sum of terms containing only monomial functions of the variables. In a first instance, for source and observation points located in different regions, the Green's function is decomposed into a polynom of the difference between coordinates, $r_i - r'_i$. In this notation, the primed coordinates correspond to sources and the unprimed ones to observation points. The index i stands for the selection of coordinate, i.e. x, y or z.
Thanks to the Newton binomial formula the powers of $r_i - r'_i$ appearing in the Green's function can be rewritten as follows:

$$(r'_i - r_i)^{\alpha_i} = \sum_{k_i=0}^{\alpha_i} \begin{pmatrix} \alpha_i \\ k_i \end{pmatrix} (r'_i)^{\alpha_i - k_i} (-r'_i)^{k_i} \quad (1)$$

where $\alpha_i, k_i \in \mathbb{N}$ and $\begin{pmatrix} \alpha_i \\ k_i \end{pmatrix} = \frac{\alpha_i!}{(\alpha_i - k_i)! k_i!}$. Let us define the following function that corresponds to the different moments of a given current $\overline{T}$ in region A

$$M_{l,k,A}(\overline{T}) \equiv \int_A f_l(\overline{T})(\overline{r}) \prod_{i=1}^{3} (r_i)^{k_i} d\overline{r} \quad (2)$$

where $k = (k_1, k_2, k_3)$. Now, the electric field due to basis function $\overline{J}$ and tested by testing function $\overline{T}$ can be rewritten as follows:

$$< \overline{T} | \overline{E}(\overline{J}) > = j\mu c \sum_{l=1}^{4} \sum_{|\alpha|=0}^{N} \sum_{k=0}^{\alpha} K^{\alpha} B^{\alpha,k} M_{l,k,A}(\overline{T}) M_{l,\alpha-k,B}(\overline{J}) \quad (3)$$

where $j \equiv \sqrt{-1}$, μ the free space permeability, c the speed of light, $|\alpha| \equiv \alpha_1 + \alpha_2 + \alpha_3$, N the Taylor expansion order, $\sum_{k=0}^{\alpha} \equiv \sum_{k_1=0}^{\alpha_1} \sum_{k_2=0}^{\alpha_2} \sum_{k_3=0}^{\alpha_3}$, $B^{\alpha,k} \equiv \prod_{i=1}^{3} \begin{pmatrix} \alpha_i \\ k_i \end{pmatrix} (-1)^{k_i}$, $K^{\alpha} \equiv \frac{1}{\alpha!} \frac{\partial^{\alpha} g}{\partial \overline{x}^{\alpha}} |_{\overline{R}_2 - \overline{R}_1}$ and $g : \mathbb{C}, \overline{x} \rightarrow \frac{e^{jk||\overline{x}||}}{||\overline{x}||}$. The local regions A and B correspond to respectively a local source region and a local test region, which must be separated by a reasonable distance in order to limit the slow convergence of the Taylor expansion close to the singularity. Therefore, a dead region must be defined around a source region in which the brute force method is applied i.e. use matrix-vector multiplication to compute the source to tested field operation. Obviously, the dead region includes the source region. Fig. 1 illustrates the definitions of the respective domains. In order to apply the approximation, the geometry of the whole MoM problem must be split into several local source domains, to which correspond respectively several local test domains. The field scattered by the current in the source domain is tested in the test region, where the approximation remains valid.

For each local source domain in the source zone,

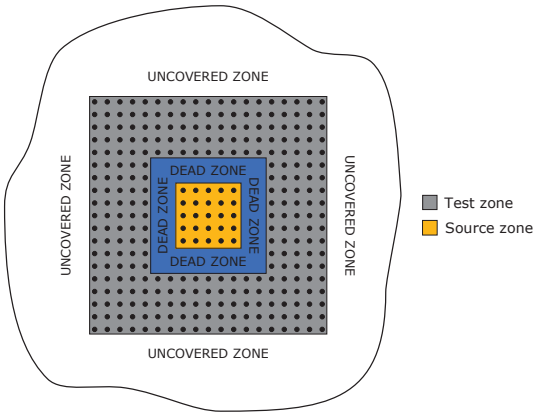

Figure 1: Definitions of the source, dead and test zones

one can use Eq. 4 to speed up the computation. The final expression reads

$$< \overline{T}, \overline{E}(\overline{S}) > \approx j\mu c \sum_{l=1}^{4} \sum_{|\alpha|=0}^{n} \sum_{k=0}^{\alpha} B^{\alpha,k} \sum_{p=1}^{P} M_{l,k,A_p}(\overline{T}) \sum_{o=1}^{O} G^{\alpha}_{p,o} M_{l,\alpha-k,B_o}(\overline{S}) \quad (4)$$

where B_o is a local source domain indexed o, A_p is a local test domain indexed p, O is the total number of local source domains in the source zone, P is the total number of local test domains in the test zone, $G^{\alpha}_{po} \equiv \frac{1}{\alpha!} \frac{\partial^{\alpha} g}{\partial \overline{x}^{\alpha}} |_{\overline{T}_p - \overline{S}_o}$, T_p is the center point of the local test domain indexed p and S_o is the center of the local source domain indexed o.

In order to further speed up the evaluation of Eq. 6, a truncated SVD decomposition [11] can be performed on the matrix G^{α}. The size of the matrix is reduced by selecting the singular values larger than a certain given threshold. Let

$p \in \mathbb{N}$ be the number of selected singular values. Let us define $U_r^\alpha \equiv [c_1, ..., c_p] \in \mathbb{C}^{M \times P}$ and $V_r^\alpha = [S_{11} l_1; ...; S_{PP} l_P] \in \mathbb{C}^{P \times N}$; then the approximate matrix reads

$$G^{\alpha,a} = U_r^\alpha V_r^\alpha \tag{5}$$

Now, one can subdivide the whole geometry into non-overlapping source zones. Each source domain receives a portion of the current living on the scatterer. For each current in each source zone, Eq. 4 is used to speed up the computation. The source zone size is predefined in order to be able to precalculate the compressed matrices. A uniform grid that covers the structure is first generated. The grid is generated in such a way that each source zone contains the same amount of sampling points. Each local region of validity of the Green's function approximation (the region of space around a sampling point) is indexed by a unique and global index. This global index is also used to point to the center of this local region which is by definition a rectangular parallelepiped. For each source domain denoted $S_{g,s}$ corresponds a test zone and a dead-zone denoted respectively by $T_{g,s}$ and $DZ_{g,s}$. The basis functions used in this paper are the Rao-Wilton-Glisson (RWG) basis functions [9]. The test of the field in a dead-zone $T_{g,s}$ is performed using a brute-force method and is given by $I_{s,2}\beta$ where β is the source current expressed through half basis function coefficients and $I_{s,2}$ is the brute force EFIE operator computed for the half source basis functions in the source zone $S_{g,s}$ and the half test basis functions in the dead-zone $DZ_{g,s}$ multiplied by an injection matrix. The injection matrix injects the current coefficients of the half basis functions into the set of half basis functions that live in $S_{g,s}$. The testing of the field in the $T_{g,s}$ zone is given by

$$I_{s,1}\beta \equiv j\mu c \sum_{l=1}^{4} \sum_{|\alpha|=0}^{n} \sum_{k=0}^{\alpha} B^{\alpha,k} M_{g,l,k} I_{T,s}^t U_r^\alpha V_r^\alpha I_{S,s} M_{g,l,\alpha-k} \beta \tag{6}$$

where $I_{S,s}$ and $I_{T,s}$ are the sparse parallel injection matrices that inject global points of the grid to local points in respectively $I_{S,s}$ and $I_{T,s}$. The matrix $M_{l,\alpha-k}$ is the global matrix of the moments and is a sparse matrice as well. The number of non-zero elements in $M_{l,\alpha-k}$ is equal to N (the number of half basis functions). The matrices U_r^α and V_r^α are the compressed matrices that remain of small fixed dimension. Finally, the contribution of all source domains to the matrix-vector product is given by

$$\sum_{s=1}^{S} I_{s,1}\beta = j\mu c \sum_{l=1}^{4} \sum_{|\alpha|=0}^{n} \sum_{k=0}^{\alpha} B^{\alpha,k} M_{g,l,k} \sum_{s=1}^{S} I_{T,s}^t U_3^\alpha V_3^\alpha I_{S,s} M_{l,\alpha-k} \beta \tag{7}$$

where S is the total number of source zones covering the whole geometry. Eq. 7 expresses the tested fields, as produced by basis functions covering the whole geometry, and tested by functions located everywhere else (excluding the dead and source zones). The limited number of columns and lines in U and V matrices, resulting from the low-rank aspect of the interactions, accelerate the matrix-vector products.

3 Numerical results

The simulation was performed on a sphere of radius one meter and at frequency of 178MHz an with 19000 (RWG) basis functions. Figure 2 shows a given source domain and its associated dead-zone domain in green and the test domain in blue. The number of source domains required to cover the whole structure is 74. In this paper, GMRES [13]

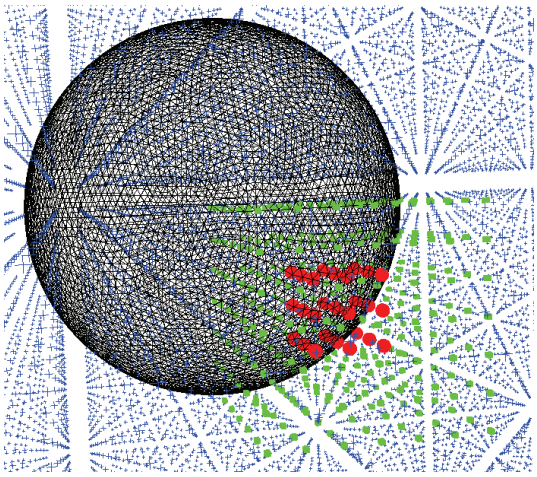

Figure 2: Presentation of a source domaine (red) and its associated dead-zone (green) and test domains (blue).

is used to solve the system of equations. In order to improve further the computational time, lower order Taylor approximation are used to calculate first-guess currents for the GMRES solver. The taylor order has been set to three and the accuracy set to a relative error of 10^{-5} on the excitation vector. The following criterion of relative error for

type \ Taylor order	0	1	2	3
FNFMMS Non pre.	250s	305s	602s	1211s
FNFMMS Pre.	49s	68s	154s	315s

Table 1: GMRES Convergence times

each guess solution has been used

$$error_n = \frac{error_{n-1}}{10} \quad (8)$$

$$error_3 = 10^{-5} \quad (9)$$

where n corresponds to the Taylor order. Fig. 3 shows that the GMRES convergence for this relative residual error criterion. In order to improve the computational time, a low-frequency preconditioner has been used [1], [12]. Table 1 shows the GMRES computation time for each order of Taylor approximation with and without the preconditioner. Table 2 shows the total computation time (GMRES and the brute force matrices computation) for the method with and without the preconditioner. The brute-force solution with the low-frequency solver has been calculated as well to show the improvement of GMRES convergence on the accurate impedance matrix. It is important to note that the low-frequency preconditioner is necessary for the good behavior of the method on such a dense mesh. When the preconditioner is used, the proposed method is more than three times faster than a brute-force approach. It is important to say that the computational times for the GMRES convergence of respectively the brute force method with and without the preconditioner are so low because an optimized LAPACK library for Intel CPU has been used. The proposed method has been implemented in C++ but without optimized instructions for the underlying hardware.

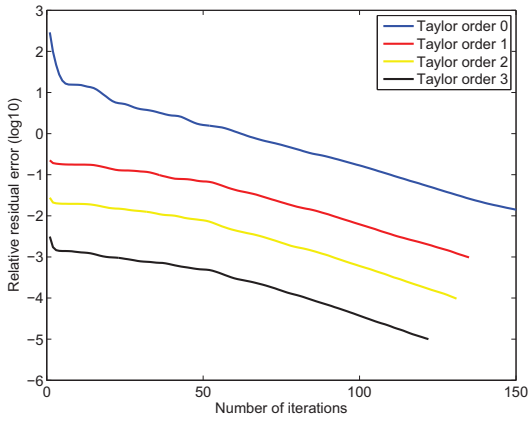

Figure 3: Number of iterations.

	GMRES	DeadZone	Total
FNFMMS Non pre.	2368s	324s	2692s
FNFMMS Pre	586	324s	910s
Brute Force	229s	2784s	3013s
Brute Force Pre.	122s	2784s	2907s

Table 2: Comparison of total simulation times

4 Conclusion

The new proposed method appears to be more than three times faster than a brute force method when combined with an efficient preconditioner. The preconditioner is necessary in order to obtain good performances. The method remains of complexity $O(N^2)$ because of the fixed size of the dead zones. However, we have been working to overcome to issue and a solution will be soon published.

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