Beyond Hilbert space: RHS, PIP and all that

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Abstract. We review the different formalisms that can be used for quantum mechanics, all of them going beyond the standard Hilbert space formulation: Rigged Hilbert Spaces (RHS), partial inner product spaces (PIP spaces), etc. In particular, we consider the spectral analysis of observables and the description of symmetries.

1. Introduction

1.1. Basic principles of quantum mechanics

Quantum Mechanics (QM) is based on two fundamental principles:

- The superposition principle: any linear combination of two states is a state, which implies that the state space \mathcal{H}_o is a vector space; thus QM has an *intrinsic* linear structure (which is not true in classical physics).
- The transition amplitude between two states $\phi_{\rm in}, \phi_{\rm out}$ is given by a sesquilinear form: $A(\phi_{\rm in} \to \phi_{\rm out}) = \langle \phi_{\rm out} | \phi_{\rm in} \rangle$. Similarly, the corresponding transition probability is given by : $P(\phi_{\rm in} \to \phi_{\rm out}) = |\langle \phi_{\rm out} | \phi_{\rm in} \rangle|^2$. It follows that the state space \mathcal{H}_o is a pre-Hilbert space. Thus, \mathcal{H}_o carries an equivalence class of Hilbertian norms and its invariance group is $GL(\mathcal{H}_o)$, the set of bounded operators with bounded inverse.

Then, for reasons of "mathematical confort", one fixes a particular Hilbertian norm and takes the completion \mathcal{H} of \mathcal{H}_o , so that \mathcal{H} is a Hilbert space (even projective) and the invariance group is $U(\mathcal{H})$, the set of unitary operators.

The problem, however, is that Hilbert space is both too small, since it lacks nonnormalizable eigenstates, and too large, since it contains nonphysical states, for instance states of infinite energy.

One way of solving this dilemma is to take for state space a Rigged Hilbert Space (RHS) or Gel'fand triplet [1]

$$\Phi \subset \mathcal{H} \subset \Phi^{\times}. \tag{1}$$

The standard example is Schwartz's space of tempered distributions

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})^{\times}$$
 (conjugate dual). (2)

The duality gives a pairing $\langle \mathcal{S}(\mathbb{R}), \mathcal{S}(\mathbb{R})^{\times} \rangle$ with the "inner product" inherited from L^2 : $\langle f|g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x) \, dx$. Thus we get a *partial inner product* on the large space $\mathcal{S}(\mathbb{R})^{\times}$.

Interpolating between the spaces of the triplet (2), one obtains a chain of Hilbert spaces, and this is the structure we will be interested in from now on.

As a matter of fact, many other function spaces that play a central role in analysis come in the form of families, indexed by one or several parameters that characterize the behavior of functions (smoothness, behavior at infinity, \ldots). The typical structure is a *chain or a lattice of* Hilbert or (reflexive) Banach spaces. Let us give some familiar examples.

- (i) The Lebesgue L^p spaces on [0, 1], $\mathcal{I} = \{L^p([0, 1], dx), 1 \leq p \leq \infty\}$.
- (ii) The scale of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ built on the powers of a positive self-adjoint operator $C \geq 1$ in a Hilbert space \mathcal{H}_0 . Here $\mathcal{H}_n = D(C^n)$, with the graph norm $||f||_n = ||C^n f||, f \in$ $D(\overline{C^n})$, for $n \in \mathbb{N}$ or $n \in \mathbb{R}^+$, and $\mathcal{H}_{-n} = \mathcal{H}_n^{\times}$ (conjugate dual). Note that the triplet (2) is of this type.
- (iii) Lattices of Hilbert or Banach spaces (total order is irrelevant, partial order suffices!), for instance, the lattice generated by the spaces $L^p(\mathbb{R}, dx)$.
- (iv) Nested Hilbert spaces [2].

In all cases, the inner product of the central Hilbert space cannot be extended to the largest space of the chain, but only on *compatible* pairs. Thus one needs a unifying formalism, which is afforded by *partial inner product spaces* (PIP-space). In this paper, we shall give a quick overview of PIP-spaces and operators on them, including their spectral properties. A complete information may be found in our monograph [3], including references to the original work.

2. Partial Inner Product spaces

How can one generate such a structure systematically? The basic question is, given a vector space V and two vectors $f, g \in V$, when does their inner product make sense?

A formal way of introducing PIP-spaces is given by the idea of *linear compatibility* [3], by which we mean a symmetric binary relation # on V which preserves linearity:

$$\begin{aligned} f \# g &\iff g \# f, \ \forall \, f, g \in V, \\ f \# g, f \# h &\implies f \# (\alpha g + \beta h), \ \forall \, f, g, h \in V, \ \forall \, \alpha, \beta \in \mathbb{C}. \end{aligned}$$

Then, for any subset $S \subset V$, the set $S^{\#} = \{g \in V : g \# f, \forall f \in S\}$ is a vector subspace of V and one has $S^{\#\#} = (S^{\#})^{\#} \supseteq S$, $S^{\#\#\#} = S^{\#}$. Thus,

$$f #g \iff f \in \{g\}^{\#} \iff \{f\}^{\#\#} \subseteq \{g\}^{\#}.$$
(3)

From now on, we will call assaying subspace of V a subspace S such that $S^{\#\#} = S$ and denote by $\mathcal{F}(V, \#)$ the family of all assaying subsets of V, ordered by inclusion. Assaying subsets will be denoted V_r, V_q, \ldots and the index set by F. By definition, $q \leq r$ if and only if $V_q \subseteq V_r$. We also write $V_{\overline{r}} = V_r^{\#}$, $r \in F$. Thus the relations (3) mean that f # g if and only if there is an index $r \in F$ such that $f \in V_r, g \in V_{\overline{r}}$. In other words, assaying subspaces are the building blocks of the whole structure. Now it is easy to see that the map $S \mapsto S^{\#\#}$ is a closure, in the sense of universal algebra, so that the assaying subspaces are precisely the "closed" subsets. Therefore one has the following standard result [4].

Theorem 1 The family $\mathcal{F}(V, \#) = \{V_r, r \in F\}$ of all assaying subspaces, ordered by inclusion, is a complete involutive lattice, under the following operations, arbitrarily iterated:

- $V_r \quad \leftrightarrow \quad V_{\overline{r}} := (V_r)^{\#},$. involution:
- $V_{p \wedge q} := V_p \wedge V_q = V_p \cap V_q, \quad (p, q, r \in F)$ $V_{p \vee q} := V_p \vee V_q = (V_p + V_q)^{\#\#}.$. infimum:
- . *supremum*:

The smallest element of $\mathcal{F}(V,\#)$ is $V^{\#} = \bigcap_r V_r$ and the greatest element is $V = \bigcup_r V_r$. We also note the following relations:

$$(V_{p\wedge q})^{\#} = V_{\overline{p}\wedge \overline{q}} = V_{\overline{p}\vee\overline{q}} = V_{\overline{p}} \vee V_{\overline{q}}.$$

A partial inner product on (V, #) is a hermitian form $\langle \cdot | \cdot \rangle$, not necessarily positive definite, defined exactly on compatible pairs of vectors. A partial inner product space (PIP-space) is a vector space V equipped with a linear compatibility and a partial inner product.

We require the PIP-space (V, #) to be *nondegenerate*, i.e., $\langle f|g \rangle = 0$ for all $f \in V^{\#}$ implies g = 0. Then $(V^{\#}, V)$ is a dual pair in the sense of topological vector spaces [5], and so is every couple $(V_r, V_{\overline{r}}), r \in F$. In the sequel, we also assume that the partial inner product is positive definite.

Since we want the topological structure to parallel the algebraic structure, we impose that $(V_r[\mathbf{t}_r])^{\times} = V_{\overline{r}}, \forall r \in F$. This implies that the topology \mathbf{t}_r of V_r must be intermediate between the weak topology $\sigma(V_r, V_{\overline{r}})$ and the Mackey topology $\tau(V_r, V_{\overline{r}})$. We assume that every V_r carries its *Mackey topology* $\tau(V_r, V_{\overline{r}})$. As a consequence, if V_r is a Hilbert space or a reflexive Banach space, then $\tau(V_r, V_{\overline{r}})$ coincides with the norm topology. Next, r < s implies $V_r \subset V_s$, and the embedding operator $E_{sr}: V_r \to V_s$ is continuous and has dense range. In particular, $V^{\#}$ is dense in every V_r .

Let us give two simple examples.

- (1) $V = \omega$, the space of *all* complex sequences $x = (x_n)$, with the compatibility relation $x \# y \Leftrightarrow \sum_{n=1}^{\infty} |x_n y_n| < \infty$, and the partial inner product $\langle x | y \rangle = \sum_{n=1}^{\infty} \overline{x_n} y_n$, for x # y. The central, self-dual Hilbert space is ℓ^2 .
- (2) $V = L^1_{\text{loc}}(\mathbb{R}, dx)$, the space of Lebesgue measurable, locally integrable functions, with the compatibility relation $f \# g \Leftrightarrow \int_{\mathbb{R}} |f(x)g(x)| dx < \infty$ and the partial inner product $\langle f|g \rangle = \int_{\mathbb{R}} \overline{f(x)}g(x) dx$, for f # g. The central, self-dual Hilbert space is $L^2(\mathbb{R}, dx)$.

The previous examples show that $\mathcal{F}(V, \#)$ is a huge lattice (it is complete!) and that assaying subspaces may be complicated, such as Fréchet spaces, non metrizable spaces, etc. This situation suggests to choose an involutive sublattice \mathcal{I} of \mathcal{F} , indexed by I, such that

- (i) \mathcal{I} is generating, that is, f #g iff $\exists r \in I$ such that $f \in V_r, g \in V_{\overline{r}}$;
- (ii) every $V_r, r \in I$, is a Hilbert space or a reflexive Banach space;
- (iii) \mathcal{I} contains a unique self-dual Hilbert space $V_o = V_{\overline{o}}$.

In that case, the structure $V_I := (V, \mathcal{I}, \langle \cdot | \cdot \rangle)$ is called, respectively, a *lattice of Hilbert spaces* (LHS) or a *lattice of Banach spaces* (LBS). Note that $V^{\#}, V$ themselves usually do not belong to the family $\{V_r, r \in I\}$, but they can be recovered as $V^{\#} = \bigcap_{r \in I} V_r, V = \sum_{r \in I} V_r$.

It may be useful to note that compatibility can be varied: coarsening is always possible, but refinement not always. An important case is the refining of a RHS into a LHS, e.g. the Schwartz triplet $S \subset L^2 \subset S^{\times}$ [3].

Let us give some examples of lattice structures.

2.1. Sequence spaces

In $V = \omega$, take for \mathcal{I} the lattice generated by $\ell^2(r) = \{x = (x_n) : \sum_{n=1}^{\infty} |x_n|^2 r_n^{-2} < \infty\}$, with $r = (r_n), r_n > 0$ a sequence of positive numbers, and

- infimum: $\ell^2(p \wedge q) = \ell^2(p) \wedge \ell^2(q) = \ell^2(r), r_n = \min(p_n, q_n)$
- supremum: $\ell^2(p \lor q) = \ell^2(p) \lor \ell^2(q) = \ell^2(s), \, s_n = \max(p_n, q_n)$
- involution : $\ell^2(r) \Leftrightarrow \ell^2(\overline{r}) = \ell^2(r)^{\times}, \ \overline{r}_n = 1/r_n$.

2.2. Spaces of locally integrable functions

In $V = L^1_{\text{loc}}(\mathbb{R}, dx)$, take for \mathcal{I} the lattice generated by $L^2(r) = \{f \in L^1_{\text{loc}}(\mathbb{R}, dx) : \int |f(x)|^2 r(x)^{-2} dx < \infty\}$, with $r \in L^2_{\text{loc}}(\mathbb{R}, dx), r(x) > 0$ a.e., and

- infimum: $L^2(p \wedge q) = L^2(p) \wedge L^2(q) = L^2(r), r(x) = \min(p(x), q(x))$
- supremum: $L^2(p \lor q) = L^2(p) \lor L^2(q) = L^2(s), \ s(x) = \max(p(x), q(x))$
- involution : $L^2(r) \Leftrightarrow L^2(\overline{r}), \ \overline{r} = 1/r.$

2.3. The spaces $L^p(\mathbb{R}, dx), 1$

This is not a chain, but a genuine lattice, since one has only $L^p \cap L^q \subset L^s$, p < s < q. The lattice operations are the following [6]:

- $L^p \wedge L^q = L^p \cap L^q$, with projective norm $||f||_{p \wedge q} = ||f||_p + ||f||_q$
- $L^p \vee L^q = L^p + L^q$, with inductive norm $||f||_{p \vee q} = \inf_{f=g+h} (||g||_p + ||h||_q), g \in L^p, f \in L^q$
- For $1 < p, q < \infty$, both spaces $L^p \wedge L^q$ and $L^p \vee L^q$ are reflexive Banach spaces and one has $(L^p \wedge L^q)^{\times} = L^{\overline{p}} \vee L^{\overline{q}}, \quad (L^p \vee L^q)^{\times} = L^{\overline{p}} \wedge L^{\overline{q}}.$

For the visualization, it is useful to introduce the following notation:

$$L^{(p,q)} = \begin{cases} L^p \wedge L^q, & \text{if } p \ge q\\ L^p \vee L^q, & \text{if } p \le q \end{cases} \quad (1 \le p, q \le \infty).$$

Then $L^{(p,q)}$ is represented by the point (1/p, 1/q) in the unit square $J = [0, 1] \times [0, 1]$.

2.4. Other examples of LBS (or containing a LBS)

(i) Amalgam spaces $W(L^p, \ell^q) \equiv (L^p, \ell^q)$: This space consists of functions on \mathbb{R} locally in L^p with ℓ^q behavior at infinity. It is a Banach space for the norm

$$||f||_{p,q} = \left\{ \sum_{n=-\infty}^{\infty} \left[\int_{n}^{n+1} |f(x)|^{p} \, \mathrm{d}x \right]^{q/p} \right\}^{1/q}, \ 1 \le p, q < \infty.$$

(ii) Mixed norm spaces

 $L_m^{p,q}(\mathbb{R}^{2d}) = \{ f \text{ measurable } : ||f||_m^{p,q} < \infty \}, \ 1 \leq p,q < \infty, \text{ where } \}$

$$\|f\|_m^{p,q} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x,\omega)|^p m(x,\omega)^p \,\mathrm{d}x\right)^{q/p} \,\mathrm{d}\omega\right)^{1/q}, m \in L^1_{\mathrm{loc}}(\mathbb{R}^{2d}, \,\mathrm{d}x \,\mathrm{d}\omega), m > 0.$$

This is a Banach space for the norm $\|\cdot\|_m^{p,q}$.

- (iii) Inhomogeneous Besov spaces $B_{pq}^{s}(\mathbb{R}^{d})$, well adapted to wavelet analysis.
- (iv) Modulation spaces $M_m^{p,q}(\mathbb{R}^d)$, well adapted to Gabor analysis.

3. Operators on PIP-spaces

The basic idea of (indexed) PIP-spaces is that vectors should not be considered individually, but only in terms of the subspaces V_r :

f # g if, and only if, there is an $r \in I$ such that $f \in V_r, g \in V_{\overline{r}}$.

For the corresponding definition of operator on a PIP-space, we impose that only *bounded* operators between Hilbert or Banach spaces are allowed, so that an operator is a *coherent collection* of bounded operators.

Thus, given a LHS-space $V_I = \{V_r, r \in I\}$, we define an *operator* on V_I as a map $A : \mathcal{D} \to V_I$, where:

- (i) \mathcal{D} is a nonempty union of assaying subspaces of $V : \mathcal{D} = \bigcup_{q \in \mathsf{d}(A)} V_q$
- (ii) For every $q \in d(A)$, there is a $p \in I$ such that the restriction of $A : V_q \to V_p$ is linear and continuous; we denote it by $A_{pq} \in \mathcal{B}(V_q, V_p)$.
- (iii) A has no proper extension satisfying (i) and (ii), i.e., it is maximal.

Examples of global operators on $L^1_{loc}(\mathbb{R}, dx)$ or $\mathcal{S}^{\times}(\mathbb{R})$ are Fourier transform, translation, modulation, scaling (dilation), ...

The linear bounded operator $A_{pq}: V_q \to V_p$ is called a *representative* of A. The operator A is characterized by the set $j(A) = \{(q, p) \in I \times I : A_{pq} \text{ exists}\}$ and it is equivalent to the collection of its representatives: $A \simeq \{A_{pq}: (q, p) \in j(A)\}$.

Given an operator A, define the following sets:

 $\mathsf{d}(A) = \{ q \in I : \text{there is a } p \text{ such that } A_{pq} \text{ exists} \};$

 $i(A) = \{ p \in I : \text{there is a } q \text{ such that } A_{pq} \text{ exists} \}.$

- d(A) is an *initial* subset of I: if $q \in d(A)$ and q' < q, then $q' \in d(A)$, and $A_{pq'} = A_{pq}E_{qq'}$, where $E_{qq'}$ = unit operator; this is what we mean by a 'coherent' collection....
- i(A) is a final subset of I: if $p \in i(A)$ and p' > p, then $p' \in i(A)$.
- $j(A) \subset d(A) \times i(A)$, with strict inclusion in general.

In the case of a scale, the properties of the operator A may be characterized by the following figure:



We denote by $Op(V_I)$ the set of all operators on V_I . A similar definition may be given for operators $A: V_I \to Y_K$ between two LHS or LBS. The resulting set is denoted by $Op(V_I, Y_K)$.

Since $V^{\#} \subset V_r$, $\forall r \in I$, with dense image, an operator on V_I may be viewed as a sesquilinear form on $V^{\#} \times V^{\#}$, and also as a linear map from $V^{\#}$ into V. But we want to keep also the algebraic operations on operators ! Thus we define:

(i) Adjoint A^{\times} : every $A \in Op(V_I)$ has a unique adjoint $A^{\times} \in Op(V_I)$:

 $\langle A^{\times}x|y\rangle = \langle x|Ay\rangle$, for $y \in V_r$, $r \in \mathsf{d}(A)$, and $x \in V_{\overline{s}}$, $s \in \mathsf{i}(A)$,

that is, $(A^{\times})_{\overline{rs}} = (A_{sr})^*$ (usual Hilbert/Banach space adjoint). Therefore, one has always $A^{\times \times} = A, \forall A \in \operatorname{Op}(V_I)$: no extension is allowed, by the maximality condition (iii).

(ii) Partial multiplication : AB is defined if, and only if, there is a $q \in i(B) \cap d(A)$, that is, if, and only if, there is a continuous factorization through some V_q :

 $V_r \xrightarrow{B} V_q \xrightarrow{A} V_s$, i.e., $(AB)_{sr} = A_{sq}B_{qr}$.

Notice that, for a LHS/LBS, the domain $\mathcal{D}(A)$ is vector subspace of V, but this is not true for a general PIP-space. In any case, $Op(V_I)$ is a vector space and a *partial* *-algebra [7].

4. Special classes of operators on PIP-spaces

In the sequel, we will need several special types of operators, that in fact generalize the corresponding ones on a Hilbert space.

4.1. Regular and totally regular operators

An operator A on V_I is regular if d(A) = i(A) = I; equivalently, $A : V^{\#} \to V^{\#}$ and $A : V \to V$ continuously. An operator A on V_I is totally regular if $j(A) \supset \text{diag}(I \times I)$; equivalently, $A_{rr}: V_r \to V_r, \forall r \in I$, continuously.

4.2. Homomorphisms

An operator $A \in Op(V_I, Y_K)$ is called a homomorphism if

- (i) for every $r \in I$ there exists $u \in K$ such that both A_{ur} and $A_{\overline{u}\overline{r}}$ exist.
- (ii) for every $u \in K$ there exists $r \in I$ such that both A_{ur} and $A_{\overline{u}\overline{r}}$ exist.

The set of all homomorphisms is denoted by $Hom(V_I, Y_K)$. Clearly, one has

- $A \in \operatorname{Hom}(V_I, Y_K)$ if and only if $A^{\times} \in \operatorname{Hom}(Y_K, V_I)$.
- If $A \in \text{Hom}(V_I, Y_K)$, then $j(A^{\times}A)$ contains the diagonal of $I \times I$ and $j(AA^{\times})$ contains the diagonal of $K \times K$.

4.3. Isomorphisms

The operator A is an *isomorphism* if $A \in \text{Hom}(V_I, Y_K)$ and there exists $B \in \text{Hom}(Y_K, V_I)$ such that $BA = I_V, AB = I_Y$ (identity operators).

4.4. Unitary operators

The operator U is unitary if $U^{\times}U$ and UU^{\times} are defined and $U^{\times}U = I_V, UU^{\times} = I_Y$ (caution: unitary operators need not be homomorphisms !).

Thus the unitary isomorphisms are the natural setting for group representations in LHS, as we shall see in Section 8.

4.5. Symmetric operators

A is said to be symmetric if $A^{\times} = A$. For such an operator, a generalized KLMN theorem holds true, stating when a symmetric operator has a self-adjoint restriction to the central Hilbert space V_o (see Section 6).

An application of this theorem is the treatment of very singular operators. The simplest case runs as follows:. $V_{-} \subset V_{-} \sim V_{-} \subset V_{-} \quad (V_{-} = \text{Hilbert})$

Then :

$$V_r \subset V_o \cong V_{\overline{o}} \subset V_s \quad (V_o \equiv \text{Inder})$$

- A_{oo} is a bounded operator $V_o \rightarrow V_o$;
- A_{or} is an unbounded operator $V_r \to V_o$, with domain $D(A) \supset V_r$;
- A_{sr} is a singular operator $V_r \to V_s$, with Hilbert space domain possibly reduced to $\{0\}$.

4.6. Orthogonal projections

Let V_I be a nondegenerate PIP-space. Then P is an orthogonal projection whenever $P \in \text{Hom}(V_I)$ and $P^2 = P = P^{\times}$. It follows that $P: V_r \to V_r, \forall r \in I$.

The main result is that W is an orthocomplemented subspace of V, i.e., $V = W \oplus Z$ if, and only if, W is the range of orthogonal projection P. There are equivalent topological conditions, so that an orthocomplemented subspace is the same thing as a PIP-subspace.

4.7. Invertible operators

Let V_J be a LBS/LHS and $A \in \operatorname{Op}(V_J)$. Then A is *invertible* if it has at least one invertible representative. Equivalently, there exists $B \in \operatorname{Op}(V_J)$ and two indexes p, q such that $(p,q) \in \mathfrak{j}(A), (q,p) \in \mathfrak{j}(B)$, and AB = BA = I. Note that the two conditions AB, BA welldefined and AB = BA = I are not sufficient by themselves. Whenever $A \in \operatorname{Op}(V_J)$ is invertible, it has a unique inverse $A^{-1} \in \operatorname{Op}(V_J)$.

Given $A \in \operatorname{Op}(V_J)$, the notion of invertibility is needed for defining the resolvent set $\rho(A)$ and the spectrum $\sigma(A)$ (with the usual analyticity properties), and also the resolvent operator $R_{\lambda}(A) := (A - \lambda I)^{-1} \in \operatorname{Op}(V_J)$ [8].

5. Spectral analysis in a RHS

Take a rigged Hilbert space (RHS) $\Phi \subset \mathcal{H} \subset \Phi^{\times}$, where Φ is dense in \mathcal{H} , with a finer topology, and Φ^{\times} is the conjugate dual of Φ . Let A be a self-adjoint operator in \mathcal{H} , such that $A : \Phi \to \Phi$, continuously. Then A has a continuous extension $A^{\dagger} : \Phi^{\times} \to \Phi^{\times}$ defined by duality:

$$\langle \phi | A^{\dagger} \xi \rangle = \langle A \phi | \xi \rangle, \forall \phi \in \Phi, \xi \in \Phi^{\times}.$$
(4)

A vector $\xi_{\lambda} \in \Phi^{\times}$ is called a *generalized eigenvector* for A, with *generalized eigenvalue* λ , if one has $A^{\dagger}\xi_{\lambda} = \lambda\xi_{\lambda}$.

Then the main result, obtained by Gel'fand and Maurin, independently, is the following theorem [1].

Theorem 2 [Gel'fand-Maurin spectral theorem] If Φ is nuclear and complete, A has a complete set of generalized eigenvectors $\xi_{\lambda} \in \Phi^{\times}$, $\lambda \in \mathbb{R}$, that is, one has, for all $\phi, \psi \in \Phi$,

$$\langle \phi | \psi \rangle = \int_{\mathbb{R}} \xi_{\lambda}(\phi) \,\overline{\xi_{\lambda}(\psi)} \, d\mu(\lambda) \tag{5}$$

$$\equiv \int_{\mathbb{R}} \langle \phi | \xi_{\lambda} \rangle \, \langle \xi_{\lambda} | \psi \rangle \, d\mu(\lambda) \tag{6}$$

for some measure μ on \mathbb{R} , where $\mu = \sum_i \delta(\cdot - \lambda_i) + \mu_c$, where $\{\lambda_i\}$ are the eigenvalues of A in \mathcal{H} and supp $\mu_c \supset \sigma_{\text{cont}}(A)$, where $\sigma_{\text{cont}}(A)$ denotes the continuous spectrum of A.

The upshot is that all eigenvalues are treated on the same footing, which is the characteristic feature of the Dirac formalism in QM [9, 10].

6. Spectral analysis in a LHS

There is another way to generate a self-adjoint operator in a Hilbert space, namely by restriction of a symmetric operator acting in a larger space. This is the content of the celebrated KLMN theorem [KLMN stands for Kato, Lax, Lions, Milgram, Nelson]. The proof of Nelson actually takes place in a LHS (or part of it, namely the central triplet of Hilbert spaces) [3, Sec.3.3.5].

Theorem 3 [KLMN theorem] Let C > I be a self-adjoint operator in \mathcal{H} and $V_J = \{\mathcal{H}_n, n \in \mathbb{Z}\}$ the scale of Hilbert spaces built on the powers of C. Let $A = A^{\times} \in V_J$ be a symmetric operator with $(m, n) \in j(A)$, where $\mathcal{H}_m \subseteq \mathcal{H}_0 \subseteq \mathcal{H}_n$.

Assume there exists a $\lambda \in \mathbb{R}$ such that $A - \lambda I$ has an invertible representative $A_{nm} - \lambda E_{nm}$: $\mathcal{H}_m \to \mathcal{H}_n$. Then A_{nm} has a unique restriction to a self-adjoint operator A_0 in the Hilbert space $\mathcal{H} = \mathcal{H}_0$, with dense domain $D(A_0) = \{f \in \mathcal{H}_m : Af \in \mathcal{H}_0\}$. The number λ does not belong to the spectrum of A_0 .

One may note that the result extends to a general LHS, not only the scale of a self-adjoint operator.

Now, the Gel'fand-Maurin spectral theorem may also be reformulated in the context of a LHS.

Theorem 4 [Gel'fand-Maurin spectral theorem] Assume C > I with C^{-1} Hilbert-Schmidt, so that $V^{\#} = D^{\infty}(C)$ is a nuclear Fréchet space. Let $A = A^{\times} \in V_J$) be a symmetric operator with $(m,\overline{m}) \in j(A)$, where $\mathcal{H}_m \subseteq \mathcal{H}_0 \subseteq \mathcal{H}_{\overline{m}}$ $(\overline{m} := -m)$. Let A_{0m} be the restriction of A to $D(A_{0m}) = \{f \in \mathcal{H}_m : Af \in \mathcal{H}\} \subset \mathcal{H}_m$ and assume it is densely defined and essentially self-adjoint. Then A has a complete set of generalized eigenvectors belonging to $\mathcal{H}_{\overline{m}}$, in the sense that the relations (5)-(6) hold true. Combining the two theorems above, we get a generalization of the preceding theorem. In the statement, the condition that C^{-1} be a Hilbert-Schmidt operator implies that the extreme space $\Phi = D^{\infty}(C) := \bigcap_{n \in \mathbb{Z}} D(\mathcal{H}_n)$ is nuclear, the crucial condition in the Gel'fand-Maurin theorem.

Theorem 5 [Generalized Gel'fand-Maurin spectral theorem] Let C > I be a self-adjoint operator in \mathcal{H} and $V_J = \{\mathcal{H}_n, n \in \mathbb{Z}\}$ the scale of Hilbert spaces built on the powers of C, with C^{-1} Hilbert-Schmidt. Let $A = A^{\times} \in V_J$) be a symmetric operator with $(m, \overline{m}) \in j(A)$, where $\mathcal{H}_m \subseteq \mathcal{H}_0 \subseteq \mathcal{H}_{\overline{m}}$. Assume there exists a $\lambda \in \mathbb{R}$ such that $A - \lambda I$ has an invertible representative $A_{\overline{m}m} - \lambda E_{\overline{m}m} : \mathcal{H}_m \to \mathcal{H}_{\overline{m}}$.

Then $A_{\overline{m}m}$ has a unique restriction to a self-adjoint operator A_0 in the Hilbert space $\mathcal{H} = \mathcal{H}_0$ with dense domain $D(A_0) = \{f \in \mathcal{H}_m : Af \in \mathcal{H}_0\}$. The operator A_0 has a complete set of generalized eigenvectors belonging to $\mathcal{H}_{\overline{m}}$. The number λ does not belong to the spectrum of A_0 .

6.1. Tight riggings

Define the extended spectrum of A_0 , $\sigma_{\text{ext}}(A_0) \supset \sigma(X_0)$, as the closure of the set of all generalized eigenvalues of A_0 . Then the rigging is said to be tight if $\sigma_{\text{ext}}(A_0) = \sigma(X_0)$, that is, there are no additional eigenvalues when going from \mathcal{H} to Φ^{\times} [11].

As an example, consider the operator $A_0 = -i \frac{\mathrm{d}}{\mathrm{d}x}$ in $L^2(\mathbb{R})$. The generalized eigenvectors are the functions $\xi_{\lambda}(x) = e^{i\lambda x}, \lambda \in \sigma_{\mathrm{ext}}(A_0)$. Then we have a tight rigging in the RHS $\mathcal{S} \subset L^2 \subset \mathcal{S}^{\times}$, since $\sigma(A_0) = \sigma_{\mathrm{ext}}(A_0) = \mathbb{R}$, but a nontight rigging in the RHS $\mathcal{D} \subset L^2 \subset \mathcal{D}^{\times}$, since now $\sigma_{\mathrm{ext}}(A_0) = \mathbb{C}$.

7. Symmetries: Representations of Lie groups and Lie algebras in a RHS

Consider first the traditional approach of Schrödinger, Dirac, von Neumann,..., where states are represented by rays in a Hilbert space \mathcal{H} and observables by self-adjoint operators in \mathcal{H} . In this context, a *symmetry* is defined as a bijection between states that preserves the absolute values of all transition amplitudes. According to Wigner, a symmetry τ is realized by a unitary or an anti-unitary operator in \mathcal{H} [12]. Then, if the system admits a *symmetry group* { $\tau_g, g \in G$ }, with G a Lie group, the latter is realized by a strongly continuous unitary (projective) representation U of G in \mathcal{H} (Wigner-Bargmann) [12, 13].

Assume all "relevant" observables have a common, dense, invariant domain in \mathcal{H} . Then one gets a *Rigged Hilbert Space (RHS)* $\Phi \subset \mathcal{H} \subset \Phi^{\times}$, where Φ describes the set of all physical states. Hence its (conjugate) dual represents pieces of equipment, i.e., measurement apparatus: the input of the linear functional is a physical state, the output is a number. The problem, of course, is how to build Φ . Several solutions have been described in the literature [9, 10, 14].

If U is a unitary representation of a symmetry group G in \mathcal{H} , it must map physical states into physical states, continuously, and similarly for the measuring devices. Thus we obtain two additional representations:

- U_{Φ} , restriction of U to Φ : this is the *active* point of view for symmetry operations;
- U_{Φ}^{\times} , extension of U^{\dagger} from \mathcal{H} to Φ^{\times} : $U_{\Phi}^{\times}(g)\xi = U^{\dagger}(g)\xi$ for $g \in G, \xi \in \mathcal{H}$: this is the passive point of view.

The equivalence of the two points of view is manifested by the requirement that U_{Φ} and U_{Φ}^{\times} are contragredient of each other, that is,

$$\langle \phi | U_{\Phi}^{\times}(g) \xi \rangle = \langle U_{\Phi}(g^{-1}) \phi | \xi \rangle, \, \forall \, g \in G, \phi \in \Phi, \xi \in \Phi^{\times},$$

or, equivalently,

$$\langle U_{\Phi}(g)\phi|U_{\Phi}^{\times}(g)\xi\rangle = \langle\phi|\xi\rangle, \,\forall\,g\in G, \phi\in\Phi, \xi\in\Phi^{\times}$$

a relation that embodies the unitarity of U in \mathcal{H} :

$$\langle U(g)f|U(g)h\rangle = \langle f|h\rangle, \,\forall g \in G, f, h \in \mathcal{H}.$$

This definition implies that U_{Φ}^{\times} is an extension of both U_{Φ} and U, as it should in view of the triplet structure of the RHS.

Similar considerations apply to the representation of elements of the *Lie algebra* \mathfrak{g} or its *universal enveloping algebra* $\mathfrak{U}(\mathfrak{g})$, which contains observables of the system.

8. Symmetries: Representations of Lie groups and Lie algebras in a LHS

Now we consider the converse problem. Given a strongly continuous unitary representation U_{00} of a *Lie group* G in a Hilbert space \mathcal{H}_0 , we want to build a PIP-space or a LHS V_I , with \mathcal{H}_0 as central Hilbert space, in such a way that U_{00} extends to a unitary representation U in V_I . Of course, we must represent also elements of the *Lie algebra* \mathfrak{g} or the *universal enveloping algebra* $\mathfrak{U}(\mathfrak{g})$, since these may be observables.

The solution is to exploit Nelson's theory of analytic vectors [15]. First some definitions. A vector $\xi \in \mathcal{H}_0$ is called a C^{∞} -vector for U_{00} (resp. an *analytic* vector) if the map $g \mapsto U_{00}(g)\xi$ of G into \mathcal{H}_0 is C^{∞} (resp. analytic). The set \mathcal{H}_0^{∞} of all C^{∞} -vectors is dense in \mathcal{H}_0 .

Of particular interest is the so-called *G*^arding domain \mathcal{H}_0^G , which consists of finite linear combinations of vectors of the form $\tilde{U}_{00}(f)\phi$, $f \in C_0^{\infty}(G)$, where $\tilde{U}_{00}(f)\phi = \int_G U_{00}(g)\phi f(g) \, \mathrm{d}g$ and $\mathrm{d}g$ is the left-invariant Haar measure on G.

The key properties of the Gårding domain are the following: $\mathcal{H}_0^G \subset \mathcal{H}_0^\infty$, and \mathcal{H}_0^G is dense in \mathcal{H}_0 ; \mathcal{H}_0^G is stable under $U_{00}(g)$, $\forall g \in G$; and \mathcal{H}_0^G is contained in the domain of the representatives of all elements of the Lie algebra \mathfrak{g} of G and stable under them.

In particular, every element $T \in \mathfrak{U}(\mathfrak{g})$ is represented in \mathcal{H}_0^G by an operator $\widetilde{U}_{00}(T)$ (often essentially self-adjoint on \mathcal{H}_0^G) defined by the relation

$$\widetilde{U}_{00}(T)\widetilde{U}_{00}(f)\xi = \widetilde{U}_{00}(Tf)\xi, \ f \in \mathbb{C}_0^{\infty}(G)(G), \ \xi \in \mathcal{H}.$$

Define the Nelson operator $\Delta := \sum_{j=1}^{n} X_j^2$, where $\{X_j, j = 1, \ldots, n\}$ are the representatives under U_{00} of a basis of \mathfrak{g} . Then Δ is essentially self-adjoint on the Gårding domain \mathcal{H}_0^G , and $\overline{\Delta}$ is self-adjoint and positive. Now we define the associated PIP-space as $V_I := \{\mathcal{H}_n, n \in \mathbb{Z}\}$, the canonical scale of Hilbert spaces generated by the powers of the operator $(\overline{\Delta} + 1)$:

$$V^{\#} := D^{\infty}(\overline{\Delta}) = \mathcal{H}_{0}^{\infty} \subset \mathcal{H}_{0} \subset V := D_{\overline{\infty}}(\overline{\Delta}).$$
⁽⁷⁾

In the scale V_I , $U_{00}(g)$ maps every \mathcal{H}_n , $n = 0, 1, 2, \ldots$, into itself continuously, for all $g \in G$. By transposition, the same is true for $U_{nn}(g^{-1}) : \mathcal{H}_{\overline{n}} \to \mathcal{H}_{\overline{n}}$, $n = 1, 2, \ldots$

$$\mathcal{H}_0^\infty \subset \ldots \subset \mathcal{H}_n \subset \ldots \subset \mathcal{H}_0 \subset \ldots \subset \mathcal{H}_{\overline{n}} \subset \ldots \subset [\mathcal{H}_0^\infty]^{\times}$$

In other words, U_{00} extends to a unitary representation U by totally regular automorphisms of the LHS V_I .

Corresponding to the triplet (7), we have now three continuous representations : $U_{\infty\infty} \subset U_{00} \subset U_{\overline{\infty\infty}}$, which are simultaneously topologically irreducible.

9. Applications of RHSs and PIP spaces

(1) In mathematical physics

The various formalisms that go beyond Hilbert space have been motivated by difficulties in the traditional approach of QM. Both RHS and PIP-spaces yield a framework suitable for the description of quantum systems and their symmetry properties. In particular, PIP-spaces generalize both the traditional Hilbert space method and the RHS approach, yet the mathematics involved are simpler, there is no need for sophisticated functional analysis concepts.

Let us quote a number of concrete applications.

- The first rigorous formulation of the Dirac formalism in QM was, of course, obtained via the RHS approach: the bra-ket formalism, generalized (nuclear) spectral theorem, ... [9, 10, 14].
- As explained in Section 4.5, PIP-spaces allow to treat rigorously singular interactions in QM. A spectacular example is the clean description of the Kronig-Penney 1D crystal model, and 2D or 3D periodic point interactions [16].
- In quantum scattering theory, a unification of the Weinberg-Van Winter approach and the Complex scaling method may be obtained with a LHS of analytic functions [3, Sec.7.2].
- Various rigorous formulations of Quantum Field Theory, via RHS or PIP-spaces : axiomatic Wightman QFT, Borchers' field algebra, Nelson's Euclidean field theory are based on a RHS or a LHS.
- PIP spaces yield a natural environment for the representations of Lie groups/algebras, in particular for QM symmetries, as seen in Section 8 above.

(2) In signal/image processing

The PIP-space formalism is ubiquitous in the formulation of signal or image processing. Indeed, many families of function or distribution spaces that underlie theory are of this type, for instance, Wiener amalgam spaces, modulation spaces, the Feichtinger algebra $\mathcal{S}_0 \subset L^2 \subset \mathcal{S}_0^{\times}$, Besov spaces, coorbit spaces, and so on [3, Chap.8].

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