# A Schumpeterian Theory of Multi-Quality Firms *

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**Abstract**

This paper introduces multi-quality firms within a Schumpeterian framework. Featuring non-homothetic preferences and income disparities in an otherwise standard quality-ladder model, it shows that the resulting differences in the willingness to pay for quality among consumers generate both positive investments in R&D by industry leaders *and* positive market shares for more than one quality, hence allowing for the emergence of multi-product firms within a vertical innovation framework. This positive investment in R&D by incumbents is obtained with complete equal treatment in the R&D field between the incumbent patent holder and the challengers: in this framework, the incentive for a leader to invest in R&D stems from a "surplus appropriation effect" specific to vertically-differentiated markets, i.e. the perspective of more efficient price discrimination when expanding the product portfolio. Such a framework makes it possible to analyse the impact of income distribution, as well as that of several possible R&D policies, both on long-term growth and on the allocation of R&D activities between challengers and incumbents.

*Keywords:* Growth, Innovation, Income inequality, Multi-Product firms.

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# 1 Introduction

The importance and specificities of multi-product firms (MPFs) have lately been exemplified by a growing body of literature.[1] In particular, recent work has emphasised MPFs' specific supply and demand linkages, and examined how those forces impact MPFs' product-market decisions such as intra-firm portfolio adjustments or investment in product innovation (Eckel and Neary, 2010; Dhingra, 2013). Dynamic R&D-driven growth models studying the behaviour and impact on aggregate innovation of MPFs have already been provided for the cases in which firms are multi-industry (Klette and Kortum, 2004; Akcigit and Kerr, 2017) or multi-variety (Minniti, 2006). However, the standard quality-ladder framework has not yet been able to account for the existence of multi-quality firms, i.e. firms selling more than one quality-differentiated version of the same good. Indeed, the creative destruction mechanism at the heart of Schumpeterian models traditionally not only deters leaders from investing in R&D, but *also* guarantees the systematic exit of any quality that has moved away from the frontier.[2]

Yet examples of firms offering more than one quality-differentiated version of the same product abound. Apple recently jointly launched its latest flagship phone, the iPhone X, along with no less than 4 lower-cost versions (iPhone 8, iPhone 7, iPhone 6s, and iPhone SE); the iPad is now available in 3 versions (iPad pro, iPad Air, and iPad mini). Similarly, Intel commercialises a whole array of microchips, selling its latest, highly efficient processors at high prices (Xeon and Core), while simultaneously offering cheaper models further from the industry frontier (Celeron and Atom). In the car industry, the seminal example of such diversification is the decision in the 1920s by General Motors to simultaneously commercialise 5 different brands ( Chevrolet, Pontiac, Oldsmobile, Buick, and Cadillac), each one explicitly designed to match a specific price range (and therefore specific consumers). A more recent example would be Renault-Nissan launching several low-cost cars in the last decade, which were specifically marketed for developing countries (Dacia's Logan was initially designed for the Eastern European markets, while Datsun's Go just recently launched on the Indian market) and re-used obsolete technologies previously featured in the manufacturer's leading brands (Renault Clio for the Logan and Nissan Micra for the Go). These examples show how firms resort to *vertical* brand diversification in order to better price discriminate among consumers displaying differences in purchasing power.

The present paper builds on this body of anecdotal evidence, and provides a model accounting for the existence of **multi-quality leaders** within a dynamic Schumpeterian framework. More precisely, along the salient features of the examples described above, I argue that as long as preferences are allowed to be non-homothetic, income distribution impacts the strength and scope of the creative destruction process at work in quality-ladder models. Income differences may then account for both the survival of more than

[1] Among others, Bernard et al. (2010) estimate that MPFs account for 41% of the total number of US firms, as well as for 91% of total output; also, they estimate that the contribution to the US output growth of MPFs' product mix decisions (i.e. product adding and dropping) is greater than that of firm entry and exit.

[2] Mussa and Rosen (1978) as well as Gabszewicz et al. (1986) study the pricing decisions of multi-quality firms, but in a static framework precluding any specific modelling of the R&D process which led to the initial design of the product line. Klette and Kortum (2004) as well as Akcigit and Kerr (2017) feature MPFs in a quality-ladder world; however, multi-product firms are also multi-industry firms in their models, with only one quality being sold within each product line.

one quality at equilibrium *and* positive investment in R&D by incumbents. The result is the endogenous emergence in a dynamic framework of multi-quality leaders whose product portfolio composition and investment in R&D activities are both influenced by the extent of income disparities.

The intuition behind this result is straightforward, and is related to the well-explored notion of second-degree price discrimination in vertically differentiated markets. For a monopolist, serving costumers who do not care much for quality creates negative externalities, since it hinders capturing consumer surplus from those who have a stronger taste for quality. Mussa and Rosen (1978) have demonstrated that, in vertically differentiated markets, a monopolist confronted with such disparities in consumers' taste for quality optimally resorts to product diversification, offering lower quality, lower priced items to the less enthusiastic consumers, so as to be able to charge higher prices to more adamant buyers of high quality units. Their microeconomic *static* set-up assumes that the monopolist has a whole product line at its disposal, while in a standard quality-ladder dynamic framework, the monopolist only has access to as many qualities as times it has innovated in a row. I demonstrate that in a dynamic set-up, the internalisation of such negative externalities leads to investment in R&D by incumbent monopolists, and in case of success, to the existence of firms simultaneously offering more than one quality of their product.

First, this surplus appropriation effect is precisely identified in a partial equilibrium framework. Modelling stochastic free-entry patenting races in a vertically differentiated market, I demonstrate that, provided there exist differences in the willingness to pay for quality among consumers, the perspective of price-discrimination when broadening the product portfolio is a powerful enough incentive for the incumbent to invest positive amounts in R&D: Arrow's negative replacement effect is counteracted by this positive surplus appropriation effect. I then integrate this mechanism in a standard Schumpeterian model: assuming non-homothetic preferences and differences in wealth endowment, I obtain different tastes for quality among consumers; modelling stochastic R&D races in which both incumbents and challengers participate so as to patent the next highest quality in vertically differentiated markets, I demonstrate that incumbents optimally invest a strictly positive amount in R&D. Such a behaviour can be explained by the fact that the profits of a successful incumbent are higher than those of a successful challenger. This results from the possibility of price discriminating when several qualities have been patented in a row.[3]

I then move on to studying the impact of both income distribution and R&D policies on challengers' and incumbents' innovation incentives, and by extension on long-term growth. First, I show that, in a quality-ladder model, the impact on growth of an increase in the inequality level depends on the nature of the shock considered. More precisely, an increase in the income *gap* has a positive impact on the R&D activities of both incumbents and challengers, hence increasing the growth rate of the economy. On the other hand, greater income *concentration* is unequivocally detrimental to growth, diminishing both the incum-

[3]Indeed, in such a framework, a *challenger* which has won the latest innovation race in a given quality-differentiated market needs to decide between two alternatives: capturing the whole market by charging a price sufficiently low to appeal to the poorest households, *or* selling its product at a higher price only to the wealthiest consumers, at the cost of abandoning the rest of the market to its direct competitor (i.e. the previous quality leader). On the other hand, an *incumbent* which has won an innovation race retains exclusive monopoly rights over two successive qualities: the firm can then efficiently discriminate between rich and poor consumers by offering two distinct price/quality bundles, capturing the whole market *and* reaping the maximum surplus from wealthy consumers at the same time.

bents' and the challengers' R&D investments. Investigating the impact of different possible R&D tax schemes, I then demonstrate that the effects of R&D policy depend on whether it is aimed at challengers or incumbents. While barriers to entry decrease challengers' R&D investment and increase incumbents', leading to a decrease in the overall rate of growth, a subsidy directed at incumbents' R&D expenditures increases both incumbents' and challengers' investments.

The main contribution of this paper is to provide a dynamic framework which endogenously accounts for the emergence of multi-quality leaders in the presence of income disparities among consumers. Beyond its novelty, such a result holds several implications. First, it exemplifies a specific motivation for incumbents to invest in R&D beyond preemptive patenting (Gilbert and Newbery, 1982) which has been overlooked so far by the literature: the perspective to more efficiently price discriminate when expanding the quality portfolio in vertically differentiated markets (surplus appropriation effect). Second, such a framework makes it possible to investigate the impact of income distribution on the intensity of both challengers' *and* incumbents' innovation activities, a feature that totally overturns the predictions obtained so far in the quality-ladder literature regarding the interactions of growth and inequality operating through the product market (Zweimuller and Brunner, 2005).

*Relation to the Literature*

This paper is first and foremost related to the broad literature (encompassing both the industrial organisation and growth theory fields) studying incumbents' specific incentives to conduct R&D in patent-protected markets where innovation is a natural barrier to entry. Section 2 below is devoted to providing a detailed discussion of this literature and of the way this paper contributes to it.

This paper also belongs to the smaller literature studying the specific R&D behaviour of multi-product firms in a dynamic, general equilibrium framework. Klette and Kortum (2004) as well as Akcigit and Kerr (2017) have already provided quality-ladder models in which industry leaders invest in exploration R&D so as to expand their activities in *other* sectors. However, these frameworks cannot account for leaders broadening their product portfolio within a *given* industry. Minniti (2006) embeds multi-product firms selling more than one horizontally differentiated variety of a given good in an endogenous growth model; however, his model is an expanding-variety one, hence precluding the emergence of multi-*quality* firms.

Finally, this paper is also related to the literature examining the relationship between long-term growth and income distribution operating through the demand side. Foellmi and Zweimuller (2006) demonstrate that in an expanding-variety framework, higher inequality levels are systematically beneficial to long-term growth. Foellmi et al. (2014) provide a model combining both product innovations (introducing new luxury goods) and process innovations (transforming those goods into necessities through mass production technologies): in such a framework, the impact of higher inequality on growth is ambiguous, and depends on the scope of the productivity gains stemming from the process innovations. However, both contributions investigate the impact of income distribution on growth in a *horizontal* differentiation framework, where firms retain *permanent* monopoly rights over their *single* product. Li (2003) and Zweimuller and Brunner (2005) on the other hand study the impact of disparities in the purchasing power of households in a quality-ladder framework. Zweimuller and Brunner (2005) in particular show that a reduction in the level of inequality through a reduction in the income gap is beneficial to innovation intensity and, hence, for growth. However, they only consider the R&D investment of challengers, and overlook the existing incentives for incumbent innovation in the presence of differences in consumers' willingness to pay.

The rest of the paper is organised as follows. In a simple partial equilibrium framework, Section 2 isolates the surplus appropriation effect present in vertically differentiated markets. Section 3 presents the structure of the general equilibrium model, while Section 4 studies its steady-state properties. Section 5 then analyses the effects of the extent of inequality and of several possible R&D policies on the innovation intensity of both challengers and incumbents, as well as on long-run growth. Section 6 concludes.

## 2 R&D Investments by Incumbents: The Surplus Appropriation Effect

In markets where innovation is a prerequisite to entry, monopolist incumbents' R&D investments have been under particular scrutiny since they may potentially result in blockaded entry and monopoly persistence. The literature has mainly identified two opposite mechanisms influencing incumbents' incentives to invest in R&D: the replacement effect (Arrow, 1962) and the efficiency effect (Gilbert and Newbery, 1982). Arrow's replacement effect is one of deterrence: when considering the expected gains from innovating, monopolists face lower R&D incentives than firms operating under perfect competition, since a successful incumbent partly replaces itself when patenting a new version of its good or a more efficient production technology (Arrow, 1962). On the other hand, Gilbert and Newbery (1982) identified a positive efficiency effect: under free entry and deterministic R&D outcomes, the entry threat of potential rivals might lead the incumbent firm to optimally resort to pre-emptive patenting, i.e. over-investing in R&D so as to protect its dominant position.

The relative strength of these two effects (and the resulting monopoly persistence or attrition) has been found to depend on the nature of the R&D process (deterministic R&D investments or stochastic patent races), the nature of the innovation considered (drastic or non-drastic), and the nature of firms' strategic interactions, both during the patent races and on the product market considered. In the case of drastic product innovations and uncertainty in free-entry R&D races, Reinganum (1983) showed that the efficiency effect tends to disappear, leading to "incumbency changing hands more often than not."[4] Schumpeterian growth models have aligned on this very specific set-up, resulting in a total absence of incumbents' R&D activity and systematic leapfrogging in the canonical model

[4]Indeed, even in the Gilbert and Newbery (1982) seminal model of deterministic R&D investments, incumbents and challengers invest an equal amount of R&D in the case of winner-takes-all drastic innovations. Under the alternative assumption of stochastic R&D races, Reinganum (1983) then showed that the pre-emption incentive is trumped by the accelerated disappearance of incumbents' current monopoly profits when investing higher amounts in R&D, leading to incumbents' under-investment in patent races. Fudenberg et al. (1983), Fudenberg and Tirole (1984, 1985), and Vives (1989, 2008) then further refined these results under several alternative set-ups.

(Aghion and Howitt, 1992).

All the quality-ladder growth models featuring innovation by incumbents proposed by the literature so far have relaxed one of the two fundamental assumptions guaranteeing the disappearance of the efficiency effect, namely (i) free-entry stochastic R&D races in which incumbents and entrants compete on equal footing (Segerstrom, 2007; Etro, 2008; Denicolo and Zanchettin, 2012; Acemoglu and Cao, 2015), and/or (ii) drastic innovations allowing the successful innovator to behave as a monopolist (Aghion et al., 2001).[5] I however argue that *provided there exist differences in consumers' taste for quality*, a further motivation exists for incumbents to invest in R&D beyond pre-emptive patenting in vertically differentiated markets: the *price discrimination* motivation. The surplus loss stemming from having to serve consumers who display the lowest taste for quality is indeed a powerful incentive for incumbents to keep expanding their quality portfolio beyond the first successful innovation, so as to price discriminate more efficiently.

In order to precisely identify and comment on this **surplus appropriation effect**, I will now quickly sketch out a patent race meeting the standard assumptions in the baseline Schumpeterian growth model (cf. Barro and i Martin (2003), Chapter 7). More precisely, consider the optimal R&D investments of an incumbent (I) and potential challengers (C) aiming at improving the quality of a vertically differentiated final good. Each innovation increases quality by a fixed rung $q$, and is assumed to be drastic: the successful firm, whatever its identity, systematically gets to behave as a monopolist. Stochastic free-entry R&D races are assumed, with the probability of innovating $p_i(n)$ depending linearly on the total expenditures over R&D $\phi_i(n)$, $i \in [I, C]$: more precisely, $p_i(n) = Q(n)\phi_i(n)$ with $Q(n)$ capturing the effect of the current technology position $n$.[6] The expected value of an innovation is then $E[v_i(n)] = \frac{\pi_i(n)}{r+p_i(n)}$, with $r$ the interest rate over time and $\pi_i(n)$ the monopoly profits of the successful innovator ($i \in [I, C]$).

In an industry where the highest quality currently available is $q^n$, the standard free-entry condition for entrants equates the costs incurred when engaging in R&D $\phi_C(n)$ to the expected value of innovating $p_C(n)E[v_C(n + 1)]$:

$$\phi_C(n)\,(1 - Q(n)E[v_C(n + 1)]) = 0 \tag{1}$$

The incumbent's Hamilton-Jacobi-Bellman equation is of the form:

$$rv_I(n) = \max_{\phi_I(n)\geq 0} \{\pi_I(n) - \phi_I(n) + Q(n)\phi_I(n)(E[v_I(n+1)] - E[v_I(n)]) - Q(n)\phi_C(n)E[v_I(n)]\}$$

[5]Segerstrom (2007), Denicolo and Zanchettin (2012), and Acemoglu and Cao (2015) have assumed that incumbents have a cost advantage in stochastic R&D races, leading to the attenuation of the replacement effect; alternatively, Etro (2004, 2008) modelled sequential patent races in which incumbents act as Stackelberg leaders, therefore allowing for the emergence of the efficiency effect. Aghion et al. (2001) on the other hand proposed a model in which innovations are non-drastic, with quality acting as a demand shifter rather than a purely vertical attribute: in a context where the number of competing firms is exogenously imposed, the perspective of lessening the competitive pressure (and broadening the market share) then provides an incentive for the industry leader to resort to incremental innovation.

[6]Hence, decreasing returns are not imposed, neither at the firm nor at the industry level; both the incumbent and the entrants face the same R&D costs.

with the first-order condition (f.o.c.):

$$\underbrace{(-1 + Q(n)E[v_I(n+1)]}_{(*)} \underbrace{-Q(n)E[v_I(n)]}_{(**)})\phi_I(n) = 0 \tag{2}$$

In the standard Schumpeterian model featuring homothetic preferences and homogeneous consumers, the monopoly profits reaped by the successful innovator are the same, whether the latter is a former incumbent or a newly entered challenger: hence, $E[v_I(n+1)] = E[v_C(n+1)]$. The free-entry condition (1) then immediately entails $(*) = 0$. The remaining term $(**)$ is negative, capturing Arrow's replacement effect. Hence, as stated above, it is optimal for incumbents to refrain from investing in R&D in a vertical framework featuring free-entry stochastic patent races, since they would only accelerate the cannibalisation of their own market (Reinganum, 1983).

Now alternatively consider a framework in which *consumers are assumed to differ in terms of taste for quality.* Mussa and Rosen (1978) and Gabszewicz et al. (1986) have demonstrated that monopolists facing heterogeneous consumers in vertically differentiated markets get to increase their profits when expanding their product line. Indeed, a firm which can produce and sell several qualities will be able to better *discriminate* among consumers who differ in their willingness to pay. It will thus capture the incremental profits generated by charging a higher price to quality-loving consumers, while still offering a lower quality (charged at a lower price) to consumers less prone to value quality. In other words, in a framework allowing for differences in the willingness to pay, the expected value of successfully innovating is higher for the current incumbent than for a new entrant: $E[v_I(n+1)] > E[v_C(n+1)]$. The negative cannibalisation term $(**)$ of the first-order condition (2) is then compensated for by a strictly positive term $(*)$, capturing the surplus appropriation effect outlined above.

This effect stems directly from the incumbent's reaction to a negative externality specific to vertically differentiated markets: in the absence of the possibility of first-degree discrimination, the existence of poor consumers prevents the monopolist from capturing the maximum consumer surplus from those who have a stronger taste for quality. In a static framework, a monopolist with a whole quality range at its disposal internalises this heterogenous taste for quality externality by inducing less enthusiastic consumers to buy lower quality items charged at a lower price, opening the possibility of charging higher prices to more adamant buyers of high quality units (Mussa and Rosen, 1978). As a consequence, a broader range of qualities than what is optimal is offered in the end. In a dynamic model explicitly modelling innovation races and allowing for the presence of a competitive fringe, the monopolist only retains exclusive patent rights for as many qualities as R&D races it has won *in a row*: the negative externalities stemming from having to serve consumers who differ in their quality valuation is then internalised by expanding the line of products towards *higher* (and not lower) qualities, i.e. through R&D investment.

After having precisely identified the mechanism at work in a partial equilibrium framework, I now present an economy displaying the required feature, i.e. differences in the willingness to pay for quality among consumers. More precisely, I model non-homothetic preferences through the unit consumption of the quality good, and incorporate this feature in an otherwise canonical quality-ladder framework displaying income inequality.

# 3 The Model

It will first prove useful to provide a quick overview of the model architecture. Consider an economy featuring heterogeneous households in terms of wealth, who consume a continuum of quality-differentiated goods and a composite standardised good. The non-homotheticity of preferences is obtained through a unit consumption requirement for the vertically differentiated goods, generating differences in the willingness to pay for quality along income. Each vertically differentiated industry is then in a situation of natural oligopoly (Shaked and Sutton, 1982), with an endogenously determined number of active firms competing through vertical product differentiation and strategic pricing. Free-entry patent races take place within each vertically differentiated industry so as to discover the next highest quality; along the lines of the example presented in Section 2 above, R&D investments by incumbents and potential entrants are influenced by the degree of price discrimination (and the resulting profits) each type of firm will be able to impose if successful. I focus on the balanced-growth path properties of such an economy, in which the extent of income inequality will have an impact on the long-run rate of growth.

I now move on to presenting the details of the model.

## 3.1 Consumers

*3.1.1. The Distribution of Endowments*

The economy is populated by a fixed number $L$ of consumers who live infinitely and supply one unit of labour each period, paid at a constant wage $w$. While all consumers are identical with respect to their preferences and their labour income, they are assumed to differ in terms of wealth, based on firms' asset ownership. More precisely, I assume a two-class society with rich (R) and poor (P) consumers distinguished by their wealth $\omega_R(t)$ and $\omega_P(t)$.[7]

The share of poor consumers within the population is denoted by $\beta$. The extent of inequality within the economy is determined by this share, as well as by the repartition between the rich and the poor of the aggregate stock of assets $\Omega(t)$. $d \in (0, 1)$ is defined as the ratio of the value of the stock of assets owned by a poor consumer *relative* to the average per-capita wealth: $d = \frac{\omega_P(t)}{\Omega(t)/L}$. The wealth position of the rich can be computed for a given $d$ and $\beta$, and lastly $\omega_P(t) = d\frac{\Omega(t)}{L}$ and $\omega_R(t) = \frac{1-\beta d}{1-\beta}\frac{\Omega(t)}{L}$.

The current income $y_i(t)$ of an individual belonging to group $i$ ($i = P, R$) is then of the form:

$$y_i(t) = w + r(t)\omega_i(t) \tag{3}$$

where $r(t)$ is the interest rate.

*3.1.2. Preferences and Consumer Choices*

[7]All the results presented in the paper pertaining to investment in R&D by incumbents are robust under the alternative specification of inequality generated through differences in income, i.e. through different endowments in labour efficiency units. As demonstrated in Online Appendix O1, the main mechanism of the model also holds under an alternative specification in which income is uniformly distributed over a given support $[a; b]$.

Households make consumption choices over two types of final goods available within the economy: a continuum of indivisible quality goods (indexed by $s \in [0,1]$) being subject to quality improvements over time, and a Hicksian standardised composite commodity serving as the numeraire.[8] The crucial assumption adopted here so as to ensure the non-homotheticity of preferences is a **unit consumption** requirement for the vertically differentiated goods, i.e. the consumption of a given quality good yields a positive utility only for the first unit, and zero utility for any additional unit.[9]

At any date $t$, $q_n(s,t) = k^{n(s,t)}q_0$ denotes the highest quality available within one industry, with $k > 1$ and $n(s,t)$ the number of innovations in industry $s \in [0,1]$ between time 0 and $t$. For the sake of simplicity, I also set $q_0 = 1 \; \forall \; s \in [0,1]$. Along the unit consumption requirement, an individual belonging to group $i$ then needs to make two distinct choices for each industry $s$ at each period $t$:

(1) whether to consume or not good $s$;

(2) *if* she does choose to consume good $s$, she then needs to determine which quality level $q_j(s,t)$ (with $j \in [0, n(s,t)]$) offers her the highest utility *given its price* $p_j(s,t)$.

At time $\tau$, the objective function of a type $i$ consumer is then given by:

$$\mathcal{U}_i(t) = \int_\tau^\infty \ln(c_i(t)Q_i(t))e^{-\rho(t-\tau)}dt \tag{4}$$

with $Q_i(t)$ the (group-specific) index of consumed qualities over industries, $c_i(t)$ denoting the consumption of the homogeneous good, and $\rho$ the rate of time preference. We have $Q_i(t) = \int_0^1 x_i(s,t)q_j^i(s,t)ds$, with $x_i(s,t)$ an indicator function that takes the value 1 if a consumer belonging to group $i$ consumes good $s$ (and 0 otherwise), and $q_j^i(s,t)$ the quality level chosen by a type $i$ consumer. Households make consumption choices both within and across periods so as to maximise the above lifetime utility subject to the lifetime budget constraint

$$\int_\tau^\infty (c_i(t) + P_i(t))e^{-R(t,\tau)}dt \leq \omega_i(\tau) + \int_\tau^\infty we^{-R(t,\tau)}dt \tag{5}$$

where $R(t,\tau) = \int_\tau^t r(s)ds$ is the cumulative discount factor between times $\tau$ and $t$, $r(t)$ is the interest rate at time $t$, $\omega_i(\tau)$ is the initial wealth level held by a type $i$ consumer, and $P_i(t) = \int_0^1 x_i(s,t)p_j(s,t)ds$ is the (group-specific) price index associated with the quality good consumption index $Q_i(t)$.

First, the households' intra-temporal consumption choices are characterised. The first-order condition associated with the consumption of the homogenous good $c_i(t)$ is:

$$\frac{1}{c_i(t)} = \lambda_i(t) \tag{6}$$

[8] More precisely, it is assumed to be produced with a unit labour input of $1/w$ and to be competitively priced.

[9] This particular way of modelling non-homotheticity is the most classic in qualitative choice models featuring firms' strategic pricing (Gabszewicz and Thisse, 1980; Shaked and Sutton, 1982). Differences in the willingness to pay could also have been obtained by imposing exogenously different tastes for quality (Glass, 1997).

where $\lambda_i(t)$ is the marginal utility of wealth at time $t$ (i.e. the current-value multiplier). The first-order conditions for the **discrete consumption choice** of the quality-differentiated goods are of the form:

$$\{x_i(s,t), q_j^i(s,t)\} \tag{7}$$
$$= \begin{cases} \{1, k^{n(s,t)}\} & \text{if } \mu_i(t)k^{n(s,t)} - p_n(s,t) \geq \max\left[\mu_i(t)k^{n(s,t)-1} - p_{n-1}(s,t), ..., \mu_i(t) - p_0(s,t), 0\right] \\ ... & \\ \{1, 1\} & \text{if } \mu_i(t) - p_0(s,t) \geq \max\left[\mu_i(t)k^{n(s,t)} - p_n(s,t), ..., \mu_i(t)k - p_1(s,t), 0\right] \\ \{0, .\} & \text{otherwise} \end{cases}$$

where $\mu_i(t) = \frac{1}{\lambda_i(t)Q_i(t)}$ is household $i$'s willingness to pay per unit of quality. These f.o.cs state that: (1) for a type $i$ consumer, the purchase of a unit of quality good $s$ needs to represent a positive utility gain; and (2) provided a utility gain is possible, good $s$ will be purchased at the quality level that offers the highest gain *given* its price and the price of every other available quality within industry $s$.[10]

I now move on to the inter-temporal aspect of the consumer problem. First, the separability of utility (both over time and across goods) guarantees that for any given foreseen time path $P_i(t)$ of expenditures devoted to the continuum of quality goods, the optimal time path of consumption expenditures $c_i(t)$ on homogenous commodities has to fulfill the standard first-order condition of such an intertemporal maximisation problem:

$$\frac{\dot{c}_i(t)}{c_i(t)} = r(t) - \rho \tag{8}$$

The time path of $Q_i(t)$, on the other hand, is the combined result of (i) the successive stochastic quality jumps occurring in every quality-differentiated industry $s$, and (ii) the resulting optimal consumer choices *given* the prices charged for the different qualities available within industry $s$. I will hence go back to fully characterising it after having described firms' pricing and R&D decisions and established the BGP properties of such a model. It is already worth mentioning that in *product*-innovation models such as this one (i.e. without any mechanism ensuring productivity improvements), the quality index is the sole variable displaying a strictly positive growth rate along the balanced growth path (BGP), resulting in a positive growth of the consumer's utility.

Since the analysis carried out in this article pertains to the BGP properties of such a model, along which all variables remain constant or grow at a constant rate, I will from now on omit the functional dependence of the different variables on time so as to simplify notations.

## 3.2 Market Structure and Pricing

Within each industry $s \in [0,1]$, firms carry out R&D in order to improve the quality of the final consumption good $s$. Two types of firms can engage in R&D races: the current quality leader (incumbent), and followers (challengers). Within each sector $s$, each type of firm chooses an optimal amount to invest in R&D, with the probability of winning the next innovation race linearly increasing along the amount invested (cf. Subsection 3.3 for a

[10] For a similar consumer choice problem in an R&D-driven growth model, see Foellmi et al. (2014).

full description of the R&D technology). Since firms implement these investment decisions *considering* their expected profits in the case of a successful innovation, I proceed by **backward induction**, first detailing the optimal pricing strategy of a successful innovator (and the corresponding profits) in this subsection, *and then* determining the corresponding optimal investment in R&D in the next one.[11] Finally, for the sake of notation brevity, sector dependence will be dropped in both subsections.

Labour is the only input, with a constant unit labour requirement $a < 1$.[12]

Since the quality goods are characterised by unit consumption and fixed quality increments, firms use prices as strategic variables. Firms know the shares of groups $P$ and $R$ in the population, the respective incomes $y_R$ and $y_P$, as well as the preference structure of the consumers, but *cannot distinguish individuals by income.* In order to describe the strategic decisions made by active firms within a given industry, it proves convenient to define the threshold price $\tilde{p}^i_{\{j,j-m\}}$ for which a consumer belonging to group $i$ is indifferent between quality $j-m$ and quality $j$ in industry $s$, *given* the price $p_{j-m}$ charged for quality $j-m$. Determining such a threshold price amounts to solving the following equality, immediately derived from condition (7):

$$\mu_i k^j - \tilde{p}^i_{\{j,j-m\}} = \mu_i k^{j-m} - p_{j-m} \tag{9}$$

Considering the fact that $q_j = k^m q_{j-m}$, solving for $p^{Ti}_{\{j,j-m\}}$ in the above equality yields:[13]

$$\tilde{p}^i_{\{j-m,j\}} = k^{j-m}(k^m - 1)\mu_i + p_{j-m} \tag{10}$$

The price $\tilde{p}^i_{\{j-m,j\}}$ is the maximum price that the firm selling quality $j$ in industry $s$ can charge to a type $i$ consumer in order to have a positive market share, when competing against the firm selling quality $j-m$. As can be seen, this threshold price positively depends on type-$i$ consumers' willingness to pay $\mu_i = \frac{1}{\lambda_i Q_i}$ (with $\mu_R > \mu_P$), as well as on the price charged by the competitor $p_{j-m}$.

It is then possible to establish that two different situations are possible for each industry $s \in [0,1]$: either the current top quality good $q_n$ is sold to both groups of consumers, *or* the top quality good $q_n$ is sold to the rich consumers, while the second-best quality good $q_{n-1}$ is sold to the poor consumers (cf. Appendix A for a rigorous demonstration). It can also be established that the **decision regarding the market structure belongs to the producer of the highest quality** $q_n$, since it operates in a purely vertical market and hence *can always* set a price that will drive its competitors out; however, depending on the degree of inequality within the economy, the highest quality producer might not find it *optimal* (Gabszewicz et al., 1986). Finally, since both incumbents and challengers can

[11] In this subsection and the following one, I will assume that both incumbents and challengers invest a positive amount in R&D. Section 4 will provide a clear discussion of the parameter conditions guaranteeing the existence and uniqueness of such a multi-quality firm equilibrium.

[12] Given that the model assumes the unit consumption of the quality goods, $a$ must be smaller than 1.

[13] I resort to the assumption (classic in the monopolistic literature) that firms within a particular sector $s$ take the economy-wide willingness to pay for one unit of quality $\mu_i$ as given in their decision-making. Indeed, because of the existence of a *continuum* of quality good industries, firms within a given sector are "small in the big, but big in the small" (Neary, 2016): even though they resort to strategic pricing within their own industry, they do not take into account the impact of their pricing decisions on economy-wide variables such as $\lambda_i$ and $Q_i$.

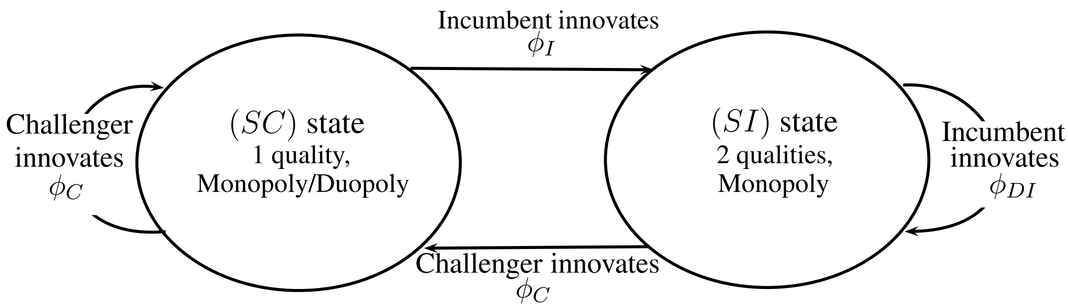
freely enter the successive patent races leading to the discovery of the next highest quality, the quality leader in a given market at a given time may either be a former incumbent (i.e. the firm to which the patent of the previously highest quality belonged ) or a new entrant: as will be shown below, the optimal pricing strategy chosen by the successful firm *also depends on its identity.*

To sum up, the pricing structure within each sector depends on two factors at any point in time: (i) the distribution of wealth across heterogeneous households, and (ii) the identity of the latest innovator (former incumbent or newcomer on the market). As a consequence, each industry $s \in [0, 1]$ actually fluctuates between two states (SC) and (SI) over time, depending on the stochastic outcome of the latest patenting race:

- **Successful Challenger (SC) state:** a *challenger* is the winner of the next R&D race, i.e. the new quality leader is a newcomer. In that case, it retains exclusive monopoly rights *only* for the highest quality $q_n$. As will be demonstrated below, *the market structure then depends on the income distribution within the economy.*

- **Successful Incumbent (SI) state:** the *current quality leader* is the winner of the next R&D race. In that case, it retains exclusive monopoly rights for *both* the highest quality $q_n$ and the second-best quality $q_{n-1}$. *The market structure is then necessarily a monopoly.*

The figure below illustrates the fluctuations between the two possible states over time. I will now discuss the market structure as well as the prices charged in the two existing states.




### 3.2.1 Prices and Profits in the (SC) State

In industries which are in the (SC) state, a challenger is the winner of the latest innovation race. The distance between this new leader and the competitive fringe (i.e. potential competitors with patent rights over lower qualities) is then only one rung along the quality ladder. That is, even when assuming that being able to produce a quality $q_j$ automatically grants the ability to produce any lower quality $q_{j-m}$ ($m \in [0; j]$), the new leader will face Bertrand competition for any quality below the frontier:[14] it will hence be able to extract monopoly rents (i.e. positive profits) solely from the sale of the highest quality $q_n$. The pricing strategy chosen by the leader for its unique quality $q_n$ then depends on the wealth distribution in the economy. The leader can decide to charge a price that enables it to

[14]Indeed, since the unit consumption of every quality good is imposed, firms necessarily use prices as strategic variables; also, the utility specification guarantees that different qualities are perfect substitutes (for an alternative set-up in which goods are imperfect substitutes and quality is considered a demand shifter rather than a purely vertical attribute, see Aghion et al., 2001).

capture the whole market (Gabszewicz et al., 1986), in which case the market structure is a **monopoly** with only the highest quality being sold. The leader may alternatively decide to charge a higher price and serve only the upper part of the market, leaving the lower part to the producer of quality $q_{n-1}$ (Gabszewicz and Thisse, 1980): the market structure is then a **duopoly**. In the main paper, I will limit myself to discussing the monopoly case at length.[15] The full discussion, exposition, and resolution of the duopoly case can be found in Online Appendix O2.

The optimal price chosen by a leader which wants to capture the whole market is $\tilde{p}^P_{\{n,n-1\}}$.[16] Assuming that the producer of quality $q_{n-1}$ engages in limit pricing (i.e. $p_{n-1} = wa$) and using (6) so as to obtain $\mu_i = \frac{c_i}{Q_i}$, the price $p_{SC}$ charged by the quality leader in a sector $s$ in the (SC) state is then of the form:

$$p_{SC} = k^n \left(\frac{k-1}{k}\right) \frac{c_P}{Q_P} + wa \tag{11}$$

The profits $\pi_{SC}(n)$ of a successful challenger in an industry $s$ where there have been $n$ successful innovations so far are then of the form:

$$\pi_{SC}(n) = k^n L \left(\frac{k-1}{k}\right) \frac{c_P}{Q_P} \tag{12}$$

### 3.2.2 Prices and Profits in the (SI) State

In an industry in the (SI) state, the former quality leader has won a second R&D race in a row, and retains exclusive monopoly rights for *both* the highest quality $q_n$ and the second-best quality $q_{n-1}$. According to Lemma 2 (cf. Appendix A), the market structure is then necessarily a *monopoly*; however, unlike the monopoly case in the (SC) state, the two highest qualities both have positive market shares. Indeed, the quality leader is two rungs above the competitive fringe along the quality ladder: facing two groups of consumers with different levels of income, the leader will hence be able to offer two distinct price-quality bundles so as to maximise its profit (Gabszewicz et al., 1986). The price charged by the monopolist for its second-best quality $q_{n-1}$ will be the maximal price enabling it to capture the poor group of consumers $\tilde{p}^P_{\{n-1,n-2\}}$, given that the producer of quality $q_{n-2}$ engages in limit pricing. Denoting this price by $p^P_{SI}$:

$$p^P_{SI} = k^{n-1} \left(\frac{k-1}{k}\right) \frac{c_P}{Q_P} + wa \tag{13}$$

[15] Indeed, as will become clear in the following sections, not only can the monopoly case be fully analytically solved and analyzed in terms of comparative statics, but it is also robust in most parametric cases (the duopoly case is actually only a possible equilibrium under some further conditions identified in Zweimuller and Brunner, 2005).

[16] Indeed, charging a price guaranteeing that poorer consumers buy the highest quality $q_n$ automatically ensures that richer consumers will consume the highest quality too, since $p^{Ti}_{\{n,n-1\}}$ is increasing along a consumer's willingness to pay $\mu_i = \frac{1}{\lambda_i Q_i}$

The price charged for the highest quality $q_n$ will then be $\tilde{p}^R_{\{n,n-1\}}$, given the price $p^P_{SI}$ charged for quality $q_{n-1}$. Denoting this price by $p^R_{SI}$:

$$p^R_{SI} = k^n \left(\frac{k-1}{k}\right) \frac{c_R}{Q_R} + k^{n-1} \left(\frac{k-1}{k}\right) \frac{c_P}{Q_P} + wa \tag{14}$$

The profits $\pi_{SI}(n)$ of a successful incumbent in an industry $s$ where there have been $n$ successful innovations so far are then of the form:

$$\pi_{SI}(n) = k^n L \left[(1-\beta)\left(\frac{k-1}{k}\right)\frac{c_R}{Q_R} + \left(\frac{k-1}{k^2}\right)\frac{c_P}{Q_P}\right] \tag{15}$$

### 3.2.3 Quality Consumption Indices

Denoting the shares of sectors respectively in the (SC) and the (SI) state by $\theta_{SC}$ and $\theta_{SI}$, it is finally possible to notice that quality consumption indices $Q_P$ and $Q_R$ entering the consumers' utility function take the following form in the case of a monopoly in the $(SC)$ state:

$$Q_P = \int_{\theta_{SC}} k^{n(s)} ds + \int_{\theta_{SI}} k^{n(s)-1} ds; \qquad Q_R = \int_0^1 k^{n(s)} ds \tag{16}$$

## 3.3 The R&D Sector

Within each industry $s \in [0,1]$, firms carry out R&D in order to discover the next quality level. Two types of firms can engage in R&D races: the current quality leader (incumbent), and followers (challengers). Free entry is assumed, with every firm having access to the same R&D technology within each sector $s$. Innovations are random, and occur for a given firm $f$ within sector $s$ according to a Poisson process of hazard rate $\phi_f(s)$. Labour is the only input, and constant returns to R&D are assumed at the firm level: in order to have an immediate probability of innovating of $\phi_f(s)$ in a sector $s$ having reached the quality level $k^{n(s)}$, a firm needs to hire $F\frac{k^{n(s)+1}}{Q}\phi_f(s)$ units of labour, with $F$ a positive constant and $Q = \int_0^1 k^{n(s)} ds$ the economy's quality index (i.e. the average quality level reached across sectors). This R&D cost function implies that R&D becomes more difficult in sectors that are too far ahead of the average technology reached in the rest of the economy.

As becomes clear when considering sector-specific profit functions (12) and (15), such a cost structure ensures that innovations become neither more profitable in sectors where there have already been more quality jumps, nor less profitable as the quality index grows: while the first case would ultimately lead to the disappearance of every sector but the most performant (and hence profitable) one, the second would lead to a no-growth steady state.[17] More precisely, this R&D sector specification guarantees that within the group of industries in a given state, the probability of innovating for a given type of firm (challengers or incumbents) will be the same in every sector, **regardless of the sector-specific rung $n(s)$ reached along the quality ladder**. This will lead to the survival of every quality

[17] This type of assumption is standard in quality-ladder models to ensure long-term balanced growth; it can be indifferently embodied in growing R&D costs for a given innovation probability *or* in a decreasing probability of innovating for given R&D costs (see Barro and i Martin (2003), Chapter 7).

sector along the BGP, and guarantees that every industry is symmetric with respect to transition probabilities from one state to the other.

For a given sector $s$ in which $n$ innovations have occurred so far, $v_C(n)$ is defined as the value of a challenger firm, $v_{SC}(n)$ as the expected present value of a quality leader having innovated once, and $v_{SI}(n)$ as the expected present value of a quality leader having innovated twice. Free entry and constant returns to scale imply that R&D challengers have no market value, whatever state the economy is in: $v_C(n) = 0$. The free entry of challengers in the successive R&D races also yields the traditional equality constraint between the expected profits of innovating for the first time $\phi_C v_{SC}(n)$ and the costs engaged $\phi_C \frac{k^{n+1}}{Q} wF$:

$$v_{SC}(n) = \frac{k^{n+1}}{Q} wF \tag{17}$$

The incumbent on the other hand participates in the race while it has already innovated at least once, and is hence the current producer of the leading quality for industries in the $(SC)$ state/of the **two** highest qualities for industries in the $(SI)$ state. In industries in the (SC) state, the incumbent faces the following Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} r v_{SC}(n) &= \max_{\phi_I \geq 0} \Big\{ \pi_{SC}(n) - wF \frac{k^{n+1}}{Q} \phi_I \\ &+ \phi_I \Big( v_{SI}(n+1) - v_{SC}(n) \Big) + \phi_C \Big( v_C - v_{SC}(n) \Big) \Big\} \end{aligned} \tag{18}$$

In an (SC)-state sector, the incumbent earns the profits $\pi_{SC}(n)$ and incurs the R&D costs $wF \frac{k^{n+1}}{Q} \phi_I$. With instantaneous probability $\phi_I$, the leader innovates once more, the industry jumps to the state (SI), and the value of the leader (now holding monopoly rights over two distinct qualities) climbs to $v_{SI}(n+1)$.[18] However, with overall instantaneous probability $\phi_C$, some R&D challenger innovates, and the quality leader falls back to being a follower: its value drops to $v_C = 0$. The industry then remains in the state (SC), and only one quality is produced.

In industries in the (SI) state, the quality-differentiated incumbent faces the following Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} r v_{SI}(n) &= \max_{\phi_{DI} \geq 0} \Big\{ \pi_{SI}(n) - wF \frac{k^{n+1}}{Q} \phi_{DI} \\ &+ \phi_{DI} \Big( v_{SI}(n+1) - v_{SI}(n) \Big) + \phi_C \Big( v_C - v_{SI}(n) \Big) \Big\} \end{aligned} \tag{19}$$

The incumbent in the (SI) state earns the profits $\pi_{SI}(n)$ of a monopolist able to discriminate between rich and poor consumers by offering two distinct price/quantity bundles. It incurs the R&D costs $wF \frac{k^{n+1}}{Q} \phi_{DI}$. With instantaneous probability $\phi_{DI}$, the incumbent innovates once more, in which case its value becomes $v_{SI}(n+1)$.[19] Hence, the incumbent will still be the producer of the two qualities sold, *but* will drive itself out of the market

[18] Accordingly to the crucial condition identified and discussed in the introduction as necessary to generate innovation by the incumbent, this expected value of innovating for a second time $v_{SI}(n+1)$ is different from the expected value of innovating for the first time $v_{SC}(n)$.

[19] It has indeed been established with Lemma 2 that at most two successive quantities are sold at equilibrium.

for the former quality $q_{n-1}$, which has become quality $q_{n-2}$ with the latest quality jump. The industry then remains in state (SI). With instantaneous probability $\phi_C$,[20] some R&D follower innovates, and the quality leader then falls back to being an R&D challenger: its value falls to $v_C = 0$. The industry then jumps to the state (SC), and only the new highest quality is sold by the latest successful innovator.

In both states, the incumbent firm chooses its R&D effort to maximise the right-hand side of its Bellman equation. (18) and (19) then yield the following first-order conditions:

$$\left(-wF\frac{k^{n+1}}{Q} + v_{SI}(n+1) - v_{SC}(n)\right)\phi_I = 0, \quad \phi_I \geq 0 \tag{20}$$

$$\left(-wF\frac{k^{n+1}}{Q} + v_{SI}(n+1) - v_{SI}(n)\right)\phi_{DI} = 0, \quad \phi_{DI} \geq 0 \tag{21}$$

For industries in the (SC) state, (20) yields a relationship between the R&D costs and the incremental value of a further innovation. Combined with (20), (21) entails either $\phi_{DI} = 0$ or $v_{SC}(n) = v_{SI}(n)$. The second possibility cannot be true, since $\pi_{SI}(n) > \pi_{SC}(n)$: hence, we necessarily have that $\phi_{DI} = 0$. Plugging (17), (20), and (21) in (18) and (19) and substituting for the profit values obtained in (12) and (15), it is possible to obtain the 2 following expressions, equating the incurred R&D costs to expected profits in both possible states:[21]

$$\frac{wF}{Q} = \frac{L\left(\frac{k-1}{k}\right)\frac{c_P}{Q_P}}{r+\phi_C} \tag{22}$$

$$\left(\frac{k+1}{k}\right)\frac{wF}{Q} = \frac{L\left[(1-\beta)\left(\frac{k-1}{k}\right)\frac{c_R}{Q_R} + \left(\frac{k-1}{k^2}\right)\frac{c_P}{Q_P}\right]}{r+\phi_C} \tag{23}$$

# 4 Balanced Growth Path Equilibrium

## 4.1 Labour Market Equilibrium

The equilibrium on the labour market can be characterised as follows. While challengers invest a total amount of $F\frac{k^{n(s)+1}}{Q}\phi_C$ in R&D in every industry $s$, incumbents only invest the amount $F\frac{k^{n(s)+1}}{Q}\phi_I$ in industries in the (SC) state. The total labour demand in the R&D sector is hence equal to $F\left(\int_0^1 \frac{k^{n(s)+1}}{Q}\phi_C ds + \int_{\theta_{SC}} \frac{k^{n(s)+1}}{Q}\phi_I ds\right)$. The unit consumption of the differentiated goods and identical marginal costs of production regardless of the quality level yield a total amount of $aL$ units of labour devoted to the production of the quality good. Finally, $(L/w)\left(\beta c_P + (1-\beta)c_R\right)$ are the units of labour devoted to the production of the standardised good.

[20] The challengers will invest the same amount in the R&D sector $\phi_C$ whether the sector considered is in state (SC) or (SI), since they face the same expected reward $v_{SC}(n+1)$ in both cases: a successful innovation by a challenger indeed always brings the industry back to state (SC).

[21] Considering (22) and (23), it is also confirmed that the conditions ruling the R&D investment decisions within one sector do not depend on any sector-specific value, but only on economy-wide variables: as announced in the beginning of the subsection, the probabilities of innovating are hence the same in every sector.

The following equation then describes the equilibrium on the labour market:

$$L = Fk\phi_C + Fk\phi_I \frac{\int_{\theta_{SC}} k^{n(s)} ds}{Q} + aL + (L/w)\left(\beta c_P + (1-\beta)c_R\right) \tag{24}$$

## 4.2 Balanced Growth Path Analysis

**Definition 1** *In the case of a monopoly market structure in the (SC) state, an equilibrium is defined by a time path for consumption of the homogenous good for both types of consumers* $\{c_i(t)\}_{i=(R,P),t=0}^{\infty}$ *satisfying* (6)*, a time path for the quality index for both types of consumers* $\{Q_i(t)\}_{i=(R,P),t=0}^{\infty}$ *satisfying* (7)*, time paths for innovation probabilities (and corresponding sector-specific R&D expenditures) by incumbents and challengers* $\{\phi_C(t), \phi_I(t)\}_{t=0}^{\infty}$ *satisfying* (17) *and* (20)*, time paths for sector-specific prices and profits* $\{p_{SC}(s,t), p_{SI}^P(s,t), p_{SI}^R(s,t), \pi_{SC}\big(n(s,t)\big), \pi_{SC}\big(n(s,t)\big)\}_{s\in(0,1),t=0}^{\infty}$ *satisfying* (11), (13), (14), (12)*, and* (15)*, and a time path for the interest rate* $\{r(t)\}_{t=0}^{\infty}$ *satisfying* (8)*.*

In addition, a balanced growth path (BGP) is defined as an equilibrium path along which every variable grows at a constant rate, either null or positive. In such a *product*-innovation model (i.e. precluding any productivity improvement) with fixed wage and population levels $w$ and $L$, the BGP is characterised by constant levels of innovation $\phi_C$ and $\phi_I$, overall wealth $\Omega$, and consumption $c_i$ $(i = R, P)$.[22] However, consumers are still better off over time due to the quality improvements of the differentiated goods and the resulting growth of individual utility. As already stated in the previous section, this paper focuses on such a BGP, whose properties will now be described.

### 4.2.1 Residual Consumption Levels $c_P$ and $c_R$ along the BGP

The R&D sector as well as the labour market equilibrium conditions (22), (23), and (24) depend on consumers' residual consumption of the standardised homogenous good $c_i$,[23] which will now be further characterised. First, constant levels of consumption imply that along the BGP, current income $y_i$ equates to current consumption $c_i + P_i$, i.e. $c_i = y_i - P_i$ once the long-run growth path is reached.

Using (11), (13), and (14), as well as the BGP properties of the random process governing the fluctuations between the two possible states (SC) and (SI) within every industry, it is then possible to obtain the following expressions for the economy-wide price indices $P_R$ and $P_P$ (see Appendix B for the demonstration and detailed computations):

$$P_P = \frac{(k-1)y_P + kwa}{2k-1} \tag{25}$$

$$P_R = \frac{(2k-1)(k-1)\theta_{SI} y_R + (k-1)k y_P + k^2 wa}{(2k-1)(k + \theta_{SI}(k-1))} \tag{26}$$

As can be seen considering (26), the economy-wide price index $P_R$ increases along the share $\theta_{SI}$ describing the proportion of industries in the (SI) state. This stems from the fact

[22] The consumption of the continuum of quality-differentiated goods is always constant, since unit consumption is imposed in this model.

[23] In the case of the R&D sector equilibrium conditions, it is through the willingness to pay $\mu_i = \frac{c_i}{Q_i}$.

that rich consumers pay *less* than their maximum threshold price in (SC)-state industries, while they are efficiently price discriminated (and hence pay a higher price) in (SI)-state industries. However (and importantly), *within* a specific sector $s$ in the (SI) state, the price $p_{SI}^R$ that the incumbent will be able to charge to the rich is *decreasing* along $\theta_{SI}$ (since $p_{SI}^R$ as presented in (14) depends positively on $\mu_R = \frac{y_R - P_R}{Q}$): the fewer the industries in the (SI) state, the higher the price the leaders will be able to charge to the rich consumers *within those industries.*

$P_P$ on the other hand does not depend on the shares $\theta_{SC}$ and $\theta_{SI}$: indeed, poor consumers pay their maximum threshold price in every industry.

Using (25) and (26), it is then finally possible to obtain the following expressions for the homogenous good consumption of both consumer groups $c_P$ and $c_R$:

$$c_P = \left(\frac{k}{2k-1}\right)(y_P - wa) \tag{27}$$

$$c_R = \frac{k\big((2k-1)y_R - (k-1)y_P - kwa\big)}{(2k-1)(k+(k-1)\theta_{SI})} \tag{28}$$

#### 4.2.2 Industry Shares along the BGP

Along the BGP equilibrium, it is possible to express $\theta_{SC}$ and $\theta_{SI}$ as functions of the innovation rates $\phi_C$ and $\phi_I$. Indeed, to ensure that $c_R$ as expressed in (28) remains constant, the share of industries in each state must remain constant. Hence, the inflows must equal the outflows of each state: the condition $\phi_C\theta_{SI} = \phi_I\theta_{SC}$ must be respected along the BGP.[24] Combining it with the fact that the two shares sum up to 1 (i.e. $\theta_{SC} + \theta_{SI} = 1$):

$$\theta_{SC} = \frac{\phi_C}{\phi_I + \phi_C}, \quad \theta_{SI} = \frac{\phi_I}{\phi_C + \phi_I} \tag{29}$$

#### 4.2.3 Labour Market Equilibrium along the BGP

The BGP properties of the random process governing the fluctuations between the two possible states (SC) and (SI) within every industry (cf. Appendix B for a full presentation ) make it possible to obtain the following expression for the long-run labour market equilibrium condition:

$$L = Fk\phi_C + \theta_{SC}Fk\phi_I + aL + (L/w)\left(\beta c_P + (1-\beta)c_R\right) \tag{30}$$

#### 4.2.4 Existence and Uniqueness of the BGP

Having now fully described the long-run properties of the economy, the parametric conditions ensuring the existence and uniqueness of a **multi-quality firm BGP** can be characterised.

**Proposition 1 (Existence and uniqueness of a steady-state equilibrium):** *For $k$ and $\beta$ sufficiently high and $d$ low enough to ensure $1-\beta-d>0$, there exists a unique*

[24] Indeed, for each industry in the (SC) state, the probability of exiting this state is equal to the probability $\phi_I(s)$ of an incumbent innovating; for each industry in the (SI) state, the probability of entering the (SC) state corresponds to the probability $\phi_C(s)$ of a challenger innovating.

*BGP along which (i) there is necessarily a monopoly in the (SC) state, (ii) both incumbents and challengers invest strictly positive amounts in R&D, and (iii) the economy-wide quality index grows at the constant rate* $\gamma = \frac{\dot{Q}}{Q} = (k-1)\phi_C(1 + \frac{\phi_I}{\phi_I + \phi_C})$.[25]

*Proof:* cf. Appendix C. □

Note that Proposition 1 implies not only that there exists a unique positive solution for the system of variables as defined in Definition 1, but also that the equilibrium with a monopoly market regime in the (SC) state is robust (existence), while its duopoly counterpart is not (uniqueness). In other words, a multi-quality firm BGP with a *monopoly market regime* in the (SC) state emerges only for parameter values respecting the conditions stated in Proposition 1 above. Online Appendix O2 similarly defines conditions under which there exists a unique and robust multi-quality firm BGP with a *duopoly market regime* in the (SC) state.[26] Outside those parameter constellations though, condition (20) yields $\phi_I = 0$ (i.e. incumbents do not find it optimal to invest positive amounts in R&D), and the model collapses to a multi-industry version of the Zweimuller and Brunner (2005) framework.

I now provide intuitions regarding the conditions on the exogenous parameters needed to obtain the result presented in Proposition 1.

Two conditions are necessary to guarantee a strictly positive amount invested in R&D by incumbents along the BGP (i.e. $\phi_I > 0$): $k$ high enough and $1 - \beta - d > 0$, which translates into d low enough when considering that $\beta$ needs to take high values (cf. below). The two conditions ensure that the gains from price discriminating are high enough to represent viable incentives for the incumbent to invest in R&D. First, $k$ represents the utility increment of consuming quality $q_n$ over quality $q_{n-1}$: the higher $k$, the higher the gap between $p_{SC}$ and $p^R_{SI}$ in any given industry. Second, $d$ is an inverse measure of the income gap between rich and poor consumers: the lower $d$, the higher the difference between $c_R$ and $c_P$, which again implies a greater profit increment for a successful incumbent.

The last condition ($\beta$ sufficiently high) is necessary to guarantee the existence and uniqueness of the obtained BGP. Regarding the existence, it is necessary to check that for the obtained equilibrium values of the endogenous variables, the monopoly market structure in the (SC) state is *robust*, i.e. the new leader does not prefer the alternative duopoly regime when comparing expected profits. Regarding the uniqueness, it is also necessary to make sure that the equilibrium values obtained when solving for a BGP with a duopoly market structure in the (SC) state (as defined in Online Appendix O2) do not define a robust equilibrium. Intuitively enough, high values of $\beta$ ensure that the monopoly is the **only** viable price regime: indeed, a leader facing a sizeable enough group of poor people will not be willing to abandon that part of the market to its direct competitor.

Proposition 1 states that in an economy where sufficiently strong disparities in purchasing power exist, the surplus appropriation effect identified in Section 2 efficiently counteracts Arrow's replacement effect, leading incumbents to invest in R&D beyond their first

[25] The BGP time paths of the quality consumption indices $Q_P$ and $Q_R$ can be directly derived from the growth rate of the quality index. We have $\frac{\dot{Q}_R}{Q_R} = \frac{\dot{Q}}{Q}$; on the other hand, along (16), we have $\frac{\dot{Q}_P}{Q_P} = (k-1)\phi_C\left(1 + \frac{\phi_I}{\phi_C + (1/k)\phi_I}\right)$.

[26] However, due to the impossibility of obtaining closed-form solutions, simulations must be used to determine the robustness conditions of the BGP in the case of a duopoly in the (SC) state.

successful innovation.[27] In this paper's general equilibrium framework, the immediate consequence of this result is the **endogenous emergence of multi-quality leaders in a dynamic quality-ladder model**, since income disparities generate both (1) the survival of more than one quality at the equilibrium, and (2) positive investment in R&D activities by incumbents.

The mechanism at the root of the surplus appropriation effect is product differentiation and the possibility of price discrimination (and hence increased profits) resulting from an expansion of the product portfolio. **Such a mechanism can only arise in vertically differentiated markets**, demonstrating the limits of the isomorphy between expanding-variety and quality-ladder growth models initially outlined by Grossman and Helpmann (1991). These authors indeed argued that in those two types of R&D growth models featuring patent-protected markets, agents invest in R&D to obtain the exclusive ability to manufacture a new product: whether this new product is a new variety or a new quality only marginally matters, *provided that firms get to behave as monopolists on the market for this newly created good.* On horizontally differentiated markets, monopolistic competition is indeed the natural outcome, with each new entrant being the sole producer of the new variety. In vertically differentiated markets, the monopoly power of the quality leader seems even more unquestionable, since firms compete head to head. As Shaked and Sutton (1983) described it: "the defining characteristic of this kind of product differentiation is that, were any two of the goods in question offered at the same price, then all consumers would agree in choosing the same one, i.e. that of higher quality."

While preserving the assumption of purely vertical competition, my model however endogenously shifts the market structure from monopolistic to potentially oligopolistic by allowing for different tastes in quality among consumers (stemming in this case from income differences). By doing so, I build on a broad industrial organisation literature demonstrating that quality-differentiated markets are in fact natural oligopolies (Shaked and Sutton, 1983) as soon as heterogeneous quality valuations are accounted for. In such markets, the optimal equilibrium outcome is then either a limited number of active firms vertically competing with each other (Gabszewicz and Thisse, 1980), or a multi-quality monopolist having optimally segmented the market (Gabszewicz et al., 1986).[28] While the first case is a possible outcome of my model (it corresponds to the duopoly in the (SC) state case, which Online Appendix O2 deals with), I focus in the main text (and in Proposition 1 above) on the second case, i.e. the monopoly case.

In both cases, the price discrimination possibilities stemming from vertical product differentiation drive R&D investment by incumbents through a surplus appropriation motivation. In the duopoly case, adding qualities to the incumbent's portfolio allows it to gain new market shares.[29] In the "monopoly case", offering a whole product range allows the

[27]This model is hence the first quality-ladder model since Aghion et al. (2001) to outline the existence of an innovation incentive stemming from the product market rather than from an incumbent advantage in R&D races. It is furthermore the first one to do so in a purely vertical framework: indeed, Aghion et al. (2001) exogenously impose a duopolistic market structure by assuming the existence of two firms producing imperfectly substitutable, quality-differentiated goods in each industry. The two competing firms are both horizontally and vertically differentiated: quality is then a demand shifter rather than a purely vertical attribute. In such a framework, the efficiency effect is at the root of incumbents' incentives to invest in R&D, since innovation can be viewed as a way to further differentiate from a direct competitor.

[28]Mussa and Rosen (1978) were the first to investigate the differentiation strategy of a multi-quality monopolist, but the existence of the latter is not an endogenous outcome of their seminal contribution.

[29]Gabszewicz and Thisse (1980) have demonstrated that in the case of constant production costs, the

quality leader to mitigate the effects of the negative heterogenous taste for quality externality identified by Mussa and Rosen (1978). Indeed, as extensively explained in Section 2, in a dynamic set-up such as mine, the negative externalities stemming from having to serve two distinct groups of consumers with different quality valuations are internalised by expanding the line of products upwards through R&D investment.

# 5 Income Distribution, Public Policies, and Long-Run Growth

The rich environment proposed by my model makes it possible to investigate the impact of several structural elements of the economic environment on the BGP growth rate. More precisely, I will first focus on the interactions between income distribution and long-run growth operating through the demand market, before considering the impact on long-run growth of R&D policies such as entry barriers and/or taxes on incumbents.

## 5.1 Income Distribution and Growth

Existing models investigating the impact of income inequality in a quality-ladder set-up pin down the R&D investment rate with a simple free-entry condition in the R&D sector, overlooking the possibility of incumbents investing in R&D, and concluding that there is an unambiguous detrimental impact of inequality on long-term growth (Zweimuller and Brunner, 2005). As will now be demonstrated, taking into account the possibility of second-degree price discrimination and the resulting participation of incumbents in innovation races overturns the predictions regarding the impact of varying inequality on R&D investment and the resulting long-run growth rate.

In the following analysis, two types of variations in the extent of wealth disparities are considered: (a) a larger income *gap* (i.e. a decrease in $d$ for a fixed level of $\beta$), and (b) a greater wealth *concentration*[30] (i.e. an increase in $\beta$ for a given $d$). The results of these comparative statics can be summarised in the following proposition:

**Proposition 2 (Wealth distribution and long-term growth)**:

(a) *Effect of a larger income gap (corresponding to a decrease in $d$): the challengers' innovation rate $\phi_C$ and the incumbent's innovation rate $\phi_I$ increase, resulting in an increase in the long-run growth rate $\gamma$. An increase in the income gap also leads to a greater share of R&D activities to be carried out by incumbents.*

(b) *Effect of a greater income concentration (corresponding to an increase in $\beta$): the challengers' innovation rate $\phi_C$ and the incumbent's innovation rate $\phi_I$ decrease, leading to a decrease in the long-run growth rate $\gamma$. An increase in income concentration also leads to a greater share of R&D activities to be carried out by challengers.*

number of firms which can coexist in a differentiated industry cannot exceed a finite value $\bar{n}$: the income distribution endogenously determines the maximal number of qualities that defines an industry, and entry from above in a fully served market (i.e. in a market in which $\bar{n}$ qualities are already offered) necessarily entails the exit of the firm selling the lowest quality, and an increase in the profits of the firm selling two of the consumed qualities.

[30] Indeed, a rise in the share of the poor population $\beta$ while keeping $d$ constant corresponds to a higher concentration of wealth within a smaller group of rich people. More precisely, it implies an increase in the *relative* income of a rich consumer ($\frac{\partial d_R}{\partial \beta} = \frac{1-d}{(1-\beta)^2} > 0$): there are more poor individuals with the same income, and fewer rich ones with more income.

*Proof*: cf. Appendix D. □

(a) Let us first comment on the effects of a larger income gap, i.e. a decrease in $d$.

To obtain intuitions regarding the variations in the R&D investment rates following a variation in the income gap, consider the impact of such a shock on the expected gains associated with successfully innovating for the first and the second time. First, since both $\beta$ and the quantities produced are fixed (the quality-differentiated industries face unit consumption), there can be no variation in the *market size* following an increase in the income gap: profit variations will derive from *price* adjustments.

First, let us examine the variation of $\phi_I$, i.e. the amount invested in R&D by the current incumbent operating in an (SC)-state industry. For this non-differentiated leader, the critical income when choosing how much to invest in R&D is that of rich households, since the *incremental* gain from innovating for a second time stems from the higher price charged to the upper end of the market. The income of rich households increases following the considered shock: hence, at a given level of wealth $\Omega$, a decrease in $d$ (i.e. a redistribution of wealth from the poor to the rich) has a *positive price effect* on the profits of a successful incumbent. The incentives to invest in R&D for an incumbent have hence become greater: $\phi_I$ increases.

The variation in $\phi_C$ is somewhat more difficult to rationalise a priori, since the exact counterpart of the above reasoning points to a *negative price effect* on the profits of a successful challenger: indeed, following a decrease in $d$, a successful challenger entering a given industry with only one quality at its disposal has to charge a *lower* price to capture the impoverished lower end of the market. This negative price effect is however counteracted by a less obvious positive price effect, linked to the variation in the share of industries $\theta_{SI}$ in the (SI) state. $\theta_{SI}$ indeed increases following the increase in $\phi_I$ discussed above. In sectors in the (SI) state, poor consumers are being sold quality $n-1$, which is one rung below the industry frontier. As a consequence, they tend to value the fewer industries in which they are sold the highest available quality more, i.e. the decreased share of sectors in the (SC) state. Successful challengers benefit from this greater appeal. This positive price effect is captured by the ratio $Q/Q_P$, present in the free-entry R&D condition (22) and increasing along $\theta_{SI}$.[31]

(b) I now move on to commenting on the effects of an increase in $\beta$ when there is a monopoly price regime in the $(SC)$ state.

Once again, let us first examine the impact of an increase in wealth concentration on the R&D investment of incumbents $\phi_I$. Following an increase in $\beta$ (and unlike a shock on $d$ which only generates variations in prices), there are both a market size effect *and* a price effect on the expected profits of a successful incumbent. Indeed, price-discriminating monopolists operating in (SI)-state industries can now charge a *higher* price, but to a *smaller* part of the population. Contrary to what happens in the horizontal differentiation case (Foellmi and Zweimuller, 2006), the negative market size effect systematically dominates here, and the incumbents' investment in R&D $\phi_I$ decreases. This difference between horizontal- and vertical-differentiation models can be rationalised through the fact that, in the case of vertical differentiation, the price effect is limited by the presence of a

[31] More precisely, we have $Q/Q_P = \frac{k}{k\theta_{SC}+\theta_{SI}}$, which can be transformed into $Q/Q_P = \frac{k}{k-(k-1)\theta_{SI}}$ using the property $\theta_{SC} + \theta_{SI} = 1$.

competitive fringe, which is not present in the case of horizontal differentiation.

Regarding variations of $\phi_C$ following a shock on $\beta$, the market size of a successful challenger is not altered by such a shock. The decrease in $\phi_C$ following an increase in wealth concentration then stems from the negative price effect resulting from a decrease in the ratio $Q/Q_P$ following the decrease in $\theta_{SI}$.

Several implications of these results can be identified. First, when asking "How does inequality affect investment in R&D and growth in a quality-ladder set-up?", the answer depends crucially on whether higher levels of inequality result from a larger income gap or from a higher income concentration. In the case of a larger income gap, only price effects are at play, leading to an increase in the R&D investment of both types of actors, and ultimately to a higher growth rate for the economy. In the case of increased wealth concentration on the other hand, the positive price effect is more than counterbalanced by a negative market size effect, leading to a decrease in the R&D investments of both types of actors. The two shocks also lead to different predictions in terms of the reallocation of the overall R&D bulk from one type of actor to the other.

Second, when comparing these predictions to those obtained in the case of expanding-variety growth models (Foellmi and Zweimuller, 2006; Foellmi et al., 2014), the nature of the differentiation considered (i.e. horizontal vs vertical) is crucial in order to predict the impact of varying inequality on R&D investment and growth. Indeed, Foellmi and Zweimuller (2006) have shown that in a horizontal-differentiation framework, higher levels of inequality are systematically positive for an economy's growth rate. The intuition pertains to the life cycle of products: lower levels of inequality induce a positive market size effect (the market for a new good develops faster into a mass market), but a negative price effect (the willingness to pay for a new product decreases with a less wealthy rich class). The latter always dominates the former, since profit flows *early* in the life cycle of the product matter more, and are lowered by a decrease in inequality.[32] However, this mechanism relies on the crucial assumption that a firm keeps *permanent monopoly rights* over a given good, without running the risk of being leapfrogged. In the case of a vertical-differentiation model in which firms compete head to head and push their competitors out of the market when innovating, the predictions are altered.

### 5.2 R&D Policies and Growth

Last, I will now analyse the effect of R&D policies on long-run growth. More precisely, it is assumed that the government can introduce two different types of taxes: a tax $\tau_c$ on the R&D expenditures of challengers (i.e. a barrier to entry), and a tax $\tau_i$ on the R&D expenditures of incumbents. In order to precisely identify the effect of these different policy instruments, three distinct scenarios are considered: (a) the taxation of R&D expenditures by challengers only ($\tau_c > 0$ and $\tau_i = 0$), (b) the taxation of R&D expenditures by incumbents only ($\tau_c = 0$ and $\tau_i > 0$), and (c) a balanced-budget policy in which the policy maker taxes R&D entrant expenditures and uses the generated revenues to subsidise R&D expenditures by incumbents ($\tau_c > 0$ and $\tau_i < 0$). As in Acemoglu (2008), this last scenario

[32] Foellmi et al. (2014) show that even when the monopolist can engage in process innovation to transform its luxury good into a product of mass consumption (hence engaging in a form of price discrimination), higher inequality levels still have a positive impact on growth *provided* the technological spillovers stemming from the introduction of mass production are not too important.

can in particular be interpreted as the enforcement of a patent policy in which entrants pay a fee to incumbents for partially benefitting from the accumulated knowledge. The impact of these different tax schemes can be summarised as follows:

**Proposition 3 (R&D policy and long-term growth)**:

(a) *Barrier to entry: following the introduction of a positive tax rate on challengers $\tau_c$, the challenger's innovation rate $\phi_C$ decreases, while the incumbent's innovation rate $\phi_I$ increases for small values of $\tau_c$. Barriers to entry lead to a lower growth rate $\gamma$, and to a higher R&D share to be carried out by incumbents.*

(b) *Taxation of incumbents: following the introduction of a positive tax rate on incumbents $\tau_i$, the challenger's innovation rate $\phi_C$ and the incumbent's innovation rate $\phi_I$ decrease, resulting in a decrease of the long-run growth rate $\gamma$. Taxes levied on incumbents also lead to a greater share of R&D activities to be carried out by challengers.*

(c) *Tax-and-subsidy scheme: taxing R&D expenditures by entrants and using the collected amounts to subsidise R&D expenditures by incumbents increases both the challenger's innovation rate $\phi_C$ and the incumbent's innovation rate $\phi_I$, resulting in an increase in the long-run growth rate $\gamma$. Such a tax scheme also leads to a greater share of R&D activities carried out by incumbents.*

*Proof*: cf. Appendix D. □

The intuitions for the effects of the different policy schemes are as follows.

(a) A positive tax rate on entrants' research activities diminishes the net value of leapfrogging for challengers, and logically leads to a decrease in their R&D investments. This lower replacement rate $\phi_C$ automatically increases both the current value of the active monopolist in every industry (since the latter now faces a lower probability of being replaced) *and* the expected value of efficient price discrimination in sectors in the (SC)-state, which leads to higher R&D investments by incumbents. A barrier-to-entry tax hence entails a reallocation of the R&D investments between the two types of innovative firms (challengers and incumbents). However, this shift of R&D expenditures from entrants to incumbents is systematically detrimental to growth. This last result can be rationalised when considering that in my model, the long-run growth rate takes the form $\gamma = (k - 1)\phi_C(1 + \frac{\phi_I}{\phi_I+\phi_C})$. Leapfrogging investments by entrants hence contribute more to the overall growth rate $\gamma$ than product differentiation investments by incumbents: indeed, while potential entrants are active in every industry, incumbents innovate only until they have reached the optimal differentiation degree, therefore leading to a sort of research attrition in (SI)-state sectors.[33]

(b) In my model, the introduction of a tax on incumbents $\tau_i$ is harmful for all actors in the economy. The negative impact of such a tax on incumbents' R&D is straightforward, since it diminishes the net gains of successful price discrimination. On the other hand,

[33]This result can be compared to those obtained by Acemoglu and Cao (2015) as well as Denicolo and Zanchettin (2012), who both consider taxes on entrants in models featuring R&D investment by both challengers and incumbents. While Denicolo and Zanchettin (2012) find a negative impact of barriers to entry on growth, Acemoglu and Cao (2015) find that the increase in incremental innovation by industry leaders more than compensates for the decrease in entrants' leapfrogging investments, leading to a positive impact on long-run growth.

such a tax also reduces the value of leapfrogging for potential entrants, not by increasing the R&D costs but by decreasing the associated expected profits. Indeed, the decrease in incumbents' R&D investment $\phi_I$ entails a negative price effect on the first-round profits of the non-discriminating monopolist, resulting from the increase in $\theta_{SC}$ (i.e. the share of industries in the (SC) state). With a higher share of industries in the (SC)-state, poor consumers are offered the highest quality available in a greater share of sectors, which reduces the price that a challenger can charge when entering a market (this effect is captured by the ratio $Q/Q_P$, present in the free-entry R&D condition (22) and decreasing along $\theta_{SC}$). Finally, this decrease in both $\phi_C$ and $\phi_I$ unambiguously leads to a lower growth rate of the economy.

A conclusion from this second exercise is that *subsidising* R&D by incumbents is systematically positive for the economy's long-run growth rate, since it increases both incumbents' and entrants' R&D investments. Such a policy however needs to be funded, and this is why the last exercise is carried out, in which the subsidies accruing to incumbents stem from a positive tax on entrants.

(c) While the impact of the third exercise on incumbents' R&D investment $\phi_I$ is straightforward (both a tax on entrants and a subsidy for incumbents' R&D lead to an increase in $\phi_I$), the overall impact of such a balanced-budget policy on both entrants' innovation *and* the overall growth rate was a priori ambiguous. Indeed, the positive effect of an incumbent subsidy on the net value of leapfrogging (i.e. the decrease in $\theta_{SC}$) competes against the negative effect of a tax on entrants. Proposition 3 shows that the positive price effect outweighs the negative cost effect, leading to an increase in entrants' R&D investment and a higher rate of long-run growth in the case of a tax-and-subsidy scheme.[34]

# 6 Conclusion

This paper accounts for the endogenous emergence of multi-quality firms within a Schumpeterian framework as soon as differences in the willingness to pay are allowed for among consumers. By doing so, it also identifies a further incentive for incumbents' R&D investments beyond the pre-emptive patenting motivation: in vertically differentiated markets, incumbents may carry out research to be able to perform more efficient price discrimination among consumers differing in their taste for quality. The paper also contributes to the analysis of the interactions between income distribution and long-term growth operating through the demand side, showing that while an increase in the income gap can be beneficial to growth, greater wealth concentration is systematically detrimental to the economy. Finally, I have demonstrated that a balanced-budget R&D policy consisting in the taxation of entrants to subsidise incumbents is beneficial to growth.

A model such as this one could naturally be applied to a two-country framework, in order to contribute to the developing literature studying the determinants and impact of vertical, intra-industrial trade (Fajgelbaum et al., 2011). Indeed, while the impact on growth of inter-industrial quality trade has already been extensively studied (product life cycle), the framework presented in this paper could be a good starting point for the elaboration of a dynamic model of intra-industrial quality trade (quality life cycle).

[34] This last result is in line with Denicolo and Zanchettin (2012), who similarly demonstrate that such a balanced-budget R&D policy of taxing entrants and subsidising incumbents leads to an increase in long-run growth.

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# A Firms' Optimal Pricing Strategy

I first establish the following lemma:

**Lemma 1:** *Within each industry $s \in [0,1]$, if $p_j \geq wa$ holds for the price of some quality $q_j$, then for the producer of any higher quality $q_{j+m}$, $1 \leq m \leq n(s) - j$, there exists a price $p_{j+m} > wa$ such that:*
***(i)*** *any consumer prefers quality $q_{j+m}$ to $q_j$,*
***(ii)*** *she makes strictly positive profits.*

*Proof:* Considering (10), it is straightforward that $\tilde{p}^i_{\{j+m,j\}} > p_j$. Hence, it is always possible for the producer of the quality $j+m$ to set a price $p_{j+m} > p_j \geq wa$ such that $p_{j+m} \leq \tilde{p}^i_{\{j+m,j\}}$, i.e. such that quality $q_{j+m}$ is preferred to quality $q_j$ by the consumers of group $i$. □

Hence, within each industry $s$, when taking for granted that a producer never sells its quality at a price below the unit production cost $wa$, it is always possible for the producer of the highest quality to drive all of its competitors out of the market while still making strictly positive profits. Along this result, any firm entering industry $s$ with a new highest quality $q_n$[35] has to consider the following trade-off concerning the pricing of its product: **setting the highest possible price for any given group of costumers vs lowering its price in order to capture a further group of consumers**.

It is then possible to show that in an economy characterised by two distinct groups of consumers (R and P), we have:

**Lemma 2:** *Within each sector $s \in [0,1]$, we have that, at equilibrium,*
***(1)*** *The highest quality is necessarily produced,*
***(2)*** *At most the two highest qualities $q_n$ and $q_{n-1}$ are actually produced,*

*Proof:* This article focuses on parameter cases in which growth occurs within the economy, i.e. it postulates that there exists at least one sector $s$ in which the quality good is consumed along the BGP. Hence, we necessarily have $\mu_i \geq wa$, with $wa$ the lowest price any producer can charge. Similar production costs across sectors, time, and quality levels then immediately entail that every quality can be produced and sold with positive profits. Along Lemma 1, higher qualities drive out lower ones: *provided the parametric conditions guarantee growth within the economy*, the producer of the highest quality within each sector $s$ will hence always be able to set a profitable price such that its quality is preferred to any other by a given consumer in group $i$. Furthermore, since there are only two consumer groups within the economy, at most two distinct qualities can be sold within each sector. This ends the proof. □

# B Computation of the Price Indices $P_R$ and $P_P$

Poor consumers pay $p_{SC}(s)$ for goods being produced in (SC)-state industries, and $p^P_{SI}(s)$ for goods being produced in (SI)-state industries. We hence have $P_P = \int_{\theta_{SC}} p_{SC}(s)ds + \int_{\theta_{SI}} p^P_{SI}(s)ds$. Rich consumers also pay $p_{SC}(s)$ in industries being in the (SC) state, and $p^R_{SI}(s)$ in industries being in the (SI) state: we have $P_R = \int_{\theta_{SC}} p_{SC}(s)ds + \int_{\theta_{SI}} p^R_{SI}(s)ds$.

[35] Along the definition of the quality ladder in a given sector provided in Subsection 2.1., $n$ indeed designates the total number of innovations that have taken place so far within a given sector $s$.

Using (11), (13) and (14), we then obtain the following expressions for the two economy-wide indices:

$$P_P = \int_{\theta_{SC}} k^{n(s)} \left(\frac{k-1}{k}\right) \frac{(y_P - P_P)}{Q_P} ds + \int_{\theta_{SI}} k^{n(s)-1} \left(\frac{k-1}{k}\right) \frac{(y_P - P_P)}{Q_P} ds$$

$$P_R = \int_{\theta_{SC}} k^{n(s)} \left(\frac{k-1}{k}\right) \frac{(y_P - P_P)}{Q_P} ds + \int_{\theta_{SI}} \left(k^{n(s)} \left(\frac{k-1}{k}\right) \frac{y_R - P_R}{Q_R} + k^{n(s)-1} \left(\frac{k-1}{k}\right) \frac{y_P - P_P}{Q_P}\right) ds$$

Considering the fact that along (16) we have $Q_P = \int_{\theta_{SC}} k^{n(s)} ds + \int_{\theta_{SI}} k^{n(s)-1} ds$, it is straightforward to simplify the first expression into $P_P = \left(\frac{k-1}{k}\right)(y_P - P_P)$, which itself yields:

$$P_P = \frac{(k-1)y_P + kwa}{2k-1}$$

Since $Q_R = \int_0^1 k^{n(s)} ds$, the above expression for $P_R$ simplifies to:

$$P_R = \left(\frac{k-1}{k}\right)(y_P - P_P) + \left(\frac{k-1}{k}\right)(y_R - P_R) \underbrace{\frac{\int_{\theta_{SI}} k^{n(s)} ds}{\int_0^1 k^{n(s)} ds}}_{=(*)} + wa$$

**Proposition 4:** *Along the BGP (i.e. for $t$ big enough), the number of quality jumps per unit of time $\frac{n(s,t)}{t}$ can be approximated by the same constant $\nu$ in every industry $s$.*

*Proof:* We focus on one sector $s$ which has just jumped back from state (SI) to state (SC). The operating firm on the market is hence a former challenger. There will then be a certain number $M-1$ (with $M \in [1, +\infty[$) of "investment races" for which the considered sector remains in the state (SC), i.e. for which the winner of the race is a challenger. At some point though, the winner of the $M$th race ends up to be an incumbent: the sector switches to state (SI). Since incumbents stop investing in R&D once having fully price-discriminated, the next jump will *necessarily* bring the sector back to state (SC), and a new cycle begins. We denote the number of innovations having occurred in this cycle as $W_c = M + 1$, where $c$ is the index of the cycle. $W_c$ can be viewed as a "reward", with $S_c$ being the holding time until this reward is reached. The couples $(S_1, W_1)$, $(S_2, W_2)$,... are i.i.d. random variables. The total number of innovations at a given time $t$ for which $X_t$ cycles have already occurred is $Y_t = \sum_{c=1}^{X_t} W_c$. $Y_t$ is a "renewal-reward process": it depends on the holding times $S_1$, $S_2$,... between two cycles, and on the rewards corresponding to each cycle $W_1$, $W_2$,... Along the BGP, we can then apply the strong law of large numbers for renewal-reward processes, which states that $\lim_{t\to\infty} \frac{Y_t}{t} = \frac{E[W_1]}{E[S_1]}$. Since every industry $s \in [0,1]$ displays exactly the same immediate innovation probabilities for incumbents ($\phi_I$) and challengers ($\phi_C$), the number of innovations $n(s,t)$ in every sector $s$ can be described by a renewal-reward process identical to the one described above. We then have that the number of quality jumps per unit of time $\frac{n(s,t)}{t}$ is equal to the same constant $\nu$ in every industry $s$ for $t$ big enough, with $\nu = \frac{E[W_1]}{E[S_1]}$. This ends the proof. □

Using Proposition 3, we have $(*) = \frac{k^{\nu T} \int_{\theta_{SI}} ds}{k^{\nu T} \int_0^1 ds} = \theta_{SI}$, which leads to finally obtaining the following expression for $P_R$:

$$P_R = \frac{(2k-1)(k-1)\theta_{SI} y_R + (k-1)k y_P + k^2 wa}{(2k-1)(k + \theta_{SI}(k-1))}$$

## C Existence and Uniqueness of the BGP in the Monopoly Case

Along the BGP, (8) implies that we have $r = \rho$. Using Proposition 3 (cf Appendix B) as well as (29), we also have $\frac{Q}{Q_P} = \frac{k(\phi_I+\phi_C)}{k\phi_C+\phi_I}$ along the BGP. Using (22) so as to substitute for $c_P$ in (23) and keeping in mind that $y_R$ and $y_P$ are linear in $\Omega$ along (3), characterising the BPG then boils down to solving the following 5 equations for values of $\phi_C$, $\phi_I$, $c_R$, $c_P$ and overall wealth $\Omega$:

$$wF(\rho+\phi_C) = L\left(\frac{k-1}{k}\right)c_P\frac{k(\phi_I+\phi_C)}{k\phi_C+\phi_I} \tag{31}$$

$$wF(\rho+\phi_C) = L(1-\beta)\left(\frac{k-1}{k}\right)c_R \tag{32}$$

$$wL = wFk\phi_C + \frac{wFk\phi_I\phi_C}{\phi_I+\phi_C} + awL + L\left[\beta c_P + (1-\beta)c_R\right] \tag{33}$$

$$c_P = \left(\frac{k}{2k-1}\right)(y_P - wa) \tag{34}$$

$$c_R = \frac{k(\phi_I+\phi_C)\big((2k-1)y_R - (k-1)y_P - kwa\big)}{(2k-1)(k\phi_C + (2k-1)\phi_I)} \tag{35}$$

We first notice that since $a < 1$ and $y_R > y_P$, positive values of $\phi_C$, $\phi_I$ and of the overall wealth $\Omega$ necessarily entail positive values for $c_P$ and $c_R$ along (34) and (35). Using (31) and (32) so as to substitute for $c_P$ and $c_R$, (33) then yields the following expression for $\phi_I$ as a function of $\phi_C$:

$$\phi_I = \phi_C\frac{Fk(\rho(1+\beta)+(k+\beta)\phi_C) - (1-a)(k-1)L}{(1-a)(k-1)L - F(k(\rho-\phi_C) + 2k^2\phi_C + \beta(\rho+\phi_C))} \tag{36}$$

Equating the right-hand sides of (31) and (32) and using (34) and (35) to substitute for $c_P$ and $c_R$, it is also possible to express $\Omega$ as a function of $\phi_C$ and $\phi_I$:

$$\Omega = \frac{w\left(F(\rho+\phi_C)(2k-1)(k\phi_C+\phi_I) - k(k-1)(1-a)L(\phi_I+\phi_C)\right)}{d(k-1)k\rho(\phi_C+\phi_I)} \tag{37}$$

Substituting for $\Omega$ and $\phi_I$ using (36) and (37), it is finally possible to transform (31) into a second-degree polynomial in $\phi_C$, which when solved yields:

$$\phi_C = \frac{A \pm \sqrt{B + A^2}}{D}$$

with $A$, $B$ and $D$ having the following analytical expressions:

$$A = (1-a)(k-1)L\Big((1-d)(k^2-1) + k(2-\beta)\Big) - Fk\rho\Big(4k^2 - 1 + d(k(2+3\beta) + 1 - 2k^2(1+\beta))\Big) \tag{38}$$

$$B = 8Fk^2\rho\big(2k-1-d(k-1)(1+\beta)\big)\Big((1-a)(k-1)L(d+k(2-\beta)-1) - Fk\rho(2k+d(1+\beta k)-1)\Big) \tag{39}$$

$$D = 4Fk^2(2k-1-d(k-1)(1+\beta)) > 0 \tag{40}$$

We now define several expressions, that will prove useful for identifying the conditions ensuring positive values for $\phi_C$, $\phi_I$ and $\Omega$.

$$\begin{aligned}
(a) &= \left(\frac{k}{k-1}\right)(\rho(1+\beta)+\phi_C(k+\beta)) \\
(b) &= (1-a)L/F \\
(c) &= \frac{\rho(k+\beta)+\phi_C(k(2k-1)+\beta)}{k-1} \\
(d) &= (\rho+\phi_C)\left(\frac{2k-1}{k(k-1)}\right)\left(\frac{k\phi_C+\phi_I}{\phi_C+\phi_I}\right) \\
(e) &= \rho\left(\frac{k}{k-1}\right)\frac{2k-1+d(1+\beta k)}{2k-1+d-\beta k}
\end{aligned}$$

$(b) > (e)$ is sufficient (but not necessary) so as to ensure that we obtain a unique positive solution for $\phi_C$, since it entails $B > 0$, which itself implies $\sqrt{A^2+B} > A$ regardless of the sign of $A$. We then have $\phi_{C1} = \frac{A-\sqrt{A^2+B}}{D} < 0$, and:

$$\phi_{C2} = \frac{A+\sqrt{A^2+B}}{D} > 0 \tag{41}$$

We can then see from (36) that a positive solution for $\phi_I$ is obtained provided that we have $\phi_C > 0$ and $(a) > (b) > (c)$. Finally, for positive values of $\phi_C$ and $\phi_I$, the condition $(d) > (b)$ is sufficient for (37) to yield a positive value for the overall wealth $\Omega$. We need to ensure that those different inequalities are compatible. So as to guarantee a unique positive solution for $\phi_C$, $\phi_I$ and $\Omega$, the following conditions are hence sufficient (but not necessary):

$$(d) > (a) > (b) > (e) > (c) \qquad (*)$$

We will now determine the conditions on the parameters of the model for this string of inequalities to be respected.

First, since none of the parameters appearing in $(b)$ are present in $(e)$, **it is always possible to choose values of $a$, $F$ and $L$ such that** $(b) > (e)$, ensuring $\phi_C > 0$. For the following demonstrations, we also establish that we have $\lim_{k\to\infty}\phi_C = \frac{(1-a)(1-d)L-2F\rho(2-d(1+\beta))+\sqrt{((1-a)(1-d)L-2F\rho(2-d(1+\beta)))^2}}{4F(2-(1+\beta)d)}$, which is equal to zero provided we have $\frac{2\rho(2-d(1+\beta))}{1-d} > (b)$. We assume for now that this condition is met, and will show below that it is in fact implied by $(*)$. The following limits immediately follow and will also prove useful: $\lim_{k\to\infty}(1/k)\phi_C = 0$, $\lim_{k\to\infty}k\phi_C = 0$, $\lim_{k\to\infty}k^2\phi_C = \infty$.

Using those results, we have that $(a) \to \rho(1+\beta)$ and $(e) \to \rho\left(\frac{2+\beta d}{2-\beta}\right)$ as $k \to \infty$. **We hence have $(a) > (e)$ for sufficiently high values of $k$ and provided we have $1+\beta > \frac{2+\beta d}{2-\beta}$, which can be rearranged as** $1-\beta-d > 0$. Again, since those two conditions do not put any structure on the parameters present in $(b)$, **it is then always possible to choose values of $a$, $F$ and $L$ such that** $(a) > (b) > (e)$.

Noticing that $(c) \to \rho$ as $k \to \infty$ and using the limit of $(e)$ established above, it immediately follows that **(e)>(c) for sufficiently high values of $k$**.

Finally, dividing both the numerator and the denominator of $\phi_I$ as given in (36) by $k$, we obtain $\phi_I = \phi_C\frac{E}{G}$ with $E = F\rho(1+\beta)-(1-a)L+Fk\phi_C+F\beta\phi_C+(1/k)(1-a)L$ and $G = (1-a)L-F\rho+F\phi_C-2Fk\phi_C-(1/k)F\beta\rho-(1/k)\phi_C F\beta$. Using the limits established above, we have $\lim_{k\to+\infty}\left(\frac{E}{G}\right) = \frac{F\rho(1+\beta)-(1-a)L}{(1-a)L-F\rho} > 0$,[36] which entails $\lim_{k\to+\infty}(\phi_I) = 0$. It then follows that $\lim_{k\to\infty}(d) = 2\rho > (1+\beta)\rho = \lim_{k\to\infty}(a)$. This entails that **we necessarily have (d)>(a) for sufficiently high values of $k$.**

The intuitions concerning those conditions on the parameter values are commented in the main text of this paper.

The set of parametric conditions $(*)$, respected for values of $\beta$ and $d$ such that $1-\beta-d > 0$ and for sufficiently high values of $k$, are hence sufficient so as to guarantee that **there exists systematically necessarily one (and no more than one) positive solution** for $\phi_C$, $\phi_I$ and $\Omega$, entailing positive solutions for $c_P$ and $c_R$ respectively given by (34) and (35).

Proving that the system of five equations (31)-(35) admits a unique and positive solution in $(\phi_I, \phi_C, \Omega, c_P, c_R)$ is however not sufficient so as to demonstrate the existence and the uniqueness of the defined BGP. Indeed, we have assumed from the beginning that the equilibrium market structure chosen by the new leader in the (SC) state is a **monopoly**. We also need to check that for the obtained values for $c_P$, $c_R$, $\phi_C$, $\phi_I$ and $\Omega$, this specific price regime indeed represents a *robust* equilibrium, i.e. **the new leader does not prefer the alternative regime when comparing expected profits**. We demonstrate in online Appendix O2 (where we formally expose and discuss the "duopoly case") that under the parametric conditions $(*)$, the positive equilibrium described by $(31)-(35)$ defines a unique and robust BGP where the market structure in the (SC) case is necessarily a monopoly. This ends the proof. □

# D Demonstration of the Comparative Statics

## Income Distribution and Growth

### Comparative Statics in the Case of a Variation in $d$

We first consider the variation of $\phi_C = \frac{A+\sqrt{B+A^2}}{D}$ (the values of $A$, $B$ and $D$ being defined by (38), (39) and (40) in Appendix C) following a shock on $d$:

$$\frac{\partial \phi_C}{\partial d} = \left( \underbrace{\left(1+\frac{A}{\sqrt{\Delta}}\right)\frac{\partial A}{\partial d}}_{T_1} + \underbrace{\frac{\partial B}{\partial d}\frac{1}{2\sqrt{\Delta}}}_{T_2} \right) D - \underbrace{(A+\sqrt{\Delta})\frac{\partial D}{\partial d}}_{T_3} \tag{42}$$

with $\Delta = B + A^2$, and the following forms for $\frac{\partial A}{\partial d}$, $\frac{\partial B}{\partial d}$ and $\frac{\partial D}{\partial d}$:

$$\frac{\partial A}{\partial d} = Fk\rho\left(2k^2(1+\beta)-1-k(2+3\beta)\right) - (1-a)(k-1)^2(k+1)L \tag{43}$$

$$\frac{\partial B}{\partial d} = 8Fk^2\rho\Big[\underbrace{(2k-1-d(k-1)(1+\beta))((1-a)(k+1)L - Fk\rho(1+\beta k))}_{B_{d1}} \underbrace{-(k-1)(1+\beta)\Big((1-a)(k-1)L(d+k(2-\beta)-1) - Fk\rho(2k+d(1+\beta k)-1)\Big)}_{B_{d2}}\Big] \tag{44}$$

$$\frac{\partial D}{\partial d} = -4dFk^2(k-1)(1+\beta) < 0 \tag{45}$$

We will now show that under the parametric conditions guaranteeing the existence and uniqueness of the BGP (cf Appendix C), the sign of (42) is ultimately determined by

[36] This expression is indeed necessarily positive under $(a) > (b) > (c)$, since the latter entails $\rho(1+\beta)F > (1-a)L > \rho F$.

the sign of $\frac{\partial B}{\partial d}$. So as to proceed to this demonstration, we need to define the following expression:

$$\begin{aligned}(f) &= \rho\left(\frac{k}{k-1}\right)\frac{2(2-d(1+\beta))k^2+d(2+3\beta)k-1+d}{(1-d)(k^2-1)+k(2-\beta)}\\ (g) &= \rho\left(\frac{k}{k-1}\right)(1+\beta k)\end{aligned}$$

As it can be seen considering (38), the sign of $A$ (that we did not need to establish so as to prove that $\phi_C > 0$ as long as $B > 0$) is given by the sign of $(b)-(f)$. As $k \to +\infty$, we have that $(f) \to \frac{2(2-d(1+\beta))}{1-d}\rho > 2\rho = \lim_{k\to\infty}(d)$. Under condition (*) (cf Appendix C), we hence necessarily have that $(f) > (d) > (b)$ for high enough values of $k$, which entails $A < 0$. Considering (38) and (39), one can furthermore see that $B$ is a polynomial of degree 5 in $k$, while $A$ is a polynomial of degree 3: for high enough values of $k$, we have that $B$ becomes negligible in $\sqrt{B+A^2}$, which can be approximated by $\sqrt{A^2} = |A|$. We hence have that as $k \to +\infty$, $\frac{A}{\sqrt{\Delta}} \to 1^-$ and $A + \sqrt{\Delta} \to 0^+$. As a consequence, the terms $T_1$ and $T_3$ become negligible, and the sign of $\frac{\partial \phi_C}{\partial d}$ is determined by the sign of $T_2$, i.e. of $\frac{\partial B}{\partial d}$ since we have $D > 0$.

We have $2k-1-d(k-1)(1+\beta) < 0$, and for $k$ sufficiently high we necessarily have $(g) > (b)$, which ensures that the term $B_{d1}$ of (44) is negative. We also have $B_{d2} < 0$ since $(b) > (e)$ (cf condition (*) in Appendix $C$). As a consequence, we have $\frac{\partial B}{\partial d} < 0$, which entails $\frac{\partial \phi_C}{\partial d} < 0$: **under the conditions ensuring the existence and uniqueness of the BGP, $\phi_C$ is decreasing along $d$.**

We then move to determining the variations of $\phi_I$ following a shock on $d$:

$$\frac{\mathrm{d}\phi_I}{\mathrm{d}d} = \frac{\partial \phi_I}{\partial d} + \frac{\partial \phi_I}{\partial \phi_C}\frac{\partial \phi_C}{\partial d}$$

We first notice that $\frac{\partial \phi_I}{\partial d} = 0$. Using (36), we then reformulate $\phi_I = \phi_C \frac{H}{I}$ with $H = F(k-1)((a)-(b)) > 0$ and $I = F(k-1)((b)-(c)) > 0$. Since we have $\frac{\partial H}{\partial \phi_C} = Fk(k+\beta)$ and $\frac{\partial I}{\partial \phi_C} = -F(k(2k-1)+\beta)$, we obtain the following expression for $\frac{\partial \phi_I}{\partial \phi_C}$:

$$\frac{\partial \phi_I}{\partial \phi_C} = \frac{HI + \phi_C IFk(k+\beta) + \phi_C HF(k(2k-1)+\beta)}{I^2} > 0 \tag{46}$$

(46), along with our above result that $\frac{\partial \phi_C}{\partial d} < 0$, yields $\frac{\mathrm{d}\phi_I}{\mathrm{d}d} < 0$: **under the conditions ensuring the existence and uniqueness of the BGP, $\phi_I$ is decreasing along $d$.**

Finally, the variation of $\theta_{SI}$ can be determined considering the equilibrium condition (31) (cf. Appendix C), which is a reformulation of the R&D free-entry condition (22) and in which $Q/Q_P = \frac{k}{k\theta_{SC}+\theta_{SI}}$ appears in the right-hand side (RHS). Using $\theta_{SC} + \theta_{SI} = 1$, we have $Q/Q_P = \frac{k}{k-(k-1)\theta_{SI}}$, and obtain that this ratio is increasing along $\theta_{SI}$. Following an increase in $d$, $c_P$ increases and $\phi_C$ decreases, moving the RHS and the LHS of (31) in two opposite directions. So as to re-establish the equality, $Q/Q_P$ needs to decrease: **we necessarily have that $\theta_{SI}$ has decreased following the positive shock on $d$.** □

**Comparative Statics in the Case of a Variation in $\beta$**

Similarly, we first consider the variation of $\phi_C$ following a shock on $\beta$:

$$\frac{\partial \phi_C}{\partial \beta} = \left(\left(1+\frac{A}{\sqrt{\Delta}}\right)\frac{\partial A}{\partial \beta} + \frac{\partial B}{\partial \beta}\frac{1}{2\sqrt{\Delta}}\right) D - (A+\sqrt{\Delta})\frac{\partial D}{\partial \beta} \tag{47}$$

with the following form for $\frac{\partial B}{\partial \beta}$:

$$\begin{aligned}\frac{\partial B}{\partial \beta} &= 8Fk^2\rho\Big[\underbrace{-k(2k-1-d(k-1)(1+\beta))((1-a)(k-1)L+dFk\rho)}_{B_{\beta 1}} \\ &\underbrace{-d(k-1)\Big((1-a)(k-1)L(d+k(2-\beta)-1)-Fk\rho(2k+d(1+\beta k)-1)\Big)}_{B_{\beta 2}}\Big]\end{aligned} \tag{48}$$

The reasoning we presented when analyzing $\frac{\partial \phi_C}{\partial d}$ still holds here: under the parametric conditions guaranteeing the existence and uniqueness of the BGP, the sign of (47) is ultimately determined by the sign of $\frac{\partial B}{\partial \beta}$. We unequivocally have $B_{\beta 1} < 0$, and since $(b) > (e)$ we have $B_{\beta 2} < 0$. As a consequence, we have $\frac{\partial B}{\partial \beta} < 0$, which entails $\frac{\partial \phi_C}{\partial \beta} < 0$: **under the conditions ensuring the existence and uniqueness of the BGP, $\phi_C$ is decreasing along $\beta$.**

We then move to determining the variations of $\phi_I$ following a shock on $\beta$:

$$\frac{\mathrm{d}\phi_I}{\mathrm{d}\beta} = \frac{\partial \phi_I}{\partial \beta} + \frac{\partial \phi_I}{\partial \phi_C}\frac{\partial \phi_C}{\partial \beta}$$

We have already established that $\frac{\partial \phi_C}{\partial \beta} < 0$ and $\frac{\partial \phi_I}{\partial \phi_C} > 0$. We finally have the following partial derivative of $\phi_I$ w.r.t. $\beta$:

$$\frac{\partial \phi_I}{\partial \beta} = \frac{F(k-1)\phi_C(\rho+\phi_C)((1-a)(k-1)L - Fk(\rho+2k\phi_C))}{\Big(F(k(\rho-\phi_C)+2k^2\phi_C+\beta(\rho+\phi_C)) - (1-a)(k-1)L\Big)^2} \tag{49}$$

For $k$ high enough and under the condition $(b) > (c)$ we have that $(1-a)L/F > \left(\frac{k}{k-1}\right)(\rho+2k\phi_C)$, which ensures that (49) is positive. The overall sign of $\frac{\mathrm{d}\phi_I}{\mathrm{d}\beta}$ is hence left to be ambiguous. This ambiguity can however be lifted considering the equilibrium condition (31) (cf. Appendix C). Following a positive shock on $\beta$, the LHS decreases through the decrease in $\phi_C$. So as to re-establish the equality, the RHS needs to decrease identically, which necessarily entails that $\theta_{SI}$ **has decreased following the shock on** $\beta$ (cf our detailed argumentation regarding the variation of $\theta_{SI}$ in the case of a positive shock on $d$ above). Since $\phi_C$ decreases along $\beta$, $\phi_I$ needs to decrease even more so as to obtain a decrease in $\theta_{SI}$: we hence necessarily have that $\phi_I$ **is also a decreasing function of** $\beta$. □

## R&D Taxation Policy

### Comparative Statics in the Case of a Variation in $\tau_c$

For the sake of concision and readability of the demonstrations, we temporarily assume that the tax rate $\tau_i$ applied to incumbents is set to zero. The equilibrium condition (31) is now of the form:

$$wF(\rho+\phi_C)(1+\tau_c) = L\left(\frac{k-1}{k}\right)c_P\frac{k(\phi_I+\phi_C)}{k\phi_C+\phi_I} \tag{50}$$

while other equilibrium conditions are left unchanged. Sufficient conditions ensuring that we have a unique and positive solution for $\phi_C$, $\phi_I$ and $\Omega$ are now of the form:

$$(d)' > (a)' > (b) > (e)' > (c)' \qquad (*)'$$

$$\begin{aligned}
(a)' &= \left(\frac{k}{k-1}\right)(\rho(1+\beta)+\phi_C(k+\beta)+\tau_c\beta(\rho+\phi_C)) \\
(c)' &= \frac{\rho(k+\beta)+\phi_C(k(2k-1)+\beta)+\tau_c\beta(\rho+\phi_C)}{k-1} \\
(d)' &= (\rho+\phi_C)\left(\frac{2k-1}{k(k-1)}\right)\left(\frac{(k\phi_C+\phi_I)(1+\tau_c)}{\phi_C+\phi_I}\right) \\
(e)' &= \rho\left(\frac{k}{k-1}\right)\frac{2k-1+d(1+\beta k)+\tau_c(2k-1-(k-1)d+\beta dk)}{2k-1+d-\beta k+\tau_c(2k-1-(k-1)d-\beta k)}
\end{aligned}$$

We have $\lim_{k\to\infty}(a)' = \rho(1+\beta(1+\tau_c))$, $\lim_{k\to\infty}(c)' = \rho$, $\lim_{k\to\infty}(d)' = 2\rho(1+\tau_c)$ and $\lim_{k\to\infty}(e)' = \rho\left(1+\frac{\beta(1+d)(1+\tau_c)-\tau_c d}{2(1+\tau_c)-\beta(1+\tau_c)-\tau_c d}\right)$. It can be demonstrated that we have $(a)' > (e)'$ for sufficiently high values of $k$ and under the condition $(2-\beta)(1+\tau_c)-(1+d) > 0$, which for positive values of $\tau_c$ is slacker than the condition $1-\beta-d > 0$ needed so as to establish the existence and uniqueness of the BGP equilibrium in Appendix C above; all the other inequalities are respected for high enough values of $k$ without any further conditions on the model's parameters.

We now first consider the **variation of $\phi_C$ following a shock on $\tau_c$.** The reasoning we presented above when considering variations in $\phi_C$ following shocks on $\beta$ and $d$ still holds here: under the parametric conditions guaranteeing the existence and uniqueness of the BGP and with $\phi_C = \frac{A_{\tau_c}+\sqrt{B_{\tau_c}+A_{\tau_c}^2}}{D_{\tau_c}}$, the sign of $\frac{\partial\phi_C}{\partial\tau_c}$ is ultimately determined by the sign of $\frac{\partial B_{\tau_c}}{\partial\tau_c}$. We have $B_{\tau_c} = \text{Cst}(1+\tau_c)(x-y\tau_c)$ with:

$$\begin{aligned}
\text{Cst} &= 8Fk^2\rho\big(2k-1-d(k-1)(1+\beta)\big) > 0 \\
x &= (1-a)(k-1)L(d+k(2-\beta)-1) - Fk\rho(2k+d(1+\beta k)-1) > 0 \;\text{ under }\; (b) > (e)' \\
y &= Fk\rho(k(2-d+\beta d)-1+d) - (1-a)(k-1)L(k(2-d-\beta)-1+d)
\end{aligned}$$

$y > 0$ can be reformulated as $\rho\left(\frac{k}{k-1}\right)\left(\frac{k(2-d+\beta d)-1+d}{k(2-d-\beta)-1+d}\right) > (1-a)L/F$. The LHS of the inequality tends to $\frac{2-d+\beta d}{2-d-\beta} = 1+\frac{\beta(1+d)}{2-d-\beta}$ as $k\to\infty$, and under the condition $(a)' > (b)$ we hence have $y > 0$ provided we have $\frac{\beta(1+d)}{2-d-\beta} > \beta(1+\tau_c)$. This inequality is respected under the sufficient (but not necessary) condition $\beta+2d-1 > 0$, which is compatible with $(*)'$ and respected for sufficiently high values of $\beta$ (needed so as to ensure the robustness of the "monopoly" equilibrium, cf Appendix C). We then have $\frac{\partial B_{\tau_c}}{\partial\tau_c} < 0$ if and only if we have $y > \frac{x}{1+2\tau_c}$. When $k\to\infty$, this inequality can be reformulated as $(1-a)L/F < \rho\left(\frac{2(1+\tau_c)(2+\beta d)-d(1+2\tau_c)}{2(1+\tau_c)(2-\beta)-d(1+2\tau_c)}\right) = \rho\left(1+\frac{2(1+\tau_c)\beta(1+d)}{2(1+\tau_c)(2-\beta)-d(1+2\tau_c)}\right)$, and holds under the condition $(a)' > (b)$ provided we have $\frac{2(1+\tau_c)\beta(1+d)}{2(1+\tau_c)(2-\beta)-d(1+2\tau_c)} > \beta(1+\tau_c)$. This last inequality is respected provided we have $2\beta+(3+2\tau_c)d-2(1+2\tau_c) > 0$, which holds for high values of $\beta$. As a consequence, we have $\frac{\partial B_{\tau_c}}{\partial\tau_c} < 0$, which entails $\frac{\partial\phi_C}{\partial\tau_c} < 0$: **under the conditions ensuring the existence and uniqueness of the BGP (i.e. high values of $\beta$ and $k$), $\phi_C$ is decreasing along $\tau_c$.**

We then move to determining the **variations of $\phi_I$ following a shock on $\tau_c$**:

$$\frac{\mathrm{d}\phi_I}{\mathrm{d}\tau_c} = \frac{\partial\phi_I}{\partial\tau_c} + \frac{\partial\phi_I}{\partial\phi_C}\frac{\partial\phi_C}{\partial\tau_c}$$

We have already established that $\frac{\partial\phi_C}{\partial\tau_c} < 0$ and $\frac{\partial\phi_I}{\partial\phi_C} > 0$. We finally have the following partial derivative of $\phi_I$ w.r.t. $\tau_c$:

$$\frac{\partial\phi_I}{\partial\tau_c} = \frac{F(k-1)\beta\phi_C(\rho+\phi_C)((1-a)(k-1)L - Fk(\rho+2k\phi_C))}{\Big(F(k(\rho-\phi_C)+2k^2\phi_C+\beta(1+\tau_c)(\rho+\phi_C)) - (1-a)(k-1)L\Big)^2} \tag{51}$$

For $k$ high enough and under the condition $(b) > (c)'$ we have that $(1-a)L/F > \left(\frac{k}{k-1}\right)(\rho + 2k\phi_C)$, which ensures that (51) is positive. The overall sign of $\frac{\mathrm{d}\phi_I}{\mathrm{d}\tau_c}$ is hence left to be ambiguous. Unfortunately, this ambiguity cannot be lifted considering the variations in $\theta_{SI}$ following a shock on $\tau_c$, as it can be seen when considering the equilibrium condition (32) (left unchanged following the introduction of a "barrier to entry" tax $\tau_c$). Following a positive shock on $\tau_c$, the LHS decreases through the decrease in $\phi_C$. So as to re-establish the equality, the RHS needs to decrease identically, with $c_R$ decreasing along $\theta_{SI}$. **We hence have that $\theta_{SI}$ necessarily increases following a positive shock on $\tau_c$**. This is however not sufficient to determine the direction of variation of $\phi_I$ (it could either increase, or decrease by a lower amount than $\phi_C$). We hence resort to simulations for a wide array of parametric values compatible with the existence and robustness of a "monopoly" equilibrium, and the following numerical finding emerges:

**Numerical finding**: *Under the parametric conditions $(*)'$, we systematically have $\frac{d\phi_I}{d\tau_c} > 0$ for relatively small values of $\tau_c$. For low values of the parameter $k$, $\frac{d\phi_I}{d\tau_c}$ becomes negative beyond a threshold value $\bar{\tau}_c$; for high enough values of $k$ however, $\frac{d\phi_I}{d\tau_c}$ remains positive.* □

### Comparative Statics in the Case of a Variation in $\tau_i$

For the sake of concision and readability of the demonstrations, we now temporarily assume that the tax rate $\tau_c$ applied to challengers is set to zero. The equilibrium condition (32) is now of the form:

$$wF(\rho+\phi_C)(1+\tau_i) = L(1-\beta)\left(\frac{k-1}{k}\right)c_R \tag{52}$$

while other equilibrium conditions are left unchanged. Sufficient conditions ensuring that we have a unique and positive solution for $\phi_C$, $\phi_I$ and $\Omega$ are now of the form:

$$(d) > (a)'' > (b) > (e)'' > (c)'' \qquad (*)''$$

$$\begin{aligned}
(a)'' &= \left(\frac{k}{k-1}\right)(\rho(1+\beta)+\phi_C(k+\beta)+\tau_i(\rho+\phi_C)) \\
(c)'' &= \frac{\rho(k+\beta)+\phi_C(k(2k-1)+\beta)+\tau_i k(\rho+\phi_C)}{k-1} \\
(e)'' &= \rho\left(\frac{k}{k-1}\right)\frac{(1+\tau_i)(2k-1+d(1+\beta k)+\tau_i dk)}{2k-1+d-\beta k+\tau_i dk}
\end{aligned}$$

We have $\lim_{k\to\infty}(a)'' = \rho(1+\beta+\tau_i)$, $\lim_{k\to\infty}(c)'' = \rho(1+\tau_i)$ and $\lim_{k\to\infty}(e)'' = \rho(1+\tau_i)\left(1+\frac{\beta(1+d)+\tau_i d}{2-\beta+\tau_i d}\right)$. It can be demonstrated that we have $(a)'' > (e)''$ for sufficiently high values of $k$ and under the condition $1-\beta-(1-\tau_i)d > 0$, which for positive values of $\tau_i$ is slacker than the condition $1-\beta-d > 0$ needed so as to establish the existence and uniqueness of the BGP equilibrium in Appendix C above; all the other inequalities are respected for high enough values of $k$ without any further conditions on the model's parameters.

We now first consider the **variation of $\phi_C$ following a shock on $\tau_i$**. Again, under the parametric conditions guaranteeing the existence and uniqueness of the BGP and with $\phi_C = \frac{A_{\tau_i}+\sqrt{B_{\tau_i}+A^2_{\tau_i}}}{D_{\tau_i}}$,, the sign of $\frac{\partial\phi_C}{\partial\tau_i}$ is ultimately determined by the sign of $\frac{\partial B_{\tau_i}}{\partial\tau_i}$. We have $\frac{\partial B_{\tau_i}}{\partial\tau_i} = 4Fk\rho\big(\text{Term1}+\text{Term2}\big)$ with:

$$\text{Term1} = \underbrace{\big(2k(2k-1-d(k-1)(1+\beta)) + \tau_i(k(2+d(\tau_i+\beta)) - 1 + d)\big)}_{>0}\Big((1-a)(k-1)Lkd - Fk\rho(2k-1+d(k(1+\beta+2\tau_i)+1))\Big)$$

$$\text{Term2} = \underbrace{(k(2+d(\beta+\tau_i)) - (1-d))}_{>0}\underbrace{\Big((1-a)(k-1)L(d+k(2-\beta)-1+\tau_i dk) - Fk\rho\big(k(2+\beta d)+d-1+\tau_i dk\big)\Big)}_{>0 \text{ under } (b)>(e)''}$$

Term1 and Term2 can be viewed as polynomials in $k$, with Term1 being of degree 4 and Term2 being only of degree 3. As $k\to\infty$, the sign of $\frac{\partial B_{\tau_i}}{\partial\tau_i}$ is hence determined by the sign of $(1-a)Ld - F\rho(2+d(1+\beta+2\tau_i))$. This expression is negative provided we have $\rho\left(\frac{2+d(1+\beta+2\tau_i)}{d}\right) > (1-a)L/F$, which is guaranteed under the condition $(d) > (b)$. **Under the conditions ensuring the existence and uniqueness of the BGP, $\phi_C$ is hence decreasing along $\tau_i$.**

We then move to determining the **variations of $\phi_I$ following a shock on $\tau_i$**:

$$\frac{\mathrm{d}\phi_I}{\mathrm{d}\tau_i} = \frac{\partial\phi_I}{\partial\tau_i} + \frac{\partial\phi_I}{\partial\phi_C}\frac{\partial\phi_C}{\partial\tau_i}$$

We have already established that $\frac{\partial\phi_C}{\partial\tau_i} < 0$ and $\frac{\partial\phi_I}{\partial\phi_C} > 0$. We finally have the following partial derivative of $\phi_I$ w.r.t. $\tau_i$:

$$\frac{\partial\phi_I}{\partial\tau_i} = \frac{F^2(k-1)k\phi_C(\rho+\phi_C)(\beta\rho-(k-\beta)\phi_C)}{\Big(F(k(\rho-\phi_C)+2k^2\phi_C+\beta(1+\tau_c)(\rho+\phi_C)) - (1-a)(k-1)L\Big)^2} \tag{53}$$

For $k$ high enough we have that $-(k-\beta)\phi_C \to 0$, which ensures that (53) is positive. The overall sign of $\frac{\mathrm{d}\phi_I}{\mathrm{d}\tau_i}$ is hence left to be ambiguous. This ambiguity can however be lifted considering the equilibrium condition (31) (left unchanged following the introduction of an "incumbent" tax $\tau_i$). Following a positive shock on $\tau_i$, the LHS decreases through the decrease in $\phi_C$. So as to re-establish the equality, the RHS needs to decrease identically, which necessarily entails that $\theta_{SI}$ **has decreased following the shock on $\tau_i$**. Since $\phi_C$ decreases along $\tau_i$, $\phi_I$ needs to decrease even more so as to obtain a decrease in $\theta_{SI}$: we hence necessarily have that $\phi_I$ **is also a decreasing function of $\tau_i$**. □

### Comparative Statics in the Case of a "Tax-and-Subsidy" Scheme

We consider now the case of a positive tax on R&D expenses by challengers used to subsidise R&D investment by incumbents (i.e. $\tau_c > 0$ and $\tau_i < 0$). Implementing such a balanced-budget policy implies the following budget constraint for the government: $\int_0^1 wF\frac{k^{n(s)+1}}{Q}\phi_C(s)\tau_c ds = -\int_{\theta_{SC}} wF\frac{k^{n(s)+1}}{Q}\phi_I(s)\tau_i ds$. Using the BGP properties of the random process governing the fluctuations between (SC) and (SI) within each industry, this constraint boils down to $\tau_c = -\theta_{SI}\tau_i$. This equality bears several implications. First, since $\theta_{SI} < 1$, it implies that the subsidy accruing to a "R&D-active" incumbent is greater than the revenues stemming from the "entrant tax" within a given industry: indeed, incumbents only invest in R&D in the share $\theta_{SC}$ of industries being in the (SC) state, while entrants are active in every sector of the economy. Second, it shows that the implementation of a "tax-and-subsidy" scheme under a strict rule of budget equilibrium completely modifies the structure of the model: the subsidy amount becomes endogenous due to the presence of the share $\theta_{SI}$, and closed-form solutions for the BGP equilibrium as well as analytical demonstrations of comparative statics are impossible to obtain.

We hence proceed to a simplified exercise that preserves the model structure and the possibility of closed-form solutions for the comparative statics, without violating the necessity for the government to be able to fund an eventual subsidy to R&D by incumbents: we simply impose $\tau_c = -\tau_i$. Indeed, as observed above, it is possible under a strictly balanced budget to subsidise the R&D activities of a "R&D-active" incumbent at a rate $\tau_i$ that is **superior** to the tax rate $\tau_c$ imposed on entrants. Imposing a strictly equivalent rate $\tau$ for both the tax and the subsidy remains therefore in the realm of budget-sound policies.

In the case of such a "tax-and-subsidy" scheme, we have that the equilibrium conditions (31) and (32) are modified in the following way:

$$wF(\rho+\phi_C)(1+\tau) = L\left(\frac{k-1}{k}\right)c_P\frac{k(\phi_I+\phi_C)}{k\phi_C+\phi_I} \quad (54)$$

$$wF(\rho+\phi_C)(1-\tau) = L(1-\beta)\left(\frac{k-1}{k}\right)c_R \quad (55)$$

while other equilibrium conditions are left unchanged. Sufficient conditions ensuring that we have a unique and positive solution for $\phi_C$, $\phi_I$ and $\Omega$ are now of the form:

$$(d)''' > (a)''' > (b) > (e)''' > (c)''' \qquad (*)'''$$

$$(a)''' = \left(\frac{k}{k-1}\right)(\rho(1+\beta)+\phi_C(k+\beta)-(1-\beta)\tau(\rho+\phi_C))$$

$$(c)''' = \frac{\rho(k+\beta)+\phi_C(k(2k-1)+\beta)-(k-\beta)\tau(\rho+\phi_C)}{k-1}$$

$$(d)''' = (\rho+\phi_C)\left(\frac{2k-1}{k(k-1)}\right)\left(\frac{(k\phi_C+\phi_I)(1+\tau_c)}{\phi_C+\phi_I}\right)$$

$$(e)''' = \rho\left(\frac{k}{k-1}\right)\frac{(1-\tau)(2k-1+d(1+\beta k)+\tau(k(2(1-d)+\beta d)-1+d))}{2k-1+d-\beta k+\tau(k(2(1-d)-\beta)-1+d)}$$

We have $\lim_{k\to\infty}(a)''' = \rho(1+\beta(1+\tau)-\tau)$, $\lim_{k\to\infty}(c)''' = \rho(1-\tau)$, $\lim_{k\to\infty}(d)''' = 2\rho(1+\tau)$ and $\lim_{k\to\infty}(e)'' = \rho(1-\tau)\left(\frac{(2+\beta d)(1+\tau)-2\tau d}{(2-\beta)(1+\tau)-2\tau d}\right)$. It can be demonstrated that we have $(a)''' > (e)'''$ for sufficiently high values of $k$ and under the condition $1-\beta-d+(2-\beta-d) > 0$, which for positive values of $\tau$ is slacker than the condition $1-\beta-d > 0$ needed so as to establish the existence and uniqueness of the BGP equilibrium in Appendix C above; all the other inequalities are respected for high enough values of $k$ without any further conditions on the model's parameters.

We now first consider the **variation of $\phi_C$ following a positive shock on $\tau$**. As in every other case we have considered, under the parametric conditions guaranteeing the existence and uniqueness of the BGP, the sign of $\frac{\partial \phi_C}{\partial \tau}$ is ultimately determined by the sign of $\frac{\partial B_\tau}{\partial \tau}$. Using the notations $B_{\tau 1}$ and $B_{\tau 2}$ introduced above, $\frac{\partial B_\tau}{\partial \tau}$ can be expressed as:

$$\begin{aligned}\frac{\partial B_\tau}{\partial \tau} \quad = \quad & 4Fk\rho\Big[B_{\tau 2}\underbrace{\Big((2k-1)(2k-1-2\tau)+d\big(k(2+\beta+2\tau(2-\beta))-1-2k^2(1+\beta)-2\tau\big)\Big)}_{>0} \\ & +B_{\tau 1}\Big(\underbrace{(1-a)(k-1)L(2k(1-d)+\beta k-1+d)}_{>0}+\underbrace{2Fk\rho((2k-1)\tau+d(k(1-(2-\beta)\tau)+\tau))}_{>0}\Big)\Big]\end{aligned}$$

Since we have $B_{\tau 2} > 0$ under the condition $(b) > (e)$, we have that $\frac{\partial B_\tau}{\partial \tau}$ is necessarily positive, entailing $\frac{\partial \phi_C}{\partial \tau} > 0$. **Under the conditions ensuring the existence and uniqueness of the BGP, $\phi_C$ is hence increasing along $\tau$.**

The variation of $\phi_I$ following a positive shock on $\tau$ is straightforward, since both an increase in the tax on entrants and an increase in the subsidy to incumbents (i.e. the two effects of an increase in $\tau$) lead to an increase of $\phi_I$. The variation of $\theta_{SI}$ can be inferred from the equilibrium condition (54). Following a positive shock on $\tau$, the LHS increases through both the increase in $\phi_C$ and the increase in $\tau$. So as to re-establish the equality, the RHS needs to increase identically, which necessarily entails that $\theta_{SI}$ **has increased following the shock on $\tau$**. □