SIZE-BIASED TRANSFORM AND CONDITIONAL MEAN RISK SHARING, WITH APPLICATION TO P2P INSURANCE AND TONTINES

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Abstract

Using risk-reducing properties of conditional expectations with respect to convex order, Denuit and Dhaene (2012) proposed the conditional mean risk sharing rule to allocate the total risk among participants to an insurance pool. This paper relates the conditional mean risk sharing rule to the size-biased transform when pooled risks are independent. A representation formula is first derived for the conditional expectation of an individual risk given the aggregate loss. This formula is then exploited to obtain explicit expressions for the contributions to the pool when losses are modeled by compound Poisson sums, compound Negative Binomial sums, and compound Binomial sums, to which Panjer recursion applies. Simple formulas are obtained when claim severities are homogeneous. A couple of applications are considered. First, to a peer-to-peer (P2P) insurance scheme where participants share the first layer of their respective risks while the higher layer is ceded to a (re)insurer. Second, to survivor credits to be shared among surviving participants in tontine schemes.

Keywords: conditional expectation, risk pooling, risk measures, compound distributions, Panjer family of distributions.

1 Introduction and motivation

Initially developed in order to unify various sampling distributions when the chance of being recorded by an observer varies, weighted distributions are closely related to weighted risk measures and weighted capital allocation rules. Among these weighted distributions, this paper considers the size-biased, or length-biased one corresponding to the identity weight function. It refers to the situation where larger observations are more likely to be recorded. Hence, the available data are of bigger size compared to the actual population values. Translated to an actuarial context, this means that claim amounts are made larger before performing actuarial calculations, which generates a safety loading. The size-biased transform has been comprehensively reviewed by Aaratia et al. (2019). It can be traced back to the late 1960s in the statistical literature and has proven to be useful in the study of risk measures after the pioneering work by Furman and Landsman (2005, 2008) and Furman and Zitikis (2008a,b). The reader is referred to Denuit (2018) for an introduction to the size-biased transform and its properties, with insurance applications.

This paper uses the size-biased transform to establish a useful representation for the conditional mean risk sharing rule proposed by Denuit and Dhaene (2012). This result is then applied to derive explicit expressions for the part of the total risk supported by each participant in a pooling scheme when individual losses are represented by compound Poisson sums, compound Negative Binomial sums, and compound Binomial sums. Panjer recursive formula can be obtained to compute the respective contributions for each participant to the pool, in function of the realized total loss retained by the pool.

As an application, the results are used to share the total loss among participants to a peer-to-peer (P2P) insurance scheme. In such a scheme, participants agree to pool the first layer of the risks they face, whereas the higher losses are still covered by a third party, typically an insurance or reinsurance company. In general, the pool keeps individual claim amounts up to a given threshold acting as a deductible and the part above the threshold is covered by the (re)-insurer with the help of an excess-of-loss treaty. The main advantage for participants to a P2P scheme is that they can access higher amounts of deductibles compared to standard covers, thanks to the risk-reducing effect of pooling.

Additional guarantees can be added to make participation to P2P schemes attractive compared to corresponding regular insurance covers. For instance, an upper bound to the individual contributions to the pool can be specified, which often corresponds to the yearly premium paid for the ordinary insurance policy covering the same peril. In such a way, participants can only win if they adhere to the pool, compared to buying traditional insurance (but of course, there is a price to pay and the amount allocated to cover the pooled losses is reduced accordingly).

P2P insurance schemes generally include cash-back mechanisms when claim amounts do not exceed premiums paid. In such a case, the contract operates as a participating policy: participants pay the regular insurance premium and can only get refund at the end of the period. This raises the following question: how to fairly share, in an understandable and transparent way, that part of the premium that has not been used to cover the claims? As everybody is well aware that insurance risks are heterogeneous, equally sharing the total losses among participants appears to be unfair. To be successful, P2P insurance schemes require an appropriate risk sharing mechanism recognizing the different distributions of the risks brought to the pool. Intuitively speaking, participants pooling smaller risks should contribute less to the realized total loss. The present paper proposes a simple and mathematically correct solution based on the conditional mean risk sharing rule proposed by Denuit and Dhaene (2012). Being based on the concept of mean value, this sharing rule turns out to be quite intuitive as most participants have at least a vague idea about averaging and its risk-reducing effect. Since participants can be informed when they enter the pool about the amount they will have to contribute as a function of the total realized loss, this approach ensures full transparency.

As a second application, we study the way survivor credits are shared under mutual inheritance rules. A tontine, or survival fund refers to a fixed-term fund that boosts the yield of investments thanks to an agreed mutual inheritance rule among participants. According to this rule agreed at inception, the initial contributions are lost in case of death during the reference period. The way survivor credits are distributed must account for the possibly unequal death probabilities and contributed amounts. In that respect, everyone intuitively feels that the participants assuming more risk, because of higher death probability or higher amount of contribution, must receive a higher share of these credits.

Often in the history of tontines, homogeneous groups of participants were created on the basis of gender, age, health status, amount of contribution, etc. This ensures that one can reasonably expect that the same death probability applies to all members inside each group. However, if the tontine mechanism is operated in each group isolatedly, the reduced number of individuals inevitably increases the volatility of the final payout to survivors. Allowing for heterogeneity in death probabilities and contributed amounts avoids this drawback which makes the participation in such investment more attractive (especially at old ages, if such tontines are used as substitutes to more conventional pension insurances), provided the survivor credits are shared in an understandable and transparent way. Again, the conditional mean risk sharing rule appears to provide an effective solution in that respect.

The remainder of this paper is organized as follows. In Section 2, we first recall the definitions of the conditional mean risk sharing rule as well as of the size-biased transform. Then, we establish a useful representation theorem for the individual contributions to the pool under conditional mean risk sharing rule. This result is applied in Sections 3-4-5 when individual contributions to the pool are described by compound sums. Compound Poisson sums are considered in Section 3, compound Negative Binomial ones in Section 4, and compound Binomial ones in Section 5. The results are applied in Section 6, first to a P2P insurance scheme and then to a tontine. The final Section 7 briefly concludes the paper.

2 Representation for conditional mean risk sharing rules

2.1 Conditional mean risk sharing rules

Consider n participants to an insurance pool, numbered $i = 1, 2, \ldots, n$. Each of them faces a risk X_i. By risk, we mean a non-negative random variable representing a monetary loss. In the remainder of this paper, we assume that $X_1, X_2, \ldots, X_n$ are independent. This application is not restrictive for applications to P2P insurance or tontines considered in this paper and an extension to conditionally independent risks can be worked out following the lines of Denuit (2018). Henceforth, we use the notation $S = \sum_{i=1}^n X_i$ for the total risk of the pool. In a risk pooling scheme, each participant contributes an amount $h_i(s)$ where $s = \sum_{i=1}^n x_i$ is the sum of the observed realizations of $X_1, X_2, \ldots, X_n$. In the design of the scheme, it is important that the sharing rule represented by the functions h_i is both intuitively acceptable and transparent.

Denuit and Dhaene (2012) studied the conditional mean risk sharing (or allocation) $h_i^\star$ defined as

$$h_i^\star(S) = \mathrm{E}[X_i|S], \;\; i = 1, 2, \ldots, n. \tag{2.1}$$

Clearly, the conditional mean risk sharing (2.1) allocates the full risk S as we obviously have

$$\sum_{i=1}^n h_i^\star(S) = \sum_{i=1}^n \mathrm{E}[X_i|S] = S.$$

In words, participant i must contribute the expected value of the risk X_i brought to the pool, given the total loss S. This rule appears to be fair as the expected contribution is equal to the expected loss, that is,

$$\mathrm{E}[h_i^\star(S)] = \mathrm{E}\big[\mathrm{E}[X_i|S]\big] = \mathrm{E}[X_i], \;\; i = 1, 2, \ldots, n.$$

Every risk-averse decision-maker prefers $h_i^\star(S)$ over the initial risk X_i so that the conditional mean risk sharing rule appears to be beneficial to all participants. As long as the volatility of the contribution $h_i^\star(S)$ is in line with the risk appetite of participant i, pooling appears to be an attractive alternative to conventional, fixed premium insurance.

Several situations where the functions $s \mapsto h_i^\star(s) = \mathrm{E}[X_i|S = s]$ are non-decreasing for every $i = 1, 2, \ldots, n$, making $\mathrm{E}[X_1|S], \ldots, \mathrm{E}[X_n|S]$ comonotonic, are considered by Denuit and Dhaene (2012). A recent paper by Saumard and Wellner (2018) establishes general conditions ensuring the non-decreasingness of $h_i^\star$. Also, Furman et al. (2018) studied the case where $h_i^\star(s) = \beta_i s$ for some β_i depending on the means of the risks under consideration (see Theorem 3.2 in that paper).

The conditional mean risk sharing rule can be justified by actuarial fairness in the long run. Assume that the same rule is applied repeatedly over time among participants in a stable pool. Formally, denote as $\boldsymbol{X}_t = (X_{1t}, X_{2t}, \ldots, X_{nt})$ the experience for year t and assume that the random vectors $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ are independent and identically distributed. Participant i then pays $\mathrm{E}[X_{it}|S_t]$ in year t and in the long run, the average of $\mathrm{E}[X_{it}|S_t]$ converges to $\mathrm{E}[X_{it}]$, that is, to the pure premium or fair price for the risk X_{it}.

2.2 Size-biased transform

Given a risk X (i.e. a non-negative random variable representing a monetary loss) with distribution function F_X, define the risk $\widetilde{X}$ with distribution function

$$\mathrm{P}[\widetilde{X} \leq t] = \frac{\mathrm{E}\big[X\mathrm{I}[X \leq t]\big]}{\mathrm{E}[X]} = \frac{1}{\mathrm{E}[X]} \int_0^t x \mathrm{d}F_X(x),$$

where $\mathrm{I}[\cdot]$ denotes the indicator function (equal to 1 if the event appearing within the brackets is realized, and to 0 otherwise). Then, $\widetilde{X}$ is said to be a size-biased version of X, and the operator mapping the distribution function F_X of X to the distribution function $F_{\widetilde{X}}$ of $\widetilde{X}$ is called the size-biased transform. Clearly, the support of $\widetilde{X}$ cannot be larger than the support of X. Alternatively, we can also see that F_X and $F_{\widetilde{X}}$ are related through the identity

$$F_{\widetilde{X}}(t) = \frac{\mathrm{E}\big[X \big| X \leq t\big]}{\mathrm{E}[X]} F_X(t).$$

A detailed account of the properties of the size-biased transform can be found in Brown (2006) and Arratia et al. (2019). Most parametric models used in actuarial applications are closed under size-biasing. See e.g. Table 1 in Patil and Rao (1978). Also, we refer the reader to Denuit (2018) for an introduction to the properties of the size-biased transform that appear to be useful to insurance applications (with special emphasis to compound sums).

The size-biased version $\widetilde{X}$ is larger compared to X, so that $\widetilde{X}$ represents a worse loss than X. In order to see why this is true, it suffices to notice that $\widetilde{X}$ is distributed as $\max\{X, Z\}$ where the random variable Z is independent of X and has distribution function

$$\mathrm{P}[Z \leq t] = \frac{\mathrm{P}[\widetilde{X} \leq t]}{\mathrm{P}[X \leq t]} = \frac{\mathrm{E}[X | X \leq t]}{\mathrm{E}[X]}.$$

In fact, the stronger likelihood ratio order holds. We refer the reader to Denuit et al. (2005, Chapter 3) for more details concerning this stochastic order relation expressing the idea of "being larger than" for random variables. This is because the function $p \mapsto F_{\widetilde{X}}\left(F_X^{-1}(p)\right)$ is convex (see Proposition 3.3.50 in Denuit et al., 2005). This can be seen from the identity

$$F_{\widetilde{X}}\left(F_X^{-1}(p)\right) = \frac{\mathrm{E}\left[X \mathrm{I}\big[X \leq F_X^{-1}(p)\big]\right]}{\mathrm{E}[X]}$$

which shows that $p \mapsto F_{\widetilde{X}}\left(F_X^{-1}(p)\right)$ is the Lorenz curve of X, which is known to be increasing and convex.

Let us briefly discuss the zero-augmented case. Insurance risks generally have a large probability of being 0. It is interesting to notice that even if $\mathrm{P}[X = 0] > 0$, we have $\widetilde{X} > 0$ with probability 1. This can easily be deduced from

$$\begin{aligned} \mathrm{P}[\widetilde{X} = 0] &= \mathrm{P}[\widetilde{X} \leq 0] = \frac{\mathrm{E}\big[X \mathrm{I}[X \leq 0]\big]}{\mathrm{E}[X]} \\ &= \frac{\mathrm{E}\big[X \mathrm{I}[X = 0]\big]}{\mathrm{E}[X]} = 0. \end{aligned}$$

The next result uses the familiar representation of individual risk theory to show that the probability mass at zero does not matter for the size-biased transform.

Property 2.1. *Consider X of the form $X = JZ$ where J is Bernoulli distributed with mean q and Z is a positive random variable, J and Z being mutually independent and*

$$P[Z \leq t] = P[X \leq t | X > 0], \;\; t \geq 0.$$

Then, we have $\widetilde{X} =_d \widetilde{Z}$ where $=_d$ means "is distributed as".

Proof. Starting from $\mathrm{E}[X] = \mathrm{E}[J]\mathrm{E}[Z]$ and

$$\mathrm{E}\big[X\mathrm{I}[X \leq t]\big] = \mathrm{E}\big[JZ\mathrm{I}[JZ \leq t]\big] = 0 \times \mathrm{P}[J=0] + \mathrm{E}\big[Z\mathrm{I}[Z \leq t]\big] \times \mathrm{P}[J=1]$$

we can write

$$\mathrm{P}[\widetilde{X} \leq t] = \frac{\mathrm{E}\big[Z\mathrm{I}[Z \leq t]\big]}{\mathrm{E}[Z]} = \mathrm{P}[\widetilde{Z} \leq t],$$

which ends the proof. □

Consider $X = JZ$ and $Y = KZ$ where J, K, and Z are mutually independent, $\mathrm{P}[Z > 0] = 1$, J and K Bernoulli distributed. By Property 2.1, we then have

$$\widetilde{X} =_d \widetilde{Y} =_d \widetilde{Z}$$

whatever the means of J and K. Thus, we see that the probability mass at zero does not matter for the size-biased transform. See also Lemma 2.6 in Arratia et al. (2019).

2.3 Representation in terms of size-biasing

The next result gives a representation formula for $\mathrm{E}[X_i|S]$ in case of independent individual risks X_i. We consider the three situations most commonly encountered in insurance loss modeling: (i) continuous losses, (ii) discrete losses, and (iii) zero-augmented losses.

Proposition 2.2. *Consider independent risks $X_1, \ldots, X_n$ and let $\widetilde{X}_1, \ldots, \widetilde{X}_n$ be their corresponding size-biased versions, assumed to be independent, and independent of $X_1, \ldots, X_n$. The following results hold:*

(i) if $X_1, \ldots, X_n$ are continuous random variables with respective probability density functions $f_{X_1}, \ldots, f_{X_n}$ then for any $s \geq 0$,

$$\begin{aligned} E[X_i|S=s] &= E[X_i]\frac{f_{S-X_i+\widetilde{X}_i}(s)}{f_S(s)} \\ &= \frac{E[X_i]}{E[S]}\frac{f_{S-X_i+\widetilde{X}_i}(s)}{f_{\widetilde{S}}(s)}s \\ &= \frac{E[X_i]f_{S-X_i+\widetilde{X}_i}(s)}{\sum_{j=1}^n E[X_j]f_{S-X_j+\widetilde{X}_j}(s)}s. \end{aligned} \tag{2.2}$$

(ii) if $X_1, \ldots, X_n$ are valued in $\{0, 1, 2, \ldots\}$ with respective probability mass functions $p_{X_1}, \ldots, p_{X_n}$ then for any $s \in \{0, 1, 2, \ldots\}$,

$$\begin{aligned} E[X_i|S=s] &= E[X_i]\frac{p_{S-X_i+\widetilde{X}_i}(s)}{p_S(s)} \\ &= \frac{E[X_i]}{E[S]}\frac{p_{S-X_i+\widetilde{X}_i}(s)}{p_{\widetilde{S}}(s)}s \\ &= \frac{E[X_i]p_{S-X_i+\widetilde{X}_i}(s)}{\sum_{j=1}^n E[X_j]p_{S-X_j+\widetilde{X}_j}(s)}s. \end{aligned}$$

(iii) if $X_1, \ldots, X_n$ are zero-augmented random variables with positive probability masses at the origin and probability density functions over $(0, \infty)$ then

$$E[X_i | S = 0] = 0$$

and for any $s > 0$, (2.2) holds true.

Proof. The random vector $(X_1, \ldots, X_{i-1}, \widetilde{X}_i, X_{i+1}, \ldots, X_n)$ has joint distribution function

$$(x_1, \ldots, x_n) \mapsto \frac{\mathrm{E}\big[X_i \mathrm{I}[X_i \leq x_i]\big]}{\mathrm{E}[X_i]} \prod_{j \neq i} \mathrm{P}[X_j \leq x_j].$$

For any measurable function g, we then have

$$\begin{aligned} \mathrm{E}[g(X_1, \ldots, X_{i-1}, \widetilde{X}_i, X_{i+1}, \ldots, X_n)] &= \int_0^\infty \ldots \int_0^\infty g(x_1, \ldots, x_n) \frac{x_i}{\mathrm{E}[X_i]} \mathrm{d}F_{X_1}(x_1) \ldots \mathrm{d}F_{X_n}(x_n) \\ &= \mathrm{E}\left[\frac{X_i}{\mathrm{E}[X_i]} g(X_1, \ldots, X_n)\right]. \end{aligned} \tag{2.3}$$

Let us apply (2.3) to the function g given by

$$g(x_1, \ldots, x_n) = \mathrm{I}\left[\sum_{j=1}^n x_j \leq s\right].$$

We then obtain the identity

$$\mathrm{P}\left[S - X_i + \widetilde{X}_i \leq s\right] = \frac{\mathrm{E}\big[X_i \mathrm{I}\left[S \leq s\right]\big]}{\mathrm{E}[X_i]}$$

which is generally valid and applies to all cases (i), (ii), and (iii).

To establish the validity of statement (i), consider the identity

$$\mathrm{E}\big[X_i \mathrm{I}[S \leq s]\big] = \int_0^s \mathrm{E}\big[X_i \big| S = t\big] f_S(t) \mathrm{d}t.$$

Taking the derivative of

$$\mathrm{E}\big[X_i \mathrm{I}\left[S \leq s\right]\big] = \mathrm{E}[X_i] \mathrm{P}\left[S - X_i + \widetilde{X}_i \leq s\right]$$

with respect to s gives the first equality in (i). The second statement is easily obtained from the very definition of the size-biased transform of S. Summing the second identity over i gives

$$\sum_{i=1}^n \mathrm{E}[X_i | S = s] = s = \sum_{i=1}^n \frac{\mathrm{E}[X_i]}{\mathrm{E}[S]} \frac{f_{S - X_i + \widetilde{X}_i}(s)}{f_{\widetilde{S}}(s)} s$$

so that we get

$$f_{\widetilde{S}}(s) = \sum_{i=1}^n \frac{\mathrm{E}[X_i]}{\mathrm{E}[S]} f_{S - X_i + \widetilde{X}_i}(s). \tag{2.4}$$

Inserting this expression for $f_{\widetilde{S}}$ in the second equality gives the last one and ends the proof of (i).

Turning to (ii), we can write

$$\mathrm{E}\big[X_i \mathrm{I}[S=s]\big] = \mathrm{E}\big[X_i \mathrm{I}[S \leq s]\big] - \mathrm{E}\big[X_i \mathrm{I}[S \leq s-1]\big] = \mathrm{E}[X_i]\mathrm{P}\left[S - X_i + \widetilde{X}_i = s\right],$$

as announced. The other parts of the statement follow as for (i).

Finally, considering (iii), it is clear that $S = 0 \Rightarrow X_j = 0$ for all j so that $\mathrm{E}[X_i|S=0] = 0$. Both X_i and S have a probability density function over $(0, \infty)$ so that we have for $s > 0$

$$\begin{aligned}
\mathrm{E}\left[X_i|S=s\right] &= \int_0^s \mathrm{P}[X_i > x|S=s]\mathrm{d}x \\
&= \int_0^s \int_x^s \frac{f_{(X_i,S)}(t,s)}{f_S(s)}\mathrm{d}t\mathrm{d}x \\
&= \frac{1}{f_S(s)} \int_0^s \int_x^s f_{X_i}(t) f_{S-X_i}(s-t)\mathrm{d}t\mathrm{d}x \\
&= \frac{1}{f_S(s)} \int_0^s t f_{X_i}(t) f_{S-X_i}(s-t)\mathrm{d}t \\
&= \frac{\mathrm{E}[X_i]}{f_S(s)} \int_0^s f_{\widetilde{X}_i}(t) f_{S-X_i}(s-t)\mathrm{d}t \\
&= \mathrm{E}\left[X_i\right] \frac{f_{S-X_i+\widetilde{X}_i}(s)}{f_S(s)}.
\end{aligned}$$

This ends the proof. $\square$

Proposition 2.2(i) shows that

$$\mathrm{E}[X_i|S=s] \text{ increases in } s \Leftrightarrow \frac{f_{S-X_i+\widetilde{X}_i}(s)}{f_S(s)} \text{ increases in } s$$

which means that S is smaller than $S - X_i + \widetilde{X}_i$ in the likelihood ratio order. We know that X_i is smaller than $\widetilde{X}_i$ in the likelihood ratio order so that this requirement seems to be reasonable. However, it is not always valid because the likelihood ratio order is not closed under convolution, in general. We refer the reader to Section 1.C in Shaked and Shanthikumar (2007) for more details. It turns out that the sum S is smaller than $S - X_i + \widetilde{X}_i$ in the likelihood ratio order when $S - X_i$ has a log-concave probability density function (Shaked and Shanthikumar, 2007, proof of Theorem 1.C.9 page 46).

In some of the applications considered in this paper, individual risks X_i are bounded. If each X_i is valued in the interval $[0, b_i]$, so that their sum S is valued in the interval $[0, b_\bullet]$ with $b_\bullet = \sum_{i=1}^n b_i$, then the function $\mathrm{E}[X_i|S=s]$ starts from 0 (for $s = 0$) and tends to b_i as s approaches $b_\bullet$.

2.4 Application to risk sharing

Assume that the conditions of Proposition 2.2(i) hold true (analogous expressions can be derived in the discrete and zero-augmented cases). In relative terms, the respective contributions for individuals i and j satisfy

$$\frac{\mathrm{E}\left[X_i \middle| S=s\right]}{\mathrm{E}\left[X_j \middle| S=s\right]} = \frac{\mathrm{E}\left[X_i\right] f_{S-X_i+\widetilde{X}_i}(s)}{\mathrm{E}\left[X_j\right] f_{S-X_j+\widetilde{X}_j}(s)}.$$

The ratio of the respective contributions for agents i and j to the pool once it is known that $S = s$ is equal to the ratio of their a priori expectations $\mathrm{E}\left[X_i\right]/\mathrm{E}\left[X_j\right]$ corrected by the ratio of the densities of the sums $S - X_i + \widetilde{X}_i$ and $S - X_j + \widetilde{X}_j$ where their respective risks X_i and X_j have been replaced with their size-biased versions.

Example 2.3. *If X_i is infinitely divisible then we know from Pakes et al. (1996) that the distributional equality $\widetilde{X}_i =_d X_i + \Delta_i$ holds for some positive random variable Δ_i, independent of X_i. Hence, $S - X_i + \widetilde{X}_i =_d S + \Delta_i$ and we have*

$$E\left[X_i \middle| S=s\right] = E\left[X_i\right] \frac{f_{S+\Delta_i}(s)}{f_S(s)}.$$

Consider the particular case where independent Gamma losses are shared among participants. Precisely, the probability density function of X_i is given by

$$f_{X_i}(x) = \begin{cases} \frac{x^{\alpha_i-1}\tau^{\alpha_i}\exp(-x\tau)}{\Gamma(\alpha_i)}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $E[X_i] = \alpha_i/\tau$ and $V[X_i] = \alpha_i/\tau^2$. This is henceforth denoted as $X_i \sim \mathcal{G}am(\alpha_i, \tau)$. Then, $S \sim \mathcal{G}am(\alpha_\bullet, \tau)$ where $\alpha_\bullet = \sum_{i=1}^n \alpha_i$. It is easy to see that

$$f_{\widetilde{X}_i}(x) = \frac{x^{\alpha_i}\tau^{\alpha_i}\exp(-x\tau)}{\Gamma(\alpha_i)\frac{\alpha_i}{\tau}} = \frac{x^{\alpha_i}\tau^{\alpha_i+1}\exp(-x\tau)}{\Gamma(\alpha_i+1)} \Leftrightarrow \widetilde{X}_i \sim \mathcal{G}am(\alpha_i+1, \tau)$$

so that $\Delta_i \sim \mathcal{G}am(1, \tau)$. Hence,

$$S - X_i + \widetilde{X}_i =_d S + \Delta_i \sim \mathcal{G}am(\alpha_\bullet + 1, \tau).$$

Also,

$$\widetilde{S} \sim \mathcal{G}am(\alpha_\bullet + 1, \tau).$$

Thus, we finally obtain the following identity for independent Gamma losses from Proposition 2.2(i):

$$E[X_i|S=s] = \frac{E[X_i]}{E[S]}s = \frac{\alpha_i}{\alpha_\bullet}s.$$

Participants pooling such independent Gamma losses and adopting the conditional mean risk sharing rule then each pay a share $\alpha_i/\alpha_\bullet$ of the total realized loss s.

3 Compound Poisson sums

3.1 Size-biased version of compound Poisson sums

Compound sums are often used to model individual insurance losses, making explicit the frequency and severity components and accounting for the large probability mass at zero. Henceforth, we consider compound sums of the form $X = \sum_{k=1}^{N} C_k$ where N is valued in $\{0, 1, 2, \ldots\}$ and the claim severities $C_1, C_2, \ldots$ are identically distributed, distributed as C, all these random variables being independent. By convention, X is set to zero when $N = 0$ so that $\mathrm{P}[X = 0] = \mathrm{P}[N = 0]$. Compound Poisson sums are particular cases where N is Poisson distributed. The next result that can be found e.g. in Denuit (2018) shows that the size-biased version of a compound Poisson sum possesses a particularly simple structure.

Property 3.1. *Let N be Poisson distributed with mean λ, which is henceforth denoted as $N \sim \mathcal{P}oi(\lambda)$. Considering the compound sum $X = \sum_{k=1}^{N} C_k$ described above,*

$$N \sim \mathcal{P}oi(\lambda) \Rightarrow \widetilde{X} =_d X + \widetilde{C}.$$

Property 3.1 shows that size-biasing a compound Poisson sum consists in supplementing it with the size-biased version of its generic term.

3.2 Conditional mean risk sharing rule for compound Poisson sums

Assume that the loss X_i for participant i can be represented as

$$X_i = \sum_{k=1}^{N_i} C_{ik} \text{ with } N_i \sim \mathcal{P}oi(\lambda_i), \tag{3.1}$$

where the claim severities C_{ik} are positive, continuous or integer-valued, distributed as C_i, all these random variables being independent. Notice that C_{ik} is not necessarily the entire cost of the kth claim, but only the amount shared among participants. Only minor losses remain within the pool whereas larger ones are generally reinsured. This can be implemented with the help of a straight deductible or of a disappearing, or franchise deductible applied to the incurred loss, for instance.

When the shared losses X_i are of the form (3.1), Property 3.1 shows that

$$S - X_i + \widetilde{X}_i =_d S + \widetilde{C}_i.$$

Considering C_i valued in $\{1, 2, 3, \ldots\}$, we have in the compound Poisson case that

$$\mathrm{E}[X_i | S = s] = \mathrm{E}[X_i] \frac{\mathrm{P}[S + \widetilde{C}_i = s]}{\mathrm{P}[S = s]},$$

with corresponding expressions for continuously distributed summands. The relative amounts paid by two participants satisfy

$$\frac{\mathrm{E}[X_i | S = s]}{\mathrm{E}[X_j | S = s]} = \frac{\mathrm{E}[X_i]}{\mathrm{E}[X_j]} \frac{\mathrm{P}[S + \widetilde{C}_i = s]}{\mathrm{P}[S + \widetilde{C}_j = s]}.$$

The ratio $\mathrm{P}[S+\widetilde{C}_i=s]/\mathrm{P}[S+\widetilde{C}_j=s]$ can be effectively computed as follows:

- first, the distribution of S is obtained with the help of the Panjer algorithm giving the values for $\mathrm{P}[S=s]$.
- then, convolution formulas can be used to get $\mathrm{P}[S+\widetilde{C}_i=s]$ and $\mathrm{P}[S+\widetilde{C}_j=s]$.

3.3 Alternative representation

Notice that S is also compound Poisson distributed. Precisely, $S =_d \sum_{k=1}^{M} Y_k$ with M Poisson distributed with mean $\lambda_\bullet = \sum_{i=1}^{n} \lambda_i$ and independent Y_k with common distribution function

$$F_Y(y) = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_\bullet} F_{C_i}(y) = \mathrm{P}[C_J \leq y]$$

where the random variable J is valued in $\{1,2,\ldots,n\}$ with probability mass function $\mathrm{P}[J=i]=\frac{\lambda_i}{\lambda_\bullet}$. Invoking Property 3.1 together with the second identity in Proposition 2.2(i), we then have

$$\mathrm{E}[X_i|S=s] = \frac{\mathrm{E}\left[X_i\right]}{\mathrm{E}[S]} \frac{\mathrm{P}[S+\widetilde{C}_i=s]}{\mathrm{P}[S+\widetilde{Y}=s]} s.$$

The random variable Y appears to obey a mixture of the claim severity distributions. This is why we discuss the size-biased transform of mixture distributions. Consider a family of non-negative random variables $\{X_\theta,\ \theta \geq 0\}$ indexed by a single, non-negative parameter θ. Let Θ be a mixing parameter with distribution function F_Θ. The corresponding mixture X_Θ has distribution function

$$\mathrm{P}[X_\Theta \leq x] = \int_0^\infty \mathrm{P}[X_\theta \leq x] \mathrm{d}F_\Theta(\theta).$$

It can be shown (see e.g. Denuit, 2108) that the size-biased transform of the mixture X_Θ corresponds to the mixture of the random variables $\{\widetilde{X}_\theta,\ \theta \geq 0\}$ with mixing parameter $\Theta^\star$ distributed according to

$$\mathrm{d}F_{\Theta^\star}(\theta) = \frac{\mathrm{E}[X_\theta]}{\mathrm{E}[X_\Theta]} \mathrm{d}F_\Theta(\theta).$$

This general result allows us to derive the size-biased version of Y. Specifically, the distribution function of $\widetilde{Y}$ is given by

$$F_{\widetilde{Y}}(y) = \mathrm{P}[\widetilde{C}_{I^\star} \leq y] \text{ where } \mathrm{P}[I^\star = i] = \frac{\mathrm{E}[C_i]}{\sum_{j=1}^{n} \frac{\lambda_j}{\lambda_\bullet} \mathrm{E}[C_j]} \frac{\lambda_i}{\lambda_\bullet} = \frac{\lambda_i \mathrm{E}[C_i]}{\sum_{j=1}^{n} \lambda_j \mathrm{E}[C_j]}.$$

Hence,

$$F_{\widetilde{Y}}(y) = \sum_{i=1}^{n} \frac{\lambda_i \mathrm{E}[C_i]}{\sum_{j=1}^{n} \lambda_j \mathrm{E}[C_j]} \mathrm{P}[\widetilde{C}_i \leq y].$$

We then have

$$\begin{aligned}
\mathrm{P}[\widetilde{S}=s] &= \mathrm{P}[S+\widetilde{Y}=s] \\
&= \sum_{i=1}^{n} \frac{\lambda_i \mathrm{E}[C_i]}{\sum_{j=1}^{n}\lambda_j \mathrm{E}[C_j]} \mathrm{P}[S+\widetilde{C}_i=s] \\
&= \sum_{i=1}^{n} \frac{\mathrm{E}[X_i]}{\mathrm{E}[S]} \mathrm{P}[S+\widetilde{C}_i=s]
\end{aligned}$$

and

$$\mathrm{E}[X_i|S=s] = \frac{\mathrm{E}[X_i]\mathrm{P}[S+\widetilde{C}_i=s]}{\sum_{j=1}^{n}\mathrm{E}[X_j]\mathrm{P}[S+\widetilde{C}_j=s]}s.$$

The calculation of $\mathrm{E}[X_i|S=s]$ then proceeds along the same lines as before. Panjer algorithm is first used to obtain the probability mass function of S which is then convoluted with $\widetilde{C}_i$ and with $\widetilde{Y}$.

3.4 Homogeneous claim sizes

Assume that the heterogeneity is confined to claim frequencies, in the sense that $C_1, C_2, \ldots, C_n$ are identically distributed, and distributed as C, say, where C is either integer-valued or continuously distributed. Then, $\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_n$ are also identically distributed and we have

$$\frac{\mathrm{E}[X_i|S=s]}{\mathrm{E}[X_j|S=s]} = \frac{\mathrm{E}\left[X_i\right]}{\mathrm{E}\left[X_j\right]} = \frac{\lambda_i}{\lambda_j} \Rightarrow \mathrm{E}[X_i|S=s] = \frac{\lambda_i}{\lambda_j}\mathrm{E}[X_j|S=s]$$

Summing this identity over i yields

$$s = \frac{\lambda_\bullet}{\lambda_j}\mathrm{E}[X_j|S=s]$$

so that we finally obtain

$$\mathrm{E}[X_i|S=s] = \frac{\lambda_i}{\lambda_\bullet}s. \tag{3.2}$$

Remark 3.2. The representation (3.2) can also be obtained from Theorem 4.1(1) in Furman et al. (2018) since

$$\frac{\ln L_{X_i}(t)}{\ln L_{\sum_{j\neq i} X_j}(t)} = \frac{\lambda_i}{\sum_{j\neq i}\lambda_j}$$

is constant, where $L_Z(\cdot)$ denotes the Laplace transform of the random variable Z.

4 Compound Negative Binomial sums

4.1 Size-biased version of compound Negative Binomial sums

Assume that the Poisson parameter λ is replaced with a random variable Λ obeying the $\mathcal{G}am(\alpha,\beta)$ distribution. Then, the counting distribution is the Negative Binomial $\mathcal{NB}in(\alpha,\beta)$

with probability mass function

$$\mathrm{P}[N=k]=\frac{\beta^{\alpha}}{(1+\beta)^{\alpha+k}}\frac{\Gamma(\alpha+k)}{k!\Gamma(\alpha)},\quad k=0,1,2,\ldots$$

The next result that can be found e.g. in Denuit (2018) shows that the size-biased version of a compound Negative Binomial sum possesses a particularly simple structure.

Property 4.1. *Consider the compound sum* $X=\sum_{k=1}^{N}C_k$. *Then,*

$$N\sim\mathcal{NB}in(\alpha,\beta)\Rightarrow\widetilde{X}=_d X+\widetilde{C}+Z$$

where $Z=\sum_{k=1}^{M}D_k$, $M\sim\mathcal{NB}in(1,\beta)$ *and* $D_1,D_2,\ldots$ *are distributed as* C, *all the random variables being independent.*

Compared to the homogeneous Poisson case where size-biasing consists in adding $\widetilde{C}$ to the compound Poisson sum, we see that in the Negative Binomial case an extra term Z enters the size-biased version. The random variable Z accounts for the heterogeneity in the Poisson mean which increases the level of riskiness compared to the homogeneous Poisson case.

4.2 Conditional mean risk sharing rule for compound Negative Binomial sums

Assume that the loss X_i for participant i can be represented as

$$X_i=\sum_{k=1}^{N_i}C_{ik}\text{ with }N_i\sim\mathcal{NB}in(\alpha_i,\beta),\tag{4.1}$$

where the claim severities C_{ik} are positive, continuous or integer-valued, distributed as C_i, all these random variables being independent. According to Property 4.1, we know that $\widetilde{X}_i=_d X_i+\widetilde{C}_i+Z_i$, all the random variables entering the decomposition being independent.

When the shared losses X_i are of the form (4.1), Property 4.1 shows that

$$S-X_i+\widetilde{X}_i=_d S+Z_i+\widetilde{C}_i.$$

Considering C_i valued in $\{1,2,3,\ldots\}$, we have

$$\mathrm{E}[X_i|S=s]=\mathrm{E}\left[X_i\right]\frac{\mathrm{P}[S+Z_i+\widetilde{C}_i=s]}{\mathrm{P}[S=s]},$$

with corresponding expressions for continuously distributed summands. The relative amounts paid by two participants satisfy

$$\frac{\mathrm{E}[X_i|S=s]}{\mathrm{E}[X_j|S=s]}=\frac{\mathrm{E}\left[X_i\right]}{\mathrm{E}\left[X_j\right]}\frac{\mathrm{P}[S+Z_i+\widetilde{C}_i=s]}{\mathrm{P}[S+Z_j+\widetilde{C}_j=s]}.$$

The probability mass function of $S+Z_i$ can be effectively computed with the help of the Panjer algorithm giving the values for $\mathrm{P}[S+Z_i=s]$. This is because $S+Z_i$ is a compound Negative Binomial sum with a number of terms obeying the $\mathcal{NB}in(\alpha_i+1,\beta)$ distribution. Then, convolution formulas can be used to get the probability mass function of the sum of $S+Z_i$ and $\widetilde{C}_i$.

4.3 Homogeneous claim sizes

Let us now consider the conditional mean risk sharing of independent compound Negative Binomial losses $X_i = \sum_{k=1}^{N_i} C_{ik}$ where $N_i \sim \mathcal{NB}in(\alpha_i, \beta)$ and $C_1, \ldots, C_n$ are identically distributed. We then have

$$S + Z_i + \widetilde{C}_i =_d S + Z_j + \widetilde{C}_j \text{ for all } i \text{ and } j$$

so that

$$\frac{\mathrm{E}[X_i|S=s]}{\mathrm{E}[X_j|S=s]} = \frac{\mathrm{E}[X_i]}{\mathrm{E}[X_j]} = \frac{\alpha_i}{\alpha_j}.$$

Similarly to the Poisson case, we get

$$\mathrm{E}[X_i|S=s] = \frac{\alpha_i}{\alpha_\bullet} s.$$

Again, the homogeneity of claim severities allows us to obtain a very simple expression of the conditional mean risk sharing.

5 Compound Binomial case

Let $\mathcal{B}in(m, q)$ denote the Binomial distribution with size m and success probability q. In order to solve the compound Binomial case, recall that any compound sum $X = \sum_{k=1}^{N} C_k$ where N obeys the $\mathcal{B}in(m, q)$ distribution can be equivalently represented as a sum of m independent and identically distributed random variables with a probability mass at zero. Precisely, defining $Y_k = J_k C_k$ where $J_1, J_2, \ldots, J_m$ are independent Bernoulli distributed random variables with common mean q, we have

$$X =_d \sum_{k=1}^{m} Y_k \tag{5.1}$$

in such a case. This equivalent representation is the key to the next result.

Property 5.1. *Consider $X = \sum_{k=1}^{N} C_k$ where $N \sim \mathcal{B}in(m, q)$ and $C_1, C_2, \ldots, C_m$ are positive random variables distributed as C, all these random variables being independent. Then,*

$$\widetilde{X} =_d \sum_{k=1}^{\widetilde{N}-1} C_k + \widetilde{C} \text{ where } \widetilde{N} - 1 \sim \mathcal{B}in(m-1, q).$$

Proof. As (5.1) involves a sum of independent random variables, the size-biased transform of such a sum is of interest. Following the lines leading to (2.4), we deduce that the size-biased version of X in (5.1) is

$$\widetilde{X} =_d \sum_{k=1}^{m-1} Y_k + \widetilde{Y}_m$$

because the random variables Y_k are identically distributed. The announced result then follows from the distributional equality (5.1), by noting that $\widetilde{JC} =_d \widetilde{C}$ from Property 2.1.

Now, the size-biased version of $N \sim \mathcal{B}in(m, q)$ has probability mass function $p_{\widetilde{N}}$ given by $p_{\widetilde{N}}(0) = 0$ and

$$p_{\widetilde{N}}(k) = \frac{kp_N(k)}{\mathrm{E}[N]} = \frac{(m-1)!}{(k-1)!(m-k)!} q^{k-1}(1-q)^{m-1-(k-1)}$$

so that $\widetilde{N} - 1 \sim \mathcal{B}in(m-1, q)$, as announced. This ends the proof. □

6 Applications

6.1 P2P insurance scheme

Consider a P2P insurance scheme involving 4 participants ($n = 4$) bringing to the pool compound Poisson losses described in the next table:

Participant i	1	2	3	4
Case 1				
λ_i	0.08	0.08	0.1	0.1
Case 2				
λ_i	0.04	0.08	0.1	0.14
$\mathrm{P}[C_i = 1]$	0.1	0.15	0.1	0.15
$\mathrm{P}[C_i = 2]$	0.2	0.25	0.2	0.25
$\mathrm{P}[C_i = 3]$	0.4	0.3	0.3	0.3
$\mathrm{P}[C_i = 4]$	0.3	0.3	0.4	0.3

Claim severities C_i expressed in some monetary unit are valued in $\{1, 2, 3, 4\}$. The claim severities are larger for X_2 and X_4, compared to X_1 and X_3. We distinguish two cases according to expected claim frequencies. In the first case, expected claim frequencies are identical for participants 1 and 2 (and equal to 8%) whereas participants 3 and 4 have higher expected claim frequencies (equal to 10% for both of them). In the second case, expected claim frequencies increase for participants 1 to 4, from 4% to 14%.

The `R` package `actur` has been used to perform the calculations; see Dutang et al. (2008). Panjer recursion is performed using the function `aggregateDist`, to obtain the distribution of $S = X_1 + \ldots + X_4$. Then, a direct convolution is achieved to obtain the distribution of $S + \widetilde{C}_i$ for $i = 1, \ldots, 4$.

Figure 1 displays the share of the total loss s to be contributed by each participant i. We display the results for s up to 15 (the 99.9% quantile of S is 11 in case 1). An horizontal line is clearly visible on Figure 1, separating the shares of participants 1-2 from the share of participants 3-4. In other words, participants 1-2 always contribute together the same percentage of the total realized loss s. The respective shares of participants 1 and 2 differ according to s, with a relatively lower share for participant 1 whose claim severity appears to be somewhat smaller compared to participant 2. Let us explain the presence of this horizontal separation in our example. Participants 1 and 3 have identical severity distributions, as well as participants 2 and 4. Thus, $X_1 + X_2$ and $X_3 + X_4$ are compound Poisson sums with identical claim severities so that we know that participants 1 and 2 together contribute to the pool a share $(\lambda_1+\lambda_2)/\lambda_\bullet$ of s whereas participants 3 and 4 together contribute to the pool the remaining $(\lambda_3+\lambda_4)/\lambda_\bullet$ of s by virtue of (3.2).

The discreteness of claim severities produces some irregularities for the ratios $\mathrm{E}[X_i|S=s]/s$ displayed in Figure 1. For $s=1$, participant 1 contributes 0.178, participant 2 contributes 0.267, participant 3 contributes 0.222, and participant 4 contributes 0.333. These values are easily recovered as follows. If $S=1$ then only one loss occurred. The probability that it comes from participant 1 is obtained by dividing

$$\mathrm{P}[N_1=1]\mathrm{P}[C_1=1]\prod_{i=2}^{4}\mathrm{P}[N_i=0]=\lambda_1\exp(-\lambda_1)\mathrm{P}[C_1=1]\exp\left(-\sum_{i=2}^{3}\lambda_i\right)$$

by

$$\mathrm{P}[S=1]=\sum_{i=1}^{4}\mathrm{P}[N_i=1]\mathrm{P}[C_i=1]\prod_{j\neq i}\mathrm{P}[N_j=0]$$

which gives 0.178. When the total realized loss s gets large, we see that the share of each participant stabilizes, around 23% for participant 1, 21.5% for participant 2, 28.5% for participant 3 and 27% for participant 4. This can be seen from

$$\frac{\mathrm{E}[X_i|S=s]}{s}=\frac{\mathrm{E}[X_i]}{\mathrm{E}[S]}\frac{\mathrm{P}[S-X_i+\widetilde{X}_i=s]}{\mathrm{P}[\widetilde{S}=s]}=\frac{\mathrm{E}[X_i]}{\mathrm{E}[S]}\frac{\mathrm{P}[S+\widetilde{C}_i=s]}{\mathrm{P}[S+\widetilde{C}_I=s]}.$$

Now,

$$\lim_{s\to\infty}\mathrm{P}[S+\widetilde{C}_i=s]=\lim_{s\to\infty}\sum_{k=1}^{4}\mathrm{P}[\widetilde{C}_i=k]\mathrm{P}[S=s-k]=\lim_{s\to\infty}\mathrm{P}[S=s]$$

so that $\frac{\mathrm{P}[S+\widetilde{C}_i=s]}{\mathrm{P}[S+\widetilde{C}_I=s]}$ tends to 1 when $s\to\infty$. We then finally get that the respective shares $\mathrm{E}[X_i|S=s]/s$ converge to $\mathrm{E}[X_i]/\mathrm{E}[S]$, which are respectively equal to $\mathrm{E}[X_1]/\mathrm{E}[S]=22.81\%$, $\mathrm{E}[X_2]/\mathrm{E}[S]=21.63\%$, $\mathrm{E}[X_3]/\mathrm{E}[S]=28.52\%$, and $\mathrm{E}[X_4]/\mathrm{E}[S]=27.04\%$ for participants 1 to 4.

Let us now turn to Case 2. Since participants 1 and 3 have identical severity distributions, as well as participants 2 and 4, we can see there that the dominant effect comes from the increasing expected claim frequency: participant 4 must contribute the most because $\lambda_4=0.14$ is larger compared to $\lambda_1=0.04$, $\lambda_2=0.08$, and $\lambda_3=0.1$. As for case 1, the shares all stabilize when the realization s of S becomes large enough.

6.2 Survivor funds

Assume that individual i contributes to a fund an amount c_i at time 0, $i=1,2,\ldots,n$. This amount is invested until maturity m. Denoting as u the accumulation factor (i.e. the value at time m of one monetary unit invested in the fund at time 0), the contributions accumulate to $u\sum_{i=1}^n c_i$. The accumulated factor u corresponds to a yield curve, to some reference interest rate, like the 10-year treasury bond, or even to a given stock index (moving to a unit-linked mechanism). Contrarily to regular investments, the terminal amount $u\sum_{i=1}^n c_i$

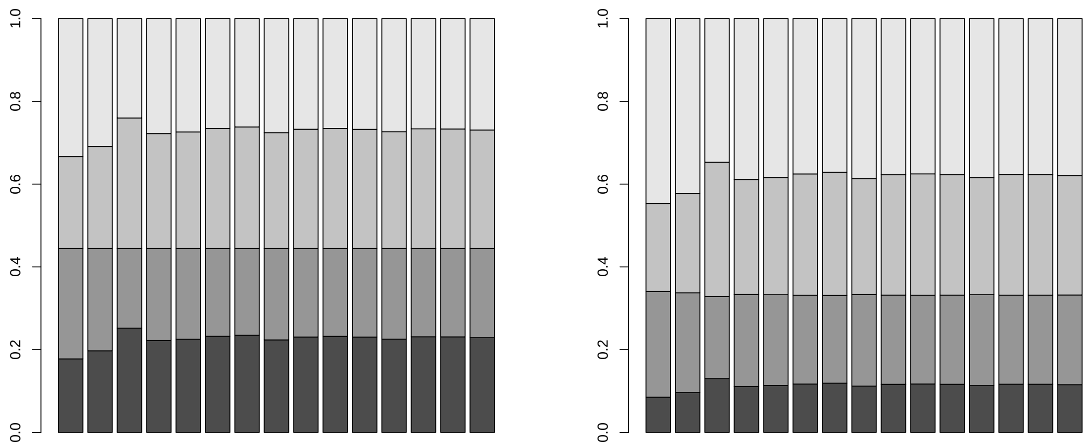

Figure 1: Respective shares $\mathrm{E}[X_i|S=s]/s$ of the total realized loss s contributed by each participant i in Case 1 (left panel) and Case 2 (right panel). From bottom to top, shares of participants 1 to 4.

is divided among participants or beneficiaries at maturity, according to some agreed mutual inheritance rule.

The respective death probabilities before time m are denoted as $q_1, q_2, \ldots, q_n$. These values are assumed to account for the effects of age, gender, socio-economic profile, etc. and all participants agree about them. Let I_i denote the survival indicator for individual i, i.e.

$$I_i = \begin{cases} 1 \text{ if individual } i \text{ survives up to time } m, \\ \\ 0 \text{ otherwise.} \end{cases}$$

Thus,

$$\mathrm{P}[I_i = 0] = q_i = 1 - \mathrm{P}[I_i = 1].$$

Participants agree about a mutual inheritance scheme defining the final payout to each of them or to their designated beneficiaries in case of death (beneficiaries receive an agreed amount in case participants die). These beneficiaries can be members of participant's family or a charity. Survivors get back their initial contribution c_i plus the interest it produced until maturity, that is, $s_i = c_i \times u$. This is similar to a classical investment, for instance a deposit on a bank account. The mutual inheritance agreement embedded in the survivor fund provides participants with an extra return above the purely financial one, i.e. a share of the accumulated amount $\sum_{j=1}^{n}(1-I_j)s_j$ left by participants who died before time m.

Several proposals for mutual inheritance schemes have be made in the literature, including Sabin (2010), Donnelly et al. (2014), Donnelly and Young (2017), and Chen et al. (2019). As shown by Donnelly (2015), the notion of fairness is very relevant for this kind of mortality pooling scheme. Fairness would here mean that the expected gain is zero for all participants.

Donnelly and Young (2017) demonstrated that, to ensure fairness $\sum_{j=1}^{n}(1-I_j)s_j$ must be shared among all participants, not only among the survivors.

Donnelly and Young (2017) proposed to distribute

$$S=\sum_{j=1}^{n}(1-I_j)s_j$$

among all individuals according to their respective levels of risk taking, as reflected by the products s_iq_i. Precisely, the final payout for participant i is then equal to

$$V_i=\begin{cases} s_i+\frac{s_iq_i}{\sum_{j=1}^{n}s_jq_j}S \text{ if participant } i \text{ survives} \\ \\ \frac{s_iq_i}{\sum_{j=1}^{n}s_jq_j}S \text{ if participant } i \text{ dies,} \end{cases}$$

with the understanding that payments for those participants who died during the reference period are made to their beneficiaries. This can be equivalently rewritten as

$$V_i=s_iI_i+\frac{s_iq_i}{\sum_{j=1}^{n}s_jq_j}S.$$

This procedure can be regarded as fair because

$$\mathrm{E}[V_i]=s_i(1-q_i)+\frac{s_iq_i}{\sum_{j=1}^{n}s_jq_j}\sum_{j=1}^{n}q_js_j=s_i.$$

This means that, on average, every participant recovers the accumulated value of his or her initial contribution. There is no risk transfer, only risk sharing as

$$\sum_{i=1}^{n}V_i=S+\sum_{i=1}^{n}s_iI_i=\sum_{i=1}^{n}s_i.$$

This means that the entire accumulated value is distributed among participants.

Notice that no guarantee is offered with respect to the life table. Probabilities q_i are only used to distribute the amount S among all participants. It is worth to mention that this mechanism allows for heterogeneity among participants as different probabilities q_i and contributions c_i are permitted. Of course, the cash flows depend on the actual mortality experience among the participants.

Even if the approach proposed by Donnelly and Young (2017) is effective, ensures fairness and conforms with intuition, it lacks of a formal justification. Here, we propose to allocate S according to the conditional mean risk sharing rule (2.1). Formally, denote as

$$X_i=(1-I_i)s_i$$

the accumulated contribution s_i lost in case of death. We then have to distribute

$$S=\sum_{i=1}^{n}X_i$$

among the n participants according to some pre-defined rule. If we adopt the conditional mean risk sharing rule then participant i receives an amount $\mathrm{E}[X_i|S]$ corresponding to his or her expected share in mortality credits. Precisely, the terminal cash-flow W_i for participant i is now equal to

$$W_i = \begin{cases} s_i + \mathrm{E}[X_i|S] \text{ if participant } i \text{ survives} \\ \\ \mathrm{E}[X_i|S] \text{ if participant } i \text{ dies.} \end{cases}$$

This can be rewritten as

$$W_i = s_i I_i + \mathrm{E}[X_i|S].$$

Compared to the system proposed by Donnelly and Young (2017), the share $\frac{s_i q_i}{\sum_{j=1}^n s_j q_j} S$ is now replaced with the conditional mean $\mathrm{E}[X_i|S]$. Notice that

$$\frac{s_i q_i}{\sum_{j=1}^n s_j q_j} S = \frac{\mathrm{E}[X_i]}{\mathrm{E}[S]} S.$$

Conditions under which $\mathrm{E}[X_i|S]$ admits this form have been studied in Furman et al. (2018). As these conditions are not fulfilled in the present case, the conditional mean risk sharing rule differs from the distribution mechanism proposed by Donnelly and Young (2017).

Clearly,

$$\begin{aligned} \sum_{i=1}^n W_i &= \sum_{i=1}^n \mathrm{E}[X_i|S] + \sum_{i=1}^n s_i I_i \\ &= \sum_{i=1}^n (1 - I_i) s_i + \sum_{i=1}^n s_i I_i \\ &= \sum_{i=1}^n s_i \end{aligned}$$

so that the entire risk is pooled within the group. Moreover,

$$\mathrm{E}[W_i] = (1 - q_i) s_i + \mathrm{E}[X_i] = s_i$$

so that the gain is zero, on average, for each participant. The game is thus fair and does not transfer money from some participants to other ones, on average.

Now, the representation derived in Proposition 2.2(ii) is still valid for general discrete random variables valued in $\{a_0, a_1, a_2, \ldots\}$ such that $a_k < a_{k+1}$. This is easy to see by substituting a_k for k in the proof. We then have

$$
\begin{aligned}
\mathrm{E}[X_i|S=s] &= \mathrm{E}[X_i]\frac{\mathrm{P}[S-X_i+\widetilde{X}_i=s]}{\mathrm{P}[S=s]} \\
&= s_i q_i \frac{\mathrm{P}\left[\sum_{j\neq i}X_j+s_i\widetilde{(1-I_i)}=s\right]}{\mathrm{P}[S=s]} \\
&= s_i q_i \frac{\mathrm{P}\left[\sum_{j\neq i}X_j=s-s_i\right]}{\mathrm{P}[S=s]} \text{ since } \widetilde{(1-I_i)}=1 \text{ by Property 2.1} \\
&= s_i \frac{q_i\mathrm{P}\left[\sum_{j\neq i}X_j=s-s_i\right]}{q_i\mathrm{P}\left[\sum_{j\neq i}X_j=s-s_i\right]+(1-q_i)\mathrm{P}\left[\sum_{j\neq i}X_j=s\right]}. \qquad (6.1)
\end{aligned}
$$

Initial anf terminal values for the conditional expectation $\mathrm{E}[X_i|S=s]$ can easily be calculated. Indeed,

$$\mathrm{E}[X_i|S=s]=0 \text{ for } s<s_i.$$

For $s=s_i$, we get

$$\mathrm{E}[X_i|S=s_i]=s_i\frac{q_i\prod_{j\neq i}(1-q_j)}{\mathrm{P}[S=s_i]}.$$

If all participants dies, so that $S=\sum_{i=1}^n s_i$, we have

$$\mathrm{E}[X_i|S=s_1+\ldots+s_n]=s_i.$$

This means that every beneficiary receives the accumulated amount s_i. Formula (6.1) can be used to calculate the conditional expectations for all $s\in\{s_i+1,\ldots,s_1+\ldots+s_n\}$.

Several recursive methods have been proposed in the actuarial literature to obtain the distribution of the yearly claim amount for life insurance portfolios. We refer the reader to the comprehensive book by Sundt and Vernic (2009) for extensive details about these recursive calculations of convolutions initiated by De Pril (1989) and Dhaene and Vandebroek (1995). The distribution of $\sum_{j\neq i}X_j$ can be obtained in this way so that the final payout to participants can be computed in function of the realized value of S, making the mechanism transparent.

As an illustration, let us consider a group of 100 participants who agree about the mutual inheritance rule described above, so that they receive a final payout W_i in exchange of an initial contribution c_i to the survivor fund. Participants are partitioned into 4 groups with death probabilities $q_1=0.05$ and $q_2=0.1$ and accumulated contributions $s_1=1$ and $s_2=2$. Precisely, n_{jk} participants are grouped in the cell (j,k) corresponding to q_j and s_k, $j,k\in\{1,2\}$. With $n_{jk}=25$ for all j and k, we get the values $\mathrm{E}[X_i|S=s]$ displayed in Figure 2. The four curves start from the origin and end at s_i, that is, at 1 for cells (1,1) and (2,1), and at 2 for cells (1,2) and (2,2).

Figure 3 compares the survivor credits according to the linear sharing rule proposed by Donnelly and Young (2017) and the conditional mean risk sharing rule. We can see that the markedly non-linear behavior of the conditional mean risk sharing rule contrasts with the linear sharing rule. For the most likely values of s, i.e. those below 20, both sharing rules

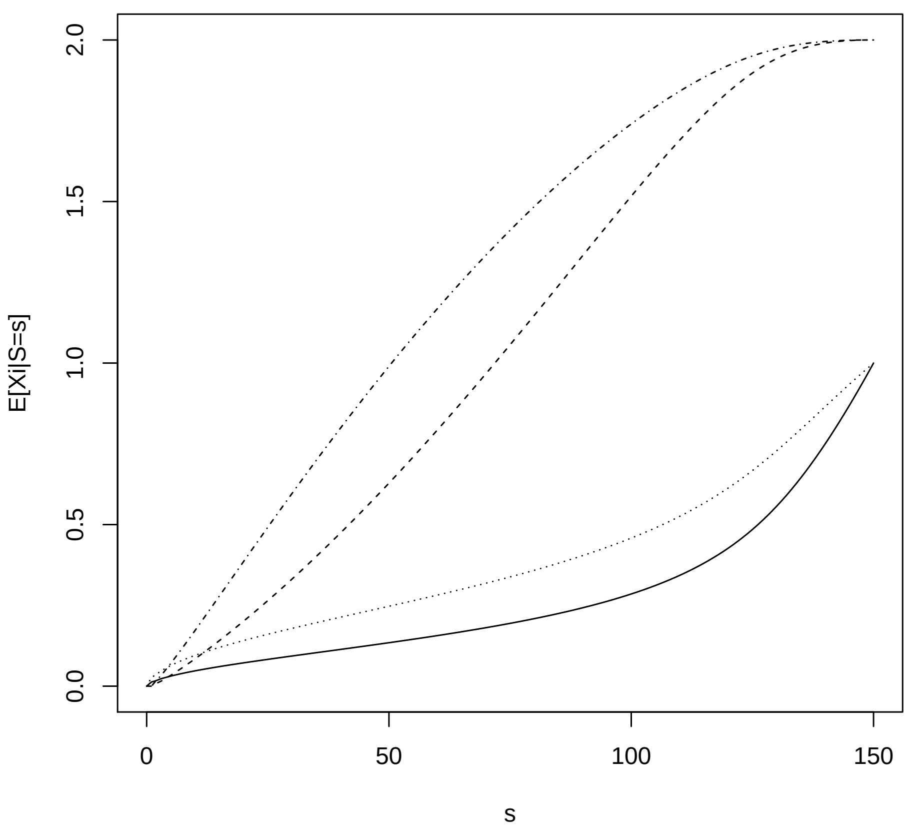

Figure 2: Respective values of $\mathrm{E}[X_i|S = s]$ received by each participant i according to group membership: solid line for cell (1,1), dashed line for cell (1,2), dotted line for cell (2,1) and dot-dash line for cell (2,2).

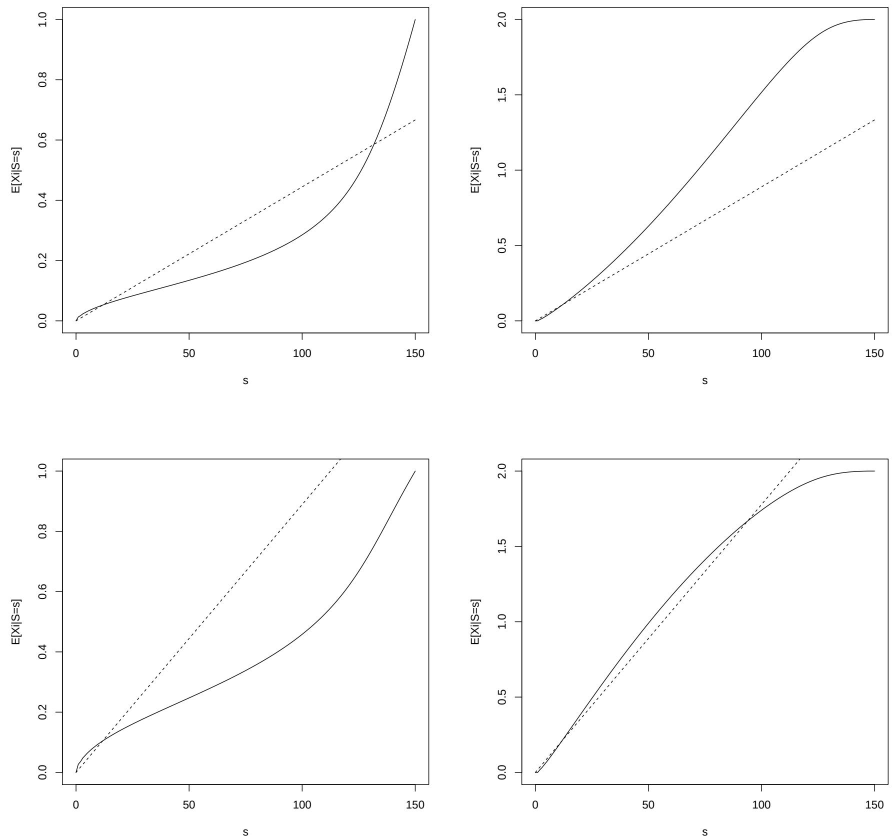

Figure 3: Comparison of $\mathrm{E}[X_i|S=s]$ (continuous line) with the share proposed by Donnelly and Young (2017) (dotted line): from upper left to lower right, cells (1,1), (1,2), (2,1), and (2,2).

nevertheless agree to a large extent. Larger differences arise when s increases, but some of them are not really material as they appear for values of s with extremely low occurrence probabilities.

Figure 4 compares the respective shares of the four groups in S for $s \geq 1$. Precisely, we plot there the shares

$$\frac{1}{s}\sum_{i\in\text{cell }(j,k)} \mathrm{E}[X_i|S=s], \quad s=1,\ldots,150,$$

for each cell (j,k) comprising 25 participants. When $s=1$ we see that only cells (1,1) and (2,1) receive some credits, as expected. Because of the larger death probabilities, participants in cell (2,1) receive a higher share of $S=1$. When s increases, we can see that the respective shares vary in different ways. For instance, the share of cell (1,1) exhibits a U-shape whereas the cumulative shares of cells (1,1) and (1,2) slowly increase in s. Considering values of s with larger occurrence probabilities (displayed in the left panel of Figure 4), we see that the shares for cells (1,1) and (2,1) dominate for small values of s but tend to decrease when s gets moderately large, favoring the high-contributions cells (1,2) and (2,2).

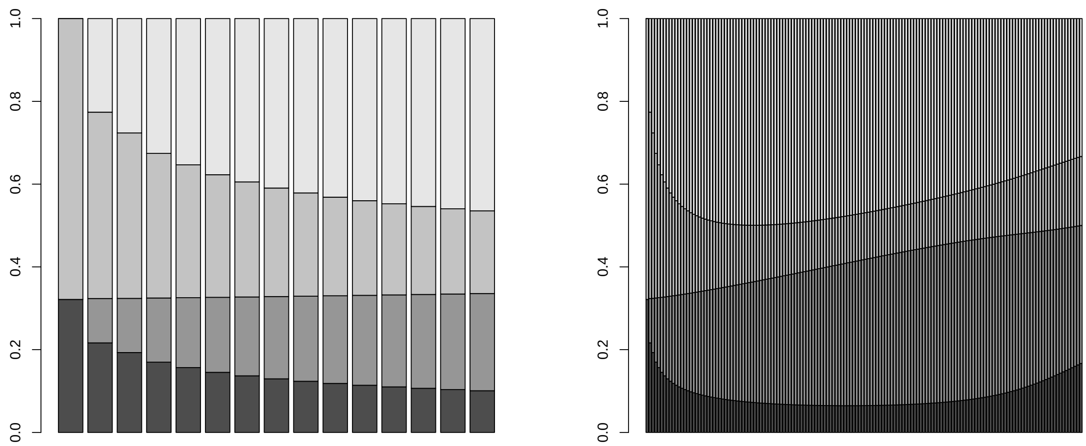

Figure 4: Respective shares of the total loss s contributed by the four groups. From bottom to top: classes (1,1), (1,2), (2,1), and (2,2). Left panel: results for s in $\{1, \ldots, 15\}$. Right panel: results for all s up to 150.

7 Discussion

The size-biased transform is known to be relevant for actuarial applications, especially in relation to risk measurement. In this paper, this transform is related to the conditional mean risk sharing rule proposed by Denuit and Dhaene (2012). Precisely, the size-biased transform is used to obtain simple expressions for contributions made by each participant to the pool as a function of the total realized loss, under the conditional mean risk sharing rule. The cases of compound Poisson, Negative Binomial and Binomial sums are studied in details, given their relevance in insurance applications.

The conditional mean risk sharing rule has been shown to be particularly effective to determine the respective contributions for participants to a P2P insurance scheme. It appears to be intuitively appealing (as it is based on mean value, a familiar concept for most people, at least to some extent) and computationally feasible. Full transparency is achieved as participants can be informed about the amount of their contribution in function of the realized total loss for the pool when they enter the P2P scheme. A certain degree of risk classification remains desirable within P2P insurance communities, to prevent adverse selection. It can also operate a posteriori, based on credibility corrections. In the Negative Binomial setting for instance, past claim numbers can easily be taken into account to re-valuate next year contributions to the pool.

The conditional mean risk sharing rule also offers a convenient solution to the distribution of survivor credits under a mutual inheritance agreement within a survivor fund. The sharing mechanism is particularly effective since the group remains open, so that each year new participants are allowed to enter the survivor fund. Also, surviving participants are free to leave the group at the end of the period because the operations are in equilibrium over each period.

Exactly as conventional life annuities can be decomposed into a sum of pure endowment contracts with increasing maturities, tontine annuities can be created by assembling a sequence of survivor schemes, with maturities matching the desired payment schedule. At some advanced age, the sponsor may replace the last payments with a regular annuity to the survivors to avoid volatility. Or this annuity can be granted once the number of participants falls below some specified threshold (but since the group is open, the number of participants should not be an issue here).

In addition to providing pension-like benefits, survivor funds also offer long-term care protections because payouts increase with the risk of dying. It is well documented that the mortality risk becomes higher after the loss of autonomy. Hence, benefits paid to dependent participants are increased if the one-year death probabilities recognize this extra risk. In such a case, the survivor fund also covers loss of autonomy by providing life-care annuity-like payouts, that is, increased payouts to dependent participants. Also, the extension to multiple lives is straightforward by defining

$$X_i = s_i \mathrm{I}[\max\{T_{i,1}, \ldots, T_{i,k_i}\} \leq m]$$

if the accumulated contribution s_i is lost only if all k_i lives $T_{i,1}, \ldots, T_{i,k_i}$ die within m years.

Longevity risk sharing offers a good compromise to retirees and meet real needs in our aging societies. To end with, let us mention that tontine schemes are applied to long-term investments, for instance by Le Conservateur based in France (`https://www.conservateur.fr/`). This mutual association was funded in 1844, with the help to promote and improve the tontine mechanism intially proposed by Lorenzo Tonti.

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