Structural assessment of masonry arches using admissible geometrical domains

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Abstract. Following Méry's thrust line approach, this paper presents an alternative method to define the structural safety of masonry arches, based on admissible geometrical domains. These are implemented in a parametric model built on the reciprocal diagrams of graphic statics. The application to a case study – a semi-circular masonry arch loaded by a central point load – helps drawing a comparison with the classical geometric safety factor as defined by Jacques Heyman. The model is also used to evaluate the impact of geometrical as well as resistance hypotheses on the structural safety level. Analyses first confirm that stereotomy only slightly influences the load bearing capacity of the arch. They also validate the common use of an infinite compressive strength for arches' constitutive material, since considering a typical value of 10 MPa reduces structural performances by less than 2%. Finally, a methodology using admissible geometrical domains is suggested to get insights on the robustness of masonry arches.

Keywords: thrust line, admissible geometrical domain, limit state analysis, masonry arches, graphic statics, structural robustness.

1 Introduction

The classical approach for evaluating the safety of masonry arches is based on the concept of thrust line, defined as "the locus of points where the resultant force passes through the joints between the *voussoirs*" [1]. As this approach only deals with equilibrium considerations, the stability of masonry arches is then ensured by the existence of at least one thrust line lying entirely inside the masonry envelope [2]. Méry [3] and Moseley [4] were the first to develop a method to construct the thrust line graphically, and to correlate its geometry with limit states occurring when the thrust line reaches the masonry envelope in an amount of points that allows the creation of a collapse mechanism. Various authors have then further developed this method for design and analysis purposes. O'Dwyer [5] proposed a new technique for the limit state analysis of masonry vaults, using a discrete network of forces in equilibrium lying inside the masonry envelope. Ochsendorf [1, 6] developed a general method to assess the stability of masonry buttresses against overturning or failure, taking into account

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the effects of leaning. Block, Ciblac and Ochsendorf [7] further developed applications for real-time limit analysis of arches and vaults.

In his *Mémoire*, Méry [3] assumed three fundamental hypotheses for masonry structures: infinite compressive strength, zero tensile strength, and no failure due to *voussoirs* sliding. Assuming the same hypotheses, Milankovitch [8] was the first to determine the correct analytical description of the thrust line for a semi-circular arch made of radial *voussoirs*. Makris and Alexakis [9] have recently discussed Milankovitch's corresponding ratio (minimum arch thickness t_{Mil} / average radius R = 0.1075) when considering vertical *voussoirs* instead of radial ones; the value obtained for the same ratio is in this case slightly conservative (t_{Mak}/R = 0.1095). Although the difference between these two ratios is very small, it shows that geometrical hypotheses have an influence on the construction of the thrust line, and consequently on the geometric assessment of the structural safety of a masonry arch. Following these considerations, Heyman [10] integrated the thrust line theory within the framework of plastic theory, considering that masonries made of rigid blocks connected by soft joints behave as a whole like a perfectly plastic material. The lower bound theorem of plasticity can then be applied to masonry arches: a given arch will stand as long as there exists a thrust line – a statically compatible distribution of internal forces – lying entirely within the masonry envelope and respecting yield conditions. As the safety of masonry arches is ensured by the increase of the arch's thickness, Heyman defined a geometric safety factor intended as the ratio between the actual thickness of the *voussoirs* and the minimum one (i.e. the one for which only one admissible thrust line exists).

2 Assessing geometrically the structural safety

Assessing the structural safety of masonry arches is not straightforward. Heyman's proposal for a geometric safety factor [10] is interesting, because it deals with the geometrical characteristics of the arch rather than with material strength. Assuming for instance a geometrical safety factor of 3 means that it should be possible to find a thrust line contained in the middle third of the arch. This criterion is commonly used to avoid tensile forces when considering linearly distributed stresses over the whole cross section [3]. However, Heyman's approach might be hard to use in practice when it comes to irregular geometries that would require to identify the critical section to which apply the safety factor. Another limitation of this method is that it cannot easily be linked with the safety factor formulations commonly used by structural engineers, which are expressed in terms of ratios between characteristic and design stresses.

2.1 Admissible geometrical domains

To overcome these limitations, the authors suggest working with graphic statics' form and force diagrams. The method presents the advantage of being simple, and of allowing the visual control of the force equilibrium [11]. The form diagram represents the path of the resulting forces acting inside the structure, consisting in a thrust line in the case of masonry arches (Fig. 1, left). The force diagram then gives the intensities of these forces through the lengths of its different bars, each segment in this diagram being related to one sole segment parallel to it in the other diagram (Fig. 1, right).

Admissible geometrical domains, intended as the locus of statically admissible positions for the different points of the force diagram [12], can be used to get information about structural safety and robustness (see Section 2.2). In the case of masonry arches, these domains show the set of admissible positions for the pole of the funicular polygon in the force diagram. These positions correspond in the form diagram to thrust lines that lie entirely within the masonry envelope. The corners of the domain (Fig. 1, right) correspond to the limit states, as the corresponding thrust line in the form diagram reaches the masonry envelope in such an amount of points that it produces a mechanism (Fig. 1, left) [13]. According to the lower bound theorem of plasticity [10], this means that the arch will only collapse if its admissible geometrical domain is an empty set of points.

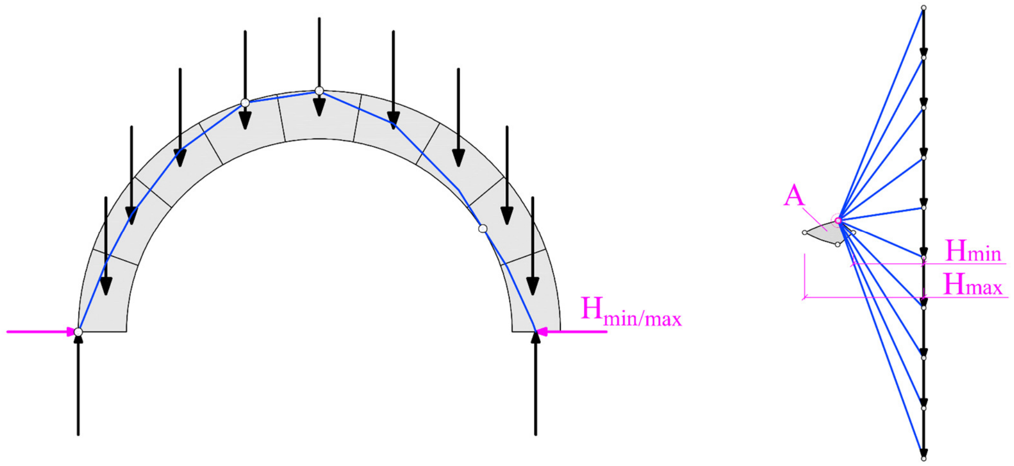

Figure 1. Graphic statics' form and force diagrams for a 9 *voussoirs* arch under self-weight

In order to apply the aforementioned geometrical approach to the analysis of masonry structures, the authors have developed a model under Rhino [14] and its parametrical algorithm editor Grasshopper [15]. The main advantage of this technique is that it provides a result both visual and interactive. Indeed, the different parameters (arch dimensions and weight, *voussoirs* geometry, applied forces, material strength) can easily be modified, the impact of these manipulations being directly visible on the size and shape of the admissible geometrical domain. These modifications are illustrated in Section 3, through the analysis of a case study.

2.2 Safety requirements

Because of the relatively low stresses acting inside historical masonry structures [10], collapse is very unlikely to occur because of a lack of resistance. Furthermore, as it is relatively irrelevant to verify the deformability of masonry structures at serviceability limit states, stiffness considerations can be neglected. The safety assessment of masonry arches is then essentially a matter of equilibrium [2]. Besides stability, another fundamental requirement of design codes against unexpected circumstances is robustness, which means that a structure has to "withstand events like fire, explosions, impact or consequences of human error, without being damaged to an extent disproportionate to the original cause" [16]. The next two subsections present a geometric approach to measure the response of masonry arches to both these criteria.

2.2.1 Stability

Stability assessment implies to compare the acting or destabilising value of a chosen parameter with the stabilising one. For masonry arches, the active thrust line is defined as the one corresponding to the minimum value of the horizontal thrust H_{min} that has to be mobilized at the arch's supports in order to find a thrust line lying entirely inside the masonry envelope. The other points of the domain then correspond to other statically admissible configurations for the thrust line, showing the reserve structural safety. H_{min} can be measured in the force diagram by the distance between the rightmost point of the domain and the external forces (Fig. 1, right). In the same way, H_{max} is the distance between the leftmost point of the domain and the external forces, characterizing the magnitude of the maximum thrust force the arch can stand without being transformed into a mechanism. A stability index λ is then defined by:

$$\lambda = \frac{H_{max}}{H_{min}} \tag{1}$$

2.2.2 Robustness

The authors also propose to use the area A of the admissible geometrical domain as a graphic indicator of structural robustness [17]. The domain indeed shows a measure of the arch's ability for plastic redistributions, corresponding to the set of possible thrust lines lying inside the masonry envelope. In case of local failure, a larger remaining domain area $A_{damaged}$ will then indicate a larger structural capacity for equilibrating the applied forces by finding another path for the thrust line. Taking inspiration from [18], a robustness index ρ can then be given by:

$$\rho = \frac{A}{A - A_{damaged}} \tag{2}$$

For example, if a movement of the abutments led to a crack – the position of which can be clearly identified –, a plastic hinge would form in the cross-section of the crack, and the thrust line must consequently pass by a well-defined point at critical stage. The ratio between domain areas before and after the crack occurred could then define the ability of the arch to survive the damage. The same kind of analysis could also be performed to evaluate the robustness of an arch against damages like *voussoir* sliding, leaning buttresses or differential settlements.

3 Impact of shape and resistance on safety assessment

The semi-circular planar masonry arch composed of nine radial *voussoirs* depicted in Fig. 1 is now evaluated in terms of stability and robustness. The arch has an internal radius of 2 m, and is 0.5 m thick and 0.5 m deep. *Voussoirs* are supposed to have an infinite compressive strength. The structure is subjected to a load case composed of its self-weight (18 kN/m^3) as well as a vertical point load F applied at the arch crown.

The impact of the magnitude of force F on the structural safety can be observed by drawing the admissible geometrical domains corresponding to different values of F (Fig. 2). As a larger value for F leads to domains moving to the left, the corresponding values of H_{max} and H_{min} are both increasing (at different rates), leading to a decreasing stability index λ. The domains also have a reducing homothetic shape, and thus a decreasing area A. The maximum point load that can be applied at the arch crown is F = 16.16 kN, leading to an admissible geometrical domain reduced to one sole point (which means $H_{max} = H_{min}$).

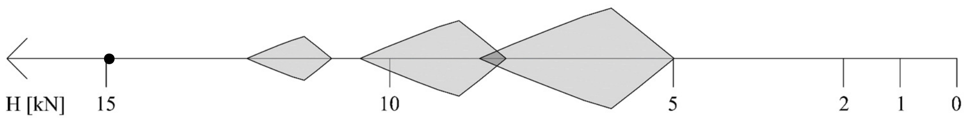

Figure 2. Admissible geometrical domains for an increasing value of *F*, namely 0, 5, 10 and 16.16[kN] (from right to left)

The arch's safety is now assessed regarding the variation of two different parameters: the shape of the *voussoirs* (Section 3.1), and the compressive strength of the constitutive material (Section 3.2).

3.1 Influence of stereotomy

Two other *voussoir* shapes have been taken into account for the analysis of the arch: *voussoirs* with vertical joints and constant width L, and *voussoirs* with vertical joints and constant weight (Fig. 3, left). Corresponding results for H_{max}, H_{min} and A are given in Fig. 3 (right), showing that the shape of the *voussoirs* has nothing but a very small influence on the arch's safety. This also confirms Makris' conclusion that the radial pattern for joints is not conservative nor safe [9], as it leads to a larger ultimate value for force F (i.e. the one for which H_{max} = H_{min} and A is reduced to one sole point).

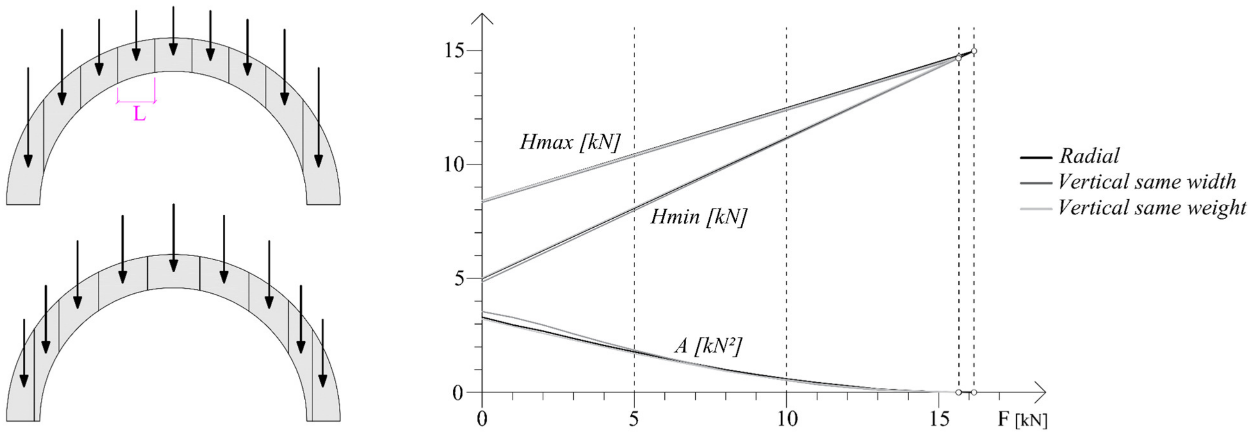

Figure 3. Other considered *voussoir* shapes (left); corresponding values for H_{max}, H_{min} and A for an increasing applied load (right)

3.2 Influence of masonry compressive strength

Another series of analyses has then been carried out taking into account a finite compressive strength for masonry. Thrust forces have then been modelled using discontinuous stress fields, a method initially developed for the structural design and analysis of reinforced concrete [19]. Discontinuous stress fields are geometric modellings of the load paths inside structures. Considering a perfectly plastic material behavior, struts and ties can be drawn with a thickness proportional to the axial load they transfer. It is then possible to take advantage of the lower bound theorem of plasticity for design purposes. When dealing with masonry arches, the aforementioned method can be applied by giving a finite thickness to the thrust line. Considering a triangular distribution of stresses in the arch [3], the thickness t_i of each section of the thrust line can be deduced from:

$$F_i = \frac{3 \cdot t_i \cdot d \cdot \sigma_c}{2} \Leftrightarrow t_i = \frac{2 \cdot F_i}{3 \cdot d \cdot \sigma_c} \tag{3}$$

with F_i the force in the ith section of the thrust line, d the constant arch's depth, and σ_c the material compressive strength assuming a uniform distribution of compressive stresses over the cross section.

Assuming no shear failure, the equilibrium conditions of the structure are that the whole surface of the discontinuous stress field has to stay inside the arch geometry. The corresponding admissible geometrical domain can then be drawn using the geometrical approach as before. Fig. 4 illustrates the variation of the domain area A for a material compressive strength decreasing from ∞ to 10 and 1 MPa [20], for an applied load F = 0 kN. Figs. 5 and 6 show the same phenomenon for an applied load of 5 and 10 kN respectively. In each of the three cases, areas A fall with the decrease of material resistance, as corresponding stress fields tend to grow larger. But this diminution is only marginal when material strength decreases to 10 MPa, leading to domain boundaries that are nearly superimposed. In opposition, the domain area reduction for σ_c = 1 MPa is clearly observable, cf. domains depicted in darker grey.

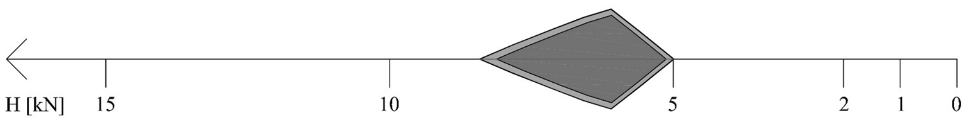

Figure 4. Admissible geometrical domains for F = 0[kN] and σ_c = ∞, 10 and 1[MPa]

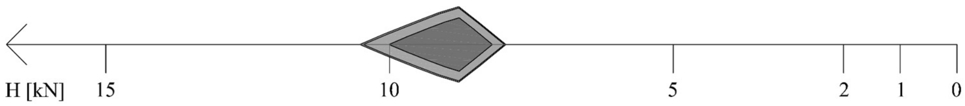

Figure 5. Admissible geometrical domains for F = 5[kN] and σ_c = ∞, 10 and 1[MPa]

Figure 6. Admissible geometrical domains for F = 10[kN] and σ_c = ∞, 10 and 1[MPa]

Fig. 7 (left) gives the complete curves for H_{max}, H_{min} and A as a function of the point load F, for the considered material strengths. As shown in Figs. 4 to 6, the two first strength values give very close results. Moreover, a larger applied force and/or a smaller material compressive strength give smaller admissible geometrical domains. The force increment also leads to a linear increase of H_{max} and H_{min}, even if at different rates. The gap between curves depicting H_{max} and H_{min} is also decreasing with the decrease of material strength.

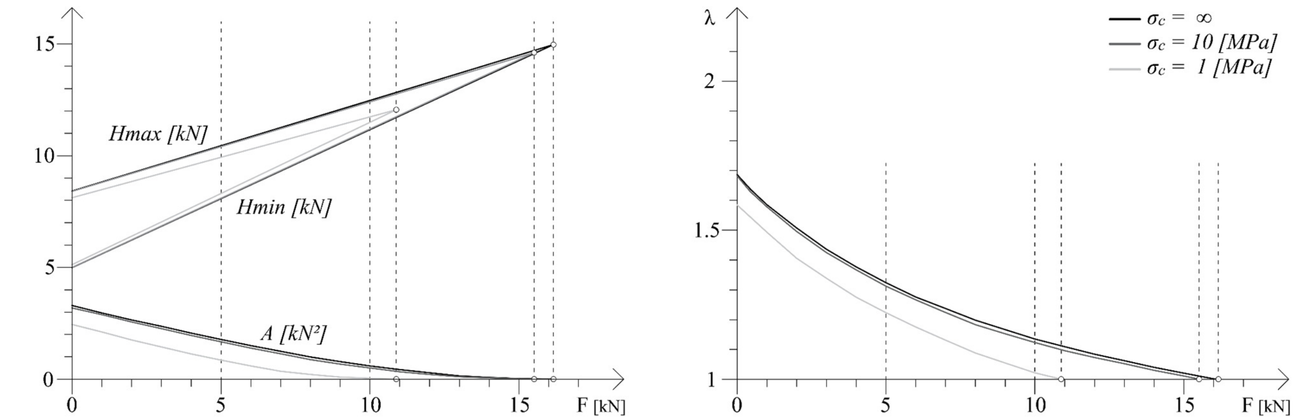

Figure 7. Values of H_{min}, H_{max} and A for a decreasing material resistance (left); stability index λ for different material compressive strengths (right)

3.3 Stability assessment

The results translated in terms of stability index (cf. Section 2.2.1) give the curves for λ as a function of the applied load F (Fig. 7, right). These curves are logically decreasing with the point load increment and/or the diminution of material compressive strength. As a summary, Table 1 gives the values for H_{max}, H_{min}, λ and A when varying the stereotomy, the material compressive strength and the magnitude of the point load F. As shown in Fig.7, both the stability index λ and the domain area A have values decreasing when increasing the point load magnitude. Table 1 confirms the slight influence of stereotomy on λ (less than 5%) and A. It also validates the common use of an infinite compressive strength for masonry, since λ is almost not affected (less than 2%) when reducing σ_c to 10 MPa. On the contrary, a compressive strength of 1 MPa reduces the structural response to a larger extent (around 10%), emphasizing the important role played by the joints on the overall stability.

Table 1. Results

Stereotomy	Value	F = 0[kN]			F = 5[kN]			F = 10[kN]		
		$\sigma_c = \infty$	$\sigma_c = 10$	$\sigma_c = 1$	$\sigma_c = \infty$	$\sigma_c = 10$	$\sigma_c = 1$	$\sigma_c = \infty$	$\sigma_c = 10$	$\sigma_c = 1$
Radial	H_{max}	8.4178	8.4065	8.1176	10.5252	10.4651	10.0101	12.5193	12.4301	11.7787
	H_{min}	4.9876	4.9984	5.1224	7.9504	7.9762	8.1869	11.0274	11.0695	11.4616
	$\boldsymbol{\lambda}$	**1.6877**	**1.6818**	**1.5847**	**1.3239**	**1.3120**	**1.2227**	**1.1353**	**1.1229**	**1.0277**
	A	3.3004	3.1820	2.4463	1.7723	1.6702	0.8481	0.5926	0.5027	0.0273
Vertical Same width	H_{max}	8.3312	8.3312	8.1238	10.5057	10.4427	9.9510	12.4463	12.3801	11.6686
	H_{min}	4.8394	4.8535	4.9760	7.7954	7.8192	8.0215	10.9878	11.0344	11.4254
	$\boldsymbol{\lambda}$	**1.7215**	**1.7165**	**1.6326**	**1.3477**	**1.3355**	**1.2405**	**1.1327**	**1.1220**	**1.0213**
	A	3.5442	3.4411	2.6770	1.8519	1.7542	0.9399	0.5331	0.4540	0.0161
Vertical same weight	H_{max}	8.4155	8.4042	8.1152	10.4923	10.4349	9.9909	12.4710	12.4216	11.7075
	H_{min}	5.0160	5.0333	5.1420	7.9890	8.0175	8.2491	11.0804	11.1223	11.5306
	$\boldsymbol{\lambda}$	**1.6777**	**1.6697**	**1.5782**	**1.3133**	**1.3015**	**1.2112**	**1.1255**	**1.1168**	**1.0153**
	A	3.2129	3.1074	2.3481	1.7038	1.6011	0.7935	0.5502	0.4533	0.0085

4 Conclusion

A fully geometrical methodology has been developed within the environment of graphic statics' reciprocal diagrams in order to assess stability and robustness of masonry arches. The method is based on admissible geometrical domains, depicting the set of positions for the pole of the funicular polygon that correspond to thrust lines lying entirely within the masonry envelope. The analysis of a case study confirms that stereotomy only has a slight influence on stability and robustness. It also shows that the arch's performance is almost not affected when considering a finite compressive strength of 10 MPa. As it is based on the lower bound theorem of plastic theory, the proposed methodology is safe and therefore suited for practical applications. Furthermore, its fully graphical interface may be a substantial advantage for practitioners when dealing with complex geometries and unknown material behavior laws. Finally, the visual and interactive environment in which it is developed gives an immediate and intuitive insight on the structural safety when dealing with historical structures.

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