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Lattices and simple groups in trees and buildings: constructions and classifications

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Table of contents

Introduction

The context of this thesis is the study of lattices in locally compact groups. A cornerstone of this subject is the seminal work of Margulis [Mar91], one of whose highlights is Margulis' arithmeticity theorem. It states that an irreducible lattice in a connected semisimple Lie group G with finite center and no compact factors is automatically arithmetic when the rank of G is at least 2. This can be viewed as a classification theorem. The set up covered by Margulis' theory encompasses also semisimple algebraic groups over non-Archimedean local fields. Bruhat–Tits theory provides a natural geometric space on which such algebraic groups act, namely the associated Euclidean buildings. It is important to emphasize that the full automorphism group of the building is in general strictly larger than the algebraic groups to which it is associated. This is especially manifest in the rank one case, where the building in question is a tree. In this way, we are naturally led to study lattices in the automorphism group of a tree, or of a Euclidean building in general, without assuming a priori that the geometric space comes from an algebraic group.

This topic received a lot of attention in the past three decades. The case of tree lattices is well understood (see [BK90], [BL01]); from the purely algebraic viewpoint those lattices are rather poor: indeed all cocompact lattices in the automorphism group of a tree are pairwise commensurable and virtually free. Burger and Mozes revealed in [BM00a] that things get much more exciting in a product of two trees. After having laid the foundations of the study of lattices in products of trees, they highlighted many new phenomena and constructed for instance lattices that are finitely presented, torsion-free and simple, thereby answering famous open problems in group theory. A product of two trees is a 2-dimensional Euclidean building of type $\tilde{A}_1 \times \tilde{A}_1$. The study of lattices in 2-dimensional Euclidean buildings of other types was also undertaken by various authors (see e.g. [CMSZ93a], [Ess13], [Wit17], [Bar00]), but focused mainly on the type $\tilde{A}_2$. There are similarities between lattices in buildings of types $\tilde{A}_1 \times \tilde{A}_1$ and $\tilde{A}_2$, but they do not actually enjoy the same qualitative properties (e.g. QI-rigidity, Kazhdan's Property (T)).

This thesis provides a contribution to these topics with the construction of numerous new examples and a detailed study of their enveloping groups. The first two chapters are concerned with trees and products of trees, the last two with $\tilde{A}_2$-buildings.

We now proceed to describe some of our main results more precisely. Each chapter has its own introduction containing more information and additional statements of independent interest.

Trees and products of trees (Chapters 1 and 2)

One of our starting points is the following question, asked by Burger, Mozes and Zimmer in [BMZ09].

Question (Burger–Mozes–Zimmer, 2009)**.** *Which groups arise as closures of projections of cocompact lattices in* $\mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$*, where* T_1 *and* T_2 *are locally finite regular trees?*

The cocompact lattices $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ of interest are those which are not commensurable to a product of lattices $\Gamma_t \leq \mathrm{Aut}(T_t)$. They are called **irreducible**. This irreducibility condition is equivalent to asking both projections on $\mathrm{Aut}(T_1)$ and $\mathrm{Aut}(T_2)$ to be non-discrete, see [BM00b, Proposition 1.2].

Under this irreducibility assumption, the above question thus asks for which pairs of non-discrete closed subgroups $H_1 \leq \mathrm{Aut}(T_1)$ and $H_2 \leq \mathrm{Aut}(T_2)$ there exists a cocompact lattice $\Gamma \leq H_1 \times H_2$ whose projections on H_1 and H_2 are dense. A first remark is that H_t must be locally topologically finitely generated for each $t \in \{1, 2\}$ (see [BMZ09, Proposition 1.1.2]), which in particular excludes the full group $\mathrm{Aut}(T_t)$.

Most known irreducible cocompact lattices in products of trees come from the algebraic world: we call them arithmetic lattices. One can for instance construct a cocompact lattice $\Gamma \leq \mathrm{PGL}(2, \mathbf{Q}_p) \times \mathrm{PGL}(2, \mathbf{Q}_{p'})$ with dense projections for each distinct odd primes p and p', see [Vig80, §4, Theorem 1.1] and [Rat04, Chapter 3]. Note that $\mathrm{PGL}(2, \mathbf{Q}_p)$ acts on its Bruhat–Tits tree T (which is $(p+1)$-regular), so it can be seen as a closed subgroup of $\mathrm{Aut}(T)$.

The first non-algebraic groups that were shown to appear as closures of projections of cocompact lattices in a product of two trees are the universal groups $U(\mathrm{Alt}(d))$ defined and studied by Burger and Mozes in [BM00a] (for sufficiently large even values of d). Given a d-regular tree T and a transitive finite permutation group $F \leq \mathrm{Sym}(d)$, the group $U(F)$ is the largest vertex-transitive subgroup of $\mathrm{Aut}(T)$ whose stabilizer of a vertex acts as F on its d neighbors (we write $\underline{G}(v) \cong F$ when the local action of a group $G \leq \mathrm{Aut}(T)$ at v is given by F). In [BM00b], the same authors indeed constructed for each $m \geq 15$ and $n \geq 19$ a cocompact lattice $\Gamma \leq U(\mathrm{Alt}(2m)) \times U(\mathrm{Alt}(2n))$ with dense projections. Later, Rattaggi constructed in his thesis [Rat04] such lattices for some smaller values of m and n, for instance $m = n = 3$. He was also able to produce a cocompact lattice $\Gamma \leq U(\mathrm{Alt}(6)) \times U(M_{12})$ with dense projections, where $M_{12} \leq \mathrm{Sym}(12)$ is the Mathieu group of degree 12. For all these examples, the density of the projections can be established thanks to the next result of Burger–Mozes [BM00a, Proposition 3.3.1].

Theorem (Burger–Mozes, 2000)**.** *Let T be the d-regular tree and let H be a non-discrete vertex-transitive closed subgroup of* $\mathrm{Aut}(T)$*. Let $F \leq \mathrm{Sym}(d)$ be such that $\underline{H}(v) \cong F$ for each $v \in V(T)$. Assume that F is 2-transitive and that the stabilizer $F(1)$ of 1 in F is simple non-abelian. Then $H = U(F)$ (up to conjugation in* $\mathrm{Aut}(T)$*).*

As soon as a projection of an irreducible cocompact lattice in a product of trees is vertex-transitive and locally isomorphic to F with F satisfying the conditions of the theorem, its closure is thus ensured to be $U(F)$. For $F = \mathrm{Alt}(d)$ with $d \geq 6$ and $F = M_{12} \leq \mathrm{Sym}(12)$, it is indeed true that F is 2-transitive and $F(1)$ is simple non-abelian.

However, as will be remarked in Chapter 2 (see for instance Tables 2.12–2.15), the local group F that is the most likely to appear when considering a generic cocompact lattice is the full group $F = \mathrm{Sym}(d)$ (at least when the degrees of the two trees are large). In that case, $F(1)$ is not simple so the above theorem does not apply. Our goal in Chapter 1 is to provide an analogous result when $F = \mathrm{Sym}(d)$ and $d \geq 6$. One cannot however expect $H = U(\mathrm{Sym}(d)) = \mathrm{Aut}(T)$ to be the unique group satisfying the hypotheses.[1] In [BM00a, Proposition 3.3.2], Burger–Mozes could already conclude in that situation that H must be 2-transitive on the set of ends ∂T of T. Our result gives a precise (infinite) list of possible groups to which H can be equal; they are all virtually simple.

Theorem (see Corollary 1.E$'$)**.** *Let T be the d-regular tree with $d \geq 6$ and let H be a non-discrete vertex-transitive closed subgroup of* $\mathrm{Aut}(T)$*. Assume that $\underline{H}(v) \cong F \geq \mathrm{Alt}(d)$ for each $v \in V(T)$. Then H has a simple subgroup of index ≤ 8 and belongs to an explicit list of examples. All groups in that list contain $U(\mathrm{Alt}(d))$ (up to conjugation in $\mathrm{Aut}(T)$).*

The context in which Chapter 1 falls is actually a bit more general: we deal with semiregular (and not only regular) trees T and closed subgroups $H \leq \mathrm{Aut}(T)$ acting 2-transitively on ∂T and with an alternating or symmetric local action at each vertex. Such a group H can be vertex-transitive (as in the statement above) but it can also have 2 orbits of vertices. The same kind of results as above is valid in that more general framework, see Theorem 1.B$'$.

The assumption that T is semiregular is not really restrictive. Indeed, given a (locally finite) tree whose vertices have valency at least 3, the existence of a group acting on it such that the induced action on the set of its ends is 2-transitive already implies that the tree must be semiregular (see Lemma 1.2.2).

Once these classification results are established, we come back in Chapter 2 to cocompact lattices in products of trees, more precisely to groups $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ acting simply transitively on the vertices

[1] Indeed, there exist many irreducible lattices $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ with T_1 and T_2 being 6-regular and such that the closure H_1 of the projection of Γ on $\mathrm{Aut}(T_1)$ is vertex-transitive and satisfies $H_1(v) \cong \mathrm{Sym}(6)$. As explained above, H_1 cannot be equal to $\mathrm{Aut}(T)$.

of $T_1 \times T_2$ where T_1 and T_2 are d_1-regular and d_2-regular respectively. We call such a group Γ a (d_1, d_2)**-group**. Our first aim is to develop tools enabling us to identify (as often as possible) the closures of the projections of a particular (d_1, d_2)-group. Under some suitable hypotheses on the local action, we develop algorithms that can be used to compute the closure of a projection.

Theorem (See Theorem 2.A)**.** *Let* $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ *be a* (d_1, d_2)*-group and let* H_1 *be the closure of the projection of* Γ *on* $\mathrm{Aut}(T_1)$*. Suppose that* $d_1 \geq 6$ *and that* $\underline{H_1}(v) \geq \mathrm{Alt}(d_1)$ *for each* $v \in V(T_1)$*.*

(i) There is an (efficient) algorithm that determines if Γ *is irreducible.*

(ii) If Γ *is irreducible and* d_1 *is even, then there is an (efficient) algorithm that computes the exact isomorphism class of* H_1*.*

Point (i) follows from results of Burger–Mozes [BM00a, Propositions 3.3.1 and 3.3.2], while (ii) requires a much more involved analysis. Those algorithms can for instance be used to give the following partial answer to the question of Burger–Mozes–Zimmer mentioned above.

Theorem (see Theorem 2.B)**.** *Let* T_1 *and* T_2 *be two* 6*-regular trees. There are exactly* 7 *groups* $H \leq \mathrm{Aut}(T_1)$ *(up to conjugation) that are transitive on* $V(T_1)$*, satisfy* $\underline{H}(v) \geq \mathrm{Alt}(6)$ *for each* $v \in V(T_1)$*, and appear as the closure of the projection on* $\mathrm{Aut}(T_1)$ *of a torsion-free irreducible* $(6,6)$*-group* $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$*.*

For each $d_1, d_2 \geq 3$ there are only finitely many conjugacy classes of (d_1, d_2)-groups, so we could not expect in the previous theorem to see infinitely many of our groups appearing as the closure of a projection of a $(6,6)$-group. However, if we fix $d_1 = 6$ and let d_2 vary, then infinitely many distinct projections on the 6-regular tree do indeed arise.

Theorem (see Theorem 2.F)**.** *Let* T_1 *be the* 6*-regular tree. There are infinitely many conjugacy classes of groups* $H \leq \mathrm{Aut}(T_1)$ *that are transitive on* $V(T_1)$*, satisfy* $\underline{H}(v) \geq \mathrm{Alt}(6)$ *for each* $v \in V(T_1)$*, and appear as the closure of the projection on* $\mathrm{Aut}(T_1)$ *of a torsion-free irreducible* $(6, d_2)$*-group* $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ *(for some* $d_2 \geq 3$*).*

After having studied projections of (d_1, d_2)-groups, we decided to use our brand-new computer programs to also go back to one of the original motivations of Burger–Mozes in [BM00b]. The following problem was asked by Peter Neumann in [Neu73].

Problem (Neumann, 1973)**.** *Let $G = F_a *_{F_c} F_b$ be a free amalgamated product of non-abelian free groups of finite rank over a subgroup of finite index. Can it happen that G is simple?*

Burger and Mozes answered this question in the positive by proving that for each $m \geq 109$ and $n \geq 150$, there exists a virtually simple $(2m, 2n)$-group $\Gamma \leq U(\mathrm{Alt}(2m)) \times U(\mathrm{Alt}(2n))$ with dense projections (see Theorem [BM00b, Theorems 5.5 and 6.4]). The simple subgroup of finite index in Γ is then isomorphic to its projection on $U(\mathrm{Alt}(2m))$, and this projection is edge-transitive. It can therefore be written as the free amalgamated product of two adjacent vertex stabilizers over an edge stabilizer (see [Ser77, Theorem 6]). Each vertex stabilizer in the first tree is a cocompact lattice in the second tree that is free, so we get a simple group of the form $F_a *_{F_c} F_b$ as wanted.

One main ingredient developed and used by Burger–Mozes for the construction of virtually simple lattices is their Normal Subgroup Theorem (NST). The NST states that if the two closures of projections of a cocompact lattice Γ are boundary-2-transitive and virtually simple, then all non-trivial normal subgroups of Γ have finite index. The strategy to find a virtually simple (d_1, d_2)-group Γ is to ensure not only that the NST applies but also that Γ is not residually finite: this is in general done by embedding another non-residually finite lattice in Γ.

The degrees of the free groups involved in the amalgams found by Burger–Mozes are however really huge, so the presentations of those groups would be too large to manipulate. Rattaggi later found in [Rat04] a $(8, 12)$-group with a simple subgroup of index 4. With the same reasoning, he observed that this simple subgroup is a free amalgamated product $F_7 *_{F_{73}} F_7$. Even more recently, Bondarenko and Kivva found in [BK17] a $(8, 8)$-group with a simple subgroup of index 4 that decomposes as $F_7 *_{F_{49}} F_7$. In all these references, no explicit presentation for the simple subgroup was computed.

In Chapter 2 and in parallel to our study of projections, we use the same techniques as the above authors to produce a $(4,5)$-group with a simple subgroup of index 4 that takes the form $F_3 *_{F_{11}} F_3$. Some explanations of how we could go from $(8,8)$ to $(4,5)$ are listed below.

- We do not restrict our attention to torsion-free (d_1, d_2)-groups. In particular d_1 and d_2 can be odd (while they are automatically even when the (d_1, d_2)-group is torsion-free);
- A non-residually finite $(3,3)$-group can be found thanks to a result of Caprace and Wesolek [CW17, Corollary 6.4] (while Bondarenko–Kiva use a non-residually finite $(4,4)$-group);
- We use the NST of Bader–Shalom [BS06] instead of the NST of Burger–Mozes. Indeed, we are not able to compute the closures H_1 and H_2 of the projections of our $(4,5)$-group; in particular we do not know *a priori* whether they are virtually simple. The NST of Bader–Shalom gives the same result as the one of Burger–Mozes but without the assumption that H_1 and H_2 are virtually simple. It will then turn out *a posteriori* from the virtual simplicity of our $(4,5)$-group that H_1 and H_2 are indeed virtually simple.

The presentation of our simple group is given below.

Theorem (see Corollary 2.D (ii))**.** *The following group is isomorphic to a free amalgamated product $F_3 *_{F_{11}} F_3$. It is simple and is an index 4 subgroup of a $(4,5)$-group.*

$$
\begin{aligned}
\langle x_1, x_2, x_3, y_1, y_2, y_3 \quad | \quad & x_1 = y_1,\ x_2^2 = y_2 y_1^{-1} y_2,\ x_3^2 = y_3^2, \\
& x_3^{-1} x_1 x_3 = y_3^{-1} y_2 y_3,\ x_3^{-1} x_2 x_3 = y_3^{-1} y_1 y_3, \\
& x_2^{-1} x_1 x_2 = y_2^{-1} y_1^{-1} y_2,\ x_2^{-1} x_3^{-2} x_2 = y_2^{-1} y_1 y_3^{-2} y_2, \\
& x_2^{-1} x_3^{-1} x_2^{-1} x_1 x_3 x_2 = y_2^{-1} y_1 y_3^{-1} y_1^{-1} y_2 y_3 y_1^{-1} y_2, \\
& x_2^{-1} x_3^{-1} x_1 x_2 x_3 x_2 = y_2^{-1} y_1 y_3^{-1} y_2 y_3^{-1} y_1 y_3 y_1^{-1} y_2, \\
& x_2^{-1} x_3^{-1} x_2^2 x_3 x_2 = y_2^{-1} y_1 y_3^{-1} y_1 y_3^{-1} y_1 y_3 y_1^{-1} y_2, \\
& x_2^{-1} x_3^{-1} x_2^{-1} x_3 x_2 x_3 x_2 = y_2^{-1} y_1 y_3^{-1} y_1^{-1} y_3^{-1} y_1 y_3 y_1^{-1} y_2 \rangle
\end{aligned}
$$

As mentioned above, Burger–Mozes and their followers were only dealing with regular trees of even degrees. In the following result, we show that any d-regular tree with $d \geq 4$ can appear as a factor of a product of two trees in which a simple cocompact lattice lives.

Theorem (see Theorem 2.E)**.** *For each $n \geq 2$, there exists a $(2n, 2n+1)$-group with a simple subgroup of index 2^n.*

We finally end Chapter 2 with a short discussion about lattices in products of three trees. This theme, mentioned as follows by Farb, Mozes and Thomas in [FMT15, Problem 8], remains a wasteland in which very few mathematicians ventured.

Problem (Farb–Mozes–Thomas, 2015)**.** *Study lattices in products of three or more trees.*

The question whether there exists a virtually simple non-arithmetic cocompact lattice in a product of more than two trees is actually still unanswered. We obtain a result about non-existence of certain lattices in the product of three 6-regular trees, see Theorem 2.H.

Buildings of type $\tilde{A}_2$ (Chapters 3 and 4)

After lattices in products of trees, we focus on lattices in buildings of type $\tilde{A}_2$. There is a general definition for a building of any type, but those of type $\tilde{A}_2$ have a very nice alternative characterization. Indeed, a (thick) $\tilde{A}_2$-building is a simply connected simplicial complex of dimension 2 such that all simplicial spheres of radius 1 around vertices are isomorphic to the incidence graph of a projective plane. Given an $\tilde{A}_2$-building, we call these projective planes the **residue planes** of the building.

Any simple algebraic group has a well-understood action on a building of some type, and these geometric objects can thus help understanding the properties of algebraic groups. In the case of locally finite $\tilde{A}_2$-buildings, the algebraic groups that are involved are the groups $\mathrm{PGL}(3, D)$ for D a finite dimensional division algebra over a local field. An $\tilde{A}_2$-building associated to such a group is called **Bruhat–Tits**, and its residue planes are all Desarguesian.

On the other hand, the existence of locally finite $\tilde{A}_2$-buildings with non-Desarguesian residue planes has been known since 1986. Ronan indeed gave in [Ron86] a general construction affording all possible $\tilde{A}_2$-buildings and from which it is clear that any projective plane can appear as a residue plane in an $\tilde{A}_2$-building. However, this point of view does not provide any information on the automorphism group of the building. In particular it does not seem fruitful for the construction of a locally non-Desarguesian $\tilde{A}_2$-building with a cocompact lattice. The problem of finding such a building was asked by Kantor in [Kan86, Page 124].

Problem (Kantor, 1986)**.** *Construct a locally finite $\tilde{A}_2$-building with non-Desarguesian residue planes and admitting a (torsion-free) cocompact lattice.*

Independently, Howie also asked the next more specific question [How89, Question 6.12]. Note that, for any $\tilde{A}_2$-building Δ, there exists a type function $t\colon V(\Delta) \to \{0, 1, 2\}$ such that each triangle in Δ has one vertex of each type. An automorphism of Δ is then called **type-rotating** if its induced action σ on the set of types $\{0, 1, 2\}$ satisfies $\sigma(t) = t + c \bmod 3$ for some $c \in \{0, 1, 2\}$.

Question (Howie, 1989)**.** *Does there exist a group acting simply transitively and by type-rotating automorphisms on the vertices of a locally finite $\tilde{A}_2$-building whose residue planes are non-Desarguesian?*

Some constructions in [VM87] and [BP07] yield $\tilde{A}_2$-buildings with exotic residues, but without any lattice (or, as the case may be, without information on its possible existence). In Chapter 3, we solve the above problem and question by proving the following result. The Hughes plane $\mathcal{H}_9$ of order 9, which was first constructed by Veblen and Wedderburn in 1907, was the first discovered finite non-Desarguesian projective plane.

Theorem (see Theorem 3.A)**.** *There exists a group Γ acting simply transitively and by type-rotating automorphisms on the vertices of an $\tilde{A}_2$-building whose residue planes are isomorphic to $\mathcal{H}_9$.*

The construction of that lattice uses the formalism developed by Cartwright, Mantero, Steger and Zappa in [CMSZ93a]. In their work, they could for instance list all groups acting simply transitively and by type-rotating automorphism on the vertices of an $\tilde{A}_2$-building whose residue planes have order 2 or 3. In our case, constructing only one lattice in some $\tilde{A}_2$-building where the Hughes plane of order 9 appears as a residue plane is already a challenge. In Chapter 3 we reveal how a computer could be programmed to search for such a lattice.

Recently, Caprace also solved the problem of Kantor by showing that any projective plane whose order is a prime power appears as a residue plane in an $\tilde{A}_2$-building admitting a cocompact lattice, see [Cap17b, Remark 8]. It is however not clear that the lattices constructed in that way are virtually torsion-free, whereas Γ in the previous theorem has a torsion-free subgroup of index 3. Also, if the chosen projective plane is non-Desarguesian, then it does not appear at all vertices of the $\tilde{A}_2$-building. In particular the lattice is not transitive on the vertices.

Our $\tilde{A}_2$-building with residue planes isomorphic to $\mathcal{H}_9$ can be shown to have a discrete automorphism group, see Theorem 3.A. This however requires the help of a computer. In general, deciding whether the automorphism group of a particular $\tilde{A}_2$-building Δ is discrete or not is a difficult task. Of course $\mathrm{Aut}(\Delta)$ is non-discrete when Δ is Bruhat–Tits, so the real question concerns **exotic** (i.e. non-Bruhat–Tits) $\tilde{A}_2$-buildings. Below we list the known sources of examples of exotic $\tilde{A}_2$-buildings. A **panel** in an $\tilde{A}_2$-building is a simplex of dimension 1 (i.e. an edge).

(1) $\tilde{A}_2$-buildings can be constructed inductively, starting from a point and gluing triangles to the ball of radius r centered at that point so as to build the ball of radius $r + 1$. This kind of construction is explained in [Ron86] and [BP07], where it was observed that $\tilde{A}_2$-buildings can be "really" exotic. It is however rather hard to have any information on the automorphism group of a building constructed in that way.

(2) $\tilde{A}_2$-buildings with lattices have been studied a lot: some of them with a panel-regular lattice (see for instance [Ess13] and [Wit17]), others with a vertex-regular lattice (as in Chapter 3, see also [CMSZ93a] and [CMSZ93b]), and also some with a lattice having two orbits of vertices (see [Bar00, §3], or [Cap17b, Remark 8] mentioned earlier). For the small examples, i.e. the ones with a small enough **thickness**

(the number of triangles adjacent to a single panel), it could be checked with a computer that the automorphism group was discrete as soon as the building was exotic.

(3) Exotic $\tilde{A}_2$-buildings can also be constructed from valuations on planar ternary rings, see [VM87]. The automorphism groups of the $\tilde{A}_2$-buildings constructed in that way in [VM90, §7] are vertex-transitive and non-discrete, but they fix a vertex at infinity, and are thus not unimodular by [CM13, Theorem M] (in particular, they cannot contain any lattice).

In Chapter 4, we show the following result. We write $\mathrm{Aut}(\Delta)^+$ for the subgroup of $\mathrm{Aut}(\Delta)$ preserving the types of vertices. Note that, similarly to vertices, there are three types of panels.

Theorem (See Theorem 4.A)**.** *Let Δ be a locally finite thick $\tilde{A}_2$-building such that $\mathrm{Aut}(\Delta)^+$ is transitive on panels of each type. Then either:*

(a) *Δ is Bruhat–Tits; or*

(b) $\mathrm{Aut}(\Delta)$ *is discrete.*

In particular, this theorem applies to all locally finite thick $\tilde{A}_2$-buildings with a panel-regular lattice (see (2) above). A natural question to ask is whether the panel-transitivity can be weakened in this theorem, and for instance replaced by vertex-transitivity. Because of the $\tilde{A}_2$-building described in [VM90, §7] (see (3) above), such a result cannot be true in these general terms. We however obtain the same conclusion if we suppose that $\mathrm{Aut}(\Delta)$ is unimodular and that Δ has thickness $p+1$ for some prime p (i.e. the local projective planes in Δ have order p), see Theorem 4.B. This other result can in particular be applied to the locally finite thick $\tilde{A}_2$-buildings Δ with a vertex-regular lattice (see (2) above) when the thickness of Δ is $p+1$ for some prime p.

As pointed out by the referee of the paper presenting the results of Chapter 4, the question whether the automorphism group of an exotic $\tilde{A}_2$-building admitting a cocompact lattice is always discrete was asked by Steger in talks given in Blaubeuren and Orléans in 2007. We provide partial answers to that question.

The theorem above can also be viewed as giving weak hypotheses on $\mathrm{Aut}(\Delta)$ under which Δ is automatically Bruhat–Tits. It was proved in [VMVS98] by Van Maldeghem and Van Steen that Δ is Bruhat–Tits as soon as $\mathrm{Aut}(\Delta)$ is *Weyl-transitive.* Our result actually shows that having $\mathrm{Aut}(\Delta)^+$ transitive on panels of each type and non-discrete (which is strictly weaker than requiring the Weyl-transitivity) is already sufficient to have the same conclusion. Our proof actually uses the machinery developed by the authors in [VMVS98].

We close Chapter 4 with a result giving a local condition under which an $\tilde{A}_2$-building is ensured to be exotic, see Theorem 4.C. It is thus somewhat complementary to the above theorem. We mention here an application of that result to the context of panel-regular lattices. Following [Wit17], a **Singer cyclic lattice** is a group $\Gamma \leq \mathrm{Aut}(\Delta)$ acting simply transitively on the panels of each type of an $\tilde{A}_2$-building Δ and such that each vertex stabilizer in Γ is cyclic. It is called **exotic** if Δ is exotic, and the **parameter** of Γ is the order of the local projective planes in Δ. The number of isomorphism classes of Singer cyclic lattices with parameter q grows super-exponentially with q (see [Wit17, Theorem B]), and we show that almost all of them are exotic.

Theorem (See Corollary 4.E)**.** *Almost all Singer cyclic lattices are exotic in the following sense:*

$$\lim_{q\to\infty} \frac{|\{\textit{exotic Singer cyclic lattices with parameter } q\}/\!\sim|}{|\{\textit{Singer cyclic lattices with parameter } q\}/\!\sim|} = 1,$$

where q ranges over prime powers and $\sim$ is the isomorphism relation.

As mentioned above, it is a consequence of our discreteness result that all exotic Singer cyclic lattices live in an $\tilde{A}_2$-building with a discrete automorphism group. Using the fact that cocompact lattices in $\tilde{A}_2$-buildings are QI-rigid [KL97], this in particular implies that they have finite index in their abstract commensurator group. This can be seen as an analog of the result of Margulis stating that an irreducible lattice in a connected semisimple Lie group G with finite center and no compact factors is arithmetic if it has infinite index in its commensurator in G, see [Mar91, Theorem IX.1.16].

Chapter 1

Boundary 2-transitive automorphism groups of trees

The main goal of this chapter is to give a full classification of boundary 2-transitive automorphism groups of trees whose local action at each vertex contains the alternating group, under the assumption that each vertex has valency at least 6. The results of this chapter have been published in [Rad17a].

1.1 Main results

Let T be the (d_0, d_1)-semiregular tree, with $d_0, d_1 \geq 4$. Let $V(T) = V_0(T) \sqcup V_1(T)$ be the canonical bipartition of the vertex set $V(T)$ so that each vertex **of type** $t \in \{0, 1\}$ (i.e. in $V_t(T)$) is incident to d_t edges. The subgroup of $\mathrm{Aut}(T)$ consisting of elements preserving this bipartition is denoted by $\mathrm{Aut}(T)^+$. We write $H \leq_{cl} G$ to mean that H is a *closed subgroup* of G and define the sets

$$\mathcal{H}_T := \{H \leq_{cl} \mathrm{Aut}(T) \mid H \text{ is 2-transitive on } \partial T\}$$

and

$$\mathcal{H}_T^+ := \{H \in \mathcal{H}_T \mid H \leq \mathrm{Aut}(T)^+\}.$$

Note that when $d_0 \neq d_1$, all automorphisms of T are type-preserving so that $\mathcal{H}_T^+ = \mathcal{H}_T$.

Consider a group $H \in \mathcal{H}_T$. For each vertex $v \in V(T)$, one can look at the action of the stabilizer $H(v)$ of v in H on the set $E(v)$ of edges incident to v. The image of $H(v)$ in $\mathrm{Sym}(E(v))$ is denoted by $\underline{H}(v)$. Since H is 2-transitive on ∂T, it is transitive on $V_0(T)$ and on $V_1(T)$ (see Lemma 1.2.2 below). Hence, all groups $\underline{H}(v)$ with $v \in V_0(T)$ (resp. $v \in V_1(T)$) are permutation isomorphic to the same group $F_0 \leq \mathrm{Sym}(d_0)$ (resp. $F_1 \leq \mathrm{Sym}(d_1)$). In this context of finite permutation groups, we use the symbol $\cong$ to mean *permutation isomorphic*. The goal of this chapter is to provide a full classification of the groups $H \in \mathcal{H}_T$ such that $F_0 \geq \mathrm{Alt}(d_0)$ and $F_1 \geq \mathrm{Alt}(d_1)$, under the assumption that $d_0, d_1 \geq 6$.

Let us first describe some key examples of groups in $\mathcal{H}_T^+$. In [BM00a, §3.2], the notion of a *legal coloring* of a d-regular tree is defined, and consists in coloring the edges of the tree with d colors. For our purposes, we need to generalize this notion to a (d_0, d_1)-semiregular tree, and a way to do so is to color the vertices instead of the edges. A **legal coloring** i of T consists of two maps $i_0 \colon V_0(T) \to \{1, \ldots, d_1\}$ and $i_1 \colon V_1(T) \to \{1, \ldots, d_0\}$ such that $i_0|_{S(v,1)} \colon S(v,1) \to \{1, \ldots, d_1\}$ is a bijection for each $v \in V_1(T)$ and $i_1|_{S(v,1)} \colon S(v,1) \to \{1, \ldots, d_0\}$ is a bijection for each $v \in V_0(T)$. Here, $S(v, r)$ is the set of vertices of T at distance r from v. The map i is defined on $V(T)$ by $i|_{V_0(T)} = i_0$ and $i|_{V_1(T)} = i_1$ (see Figure 1.1). Given $g \in \mathrm{Aut}(T)$ and $v \in V(T)$, one can look at the **local**

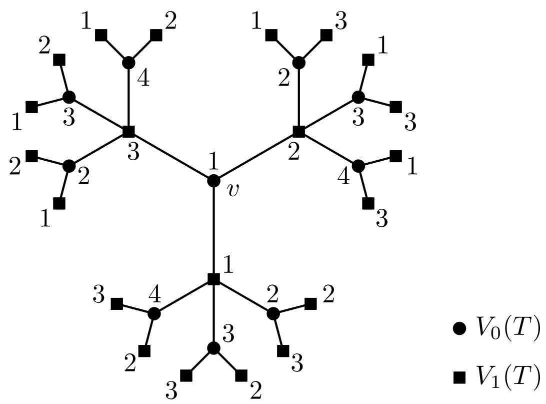


Figure 1.1: A legal coloring of $B(v, 3)$ in the $(3, 4)$-semiregular tree.

action of g at the vertex v by defining

$$\sigma_{(i)}(g,v) := i|_{S(g(v),1)} \circ g \circ i|_{S(v,1)}^{-1} \in \begin{cases} \mathrm{Sym}(d_0) & \text{if } v \in V_0(T), \\ \mathrm{Sym}(d_1) & \text{if } v \in V_1(T). \end{cases}$$

In the particular case where $d_0 = d_1$, there is a natural correspondence between our definition of a legal coloring and the definition given in [BM00a]. One should however note that, with our definition, the group of all automorphisms $g \in \mathrm{Aut}(T)$ such that $\sigma_{(i)}(g,v) = \mathrm{id}$ for each $v \in V(T)$ is *not* vertex-transitive (and even not transitive on $V_0(T)$), while the universal group $U(\mathrm{id})$ defined in the same way in [BM00a, §3.2] is vertex-transitive. One must therefore be careful when comparing [BM00a] with the present text. Another definition of legal colorings for semiregular trees was given by Smith in [Smi17, §3]: it is equivalent to ours.

The notion of a legal coloring allows us to define the following groups.

Definition. Let T be the (d_0, d_1)-semiregular tree and let i be a legal coloring of T. When $v \in V(T)$ and Y is a subset of $\mathbf{Z}_{\geq 0}$, we set $S_Y(v) := \bigcup_{r\in Y} S(v,r)$. For all (possibly empty) finite subsets Y_0 and Y_1 of $\mathbf{Z}_{\geq 0}$, define the group

$$G_{(i)}^{+}(Y_0, Y_1) := \left\{ g \in \mathrm{Aut}(T)^{+} \;\middle|\; \begin{array}{c} \prod_{w\in S_{Y_0}(v)} \mathrm{sgn}(\sigma_{(i)}(g,w)) = 1 \ \forall v \in V_{t_0}(T), \\ \prod_{w\in S_{Y_1}(v)} \mathrm{sgn}(\sigma_{(i)}(g,w)) = 1 \ \forall v \in V_{t_1}(T) \end{array} \right\},$$

where $t_0 := (\max Y_0) \bmod 2$, $t_1 := (1+\max Y_1) \bmod 2$ and $\max(\varnothing) := 0$.

The choice of t_0 and t_1 in this definition is made in such a way that, in each set $S_{Y_t}(v)$ under consideration, the vertices at maximal distance from v are of type t (i.e. $S(v, \max Y_t) \subseteq V_t(T)$), for $t \in \{0,1\}$.

Remark that $G_{(i)}^{+}(\varnothing,\varnothing) = \mathrm{Aut}(T)^{+}$ and that all groups $G_{(i)}^{+}(Y_0,Y_1)$ contain the group $G_{(i)}^{+}(\{0\},\{0\})$, which we also denote by $\mathrm{Alt}_{(i)}(T)^{+}$ and satisfies

$$\mathrm{Alt}_{(i)}(T)^{+} = \{g \in \mathrm{Aut}(T)^{+} \mid \sigma_{(i)}(g,v) \text{ is even for each } v \in V(T)\}.$$

When T is the d-regular tree, i.e. when $d_0 = d_1 = d$, it can be seen that $\mathrm{Alt}_{(i)}(T)^+$ is conjugate to the universal group $U(\mathrm{Alt}(d))^+$ of Burger–Mozes [BM00a, §3.2].

Our first result describes various properties of the groups defined above. We denote by $N_G(H)$ the normalizer of H in G and write $\mathbf{C}_2$ and $\mathbf{D}_8$ for the cyclic group of order 2 and the dihedral group of order 8, respectively.

Theorem 1.A. *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 4$ and let i be a legal coloring of T. Let Y_0 and Y_1 be finite subsets of $\mathbf{Z}_{\geq 0}$.*

(i) $G^+_{(i)}(Y_0, Y_1)$ belongs to $\mathcal{H}^+_T$.

(ii) $G^+_{(i)}(Y_0, Y_1)$ is abstractly simple.

(iii) We have

$$N_{\mathrm{Aut}(T)^+}(G^+_{(i)}(Y_0, Y_1)) \Big/ G^+_{(i)}(Y_0, Y_1) \cong (\mathbf{C}_2)^k$$

with $k = |\{t \in \{0,1\} \mid Y_t \neq \varnothing\}|$.
If $d_0 = d_1$ and $Y_0 = Y_1 =: Y$ with $Y \neq \varnothing$, then

$$N_{\mathrm{Aut}(T)}(G^+_{(i)}(Y, Y)) \Big/ G^+_{(i)}(Y, Y) \cong \mathbf{D}_8 \,.$$

Using the fact that the pointwise stabilizers of half-trees are non-trivial in these groups $G^+_{(i)}(Y_0, Y_1)$, one can also show that they are not linear over a local field (see [CRW17, Corollary R]), and even not locally linear (as defined in [CS15]).

For any group $H \in \mathcal{H}_T$, Burger and Mozes proved that the subgroup $H^{(\infty)}$ of H defined as the intersection of all finite index open subgroups of H is such that $H^{(\infty)} \in \mathcal{H}^+_T$ and $H^{(\infty)}$ is topologically simple (see [BM00a, Proposition 3.1.2]). Our main classification theorem reads as follows. Note that two groups in $\mathcal{H}_T$ are topologically isomorphic if and only if they are conjugate in $\mathrm{Aut}(T)$ (see Proposition 1.I.1 in Appendix 1.I), so this is a classification up to topological isomorphism.

Theorem 1.B (Classification)**.** *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 4$ and let i be a legal coloring of T. Let $\underline{\mathcal{S}}_{(i)}$ be the set of* *groups* $G^+_{(i)}(Y_0, Y_1)$ *where* Y_0 *and* Y_1 *are finite subsets of* $\mathbf{Z}_{\geq 0}$ *satisfying the following condition: if* Y_0 *and* Y_1 *are both non-empty, then for each* $y \in Y_t$ *(with* $t \in \{0, 1\}$*), if* $y \geq \max Y_{1-t}$ *then* $y \equiv \max Y_t \bmod 2$.

(i) Two groups $G^+_{(i)}(Y_0, Y_1)$ *and* $G^+_{(i)}(Y'_0, Y'_1)$ *belonging to* $\underline{\mathcal{S}}_{(i)}$ *are conjugate in* $\mathrm{Aut}(T)$ *if and only if* $(Y_0, Y_1) = (Y'_0, Y'_1)$ *or* $d_0 = d_1$ *and* $(Y_0, Y_1) = (Y'_1, Y'_0)$.

(ii) Suppose that $d_0, d_1 \geq 6$*. Let* $H \in \mathcal{H}^+_T$ *be such that* $\underline{H}(x) \cong F_0 \geq \mathrm{Alt}(d_0)$ *for each* $x \in V_0(T)$ *and* $\underline{H}(y) \cong F_1 \geq \mathrm{Alt}(d_1)$ *for each* $y \in V_1(T)$*. Then* $[H : H^{(\infty)}] \in \{1, 2, 4\}$ *and* $H^{(\infty)}$ *is conjugate in* $\mathrm{Aut}(T)^+$ *to a group belonging to* $\underline{\mathcal{S}}_{(i)}$.

We actually give, in the text, the exact description of all groups $H \in \mathcal{H}^+_T$ satisfying the hypotheses of Theorem 1.B (ii) (see Theorem 1.B′). The condition $d_0, d_1 \geq 6$ is used several times in our proof and is actually necessary. Indeed, due to the exceptional isomorphisms $\mathrm{PSL}(2, \mathbf{F}_3) \cong \mathrm{Alt}(4)$ and $\mathrm{PSL}(2, \mathbf{F}_4) = \mathrm{SL}(2, \mathbf{F}_4) \cong \mathrm{Alt}(5)$, the linear groups $\mathrm{PSL}(2, \mathbf{F}_3(\!(X)\!))$ and $\mathrm{PSL}(2, \mathbf{F}_4(\!(X)\!))$, which act on their respective Bruhat–Tits trees T_4 and T_5 (where T_d is the d-regular tree), are elements of $\mathcal{H}^+_{T_4}$ and $\mathcal{H}^+_{T_5}$ respectively whose local action at each vertex is the alternating group. This shows that Theorem 1.B (ii) fails when $d_0 = d_1 \in \{4, 5\}$. In §1.6, we also give a non-linear counterexample when $d_0 = 4$ and $d_1 \geq 4$.

As a corollary of Theorem 1.B, we find the corresponding result for $H \in \mathcal{H}_T \setminus \mathcal{H}^+_T$ (when $d_0 = d_1$, so that $\mathcal{H}^+_T \subsetneq \mathcal{H}_T$). In this case, H is automatically transitive on $V(T)$.

Corollary 1.C. *Let* T *be the* d*-regular tree with* $d \geq 6$ *and let* i *be a legal coloring of* T*. Let* $H \in \mathcal{H}_T \setminus \mathcal{H}^+_T$ *be such that* $\underline{H}(v) \cong F \geq \mathrm{Alt}(d)$ *for each* $v \in V(T)$*. Then* $[H : H^{(\infty)}] \in \{2, 4, 8\}$ *and* $H^{(\infty)}$ *is conjugate in* $\mathrm{Aut}(T)^+$ *to* $G^+_{(i)}(Y, Y)$ *for some finite subset* Y *of* $\mathbf{Z}_{\geq 0}$.

Here again, a full description of all groups $H \in \mathcal{H}_T \setminus \mathcal{H}^+_T$ satisfying the hypotheses of Corollary 1.C is given in the text (see Corollary 1.C′).

When $H \in \mathcal{H}_T$, the 2-transitivity of H on ∂T implies that $\underline{H}(v)$ is a 2-transitive permutation group for each $v \in V(T)$ (see Lemma 1.2.2).

The finite 2-transitive permutation groups have been classified, using the Classification of the Finite Simple Groups, and the set of integers

$$\Theta := \{m \geq 6 \mid \text{each 2-transitive subgroup of } \mathrm{Sym}(m) \text{ contains } \mathrm{Alt}(m)\}$$

is known (see Proposition 1.II.1 in Appendix 1.II). The ten smallest numbers in Θ are 34, 35, 39, 45, 46, 51, 52, 55, 56 and 58. Moreover, Θ is asymptotically dense in $\mathbf{Z}_{>0}$ (see Corollary 1.II.2). When $d_0, d_1 \in \Theta$, the hypotheses of Theorem 1.B (ii) and Corollary 1.C (if $d_0 = d_1$) are always satisfied (by definition) and we get the following result.

Corollary 1.D. *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \in \Theta$, let i be a legal coloring of T and let $H \in \mathcal{H}_T$. If $H \in \mathcal{H}_T^+$, then $[H : H^{(\infty)}] \in \{1, 2, 4\}$ and $H^{(\infty)}$ is conjugate in $\mathrm{Aut}(T)^+$ to a group belonging to $\underline{\mathcal{S}}_{(i)}$ (as defined in Theorem 1.B). If $d_0 = d_1$ and $H \notin \mathcal{H}_T^+$, then $[H : H^{(\infty)}] \in \{2, 4, 8\}$ and $H^{(\infty)}$ is conjugate in $\mathrm{Aut}(T)^+$ to $G_{(i)}^+(Y, Y)$ for some finite subset Y of $\mathbf{Z}_{\geq 0}$.*

It has also been proved by Burger and Mozes in [BM00a, Propositions 3.3.1 and 3.3.2] that if $H \leq_{cl} \mathrm{Aut}(T)$ is vertex-transitive and if $\underline{H}(v) \cong F \geq \mathrm{Alt}(d)$ with $d \geq 6$, then H is either discrete or 2-transitive on ∂T. We can therefore combine this result with Corollary 1.C to obtain the following.

Corollary 1.E. *Let T be the d-regular tree with $d \geq 6$, let i be a legal coloring of T and let H be a vertex-transitive closed subgroup of $\mathrm{Aut}(T)$. If $\underline{H}(v) \cong F \geq \mathrm{Alt}(d)$ for each $v \in V(T)$, then either H is discrete or $[H : H^{(\infty)}] \in \{2, 4, 8\}$ and $H^{(\infty)}$ is conjugate in $\mathrm{Aut}(T)^+$ to $G_{(i)}^+(Y, Y)$ for some finite subset Y of $\mathbf{Z}_{\geq 0}$.*

For $d \in \Theta$, the condition $\underline{H}(v) \cong F \geq \mathrm{Alt}(d)$ can be replaced by requiring $\underline{H}(v)$ to be 2-transitive. Note that the result of Burger–Mozes stated above is not true if we replace vertex-transitivity by edge-transitivity. Indeed, the group

$$H = \left\{ g \in \mathrm{Aut}(T)^+ \;\middle|\; \begin{array}{c} \forall v \in V_0(T),\, \forall x, y \in S(v, 2): \\ i(x) = i(y) \Rightarrow i(g(x)) = i(g(y)) \end{array} \right\}$$

where i is a legal coloring of T is an example of a closed subgroup of $\mathrm{Aut}(T)$ with $\underline{H}(v) \cong \mathrm{Sym}(d)$ for each $v \in V(T)$ and which is edge-transitive but such that H is non-discrete and the action of H on ∂T is not 2-transitive.

Remark. Due to a result of Trofimov, the hypothesis that T is a tree is not even necessary in Corollary 1.E (see [Tro07, Proposition 3.1]).

In order to prove the classification, we first need to generalize some results of [BM00a] to the case of non-vertex-transitive groups. This leads us to the following side result, which is an analog of [BM00a, Proposition 3.3.1].

Theorem 1.F. *Let T be the (d_0, d_1)-semiregular tree and let $F_0 \leq \mathrm{Sym}(d_0)$ and $F_1 \leq \mathrm{Sym}(d_1)$. Let $H \in \mathcal{H}_T^+$ be such that $\underline{H}(x) \cong F_0$ for each $x \in V_0(T)$ and $\underline{H}(y) \cong F_1$ for each $y \in V_1(T)$. Suppose that, for each $t \in \{0,1\}$, the stabilizer $F_t(1)$ of 1 in F_t is simple non-abelian. Then there exists a legal coloring i of T such that H is equal to the group*

$$U_{(i)}^+(F_0, F_1) := \left\{ g \in \mathrm{Aut}(T)^+ \;\middle|\; \begin{array}{c} \sigma_{(i)}(g,x) \in F_0 \text{ for each } x \in V_0(T), \\ \sigma_{(i)}(g,y) \in F_1 \text{ for each } y \in V_1(T) \end{array} \right\}.$$

Notice that $U_{(i)}^+(\mathrm{Alt}(d_0), \mathrm{Alt}(d_1)) = \mathrm{Alt}_{(i)}(T)^+$.

Structure of the chapter

The proof of the classification is divided into different main steps. The first step, which is the subject of §1.2 (where Theorem 1.F is also proved) and §1.3, consists in showing that the groups satisfying the hypotheses of Theorem 1.B (ii) all contain, up to conjugation, the group $\mathrm{Alt}_{(i)}(T)^+$:

Theorem 1.G. *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 6$. Let $H \in \mathcal{H}_T^+$ be such that $\underline{H}(x) \cong F_0 \geq \mathrm{Alt}(d_0)$ for each $x \in V_0(T)$ and $\underline{H}(y) \cong F_1 \geq \mathrm{Alt}(d_1)$ for each $y \in V_1(T)$. Then there exists a legal coloring i of T such that $H \supseteq \mathrm{Alt}_{(i)}(T)^+$.*

Note that Theorem 1.G is already sufficient to obtain meaningful information on the groups H satisfying the hypotheses. For instance, it follows from Theorem 1.G that the pointwise stabilizer of a half-tree in such a group H is never trivial.

For a fixed legal coloring i, we then find in §1.5 all the groups $H \in \mathcal{H}_T^+$ containing $\mathrm{Alt}_{(i)}(T)^+$. The strategy adopted to do so is somewhat involved and what follows is a rough description of it. Following [BEW15], the n**-closure** $J^{(n)}$ of an arbitrary group $J \leq \mathrm{Aut}(T)$ is defined by

$$J^{(n)} := \{g \in \mathrm{Aut}(T) \mid \forall v \in V(T), \exists h \in J : g|_{B(v,n)} = h|_{B(v,n)}\}.$$

The next important step in our proof then reads as follows.

Theorem 1.H. *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 6$, let i be a legal coloring of T and let $H \in \mathcal{H}_T^+$ be such that $H \supseteq \mathrm{Alt}_{(i)}(T)^+$. Then there exists $K \in \mathbf{Z}_{\geq 0}$ such that $H = H^{(K)}$.*

Theorem 1.H is crucial, since it means that H is completely determined by its local action on T on a sufficiently large scale. In particular, observe that for each $K \in \mathbf{Z}_{\geq 0}$ there is only a finite number of groups $H \in \mathcal{H}_T^+$ with $H \supseteq \mathrm{Alt}_{(i)}(T)^+$ and such that $H = H^{(K)}$. This already implies that the classification will lead to a countable family of groups. The idea to complete the classification is finally to fix K, to find an upper bound to the number of groups H satisfying the hypotheses and such that $H = H^{(K)}$, and to show that this upper bound is achieved by the various groups from the explicit list described beforehand. These groups are all defined in §1.4, where Theorem 1.A is also proved (see Lemma 1.4.2, Theorem 1.4.6 and Lemma 1.4.10).

1.2 From local to global structure

In this section, we consider a group $H \in \mathcal{H}_T^+$ and analyze how the knowledge of $\underline{H}(v)$ for each $v \in V(T)$ has an impact on the global structure of H. This section is largely inspired from the work of Burger–Mozes [BM00a]. Our goal is to generalize several of their results to the situation where the groups are not vertex-transitive.

Most of our notations come from [BM00a]. If $x \in V(T)$, then $S(x, n)$ (resp. $B(x, n)$) is the set of vertices of T at distance exactly n (resp.

at most n) from x. We also set $c(x,n) := |S(x,n)|$. If $n \geq 0$ and $x_1, \dots, x_k \in V(T)$, then define $H_n(x_1, \dots, x_k)$ to be the pointwise stabilizer of $\bigcup_{i=1}^k B(x_i, n)$. In the particular case where $n = 0$, we write $H(x_1, \dots, x_k)$ instead of $H_0(x_1, \dots, x_k)$ as it is simply the stabilizer of vertices $x_1, \dots, x_k$. For $x \in V(T)$, set

$$\underline{H}_n(x) := H_n(x) \Big/ H_{n+1}(x) \,.$$

Once again, for $n = 0$ we write $\underline{H}(x)$ instead of $\underline{H}_0(x)$ and this exactly corresponds to the definition of $\underline{H}(x)$ given in §1.1.

We start by giving the following results which will be used throughout this chapter.

Lemma 1.2.1. *Let T be a locally finite tree whose vertices have valency at least 3 and let $H \leq_{cl} \mathrm{Aut}(T)$. Then H is 2-transitive on ∂T if and only if $H(v)$ is transitive on ∂T for each $v \in V(T)$.*

Proof. See [BM00a, Lemma 3.1.1]. □

Lemma 1.2.2. *Let T be a locally finite tree whose vertices have valency at least 3 and let $H \leq_{cl} \mathrm{Aut}(T)$ be acting 2-transitively on ∂T. Then T is semiregular and, for each $x, x', y, y' \in V(T)$ such that x and x' have the same type and $d(x,y) = d(x',y')$, there exists $h \in H$ such that $h(x) = x'$ and $h(y) = y'$.*

Proof. This is a direct consequence of Lemma 1.2.1. □

1.2.1 Subgroups of products of finite simple groups

Lemma 1.2.5 below is a basic result about finite groups and will play a fundamental role in the sequel. Its statement comes from [BM00a, Lemma 3.4.3], but the proof therein requires supplementary details because the definition of a *product of subdiagonals* needs to be amended (probably due to a misnomer). The result could be deduced from Goursat's Lemma (see [Gou89, §11–12] and [Lan02, Exercise I.5]); we provide a self-contained proof for the reader's convenience.

Given a product of groups $G_1 \times \cdots \times G_n$ and $i_1, \ldots, i_m \in \{1, \ldots, n\}$, we write $\mathrm{proj}_{i_1,\ldots,i_m} \colon G_1 \times \cdots \times G_n \to G_{i_1} \times \cdots \times G_{i_m}$ for the projection on factors $G_{i_1}, \ldots, G_{i_m}$.

Lemma 1.2.3. *Let S be a finite simple non-abelian group and let $G \leq S^n$ $(n \geq 2)$. If $\mathrm{proj}_{i,j}(G) = S^2$ for each $1 \leq i < j \leq n$, then $G = S^n$.*

Proof. We prove by induction on m that $\mathrm{proj}_{i_1,\ldots,i_m}(G) = S^m$ for each $1 \leq i_1 < \cdots < i_m \leq n$. By hypothesis this is true for $m = 2$. Now let $m \geq 3$ and suppose it is true for $m - 1$. Given $1 \leq i_1 < \cdots < i_m \leq n$, we need to show that $\mathrm{proj}_{i_1,\ldots,i_m}(G) = S^m$. For any $k \in \{1, \ldots, m\}$, we have that $\mathrm{proj}_{i_k}(\ker(\mathrm{proj}_{i_1,\ldots,\widehat{i_k},\ldots i_m})) \trianglelefteq S$, so it is either trivial or equal to S (since S is simple). In the latter case, since $\mathrm{proj}_{i_1,\ldots,\widehat{i_k},\ldots i_m}(G) = S^{m-1}$ by induction hypothesis, we directly get that $\mathrm{proj}_{i_1,\ldots,i_m}(G) = S^m$ and we are done. Now we assume that $\mathrm{proj}_{i_k}(\ker(\mathrm{proj}_{i_1,\ldots,\widehat{i_k},\ldots i_m}))$ is trivial for all k $(*)$. Taking $k = m$ in $(*)$, we get that there exists $\alpha \colon S^{m-1} \to S$ such that $\mathrm{proj}_{i_m}(g) = \alpha(\mathrm{proj}_{i_1,\ldots,i_{m-1}}(g))$ for all $g \in G$. Moreover, $k = 1$ in $(*)$ implies that the map $\beta \colon S \to S$ defined by $\beta(s) = \alpha(s, 1, \ldots, 1)$ is injective, and hence surjective. For the same reason with $k = 2$, the map $\gamma \colon S \to S$ defined by $\gamma(s) = \alpha(1, s, 1, \ldots, 1)$ is surjective. Since $\beta(s) \cdot \gamma(s') = \alpha(s, s', 1, \ldots, 1) = \gamma(s') \cdot \beta(s)$ for all $s, s' \in S$, we get that S is abelian, a contradiction. $\square$

Given a group S and a positive integer n, a **product of subdiagonals** of S^n is a subgroup of S^n of the form $(\alpha_1 \times \cdots \times \alpha_n)(\Delta_{I_1} \cdots \Delta_{I_r})$, where $\{I_j \mid 1 \leq j \leq r\}$ is a partition of $\{1, \ldots, n\}$, Δ_J is defined by $\Delta_J := \{(s_1, \ldots, s_n) \in S^n \mid s_i = 1 \ \forall i \notin J \text{ and } s_k = s_\ell \ \forall k, \ell \in J\}$ for each subset $J \subseteq \{1, \ldots, n\}$, and $\alpha_1, \ldots, \alpha_n \in \mathrm{Aut}(S)$. Here, $\alpha_1 \times \cdots \times \alpha_n \in \mathrm{Aut}(S^n)$ is the Cartesian product of $\alpha_1, \ldots, \alpha_n$.

Lemma 1.2.4. *Let S be a finite simple non-abelian group and let $G \leq S^n$ $(n \geq 1)$. If $\mathrm{proj}_i(G) = S$ for each $i \in \{1, \ldots, n\}$, then G is a product of subdiagonals of S^n.*

Proof. For any $i, j \in \{1, \ldots, n\}$, we have $\mathrm{proj}_j(\ker(\mathrm{proj}_i)) \trianglelefteq S$, so it is either trivial or equal to S. Let us define the relation $\sim$ on $\{1, \ldots, n\}$ by $i \sim j$ if and only if $\mathrm{proj}_j(\ker(\mathrm{proj}_i))$ is trivial. We claim that $\sim$ is an equivalence relation. Reflexivity and transitivity are clear. Let us prove that it is also symmetric. For $i, j \in \{1, \ldots, n\}$, write $G_{i,j} := \mathrm{proj}_{i,j}(G)$. Then $\mathrm{proj}_i|_{G_{i,j}} : G_{i,j} \to S$ has image S by hypothesis, and its kernel is trivial if and only if $i \sim j$. So $|G_{i,j}| = |S|$ if and only if $i \sim j$. It follows directly that $\sim$ is symmetric and hence an equivalence relation.

Now let $I_1, \ldots, I_r$ be the equivalence classes of $\sim$: they form a partition of $\{1, \ldots, n\}$. For each $1 \leq j \leq r$, choose $x_j \in I_j$. For such a j and for $y \in I_j$, we have $x_j \sim y$ and thus $\ker(\mathrm{proj}_{x_j}) = \ker(\mathrm{proj}_y)$. As a consequence, there exists $\alpha_y \in \mathrm{Aut}(S)$ such that $\mathrm{proj}_y(g) = \alpha_y(\mathrm{proj}_{x_j}(g))$ for all $g \in G$. Combined with the fact that $\mathrm{proj}_{x_1,\ldots,x_r}(G) = S^r$ (by Lemma 1.2.3, because $x_i \not\sim x_j$ implies $\mathrm{proj}_{x_i,x_j}(G) = S^2$), we obtain that G is a product of subdiagonals of S^n whose underlying partition is $\{I_j \mid 1 \leq j \leq r\}$. ☐

Lemma 1.2.5. *Let $S \trianglelefteq L$ be finite groups, where L/S is solvable and S is simple non-abelian. Let $G \leq L^n$ ($n \geq 1$) be such that* $\mathrm{proj}_i(G) \geq S$ *for all $i \in \{1, \ldots, n\}$. Then $G \cap S^n$ is a product of subdiagonals of S^n.*

Proof. In view of Lemma 1.2.4, it suffices to show that $\mathrm{proj}_i(G \cap S^n) = S$ for each $i \in \{1, \ldots, n\}$. Given a group H, we write $H^{(0)} = H$ and $H^{(k)} = [H^{(k-1)}, H^{(k-1)}]$ for each $k \geq 1$. Since L/S is solvable, there exists k such that $(L/S)^{(k)}$ is trivial. This implies that $L^{(k)} \leq S$. Hence, we obtain $G^{(k)} \leq (L^n)^{(k)} = (L^{(k)})^n \leq S^n$ and

$$\mathrm{proj}_i(G \cap S^n) \geq \mathrm{proj}_i(G^{(k)}) = \mathrm{proj}_i(G)^{(k)} \geq S^{(k)} = S.$$ ☐

1.2.2 Kernel of the action on balls

We can now start to adapt the results [BM00a, Lemmas 3.4.2, 3.5.1 and 3.5.3] to the case of groups that are not vertex-transitive. Note that the proofs of some of our results are significantly more complicated because of this missing hypothesis.

Lemma 1.2.6. *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 3$ and let $H \in \mathcal{H}_T^+$. Let x and y be adjacent vertices of T and let $k \geq 1$. Then $H_k(x) \neq H_k(y)$. In particular, $\underline{H}_{k-1}(x)$ or $\underline{H}_{k-1}(y)$ is non-trivial.*

Proof. Assume for a contradiction that $H_k(x) = H_k(y)$. Since $H_k(x) \trianglelefteq H(x)$ and $H_k(y) \trianglelefteq H(y)$, we get $H_k(x) \trianglelefteq \langle H(x), H(y)\rangle = H$. As H is transitive on $V_0(T)$ and $V_1(T)$, this means that $H_k(x) = H_k(x')$ for each $x' \in V(T)$, implying that $H_k(x)$ is trivial. This is impossible as H would then be countable, which contradicts its 2-transitivity on ∂T.

In particular, $H_k(x)\setminus H_k(y)$ or $H_k(y)\setminus H_k(x)$ is non-empty. If $H_k(x)\setminus H_k(y) \neq \varnothing$, then there exists $h \in H_k(x) \setminus H_k(y) \subseteq H_{k-1}(y) \setminus H_k(y)$ and hence $\underline{H}_{k-1}(y)$ is non-trivial. If $H_k(y) \setminus H_k(x) \neq \varnothing$ then we get that $\underline{H}_{k-1}(x)$ is non-trivial. $\square$

Recall that the **socle** of a group G is the subgroup generated by the minimal non-trivial normal subgroups of G. In the next results, we will often use the easy fact that if G is a finite group whose socle S is simple and of index at most 2 in G, then S is the only non-trivial proper normal subgroup of G. If, moreover, S is non-abelian, then it follows that the center $Z(G)$ of G is trivial.

Lemma 1.2.7. *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 3$ and let $F_1 \leq \mathrm{Sym}(d_1)$. Let $H \in \mathcal{H}_T^+$ be such that $\underline{H}(y) \cong F_1$ for each $y \in V_1(T)$. Suppose that the socle S_1 of the stabilizer $F_1(1)$ of 1 in F_1 is simple non-abelian and of index ≤ 2. Then for each $x \in V_0(T)$, one of the following holds.*

(A) $H_1(x, y) = H_2(x)$ for each $y \in S(x, 1)$.

(B) $\underline{H}_1(x) \supseteq (S_1)^{d_0}$, where $\underline{H}_1(x)$ is seen in the natural way as a subgroup of $(F_1(1))^{d_0}$.

Proof. Fix $x \in V_0(T)$. For each vertex $y \in S(x, 1)$, the inclusion $H_1(x) \subseteq H(x, y)$ induces a homomorphism $\varphi_y \colon H_1(x) \to H(x, y) \big/ H_1(y) =: H_{x,y}$ which is such that $\varphi_y(H_1(x)) \trianglelefteq H_{x,y}$. Note that $H_{x,y} \cong F_1(1)$. This also gives rise to an injective homomorphism

$$\varphi \colon \underline{H}_1(x) \to \prod_{y \in S(x,1)} H_{x,y} \cong (F_1(1))^{d_0}.$$

As $\varphi_y(H_1(x)) \trianglelefteq H_{x,y}$ and $H_1(x) \trianglelefteq H(x)$, there are only two possibilities: either $\varphi_y(H_1(x))$ is trivial for each $y \in S(x, 1)$, or $\varphi_y(H_1(x)) \supseteq S_1$

(via the isomorphism $H_{x,y} \cong F_1(1)$) for each $y \in S(x,1)$. In the first case, we directly get $H_1(x) = H_1(x,y)$ for each $y \in S(x,1)$, which implies $H_1(x) = H_2(x)$ and in particular $H_1(x,y) = H_2(x)$ for each $y \in S(x,1)$. In the second case, by Lemma 1.2.5 the group $\varphi(\underline{H}_1(x)) \cap (S_1)^{d_0}$ is a product of subdiagonals. These subdiagonals determine a bloc decomposition for the $H(x)$-action on $S(x,1)$. As this action is 2-transitive (by Lemma 1.2.2), there are two options: it is either the full group $(S_1)^{d_0}$ or a full diagonal $(\alpha_1 \times \cdots \times \alpha_{d_0})(\Delta_{\{1,\ldots,d_0\}})$ (with the notation given in §1.2.1). If it is the full group, then $\varphi(\underline{H}_1(x)) \supseteq (S_1)^{d_0}$ as wanted. Otherwise, $H_1(x,y) \big/ H_2(x)$ is a 2-group for each $y \in S(x,1)$. In particular, if $z \in S(x,1)$ with $z \neq y$ then the image I of $H_1(x,y)$ in $H_{x,z} = H(x,z) \big/ H_1(z) \cong F_1(1)$ is a subnormal 2-group of $H_{x,z}$ (because $H_1(x,y) \trianglelefteq H_1(x) \trianglelefteq H(x,z)$). Since S_1 is not a 2-group, the only possibility for I is to be trivial. We thus have $H_1(x,y) \subseteq H_1(z)$ for each $z \in S(x,1)$, which means that $H_1(x,y) = H_2(x)$. $\square$

Lemma 1.2.8. *Let T be the (d_0,d_1)-semiregular tree with $d_0,d_1 \geq 3$ and let $F_0 \leq \mathrm{Sym}(d_0)$ and $F_1 \leq \mathrm{Sym}(d_1)$. Let $H \in \mathcal{H}_T^+$ be such that $\underline{H}(x) \cong F_0$ for each $x \in V_0(T)$ and $\underline{H}(y) \cong F_1$ for each $y \in V_1(T)$. Suppose that, for each $t \in \{0,1\}$, the socle S_t of $F_t(1)$ is simple non-abelian and of index ≤ 2. Fix two adjacent vertices $x \in V_0(T)$ and $y \in V_1(T)$ and let $k \geq 1$. Assume that $\underline{H}_k(x) \supseteq (S_{k \bmod 2})^{c(x,k)}$ and, if $k \neq 1$, that $\underline{H}_{k-1}(x) \supseteq (S_{(k-1) \bmod 2})^{c(x,k-1)}$. Then $\underline{H}_k(y)$ is non-trivial.*

Proof. For $z \in S(x,n)$, let $p(z)$ be the vertex at distance $n-1$ from x that is adjacent to z and $H_{x,z} := H(z,p(z)) \big/ H_1(z)$. Define also $S_n(x,y)$ to be the set of vertices of $S(x,n)$ that are at distance $n-1$ from y and $a(x,n) := |S_n(x,y)|$.

For simplicity, we set $s := k \bmod 2$ and $t := (k-1) \bmod 2$. We first claim that there exists $g \in H_{k-1}(x,y) \setminus H_k(x)$ whose image $\sigma(g)$ in $\prod_{z \in S_k(x,y)} H_{x,z} \cong (F_s(1))^{a(x,k)}$ is contained in $(S_s)^{a(x,k)}$. First remark that $H_{k-1}(x,y) \setminus H_k(x)$ is non-empty in view of the hypothesis $\underline{H}_{k-1}(x) \supseteq (S_t)^{c(x,k-1)}$ (if $k=1$, use $H(x,y) \big/ H_1(x) \cong F_t(1) \supseteq S_t$). Hence, if $F_s(1) = S_s$ the claim is trivially true. On the other hand, if $[F_s(1) : S_s] = 2$ then take $h \in H_{k-1}(x,y)$ such that $h^2 \in H_{k-1}(x,y) \setminus H_k(x)$. Such an element exists as S_t is not a 2-group. Then $g = h^2$

satisfies the claim.

Now take $g' \in H_k(x)$ such that $\sigma(g') = \sigma(g)$, whose existence is ensured by the fact that $\underline{H}_k(x) \supseteq (S_s)^{c(x,k)}$. Then the element $g'g^{-1}$ is contained in $H_k(y)$ but not in $H_{k+1}(y)$ (by construction), so $\underline{H}_k(y)$ is non-trivial. $\square$

In the proof of the following lemma, we use the Schreier conjecture stating that $\mathrm{Out}(S)$ is solvable for each finite simple group S. This conjecture has been proved using the Classification of the Finite Simple Groups. Note however that, except for Theorem 1.F, we will only use Lemma 1.2.9 with $S_0 = \mathrm{Alt}(d_0)$ and $S_1 = \mathrm{Alt}(d_1)$, in which case the solvability of $\mathrm{Out}(S_0)$ and $\mathrm{Out}(S_1)$ is clear.

Lemma 1.2.9. *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 3$ and let $F_0 \leq \mathrm{Sym}(d_0)$ and $F_1 \leq \mathrm{Sym}(d_1)$. Let $H \in \mathcal{H}_T^+$ be such that $\underline{H}(x) \cong F_0$ for each $x \in V_0(T)$ and $\underline{H}(y) \cong F_1$ for each $y \in V_1(T)$. Suppose that, for each $t \in \{0,1\}$, the socle S_t of $F_t(1)$ is simple non-abelian, of index ≤ 2 and transitive but not simply transitive on $\{2,\dots,d_t\}$. Then $\underline{H}_1(x) \supseteq (S_1)^{d_0}$ for each $x \in V_0(T)$ and $\underline{H}_1(y) \supseteq (S_0)^{d_1}$ for each $y \in V_1(T)$.*

Proof. For each $v \in V(T)$, we can apply Lemma 1.2.7. This gives two possibilities ((A) or (B)) at each vertex of T. As H is transitive on $V_0(T)$ and $V_1(T)$, the situation must be identical at all vertices of the same type. In total, there are three possible situations: (A) for all vertices, (A) for one type of vertices and (B) for the other, or (B) for all vertices. To prove the statement, we must show that the only situation that really occurs is the last one. To do so, we prove that the two other situations are impossible.

We already know that we cannot have (A) for all vertices, since it would imply that $H_2(x) = H_1(x,y) = H_2(y)$ for two adjacent vertices x and y, contradicting Lemma 1.2.6.

Now assume for a contradiction that we have (A) for $V_0(T)$ and (B) for $V_1(T)$ (the reverse situation being identical). If $x \in V_0(T)$ and $y \in S(x,1)$, then (A) means that $H_1(x,y) = H_2(x)$. The homomorphism $\varphi_y \colon H_1(x) \to H(x,y) \big/ H_1(y) \cong F_1(1)$ has a normal image and its kernel is exactly $H_1(x,y) = H_2(x)$. Hence, $\underline{H}_1(x) = {H_1(x)} \big/ {H_2(x)}$ is isomorphic to a normal subgroup of $F_1(1)$: it is either trivial or isomorphic to S_1 or $F_1(1)$. By Lemma 1.2.8 (with $k = 1$), since $\underline{H}_1(y) \supseteq (S_0)^{d_1}$, $\underline{H}_1(x)$ cannot be trivial.

For the sake of brevity, set $\tilde{H} := \underline{H}_1(x)$ and $G := {H(x)} \big/ {H_2(x)}$. We have shown that $\tilde{H}$ is isomorphic to S_1 or $F_1(1)$, which implies that the center $Z(\tilde{H})$ of $\tilde{H}$ is trivial, and $\tilde{H}$ is a normal subgroup of G. Hence, G contains the direct product of $\tilde{H}$ and its centralizer $C_G(\tilde{H})$ (the intersection of these two normal subgroups being $Z(\tilde{H})$).

Claim. *The product $\tilde{H} \cdot C_G(\tilde{H})$ is a subgroup of index at most 2 of G.*

Proof of the claim: Consider the homomorphism

$$\alpha \colon G \to \mathrm{Out}(\tilde{H}) \colon g \mapsto [h \in \tilde{H} \mapsto ghg^{-1} \in \tilde{H}].$$

An element $g \in G$ is in the kernel of α if and only if there exists $k \in \tilde{H}$ such that $ghg^{-1} = khk^{-1}$ for all $h \in \tilde{H}$, which is equivalent to saying that $k^{-1}g \in C_G(\tilde{H})$. Hence, $\ker(\alpha) = \tilde{H} \cdot C_G(\tilde{H})$. We write $K := \tilde{H} \cdot C_G(\tilde{H})$ and want to show that $[G : K] \leq 2$. Since $K = \ker(\alpha)$, the quotient G/K can be embedded into $\mathrm{Out}(\tilde{H})$. By the Schreier conjecture (see [DM96, Appendix A]), $\mathrm{Out}(S_1)$ is solvable. As $\tilde{H} \cong S_1$ or $F_1(1)$, it implies that $\mathrm{Out}(\tilde{H})$ is solvable. Indeed, if $[F_1(1) : S_1] = 2$ then there is a natural map $j \colon \mathrm{Aut}(F_1(1)) \to \mathrm{Out}(S_1)$, and one can show that $\ker(j) \subseteq \mathrm{Inn}(F_1(1))$, so that ${\mathrm{Aut}(F_1(1))} \big/ {\ker(j)} \cong \mathrm{im}(j) \leq \mathrm{Out}(S_1)$ surjects onto $\mathrm{Out}(F_1(1))$, making it solvable.

We just proved that G/K is solvable. By the third isomorphism theorem, we have

$$\left(G/\tilde{H}\right) \big/ \left(K/\tilde{H}\right) \cong G/K.$$

Since $G/\tilde{H} \cong F_0$, this means that G/K is isomorphic to a quotient of F_0, let us say F_0/N with $N \trianglelefteq F_0$. There remains to show that $[F_0 : N] \leq 2$, using the fact that F_0/N is solvable. Consider the injective map $i \colon {F_0(1)} \big/ {N(1)} \hookrightarrow F_0/N$ where $N(1)$ is the stabilizer of 1 in N. Since F_0/N is solvable, ${F_0(1)} \big/ {N(1)}$ is also solvable. However, $N(1)$

can only be trivial or equal to $F_0(1)$ or S_0. It cannot be trivial as $F_0(1)$ is not solvable, so $\left|F_0(1) \big/ N(1)\right| \leq 2$. In particular, N is a non-trivial normal subgroup of the 2-transitive group F_0, which implies that N is transitive. Hence, the map i defined above is an isomorphism, and $\left|F_0 \,/ N\right| = \left|F_0(1) \big/ N(1)\right| \leq 2$ as wanted. ∎

Using the fact that $\tilde{H} \cdot C_G(\tilde{H})$ is a subgroup of index 1 or 2 of G, one can find a contradiction. Denote by $v_1, \dots, v_{d_0}$ the vertices adjacent to x and by $a_1^{(1)}, \dots, a_{d_1-1}^{(1)}$ the vertices adjacent to v_1 different from x (see Figure 1.2). As a corollary of the claim, the group $C_G(\tilde{H})$ acts non-trivially and therefore transitively on $S(x,1) = \{v_1, \dots, v_{d_0}\}$. Hence, there exist $c_2, \dots, c_{d_0} \in C_G(\tilde{H})$ such that $c_k(v_1) = v_k$ for each $k \in \{2, \dots, d_0\}$. Define $a_i^{(k)} = c_k(a_i^{(1)})$ for each $k \in \{2, \dots, d_0\}$ and $i \in \{1, \dots, d_1 - 1\}$. In this way, for each k the vertices $a_1^{(k)}, \dots, a_{d_1-1}^{(k)}$ are the vertices adjacent to v_k different from x. Thanks to this choice, if $h \in \tilde{H}$ satisfies $h(a_i^{(1)}) = a_j^{(1)}$ for some i and j then the fact that $hc_k = c_k h$ directly implies that $h(a_i^{(k)}) = a_j^{(k)}$ for each $k \in \{2, \dots, d_0\}$. In other words, as soon as the action of $h \in \tilde{H}$ on the vertices adjacent to v_1 is known, its action on the vertices adjacent to v_k is also known for each $k \in \{2, \dots, d_0\}$.

Now consider $c \in C_G(\tilde{H})$ with $c(v_k) = v_\ell$, for some $k, \ell \in \{1, \dots, d_0\}$. If we write $c(a_i^{(k)}) = a_{\sigma(i)}^{(\ell)}$ (for all i) with $\sigma \in \mathrm{Sym}(d_1 - 1)$, then the fact that c centralizes $\tilde{H}$ implies that σ centralizes S_1. Denote by $O_1, \dots, O_r$ the distinct orbits of $C_{\mathrm{Sym}(d_1-1)}(S_1)$, forming a partition of $\{1, \dots, d_1 - 1\}$. Since S_1 is transitive on $\{1, \dots, d_1 - 1\}$, we directly get that $|O_1| = \dots = |O_r|$ and that S_1 preserves the partition $O_1 \sqcup \dots \sqcup O_r$. If $r = 1$, then

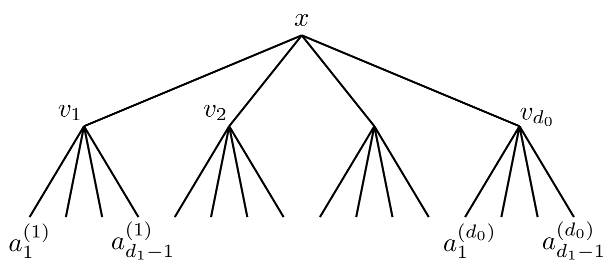


Figure 1.2: Illustration of Lemma 1.2.9.

$C_{\mathrm{Sym}(d_1-1)}(S_1)$ is transitive and hence S_1 is simply transitive, which is impossible by hypothesis. If $r = 2$, then $\{s \in S_1 \mid s(O_1) = O_1\}$ is a subgroup of index 2 of S_1, which contradicts its simplicity. Hence, we must have $r \geq 3$.

We now explain how this contradicts the 2-transitivity of H. Let us look at the possible images of the ordered pair $(a_1^{(1)}, a_1^{(2)})$ by elements of G. In view of Lemma 1.2.2, for all distinct $k, \ell \in \{1, \dots, d_0\}$ and all $i, j \in \{1, \dots, d_1 - 1\}$ there should exist some element $g \in G$ such that $g((a_1^{(1)}, a_1^{(2)})) = (a_i^{(k)}, a_j^{(\ell)})$. This means that $\left|G \cdot (a_1^{(1)}, a_1^{(2)})\right| = d_0(d_0 - 1)(d_1 - 1)^2$. However, in view of what has been observed above, the image of $(a_1^{(1)}, a_1^{(2)})$ by an element of $\tilde{H}$ is always of the form $(a_i^{(1)}, a_i^{(2)})$, and the image of $(a_i^{(1)}, a_i^{(2)})$ by an element of $C_G(\tilde{H})$ is always of the form $(a_j^{(k)}, a_{j'}^{(\ell)})$ with j and j' in the orbit $O_a \ni i$. Consequently, we have $|(\tilde{H} \cdot C_G(\tilde{H})) \cdot (a_1^{(1)}, a_1^{(2)})| \leq d_0(d_0 - 1)r\left(\frac{d_1-1}{r}\right)^2$ (because there are r orbits, each of size $\frac{d_1-1}{r}$). Since $[G : (\tilde{H} \cdot C_G(\tilde{H}))] \leq 2$, this implies that $\left|G \cdot (a_1^{(1)}, a_1^{(2)})\right| \leq \frac{2}{r} \cdot d_0(d_0 - 1)(d_1 - 1)^2$, which contradicts the fact that $r \geq 3$. □

Proposition 1.2.10. *Under the assumptions of Lemma 1.2.9 and for each $x \in V(T)$ and each $k \in \mathbf{Z}_{>0}$, we have*

$$\underline{H}_k(x) \supseteq (S_{(t+k) \bmod 2})^{c(x,k)},$$

where $t \in \{0, 1\}$ is the type of x.

Proof. For $x \in V(T)$ and $z \in S(x, n)$, set $H_{x,z} := H(z, p(z)) \big/ H_1(z)$ where $p(z)$ is the vertex at distance $n - 1$ from x that is adjacent to z. For $y \in S(x, 1)$, let also $S_n(x, y)$ be the set of vertices of $S(x, n)$ that are at distance $n - 1$ from y and $a(x, n) := |S_n(x, y)|$.

We prove the result by induction on k. For $k = 1$, this is exactly Lemma 1.2.9. Now let $k \geq 2$ and assume the result is proved for $k - 1$ (and for all vertices). We show that it is therefore also true for k. By Lemma 1.2.6 and since H is transitive on $V_0(T)$ and $V_1(T)$, $\underline{H}_k(x)$ is non-trivial for each $x \in V_0(T)$ or $\underline{H}_k(y)$ is non-trivial for each $y \in V_1(T)$. Assume without loss of generality that $\underline{H}_k(x)$ is non-trivial for each $x \in V_0(T)$. We first prove that $\underline{H}_k(x) \supseteq (S_s)^{c(x,k)}$ for each $x \in V_0(T)$, where $s := k \bmod 2$.

Fix $x \in V_0(T)$. For each $y \in S(x,1)$, let I_y be the image of $H_{k-1}(y)$ in the product $\prod_{z\in S_k(x,y)} H_{x,z}$. By the induction hypothesis, we have $I_y \supseteq (S_s)^{a(x,k)}$. But $H_k(x) \trianglelefteq H_{k-1}(y)$, so if I'_y is the image of $H_k(x)$ in this product, then $I'_y \trianglelefteq I_y$ and $I'_y \cap (S_s)^{a(x,k)} \trianglelefteq (S_s)^{a(x,k)}$. The only normal subgroups of $(S_s)^{a(x,k)}$ are the products made from the trivial group and the full group S_s. By transitivity of $H(x)$ on $S(x,k)$ (see Lemma 1.2.1), $I'_y \cap (S_s)^{a(x,k)}$ must be either trivial or equal to $(S_s)^{a(x,k)}$. Suppose that $I'_y \cap (S_s)^{a(x,k)}$ is trivial. Then I'_y is trivial, since the contrary and the fact that $I'_y \trianglelefteq I_y$ would imply that $F_s(1)$ has a normal subgroup of order 2, which is not the case. Then, by transitivity of $H(x)$ on $S(x,1)$, I'_y must be trivial for each $y \in S(x,1)$. This is impossible as $\underline{H}_k(x)$ is non-trivial. Hence, I'_y contains $(S_s)^{a(x,k)}$.

Now $\underline{H}_k(x)$ is the image of $H_k(x)$ in $\prod_{y\in S(x,1)} \prod_{z\in S_k(x,y)} H_{x,z}$, so from Lemma 1.2.5 we deduce that $\underline{H}_k(x) \cap (S_s)^{c(x,k)}$ is a product of subdiagonals in $(S_s)^{c(x,k)}$. We claim that it must be the full group $(S_s)^{c(x,k)}$. By contradiction, suppose it is not the case. Then the product of subdiagonals induces a bloc decomposition $\{B_i\}_{1\le i\le r}$ for the $H(x)$-action on $S(x,k)$ with $|B_{i_0}| \geq 2$ for some i_0 and $|B_i \cap S_k(x,y)| \leq 1$ for all i and $y \in S(x,1)$ (because $I'_y \supseteq (S_s)^{a(x,k)}$). Choose $y \neq y'$ in $S(x,1)$ such that $B_{i_0} \cap S_k(x,y) = \{z\}$ and $B_{i_0} \cap S_k(x,y') = \{z'\}$. Take $w \in S_k(x,y')$ with $w \neq z'$. By Lemma 1.2.2, there exists $g \in H(x)$ such that $g(z) = z$ and $g(z') = w$, which is a contradiction with the bloc decomposition. Therefore, we have $\underline{H}_k(x) \supseteq (S_s)^{c(x,k)}$ as wanted.

We are done for each $x \in V_0(T)$. Now if we try to do the same reasoning for $y \in V_1(T)$, the only issue is that $\underline{H}_k(y)$ could a priori be trivial. However, since $\underline{H}_k(x) \supseteq (S_s)^{c(x,k)}$ for each $x \in V_0(T)$ and as $\underline{H}_{k-1}(x) \supseteq (S_{1-s})^{c(x,k-1)}$ by induction hypothesis, Lemma 1.2.8 precisely tells us that $\underline{H}_k(y)$ is non-trivial. Hence, we also get $\underline{H}_k(y) \supseteq (S_{1-s})^{c(y,k)}$ in the same way. □

1.2.3 A global result

In the particular case where $F_0(1)$ and $F_1(1)$ are simple non-abelian, we can deduce from Proposition 1.2.10 that there is, up to conjugation, only one group $H \in \mathcal{H}_T^+$ such that $\underline{H}(x) \cong F_0$ for each $x \in V_0(T)$ and $\underline{H}(y) \cong F_1$ for each $y \in V_1(T)$. This is the subject of Theorem 1.F whose statement is recalled below.

Recall that a legal coloring i of T is a map defined piecewise by $i|_{V_0(T)} = i_0$ and $i|_{V_1(T)} = i_1$ where, for each $t \in \{0,1\}$, the map $i_t\colon V_t(T) \to \{1,\dots,d_{1-t}\}$ is such that $i_t|_{S(v,1)}\colon S(v,1) \to \{1,\dots,d_{1-t}\}$ is a bijection for each $v \in V_{1-t}(T)$. For $g \in \mathrm{Aut}(T)$ and $v \in V(T)$, the local action of g at v is $\sigma_{(i)}(g,v) := i|_{S(g(v),1)} \circ g \circ i|_{S(v,1)}^{-1}$. Given $F_0 \leq \mathrm{Sym}(d_0)$ and $F_1 \leq \mathrm{Sym}(d_1)$, the group $U_{(i)}^+(F_0,F_1)$ is defined by

$$U_{(i)}^+(F_0,F_1) := \left\{ g \in \mathrm{Aut}(T)^+ \;\middle|\; \begin{array}{c} \sigma_{(i)}(g,x) \in F_0 \text{ for each } x \in V_0(T), \\ \sigma_{(i)}(g,y) \in F_1 \text{ for each } y \in V_1(T) \end{array} \right\}.$$

The following basic result will be used constantly in the next sections.

Lemma 1.2.11. *Let T be the (d_0,d_1)-semiregular tree and let i be a legal coloring of T.*

- *If $g,h \in \mathrm{Aut}(T)$ and $v \in V(T)$, then $\sigma_{(i)}(gh,v) = \sigma_{(i)}(g,h(v)) \circ \sigma_{(i)}(h,v)$.*
- *If $g \in \mathrm{Aut}(T)$ and $v \in V(T)$, then $\sigma_{(i)}(g^{-1},v) = \sigma_{(i)}(g,g^{-1}(v))^{-1}$.*

Proof. This directly follows from the definition of $\sigma_{(i)}(g,v)$. ☐

The next result is the edge-transitive version of [BM00a, Proposition 3.2.2].

Lemma 1.2.12. *Let T be the (d_0,d_1)-semiregular tree with $d_0,d_1 \geq 3$ and let $F_0 \leq \mathrm{Sym}(d_0)$ and $F_1 \leq \mathrm{Sym}(d_1)$. Let H be an edge-transitive subgroup of $\mathrm{Aut}(T)^+$ such that $\underline{H}(x) \cong F_0$ for each $x \in V_0(T)$ and $\underline{H}(y) \cong F_1$ for each $y \in V_1(T)$. Then there exists a legal coloring i of T such that $H \subseteq U_{(i)}^+(F_0,F_1)$.*

Proof. Fix $x \in V_0(T)$ and, for each $v \in V_0(T)$ different from x, let $p(v)$ be the vertex of $S(v,1)$ the closest to x and $q(v)$ be the vertex of $S(p(v),1)$ the closest to x. For each such v, since H is edge-transitive, we can choose an element $h_v \in H$ that fixes $p(v)$ and sends v to $q(v)$. We define an appropriate map $i_1\colon V_1(T) \to \{1,\dots,d_0\}$ inductively on

$X_n := V_1(T) \cap B(x, 2n-1)$. For $n = 1$, we choose a bijection $i_x\colon X_1 = S(x,1) \to \{1,\dots,d_0\}$ such that $i_x\underline{H}(x)i_x^{-1} = F_0$ and set $i_1|_{X_1} = i_x$. Now assume that i_1 is defined on X_n. To extend i_1 to X_{n+1}, we set $i_1|_{S(v,1)} = i_1|_{S(q(v),1)} h_v|_{S(v,1)}$ for each $v \in S(x,2n)$. The map $i_0\colon V_0(T) \to \{1,\dots,d_1\}$ is defined in the same way by fixing $y \in V_1(T)$ and choosing $h_v \in H$ for each $v \in V_1(T)$ different from y as above. Define finally i by $i|_{V_0(T)} = i_0$ and $i|_{V_1(T)} = i_1$.

Given $v \in V_0(T)$ different from x, our construction is such that $\sigma_{(i)}(h_v, v) = \mathrm{id}$. Hence, if v is at distance $2n$ from x, the element $\tilde{h}_v = h_{q^{n-1}(v)} \cdots h_{q(v)} h_v \in H$ satisfies $\tilde{h}_v(v) = x$ and $\sigma_{(i)}(\tilde{h}_v, v) = \mathrm{id}$ (by Lemma 1.2.11). Now if we consider $g \in H$ and $v \in V_0(T)$, the element $\tilde{h}_{g(v)} g \tilde{h}_v^{-1} \in H$ fixes x and is therefore such that $\sigma_{(i)}(\tilde{h}_{g(v)} g \tilde{h}_v^{-1}, x) \in F_0$. Using Lemma 1.2.11, we obtain that $\sigma_{(i)}(g, v) \in F_0$. In the same way, for $v \in V_1(T)$ we get $\sigma_{(i)}(g, v) \in F_1$. We thus have $g \in U^+_{(i)}(F_0, F_1)$ and hence $H \subseteq U^+_{(i)}(F_0, F_1)$. ☐

Let us now prove Theorem 1.F. Note that the fact that $F_t(1)$ is simple non-abelian implies that $|F_t(1)| \geq 60$ and hence that $d_t \geq 6$ for each $t \in \{0,1\}$.

Theorem 1.F. *Let T be the (d_0,d_1)-semiregular tree and let $F_0 \leq \mathrm{Sym}(d_0)$ and $F_1 \leq \mathrm{Sym}(d_1)$. Let $H \in \mathcal{H}_T^+$ be such that $\underline{H}(x) \cong F_0$ for each $x \in V_0(T)$ and $\underline{H}(y) \cong F_1$ for each $y \in V_1(T)$. Suppose that, for each $t \in \{0,1\}$, $F_t(1)$ is simple non-abelian. Then there exists a legal coloring i of T such that $H = U^+_{(i)}(F_0, F_1)$.*

Proof. By Lemma 1.2.12, there exists a legal coloring i of T such that $H \subseteq U^+_{(i)}(F_0, F_1)$. For each $t \in \{0,1\}$, F_t is 2-transitive and hence $F_t(1)$ is transitive on $\{2,\dots,d_t\}$. Moreover, $F_t(1)$ is never simply transitive. Indeed, if it was the case then F_t would be sharply 2-transitive, but the finite sharply 2-transitive permutation groups have been classified and they never have a simple non-abelian point stabilizer (see [Zas35], [DM96, §7.6]). We can therefore apply Proposition 1.2.10 and directly obtain, since H is closed in $\mathrm{Aut}(T)$, that for each $v \in V(T)$ the stabilizer $H(v)$ is equal to $U^+_{(i)}(F_0, F_1)(v)$. As H is generated by its vertex stabilizers, the conclusion follows. ☐

1.3 A common subgroup

We assume in this section that $H \in \mathcal{H}_T^+$ satisfies $\underline{H}(x) \cong F_0 \geq \mathrm{Alt}(d_0)$ for each $x \in V_0(T)$ and $\underline{H}(y) \cong F_1 \geq \mathrm{Alt}(d_1)$ for each $y \in V_1(T)$. Our goal is to prove, under this hypothesis and when $d_0, d_1 \geq 6$, that there always exists a legal coloring i of T such that $H \supseteq \mathrm{Alt}_{(i)}(T)^+$. Recall that $\mathrm{Alt}_{(i)}(T)^+ = U_{(i)}^+(\mathrm{Alt}(d_0), \mathrm{Alt}(d_1))$, i.e.

$$\mathrm{Alt}_{(i)}(T)^+ = \{g \in \mathrm{Aut}(T)^+ \mid \sigma_{(i)}(g, v) \text{ is even for each } v \in V(T)\}.$$

Under these assumptions, we will apply Proposition 1.2.10. Indeed, when $F_t \supseteq \mathrm{Alt}(d_t)$ with $d_t \geq 6$ (for $t \in \{0, 1\}$), the socle S_t of $F_t(1)$ is $\mathrm{Alt}(d_t - 1)$ which is simple non-abelian, of index at most 2 in $F_t(1)$, and transitive but not simply transitive on $\{2, \ldots, d_t\}$.

Remark that, if $F_0 = \mathrm{Alt}(d_0)$ and $F_1 = \mathrm{Alt}(d_1)$, then we already know by Theorem 1.F that $H = \mathrm{Alt}_{(i)}(T)^+$ for some legal coloring i. The task is however surprisingly more difficult when $F_0 = \mathrm{Sym}(d_0)$ or $F_1 = \mathrm{Sym}(d_1)$.

1.3.1 Finding good colorings of rooted trees

For our next results, we denote by $T_{d_0,d_1,n}$ the rooted tree of depth n where the root v_0 has d_0 children, the vertices at positive even distance from v_0 (except the leaves) have $d_0 - 1$ children, and the vertices at odd distance from v_0 (except the leaves) have $d_1 - 1$ children. Similarly, $T'_{d_0,d_1,n}$ is the rooted tree of depth n where v_0 and all the vertices at even distance from v_0 have $d_0 - 1$ children while the vertices at odd distance from v_0 have $d_1 - 1$ children. Remark that, in the (d_0, d_1)-semiregular tree T, a ball $B(v, n)$ around a vertex v of type 0 is isomorphic to $T_{d_0,d_1,n}$. The intersection of $B(v, n)$ with a half-tree of T rooted in v is isomorphic to $T'_{d_0,d_1,n}$.

The notion of a legal coloring of $T_{d_0,d_1,n}$, as well as the permutations $\sigma_{(i)}(g, v)$ for $g \in \mathrm{Aut}(T_{d_0,d_1,n})$ and $v \notin \partial T_{d_0,d_1,n}$ (i.e. v is not a leaf), are defined as for semiregular trees. We can also define a legal coloring of $T'_{d_0,d_1,n}$: it suffices to precise that only $d_0 - 1$ colors are used for the vertices adjacent to v_0. The notation $\sigma_{(i)}(g, v)$ has also a meaning, but

$\sigma_{(i)}(g, v_0) \in \mathrm{Sym}(d_0 - 1)$ instead of $\mathrm{Sym}(d_0)$. Given $\tilde{T} = T_{d_0,d_1,n}$ or $T'_{d_0,d_1,n}$ with a legal coloring i, we finally define

$$\mathrm{Alt}_{(i)}(\tilde{T}) := \{g \in \mathrm{Aut}(\tilde{T}) \mid \sigma_{(i)}(g, v) \text{ is even for each } v \notin \partial\tilde{T}\}.$$

In the rest of this section and for the sake of brevity, we will sometimes forget the word *legal* and write *coloring* instead of *legal coloring*.

Lemma 1.3.1. *Let $\tilde{T} = T'_{d_0,d_1,n}$ with $d_0, d_1 \geq 3$ and let i be a legal coloring of $\tilde{T}$. Then $\mathrm{Alt}_{(i)}(\tilde{T})$ is generated by the set $\{g^2 \mid g \in \mathrm{Alt}_{(i)}(\tilde{T})\}$.*

Proof. We proceed by induction on n. For $n = 0$, the tree $T'_{d_0,d_1,0}$ has only one vertex and there is nothing to prove. Now let $n \geq 1$ and assume the result is proved for $n-1$. The set $\left\{g|^2_{B(v_0,n-1)} \,\middle|\, g \in \mathrm{Alt}_{(i)}(\tilde{T})\right\}$ thus generates $\mathrm{Alt}_{(i)}(B(v_0, n-1))$, where v_0 is the root of $\tilde{T}$. Hence, it suffices to show that $\left\{g^2 \,\middle|\, g \in \mathrm{Fix}_{\mathrm{Alt}_{(i)}(\tilde{T})}(B(v_0, n-1))\right\}$ generates $\mathrm{Fix}_{\mathrm{Alt}_{(i)}(\tilde{T})}(B(v_0, n-1))$. Since alternating groups are generated by 3-cycles, the group $\mathrm{Fix}_{\mathrm{Alt}_{(i)}(\tilde{T})}(B(v_0, n-1))$ is generated by the elements $f \in \mathrm{Alt}_{(i)}(\tilde{T})$ fixing $\tilde{T} \setminus \{a, b, c\}$ and such that $f(a) = b$, $f(b) = c$ and $f(c) = a$ where $a, b, c \in S(v_0, n)$ have the same parent. The conclusion simply follows from the fact that each such element f is the square of $f^{-1} \in \mathrm{Alt}_{(i)}(\tilde{T})$. $\square$

In the following, if v is a vertex in a tree $\tilde{T}$ with root v_0, then X_v is the branch of v, i.e. the subtree of $\tilde{T}$ spanned by v and all its descendants. For $G \leq \mathrm{Aut}(\tilde{T})$, $\mathrm{Rist}_G(v)$ is the pointwise stabilizer in G of $\tilde{T} \setminus X_v$. We will generally see $\mathrm{Rist}_G(v)$ as a subgroup of $\mathrm{Aut}(X_v)$. Finally, G_k is the pointwise stabilizer in G of $B(v_0, k)$ for $k \geq 0$.

Lemma 1.3.2. *Let $\tilde{T} = T_{d_0,d_1,n}$ or $T'_{d_0,d_1,n}$ with $d_0, d_1 \geq 6$ (and $n \geq 1$), let v_0 be the root of $\tilde{T}$ and let i be a legal coloring of $B(v_0, n-1)$. Let $G \leq \mathrm{Aut}(\tilde{T})$ be such that $G_{n-1} \supseteq \mathrm{Alt}(d_0 - 1)^{c(v_0,n-1)}$ (or $\mathrm{Alt}(d_1 - 1)^{c(v_0,n-1)}$ or $\mathrm{Alt}(d_0)$, depending on n) and $G|_{B(v_0,n-1)} \supseteq \mathrm{Alt}_{(i)}(B(v_0, n-1))$. Then there exists a legal coloring $\bar{i}$ of $\tilde{T}$ extending i such that $G \supseteq \mathrm{Alt}_{(\bar{i})}(\tilde{T})$. Moreover, if for some vertex $y_0 \in S(v_0, 1)$ we already had a legal coloring i' of X_{y_0} coinciding with i on $X_{y_0} \cap B(v_0, n-1)$ and such that $\mathrm{Rist}_G(y_0) \supseteq \mathrm{Alt}_{(i')}(X_{y_0})$, then $\bar{i}$ can be chosen to extend i' too.*

Proof. Define $e = d_0$ if $\tilde{T} = T_{d_0,d_1,n}$ and $e = d_0 - 1$ if $\tilde{T} = T'_{d_0,d_1,n}$, so that the root v_0 of $\tilde{T}$ has exactly e neighbors. We proceed by induction on n. For $n = 1$, we have $G = G_0 \supseteq \mathrm{Alt}(e)$ by hypothesis, and thus any coloring $\bar{i}$ of $\tilde{T}$ is such that $G \supseteq \mathrm{Alt}_{(\bar{i})}(\tilde{T})$. Now let $n \geq 2$ and assume the lemma is true for $n-1$. We show it is also true for n.

Let $y_1, \ldots, y_e$ be the vertices of $S(v_0, 1)$. By hypothesis, $G_{n-1} \supseteq \mathrm{Alt}(\tilde{d}-1)^{c(v_0,n-1)}$ where $\tilde{d} = d_0$ if n is odd and $\tilde{d} = d_1$ if n is even. This implies that $\mathrm{Aut}(X_{y_1}) \geq \mathrm{Rist}_G(y_1)_{n-2} \supseteq \mathrm{Alt}(\tilde{d}-1)^{c(y_1,n-2)}$ (where $c(y_1, n-2)$ counts the vertices of X_{y_1} at distance $n-2$ from y_1 and $\mathrm{Rist}_G(y_1)$ is seen as a subgroup of $\mathrm{Aut}(X_{y_1})$). We also claim that $\mathrm{Rist}_G(y_1)|_{B(y_1,n-2)} \supseteq \mathrm{Alt}_{(i)}(X_{y_1} \cap B(y_1, n-2))$. Indeed, as $G|_{B(v_0,n-1)} \supseteq \mathrm{Alt}_{(i)}(B(v_0, n-1))$, for each $h \in \mathrm{Alt}_{(i)}(X_{y_1} \cap B(y_1, n-2))$ there exists $g \in G$ fixing $(\tilde{T} \setminus X_{y_1}) \cap B(v_0, n-1)$ and acting as h on $X_{y_1} \cap B(y_1, n-2)$. Then $g^2 \in G$ acts as h^2 on this set, and has the advantage that $g^2|_{E(x)}$ is an even permutation of $E(x)$ for each $x \in (\tilde{T} \setminus X_{y_1}) \cap S(v_0, n-1)$. As $G_{n-1} \supseteq \mathrm{Alt}(\tilde{d}-1)^{c(v_0,n-1)}$, there exists $f \in G_{n-1}$ such that $f|_{E(x)} = g^2|_{E(x)}$ for all those x. Then $f^{-1}g^2$ acts as h^2 on $X_{y_1} \cap B(y_1, n-2)$ and belongs to $\mathrm{Rist}_G(y_1)$. This means that $\mathrm{Rist}_G(y_1)|_{B(y_1,n-2)}$ contains $\{h^2 \mid h \in \mathrm{Alt}_{(i)}(X_{y_1} \cap B(y_1, n-2))\}$. By Lemma 1.3.1, we obtain $\mathrm{Rist}_G(y_1)|_{B(y_1,n-2)} \supseteq \mathrm{Alt}_{(i)}(X_{y_1} \cap B(y_1, n-2))$. We can now use our induction hypothesis on $\mathrm{Rist}_G(y_1) \leq \mathrm{Aut}(X_{y_1})$ to get a coloring i_1 of X_{y_1} extending i and such that $\mathrm{Rist}_G(y_1) \supseteq \mathrm{Alt}_{(i_1)}(X_{y_1})$. In the particular case where we are given a vertex y_0 and a coloring i' of X_{y_0} as in the statement, we set $y_1 = y_0$ and rather define $i_1 = i'$.

Now take $g_1 \in G$ with $g_1|_{B(v_0,n-1)} \in \mathrm{Alt}_{(i)}(B(v_0, n-1))$ such that the induced action of g_1 on $S(v_0, 1)$ is the 3-cycle $(y_1\ y_3\ y_2)$, g_1 fixes $X_y \cap B(v_0, n-1)$ for each $y \in S(v_0, 1) \setminus \{y_1, y_2, y_3\}$, and $g_1^3|_{B(v_0,n-1)} = \mathrm{id}|_{B(v_0,n-1)}$. Such an element g_1 exists as $G|_{B(v_0,n-1)} \supseteq \mathrm{Alt}_{(i)}(B(v_0, n-1))$. The element $h_1 = g_1^2$ acts as the 3-cycle $(y_1\ y_2\ y_3)$ on $S(v_0, 1)$, fixes $X_y \cap B(v_0, n-1)$ for each $y \in S(v_0, 1) \setminus \{y_1, y_2, y_3\}$ and also satisfies $h_1^3|_{B(v_0,n-1)} = \mathrm{id}|_{B(v_0,n-1)}$. In addition, $h_1|_{E(x)}$ is an even permutation of $E(x)$ for each $x \in (\tilde{T} \setminus (X_{y_1} \cup X_{y_2} \cup X_{y_3})) \cap S(v_0, n-1)$ (because $h_1 = g_1^2$). From i_1, construct a coloring i_2 of X_{y_2} (coinciding with i) such that $i_2|_{S(h_1(x),1)} \circ h_1 \circ i_1|_{S(x,1)}^{-1}$ is even for each $x \in X_{y_1} \cap S(v_0, n-1)$. In the same way, from i_2, construct a coloring i_3 of X_{y_3} (coinciding with

i) such that $i_3|_{S(h_1(x),1)} \circ h_1 \circ i_2|_{S(x,1)}^{-1}$ is even for each $x \in X_{y_2} \cap S(v_0, n-1)$. As $h_1 = g_1^2$, we also obtain that $i_1|_{S(h_1(x),1)} \circ h_1 \circ i_3|_{S(x,1)}^{-1}$ is even for each $x \in X_{y_3} \cap S(v_0, n-1)$. This exactly means that, for any coloring $\bar{i}$ of $\tilde{T}$ extending i, i_1, i_2 and i_3, it will be true that $h_1 \in \mathrm{Alt}_{(\bar{i})}(\tilde{T})$.

In the case where e is odd, the proof is almost finished. Indeed, repeat this process to get $h_3 \in G$ inducing $(y_3\ y_4\ y_5)$ on $S(v_0, 1)$ and colorings i_4 of X_{y_4} and i_5 of X_{y_5}, and so on until $h_{e-2} \in G$ inducing $(y_{e-2}\ y_{e-1}\ y_e)$ on $S(v_0, 1)$ and colorings i_{e-1} of $X_{y_{e-1}}$ and i_e of X_{y_e}. Then define $\bar{i}$ as the unique coloring extending $i, i_1, \ldots, i_e$. In view of our construction, $\bar{i}$ is such that $h_1, h_3, \ldots, h_{e-2} \in \mathrm{Alt}_{(\bar{i})}(\tilde{T})$. What is interesting about $h_1, h_3, \ldots, h_{e-2}$ is the fact that the permutations $(y_1\ y_2\ y_3), (y_3\ y_4\ y_5), \ldots, (y_{e-2}\ y_{e-1}\ y_e)$ generate $\mathrm{Alt}(e)$. In particular, as $\mathrm{Rist}_G(y_1) \supseteq \mathrm{Alt}_{(i_1)}(X_{y_1})$ we see by conjugating this inclusion with an element of $\langle h_1, h_3, \ldots, h_{e-2}\rangle$ sending y_1 on y_k that $\mathrm{Rist}_G(y_k) \supseteq \mathrm{Alt}_{(i_k)}(X_{y_k})$ for each $k \in \{1, \ldots, e\}$. This means that G contains all elements of $\mathrm{Alt}_{(\bar{i})}(\tilde{T})$ fixing $S(v_0, 1)$. But it also contains $h_1, h_3, \ldots, h_{e-2} \in \mathrm{Alt}_{(\bar{i})}(\tilde{T})$ whose induced actions on $S(v_0, 1)$ generate $\mathrm{Alt}(e)$, so $G \supseteq \mathrm{Alt}_{(\bar{i})}(\tilde{T})$.

If e is even, then the exact same reasoning gives us $h_3, h_5, \ldots, h_{e-3}$ and colorings $i_4, \ldots, i_{e-1}$. At the end, there is no coloring of X_{y_e} yet and the permutations $(y_1\ y_2\ y_3), (y_3\ y_4\ y_5), \ldots, (y_{e-3}\ y_{e-2}\ y_{e-1})$ only generate the even permutations of $S(v_0, 1)$ fixing y_e. So as to conclude, take $g_{e-2} \in G$ as before so that the induced action on $S(v_0, 1)$ is $(y_{e-2}\ y_e\ y_{e-1})$ and define $h_{e-2} = g_{e-2}^2$. For simplicity, we write $h := h_{e-2}$. Here, the colorings i_{e-2} and i_{e-1} are already fixed and we can only choose a coloring i_e of X_{y_e}. Choose i_e so that $i_e|_{S(h(x),1)} \circ h \circ i_{e-1}|_{S(x,1)}^{-1}$ is even for each $x \in X_{y_{e-1}} \cap S(v_0, n-1)$, and define $\bar{i}$ as the unique coloring extending $i, i_1, \ldots, i_e$. The only issue preventing us from concluding as above is that it is not sure if $h \in \mathrm{Alt}_{(\bar{i})}(\tilde{T})$. The permutation $\sigma_{(\bar{i})}(h, x)$ could indeed be odd for some $x \in (X_{y_{e-2}} \cup X_{y_e}) \cap S(v_0, n-1)$. More precisely, these are the only vertices for which $\sigma_{(\bar{i})}(h, x)$ could be odd and we even know (because $h = g_{e-2}^2$) that $\sigma_{(\bar{i})}(h, x)$ with $x \in X_{y_e} \cap S(v_0, n-1)$ is odd if and only if $\sigma_{(\bar{i})}(h, h(x))$ is odd. We therefore define

$$O := \{x \in X_{y_e} \cap S(v_0, n-1) \mid \sigma_{(\bar{i})}(h, x) \text{ is odd}\},$$

so that $O \cup h(O)$ is exactly the set of vertices at which there is an odd permutation.

To finish the proof, we show that there exists $h' \in G \cap \mathrm{Alt}_{(\bar{i})}(\tilde{T})$ with $h'|_{B(v_0,n-1)} = h|_{B(v_0,n-1)}$. Denote by $a_1^{(e-2)}, \dots, a_m^{(e-2)}$ the vertices of $X_{y_{e-2}} \cap S(v_0, n-1)$. Then define $a_j^{(e-1)} = h(a_j^{(e-2)})$ and $a_j^{(e)} = h(a_j^{(e-1)})$ for each $j \in \{1, \dots, m\}$. Finally, for each $k \in \{1, \dots, e-3\}$ choose $r_k \in G \cap \mathrm{Alt}_{(\bar{i})}(\tilde{T})$ such that $r_k(y_{e-2}) = y_k$ and define $a_j^{(k)} = r_k(a_j^{(e-2)})$ for all j. We say that $f \in \mathrm{Aut}(\tilde{T})$ *preserves the labelling* if $f(y_k) = y_\ell$ implies $f(a_j^{(k)}) = a_j^{(\ell)}$ for all j. One sees that if f preserves the labelling and if $\sigma_{(\bar{i})}(f, v_0)$ is even, then $f|_{B(v_0,n-1)} \in \mathrm{Alt}_{(\bar{i})}(B(v_0, n-1))$.

Choose $f_1, f_2 \in \mathrm{Alt}_{(\bar{i})}(\tilde{T})$ preserving the labelling, fixing X_{y_e} and such that the induced action of f_1 (resp. f_2) on $S(v_0, 1)$ is the permutation $(y_1\ y_2\ y_{e-1})$ (resp. $(y_1\ y_{e-2}\ y_{e-1})$). Such elements exist by the previous remark, and they are contained in G. Note that $d_0 \geq 6$, so $e \geq 5$ and $2 < e-2$. Let us look at the element $\tau = (f_1 \circ h \circ f_2)^2 \in G$. Clearly, τ preserves the labelling and it suffices to look at its action on $S(v_0, 1)$ to know its action on $S(v_0, n-1)$. The action of τ on $S(v_0, 1)$ is given by

$$[(y_1\ y_2\ y_{e-1})(y_{e-2}\ y_{e-1}\ y_e)(y_1\ y_{e-2}\ y_{e-1})]^2$$

which is exactly the trivial permutation. Hence, τ acts trivially on $B(v_0, n-1)$. We should now observe with the help of Lemma 1.2.11 if $\sigma_{(\bar{i})}(\tau, x)$ is even or odd, for each $x \in S(v_0, n-1)$. As $f_1, f_2 \in \mathrm{Alt}_{(\bar{i})}(\tilde{T})$, all the permutations they induce are even. Using that $\sigma_{(\bar{i})}(h, x)$ is odd if and only if $x \in O \cup h(O)$, we actually obtain that $\sigma_{(\bar{i})}(\tau, x)$ is odd if and only if $x \in O \cup h(O)$. This means that $h' = h \circ \tau \in G$, which acts as h on $B(v_0, n-1)$, is such that $\sigma_{(\bar{i})}(h', x)$ is always even, i.e. $h' \in \mathrm{Alt}_{(\bar{i})}(\tilde{T})$. $\square$

1.3.2 The common subgroup $\mathrm{Alt}_{(i)}(T)^+$

We are now ready to complete the proof of Theorem 1.G from §1.1. For the reader's convenience we reproduce its statement.

Theorem 1.G. *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 6$. Let $H \in \mathcal{H}_T^+$ be such that $\underline{H}(x) \cong F_0 \geq \mathrm{Alt}(d_0)$ for each $x \in V_0(T)$*

and $\underline{H}(y) \cong F_1 \geq \mathrm{Alt}(d_1)$ *for each* $y \in V_1(T)$*. Then there exists a legal coloring* i *of* T *such that* $H \supseteq \mathrm{Alt}_{(i)}(T)^+$*.*

Proof. Given $v \in V(T)$ and a coloring i of T, we say that i is n**-valid at** v (with $n \in \mathbf{Z}_{>0}$) if the natural image of $H(v)$ in $\mathrm{Aut}(B(v,n))$ contains $\mathrm{Alt}_{(i)}(B(v,n))$. If $H(v) \supseteq \mathrm{Alt}_{(i)}(T)^+(v)$, i is said to be ∞**-valid at** v. As H is closed in $\mathrm{Aut}(T)$, a coloring is ∞-valid at v if and only if it is n-valid at v for all $n \in \mathbf{Z}_{>0}$.

We first claim that if i is a coloring of T that is ∞-valid at v_1 and n-valid at v_2 where v_1 and v_2 are adjacent vertices (with $n \in \mathbf{Z}_{>0}$), then there exists a coloring $\tilde{i}$ of T such that $\tilde{i}|_{B(v_1,n)\cup B(v_2,n)} = i|_{B(v_1,n)\cup B(v_2,n)}$ and that is $(n+1)$-valid at v_1 and ∞-valid at v_2. To prove the claim, first define $\tilde{i}$ on $B(v_2,n) \cup T_{v_1}$ by $\tilde{i}|_{B(v_2,n)\cup T_{v_1}} = i|_{B(v_2,n)\cup T_{v_1}}$, where T_{v_1} is the subtree of T spanned by the vertices that are closer to v_1 than to v_2. This is already sufficient for $\tilde{i}$ to be $(n+1)$-valid at v_1 and n-valid at v_2. Now suppose that $\tilde{i}$ is defined on $B(v_2,k) \cup T_{v_1}$ for some $k \geq n$. We explain how to extend it to $B(v_2,k+1) \cup T_{v_1}$ so that it becomes $(k+1)$-valid at v_2. Define $\tilde{T} = B(v_2,k+1)$ and denote by G the image of $H(v_2)$ in $\mathrm{Aut}(\tilde{T})$. We have $G_k \supseteq \mathrm{Alt}(\tilde{d}-1)^{c(v_2,k)}$ (where $\tilde{d} = d_0$ or d_1) in view of Proposition 1.2.10 and $G|_{B(v_2,k)} \supseteq \mathrm{Alt}_{(\tilde{i})}(B(v_2,k))$ since $\tilde{i}$ is k-valid at v_2. Moreover, X_{v_1} is already colored (by $\tilde{i}$ too) and $\mathrm{Rist}_G(v_1) \supseteq \mathrm{Alt}_{(\tilde{i})}(X_{v_1})$ (because i is ∞-valid at v_1 and $\tilde{i}|_{X_{v_1}} = i|_{X_{v_1}}$). Lemma 1.3.2 thus gives an extension of $\tilde{i}$ to $\tilde{T}$ making it $(k+1)$-valid at v_2. The coloring $\tilde{i}$ of T defined in this way by induction is $(n+1)$-valid at v_1 and ∞-valid at v_2.

To prove the theorem, fix $x \in V_0(T)$ and $y \in V_1(T)$ two adjacent vertices of T. As $\mathrm{Alt}_{(i)}(T)^+(x)$ and $\mathrm{Alt}_{(i)}(T)^+(y)$ generate $\mathrm{Alt}_{(i)}(T)^+$, a coloring i of T is such that $H \supseteq \mathrm{Alt}_{(i)}(T)^+$ if and only if i is ∞-valid at x and y. Let us construct such a coloring. By Proposition 1.2.10 and Lemma 1.3.2, there exists a coloring i_1 of T that is ∞-valid at x. As all colorings, i_1 is 1-valid at y. Using the claim, construct i_{n+1} from i_n with $i_{n+1}|_{B(x,n)\cup B(y,n)} = i_n|_{B(x,n)\cup B(y,n)}$ for each $n \geq 1$. For n odd, i_n is ∞-valid at x and n-valid at y; while for n even, i_n is n-valid at x and ∞-valid at y. There is now a natural way to define our coloring i of T: for each $v \in V(T)$, set $i(v) = i_n(v)$ where n is such that $v \in B(x,n) \cup B(y,n)$. By construction, i is ∞-valid at x and y. ∎

1.4 Examples, simplicity and normalizers

In this section, we define all the groups that will appear in our classification theorems and analyze some of their properties.

1.4.1 Definition of the examples

We first recall the definitions of the groups appearing in §1.1 and also define new similar groups. The fact that they are indeed groups follows from Lemma 1.2.11.

Definition 1.4.1. Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 4$ and let i be a legal coloring of T. When $v \in V(T)$ and X is a subset of $\mathbf{Z}_{\geq 0}$, we set $S_X(v) := \bigcup_{r \in X} S(v, r)$. The notation $X \subset_f \mathbf{Z}_{\geq 0}$ means that X is a *non-empty finite subset* of $\mathbf{Z}_{\geq 0}$. We also write $\mathrm{Sgn}_{(i)}(g, A) := \prod_{w \in A} \mathrm{sgn}(\sigma_{(i)}(g, w))$ when A is a finite subset of $V(T)$ and $g \in \mathrm{Aut}(T)$.

First set $G^+_{(i)}(\varnothing, \varnothing) := \mathrm{Aut}(T)^+$. Then, for $X \subset_f \mathbf{Z}_{\geq 0}$, define

$$G^+_{(i)}(X, \varnothing) := \left\{ g \in \mathrm{Aut}(T)^+ \;\middle|\; \mathrm{Sgn}_{(i)}(g, S_X(v)) = 1 \text{ for each } v \in V_t(T) \right\}$$

and

$$G^+_{(i)}(X^*, \varnothing) := \left\{ g \in \mathrm{Aut}(T)^+ \;\middle|\; \begin{array}{c} \text{All } \mathrm{Sgn}_{(i)}(g, S_{X_0}(v)) \\ \text{with } v \in V_t(T) \text{ are equal} \end{array} \right\},$$

where $t = (\max X) \bmod 2$. The groups $G^+_{(i)}(\varnothing, X)$ and $G^+_{(i)}(\varnothing, X^*)$ are defined in the same way but with $t = (1 + \max X) \bmod 2$. For $X_0, X_1 \subset_f \mathbf{Z}_{\geq 0}$ and $Y_0 \in \{X_0, X_0^*\}$, $Y_1 \in \{X_1, X_1^*\}$, define

$$G^+_{(i)}(Y_0, Y_1) := G^+_{(i)}(Y_0, \varnothing) \cap G^+_{(i)}(\varnothing, Y_1).$$

Finally, for $X_0, X_1 \subset_f \mathbf{Z}_{\geq 0}$, set

$$G^+_{(i)}(X_0, X_1)^* := \left\{ g \in \mathrm{Aut}(T)^+ \;\middle|\; \begin{array}{c} \text{All } \mathrm{Sgn}_{(i)}(g, S_{X_0}(v)) \text{ with } v \in V_{t_0}(T) \text{ and} \\ \mathrm{Sgn}_{(i)}(g, S_{X_1}(v)) \text{ with } v \in V_{t_1}(T) \text{ are equal} \end{array} \right\},$$

where $t_0 = (\max X_0) \bmod 2$ and $t_1 = (1 + \max X_1) \bmod 2$.

We write $\mathcal{G}_{(i)}$ for the set of all these groups. Two groups are considered as different in this definition as soon as they have a different name, but two different groups may have exactly the same elements. We also define the following subsets $\mathcal{S}_{(i)}$ and $\mathcal{N}_{(i)}$ of $\mathcal{G}_{(i)}$, so that $\mathcal{G}_{(i)} = \mathcal{S}_{(i)} \sqcup \mathcal{N}_{(i)}$:

$$\mathcal{S}_{(i)} := \left\{ \begin{array}{c} G^+_{(i)}(\varnothing,\varnothing), G^+_{(i)}(X_0,\varnothing), \\ G^+_{(i)}(\varnothing,X_1), G^+_{(i)}(X_0,X_1) \end{array} \middle| X_0, X_1 \subset_f \mathbf{Z}_{\geq 0} \right\},$$

and

$$\mathcal{N}_{(i)} := \left\{ \begin{array}{c} G^+_{(i)}(X_0^*,\varnothing), G^+_{(i)}(\varnothing,X_1^*), \\ G^+_{(i)}(X_0^*,X_1), G^+_{(i)}(X_0,X_1^*), \\ G^+_{(i)}(X_0^*,X_1^*), G^+_{(i)}(X_0,X_1)^* \end{array} \middle| X_0, X_1 \subset_f \mathbf{Z}_{\geq 0} \right\}.$$

Finally, denote by $s\colon \mathcal{N}_{(i)} \to \mathcal{S}_{(i)}$ the map that simply erases the stars *. Our remark on the groups which are considered as different in $\mathcal{G}_{(i)}$ is essential for s to be well-defined.

Lemma 1.4.2 (Theorem 1.A (i))**.** *Let $H \in \mathcal{G}_{(i)}$. Then $H \in \mathcal{H}_T^+$.*

Proof. All the groups $H \in \mathcal{G}_{(i)}$ contain $\mathrm{Alt}_{(i)}(T)^+$ and are closed in $\mathrm{Aut}(T)$, so it suffices to prove that $\mathrm{Alt}_{(i)}(T)^+$ is 2-transitive on ∂T. By Lemma 1.2.1, it is equivalent to showing that $\mathrm{Alt}_{(i)}(T)^+(v)$ is transitive on ∂T for each $v \in V(T)$. As $\mathrm{Alt}_{(i)}(T)^+$ is closed, we can just show that the fixator in $\mathrm{Alt}_{(i)}(T)^+$ of a geodesic (v,w) with $v, w \in V(T)$ always acts transitively on $E(w) \setminus \{e\}$, where e is the edge of (v,w) adjacent to w. This is immediate, since $\mathrm{Alt}(d-1)$ is transitive when $d \geq 4$. $\square$

Given $H \in \mathcal{G}_{(i)}$ and $h \in \mathrm{Aut}(T)^+$, it is not hard to determine whether h belongs to H. Indeed, one can simply draw the tree T and label each vertex v of T with the letter e (for *even*) or o (for *odd*) depending on the parity of $\sigma_{(i)}(h,v)$. A condition on the value of $\mathrm{Sgn}_{(i)}(h, S_X(v))$ then translates in a condition on the parity of the number of vertices labelled by o in $S_X(v)$.

Using this observation, we can easily construct elements of H step by step. For example, consider $H = G^+_{(i)}(X_0, X_1^*)$. Let us observe how one can construct any labelling of T that satisfies the condition of being in H, i.e. such that if $h \in \mathrm{Aut}(T)^+$ realizes this labelling, then $h \in H$. First fix a vertex $v_0 \in V(T)$. For $n \in \mathbf{Z}_{\geq 0}$ and given a labelling of $B(v_0, n-1)$ (if $n \neq 0$), we look at how it can be extended to a labelling of $B(v_0, n)$ while satisfying the conditions for being in H. Suppose we already have a labelling of $B(v_0, n-1)$ not contradicting any of the conditions. Let $t \in \{0, 1\}$ be the type of the vertices of $S(v_0, n)$. If $n < \max X_t$, then there is no set $S_{X_0}(v)$ or $S_{X_1}(v)$ contained in $B(v_0, n)$ but not already contained in $B(v_0, n-1)$, so the labelling can be extended with no restriction. On the contrary, if $n \geq \max X_t$, then our new labelling must satisfy some additional conditions: the ones on the set $S_{X_t}(v)$ where v is a vertex at distance $n - \max X_t$ from v_0. But $\{S_{X_t}(v) \cap S(v_0, n) \mid d(v, v_0) = n - \max X_t\}$ is a partition of $S(v_0, n)$, so there is only one condition on the parity of the number of labels o on each set $S_{X_t}(v) \cap S(v_0, n)$. If $t = 0$ (recall that we consider $H = G_{(i)}^{+}(X_0, X_1^*)$), we just have to make sure that there is an even number of vertices labelled by o in $S_{X_0}(v)$. If $t = 1$, then we distinguish the following two cases. If this is the first time (of the whole process) that we observe a set of the form $S_{X_1}(v)$, then we can still make the choice of the parity of the number of labels o in $S_{X_1}(v)$. Otherwise, this parity must be the same as for this first choice. In all cases, we still have a lot of freedom in our choice of the new labelling. A labelling of T constructed in this way will always be suitable, since everything was made for the conditions to be met.

1.4.2 Simplicity

It is clear that each group $H \in \mathcal{N}_{(i)}$ has $s(H)$ as a proper normal subgroup, and is therefore not simple. Our next goal is to prove that the groups in $\mathcal{S}_{(i)}$ are simple. Banks, Elder and Willis [BEW15] provided tools to show that a group of automorphisms of trees is simple. Those happen to be exactly what we need. Note that their work is based on a generalization of Tits' Property P (see [Tit70]). For $G \leq_{cl} \mathrm{Aut}(T)$ and $k \in \mathbf{Z}_{>0}$, define

$$G^{+_k} := \langle G_{k-1}(v, w) \mid [v, w] \in E(T) \rangle.$$

The next proposition is a combination of results of [BEW15].

Proposition 1.4.3. *Let Y_0 and Y_1 be (possibly empty) finite subsets of $\mathbf{Z}_{\geq 0}$, let $H = G^+_{(i)}(Y_0, Y_1) \in \mathcal{S}_{(i)}$ and let $M = \max(\max Y_0, \max Y_1) + 1$, where we set $\max(\varnothing) = 0$ by convention. Then H^{+_M} is abstractly simple.*

Proof. Recall the following definition for $n \in \mathbf{Z}_{\geq 0}$:

$$H^{(n)} := \{g \in \mathrm{Aut}(T) \mid \forall v \in V(T), \exists h \in H : g|_{B(v,n)} = h|_{B(v,n)}\}.$$

In our case, it is clear from the definition of H that $H^{(M)} = H$. Hence, by [BEW15, Proposition 5.2], H has Property IP_M (as defined in [BEW15, Definition 5.1]). Since H is a closed subgroup of $\mathrm{Aut}(T)$, we deduce from [BEW15, Corollary 6.4] that H has Property P_M (as defined in [BEW15, Definition 6.2]). We can therefore apply [BEW15, Theorem 7.3] that asserts that H^{+_M} is abstractly simple or trivial. Since there exist non-trivial elements in $\mathrm{Alt}_{(i)}(T)^+ \subseteq H$ fixing arbitrarily large balls, we conclude that H^{+_M} is abstractly simple. □

In order to prove that a group $H \in \mathcal{S}_{(i)}$ is simple, we therefore only need to prove that $H = H^{+_M}$, where $M = \max(\max Y_0, \max Y_1) + 1$. We first assert that $H^{+_1} = H$. Note that H^{+_1} is the subgroup of H generated by the elements fixing an edge of T.

Lemma 1.4.4. *Let $H \in \mathcal{H}_T^+$ (with $d_0, d_1 \geq 3$). Then $H^{+_1} = H$.*

Proof. The result readily follows from the fact that the fixator of an edge $e = [v, w]$ in H is transitive on $E(v) \setminus \{e\}$ (by Lemma 1.2.2). □

For $[v, w] \in E(T)$, we write $T_{v,w}$ for the subtree of T spanned by the vertices that are closer to v than to w. Such a subtree is called a **half-tree**.

Lemma 1.4.5. *Let $H \in \mathcal{S}_{(i)}$. Then H is generated by the elements of H fixing a half-tree of T. In particular, $H = H^{+_k}$ for any $k \in \mathbf{Z}_{>0}$.*

Proof. We already know by Lemma 1.4.4 that $H = \langle H(v, w) \mid [v, w] \in E(T)\rangle$. Let us now prove that each $h \in H(v, w)$ (for some $[v, w] \in E(T)$) is generated by elements of H fixing a half-tree of T. We construct an element $g \in H$ such that $g|_{T_{v,w}} = h|_{T_{v,w}}$ and g fixes some half-tree of T. This will prove the statement as $h = (hg^{-1})g$.

First define g on $T_{v,w}$ by declaring that $g|_{T_{v,w}} = h|_{T_{v,w}}$. Now look at the labelled tree associated to g: for the moment, all the vertices of $T_{v,w}$ are labelled by e or o. We also label all the vertices of $T_{w,v} \cap B(w, M-1)$ where $M = \max(\max Y_0, \max Y_1) + 1$ exactly as in the labelled tree associated to h. Since $h \in H$, all the conditions to be in H that concern $T_{v,w} \cup B(w, M-1)$ are satisfied.

We now want to put new labellings on $S(w,n) \cap T_{w,v}$ for each $n \geq M$. Before doing so, we number the edges of $T_{w,v}$ in the following way: if x is a vertex at distance D from w, the edges from x to a vertex at distance $D+1$ from w are numbered with $1, 2, \dots, d_0 - 1$ (or $d_1 - 1$). Let us now label the whole tree with e and o. As already explained in §1.4.1, at each step there will be conditions on the parity of the number of labels o in sets of the form $S_X(x)$. More precisely, if we look at $S(w,n)$ (for $n \geq M$), then either there is no new condition to satisfy (because of the symbol $\varnothing$ in H), or there is a condition on each set of the form $S(w,n) \cap S(x, \max X)$ (where $X = X_0$ or $X = X_1$) with x is at distance $n - \max X$ from w. If there is no condition then we label all the vertices of $S(w,n) \cap T_{w,v}$ by e. Otherwise, in each set $S(w,n) \cap S(x, \max X)$ with x at distance $n - \max X$ from w, the number of vertices labelled by o must be either even or odd (depending on the previous labellings). If it must be even, we label all the vertices of $S(w,n) \cap S(x, \max X)$ by e. If it must be odd, we label by o the vertex z of $S(w,n) \cap S(x, \max X)$ such that the path from x to z only contains edges numbered 1. All the other vertices are labelled by e (see Figure 1.3 where $n = 3$ and $\max X = 2$).

We claim that, after having followed these rules to label the whole tree, there will always exists a half-tree $T_{s,t}$ whose vertices are all labelled by e (with $s, t \in T_{w,v}$ and t closer to w than s). This will complete the proof, since it is always possible to define g on $T_{w,v}$ so that g fixes the whole path from w to t, fixes $T_{s,t}$, and realizes the labelled tree that we just constructed. (Note that we need $d_0, d_1 \geq 4$ here.)

Let us prove the claim. Let s_0 be a vertex of $T_{w,v} \cap S(w, M)$ labelled by e. Define $(s_n)_{n \in \mathbf{Z}_{\geq 0}}$ by saying that s_j is the vertex adjacent to s_{j-1} farther from w than s_j and such that $[s_{j-1}, s_j]$ is numbered 2. We show by induction that, for each $j \in \mathbf{Z}_{\geq 0}$, the ball $B(s_j, j)$ only contains vertices labelled by e. For $j = 0$ this is clear. Now assume that all the

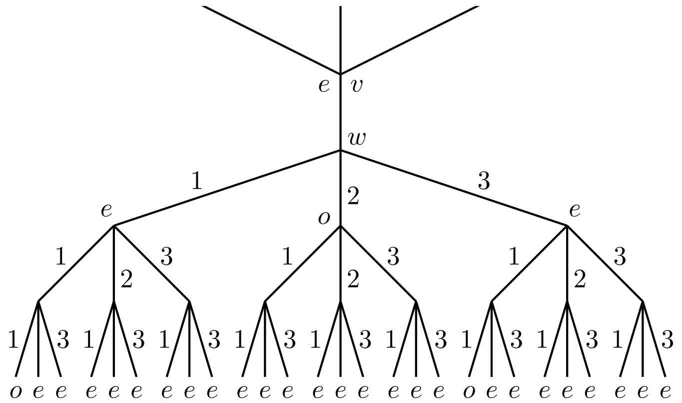


Figure 1.3: Illustration of Lemma 1.4.5 for $H = G^+_{(i)}(\{2\}, \{2\})$.

vertices of $B(s_j, j)$ are labelled by e and look at the ball $B(s_{j+1}, j+1)$. All the vertices of $B(s_{j+1}, j+1) \cap B(s_j, j)$ are labelled by e, so we only need to observe $B(s_{j+1}, j+1) \setminus B(s_j, j) = T_{s_{j+1},s_j} \cap (S(s_{j+1}, j) \cup S(s_{j+1}, j+1))$. The labels of the vertices of $T_{s_{j+1},s_j} \cap S(s_{j+1}, j) =: A$ were determined according to some eventual conditions on sets of the form $S_X(x)$. If there are no such conditions, then all the vertices of A were labelled by e as wanted. Otherwise, there are two cases: either $\max X \leq j$ or $\max X > j$. If $\max X \leq j$, then $S_X(x) \setminus A \subseteq B(s_j, j)$ so all the vertices of $S_X(x) \setminus A$ are labelled by e and the vertices of A were therefore also labelled by e. If $\max X > j$, then the condition on $S_X(x)$ may have been to put a label o somewhere, but in any case this label o was not put in A since $[s_j, s_{j+1}]$ is numbered 2 (and not 1). So all the vertices of A were labelled by e. The reasoning is exactly the same for $T_{s_{j+1},s_j} \cap S(s_{j+1}, j+1) =: A'$. This means that $B(s_M, M)$ only contains vertices labelled by e. Hence, in $T_{s_M,s_{M-1}}$ there is no condition on a set $S_X(x)$ asking to label a vertex by o (because $\max X < M$). All the vertices of the half-tree $T_{s_M,s_{M-1}}$ are thus labelled by e. □

The previous results together imply that groups in $\mathcal{S}_{(i)}$ are simple.

Theorem 1.4.6 (Theorem 1.A (ii))**.** *Let $H \in \mathcal{S}_{(i)}$. Then H is simple.*

Proof. This follows from Proposition 1.4.3 and Lemma 1.4.5. □

1.4.3 Are these examples pairwise distinct?

As highlighted in Definition 1.4.1, it is not clear for the moment if the members of $\mathcal{G}_{(i)}$ are pairwise different. One can actually remark that this is not the case: for instance, if $X_0, X_1 \subset_f \mathbf{Z}_{\geq 0}$ are such that $\max X_0 \not\equiv \max X_1 \bmod 2$ and $\max X_0 < \max X_1$, then $G^+_{(i)}(X_0, X_1) = G^+_{(i)}(X_0, X_1 \triangle X_0)$ and $G^+_{(i)}(X_0, X_1)^* = G^+_{(i)}(X_0^*, X_1 \triangle X_0)$, where $\triangle$ denotes the symmetric difference. For this reason, we introduce the following definition.

Definition 1.4.7. Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 4$ and let i be a legal coloring of T. Say that $X_0, X_1 \subset_f \mathbf{Z}_{\geq 0}$ are **compatible** if for each $x \in X_t$ (with $t \in \{0, 1\}$), if $x \geq \max X_{1-t}$ then $x \equiv \max X_t \bmod 2$. Define $\underline{\mathcal{G}}_{(i)}$ to be the set containing the following groups:

- $G^+_{(i)}(Y_0, Y_1)$, where $Y_0 \in \{\varnothing, X_0, X_0^*\}$, $Y_1 \in \{\varnothing, X_1, X_1^*\}$, $X_0, X_1 \subset_f \mathbf{Z}_{\geq 0}$ and, if $Y_0 \neq \varnothing$ and $Y_1 \neq \varnothing$, then X_0 and X_1 are compatible;

- $G^+_{(i)}(X_0, X_1)^*$, where $X_0, X_1 \subset_f \mathbf{Z}_{\geq 0}$ are compatible.

We then have the following result.

Proposition 1.4.8. *The members of $\underline{\mathcal{G}}_{(i)}$ are pairwise different.*

Proof. The groups in $\mathcal{S}_{(i)}$ are simple (Theorem 1.4.6) while those in $\mathcal{N}_{(i)}$ are not, so a group in $\mathcal{S}_{(i)}$ is never equal to a group in $\mathcal{N}_{(i)}$.

Let us now prove that two groups $G^+_{(i)}(Y_0, Y_1)$ and $G^+_{(i)}(Y_0', Y_1')$ in $\mathcal{S}_{(i)} \cap \underline{\mathcal{G}}_{(i)}$ with $(Y_0, Y_1) \neq (Y_0', Y_1')$ are always different. If $Y_0 = \varnothing$ but $Y_0' \neq \varnothing$, then $G^+_{(i)}(\varnothing, Y_1) \not\subseteq G^+_{(i)}(Y_0', Y_1')$. Indeed, for each ball $B(v, n)$ in T such that $S(v, n) \subseteq V_0(T)$, the fixator of $B(v, n)$ in $G^+_{(i)}(\varnothing, Y_1)$ can act in any manner on $B(v, n+1)$. This is not true for $G^+_{(i)}(Y_0', Y_1')$ when $n \geq \max Y_0'$. This reasoning works whenever exactly one of the two sets Y_t and Y_t' is empty for some $t \in \{0, 1\}$.

We now consider $X_0 \neq X_0'$ and show that $G^+_{(i)}(X_0, \varnothing) \neq G^+_{(i)}(X_0', \varnothing)$ (the proof is the same for $G^+_{(i)}(\varnothing, X_1) \neq G^+_{(i)}(\varnothing, X_1')$ with $X_1 \neq X_1'$). If $\max X_0 < \max X_0'$, then fix v a vertex of type $(\max X_0) \bmod 2$ and construct (as explained in §1.4.1 with the labellings, starting from v) an element $h \in G^+_{(i)}(X'_0, \varnothing)$ such that there is exactly one vertex labelled by o in $S_{X_0}(v)$. This is possible because $\max X_0 < \max X'_0$. By definition, $h \notin G^+_{(i)}(X_0, \varnothing)$. The reasoning is the same when $\max X_0 > \max X'_0$. Now assume that $\max X_0 = \max X'_0$. Suppose without loss of generality that $X'_0 \not\subseteq X_0$ and take $r \in X'_0 \setminus X_0$. Then, if v is a vertex of type $(\max X_0) \bmod 2$, there exists $h \in G^+_{(i)}(X_0, \varnothing)$ such that there is exactly one vertex labelled by o in $S(v, r)$ and all the other vertices of $B(v, \max X_0)$ are labelled by e. This element h is not in $G^+_{(i)}(X'_0, \varnothing)$ because it does not satisfy the condition on $S_{X'_0}(v)$ (since $r \in X'_0$).

Finally, let $(X_0, X_1) \neq (X'_0, X'_1)$ be such that X_0 and X_1 (resp. X'_0 and X'_1) are compatible. We show that $G^+_{(i)}(X_0, X_1) \neq G^+_{(i)}(X'_0, X'_1)$. As in the previous case, if $\max X_0 < \max X'_0$ then we can construct an element $h \in G^+_{(i)}(X'_0, X'_1)$ that is not in $G^+_{(i)}(X_0, X_1)$. The same reasoning works when $\max X_0 > \max X'_0$ or $\max X_1 \neq \max X'_1$. Now assume that $\max X_0 = \max X'_0$ and $\max X_1 = \max X'_1$, and without loss of generality that $\max X_0 \leq \max X_1$. If $X_0 \neq X'_0$ then as before we obtain an element that is in exactly one of the two groups $G^+_{(i)}(X_0, X_1)$ and $G^+_{(i)}(X'_0, X'_1)$. Now suppose that $X_0 = X'_0$ and $X_1 \neq X'_1$. Once again, assume without loss of generality that $X'_1 \not\subseteq X_1$. Let r be the greatest element of $X'_1 \setminus X_1$. Since X'_0 and X'_1 are compatible, we have $r < \max X_0$ or $r \equiv \max X_1 \bmod 2$. If v is a vertex of type $(1 + \max X_1) \bmod 2$, this means that there is no set of the form $S_{X_0}(x)$ (with x of type $(\max X_0) \bmod 2$) that is contained in $B(v, r)$ but not already in $B(v, r-1)$. Hence, there exists $h \in G^+_{(i)}(X_0, X_1)$ with exactly one vertex labelled by o in $S(v, r)$ and all the other vertices of $B(v, r)$ labelled by e. Our choice for r is such that h cannot also be an element of $G^+_{(i)}(X'_0, X'_1)$.

We proved that the groups in $\mathcal{S}_{(i)} \cap \underline{\mathcal{G}}_{(i)}$ are pairwise different. Let us now do it for $\mathcal{N}_{(i)} \cap \underline{\mathcal{G}}_{(i)}$. Take $H, H' \in \mathcal{N}_{(i)} \cap \underline{\mathcal{G}}_{(i)}$ with different names. If H and H' have exactly the same sets X_0 and/or X_1 in their name, i.e. if $s(H) = s(H')$, then it is clear from the definitions and the constructions with the labellings (see §1.4.1) that $H \neq H'$. We can therefore assume that $s(H) \neq s(H')$ (and these groups are really different because of our work for $\mathcal{S}_{(i)} \cap \underline{\mathcal{G}}_{(i)}$). Suppose for a contradiction that $H = H'$. Recall that $s(H) \trianglelefteq H$, $s(H') \trianglelefteq H'$ and $s(H)$ and $s(H')$ are simple. Hence, the group $H = H'$ has two different simple normal subgroups having a non-trivial intersection ($s(H)$ and $s(H')$ both contain $\mathrm{Alt}_{(i)}(T)^+$), which is impossible. □

As a corollary of the classification, we will also see that each group in $\mathcal{G}_{(i)}$ is equal to a group in $\underline{\mathcal{G}}_{(i)}$, which makes the definition of $\underline{\mathcal{G}}_{(i)}$ completely natural.

1.4.4 Normalizers

We are now interested in the normalizers of all our groups. Before giving them, we need to define the groups that will appear in the classification result for groups in $\mathcal{H}_T \setminus \mathcal{H}_T^+$.

Definition 1.4.9. Let T be the d-regular tree with $d \geq 4$ and let i be a legal coloring of T. With the same notation as in Definition 1.4.1, set $G_{(i)}(\varnothing, \varnothing) := \mathrm{Aut}(T)$ and define, for $X \subset_f \mathbf{Z}_{\geq 0}$,

$$G_{(i)}(X, X) := \left\{ g \in \mathrm{Aut}(T) \;\middle|\; \mathrm{Sgn}_{(i)}(g, S_X(v)) = 1 \text{ for each } v \in V(T) \right\},$$

$$G_{(i)}(X^*, X^*) := \left\{ g \in \mathrm{Aut}(T) \;\middle|\; \begin{array}{c} \text{All } \mathrm{Sgn}_{(i)}(g, S_X(v)) \text{ with } v \in V_0(T) \\ \text{are equal and all } \mathrm{Sgn}_{(i)}(g, S_X(v)) \\ \text{with } v \in V_1(T) \text{ are equal} \end{array} \right\},$$

$$G_{(i)}(X, X)^* := \left\{ g \in \mathrm{Aut}(T) \;\middle|\; \begin{array}{c} \text{All } \mathrm{Sgn}_{(i)}(g, S_X(v)) \\ \text{with } v \in V(T) \text{ are equal} \end{array} \right\},$$

and

$$G'_{(i)}(X, X)^* := \left\{ g \in \mathrm{Aut}(T) \;\middle|\; \begin{array}{c} \text{All } \mathrm{Sgn}_{(i)}(g, S_X(v)) \text{ with } v \in V_0(T) \\ \text{are equal to } p_0, \text{ all } \mathrm{Sgn}_{(i)}(g, S_X(v)) \\ \text{with } v \in V_1(T) \text{ are equal to } p_1, \text{ and} \\ p_0 = p_1 \text{ if and only if } g \in \mathrm{Aut}(T)^+ \end{array} \right\}.$$

We write $\mathcal{G}'_{(i)}$ for the set of all these groups.

The normalizers are then given in the following lemma.

Lemma 1.4.10 (Theorem 1.A (iii))**.** *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 4$ and let i be a legal coloring of T.*

(i) Define the map $n^+ : \mathcal{G}_{(i)} \to \mathcal{G}_{(i)}$ *by* $n^+(G^+_{(i)}(\varnothing,\varnothing)) = G^+_{(i)}(\varnothing,\varnothing)$, $n^+(G^+_{(i)}(X_0,\varnothing)) = G^+_{(i)}(X_0^*,\varnothing)$, $n^+(G^+_{(i)}(\varnothing,X_1)) = G^+_{(i)}(\varnothing,X_1^*)$, $n^+(G^+_{(i)}(X_0,X_1)) = G^+_{(i)}(X_0^*,X_1^*)$ *and* $n^+(H) = n^+(s(H))$ *for* $H \in \mathcal{N}_{(i)}$. *Let* $H \in \mathcal{G}_{(i)}$.

(a) If $\tau \in \mathrm{Aut}(T)^+$ *is such that* $\tau H \tau^{-1} \supseteq \mathrm{Alt}_{(i)}(T)^+$, *then* $\tau \in n^+(H)$.

(b) $n^+(H)$ *is the normalizer of* H *in* $\mathrm{Aut}(T)^+$.

(ii) If $d_0 = d_1$ *then for each* $X \subset_f \mathbf{Z}_{\geq 0}$ *the normalizer of* $G^+_{(i)}(X,X)$ *(resp.* $G^+_{(i)}(X^*,X^*)$ *and* $G^+_{(i)}(X,X)^*$*) in* $\mathrm{Aut}(T)$ *is* $G_{(i)}(X^*,X^*)$.

Proof.

(i) We first prove (a). Since $n^+(H) \supseteq H$, having $\tau H \tau^{-1} \supseteq \mathrm{Alt}_{(i)}(T)^+$ implies $\tau n^+(H)\tau^{-1} \supseteq \mathrm{Alt}_{(i)}(T)^+$. As $n^+(n^+(H)) = n^+(H)$, this means that we can just prove the statement for $H = G^+_{(i)}(\varnothing,\varnothing)$, $G^+_{(i)}(X_0^*,\varnothing)$, $G^+_{(i)}(\varnothing,X_1^*)$ and $G^+_{(i)}(X_0^*,X_1^*)$. If $H = G^+_{(i)}(\varnothing,\varnothing) = \mathrm{Aut}(T)^+$ then there is nothing to prove.

Now consider $H = G^+_{(i)}(X_0^*,\varnothing)$. Let $\tau \in \mathrm{Aut}(T)^+$ be such that $\tau G^+_{(i)}(X_0^*,\varnothing)\tau^{-1} \supseteq \mathrm{Alt}_{(i)}(T)^+$. Remind that

$$G^+_{(i)}(X_0^*,\varnothing) := \left\{ g \in \mathrm{Aut}(T)^+ \;\middle|\; \begin{array}{c} \text{All } \mathrm{Sgn}_{(i)}(g, S_{X_0}(v)) \\ \text{with } v \in V_t(T) \text{ are equal} \end{array} \right\},$$

where $t = (\max X_0) \bmod 2$. We therefore directly obtain

$$\tau G^+_{(i)}(X_0^*,\varnothing)\tau^{-1} = \left\{ g \in \mathrm{Aut}(T)^+ \;\middle|\; \begin{array}{c} \text{All } \mathrm{Sgn}_{(i)}(\tau^{-1}g\tau, S_{X_0}(v)) \\ \text{with } v \in V_t(T) \text{ are equal} \end{array} \right\}.$$

By Lemma 1.2.11, we have $\sigma_{(i)}(\tau^{-1}g\tau, w) = \sigma_{(i)}(\tau^{-1}, g\tau(w)) \circ \sigma_{(i)}(g,\tau(w)) \circ \sigma_{(i)}(\tau,w)$ and $\sigma_{(i)}(\tau^{-1}, g\tau(w)) = \sigma_{(i)}(\tau, \tau^{-1}g\tau(w))^{-1}$, so $\mathrm{Sgn}_{(i)}(\tau^{-1}g\tau, S_{X_0}(v))$ is equal to

$$\mathrm{Sgn}_{(i)}(\tau, S_{X_0}(\tau^{-1}g\tau(v))) \cdot \mathrm{Sgn}_{(i)}(g, S_{X_0}(\tau(v))) \cdot \mathrm{Sgn}_{(i)}(\tau, S_{X_0}(v)).$$

In order to show that $\tau \in G^+_{(i)}(X_0^*,\varnothing)$, we need to prove that all $\mathrm{Sgn}_{(i)}(\tau, S_{X_0}(v))$ with $v \in V_t(T)$ are equal. It suffices to show that $\mathrm{Sgn}_{(i)}(\tau, S_{X_0}(x)) = \mathrm{Sgn}_{(i)}(\tau, S_{X_0}(y))$ when $x, y \in V_t(T)$ and $d(x, y) = 2$. Fix such x and y and consider $z \in V_t(T)$ such that $d(x, z) = d(y, z) = 2$. Take $g \in \mathrm{Alt}_{(i)}(T)^+$ such that $g(\tau(x)) = \tau(z)$ and $g(\tau(z)) = \tau(y)$. By hypothesis, we have $g \in \tau G^+_{(i)}(X_0^*, \varnothing)\tau^{-1}$ so the two values

$$\mathrm{Sgn}_{(i)}(\tau, S_{X_0}(z)) \cdot \mathrm{Sgn}_{(i)}(g, S_{X_0}(\tau(x))) \cdot \mathrm{Sgn}_{(i)}(\tau, S_{X_0}(x))$$

and

$$\mathrm{Sgn}_{(i)}(\tau, S_{X_0}(y)) \cdot \mathrm{Sgn}_{(i)}(g, S_{X_0}(\tau(z))) \cdot \mathrm{Sgn}_{(i)}(\tau, S_{X_0}(z))$$

are equal. As $g \in \mathrm{Alt}_{(i)}(T)^+$, we have $\mathrm{Sgn}_{(i)}(g, A) = 1$ for each finite set $A \subseteq V(T)$ and hence $\mathrm{Sgn}_{(i)}(\tau, S_{X_0}(x)) = \mathrm{Sgn}_{(i)}(\tau, S_{X_0}(y))$.

For $H = G^+_{(i)}(\varnothing, X_1^*)$, the reasoning is the same.

For $H = G^+_{(i)}(X_0^*, X_1^*)$, the inclusion $\tau H \tau^{-1} \supseteq \mathrm{Alt}_{(i)}(T)^+$ implies in particular that $\tau G^+_{(i)}(X_0^*, \varnothing)\tau^{-1} \supseteq \mathrm{Alt}_{(i)}(T)^+$ and that $\tau G^+_{(i)}(\varnothing, X_1^*)\tau^{-1} \supseteq \mathrm{Alt}_{(i)}(T)^+$. By the previous work, we therefore obtain $\tau \in G^+_{(i)}(X_0^*, \varnothing) \cap G^+_{(i)}(\varnothing, X_1^*) = G^+_{(i)}(X_0^*, X_1^*)$.

Part (b) follows from (a). Indeed, the normalizer of H in $\mathrm{Aut}(T)^+$ is contained in $n^+(H)$ by (a), and one readily checks that $n^+(H)$ normalizes H for each $H \in \mathcal{G}_{(i)}$.

(ii) Let H be one of $G^+_{(i)}(X, X)$, $G^+_{(i)}(X^*, X^*)$ and $G^+_{(i)}(X, X)^*$. By (i), the normalizer of H in $\mathrm{Aut}(T)^+$ is $n^+(H) = G^+_{(i)}(X^*, X^*)$. Consider $\nu \in \mathrm{Aut}(T) \setminus \mathrm{Aut}(T)^+$ not preserving the types but preserving the colors, i.e. such that $i \circ \nu = i$. It is clear that ν normalizes H, and hence the normalizer of H in $\mathrm{Aut}(T)$ is exactly $n^+(H) \cup n^+(H)\nu = G^+_{(i)}(X^*, X^*) \cup G^+_{(i)}(X^*, X^*)\nu = G_{(i)}(X^*, X^*)$. $\square$

1.5 The classification

Throughout this section, we let i be a legal coloring of T and fix $d_0, d_1 \geq 6$. Our goal is to find all groups $H \in \mathcal{H}_T^+$ satisfying $H \supseteq \mathrm{Alt}_{(i)}(T)^+$. Our strategy consists in first observing the groups $H \in \mathcal{H}_T^+$ with this property and in defining some invariants (namely $c(t)$, $K(t)$ and f_v^t). We will then see that these invariants form a complete set of invariants, i.e. that they completely characterize the group H. This is the subject of Theorem 1.H$'$, which is the precise formulation of Theorem 1.H mentioned in §1.1. The idea is then simple: compute these invariants for the groups in $\underline{\mathcal{G}}_{(i)}$ and prove that these are the only values that our invariants can take. This task will turn out to be lengthy and technical because of the algebraic invariants f_v^t which are not easy to manipulate.

1.5.1 Evens and odds diagrams

Let us first fix some $v \in V(T)$ and $k \in \mathbf{Z}_{\geq 0}$. The colored rooted tree $B(v,k)$ where each vertex is additionally labelled by e or o is called a **diagram supported by** $B(v,k)$. We write $\Delta_{v,k}$ for the set of all these diagrams. Remark that $|\Delta_{v,k}| = 2^{|V(B(v,k))|}$. There is now a natural way to define the surjective map

$$\mathcal{D}\colon \mathrm{Aut}(B(v,k+1)) \to \Delta_{v,k}$$

where $B(v,k+1)$ is seen as a colored rooted tree. Indeed, given $\tilde{h} \in \mathrm{Aut}(B(v,k+1))$ we can define $\mathcal{D}(\tilde{h})$ (which we call the **diagram of** $\tilde{h}$) to be the rooted tree $B(v,k)$ where each vertex w is labelled by the parity (e for *even* or o for *odd*) of $\sigma_{(i)}(\tilde{h},w)$. We highlight the fact that $\mathcal{D}$ associates a diagram supported by $B(v,k)$ to an automorphism of the larger ball $B(v,k+1)$. In this section, we will often deal with such diagrams. For this reason, the next lemma must be well understood.

Lemma 1.5.1. *Let $\tilde{g},\tilde{h} \in \mathrm{Aut}(B(v,k+1))$ and let w be a vertex of $B(v,k)$.*

- *The label of w in $\mathcal{D}(\tilde{g}\tilde{h})$ is e if and only if the label of w in $\mathcal{D}(\tilde{h})$ and the label of $\tilde{h}(w)$ in $\mathcal{D}(\tilde{g})$ are identical.*
- *The label of w in $\mathcal{D}(\tilde{g}^{-1})$ is equal to the label of $\tilde{g}^{-1}(w)$ in $\mathcal{D}(\tilde{g})$.*

Proof. This is a corollary of Lemma 1.2.11. □

We now fix $H \in \mathcal{H}_T^+$ such that $H \supseteq \mathrm{Alt}_{(i)}(T)^+$ and denote by $\tilde{H}^k(v)$ the natural image of $H(v)$ in $\mathrm{Aut}(B(v,k+1))$. Since H is closed in

$\mathrm{Aut}(T)$ and generated by its vertex stabilizers, it is entirely described by the groups $\tilde{H}^k(v)$ with $v \in V(T)$ and $k \in \mathbf{Z}_{\geq 0}$. The next lemma shows that knowing the diagrams of elements of $\tilde{H}^k(v)$, i.e. $\mathcal{D}(\tilde{H}^k(v))$, suffices to fully know $\tilde{H}^k(v)$.

Lemma 1.5.2. *We have* $\tilde{H}^k(v) = \mathcal{D}^{-1}(\mathcal{D}(\tilde{H}^k(v)))$.

Proof. Take $\tilde{h} \in \tilde{H}^k(v)$ and $\tilde{g} \in \mathrm{Aut}(B(v,k+1))$ such that $\mathcal{D}(\tilde{g}) = \mathcal{D}(\tilde{h})$. We want to show that $\tilde{g} \in \tilde{H}^k(v)$. As $\mathcal{D}(\tilde{g}) = \mathcal{D}(\tilde{h})$, Lemma 1.5.1 directly implies that all the vertices of $\mathcal{D}(\tilde{g}\tilde{h}^{-1})$ are labelled by e, i.e. $\tilde{g}\tilde{h}^{-1}$ is an element of $\mathrm{Alt}_{(i)}(B(v,k+1))$. Since $H \supseteq \mathrm{Alt}_{(i)}(T)^+$, we have $\tilde{H}^k(v) \supseteq \mathrm{Alt}_{(i)}(B(v,k+1))$ and hence $\tilde{g} = (\tilde{g}\tilde{h}^{-1})\tilde{h} \in \tilde{H}^k(v)$. □

In view of the previous lemma, we only need to describe $\mathcal{D}(\tilde{H}^k(v))$ to entirely describe $\tilde{H}^k(v)$. We are first interested in the diagrams of $\mathcal{D}(\tilde{H}^k(v))$ where all the vertices of $B(v,k-1)$ are labelled by e. Let us call ***e*-diagram** a diagram in $\Delta_{v,k}$ with this property, and remark that $\tilde{g} \in \mathrm{Aut}(B(v,k+1))$ is such that $\mathcal{D}(\tilde{g})$ is an e-diagram if and only if $\tilde{g}|_{B(v,k)} \in \mathrm{Alt}_{(i)}(B(v,k))$. We denote by $\tilde{H}^k(v)_e$ the subgroup of $\tilde{H}^k(v)$ consisting of elements whose diagram is an e-diagram. If $\delta \in \Delta_{v,k}$ and if w is a vertex of δ then the subtree of δ spanned by w and all its descendants is called the **branch of** w. For $0 \leq r \leq k$, an ***r*-branch** of δ is a branch of a vertex at distance $k-r$ from v. The only k-branch is thus the full tree δ and the 0-branches all consist of a single leaf of δ.

Lemma 1.5.3. *Let* $v \in V(T)$ *and* $k \in \mathbf{Z}_{\geq 0}$. *Exactly one of the following assertions holds.*

1. $\mathcal{D}(\tilde{H}^k(v)_e)$ *contains all the e-diagrams.*

2. *There exists* $0 \leq r \leq k$ *such that* $\mathcal{D}(\tilde{H}^k(v)_e)$ *exactly contains the e-diagrams with an even number of labels o in each r-branch.*

3. $\mathcal{D}(\tilde{H}^k(v)_e)$ *exactly contains the e-diagrams with an even number of labels o in each* $(k-1)$*-branch and the e-diagrams with an odd number of labels o in each* $(k-1)$*-branch. (This case only occurs if* $k \geq 1$*.)*

Proof. For each e-diagram δ, label each branch of δ with E or O depending on whether it contains an even or an odd number of vertices labelled by o. Denote by $\mathcal{D}_s$ the set of e-diagrams whose s-branches are all labelled by E. By definition, $\mathcal{D}(\mathrm{Alt}_{(i)}(B(v,k+1))) = \mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots \subset \mathcal{D}_k$. Since $\tilde{H}^k(v) \supseteq \mathrm{Alt}_{(i)}(B(v,k+1))$, we have $\mathcal{D}(\tilde{H}^k(v)_e) \supseteq \mathcal{D}_0$.

Claim 1. *Let $0 \leq s \leq k-1$. If $\mathcal{D}(\tilde{H}^k(v)_e) \supseteq \mathcal{D}_s$, then $\mathcal{D}(\tilde{H}^k(v)_e) \supseteq \mathcal{D}_{s+1}$ or for every diagram $\delta \in \mathcal{D}(\tilde{H}^k(v)_e)$ and every $(s+1)$-branch b of δ, all the s-branches in b have the same label.*

Proof of the claim: Suppose there exist a diagram $\mathcal{D}(\tilde{h}) \in \mathcal{D}(\tilde{H}^k(v)_e)$ and an $(s+1)$-branch b of $\mathcal{D}(\tilde{h})$ containing both an s-branch b_1 labelled by E and an s-branch b_2 labelled by O. Let b_3 and b_4 be two other s-branches in b with the same label (such branches exist because $d_0, d_1 \geq 6$). Consider $\tilde{\tau} \in \mathrm{Alt}_{(i)}(B(v,k+1)) \subseteq \tilde{H}^k(v)$ an element interchanging b_1 and b_2, interchanging b_3 and b_4, and stabilizing all the other s-branches. In this way, $\tilde{h}\tilde{\tau}\tilde{h}^{-1} \in \tilde{H}^k(v)$ is such that the only s-branches of $\mathcal{D}(\tilde{h}\tilde{\tau}\tilde{h}^{-1})$ labelled by O are b_1 and b_2 (see Lemma 1.5.1). Conjugating this element by adequate elements of $\mathrm{Alt}_{(i)}(B(v,k+1))$ and combining them, we deduce (once again by using Lemma 1.5.1) that $\mathcal{D}(\tilde{H}^k(v)_e)$ contains all the e-diagrams where each $(s+1)$-branch contains an even number of s-branches labelled by O. These are exactly the e-diagrams with each $(s+1)$-branch labelled by E, so $\mathcal{D}(\tilde{H}^k(v)_e) \supseteq \mathcal{D}_{s+1}$. ■

Claim 2. *Let $0 \leq s \leq k-2$. If $\mathcal{D}(\tilde{H}^k(v)_e) \supseteq \mathcal{D}_s$ but $\mathcal{D}(\tilde{H}^k(v)_e) \not\supseteq \mathcal{D}_{s+1}$, then $\mathcal{D}(\tilde{H}^k(v)_e) = \mathcal{D}_s$.*

Proof of the claim: By Claim 1, the fact that $\mathcal{D}(\tilde{H}^k(v)_e) \supseteq \mathcal{D}_s$ but $\mathcal{D}(\tilde{H}^k(v)_e) \not\supseteq \mathcal{D}_{s+1}$ implies that for every diagram $\delta \in \mathcal{D}(\tilde{H}^k(v)_e)$ and every $(s+1)$-branch b of δ, all the s-branches in b have the same label $(*)$. In order to show that $\mathcal{D}(\tilde{H}^k(v)_e) = \mathcal{D}_s$, it suffices to prove that it is impossible to have a diagram with an $(s+1)$-branch only containing s-branches labelled by O. By contradiction, suppose there exist $\tilde{h} \in \tilde{H}^k(v)_e$ and some $(s+1)$-branch b of $\mathcal{D}(\tilde{h})$ all whose s-branches are labelled by O. In view of Lemma 1.5.2, we can assume that $\tilde{h}$ fixes $B(v,k)$. Let us say that b is the branch of the vertex w. Denote by $x_1, \ldots, x_r$ the children of w, by $b_1, \ldots, b_r$ the corresponding s-branches,

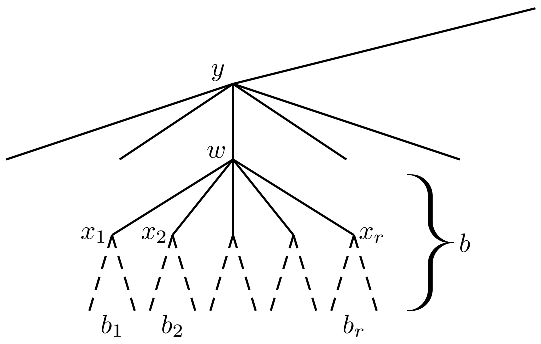


Figure 1.4: Illustration of Lemma 1.5.3, Claim 2.

and by y the parent of w (note that $w \neq v$ since $s \leq k-2$), see Figure 1.4. Let $h \in H$ be an element whose image in $\tilde{H}^k(v)$ is $\tilde{h}$ and consider an element $g \in \mathrm{Alt}_{(i)}(T)^+$ that fixes w and interchanges x_1 and y. Then $f = ghg^{-1} \in H$ is an element fixing $B(w,1)$. Now observe the image of f in $\tilde{H}^{s+1}(w)$: it is contained in $\tilde{H}^{s+1}(w)_e$ and the branches $b_2, \ldots, b_r$ are labelled by O while b_1 is labelled by E (see Lemma 1.5.1). Consider an element $\tau \in \mathrm{Alt}_{(i)}(T)^+$ that fixes w and all the vertices that are closer to y than to w, interchanges x_1 and x_2 and interchanges x_3 and x_4. Then $f\tau f^{-1} \in H$ is an element that also fixes w and all the vertices that are closer to y than to w and, if we look at its image in $\tilde{H}^k(v)$, it is contained in $\tilde{H}^k(v)_e$ and the branches b_1 and b_2 are labelled by O while the branches $b_3, \ldots, b_r$ are labelled by E. This contradicts $(*)$. ■

If $\mathcal{D}(\tilde{H}^k(v)_e) \supseteq \mathcal{D}_k$ then there are two options: either $\mathcal{D}(\tilde{H}^k(v)_e) = \mathcal{D}_k$ (i.e. we are in the second case with $r = k$) or there exists a diagram in $\mathcal{D}(\tilde{H}^k(v)_e)$ whose k-branch is labelled by O. In the latter case, $\mathcal{D}(\tilde{H}^k(v)_e)$ contains all the e-diagrams.

Suppose now that $\mathcal{D}(\tilde{H}^k(v)_e) \not\supseteq \mathcal{D}_k$. Then there exists $0 \leq s \leq k-1$ such that $\mathcal{D}(\tilde{H}^k(v)_e) \supseteq \mathcal{D}_s$ but $\mathcal{D}(\tilde{H}^k(v)_e) \not\supseteq \mathcal{D}_{s+1}$. If $s \neq k-1$ then by Claim 2 we have $\mathcal{D}(\tilde{H}^k(v)_e) = \mathcal{D}_s$, i.e. we are in the second case with $r = s$. If $s = k-1$, then by Claim 1 we know that each diagram in $\mathcal{D}(\tilde{H}^k(v)_e)$ either has all its $(k-1)$-branches labelled by E or all its $(k-1)$-branches labelled by O. If there is no diagram with labels O, then $\mathcal{D}(\tilde{H}^k(v)_e) = \mathcal{D}_{k-1}$ (i.e. we are in the second case with $r = k-1$). On the contrary, if there exists such a diagram, then $\mathcal{D}(\tilde{H}^k(v)_e)$ also contains the e-diagrams with an odd number of labels o in each $(k-1)$-branch (i.e. we are in the third case). $\square$

1.5.2 Four possible shapes for $H(v)$

For $v \in V(T)$ and $k \in \mathbf{Z}_{\geq 0}$, Lemma 1.5.3 gives different shapes that $\mathcal{D}(\tilde{H}^k(v)_e)$ can take. We now associate a symbol $\alpha_k(v)$ to each v and k by defining $\alpha_k(v) = \infty$ in the first case, $\alpha_k(v) = r$ in the second case and $\alpha_k(v) = (k-1)^*$ in the third case. A natural total order on the set of symbols $\{\infty, 0, 0^*, 1, 1^*, \ldots\}$ is given by $0 < 0^* < 1 < 1^* < \cdots$ and $x < \infty$ for each $x \neq \infty$.

Lemma 1.5.4. *For $x \in \{1, 2, \ldots, k\}$, we have $\alpha_k(v) \geq x$ if and only if there exists a diagram in $\mathcal{D}(\tilde{H}^k(v)_e)$ with exactly two vertices labelled by o, situated in the same x-branch but in different $(x-1)$-branches.*

Proof. This is a consequence of the definition of $\alpha_k(v)$. $\square$

Clearly, since $\mathrm{Alt}_{(i)}(T)^+$ is transitive on $V_0(T)$ and $V_1(T)$, we have $\alpha_k(v) = \alpha_k(v')$ when v and v' have the same type. For this reason, for $t \in \{0, 1\}$ we define α_k^t to be equal to $\alpha_k(v)$ where v is a vertex of type $(t+k) \bmod 2$. In this way, α_k^t tells us the labels that can appear in $S(v, k)$, which is a sphere containing vertices of type t.

We are now interested in how the sequences $(\alpha_k^0)_{k\in\mathbf{Z}_{\geq 0}}$ and $(\alpha_k^1)_{k\in\mathbf{Z}_{\geq 0}}$ can look like.

Lemma 1.5.5. *Let $t \in \{0, 1\}$. Either $\alpha_k^t = \infty$ for all $k \in \mathbf{Z}_{\geq 0}$ (case #0), or there exists $K \in \mathbf{Z}_{\geq 0}$ such that the sequence $(\alpha_k^t)_{k\in\mathbf{Z}_{\geq 0}}$ takes one of the following three shapes. (For cases #2 and #3, K cannot be equal to 0.)*

#	α_0^t	$\cdots$	α_{K-1}^t	α_K^t	α_{K+1}^t	α_{K+2}^t	$\cdots$
1	∞	$\cdots$	∞	K	K	K	$\cdots$
2	∞	$\cdots$	∞	$K-1$	$K-1$	$K-1$	$\cdots$
3	∞	$\cdots$	∞	$(K-1)^*$	$K-1$	$K-1$	$\cdots$

Proof. We prove this result by giving two rules that $(\alpha_k^t)_{k\in\mathbf{Z}_{\geq 0}}$ satisfies.

Claim 1. *The sequence $(\alpha_k^t)_{k\in\mathbf{Z}_{\geq 0}}$ is non-increasing.*

Proof of the claim: Let $k \in \mathbf{Z}_{\geq 0}$, let v be a vertex of type $(t+k+1)$ mod 2 and let w be a vertex adjacent to v. Given a diagram $\delta \in \mathcal{D}(\tilde{H}^{k+1}(v)_e)$, Lemma 1.5.2 tells us that it is realized by an element $\tilde{h} \in \tilde{H}^{k+1}(v)_e$ that fixes w. Hence, $\tilde{h}$ has a natural image in $\tilde{H}^k(w)_e$ and the diagram of this image is exactly the restriction of δ to $B(w,k)$. Hence, $\mathcal{D}(\tilde{H}^k(w)_e)$ contains the restriction of each element of $\mathcal{D}(\tilde{H}^{k+1}(v)_e)$ to $B(w,k)$. Observing the different possibilities for α^t_{k+1}, this always implies that $\alpha^t_k \geq \alpha^t_{k+1}$. ■

Claim 2. *If $\alpha^t_k \geq x$ with $x \in \{0, 1, \dots, k\}$, then $\alpha^t_{k+1} \geq x$.*

Proof of the claim: If $x = 0$ then the claim is trivial, so suppose that $x > 0$. Let w be a vertex of type $(t+k)$ mod 2. Since $\alpha^t_k \geq x$, there exists $h \in H(w)$ whose image in $\tilde{H}^k(w)$ has a diagram which is an e-diagram with exactly two vertices labelled by o, say a and b, in the same x-branch but in different $(x-1)$-branches (see Lemma 1.5.4). Take $c \in S(w,k)$ a vertex in this same x-branch but in a third $(x-1)$-branch and v a vertex adjacent to w such that a is closer to w than to v (see Figure 1.5). By Lemma 1.5.2, we can assume that h fixes v. Consider $\tau \in \mathrm{Alt}_{(i)}(T)^+$ an element fixing all the vertices closer to v than to w, stabilizing a and interchanging b and c. Then by Lemma 1.5.1 the image of $h\tau h^{-1}$ in $\tilde{H}^{k+1}(v)$ has a diagram which is an e-diagram having exactly two vertices labelled by o, namely b and c. By Lemma 1.5.4, this implies that $\alpha^t_{k+1} \geq x$. ■

These two claims suffice to get the result. Indeed, we either have

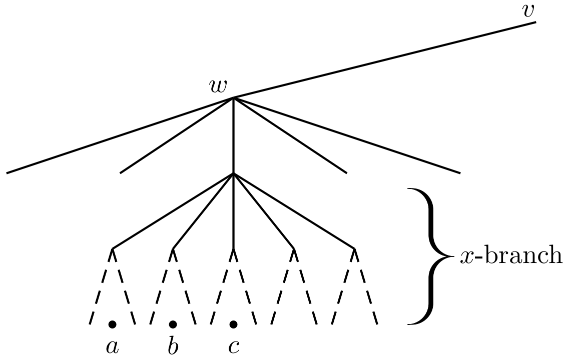


Figure 1.5: Illustration of Lemma 1.5.5, Claim 2.

$\alpha_0^t = 0$ or $\alpha_0^t = \infty$. If $\alpha_0^t = 0$ then by Claim 1 we get case #1. If $\alpha_0^t = \infty$, then either $\alpha_k^t = \infty$ for all $k \in \mathbf{Z}_{\geq 0}$, or there exists a smallest K such that $\alpha_K^t < \infty$. In the latter case, Claim 2 with $k = x = K - 1$ gives $\alpha_K^t \geq K - 1$, so $\alpha_K^t \in \{K-1, (K-1)^*, K\}$. If $\alpha_K^t \in \{K-1, K\}$, then the two claims imply that $(\alpha_k^t)_{k \geq K}$ is constant and we get cases #1 and #2. If $\alpha_K^t = (K-1)^*$, then Claim 1 and Claim 2 with $k = K$ and $x = K - 1$ give $(K-1) \leq \alpha_{K+1}^t \leq (K-1)^*$. Since α_{K+1}^t is never equal to $(K-1)^*$, we must have $\alpha_{K+1}^t = K - 1$ and then get the constant sequence as above, which gives case #3. ☐

1.5.3 The numerical invariants $c(t)$ and $K(t)$

For $t \in \{0, 1\}$, denote by $c(t) \in \{0, 1, 2, 3\}$ the case that was encountered in Lemma 1.5.5 and by $K(t)$ the smallest integer such that $\alpha_{K(t)}^t < \infty$, as in the lemma (if $c(t) = 0$, define $K(t) = \infty$). The value

$$K'(t) := \lim_{k \to \infty} \alpha_k^t$$

will also be useful for our proofs. Note that $c(t)$ and $K(t)$ completely determine $K'(t)$. Similarly, $c(t)$ and $K'(t)$ determine $K(t)$.

These invariants can be computed for each of our key examples. To simplify the notations, we define the operation $\boxplus\colon \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ by

$$a \boxplus b := a + b - \left\lceil \frac{|a-b|}{2} \right\rceil .$$

Proposition 1.5.6. *The values of $c(0)$, $c(1)$, $K'(0)$ and $K'(1)$ for the members of $\underline{\mathcal{G}}_{(i)}$ are given in Table 1.1. The last column of Table 1.1 gives, for fixed $c(0)$, $c(1)$, $K'(0)$ and $K'(1)$, the exact number of groups (in $\underline{\mathcal{G}}_{(i)}$) in the corresponding line.*

Proof. The values of the different invariants can be computed only using the definitions of the groups and the construction explained in §1.4.1 with labellings e and o. We suggest the reader to compute the invariants for some of the groups to become familiar with the definitions.

The value $2^{K'(0)}$ in the last column for lines 2 and 5 is simply equal to the number of sets $X_0 \subset_f \mathbf{Z}_{\geq 0}$ such that $\max X_0 = K'(0)$. The

		$c(0)$	$K'(0)$	$c(1)$	$K'(1)$	#
1	$G^+_{(i)}(\varnothing,\varnothing)$	0	∞	0	∞	1
2	$G^+_{(i)}(X_0,\varnothing)$	1	$\max X_0$	0	∞	$2^{K'(0)}$
3	$G^+_{(i)}(\varnothing,X_1)$	0	∞	1	$\max X_1$	$2^{K'(1)}$
4	$G^+_{(i)}(X_0,X_1)$	1	$\max X_0$	1	$\max X_1$	$2^{K'(0)\boxplus K'(1)}$
5	$G^+_{(i)}(X_0^*,\varnothing)$	3	$\max X_0$	0	∞	$2^{K'(0)}$
6	$G^+_{(i)}(\varnothing,X_1^*)$	0	∞	3	$\max X_1$	$2^{K'(1)}$
7	$G^+_{(i)}(X_0,X_1^*)$	1	$\max X_0$	3	$\max X_1$	$2^{K'(0)\boxplus K'(1)}$
8	$G^+_{(i)}(X_0^*,X_1)$	3	$\max X_0$	1	$\max X_1$	$2^{K'(0)\boxplus K'(1)}$
9	$G^+_{(i)}(X_0^*,X_1^*)$	3	$\max X_0$	3	$\max X_1$	$2^{K'(0)\boxplus K'(1)}$
10	$G^+_{(i)}(X_0,X_1)^*$ ($\max X_0=\max X_1$)	2	$\max X_0$	2	$\max X_1$	$2^{K'(0)\boxplus K'(1)}$
11	$G^+_{(i)}(X_0,X_1)^*$ ($\max X_0>\max X_1$)	1	$\max X_0$	3	$\max X_1$	$2^{K'(0)\boxplus K'(1)}$
12	$G^+_{(i)}(X_0,X_1)^*$ ($\max X_0<\max X_1$)	3	$\max X_0$	1	$\max X_1$	$2^{K'(0)\boxplus K'(1)}$

Table 1.1: Values of the invariants for the groups in $\underline{\mathcal{G}}_{(i)}$.

reasoning is the same for lines 3 and 6. Concerning line 4 and lines 7–12, the value $2^{K'(0)\boxplus K'(1)}$ corresponds to the number of pairs (X_0,X_1) with $X_0,X_1\subset_f \mathbf{Z}_{\geq 0}$ such that X_0 and X_1 are compatible (as defined in Definition 1.4.7), $\max X_0=K'(0)$ and $\max X_1=K'(1)$. Note that we do not count a group twice as all the groups in $\underline{\mathcal{G}}_{(i)}$ are pairwise different (see Proposition 1.4.8). $\square$

In Table 1.1, it is remarkable that having $c(0)=2$ also implies $c(1)=2$ and $K(0)=K(1)$. This is actually a general fact for any $H\in\mathcal{H}_T^+$ such that $H\supseteq \mathrm{Alt}_{(i)}(T)^+$.

Lemma 1.5.7. *If $c(t)=2$ for some $t\in\{0,1\}$, then $c(1-t)=2$ and $K(0)=K(1)$.*

Proof. Assume without loss of generality that $t=0$ and let v be a vertex of type $(K(0)-1)$ mod 2. Since $\alpha^0_{K(0)-1}=\infty$, there exists $h\in H$ fixing

$B(v, K(0)-1)$ and such that the diagram of its image in $\tilde{H}^{K(0)-1}(v)$ has exactly one vertex a labelled by o. Let w be a vertex adjacent to v such that a is not in the branch of w.

We first show that $\alpha^1_{K(0)} \leq K(0)-1$, which will in particular imply that $K(1) \leq K(0)$. Suppose for a contradiction that $\alpha^1_{K(0)} \geq (K(0)-1)^*$. Then there exists $g \in H$ fixing $B(v, K(0))$ and such that the diagram of its image $\tilde{g}$ in $\tilde{H}^{K(0)}(v)$ and the diagram of the image of h in $\tilde{H}^{K(0)}(v)$ coincide on the branch of w. Indeed, the condition $\alpha^1_{K(0)} \geq (K(0)-1)^*$ gives us sufficient freedom to choose the labels of $\mathcal{D}(\tilde{g})$ in the branch of w. Hence, the diagram of the image of hg^{-1} in $\tilde{H}^{K(0)}(w)$ is an e-diagram with a $(K(0)-1)$-branch (the branch of v) containing exactly one vertex labelled by o, contradicting $\alpha^0_{K(0)} = K(0)-1$.

We now prove that $K(1) \geq K(0)$, once again by contradiction, assuming that $K(1) < K(0)$. If $\tilde{h}$ is the image of h in $\tilde{H}^{K(0)}(v)$, then since $\alpha^1_{K(1)} \in \{K'(1), K'(1)^*\}$ the $K'(1)$-branches of $\mathcal{D}(\tilde{h})$ contained in the branch of w all contain an even number of vertices labelled by o. But $\alpha^1_{K(0)} = K'(1)$ (because $K(0) > K(1)$), so there exists $g \in H$ fixing $B(v, K(0))$ and such that the diagram of its image $\tilde{g}$ in $\tilde{H}^{K(0)}(v)$ and the diagram of $\tilde{h}$ coincide on the branch of w. We therefore have the same contradiction as above by considering the image of hg^{-1} in $\tilde{H}^{K(0)}(w)$.

As a conclusion, $K(1) = K(0)$ and $\alpha^1_{K(1)} \leq K(1)-1$ so $c(1) = 2$. $\square$

1.5.4 The algebraic invariants f^t_v

Our next goal is to understand the relationship between $\mathcal{D}(\tilde{H}^{k-1}(v))$ and $\mathcal{D}(\tilde{H}^{k}(v))$ (for fixed v and k). The first result in this direction is the following. We identify the group $\mathbf{C}_2$ of order 2 with $\{E, O\}$, where E is the neutral element. By convention, we say that $B(v,-1) = \varnothing$ and that $\mathcal{D}(\tilde{H}^{-1}(v))$ only contains the empty diagram ε.

Lemma 1.5.8. *Let $v \in V(T)$, let $k \in \mathbf{Z}_{\geq 0}$ and let $\delta \in \mathcal{D}(\tilde{H}^{k-1}(v))$.*

(i) If $\alpha_k(v) = \infty$, then $\mathcal{D}(\tilde{H}^k(v))$ contains all the diagrams of $\Delta_{v,k}$ whose intersection with $B(v, k-1)$ is δ.

(ii) If $\alpha_k(v) = x \in \{0, 1, \ldots, k\}$, denote by $b_1, \ldots, b_m$ the x-branches of $B(v, k)$. Then there exists a unique element $(p_1, \ldots, p_m) \in$

$\{E,O\}^m$ such that the following holds. For each $\hat{\delta} \in \Delta_{v,k}$ with $\hat{\delta} \cap B(v,k-1) = \delta$, if $q_i \in \{E,O\}$ is the parity of the number of vertices labelled by o in $\hat{\delta} \cap b_i \cap S(v,k)$, then $\hat{\delta}$ is contained in $\mathcal{D}(\tilde{H}^k(v))$ if and only if $(q_1,\dots,q_m) = (p_1,\dots,p_m)$.

(iii) If $\alpha_k(v) = (k-1)^*$, denote by $b_1,\dots,b_m$ the $(k-1)$-branches of $B(v,k)$. Then there exists a unique element $[(p_1,\dots,p_m)] \in \{E,O\}^m \big/ \langle (O,\dots,O)\rangle$ such that the following holds. For each $\hat{\delta} \in \Delta_{v,k}$ with $\hat{\delta} \cap B(v,k-1) = \delta$, if $q_i \in \{E,O\}$ is the parity of the number of vertices labelled by o in $\hat{\delta} \cap b_i \cap S(v,k)$, then $\hat{\delta}$ is contained in $\mathcal{D}(\tilde{H}^k(v))$ if and only if $[(q_1,\dots,q_m)] = [(p_1,\dots,p_m)]$.

Proof. Let $\tilde{h} \in \tilde{H}^k(v)$ be such that $\mathcal{D}(\tilde{h}) \cap B(v,k-1) = \delta$.

(i) Let $\tilde{g} \in \mathrm{Aut}(B(v,k+1))$ be such that $\mathcal{D}(\tilde{g}) \cap B(v,k-1) = \delta$. Using Lemma 1.5.1, we see that having $\mathcal{D}(\tilde{g}) \cap B(v,k-1) = \mathcal{D}(\tilde{h}) \cap B(v,k-1)$ implies that $\mathcal{D}(\tilde{g}\tilde{h}^{-1})$ is an e-diagram. As $\alpha_k(v) = \infty$, we get $\tilde{g}\tilde{h}^{-1} \in \tilde{H}^k(v)$ and thus $\tilde{g} = (\tilde{g}\tilde{h}^{-1})\tilde{h} \in \tilde{H}^k(v)$.

(ii),(iii) For each $i \in \{1,\dots,m\}$, let $p_i \in \{E,O\}$ be the parity of the number of vertices labelled by o in $\mathcal{D}(\tilde{h}) \cap b_i \cap S(v,k)$. We prove that $(p_1,\dots,p_m)$ (resp. $[(p_1,\dots,p_m)]$) satisfies the statement (and it is clear that it is unique). Let $\tilde{g} \in \mathrm{Aut}(B(v,k+1))$ be such that $\mathcal{D}(\tilde{g}) \cap B(v,k-1) = \delta$ and let $q_i \in \{E,O\}$ be the parity of the number of vertices labelled by o in $\mathcal{D}(\tilde{g}) \cap b_i \cap S(v,k)$. We have $\tilde{g} \in \tilde{H}^k(v)$ if and only if $\tilde{g}\tilde{h}^{-1} \in \tilde{H}^k(v)$, and $\mathcal{D}(\tilde{g}\tilde{h}^{-1})$ is an e-diagram. The value of $\alpha_k(v)$ and Lemma 1.5.1 then imply that $\tilde{g}\tilde{h}^{-1} \in \tilde{H}^k(v)_e$ if and only if $(q_1,\dots,q_m) = (p_1,\dots,p_m)$ (resp. $[(q_1,\dots,q_m)] = [(p_1,\dots,p_m)]$). □

Fix $t \in \{0,1\}$ such that $c(t) \neq 0$. For $k < K(t)$, if v is a vertex of type $(t+k)$ mod 2 then the fact that $\alpha_k^t = \infty$ implies by Lemma 1.5.8 (i) that $\mathcal{D}(\tilde{H}^k(v))$ exactly contains the diagrams of $\Delta_{v,k}$ whose intersection with $B(v,k-1)$ is a diagram in $\mathcal{D}(\tilde{H}^{k-1}(v))$.

On the other hand, if v is a vertex of type $(t+K(t))$ mod 2 then the shape of $\mathcal{D}(\tilde{H}^{K(t)}(v))$ cannot be directly deduced from $\mathcal{D}(\tilde{H}^{K(t)-1}(v))$. In view of Lemma 1.5.8 (ii),(iii), we can however define a map f_v^t to encode this information. The domain of f_v^t will be $\mathcal{D}(\tilde{H}^{K(t)-1}(v))$ while its codomain J^t will depend on the value of $c(t)$. Given a diagram $\delta \in \mathcal{D}(\tilde{H}^{K(t)-1}(v))$, the value of $f_v^t(\delta)$ will exactly give what is the condition on a diagram of $\Delta_{v,K(t)}$ whose intersection with $B(v, K(t)-1)$ is δ for being contained in $\mathcal{D}(\tilde{H}^{K(t)}(v))$. Let us denote by $b_1, \dots, b_{\tilde{d}}$ the branches of the vertices adjacent to v.

- If $c(t) = 1$, then $\alpha^t_{K(t)} = K(t)$ and we can apply Lemma 1.5.8 (ii) with $k = x = K(t)$. We then set $J^t := \{E, O\}$ and define $f_v^t \colon \mathcal{D}(\tilde{H}^{K(t)-1}(v)) \to J^t$ naturally: $f_v^t(\delta)$ is the unique element $p \in J^t$ given by the lemma (note that $m = 1$).
- If $c(t) = 2$, then $\alpha^t_{K(t)} = K(t) - 1$ and we can apply Lemma 1.5.8 (ii) with $k = K(t)$ and $x = K(t) - 1$. We set $J^t := \{E, O\}^{\tilde{d}}$ and define $f_v^t \colon \mathcal{D}(\tilde{H}^{K(t)-1}(v)) \to J^t$ naturally: $f_v^t(\delta)$ is the unique element $(p_1, \dots, p_{\tilde{d}}) \in J_t$ given by the lemma.
- If $c(t) = 3$, then $\alpha^t_{K(t)} = (K(t)-1)^*$ and we can apply Lemma 1.5.8 (iii) with $k = K(t)$. We set $J^t := \{E, O\}^{\tilde{d}} \big/ \langle (O, \dots, O) \rangle$ and define $f_v^t \colon \mathcal{D}(\tilde{H}^{K(t)-1}(v)) \to J^t$ naturally: $f_v^t(\delta)$ is the unique element $[(p_1, \dots, p_{\tilde{d}})] \in J_t$ given by the lemma.

The next result directly follows from the definition of f_v^t.

Lemma 1.5.9. *Let $\delta_e \in \Delta_{v,K(t)-1}$ be the diagram with all vertices labelled by e. Then δ_e belongs to $\mathcal{D}(\tilde{H}^{K(t)-1}(v))$ and $f_v^t(\delta_e)$ is the trivial element of J^t. Moreover, for $\tilde{g}, \tilde{h} \in \tilde{H}^{K(t)-1}(v)$, we have the following.*

- *If $c(t) = 1$, then $f_v^t(\mathcal{D}(\tilde{g}\tilde{h})) = f_v^t(\mathcal{D}(\tilde{g})) \cdot f_v^t(\mathcal{D}(\tilde{h}))$.*
- *If $c(t) \in \{2, 3\}$, then $f_v^t(\mathcal{D}(\tilde{g}\tilde{h})) = \sigma(f_v^t(\mathcal{D}(\tilde{g}))) \cdot f_v^t(\mathcal{D}(\tilde{h}))$, where $\sigma \colon J^t \to J^t$ permutes the coordinates in the same way as $\tilde{h}$ permutes the branches $b_1, \dots, b_{\tilde{d}}$.*

Proof. For each $k \geq 0$, the diagram in $\Delta_{v,k}$ with all vertices labelled by e is always contained in $\mathcal{D}(\tilde{H}^k(v))$ (because $H \supseteq \mathrm{Alt}_{(i)}(T)^+$). In particular, we have $\delta_e \in \mathcal{D}(\tilde{H}^{K(t)-1}(v))$ and $f_v^t(\delta_e)$ must be the trivial element of J_t.

The formula for $f_v^t(\mathcal{D}(\tilde{g}\tilde{h}))$ can be obtained from Lemma 1.5.1. $\square$

1.5.5 The invariants form a complete system

By definition, the map f_v^t defined above fully describes the shape of $\mathcal{D}(\tilde{H}^{K(t)}(v))$ from $\mathcal{D}(\tilde{H}^{K(t)-1}(v))$. A priori, it seems that we also need similar maps to deduce the shape of $\mathcal{D}(\tilde{H}^k(v))$ from $\mathcal{D}(\tilde{H}^{k-1}(v))$ for each $k > K(t)$ (where v is of type $(t+k) \bmod 2$). However, as the following lemma shows, this is not the case.

Lemma 1.5.10. *Let $t \in \{0,1\}$ be such that $c(t) \neq 0$, let $k > K(t)$ and let v be a vertex of type $(t+k) \bmod 2$. Let also $\delta \in \mathcal{D}(\tilde{H}^{k-1}(v))$. Consider $\hat{\delta} \in \Delta_{v,k}$ with $\hat{\delta} \cap B(v, k-1) = \delta$. Then $\hat{\delta}$ belongs to $\mathcal{D}(\tilde{H}^k(v))$ if and only if, for each vertex w at distance $k - K(t)$ from v, the diagram $\hat{\delta} \cap B(w, K(t))$ belongs to $\mathcal{D}(\tilde{H}^{K(t)}(w))$.*

Proof. If $\hat{\delta} \in \mathcal{D}(\tilde{H}^k(v))$ and w is a vertex at distance $k - K(t)$ from v, then by Lemma 1.5.2 the diagram $\hat{\delta}$ is realized by an element $\tilde{h}$ of $\tilde{H}^k(v)$ fixing w. Hence, the diagram of $\tilde{h}|_{B(w,K(t)+1)}$, which is $\hat{\delta} \cap B(w, K(t))$, is contained in $\mathcal{D}(\tilde{H}^{K(t)}(w))$.

Now take $\hat{\delta} \in \Delta_{v,k}$ with $\hat{\delta} \cap B(v, k-1) = \delta$ such that $\hat{\delta} \cap B(w, K(t)) \in \mathcal{D}(\tilde{H}^{K(t)}(w))$ for each vertex w at distance $k - K(t)$ from v. Consider also $\hat{\delta}' \in \mathcal{D}(\tilde{H}^k(v))$ with $\hat{\delta}' \cap B(v, k-1) = \delta$. In view of the first part of the proof, we have $\hat{\delta}' \cap B(w, K(t)) \in \mathcal{D}(\tilde{H}^{K(t)}(w))$ for each w at distance $k - K(t)$ from v. Denote by $b_1, \dots, b_m$ the $K'(t)$-branches of $B(v, k)$, and let p_i (resp. p_i') be the parity of the number of vertices labelled by o in $\hat{\delta} \cap b_i \cap S(v, k)$ (resp. in $\hat{\delta}' \cap b_i \cap S(v, k)$). In view of Lemma 1.5.8 (ii), it suffices to show that $(p_1, \dots, p_m) = (p_1', \dots, p_m')$ in order to prove that $\hat{\delta} \in \mathcal{D}(\tilde{H}^k(v))$. Let $j \in \{1, \dots, m\}$ and let w be the vertex at distance $k - K(t)$ from v whose branch b contains b_j. The diagrams $\delta_0 = \hat{\delta} \cap B(w, K(t))$ and $\delta_0' = \hat{\delta}' \cap B(w, K(t))$, which both belong to $\mathcal{D}(\tilde{H}^{K(t)}(v))$, coincide on $B(w, K(t)-1) \cup (S(w, K(t)) \setminus b)$. In particular, we have $\delta_0 \cap B(w, K(t)-1) = \delta_0' \cap B(w, K(t)-1)$ and the two diagrams must therefore satisfy the condition given by $f_w^t(\delta_0 \cap B(w, K(t)-1)) \in J^t$ $(*)$. Given a part X of a diagram, let us write $P(X)$ for the parity of the number of vertices labelled by o in X.

If $c(t) = 1$ then $b = b_j$ and $(*)$ means that $P(\delta_0 \cap S(w, K(t))) = P(\delta_0' \cap S(w, K(t)))$. Since δ_0 and δ_0' coincide on $S(w, K(t)) \setminus b$, this means that $P(\delta_0 \cap S(w, K(t)) \cap b) = P(\delta_0' \cap S(w, K(t)) \cap b)$, i.e. we have $p_j = p_j'$. If $c(t) \in \{2, 3\}$ then let $\tilde{b}_1, \dots, \tilde{b}_{\tilde{d}}$ be the branches (seen in $B(w, K(t))$) of the vertices adjacent to w. One of these branches is equal to b_j, say $\tilde{b}_1$, and another of these branches is the branch of the parent of w (in $B(v, k)$), say $\tilde{b}_2$. If $c(t) = 2$ then $(*)$ means that $P(\delta_0 \cap \tilde{b}_i \cap S(w, K(t))) = P(\delta_0' \cap \tilde{b}_i \cap S(w, K(t)))$ for each $i \in \{1, \dots, \tilde{d}\}$, and $i = 1$ directly gives $p_j = p_j'$. If $c(t) = 3$, then $(*)$ means that either $P(\delta_0 \cap \tilde{b}_i \cap S(w, K(t))) = P(\delta_0' \cap \tilde{b}_i \cap S(w, K(t)))$ for each i or $P(\delta_0 \cap \tilde{b}_i \cap S(w, K(t))) \neq P(\delta_0' \cap \tilde{b}_i \cap S(w, K(t)))$ for each i. But δ_0 and δ_0' coincide on $S(w, K(t)) \setminus b = S(w, K(t)) \cap \tilde{b}_2$, so we must have the equality for each i. In particular, $i = 1$ gives $p_j = p_j'$. □

As a consequence of the previous lemma, we find that the invariants $c(t)$, $K(t)$ and f_v^t (for $t \in \{0, 1\}$ and $v \in V(T)$ such that f_v^t is defined) fully describe the entire group H. Note that, since $\mathrm{Alt}_{(i)}(T)^+$ is transitive on $V_0(T)$ and $V_1(T)$, if $c(t) \neq 0$ then knowing f_v^t for a fixed vertex v of type $(t + K(t)) \bmod 2$ suffices to get each f_w^t.

Theorem 1.H′. *If $H, H' \in \mathcal{H}_T^+$ satisfy $H, H' \supseteq \mathrm{Alt}_{(i)}(T)^+$ and have the same invariants $c(t)$, $K(t)$ and f_v^t (for $t \in \{0, 1\}$ and $v \in V(T)$ such that f_v^t is defined), then $H = H'$.*

Proof. We fix one group $H \in \mathcal{H}_T^+$ with $H \supseteq \mathrm{Alt}_{(i)}(T)^+$ and show that, for each $v \in V(T)$ and $k \in \mathbf{Z}_{\geq 0}$, the set $\mathcal{D}(\tilde{H}^k(v))$ can be described only using the invariants $c(t)$, $K(t)$ and f_v^t. By Lemma 1.5.2 and the fact that H is generated by point stabilizers, this will prove the statement.

Let us do it by induction on k. For $k = -1$, $\mathcal{D}(\tilde{H}^{-1}(v))$ only contains the empty diagram ε for each $v \in V(T)$. Now fix $k \geq 0$ and assume that $\mathcal{D}(\tilde{H}^{k-1}(v))$ is known for each $v \in V(T)$. Let $v \in V(T)$ and define $t \in \{0, 1\}$ to be such that v is of type $(t + k) \bmod 2$. If $k < K(t)$, then $\alpha_k^t = \infty$ and we know that $\mathcal{D}(\tilde{H}^k(v))$ exactly contains the diagrams of $\Delta_{v,k}$ whose intersection with $B(v, k-1)$ is contained in $\mathcal{D}(\tilde{H}^{k-1}(v))$ (see Lemma 1.5.8 (i)). If $k = K(t)$, then f_v^t fully describes $\mathcal{D}(\tilde{H}^{K(t)}(v))$ from $\mathcal{D}(\tilde{H}^{K(t)-1}(v))$. Finally, if $k > K(t)$, then Lemma 1.5.10 shows that $\mathcal{D}(\tilde{H}^k(v))$ can be deduced from $\mathcal{D}(\tilde{H}^{k-1}(v))$ and each $\mathcal{D}(\tilde{H}^{K(t)}(w))$ with w at distance $k - K(t)$ from v. □

Theorem 1.H formulated in §1.1 is a consequence of Theorem 1.H′. Indeed, for $H \in \mathcal{H}_T^+$ satisfying $H \supseteq \mathrm{Alt}_{(i)}(T)^+$, take $K \in \mathbf{Z}_{>0}$ strictly greater than the finite numbers among $\{K(0), K(1)\}$. Then the K-closure $H^{(K)}$ of H also satisfies $H^{(K)} \in \mathcal{H}_T^+$ and $H^{(K)} \supseteq \mathrm{Alt}_{(i)}(T)^+$ and have the same invariants as H, so that $H = H^{(K)}$.

1.5.6 Possible shapes for f_v^t when $K(t) \leq K(1-t)$

We now observe which shapes f_v^t can take when $c(t) \neq 0$ and $K(t) \leq K(1-t)$. Recall that $S_X(v) := \bigcup_{r \in X} S(v,r)$ when $X \subseteq \mathbf{Z}_{\geq 0}$.

Lemma 1.5.11. *Suppose that $c(t) \neq 0$ and $K(t) \leq K(1-t)$ and let v be a vertex of type $(t + K(t)) \bmod 2$. Then the possible shapes for f_v^t are given as follows. Here, $b_1, \dots, b_{\tilde{d}}$ denote the branches of the vertices adjacent to v, as in the definition of J^t.*

- *If $c(t) = 1$, then there exists $A \subseteq \{0, 1, \dots, K(t) - 1\}$ such that $f_v^t(\delta)$ is equal to the parity of the number of vertices labelled by o in $\delta \cap S_A(v)$.*
- *If $c(t) = 2$, then there exist $A \subseteq \{1, 2, \dots, K(t) - 1\}$ and $B \subseteq \{0, 1, \dots, K(t) - 1\}$ such that $f_v^t(\delta) = (p_1, \dots, p_{\tilde{d}})$ where p_i is the parity of the number of vertices labelled by o in $\delta \cap ((S_A(v) \cap b_i) \cup (S_B(v) \setminus b_i))$.*
- *If $c(t) = 3$, then there exists $A \subseteq \{1, 2, \dots, K(t) - 1\}$ such that $f_v^t(\delta) = [(p_1, \dots, p_{\tilde{d}})]$ where p_i is the parity of the number of vertices labelled by o in $\delta \cap (S_A(v) \cap b_i)$.*

Proof. Since $K(t) \leq K(1-t)$, we have $\alpha_k^t = \alpha_k^{1-t} = \infty$ for each $k < K(t)$ and hence $\mathcal{D}(\tilde{H}^{K(t)-1}(v)) = \Delta_{v,K(t)-1}$. Now, for each $r \in \{0, 1, \dots, K(t) - 1\}$, fix a diagram $\delta_r \in \mathcal{D}(\tilde{H}^{K(t)-1}(v))$ having exactly one vertex w labelled by o, with $w \in S(v,r)$ and, if $r \geq 1$, $w \in b_1$. In view of Lemma 1.5.9, it is clear that the image of any diagram by f_v^t can always be recovered from the values that f_v^t takes on $\{\delta_0, \delta_1, \dots, \delta_{K(t)-1}\}$.

- If $c(t) = 1$, then define $A = \{r \in \{0, 1, \dots, K(t) - 1\} \mid f_v^t(\delta_r) = O\}$. Then f_v^t is exactly of the shape given in the statement.

- If $c(t) = 2$, then for each $r \in \{0, \dots, K(t) - 1\}$ we have $f_v^t(\delta_r) = (p_1^r, \dots, p_{\tilde{d}}^r)$ for some $p_i^r \in \{E, O\}$. For $r \geq 1$, considering elements $\tilde{g} \in \mathrm{Alt}_{(i)}(B(v, K(t)))$ stabilizing the branch b_1 but permuting the other branches, we directly obtain using Lemma 1.5.9 that $p_2^r = p_3^r = \cdots = p_{\tilde{d}}^r$. We can therefore write $f_v^t(\delta_r) = (x_r, y_r, \dots, y_r)$ with $x_r, y_r \in \{E, O\}$. For $r = 0$, we obtain in the same way that $p_1^0 = p_2^0 = \cdots = p_{\tilde{d}}^0$, and we write $f_v^t(\delta_0) = (y_0, \dots, y_0)$. Now if we define $A = \{r \in \{1, 2, \dots, K(t) - 1\} \mid x_r = O\}$ and $B = \{r \in \{0, 1, \dots, K(t) - 1\} \mid y_r = O\}$, then we exactly get the shape given in the statement.

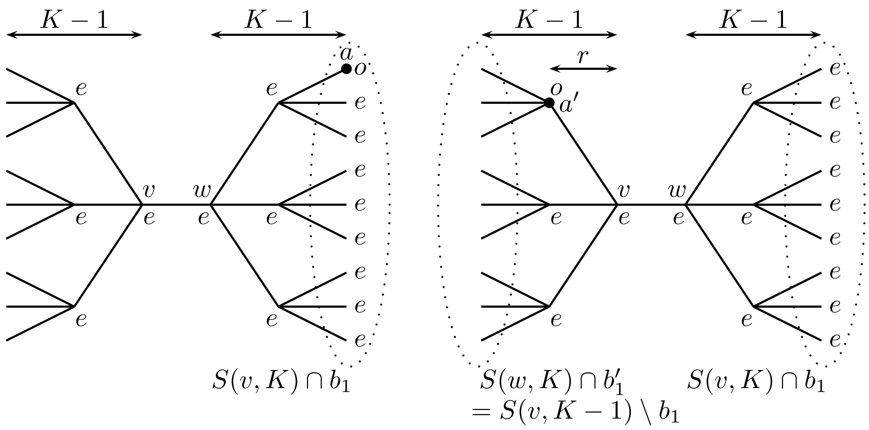
- If $c(t) = 3$, then the same reasoning as in the previous case works but it must be remembered that the values are taken modulo $(O, \dots, O)$. We thus get $f_v^t(\delta_r) = [(x_r, y_r, \dots, y_r)]$ for $r \geq 1$ and $f_v^t(\delta_0) = [(y_0, \dots, y_0)]$, but it can be assumed that all the y_r are equal to E. Defining $A = \{r \in \{1, 2, \dots, K(t) - 1\} \mid x_r = O\}$, we obtain the shape given in the statement. $\square$

When $c(t) = 2$, we also have $c(1 - t) = 2$ and $K(0) = K(1)$ by Lemma 1.5.7. In this case, Lemma 1.5.11 can be applied with $t = 0$ and $t = 1$. It is however important to note the following result. Remark that, as $\mathrm{Alt}_{(i)}(T)^+$ is transitive on $V_0(T)$ and $V_1(T)$, the sets A and B given by Lemma 1.5.11 depend on $t \in \{0, 1\}$ but not on v.

Lemma 1.5.12. *Suppose that $c(0) = c(1) = 2$ and $K(0) = K(1) =: K$. For each $t \in \{0, 1\}$, let A_t and B_t be the sets given by Lemma 1.5.11. For each $t \in \{0, 1\}$, we have $K - 1 \in B_t$ and, if $r \in \{0, \dots, K - 2\}$, then $r \in B_t$ if and only if $r + 1 \in A_{1-t}$.*

Proof. Let $t \in \{0, 1\}$ and let v and w be adjacent vertices with v of type $(t + K) \bmod 2$.

We first assume for a contradiction that $K - 1 \notin B_t$. Let a be a vertex of $S(w, K - 1)$ that is not in the branch of v (see Figure 1.6a). Since $\alpha_{K-1}^t = \infty$, there exists $h \in H$ such that a is the only vertex labelled by o in the diagram of the image of h in $\tilde{H}^{K-1}(w)$. Now if we look at the image $\tilde{h}$ of h in $\tilde{H}^{K-1}(v)$, Lemma 1.5.11 and the fact that $K - 1 \notin B_t$ imply that $f_v^t(\mathcal{D}(\tilde{h})) = (E, *, \dots, *)$, where the first branch



(a) First part. (b) Second part.

Figure 1.6: Illustration of Lemma 1.5.12.

b_1 is the branch of w. This means that the number of vertices labelled by o in $S(v,K) \cap b_1$ should be even, but this is a contradiction with the fact that a is the only vertex of $S(v,K) \cap b_1$ labelled by o.

We now show the second part of the statement. Let $r \in \{0, \dots, K-2\}$ and let a' be a vertex of $S(w, r+1)$ in the branch of v, which we denote by b_1' (see Figure 1.6b). Since $K(0) = K(1) = K$, there exists $h \in H$ such that a' is the only vertex labelled by o in the diagram of the image of h in $\tilde{H}^{K-1}(w)$. By Lemma 1.5.11 (with $1-t$ instead of t), the number of vertices labelled by o in $S(w,K) \cap b_1'$ is odd if and only if $r+1 \in A_{1-t}$. Now we observe the diagram of the image $\tilde{h}$ of h in $\tilde{H}^{K-1}(v)$. Lemma 1.5.11 tells us that $f_v^t(\mathcal{D}(\tilde{h})) = (p_1, *, \dots, *)$ where p_1 is the parity of the number of vertices labelled by o in $(S_{A_t}(v) \cap b_1) \cup (S_{B_t}(v) \setminus b_1)$. But all the vertices of $S(v,K) \cap b_1$ are labelled by e, so $p_1 = E$. Hence, there is an even number of vertices labelled by o in $(S_{A_t}(v) \cap b_1) \cup (S_{B_t}(v) \setminus b_1)$. As $K-1 \in B_t$ and a' is the only vertex of $B(v, K-2)$ labelled by o, this means that the number of vertices labelled by o in $S(v, K-1) \setminus b_1$ is odd if and only if $r \in B_t$. Since $S(w,K) \cap b_1' = S(v,K-1) \setminus b_1$, we obtained that $r+1 \in A_{1-t}$ if and only if $r \in B_t$. $\square$

1.5.7 Possible shapes for f_v^t when $K(t) > K(1-t)$

In the case where $c(0), c(1) \in \{1, 3\}$, it can happen that $K(t) > K(1-t)$ and Lemma 1.5.11 cannot be applied. Indeed, $\mathcal{D}(\tilde{H}^{K(t)-1}(v))$ does not contain all the diagrams, which prevents us from using the diagrams δ_r as above. To deal with this case, we therefore need to better understand $\mathcal{D}(\tilde{H}^{K(t)-1}(v))$. This is the subject of the following result, which is illustrated in Figure 1.7.

Lemma 1.5.13. *Suppose that $c(t) \neq 0$ and $K(t) > K(1-t)$ and let v be a vertex of type $(t + K(t)) \bmod 2$. Denote by $b_1, \ldots, b_{\tilde{d}}$ the branches of the vertices adjacent to v. For each $j \in \{1, \ldots, \tilde{d} - 1\}$, fix $\tilde{\gamma}_j \in \mathrm{Alt}_{(i)}(B(v, K(t)))$ sending b_j to $b_{\tilde{d}}$.*

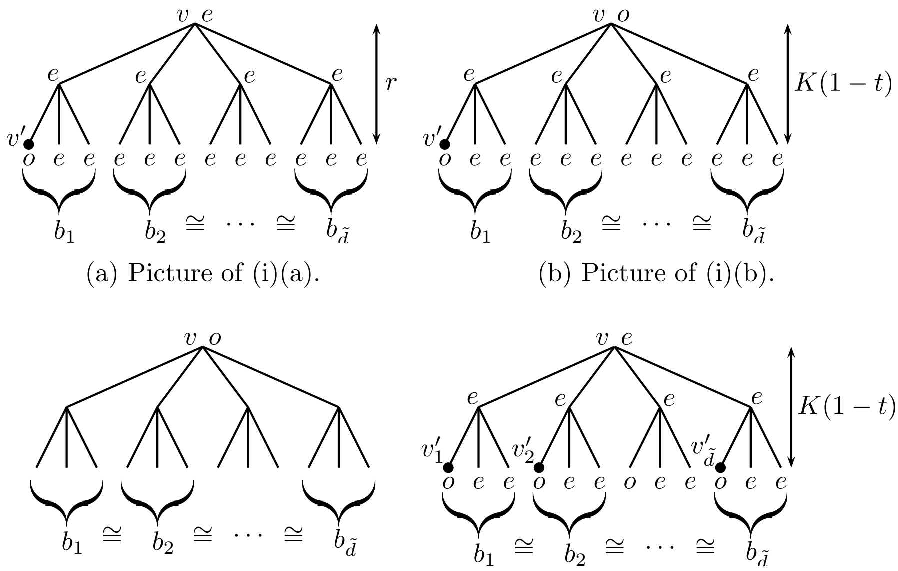
(i) Let r be an element of $\{0, \ldots, K(t) - 1\}$ such that $r < K(1-t)$ or $r \equiv K(t) \bmod 2$.

(a) If $r \geq 1$, then there exist a diagram $\delta \in \mathcal{D}(\tilde{H}^{K(t)-1}(v))$ and a vertex $v' \in S(v, r) \cap b_1$ such that v' is the only vertex labelled by o in $\delta \cap B(v, r)$ and, for each $j \in \{2, \ldots, \tilde{d} - 1\}$ and each $w \in b_j$, $\tilde{\gamma}_j(w)$ has the same label as w.

(b) If $r = 0$, $c(1-t) = 1$, $K(t) \not\equiv K(1-t) \bmod 2$ and the set A associated to $1-t$ in Lemma 1.5.11 contains 0, then there exist a diagram $\delta \in \mathcal{D}(\tilde{H}^{K(t)-1}(v))$ and a vertex $v' \in S(v, K(1-t)) \cap b_1$ such that the only vertices labelled by o in $\delta \cap B(v, K(1-t))$ are v and v' and, for each $j \in \{2, \ldots, \tilde{d} - 1\}$ and each $w \in b_j$, $\tilde{\gamma}_j(w)$ has the same label as w.

(c) If $r = 0$ and we are not in (b), then there exists a diagram $\delta \in \mathcal{D}(\tilde{H}^{K(t)-1}(v))$ in which v is labelled by o and such that, for each $j \in \{1, \ldots, \tilde{d} - 1\}$ and each $w \in b_j$, $\tilde{\gamma}_j(w)$ has the same label as w.

(ii) Suppose that $c(1-t) = 3$ and $K(t) \not\equiv K(1-t) \bmod 2$. Then there exist a diagram $\sigma \in \mathcal{D}(\tilde{H}^{K(t)-1}(v))$ and a vertex $v'_j \in S(v, K(1-t)) \cap b_j$ for each $j \in \{1, \ldots, \tilde{d}\}$ such that the only vertices labelled by o in $\sigma \cap B(v, K(1-t))$ are $v'_1, \ldots, v'_{\tilde{d}}$ and, for each $j \in \{1, \ldots, \tilde{d} - 1\}$ and each $w \in b_j$, $\tilde{\gamma}_j(w)$ has the same label as w.



(a) Picture of (i)(a). (b) Picture of (i)(b).

(c) Picture of (i)(c). (d) Picture of (ii).

Figure 1.7: Illustration of Lemma 1.5.13.

Proof. We prove all three cases of (i) simultaneously, the proof of (ii) being similar. Let us say that a diagram $\delta \in \Delta_{v,i}$ (with $0 \leq i \leq K(t) - 1$) is *suitable* if $\delta \in \mathcal{D}(\tilde{H}^i(v))$ and if δ satisfies the conditions of the statement that concern the ball $B(v,i)$.

First remark that there exists a suitable diagram $\delta \in \Delta_{v,r}$. Indeed, it suffices to label all the vertices of $\delta \cap B(v, r-1)$ by e and then to label exactly one vertex of $S(v,r)$ by o, placed in b_1 if $r \geq 1$. This always gives a diagram in $\mathcal{D}(\tilde{H}^r(v))$ because the assumption on r is made so that $\alpha_r(v) = \infty$.

We now prove that, for each $r+1 \leq i \leq K(t) - 1$, if $\delta \in \Delta_{v,i-1}$ is suitable then there exists $\hat{\delta} \in \Delta_{v,i}$ suitable and extending δ. We obviously start by defining $\hat{\delta} \cap B(v, i-1) = \delta$, and there remains to give the labels of the vertices in $S(v,i)$. If $i < K(1-t)$ or $i \equiv K(t) \bmod 2$, we have $\alpha_i(v) = \infty$ and by Lemma 1.5.8 (i) we can simply label all the vertices of $S(v,i)$ by e. Now if $i \geq K(1-t)$ and $i \not\equiv K(t) \bmod 2$, then v is of type $(1 - t + i) \bmod 2$ and we know that a diagram $\hat{\delta}$ with $\hat{\delta} \cap B(v, i-1) = \delta$ is contained in $\mathcal{D}(\tilde{H}^i(v))$ if and only if $\hat{\delta} \cap B(w, K(1-t)) \in \mathcal{D}(\tilde{H}^{K(1-t)}(w))$ for each w at distance $i - K(1-t)$ from v (see

Lemma 1.5.10 with $1-t$). In other words, the only restrictions for being in $\mathcal{D}(\tilde{H}^i(v))$ are given by the maps f_w^{1-t}. These are always restrictions on the parity of the number of vertices labelled by o in the $K'(1-t)$-branches. When $K'(1-t) < i$, these branches are smaller than the whole ball $B(v,i)$. In this case, since δ is suitable, a labelling of the vertices of $S(v,i) \cap b_{\tilde{d}}$ satisfying the restrictions that concern them can be pulled back by $\tilde{\gamma}_j$ in a labelling of the vertices of $S(v,i) \cap b_j$ also satisfying the restrictions (for each $j \in \{2,\dots,\tilde{d}-1\}$ in cases (a) and (b) and for each $j \in \{1,\dots,\tilde{d}-1\}$ in case (c)). The only case where $K'(1-t) = i$ is the case where $i = K(1-t)$, $c(1-t) = 1$, and $K(1-t) \not\equiv K(t) \bmod 2$. If $r \geq 1$, we want to prove (a) and there is no problem: we can label all the vertices of $S(v,i) \setminus b_1$ with e and adapt the labelling of $S(v,i) \cap b_1$. If $r = 0$, and if the set A associated to $1-t$ in Lemma 1.5.11 does not contain 0, then since all the vertices of $B(v,i-1) \setminus \{v\}$ are labelled by e all the vertices of $S(v,i)$ can be labelled by e. If A contains 0, then the restriction given by f_v^{1-t} imposes the number of vertices of $S(v,i)$ labelled by o to be odd. As we want to prove (b), we label one vertex of $S(v,i) \cap b_1$ by o and the other vertices of $S(v,i)$ by e. In any cases, the diagram $\hat{\delta} \in \Delta_{v,i}$ constructed in this way is suitable. ☐

Thanks to the previous lemma, we can now look at the shapes that f_v^t can take when $K(t) > K(1-t)$. It is actually sufficient for our classification to count how many shapes are possible.

Lemma 1.5.14. *Let $t \in \{0,1\}$. Fix $c(0), c(1) \in \{1,3\}$ and $K(t) > K(1-t)$ and let v be a vertex of type $(t+K(t)) \bmod 2$. Let N be the number of maps f_v^t that can be observed for at least one $H \in \mathcal{H}_T^+$ satisfying $H \supseteq \mathrm{Alt}_{(i)}(T)^+$ and with these invariants $c(0)$, $c(1)$, $K(0)$ and $K(1)$. Define*

$$R = \{r \in \{0,1,\dots,K(t)-1\} \mid r < K(1-t) \text{ or } r \equiv K(t) \bmod 2\}.$$

- *If $c(t) = 1$, then $N \leq 2^{|R|+\varepsilon}$ where $\varepsilon = 1$ if $c(1-t) = 3$ and $K(t) \not\equiv K(1-t) \bmod 2$, and $\varepsilon = 0$ otherwise.*
- *If $c(t) = 3$, then $N \leq 2^{|R\setminus\{0\}|+\varepsilon'}$ where $\varepsilon' = 1$ if $c(1-t) = 1$, $K(1-t) \neq 0$ and $K(t) \not\equiv K(1-t) \bmod 2$, and $\varepsilon' = 0$ otherwise.*

Proof. First suppose that $c(t) = 1$. In order find an upper bound on N, we start by giving a set of diagrams of $\mathcal{D}(\tilde{H}^{K(t)-1}(v))$ generating (via Lemma 1.5.9) all the diagrams of $\mathcal{D}(\tilde{H}^{K(t)-1}(v))$.

For $r \in R$, take $\delta_r \in \mathcal{D}(\tilde{H}^{K(t)-1}(v))$ as in Lemma 1.5.13 (i). In this case ($c(t) = 1$), we actually do not care of the condition with the $\tilde{\gamma}_j$.

For $r \in \{0, 1, \ldots, K(t) - 1\} \setminus R$, an element δ_r with all vertices of $B(v, r-1)$ labelled by e and exactly one vertex labelled by o in $S(v, r)$ does not exist. Instead, and if $K'(1-t) > 0$, we consider an element $\rho_r \in \mathcal{D}(\tilde{H}^{K(t)-1}(v))$ with all vertices of $B(v, r-1)$ labelled by e and exactly two vertices labelled by o in $S(v, r)$, placed such that the minimal branch containing them is a $K'(1-t)$-branch. This diagram can be used to generate, via Lemma 1.5.9, all the possible labellings of $S(v, r)$ with an even number of vertices labelled by o in each $K'(1-t)$-branch.

In the particular case where $c(1-t) = 3$ and $r = K(1-t)$, as $\alpha_r(v) = (K(1-t)-1)^*$ we also need to add a diagram $\sigma \in \mathcal{D}(\tilde{H}^{K(t)-1}(v))$ as in Lemma 1.5.13 (ii). Note that this element σ is considered if and only if $c(1-t) = 3$ and $K(t) \not\equiv K(1-t) \bmod 2$ (so that $K(1-t) \notin R$). We write $\varepsilon = 1$ in this case and $\varepsilon = 0$ otherwise.

By construction, $\mathcal{D}(\tilde{H}^{K(t)-1}(v))$ can be generated using δ_r, ρ_r and σ (if $\varepsilon = 1$). It is however not hard to convince oneself that the diagrams ρ_r can be chosen so that $f_v^t(\rho_r)$ must always be equal to E. Indeed, take ρ_r as above with vertices $a, b \in S(v, r)$ labelled by o and let $\tilde{h}$ be an element realizing this diagram. Let $\tilde{\tau} \in \mathrm{Alt}_{(i)}(B(v, K(t)))$ be an element that stabilizes a while sending the $(K(1-t)-1)$-branch containing b to another branch. In this way, $\tilde{h}\tilde{\tau}\tilde{h}^{-1}$ has a diagram ρ'_r satisfying the same property as ρ_r but it is now sure by Lemma 1.5.9 that $f_v^t(\rho'_r) = E$. Hence, a map f_v^t is fully characterized by its values $f_v^t(\delta_r)$ (for each $r \in R$) and $f_v^t(\sigma)$ (if $\varepsilon = 1$), which leaves at most $2^{|R|+\varepsilon}$ options for f_v^t.

For $c(t) = 3$, the idea is exactly the same. The only difference is that f_v^t takes its values in $\{E, O\}^{\tilde{d}} \big/ \langle (O, \ldots, O) \rangle$. The diagrams δ_r, ρ_r and σ with the same properties as above once again generate all the diagrams. Denote by $b_1, \ldots, b_{\tilde{d}}$ the branches of the vertices adjacent to v, as for the definition of J^t.

This time, we fix $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{\tilde{d}-1}$ as in Lemma 1.5.13 and really want δ_r

to satisfy the conditions given in this same lemma. In this way, for $r > 0$ we obtain using Lemma 1.5.9 that $f_v^t(\delta_r)$ is of the form $[(x_r, y_r, \dots, y_r)]$, and we can assume that $y_r = E$. For $r = 0$, if $c(1-t) = 1$, $K(t) \not\equiv K(1-t) \bmod 2$ and if the set A associated to $1-t$ in Lemma 1.5.11 contains 0, then we define $\varepsilon = 1$ and also get $f_v^t(\delta_0) = [(x_0, y_0, \dots, y_0)]$ (and can assume that $y_0 = E$). Otherwise, we set $\varepsilon = 0$ and find that $f_v^t(\delta_0) = [(E, \dots, E)]$.

For ρ_r, as in the first case they can generally be chosen so that $f_v^t(\rho_r)$ must always be equal to $[(E, \dots, E)]$. Indeed, if there exist $\tilde{h}$ and $\tilde{\tau}$ as above and stabilizing the branches $b_1, \dots, b_{\tilde{d}}$ then the same reasoning works. This is always possible, unless $c(1-t) = 1$ and $r = K(1-t) \neq 0$. This only happens when $c(1-t) = 1$, $K(t) \not\equiv K(1-t) \bmod 2$ and $K(1-t) \neq 0$, in which case we set $\varepsilon' = 1$. Otherwise, set $\varepsilon' = 0$. If $\varepsilon' = 1$, then the diagram $\rho_{K(1-t)}$ has two vertices in $S(v, K(1-t))$ labelled by o and they are in different branches, say b_1 and b_2. Let $\tilde{h}$ be an element realizing this diagram and let $\tilde{\tau} \in \mathrm{Alt}_{(i)}(B(v, K(t)))$ be an element interchanging b_2 and b_3, interchanging b_4 and b_5, and fixing $b_1, b_6, \dots, b_{\tilde{d}}$. In this way, $\tilde{h}' = \tilde{h}\tilde{\tau}\tilde{h}^{-1}$ has a diagram $\rho'_{K(1-t)}$ with two vertices in $S(v, K(1-t))$ labelled by o: one in b_2 and one in b_3. Moreover, we know that $f_v^t(\rho'_{K(1-t)}) = [(E, x, x, y, y, E \dots, E)]$. Now let $\tilde{\tau}' \in \mathrm{Alt}_{(i)}(B(v, K(t)))$ be an element interchanging b_1 and b_3, interchanging b_4 and b_5, and fixing $b_2, b_6, \dots, b_{\tilde{d}}$. Then $\tilde{h}'' = \tilde{h}'\tilde{\tau}'\tilde{h}'^{-1}$ has a diagram $\rho''_{K(1-t)}$ with two vertices in $S(v, K(1-t))$ labelled by o: one in b_1 and one in b_2. This time, we know that $f_v^t(\rho''_{K(1-t)}) = [(x, x, E, \dots, E)]$.

Concerning σ (if it must be considered), take it as in Lemma 1.5.13 (ii) so that all its branches $b_1, \dots, b_{\tilde{d}}$ are identical (via $\tilde{\gamma}_1, \dots, \tilde{\gamma}_{\tilde{d}-1}$). We obtain $f_v^t(\sigma) = [(E, \dots, E)]$.

In total, there are at most $2^{|R\setminus\{0\}|+\varepsilon+\varepsilon'}$ possibilities for f_v^t. However, having $\varepsilon = 1$ also implies $\varepsilon' = 1$ and the diagram δ_0 that we chose when $\varepsilon = 1$ can be used to generate an element with the properties of $\rho''_{K(1-t)}$. We can therefore forget $\rho''_{K(1-t)}$ when $\varepsilon = 1$, which leaves at most $2^{|R\setminus\{0\}|+\varepsilon'}$ possibilities for f_v^t. $\square$

1.5.8 Computing upper bounds

For each fixed values of $c(0)$, $c(1)$, $K'(0)$, $K'(1)$, we can now compute an upper bound on the number of groups $H \in \mathcal{H}_T^+$ satisfying $H \supseteq \mathrm{Alt}_{(i)}(T)^+$ and with these invariants. Recall that $a \boxplus b := a + b - \left\lceil \frac{|a-b|}{2} \right\rceil$ for $a, b \in \mathbf{Z}_{\geq 0}$.

Proposition 1.5.15. *Fix $c(0), c(1) \in \{0,1,2,3\}$ such that $c(0) = 2$ if and only if $c(1) = 2$ and fix $K'(0), K'(1) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$ such that $K'(t) = \infty$ if and only if $c(t) = 0$, and $K'(0) = K'(1)$ if $c(0) = c(1) = 2$. Let N be the number of groups $H \in \mathcal{H}_T^+$ satisfying $H \supseteq \mathrm{Alt}_{(i)}(T)^+$ and with these invariants $c(0)$, $c(1)$, $K'(0)$ and $K'(1)$.*

(i) If $c(0) = c(1) = 0$ then $N \leq 1$.

(ii) If $c(t) \in \{1,3\}$ and $c(1-t) = 0$ then $N \leq 2^{K'(t)}$.

(iii) If $c(t) = 1$, $c(1-t) = 3$ and $K'(t) > K'(1-t)$, then $N \leq 2 \cdot 2^{K'(0) \boxplus K'(1)}$.

(iv) If $c(0) \neq 0$, $c(1) \neq 0$ and we are not in (iii), then $N \leq 2^{K'(0) \boxplus K'(1)}$.

Proof. In view of Theorem 1.H′, we simply need to give in each case an upper bound on the number of ordered pairs $(f^0_{v_0}, f^1_{v_1})$ that can be observed, where v_0 and v_1 are fixed (when $c(t) = 0$, we say that there is only one possibility for $f^t_{v_t}$ (which was not defined)). Recall that the values of $c(t)$ and $K'(t)$ completely determine the value of $K(t)$.

- If $c(0) = c(1) = 0$ then we trivially have $N \leq 1$.
- If $c(t) = 1$ and $c(1-t) = 0$, then we get by Lemma 1.5.11 that $N \leq 2^{K(t)} = 2^{K'(t)}$, because $2^{K(t)}$ is the number of subsets of $\{0, \ldots, K(t) - 1\}$.
- If $c(t) = 3$ and $c(1-t) = 0$, then we get by Lemma 1.5.11 that $N \leq 2^{K(t)-1} = 2^{K'(t)}$, because $2^{K(t)-1}$ is the number of subsets of $\{1, \ldots, K(t) - 1\}$.
- If $c(0) \neq 0$ and $c(1) \neq 0$ then we must distinguish some cases:

- If $c(0) = c(1) = 2$ then by Lemma 1.5.12 the shape of $f^1_{v_1}$ is fully determined by the shape of $f^0_{v_0}$. Let A and B be the sets given by Lemma 1.5.11 for $t = 0$. There are $2^{K'(0)}$ possibilities for A and $2^{K'(1)}$ possibilities for B (since $K(1)-1$ must always be contained in B by Lemma 1.5.12). Hence, $N \leq 2^{K'(0)+K'(1)} = 2^{K'(0)\boxplus K'(1)}$ (recall that $K'(0) = K'(1)$ when $c(0) = c(1) = 2$).
- If $c(0), c(1) \in \{1, 3\}$ and $K(0) = K(1)$, then Lemma 1.5.11 can be applied twice to get $N \leq 2^{K'(0)+K'(1)}$. If $c(0) = c(1)$ then $K'(0) = K'(1)$ so $2^{K'(0)+K'(1)} = 2^{K'(0)\boxplus K'(1)}$. If $c(0) \neq c(1)$ then $|K'(0) - K'(1)| = 1$ and $2^{K'(0)+K'(1)} = 2^{(K'(0)\boxplus K'(1))+1}$.
- If $c(0), c(1) \in \{1, 3\}$ and $K(t) > K(1-t)$ (for some $t \in \{0, 1\}$), then by Lemma 1.5.11 there are at most $2^{K'(1-t)}$ possibilities for $f^{1-t}_{v_{1-t}}$. The number of possibilities for $f^t_{v_t}$ is given by Lemma 1.5.14. Remark that $|R| = K(t) - \left\lceil \frac{K(t)-K(1-t)}{2} \right\rceil$ (where R is defined as in Lemma 1.5.14) and that 0 does not belong to R if and only if $K(1-t) = 0$ and $K(t)$ is odd.
  * If $c(t) = c(1-t) = 1$, then there are at most $2^{|R|}$ possibilities for $f^t_{v_t}$ and we directly get $N \leq 2^{K(0)\boxplus K(1)} = 2^{K'(0)\boxplus K'(1)}$.
  * If $c(t) = c(1-t) = 3$, then $K(1-t) \neq 0$ and 0 is never contained in R, so there are at most $2^{|R|-1}$ possibilities for $f^t_{v_t}$ and $N \leq 2^{K'(0)\boxplus K'(1)}$.
  * If $c(t) = 1$ and $c(1-t) = 3$, then there are at most $2^{|R|+\varepsilon}$ possibilities for $f^t_{v_t}$ where $\varepsilon = 1$ if $K(t) \not\equiv K(1-t) \bmod 2$ and $\varepsilon = 0$ otherwise. As $K'(t) = K(t)$ and $K'(1-t) = K(1-t) - 1$, we see that ε is exactly equal to $1 + \left\lceil \frac{K(t)-K(1-t)}{2} \right\rceil - \left\lceil \frac{K'(t)-K'(1-t)}{2} \right\rceil$, so $N \leq 2^{(K'(0)\boxplus K'(1))+1}$.
  * If $c(t) = 3$ and $c(1-t) = 1$, then there are at most $2^{|R\setminus\{0\}|+\varepsilon'}$ possibilities for $f^t_{v_t}$ where $\varepsilon' = 1$ if $K(1-t) \neq 0$ and $K(t) \not\equiv K(1-t) \bmod 2$, and $\varepsilon' = 0$ otherwise. Moreover, the number $K'(1-t) + |R \setminus \{0\}| + \varepsilon'$ is equal to $K'(1-t) + K(t) - \left\lceil \frac{K(t)-K(1-t)}{2} \right\rceil - 1 + \eta + \varepsilon'$, where

$\eta = 1$ if $0 \notin R$, i.e. if $K(1-t) = 0$ and $K(t)$ is odd, and $\eta = 0$ otherwise. By definition of η and ε', we see that $\eta + \varepsilon' = 1$ if $K(t)$ and $K(1-t)$ have a different parity and $\eta + \varepsilon' = 0$ otherwise. Hence, $\eta + \varepsilon'$ is exactly equal to $\left\lceil \frac{K(t)-K(1-t)}{2} \right\rceil - \left\lceil \frac{K'(t)-K'(1-t)}{2} \right\rceil$ so that we obtain $N \leq 2^{K'(0) \boxplus K'(1)}$. □

1.5.9 The classification theorem

The following main theorem readily follows from the previous results.

Theorem 1.5.16. *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 6$ and let i be a legal coloring of T. Let $H \in \mathcal{H}_T^+$ be such that $H \supseteq \mathrm{Alt}_{(i)}(T)^+$. Then H belongs to $\underline{\mathcal{G}}_{(i)}$.*

Proof. This comes from the fact that the upper bounds given in Proposition 1.5.15 are all reached by the members of $\underline{\mathcal{G}}_{(i)}$ (see Proposition 1.5.6). Remark that, in Table 1.1, the lines 7 and 11 (and the lines 8 and 12) give the same $c(0)$ and $c(1)$, so their total number add up, thereby matching the factor 2 in the upper bound given by Proposition 1.5.15 (iii). □

We can now prove the next explicit formulation of Theorem 1.B.

Theorem 1.B′ (Classification)**.** *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 4$ and let i be a legal coloring of T.*

(i) Two groups $H, H' \in \underline{\mathcal{G}}_{(i)}$ are conjugate in $\mathrm{Aut}(T)$ if and only if $H = H'$ or $d_0 = d_1$ and either $H = G_{(i)}^+(Y_0, Y_1)$ and $H' = G_{(i)}^+(Y_0', Y_1')$ with $(Y_0, Y_1) = (Y_1', Y_0')$ or $H = G_{(i)}^+(X_0, X_1)^*$ and $H' = G_{(i)}^+(X_0', X_1')^*$ with $(X_0, X_1) = (X_1', X_0')$.

(ii) Suppose that $d_0, d_1 \geq 6$. Let $H \in \mathcal{H}_T^+$ be such that $\underline{H}(x) \cong F_0 \geq \mathrm{Alt}(d_0)$ for each $x \in V_0(T)$ and $\underline{H}(y) \cong F_1 \geq \mathrm{Alt}(d_1)$ for each $y \in V_1(T)$. Then H is conjugate in $\mathrm{Aut}(T)^+$ to a group in $\underline{\mathcal{G}}_{(i)}$.

Proof.

(i) It is a direct consequence of Lemma 1.4.10 (i) that two different groups in $\underline{\mathcal{G}}_{(i)}$ can never be conjugate in $\mathrm{Aut}(T)^+$. If $d_0 \neq d_1$

then $\mathrm{Aut}(T) = \mathrm{Aut}(T)^+$ and we are done. Now suppose that $d_0 = d_1$. Then there exists $\nu \in \mathrm{Aut}(T) \setminus \mathrm{Aut}(T)^+$ not preserving the types but preserving the colors, i.e. such that $i \circ \nu = i$. Every automorphism $\tau \in \mathrm{Aut}(T) \setminus \mathrm{Aut}(T)^+$ can be written as $\tau = \mu \circ \nu$ with $\mu \in \mathrm{Aut}(T)^+$. The statement then follows from the fact that $\nu G^+_{(i)}(Y_0, Y_1)\nu^{-1} = G^+_{(i)}(Y_1, Y_0)$ and $\nu G^+_{(i)}(X_0, X_1)^*\nu^{-1} = G^+_{(i)}(X_1, X_0)^*$.

(ii) By Theorem 1.G, there exists a legal coloring i' of T such that $H \supseteq \mathrm{Alt}_{(i')}(T)^+$. Hence, by Theorem 1.5.16, H belongs to $\underline{\mathcal{G}}_{(i')}$. But $\mathrm{Aut}(T)^+$ is transitive on the set of legal colorings of T, so each member of $\underline{\mathcal{G}}_{(i')}$ is conjugate in $\mathrm{Aut}(T)^+$ to its counterpart in $\underline{\mathcal{G}}_{(i)}$ and the conclusion follows. □

1.5.10 Proofs of the corollaries

We can now prove the different corollaries mentioned in §1.1. We actually give, for each one, a more precise formulation than its version in §1.1. For the definition of the set $\mathcal{G}'_{(i)}$, see Definition 1.4.9.

Corollary 1.C′. *Let T be the d-regular tree with $d \geq 4$ and let i be a legal coloring of T.*

(i) The members of $\mathcal{G}'_{(i)}$ are pairwise non-conjugate in $\mathrm{Aut}(T)$.

(ii) Suppose that $d \geq 6$. Let $H \in \mathcal{H}_T \setminus \mathcal{H}_T^+$ be such that $\underline{H}(v) \cong F \geq$ $\mathrm{Alt}(d)$ *for each $v \in V(T)$. Then H is conjugate in* $\mathrm{Aut}(T)^+$ *to a group belonging to $\mathcal{G}'_{(i)}$.*

Proof. We start by proving (ii).

(ii) Clearly, $H^+ := H \cap \mathrm{Aut}(T)^+$ is a subgroup of index 2 of H. Since $H^+(v) = H(v)$ for each $v \in V(T)$, we deduce by Lemma 1.2.1 that H^+ is also 2-transitive on ∂T, i.e. $H^+ \in \mathcal{H}_T^+$. Moreover, $\underline{H}^+(v) \cong F \geq \mathrm{Alt}(d)$ for each $v \in V(T)$, so Theorem 1.B′ can be applied to find the shapes that H^+ can take. It is however important to note that, if $\nu \in H \setminus H^+$, then $\nu H^+ \nu^{-1} = H^+$ while ν does not preserve the types. This means that in H^+ the situation must be the same for the vertices of type 0 and the vertices of type 1. As a consequence, H^+ can only be conjugate in $\mathrm{Aut}(T)^+$ to one of the groups $G_{(i)}^+(Y,Y)$ with $Y \in \{\varnothing, X, X^*\}$ and $G_{(i)}^+(X,X)^*$ (with $X \subset_f \mathbf{Z}_{\geq 0}$). In other words, there exists a legal coloring i' of T such that H^+ is equal to one of the groups $G_{(i')}^+(Y,Y)$ and $G_{(i')}^+(X,X)^*$.

Since H^+ is normal in H, H is contained in the normalizer of H^+ in $\mathrm{Aut}(T)$. By Lemma 1.4.10 (ii), the normalizer in $\mathrm{Aut}(T)$ of $G_{(i')}^+(\varnothing,\varnothing)$ (resp. $G_{(i')}^+(X,X)$, $G_{(i')}^+(X^*,X^*)$ and $G_{(i')}^+(X,X)^*$) is $G_{(i')}(\varnothing,\varnothing)$ (resp. $G_{(i')}(X^*,X^*)$, $G_{(i')}(X^*,X^*)$ and $G_{(i')}(X^*,X^*)$). Using the fact that H^+ is a subgroup of index 2 of H, we directly get that H is equal to $G_{(i')}(\varnothing,\varnothing)$ when $H^+ = G_{(i')}^+(\varnothing,\varnothing)$ and that H is equal to $G_{(i')}(X^*,X^*)$ when $H^+ = G_{(i')}^+(X^*,X^*)$. For the other cases, we have:

- If $H^+ = G_{(i')}^+(X,X)$, the normalizer of H^+ is $G_{(i')}(X^*,X^*)$. To get H, we must observe the extensions $H^+(\nu)$ of H^+ by an element $\nu \in G_{(i')}(X^*,X^*)$ that does not preserve the types and such that $\nu^2 \in H^+$. There are two possibilities: either $\mathrm{Sgn}_{(i')}(\nu, S_X(v)) = 1$ for each $v \in V(T)$ or $\mathrm{Sgn}_{(i')}(\nu, S_X(v)) = -1$ for each $v \in V(T)$ (we cannot have $\mathrm{Sgn}_{(i')}(\nu, S_X(v)) = 1$ for each $v \in V_0(T)$ and $\mathrm{Sgn}_{(i')}(\nu, S_X(v)) = -1$ for each $v \in V_1(T)$ since this would imply that $\nu^2 \notin H^+$). In the first case we get $H^+(\nu) = G_{(i)}(X,X)$. In the second case, define a new legal coloring i'' by $i''|_{V_0(T)} = i'|_{V_0(T)}$ and $i''|_{V_1(T)} = \tau \circ i'|_{V_1(T)}$ where $\tau \in \mathrm{Sym}(d)$ is an odd permutation. In this way, $H^+ = G_{(i')}^+(X,X) = G_{(i'')}^+(X,X)$ and $H^+(\nu) = G_{(i'')}(X,X)$.
- If $H^+ = G_{(i')}^+(X,X)^*$, the normalizer of H^+ is $G_{(i')}(X^*,X^*)$. Here also, we observe the extensions $H^+(\nu)$. In this case, all $\mathrm{Sgn}_{(i')}(\nu, S_X(v))$ with $v \in V_0(T)$ must be equal and all $\mathrm{Sgn}_{(i')}(\nu, S_X(v))$ with $v \in V_1(T)$ must be equal, but there is no additional condition since each such ν satisfies $\nu^2 \in H^+$. Replacing i' by i'' as above if necessary, we can assume that $\mathrm{Sgn}_{(i')}(\nu, S_X(v)) = 1$ for each $v \in V_0(T)$. Then, if $\mathrm{Sgn}_{(i')}(\nu, S_X(v)) = 1$ for each $v \in V_1(T)$ we obtain $H^+(\nu) =$

$G_{(i')}(X,X)^*$. On the contrary, if $\mathrm{Sgn}_{(i')}(\nu, S_X(v)) = -1$ for each $v \in V_1(T)$, then we get $H^+(\nu) = G'_{(i')}(X,X)^*$.

In any case, H is conjugate in $\mathrm{Aut}(T)^+$ to a group in $\mathcal{G}'_{(i)}$.

(i) Suppose that there exist two different groups $H, H' \in \mathcal{G}'_{(i)}$ that are conjugate in $\mathrm{Aut}(T)$. Then H^+ and H'^+ are also conjugate, and by Theorem 1.B′ (i) this implies that $H^+ = H'^+$. Since the groups in $\underline{\mathcal{G}}_{(i)}$ are pairwise distinct (Proposition 1.4.8), the only possibility is to have $H = G_{(i)}(X,X)^*$ and $H' = G'_{(i)}(X,X)^*$ (or the contrary) for some $X \subset_f \mathbf{Z}_{\geq 0}$. However, $G_{(i)}(X,X)^*$ and $G'_{(i)}(X,X)^*$ are not conjugate in $\mathrm{Aut}(T)$. Indeed, if $H^{(\infty)}$ denotes the intersection of all normal cocompact closed subgroups of H, then $(G_{(i)}(X,X)^*)^{(\infty)} = (G'_{(i)}(X,X)^*)^{(\infty)} = G^+_{(i)}(X,X)$ but $G_{(i)}(X,X)^* \big/ G^+_{(i)}(X,X) \cong (\mathbf{C}_2)^2$ while $G'_{(i)}(X,X)^* \big/ G^+_{(i)}(X,X) \cong \mathbf{C}_4$. ☐

Before proving Corollary 1.D′, recall that $\Theta \subset \mathbf{Z}_{>0}$ is the set of integers $m \geq 6$ such that each finite 2-transitive group on $\{1, \dots, m\}$ contains $\mathrm{Alt}(m)$.

Corollary 1.D′. *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \in \Theta$, let i be a legal coloring of T and let $H \in \mathcal{H}_T$. Then H is conjugate in* $\mathrm{Aut}(T)^+$ *to a group belonging to $\underline{\mathcal{G}}_{(i)}$ or $\mathcal{G}'_{(i)}$ (when $d_0 = d_1$).*

Proof. Since H is 2-transitive on ∂T, $H(v)$ is 2-transitive on $E(v)$ for each $v \in V(T)$ (see Lemma 1.2.2). By definition of Θ, this implies that $\underline{H}(x) \cong F_0 \geq \mathrm{Alt}(d_0)$ for each $x \in V_0(T)$ and $\underline{H}(y) \cong F_1 \geq \mathrm{Alt}(d_1)$ for each $y \in V_1(T)$. The conclusion follows from Theorem 1.B′ (ii) and Corollary 1.C′ (ii). ☐

Corollary 1.E′. *Let T be the d-regular tree with $d \geq 6$, let i be a legal coloring of T and let H be a vertex-transitive closed subgroup of* $\mathrm{Aut}(T)$. *If $\underline{H}(v) \cong F \geq \mathrm{Alt}(d)$ for each $v \in V(T)$, then H is discrete or H is conjugate in* $\mathrm{Aut}(T)^+$ *to a group belonging to $\mathcal{G}'_{(i)}$.*

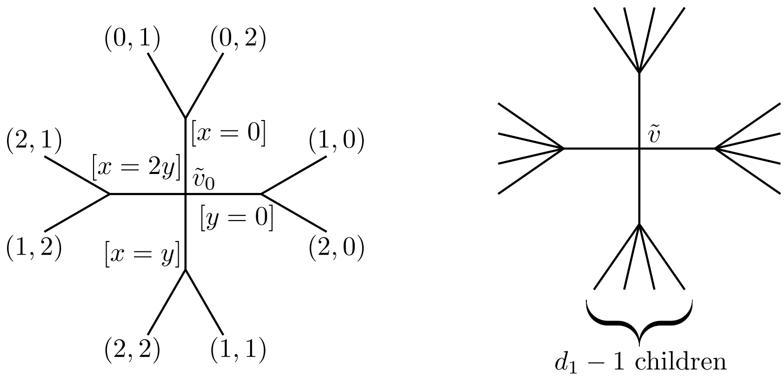
Proof. By [BM00a, Propositions 3.3.1 and 3.3.2], the hypotheses directly imply that H is discrete or 2-transitive on ∂T. The conclusion follows from Corollary 1.C′ (ii). ☐

1.6 Another example when $d_0 = 4$

Let T be the $(4, d_1)$-semiregular tree with $d_1 \geq 4$. In this section, we construct a (non-linear) group $G \in \mathcal{H}_T^+$ for which there is no legal coloring i of T such that $G \supseteq \mathrm{Alt}_{(i)}(T)^+$. This group will therefore be different from all the groups defined in §1.4.

To avoid any confusion, we use the letter j for the legal colorings of trees that will help to construct our group and the letter i will only be used for other legal colorings appearing in the results.

First consider the rooted tree $\tilde{T}_0 = T_{4,3,2}$, i.e. the rooted tree of depth 2 where the root v_0 has 4 children and each child of v_0 has $3-1=2$ children. Let ψ be a bijection between $(\mathbf{F}_3)^2 \setminus \{(0,0)\}$ and the set of eight vertices of $\tilde{T}_0$ at distance 2 from v_0, such that two such vertices x and y have the same parent if and only if $\psi^{-1}(x)$ and $\psi^{-1}(y)$ are a multiple of one another (see Figure 1.8a). The four children of v_0 thus correspond to the four lines of $(\mathbf{F}_3)^2$, or in other words to the four elements of the projective line over $\mathbf{F}_3$. The natural action of $\mathrm{SL}(2, \mathbf{F}_3)$ on $(\mathbf{F}_3)^2 \setminus \{(0,0)\}$ induces via ψ an action of $\mathrm{SL}(2, \mathbf{F}_3)$ on $\tilde{T}_0$. Let $\tilde{G}_0$ be the image of $\mathrm{SL}(2, \mathbf{F}_3)$ in $\mathrm{Aut}(\tilde{T}_0)$ defined in this way. It is clear that the pointwise stabilizer of $B(v_0, 1)$ in $\tilde{G}_0$ corresponds to the two matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Hence, $\underline{\tilde{G}_0}(v_0) \cong \mathrm{PSL}(2, \mathbf{F}_3)$ which is in turn isomorphic to $\mathrm{Alt}(4)$.



(a) $\tilde{T}_0 = T_{4,3,2}$. (b) $\tilde{T} = T_{4,d_1,2}$.

Figure 1.8: Construction of the group $\tilde{G}$.

Now consider the rooted tree $\tilde{T} = T_{4,d_1,2}$, i.e. the rooted tree of depth 2 where the root v has 4 children and each child of v has $d_1 - 1$ children (see Figure 1.8b). Fix a legal coloring $\tilde{j}$ of $\tilde{T}$ and a legal coloring $\tilde{j}_0$ of $\tilde{T}_0$ with $\tilde{j}_0(v_0) = \tilde{j}(v)$ and let α be the bijection from $B(v,1)$ to $B(v_0,1)$ preserving the colorings. We define the map $f\colon \mathrm{Aut}(\tilde{T}) \to \mathrm{Aut}(\tilde{T}_0)$ by saying that, if $g \in \mathrm{Aut}(\tilde{T})$, then $f(g) \in \mathrm{Aut}(\tilde{T}_0)$ is the unique automorphism of $\tilde{T}_0$ such that $f(g)(\alpha(x)) = \alpha(g(x))$ for each $x \in B(v,1)$ and $\sigma_{(\tilde{j}_0)}(f(g),\alpha(x))$ has the same parity as $\sigma_{(\tilde{j})}(g,x)$ for each $x \in S(v,1)$. Then consider $\tilde{G} = f^{-1}(\tilde{G}_0) \leq \mathrm{Aut}(\tilde{T})$.

It is clear from the definition of $\tilde{G}$ that $\underline{\tilde{G}}(v) \cong \mathrm{Alt}(4)$, and the next lemma shows that $\tilde{G}$ never contains $\mathrm{Alt}_{(\tilde{i})}(\tilde{T})$ for a legal coloring $\tilde{i}$ of $\tilde{T}$.

Lemma 1.6.1. *There does not exist a legal coloring $\tilde{i}$ of $\tilde{T}$ such that $\tilde{G} \supseteq \mathrm{Alt}_{(\tilde{i})}(\tilde{T})$.*

Proof. By contradiction, assume that such a coloring exists. From this one, we can construct a legal coloring $\tilde{i}_0$ of $\tilde{T}_0$ such that $\tilde{G}_0 \supseteq \mathrm{Alt}_{(\tilde{i}_0)}(\tilde{T}_0)$. Indeed, it suffices to set $\tilde{i}_0|_{B(v_0,1)} = \tilde{j}_0|_{B(v_0,1)}$ and then, for each $x \in S(v_0,1)$, to define $\tilde{i}_0$ on $S(x,1) \setminus \{v_0\}$ such that $\tilde{i}_0\tilde{j}_0|_{S(x,1)}^{-1} \in \mathrm{Sym}(3)$ has the same parity as $\tilde{i}\tilde{j}|_{S(\alpha^{-1}(x),1)}^{-1} \in \mathrm{Sym}(d_1)$. In this way, $f(\mathrm{Alt}_{(\tilde{i})}(\tilde{T})) = \mathrm{Alt}_{(\tilde{i}_0)}(\tilde{T}_0)$ and thus $\tilde{G}_0 = f(\tilde{G}) \supseteq f(\mathrm{Alt}_{(\tilde{i})}(\tilde{T})) = \mathrm{Alt}_{(\tilde{i}_0)}(\tilde{T}_0)$.

Let us name each vertex of $S(v_0,1)$ with the corresponding line in $(\mathbf{F}_3)^2 \setminus \{0,0\}$, i.e. with $[x=0]$, $[y=0]$, $[x=y]$ or $[x=2y]$. Let $g \in \mathrm{Alt}_{(\tilde{i}_0)}(\tilde{T}_0)$ be such that g interchanges $[x=0]$ and $[y=0]$ and interchanges $[x=y]$ and $[x=2y]$. Since $g \in \mathrm{Alt}_{(\tilde{i}_0)}(\tilde{T}_0) \subseteq \tilde{G}_0$, g acts as $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ or as $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ on $\tilde{T}_0$ (these are the only elements of $\tilde{G}_0$ acting as g on $B(v_0,1)$). In both cases, g^2 acts as $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ on $\tilde{T}_0$. But $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ fixes $B(v_0,1)$ and acts as a transposition at each vertex of $S(v_0,1)$, so it cannot be contained in $\mathrm{Alt}_{(\tilde{i}_0)}(\tilde{T}_0)$. This is a contradiction with the fact that $g^2 \in \mathrm{Alt}_{(\tilde{i}_0)}(\tilde{T}_0)$. $\square$

Using the group $\tilde{G} \leq \mathrm{Aut}(\tilde{T})$, we can now construct a group $G \in \mathcal{H}_T^+$ that acts locally as $\tilde{G}$. For each $x \in V_0(T)$, fix a map $J_x\colon B(x,2) \to \tilde{T}$. In this way, for each $x \in V_0(T)$ and each $g \in \mathrm{Aut}(T)^+$, we can define

$$\Sigma_{(J)}(g,x) = J_{g(x)} \circ g \circ J_x^{-1} \in \mathrm{Aut}(\tilde{T}).$$

This allows us to define $G \leq_{cl} \mathrm{Aut}(T)^+$ by

$$G := \{g \in \mathrm{Aut}(T)^+ \mid \Sigma_{(J)}(g, x) \in \tilde{G} \text{ for each } x \in V_0(T)\}.$$

The fact that G is 2-transitive on ∂T is not completely immediate.

Lemma 1.6.2. *The group G belongs to $\mathcal{H}_T^+$.*

Proof. By Lemma 1.2.1, it suffices to prove that $G(v)$ is transitive on ∂T for each $v \in V(T)$. As G is closed in $\mathrm{Aut}(T)$, we can simply show that the fixator in G of a geodesic (v, w) with $v, w \in V(T)$ always acts transitively on $E(w) \setminus \{e\}$, where e is the edge of (v, w) adjacent to w. If x and y are two vertices adjacent to w but not on (v, w), then we must find $g \in G$ fixing (v, w) and such that $g(x) = y$. We can simply construct such an element g by defining $g(x) = y$ and $g(e) = e$, and then by extending g to larger and larger balls, so that g fixes (v, w) and $\Sigma_{(J)}(g, z) \in \tilde{G}$ for each $z \in V_0(T)$. One easily checks, using the fact that $d_1 \geq 4$, that there is sufficient freedom in $\tilde{G}$ to do so. $\square$

Finally, as a corollary of Lemma 1.6.1 we find that G is indeed not isomorphic to any group defined in §1.4.

Proposition 1.6.3. *We have $\underline{G}(x) \cong \mathrm{Alt}(4)$ for each $x \in V_0(T)$ and $\underline{G}(y) \cong \mathrm{Sym}(d_1)$ for each $y \in V_1(T)$, but there does not exist a legal coloring i of T such that $G \supseteq \mathrm{Alt}_{(i)}(T)^+$.*

Proof. The fact that $\underline{G}(x) \cong \mathrm{Alt}(4)$ for $x \in V_0(T)$ and $\underline{G}(y) \cong \mathrm{Sym}(d_1)$ for $y \in V_1(T)$ readily follows from the definition of G. Now consider a legal coloring i of T. We show that $G \not\supseteq \mathrm{Alt}_{(i)}(T)^+$. Fix $x \in V_0(T)$ and consider the legal coloring $\tilde{i} = i \circ J_x^{-1}$ of $\tilde{T}$. By Lemma 1.6.1, there exists $\tilde{g} \in \mathrm{Alt}_{(\tilde{i})}(\tilde{T})$ such that $\tilde{g} \notin \tilde{G}$. One can then construct an element $g \in \mathrm{Alt}_{(i)}(T)^+$ fixing x and with $\Sigma_{(J)}(g, x) = \tilde{g}$, which is therefore such that $g \in \mathrm{Alt}_{(i)}(T)^+ \setminus G$. Hence, we have $G \not\supseteq \mathrm{Alt}_{(i)}(T)^+$. $\square$

1.I Topologically isomorphic groups in $\mathcal{H}_T$

We show in this appendix the following proposition, stating that two groups in $\mathcal{H}_T$ are topologically isomorphic if and only if they are conjugate in $\mathrm{Aut}(T)$. This is a folklore result but, because of the lack in finding a suitable reference, we give its full proof here.

Proposition 1.I.1. *Let T be the (d_0, d_1)-semiregular tree with $d_0, d_1 \geq 3$ and let $H, H' \in \mathcal{H}_T$ be isomorphic as topological groups. Then H and H' are conjugate in* $\mathrm{Aut}(T)$.

Proof. Since H acts edge-transitively on T (see Lemma 1.2.2), the vertex stabilizers H_v and edge stabilizers H_e in H are all pairwise distinct. Moreover, H_v is a maximal compact subgroup of H for each $v \in V(T)$, H_e is a maximal compact subgroup of H for each $e \in E(T)$ if and only if $H \notin \mathcal{H}_T^+$, and these are the only maximal compact subgroups of H (see [FTN91, Theorem 5.2]). Given the group H and all its compact maximal subgroups, one can also recognize which of them must be vertex stabilizers. Indeed, if $H \notin \mathcal{H}_T^+$ and $K = H_e$ is an edge stabilizer in H then there exists another maximal compact subgroup K' of H such that $[K : K \cap K'] = 2$ (namely $K' = H_v$ where v is a vertex of e). On the contrary, if $K = H_v$ is a vertex stabilizer in H (we do not suppose $H \notin \mathcal{H}_T^+$ here), then there exists no maximal compact subgroup K' of H such that $[K : K \cap K'] = 2$, because $d_0, d_1 \geq 3$ and H is edge-transitive. The vertex stabilizers in H can thus be exactly identified among the subgroups of H, without knowing anything about the action of H on T. The same is true for H'.

Now let $\varphi\colon H \to H'$ be an isomorphism of topological groups. For each $v \in V(T)$, the previous discussion shows that there is a unique vertex $\tau(v) \in V(T)$ such that $\varphi(H_v) = H'_{\tau(v)}$ and that the map $\tau\colon V(T) \to V(T)$ is a bijection. Moreover, two vertices $v, v' \in V(T)$ are neighbors in T if and only if $[H_v : H_{v'} \cap H_v] \leq [H_v : H_w \cap H_v]$ for all vertices $w \neq v$. This indeed follows easily from Lemma 1.2.2. In view of the definition of τ, this implies that v and v' are adjacent if and only if $\tau(v)$ and $\tau(v')$ are adjacent. In other words, τ is an automorphism of T. We finally claim that $\tau h \tau^{-1} = \varphi(h)$ for all $h \in H$. Indeed, we have $H'_{\tau h \tau^{-1}(v)} =$

$$\varphi(H_{h\tau^{-1}(v)}) = \varphi(hH_{\tau^{-1}(v)}h^{-1}) = \varphi(h)H'_v\varphi(h)^{-1} = H'_{\varphi(h)(v)}. \qquad \square$$

1.II Asymptotic density of the set Θ

In this appendix, we give an explicit expression for Θ and show that it is asymptotically dense in $\mathbf{Z}_{>0}$.

Proposition 1.II.1. *The set Θ is equal to*

$$\begin{aligned}\Theta = \{m \in \mathbf{Z}_{>0} \mid m \geq 6\}\\ \setminus \left(\left\{p^d \,\middle|\, p \text{ prime}, d \geq 1\right\} \cup \left\{\frac{p^{dr}-1}{p^d-1} \,\middle|\, p \text{ prime}, d \geq 1, r \geq 2\right\}\right.\\ \left.\cup \left\{2^{2d-1} \pm 2^{d-1} \,\middle|\, d \geq 3\right\} \cup \{22, 176, 276\}\right)\end{aligned}$$

Proof. This is a consequence of the classification of finite 2-transitive groups, see [Cam99, Tables 7.3 and 7.4]. Note that there exist some sporadic 2-transitive groups with $m \notin \{22, 176, 276\}$, but we did not write these values for m since they are already contained in at least one of the infinite families. $\square$

Corollary 1.II.2. *The asymptotic density $D(\Theta)$ of Θ in $\mathbf{Z}_{>0}$ is equal to 1, i.e.*

$$\lim_{n\to\infty} \frac{|\Theta \cap \{1, \ldots, n\}|}{n} = 1.$$

Proof. It suffices to prove that the asymptotic density of each of the three infinite families is equal to 0. First, we have

$$\begin{aligned}\left|\left\{2^{2d-1} \pm 2^{d-1} \,\middle|\, d \geq 3\right\} \cap \{1, \ldots, n\}\right| &\leq 2 \cdot \left|\left\{d \geq 3 \,\middle|\, 2^{2d-2} \leq n\right\}\right|\\ &= 2 \cdot |\{d \geq 3 \mid 2d-2 \leq \log_2(n)\}|\\ &\leq 2 \cdot \log_2(n)\end{aligned}$$

This directly implies that $D\left(\left\{2^{2d-1} \pm 2^{d-1} \,\middle|\, d \geq 3\right\}\right) = 0$.

We now show that the density of $\left\{\frac{p^{dr}-1}{p^d-1} \,\middle|\, p \text{ prime}, d \geq 1, r \geq 2\right\}$ is zero. The proof that the density of $\left\{p^d \,\middle|\, p \text{ prime}, d \geq 1\right\}$ is zero is similar and even easier. To simplify the notation, define $R(n) :=$

$\left|\left\{\frac{p^{dr}-1}{p^d-1}\;\middle|\; p \text{ prime}, d \geq 1, r \geq 2\right\} \cap \{1, \ldots, n\}\right|$ so that we need to compute $\lim_{n\to\infty} \frac{R(n)}{n}$. Since $\frac{p^{dr}-1}{p^d-1} \geq p^{d(r-1)}$, we have

$$\begin{aligned}
R(n) &\leq \left|\left\{(p, d, r) \;\middle|\; p \text{ prime}, d \geq 1, r \geq 2, p^{d(r-1)} \leq n\right\}\right| \\
&\leq \sum_{d=1}^{\infty}\sum_{r=2}^{\infty} \left|\left\{p \text{ prime} \;\middle|\; p \leq n^{\frac{1}{d(r-1)}}\right\}\right| \\
&= \sum_{d=1}^{\infty}\sum_{r=2}^{\infty} \pi(n^{\frac{1}{d(r-1)}})
\end{aligned}$$

where $\pi(x)$ is the number of prime numbers less or equal to x. When $d(r-1) > \log_2(n)$, we have $n^{\frac{1}{d(r-1)}} < 2$ and hence $\pi(n^{\frac{1}{d(r-1)}}) = 0$. If $L(n) := \lfloor \log_2(n) \rfloor$, we therefore have

$$R(n) \leq \sum_{d=1}^{L(n)} \sum_{r=2}^{L(n)+1} \pi(n^{\frac{1}{d(r-1)}})$$

By the prime number theorem, we have $\lim\limits_{x\to\infty} \dfrac{\pi(x)\ln(x)}{x} = 1$, so there exists $C > 0$ such that $\pi(x) \leq C\dfrac{x}{\ln(x)}$ for all $x > 0$. We therefore get

$$\begin{aligned}
R(n) &\leq C \sum_{d=1}^{L(n)} \sum_{r=2}^{L(n)+1} \frac{n^{\frac{1}{d(r-1)}}}{\ln(n^{\frac{1}{d(r-1)}})} \\
&\leq \frac{C}{\ln(n)} \sum_{d=1}^{L(n)} \sum_{r=2}^{L(n)+1} d(r-1) \cdot n^{\frac{1}{d(r-1)}}
\end{aligned}$$

Separating the case $(d, r) = (1, 2)$ from the $(L(n)^2 - 1)$ other cases gives

$$R(n) \leq \frac{C}{\ln(n)} \left(n + (L(n)^2 - 1)L(n)^2 \cdot n^{\frac{1}{2}}\right)$$

Hence,

$$\frac{R(n)}{n} \leq \frac{C}{\ln(n)} \left(1 + \frac{L(n)^4}{\sqrt{n}}\right) \to 0 \qquad \square$$

Chapter 2

Lattices in products of trees

In Chapter 1 we gave a classification result for some non-discrete automorphism groups of trees. Following ideas developed by Burger and Mozes in their seminal work [BM00a], these results can be used to study lattices in products of trees. There indeed are numerous examples of lattices in a product of two trees whose projections on each tree is non-discrete. Under some local hypotheses, the closures of those projections thus fit perfectly into the context of our classification.

2.1 Main results

Given two integers $d_1, d_2 \geq 3$, we consider the product $T_1 \times T_2$ of the d_1-regular tree T_1 and the d_2-regular tree T_2. It can be seen as a square-complex whose vertex set, edge set and square set are given by

$$\begin{aligned} V &= V(T_1) \times V(T_2), \\ E &= (E(T_1) \times V(T_2)) \cup (V(T_1) \times E(T_2)), \\ S &= E(T_1) \times E(T_2); \end{aligned}$$

with the natural incidence relations. We generally call T_1 the *horizontal* tree and T_2 the *vertical* tree, so that the edge set E decomposes as $E = E_h \cup E_v$ with $E_h = E(T_1) \times V(T_2)$ being the *horizontal edge set* and $E_v = V(T_1) \times E(T_2)$ the *vertical edge set*.

In this chapter, we will mainly be interested in groups Γ acting simply transitively on the set of vertices of $T_1 \times T_2$ and that do not exchange the two trees (i.e. such that $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$). Such a group Γ will be called a **(d_1, d_2)-group**. In [BM00b, Chapter 6] and [Rat04], the authors studied those groups with the additional property that they are torsion-free. We do not make this assumption here; authorizing torsion will actually lead us to interesting examples. A (d_1, d_2)-group Γ is said to be **reducible** if it is commensurable to a product $\Gamma_1 \times \Gamma_2$ of lattices $\Gamma_t \leq \mathrm{Aut}(T_t)$. It is called **irreducible** if it is not reducible, which is equivalent to asking that $H_1 = \overline{\mathrm{proj}_1(\Gamma)} \leq \mathrm{Aut}(T_1)$ and $H_2 = \overline{\mathrm{proj}_2(\Gamma)} \leq \mathrm{Aut}(T_2)$ are both non-discrete [BM00b, Proposition 1.2]. (Note that H_1 and H_2 are either both discrete or both non-discrete.)

Given a (d_1, d_2)-group Γ satisfying $\underline{H_t}(v_t) \geq \mathrm{Alt}(d_t)$ and $d_t \geq 6$ for some $t \in \{1, 2\}$ (and some $v_t \in V(T_t)$), it is not so hard to see if Γ is irreducible, see (i) below. Moreover, if it is irreducible, then we know by Corollary 1.E′ that H_t belongs to the collection $\mathcal{G}'_{(i)}$ for some legal coloring i of T_t. In the first part of this chapter, we develop tools allowing us in such a case to determine which group H_t is (when d_t is even), see (ii) below.

Theorem 2.A. *Let $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ be a (d_1, d_2)-group, let $t \in \{1, 2\}$ and let i be a legal coloring of T_t. Suppose that $d_t \geq 6$ and that $\underline{H_t}(v_t) \geq \mathrm{Alt}(d_t)$ (for some $v_t \in V(T_t)$). Then:*

(i) There is an (efficient) algorithm that determines if Γ is irreducible.

(ii) If Γ is irreducible and d_t is even, then there is an (efficient) algorithm that computes the group from $\mathcal{G}'_{(i)}$ to which H_t is isomorphic.

In the following theorem we gather everything we can say about torsion-free $(6, 6)$-groups, notably thanks to our algorithms above. This is a preview of what is done in §2.3. Two (d_1, d_2)-groups are called **equivalent** if they are conjugate in $\mathrm{Aut}(T_1 \times T_2)$.

Theorem 2.B. *There are 32062 torsion-free $(6, 6)$-groups up to equivalence. At least 18426 of them are reducible, and at least 8227 of them are irreducible. Moreover, given a legal coloring i of the 6-regular tree T_1, there are exactly 7 groups in $\mathcal{G}'_{(i)}$ that are equal to $\overline{\mathrm{proj}_1(\Gamma)}$ for some torsion-free (irreducible) $(6, 6)$-group Γ.*

The next goal of this chapter is to construct (d_1, d_2)-groups that are virtually simple. In order to have short presentations for these groups, we want d_1 and d_2 to be as small as possible.

Theorem 2.C. *There exist (at least):*

(i) 160 *pairwise non-commensurable virtually simple* $(6,6)$*-groups: two of them have a simple subgroup of index* 12*, and the other* 158 *have a simple subgroup of index* 4*;*

(ii) 60 *pairwise non-isomorphic virtually simple* $(4,5)$*-groups:* 12 *of them have a simple subgroup of index* 8*, and the other* 48 *have a simple subgroup of index* 4*.*

We record the following down-to-earth illustration.

Corollary 2.D.

(i) The following group, presented by 6 *generators and* 10 *relators, is a* $(6,6)$*-group with a simple subgroup of index* 4*:*

$$\begin{aligned}\langle a_1, a_2, a_3, b_1, b_2, b_3 \mid\ & a_1 b_1 a_2^{-1} b_1,\ a_1 b_2 a_2 b_2^{-1},\ a_1 b_2^{-1} a_2^{-1} b_1^{-1},\ a_1 b_1^{-1} a_2^{-1} b_2,\\ & a_1 b_3 a_1 b_3^{-1},\ a_2 b_3 a_2 b_3,\ a_2 b_3^{-1} a_3 b_3^{-1},\ a_3 b_1 a_3^{-1} b_1^{-1},\\ & a_3 b_2 a_3 b_3,\ a_3 b_2^{-1} a_3 b_2^{-1} \rangle\end{aligned}$$

(ii) The following group is isomorphic to a free amalgamated product $F_3 *_{F_{11}} F_3$*, is simple, and is an index* 4 *subgroup of a* $(4,5)$*-group:*

$$\begin{aligned}\langle x_1, x_2, x_3, y_1, y_2, y_3 \mid\ & x_1 = y_1,\\ & x_2^2 = y_2 y_1^{-1} y_2,\\ & x_3^2 = y_3^2,\\ & x_3^{-1} x_1 x_3 = y_3^{-1} y_2 y_3,\\ & x_3^{-1} x_2 x_3 = y_3^{-1} y_1 y_3,\\ & x_2^{-1} x_1 x_2 = y_2^{-1} y_1^{-1} y_2,\\ & x_2^{-1} x_3^{-2} x_2 = y_2^{-1} y_1 y_3^{-2} y_2,\\ & x_2^{-1} x_3^{-1} x_2^{-1} x_1 x_3 x_2 = y_2^{-1} y_1 y_3^{-1} y_1^{-1} y_2 y_3 y_1^{-1} y_2,\\ & x_2^{-1} x_3^{-1} x_1 x_2 x_3 x_2 = y_2^{-1} y_1 y_3^{-1} y_2 y_3^{-1} y_1 y_3 y_1^{-1} y_2,\\ & x_2^{-1} x_3^{-1} x_2^2 x_3 x_2 = y_2^{-1} y_1 y_3^{-1} y_1 y_3^{-1} y_1 y_3 y_1^{-1} y_2,\\ & x_2^{-1} x_3^{-1} x_2^{-1} x_3 x_2 x_3 x_2 = y_2^{-1} y_1 y_3^{-1} y_1^{-1} y_3^{-1} y_1 y_3 y_1^{-1} y_2 \rangle\end{aligned}$$

We also show that for any $d_1 \geq 4$, there exists a virtually simple (d_1, d_2)-group for some d_2. This is a consequence of the following result.

Theorem 2.E. *For each $n \geq 2$, there exists a $(2n, 2n+1)$-group with a simple subgroup of index 2^n.*

For further information about these virtually simple groups, see Theorem 2.4.7. Our next result provides a family of virtually simple $(6, d_2)$-groups with arbitrarily large d_2, so that the projection on the 6-regular tree becomes larger and larger when $d_2 \to \infty$.

Theorem 2.F. *There exists a virtually simple $(6, 4n)$-group $\Gamma_{6,4n}$ for each $n \geq 2$, such that $\overline{\mathrm{proj}_1(\Gamma_{6,4n})} \to \mathrm{Aut}(T_1)$ in the Chabauty topology of $\mathrm{Aut}(T_1)$ when $n \to \infty$ (where T_1 is the 6-regular tree).*

We give an explicit presentation for $\Gamma_{6,4n}$ in Theorem 2.4.8, where the closures of the two projections are also computed. This theorem can be used to prove the next statement. It was already established in much greater generality in [BK90, Corollary 4.25] and [Liu94] with a completely different approach.

Corollary 2.G. *Let F_3 be the free group on 3 generators and let T be the usual Cayley graph of F_3, i.e. the 6-regular tree. Then the commensurator of F_3 in $\mathrm{Aut}(T)$ is dense in $\mathrm{Aut}(T)$.*

We finally close the chapter with an experimental study of lattices in products of three trees. While our previous results show the existence of many irreducible (d_1, d_2)-groups, things are apparently different when a third tree pops up.

Theorem 2.H. *Let T_1, T_2, T_3 be 6-regular trees and let $v_t \in V(T_t)$ for each $t \in \{1, 2, 3\}$. There is no subgroup $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2) \times \mathrm{Aut}(T_3)$ acting simply transitively on the vertices of $T_1 \times T_2 \times T_3$ and such that the following conditions hold, where $H_t = \overline{\mathrm{proj}_t(\Gamma)} \leq \mathrm{Aut}(T_t)$:*

- *H_1, H_2, H_3 are non-discrete and $\underline{H_1}(v_1), \underline{H_2}(v_2), \underline{H_3}(v_3) \geq \mathrm{Alt}(6)$;*
- *$\mathrm{proj}_{1,3}(\Gamma)$ is dense in $H_1 \times H_3$ and $\mathrm{proj}_{2,3}(\Gamma)$ is dense in $H_2 \times H_3$;*
- *the stabilizers $\Gamma(v_1, v_3)$ and $\Gamma(v_2, v_3)$ are torsion-free.*

2.2 Structure theory of (d_1, d_2)-groups

2.2.1 (d_1, d_2)-complexes and (d_1, d_2)-data

Let Γ be some (d_1, d_2)-group. Recall from §1.1 that $V(T_1) = V_0(T_1) \sqcup V_1(T_1)$ and $V(T_2) = V_0(T_2) \sqcup V_1(T_2)$. Hence, the vertex set V of $T_1 \times T_2$ can naturally be partitioned as $V = V_{00} \sqcup V_{01} \sqcup V_{10} \sqcup V_{11}$ where $V_{ij} = V_i(T_1) \times V_j(T_2)$ for each $i, j \in \{0, 1\}$. We say that a vertex in V_{ij} is of type (i, j). Each element of Γ can be type-preserving or not on each of the two trees T_1 and T_2, so that this induces a natural (surjective) homomorphism $\Gamma \to \mathbf{C}_2 \times \mathbf{C}_2$. The kernel of this homomorphism, which we denote by Γ^+, is an index 4 normal subgroup of Γ and consists of elements of Γ that preserve the types in $T_1 \times T_2$.

Let us focus for a moment on those groups $\Lambda \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ that preserve the types and act simply transitively on the vertices of each type, as Γ^+. In the next lemma, we show that those Λ are always torsion-free.

Lemma 2.2.1. *Let $\Lambda \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ be a type-preserving group acting freely on the vertices of $T_1 \times T_2$. Then Λ is torsion-free.*

Proof. Let g be a torsion element in Λ, i.e. such that $g^n = 1$ for some $n \geq 1$. For each $t \in \{1, 2\}$, we deduce from [Tit70, Proposition 3.2] that $\mathrm{proj}_{\mathrm{Aut}(T_t)}(g)$ fixes a vertex of T_t or inverses an edge of T_t. It cannot be an inversion since it is type-preserving, so it fixes a vertex of T_t. Hence, g fixes a vertex of $T_1 \times T_2$. As Λ acts freely on the vertices of $T_1 \times T_2$, this means that $g = 1$. □

When Λ acts simply transitively on vertices of each type of $T_1 \times T_2$, the quotient square-complex $X_\Lambda = \Lambda \backslash (T_1 \times T_2)$ has four vertices. We denote them by v_{00}, v_{10}, v_{11} and v_{01}, so that the projection of a vertex in V_{ij} is v_{ij}. For each $j \in \{0, 1\}$, there are d_1 edges between v_{0j} and v_{1j} (call them *horizontal*), and for each $i \in \{0, 1\}$, there are d_2 edges between v_{i0} and v_{i1} (call them *vertical*). Also, there are exactly $d_1 d_2$ squares in X_Λ, attached to the four vertices and such that the link of each vertex is a complete bipartite graph. We call such a finite square-complex a (d_1, d_2)-*complex*, see Definition 2.2.2 below (and Figure 2.1).

Be aware that, in [Rat04], the author uses the same term for a similar (but different) square-complex.

Definition 2.2.2. A (d_1, d_2)**-complex** is a square-complex with:

- four vertices v_{00}, v_{10}, v_{11} and v_{01};
- d_1 edges between v_{00} and v_{10} and d_1 edges between v_{11} and v_{01};
- d_2 edges between v_{10} and v_{11} and d_2 edges between v_{01} and v_{00};
- $d_1 d_2$ squares attached to the four vertices, such that for each pair (e_h, e_v) of horizontal and vertical edges, there is exactly one square adjacent to both e_h and e_v.

We saw above how to go from Λ to a (d_1, d_2)-complex. Conversely, given a (d_1, d_2)-complex X, the universal cover $\tilde{X}$ of X is the product of the d_1-regular tree and the d_2-regular tree, and the fundamental group $\pi_1(X)$ of X acts simply transitively on the vertices of each type of $\tilde{X}$. One easily checks that this gives a bijective correspondence between conjugacy classes of such groups Λ (in $\mathrm{Aut}(T_1 \times T_2)$) and isomorphism classes of (d_1, d_2)-complexes.

Now if we come back to our (d_1, d_2)-group Γ, then we can consider the (d_1, d_2)-complex $X_{\Gamma^+} = \Gamma^+ \backslash (T_1 \times T_2)$. In addition, the action of Γ on $T_1 \times T_2$ induces an action of $\mathbf{C}_2 \times \mathbf{C}_2$ on X_{Γ^+}. The three non-trivial elements of $\mathbf{C}_2 \times \mathbf{C}_2$ permute the vertices of X_{Γ^+} with the three permutations $(v_{00}\ v_{10})(v_{01}\ v_{11})$, $(v_{00}\ v_{01})(v_{10}\ v_{11})$ and $(v_{00}\ v_{11})(v_{10}\ v_{01})$. We say that an action of $\mathbf{C}_2 \times \mathbf{C}_2$ on a (d_1, d_2)-complex is **good** if it induces those permutations on the vertices of that complex. Conversely,

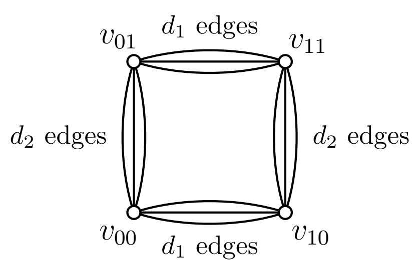


Figure 2.1: The 1-skeleton of a (d_1, d_2)-complex.

given a good action of $\mathbf{C}_2 \times \mathbf{C}_2$ on a (d_1, d_2)-complex X, we can consider the projection $p: T_1 \times T_2 \cong \tilde{X} \to X$ and lift $\mathbf{C}_2 \times \mathbf{C}_2$ to

$$\widetilde{\mathbf{C}_2 \times \mathbf{C}_2} = \{g \in \mathrm{Aut}(\tilde{X}) \mid \exists h \in \mathbf{C}_2 \times \mathbf{C}_2 : p \circ g = h \circ p\}.$$

This group $\widetilde{\mathbf{C}_2 \times \mathbf{C}_2}$ acts simply transitively on the vertices of $T_1 \times T_2$ and is thus a (d_1, d_2)-group. So we now have a bijective correspondence between (d_1, d_2)-groups (up to equivalence) and good $\mathbf{C}_2 \times \mathbf{C}_2$-actions on (d_1, d_2)-complexes (up to equivariant isomorphism).

A (d_1, d_2)-complex with a good $\mathbf{C}_2 \times \mathbf{C}_2$-action can be encoded via what we call a (d_1, d_2)-*datum*. The following definition is inspired from the definition of a *VH-datum* in [BM00b, §6]. The differences between the two notions come from the fact that we allow torsion. What the authors call a VH-datum will here be called a *torsion-free* (d_1, d_2)-*datum*, see Definition 2.2.5 below.

Definition 2.2.3. A (d_1, d_2)**-datum** $(A, B, \varphi_A, \varphi_B, R)$ consists of two finite sets A, B with $|A| = d_1$ and $|B| = d_2$, two involutions $\varphi_A: A \to A$, $\varphi_B: B \to B$ and a subset $R \subset A \times B \times A \times B$ satisfying conditions (1) and (2) below. Write $a^{-1} = \varphi_A(a)$ for $a \in A$ and $b^{-1} = \varphi_B(b)$ for $b \in B$. The two maps $\sigma, \rho: A \times B \times A \times B \to A \times B \times A \times B$ are defined by

$$\begin{aligned} \sigma(a, b, a', b') &= (a'^{-1}, b^{-1}, a^{-1}, b'^{-1}), \\ \rho(a, b, a', b') &= (a', b', a, b). \end{aligned}$$

(1) Each of the four projections of R onto the subproducts of the form $A \times B$ or $B \times A$ are bijective;

(2) R is invariant under the action of the group $\langle \sigma, \rho \rangle \cong \mathbf{C}_2 \times \mathbf{C}_2$.

Definition 2.2.4. Given $(A, B, \varphi_A, \varphi_B, R)$ and $(A', B', \varphi_{A'}, \varphi_{B'}, R')$ two (d_1, d_2)-data, an object of one of the following forms is called an **equivalence** between the two data.

- A pair (α, β) of bijections $\alpha: A \to A'$ and $\beta: B \to B'$ such that $\varphi_{A'} = \alpha \varphi_A \alpha^{-1}$, $\varphi_{B'} = \beta \varphi_B \beta^{-1}$, and $R' = (\alpha \times \beta \times \alpha \times \beta)(R)$.
- A pair (α, β) of bijections $\alpha: A \to B'$ and $\beta: B \to A'$ such that

$\varphi_{A'} = \beta\varphi_B\beta^{-1}$, $\varphi_{B'} = \alpha\varphi_A\alpha^{-1}$, and

$$R' = \{(\beta(b), \alpha(a'), \beta(b'), \alpha(a)) \mid (a, b, a', b') \in R\}.$$

This defines an equivalence relation on the set of (d_1, d_2)-data.

Given a (d_1, d_2)-datum $(A, B, \varphi_A, \varphi_B, R)$, one can build a (d_1, d_2)-complex with a good $\mathbf{C}_2 \times \mathbf{C}_2$-action as follows. Fix four vertices v_{00}, v_{10}, v_{11} and v_{01}. For each $a \in A$, draw an horizontal edge e_a between v_{00} and v_{10} and an horizontal edge e'_a between v_{11} and v_{01}. Also, for each $b \in B$, draw a vertical edge f_b between v_{10} and v_{11} and a vertical edge f'_b between v_{01} and v_{00}. Then, for each $(a, b, a', b') \in R$ we glue a square to the edges $e_a, f_b, e'_{a'}$ and $f'_{b'}$. Condition (1) in Definition 2.2.3 ensures that this square complex X is a (d_1, d_2)-datum, and (2) enables us to define a good $\mathbf{C}_2 \times \mathbf{C}_2$-action on it. Indeed, we can define $\tilde{\sigma} \in \mathrm{Aut}(X)$ by $\tilde{\sigma}\colon v_{00} \leftrightarrow v_{01}$, $v_{10} \leftrightarrow v_{11}$, $e_a \leftrightarrow e'_{a^{-1}}$, $f_b \leftrightarrow f_{b^{-1}}$, $f'_b \leftrightarrow f'_{b^{-1}}$ and $\tilde{\rho} \in \mathrm{Aut}(X)$ by $\tilde{\rho}\colon v_{00} \leftrightarrow v_{11}$, $v_{10} \leftrightarrow v_{01}$, $e_a \leftrightarrow e'_a$, $f_b \leftrightarrow f'_b$. (The actions on the squares are then clearly defined.) They are automorphisms of X because R is invariant under the action of $\langle \sigma, \rho \rangle$.

Conversely, from a good $\mathbf{C}_2 \times \mathbf{C}_2$-action on a (d_1, d_2)-complex we can come back to a (d_1, d_2)-datum. We can indeed consider two finite sets A and B with $|A| = d_1$ and $|B| = d_2$, denote the edges between v_{00} and v_{10} by e_a with $a \in A$ (arbitrarily) and those between v_{10} and v_{11} by f_b with $b \in B$ (arbitrarily). Then, if $\tilde{\rho}$ is the element of $\mathbf{C}_2 \times \mathbf{C}_2$ that exchanges v_{00} and v_{11}, we write $e'_a = \tilde{\rho}(e_a)$ for each $a \in A$ and $f'_b = \tilde{\rho}(f_b)$ for each $b \in B$. If $\tilde{\sigma}$ is the element of $\mathbf{C}_2 \times \mathbf{C}_2$ that exchanges v_{00} and v_{01}, then we define $\varphi_B\colon B \to B$ such that $\tilde{\sigma}(f_b) = f_{\varphi_B(b)}$ for each $b \in B$. Similarly, if $\tilde{\sigma}'$ exchanges v_{00} and v_{10} (i.e. $\tilde{\sigma}' = \tilde{\sigma}\tilde{\rho}$), then we define $\varphi_A\colon A \to A$ such that $\tilde{\sigma}'(e_a) = e_{\varphi_A(a)}$. Finally, we define $R \subset A \times B \times A \times B$ as the set of all (a, b, a', b') such that $e_a, f_b, e'_{a'}$ and $f'_{b'}$ are the four edges of some square in the (d_1, d_2)-complex. One easily checks that $(A, B, \varphi_A, \varphi_B, R)$ is a (d_1, d_2)-datum.

Recalling that (d_1, d_2)-groups and good $\mathbf{C}_2 \times \mathbf{C}_2$-actions on (d_1, d_2)-complexes are in correspondence, we now have a bijective correspondence between equivalence classes of (d_1, d_2)-groups and equivalence classes of (d_1, d_2)-data. A presentation for the (d_1, d_2)-group Γ corresponding to some (d_1, d_2)-datum $(A, B, \varphi_A, \varphi_B, R)$ is given by

$$\Gamma = \langle A \cup B \mid xx^{-1} = 1 \ \forall x \in A \cup B,\ aba'b' = 1 \ \forall (a, b, a', b') \in R \rangle.$$

Indeed, the group Γ acts simply transitively on the vertices of $T_1 \times T_2$, so the 1-skeleton of $T_1 \times T_2$ is a Cayley graph for Γ. Moreover, the edges of this Cayley graph are labelled by $A \cup B$ so that the four edges of each square in the graph correspond to elements of R.

Definition 2.2.5. A **torsion-free** (d_1, d_2)**-datum** is a (d_1, d_2)-datum $(A, B, \varphi_A, \varphi_B, R)$ where φ_A and φ_B have no fixed points and the action of $\langle \sigma, \rho \rangle$ on R is free.

The above correspondence between equivalence classes of (d_1, d_2)-groups and equivalence classes of (d_1, d_2)-data then restricts to a correspondence between equivalence classes of torsion-free (d_1, d_2)-groups and equivalence classes of torsion-free (d_1, d_2)-data. Note that, since φ_A and φ_B cannot have any fixed point in Definition 2.2.5, there does not exist any torsion-free (d_1, d_2)-group when d_1 or d_2 is odd.

In the following, we will always consider (d_1, d_2)-data up to equivalence. We write $A = \{a_1, \ldots, a_{d_1}\}$ and $B = \{b_1, \ldots, b_{d_2}\}$. Also, we denote by τ_1 (resp. τ_2) the number of fixed points of φ_A (resp. φ_B) in some (d_1, d_2)-data, and we assume (without losing any generality) that $\varphi_A(a_i) = a_{d_1+1-i}$ for each $i \in \{1, \ldots, \frac{d_1-\tau_1}{2}\}$ and $\varphi_A(a_i) = a_i$ for each $i \in \{\frac{d_1-\tau_1}{2}+1, \ldots, \frac{d_1+\tau_1}{2}\}$ (resp. $\varphi_B(b_i) = b_{d_2+1-i}$ for each $i \in \{1, \ldots, \frac{d_2-\tau_2}{2}\}$ and $\varphi_B(b_i) = b_i$ for each $i \in \{\frac{d_2-\tau_2}{2} + 1, \ldots, \frac{d_2+\tau_2}{2}\}$). We will sometimes write A_i instead of a_i when $\varphi_A(a_i) = a_i$ (i.e. $a_i = a_i^{-1}$), and B_i instead of b_i when $\varphi_B(b_i) = b_i$ (i.e. $b_i = b_i^{-1}$). Note that, under these assumptions, τ_1, τ_2 and R fully determine the (d_1, d_2)-datum.

2.2.2 Geometric squares

Consider some (d_1, d_2)-datum with associated parameters τ_1, τ_2 and R. Given $(a, b, a', b') \in R$, we write $[a, b, a', b']$ for the set

$$\{(a, b, a', b'), (a', b', a, b), (a'^{-1}, b^{-1}, a^{-1}, b'^{-1}), (a^{-1}, b'^{-1}, a'^{-1}, b^{-1})\} \subseteq R.$$

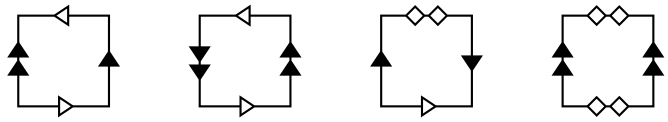


Figure 2.2: An example of a $(3,4)$-datum.

This is the $\mathbf{C}_2 \times \mathbf{C}_2$-orbit of (a, b, a', b'), we call it a **geometric square**. If the (d_1, d_2)-datum is torsion-free, then the action of $\mathbf{C}_2 \times \mathbf{C}_2$ on R is free so each geometric square contains exactly four elements. In this particular case, we have exactly $\frac{d_1 d_2}{4}$ geometric squares (remember that d_1 and d_2 must be even). When allowing torsion, we can actually have up to $d_1 d_2$ geometric squares (in the particular case where $\tau_1 = d_1$, $\tau_2 = d_2$ and $R = \{(a, b, a, b) \mid a \in A,\ b \in B\}$).

An easy way to define some particular (d_1, d_2)-datum is then to draw its geometric squares. We explain how the drawing works by giving an example. Consider the $(3,4)$-datum defined by the geometric squares $[a_1, b_1, a_1, b_2^{-1}]$, $[a_1, b_2, a_1, b_2]$, $[a_1, b_1^{-1}, A_2, b_1^{-1}]$ and $[A_2, b_2, A_2, b_2^{-1}]$. Note that the values $\tau_1 = 1$ and $\tau_2 = 0$ can be understood from the squares. Then we draw this $(3,4)$-datum as in Figure 2.2. Each square can be read counterclockwise, starting from the bottom edge. The white symbols thus represent elements of A, while the black ones represent elements of B. A single arrow with the forward orientation means a_1 (or b_1), a double arrow with the forward orientation means a_2 (or b_2), etc. A single arrow with the backward orientation means a_1^{-1} (or b_1^{-1}), a double arrow with the backward orientation means a_2^{-1} (or b_2^{-1}), etc. Finally, a lozenge (that does not have any orientation) means A_1 (or B_1), two lozenges mean A_2 (or B_2), etc. Note that we can actually read each square from the bottom or from the top edge, and clockwise or counterclockwise. These four ways of reading give the four (possibly equal) elements of R defined by the geometric square. In our example, the first and third geometric squares give four distinct elements of R, while the second and fourth geometric squares give two distinct elements of R. Note that $|R| = 4 + 2 + 4 + 2 = 12 = 3 \cdot 4$ as is needed for a $(3,4)$-datum.

One can quickly check if some set of geometric squares satisfies the hypotheses for representing a (d_1, d_2)-datum. For a torsion-free (d_1, d_2)-

datum, it suffices to verify that each possible *corner* appears exactly once in the geometric squares. By a **corner**, we mean a vertex of a square together with the two labelled edges adjacent to it, where the orientation of the arrows matters. There are $d_1 d_2$ possible corners, and they must all appear once (in one of the $\frac{d_1 d_2}{4}$ geometric squares). Now the condition is almost the same for a general (d_1, d_2)-datum: we just need to take into account that some geometric squares can have non-trivial automorphisms. So the condition is now that each of the $d_1 d_2$ possible corners must appear exactly once, up to these square automorphisms. Note that a geometric square as $[a_1, b_1, A_2, b_1^{-1}]$ cannot appear in a (d_1, d_2)-datum, since some corner appears twice while the square has no automorphism.

2.3 Projections on each factor

Consider a (d_1, d_2)-group $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$, associated to some set of geometric squares. Define $H_1 = \overline{\mathrm{proj}_1(\Gamma)} \leq \mathrm{Aut}(T_1)$ and $H_2 = \overline{\mathrm{proj}_2(\Gamma)} \leq \mathrm{Aut}(T_2)$ as the closures of the projections of Γ on $\mathrm{Aut}(T_1)$ and $\mathrm{Aut}(T_2)$. The goal of this section is to analyze which pairs of groups (H_1, H_2) can be obtained from a (d_1, d_2)-group Γ which is irreducible.

2.3.1 Action of Γ on $T_1 \times T_2$

Let us see $a_1, \ldots, a_{d_1}, b_1, \ldots, b_{d_2}$ as the generators of Γ (recall from §2.2.1 that the natural presentation of Γ has generators $a_1, \ldots, a_{d_1}, b_1, \ldots, b_{d_2}$ and relators given by the geometric squares). In $T_1 \times T_2$, whose 1-skeleton is the Cayley graph of Γ, the vertex $(v_1, v_2) \in V(T_1) \times V(T_2)$ corresponding to the identity element is such that all generators $b_1, \ldots, b_{d_2}$ fix v_1 and all generators $a_1, \ldots, a_{d_1}$ fix v_2. Also, $b_1, \ldots, b_{d_2}$ send v_2 to its d_2 neighboring vertices and $a_1, \ldots, a_{d_1}$ send v_1 to its d_1 neighboring vertices. This means that $\langle b_1, \ldots, b_{d_2} \rangle = \Gamma(v_1)$ and $\langle a_1, \ldots, a_{d_1} \rangle = \Gamma(v_2)$, where $\Gamma(v)$ denotes the fixator of v in Γ. We now explain how the action of a particular element of Γ on $T_1 \times T_2$ can be computed from the geometric squares. These explanations are similar to those in [Rat04, §1.4], but we recall them here so as to remain self-contained.

The fact that the 1-skeleton of $T_1 \times T_2$ is the Cayley graph of Γ gives a labelling of the edges of $T_1 \times T_2$ by the generators $a_1, \dots, a_{d_1}, b_1, \dots, b_{d_2}$. As for the geometric squares, we can thus associate a (white or black) symbol to each edge of $T_1 \times T_2$. This labelling is actually such that each square in $T_1 \times T_2$ corresponds to one of the geometric squares associated to Γ. We also have natural embeddings $T_1 \hookrightarrow T_1 \times T_2 \colon w \in V(T_1) \mapsto (w, v_2)$ and $T_2 \hookrightarrow T_1 \times T_2 \colon w \in V(T_2) \mapsto (v_1, w)$ from which we get a labelling of T_1 with white symbols and a labelling of T_2 with black symbols. This is actually equivalent to seeing T_1 as the Cayley graph of $\langle a_1, \dots, a_{d_1} \rangle$ and T_2 as the Cayley graph of $\langle b_1, \dots, b_{d_2} \rangle$.

The image of (v_1, v_2) by some element $g \in \Gamma$ is easy to get: it suffices to write g as a product of the generators and then follow (from the vertex (v_1, v_2)) the sequence of symbols in $T_1 \times T_2$ corresponding to these generators. The vertex at the end of the path will be $g(v_1, v_2)$. Note however that this only works because (v_1, v_2) is the vertex associated to the identity element of Γ in the Cayley graph $T_1 \times T_2$. Given some $g \in \Gamma$ and another vertex $(w_1, w_2) \in V(T_1 \times T_2)$, the way to obtain $g(w_1, w_2)$ is to first localize $g(v_1, v_2)$ with the above procedure, and then to recall that Γ preserves the symbols in $T_1 \times T_2$. Hence, it suffices to look at the symbols on some path from (v_1, v_2) to (w_1, w_2) and to follow the same symbols from $g(v_1, v_2)$ so as to arrive at $g(w_1, w_2)$.

In particular, the action of an element $g \in \langle a_1, \dots, a_{d_1} \rangle = \Gamma(v_2)$ on T_2 can be obtained by doing the following. In order to compute $g(w)$ for some $w \in V(T_2)$, we draw a rectangle whose bottom side is labelled by the sequence of white symbols corresponding to g (from left to right) and whose right side is labelled by the sequence of black symbols on the path from v_2 to w in T_2 (from bottom to top), see Figure 2.3. Then we fill

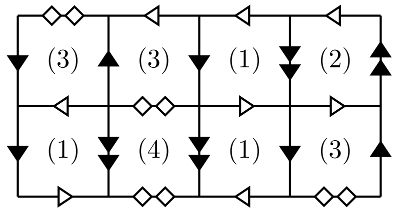


Figure 2.3: Example of a computation.

in the rectangle with the appropriate geometric squares (starting from the bottom-right corner). The rectangle that we obtain corresponds to a subcomplex of $T_1 \times T_2$: the bottom-left corner is (v_1, v_2), the bottom-right corner is $(g(v_1), v_2)$, and the top-right corner is $(g(v_1), g(w))$. Now T_2 corresponds to $v_1 \times T_2$ in $T_1 \times T_2$, so we can read $g(w)$ by looking at the left side of our rectangle. Indeed, the symbols on the path from v_2 to $g(w)$ in T_2 are exactly those on the left side of the rectangle (from bottom to top). Another way to explain why this idea works is to write h for the element of $\langle b_1, \ldots, b_{d_2} \rangle$ corresponding to the right side of the rectangle (i.e. $w = h(v_2)$ in T_2), h' for the element of $\langle b_1, \ldots, b_{d_2} \rangle$ corresponding to the left side, and g' for the element of $\langle a_1, \ldots, a_{d_1} \rangle$ corresponding to the top side. From (v_1, v_2), following g and then h leads to the same vertex as following h' and then g', so $gh = h'g'$. In particular we have $gh(v_2) = h'g'(v_2)$, which reduces to $g(w) = h'(v_2)$ as wanted. Similarly, the action of an element $g \in \langle b_1, \ldots, b_{d_2} \rangle = \Gamma(v_1)$ on T_1 can be obtained with the same idea.

This method is illustrated on Figure 2.3, with the $(3, 4)$-group Γ defined by the squares of Figure 2.2. On this figure we computed the image of vertex $b_1 b_2(v_2)$ by $a_1 A_2 a_1^{-1} A_2 \in \Gamma(v_2)$. The bottom side of the rectangle is indeed labelled by the symbols of $a_1 A_2 a_1^{-1} A_2$, and the right side by the symbols of $b_1 b_2$. After filling in the rectangle with the squares of Figure 2.2 (the numbers (1), (2), (3), (4) indicate which squares were used), it appears on the left side that the image of $b_1 b_2(v_2)$ is $b_1^{-2}(v_2)$. Note that this rectangle also shows, for instance, that the image of $A_2 a_1^{-3}(v_1)$ by $b_1^{-2} \in \Gamma(v_1)$ is $a_1 A_2 a_1^{-1} A_2(v_1)$.

2.3.2 Irreducibility

Recall that a (d_1, d_2)-group Γ is said to be reducible if it is commensurable to a product $\Gamma_1 \times \Gamma_2$ of lattices $\Gamma_t \leq \mathrm{Aut}(T_t)$. By [BM00b, Proposition 1.2], Γ is reducible if and only if $\mathrm{proj}_1(\Gamma)$ or $\mathrm{proj}_2(\Gamma)$ is discrete. If Γ is reducible, both projections are actually discrete. In some sense, the irreducible (d_1, d_2)-groups are the interesting ones. There is no known general algorithm deciding if a (d_1, d_2)-group is irreducible, but such an algorithm exists under suitable assumption on the local action.

We saw in the previous section that, for each $n \in \mathbf{Z}_{\geq 0}$, the action of all b_i (resp. a_i) on the ball $B(v_1, n)$ (resp. $B(v_2, n)$) of T_1 (resp. T_2) can be computed from the geometric squares. The group $\mathrm{proj}_t(\Gamma) \leq \mathrm{Aut}(T_t)$ is vertex-transitive, so it is discrete if and only if $\mathrm{Fix}_{\mathrm{proj}_t(\Gamma)}(B(v_t, n)) = \mathrm{Fix}_{\mathrm{proj}_t(\Gamma)}(B(v_t, n+1))$ for some $n \in \mathbf{Z}_{\geq 0}$. A way to show that some Γ is reducible thus consists in finding some $t \in \{1, 2\}$ and some $n \in \mathbf{Z}_{\geq 0}$ for which the latter equality is true.

Proving that a (d_1, d_2)-group Γ is irreducible is not easy in general, but there is a case where a good criterion exists: when $d_t \geq 6$ and $\underline{H_t}(v_t) \geq \mathrm{Alt}(d_t)$ for some $t \in \{1, 2\}$. Indeed, in this case it follows from [BM00a, Propositions 3.3.1 and 3.3.2] that H_t is non-discrete if and only if the image of $H_t(v_t)$ in $\mathrm{Aut}(B(v_t, 2))$ has order $\geq \frac{d_t!}{2} \left(\frac{(d_t-1)!}{2}\right)^{d_t}$. (The result is actually more subtle but we only need this lower bound here.) Moreover, once we know that H_t is non-discrete and that $\underline{H_t}(v_t) \geq \mathrm{Alt}(d_t)$ with $d_t \geq 6$, we actually get from Corollary 1.E$'$ that H_t is a member of $\mathcal{G}'_{(i)}$ for some legal coloring i of T_t (as defined in Definition 1.4.9). In the next section, we investigate the question of determining from the geometric squares which group of $\mathcal{G}'_{(i)}$ it actually is.

2.3.3 Recognizing a group in $\mathcal{G}'_{(i)}$

Let T be the d-regular tree for some $d \geq 3$, let i be a legal coloring of T and let H be an element of $\mathcal{G}'_{(i)}$ different from $\mathrm{Aut}(T)$. In particular H is transitive on $V(T)$. We fix some vertex $v \in V(T)$ and denote by K the smallest non-negative integer such that the homomorphism from $H(v)$ to $\mathrm{Aut}(B(v, K+1))$ is not surjective. Such an integer exists because $H \neq \mathrm{Aut}(T)$. It is equal to the two invariants $K(0)$ and $K(1)$ defined in §1.5.3. We also define $\rho = \frac{|\mathrm{Aut}(B(v,K+1))|}{|\tilde{H}^K(v)|}$, where $\tilde{H}^K(v)$ is the image of $H(v)$ in $\mathrm{Aut}(B(v, K+1))$ (as in §1.5.1). Let us finally define the homomorphism $s\colon H(v) \to (\mathbf{C}_2)^{K+1}$ by $s(h) = (s_0(h), \ldots, s_K(h))$ where $s_k(h) = \mathrm{Sgn}_{(i)}(h, S(v, k)) = \prod_{w \in S(v,k)} \mathrm{sgn}(\sigma_{(i)}(h, w))$, as in Definition 1.4.1. As Lemma 2.3.1 below shows, the value of $s_k(h)$ does not depend on the coloring i.

Lemma 2.3.1. *Let v be a vertex of the d-regular tree T and let $k \in \mathbf{Z}_{\geq 0}$. Consider some $h \in \mathrm{Aut}(T)(v)$. For each $w \in S(v, k)$, fix a bijection*

$\iota_w \colon S(w,1) \to \{1,\dots,d\}$. *Then the value of*

$$\prod_{w\in S(v,k)} \operatorname{sgn}(\iota_{h(w)} \circ h \circ \iota_w^{-1})$$

does not depend on the choices for the bijections ι_w.

Proof. Fix some other bijections $\iota'_w \colon S(w,1) \to \{1,\dots,d\}$, for each $w \in S(v,k)$. Then we have

$$\begin{aligned}
&\prod_{w\in S(v,k)} \operatorname{sgn}(\iota'_{h(w)} \circ h \circ \iota'^{-1}_w)\\
=&\prod_{w\in S(v,k)} \operatorname{sgn}(\iota'_{h(w)} \circ \iota^{-1}_{h(w)} \circ \iota_{h(w)} \circ h \circ \iota_w^{-1} \circ \iota_w \circ \iota'^{-1}_w)\\
=&\prod_{w\in S(v,k)} \operatorname{sgn}(\iota'_{h(w)} \circ \iota^{-1}_{h(w)}) \cdot \operatorname{sgn}(\iota_{h(w)} \circ h \circ \iota_w^{-1}) \cdot \operatorname{sgn}(\iota_w \circ \iota'^{-1}_w)\\
=&\prod_{w\in S(v,k)} \operatorname{sgn}(\iota_{h(w)} \circ \iota'^{-1}_{h(w)}) \cdot \operatorname{sgn}(\iota_{h(w)} \circ h \circ \iota_w^{-1}) \cdot \operatorname{sgn}(\iota_w \circ \iota'^{-1}_w)\\
=&\prod_{w\in S(v,k)} \operatorname{sgn}(\iota_{h(w)} \circ h \circ \iota_w^{-1}).
\end{aligned}$$

The last equality holds because, for each $w \in S(v,k)$, the term $\operatorname{sgn}(\iota_w \circ \iota'^{-1}_w)$ also appears as $\operatorname{sgn}(\iota_{h(h^{-1}(w))} \circ \iota'^{-1}_{h(h^{-1}(w))})$ in the product. □

We now prove in the following results that, when d is even, computing K, ρ and $s(H(v)) \leq (\mathbf{C}_2)^{K+1}$ (almost) suffices to recognize which group of $\mathcal{G}'_{(i)}$ we actually have. These invariants do not depend on the coloring i of T, which means that they can be computed without knowing for which coloring i the group H is contained in $\mathcal{G}'_{(i)}$.

Lemma 2.3.2. *Let $H = G_{(i)}(X,X)$ for some $X \subset_f \mathbf{Z}_{\geq 0}$. Then $K = \max X$, $\rho = 2$ and $s(H(v)) = \{(s_0,\dots,s_K) \in (\mathbf{C}_2)^{K+1} \mid \prod_{r\in X} s_r = 1\}$.*

Proof. This follows immediately from the definition of $G_{(i)}(X,X)$. □

For each $X \subset_f \mathbf{Z}_{\geq 0}$, we define $\alpha(X)$ as the subset of $\mathbf{Z}_{\geq 0}$ such that $S_{\alpha(X)}(v)$ is the set of vertices of T that appear in an odd number of sets $S_X(w_1),\dots,S_X(w_d)$, where $w_1,\dots,w_d$ are the d neighbors of v in T. In other words, $S_{\alpha(X)}(v)$ is the support of $\mathbf{1}_{S_X(w_1)} + \cdots + \mathbf{1}_{S_X(w_d)}$ mod 2 where $\mathbf{1}$ denotes the characteristic function. It is clear from its definition that this set of vertices indeed takes the form $S_{\alpha(X)}(v)$ for some $\alpha(X)$. In the next lemma we give an explicit expression for $\alpha(X)$, depending on the parity of d.

Lemma 2.3.3. *Let $X \subset_f \mathbf{Z}_{\geq 0}$. We have the following expressions for $\alpha(X)$, where $\triangle$ denotes the symmetric difference.*

$$\alpha(X) = \begin{cases} \{x+1 \mid x \in X\} \triangle \{x-1 \mid x \in X, x \geq 2\} & \text{if } d \text{ is even,} \\ \{x+1 \mid x \in X\} \cup (\{x-1 \mid x \in X\} \cap \{0\}) & \text{if } d \text{ is odd.} \end{cases}$$

Proof. For each $j \in \{1, \ldots, d\}$, we write $S_X(w_j) = S_X^+(w_j) \sqcup S_X^-(w_j)$, where $S_X^+(w_j)$ is the set of vertices of $S_X(w_j)$ that are further from v than from w_j, and $S_X^-(w_j) = S_X(w_j) \backslash S_X^+(w_j)$. Then, all the sets $S_X^+(w_j)$ with $j \in \{1, \ldots, d\}$ are disjoint and their union is $S_{\{x+1|x \in X\}}(v)$. Now if we look at the sets $S_X^-(w_j)$, they only contain vertices that are at distance $x-1$ from v for some $x \in X$ ($x \geq 1$). More precisely, if $x \in X$ and $x \geq 2$ then each vertex at distance $x-1$ from v is contained in exactly $d-1$ of the sets $S_X^-(w_j)$, $j \in \{1, \ldots, d\}$. Also, if $x = 1 \in X$, then v is contained in all d sets $S_X^-(w_j)$, $j \in \{1, \ldots, d\}$. These affirmations directly lead to the expressions given in the statement. $\square$

The next lemma then follows almost immediately.

Lemma 2.3.4. *We have $\alpha(X) \subset_f \mathbf{Z}_{\geq 0}$ for each $X \subset_f \mathbf{Z}_{\geq 0}$, and the map $\alpha\colon \{X \subset_f \mathbf{Z}_{\geq 0}\} \to \{X \subset_f \mathbf{Z}_{\geq 0}\}$ is injective. Moreover, we have*

$$\alpha(\{X \subset_f \mathbf{Z}_{\geq 0}\}) = \begin{cases} \{X \subset_f \mathbf{Z}_{\geq 0} \mid 0 \notin X\} & \text{if } d \text{ is even,} \\ \{X \subset_f \mathbf{Z}_{\geq 0} \mid 0 \in X \Leftrightarrow 2 \in X\} & \text{if } d \text{ is odd.} \end{cases}$$

Proof. From Lemma 2.3.3 we see that $\alpha(X)$ is finite and non-empty for each $X \subset_f \mathbf{Z}_{\geq 0}$ (because $\max X + 1 \in \alpha(X)$), i.e. $\alpha(X) \subset_f \mathbf{Z}_{\geq 0}$. Now remark from the definition of α that $\alpha(X \triangle X') = \alpha(X) \triangle \alpha(X')$ for each $X, X' \subset_f \mathbf{Z}_{\geq 0}$, where we define $\alpha(\varnothing) = \varnothing$. Therefore, if $\alpha(X) = \alpha(X')$, then $\alpha(X \triangle X') = \varnothing$ and hence $X \triangle X' = \varnothing$, i.e. $X = X'$.

The expressions for $\alpha(\{X \subset_f \mathbf{Z}_{\geq 0}\})$ can be found directly by examining Lemma 2.3.3. $\square$

Lemma 2.3.5 is then the reason why α was defined above.

Lemma 2.3.5. *Suppose that d is even and let H be one of $G_{(i)}(X^*, X^*)$, $G_{(i)}(X,X)^*$ and $G'_{(i)}(X,X)^*$ for some $X \subset_f \mathbf{Z}_{\geq 0}$. Then H is contained in $G_{(i)}(\alpha(X), \alpha(X))$.*

Proof. For any $h \in H$ and $v \in V(T)$, all $\mathrm{Sgn}_{(i)}(h, S(w_j, X))$ with $j \in \{1, \ldots, d\}$ are equal, where $w_1, \ldots, w_d$ are the d neighbors of v in T. Since d is even, the product of these d signatures is 1. This product is also equal to $\mathrm{Sgn}_{(i)}(h, S(v, \alpha(X)))$ (by definition of α), so we deduce that $h \in G_{(i)}(\alpha(X), \alpha(X))$. $\square$

From the previous lemma we can now compute $s(H(v))$ for other groups H in $\mathcal{G}'_{(i)}$ (when d is even).

Lemma 2.3.6. *Suppose that d is even and let H be one of $G_{(i)}(X^*, X^*)$, $G_{(i)}(X,X)^*$ and $G'_{(i)}(X,X)^*$ for some $X \subset_f \mathbf{Z}_{\geq 0}$. Then $K = \max X + 1$ and $s(H(v)) = \{(s_0, \ldots, s_K) \in (\mathbf{C}_2)^{K+1} \mid \prod_{r \in \alpha(X)} s_r = 1\}$. Moreover, we have $\rho = 2^{d-1}$ if $H = G_{(i)}(X^*, X^*)$ and $\rho = 2^d$ if $H = G_{(i)}(X,X)^*$ or $G'_{(i)}(X,X)^*$.*

Proof. The values of K and ρ can be directly deduced from the definitions of the groups. By definition of K, the homomorphism from $H(v)$ to $\mathrm{Aut}(B(v,K))$ is surjective. Hence, for each $(s_0, \ldots, s_{K-1}) \in (\mathbf{C}_2)^K$, there exists $s_K \in \mathbf{C}_2$ such that $(s_0, \ldots, s_K) \in s(H(v))$. If $H' = G_{(i)}(\alpha(X), \alpha(X))$, then Lemma 2.3.5 states that $H \subseteq H'$, so $s(H(v)) \subseteq s'(H'(v))$ (where $s' \colon H'(v) \to (\mathbf{C}_2)^{K+1}$ is the map associated to H'). Note that H and H' share the same K because $\max(\alpha(X)) = \max X + 1$. Now $s'(H'(v)) = \{(s_0, \ldots, s_K) \in (\mathbf{C}_2)^{K+1} \mid \prod_{r \in \alpha(X)} s_r = 1\}$ by Lemma 2.3.2, and in particular for each $(s_0, \ldots, s_{K-1}) \in (\mathbf{C}_2)^K$ there is a unique $s_K \in \mathbf{C}_2$ such that $(s_0, \ldots, s_K) \in s'(H'(v))$. From all this information it follows that $s(H(v)) = s'(H'(v))$. $\square$

When d is even, we see from Lemmas 2.3.2 and 2.3.6 that the groups in $\mathcal{G}'_{(i)}$ can be differentiated by computing K, ρ and the image of the map s, with one exception: $G_{(i)}(X,X)^*$ and $G'_{(i)}(X,X)^*$ have the same invariants (for a fixed $X \subset_f \mathbf{Z}_{\geq 0}$). This is due to the facts that their type-preserving subgroups are both equal to $G^+_{(i)}(X,X)^*$ and that all invariants are computed from vertex stabilizers. We will however be able to differentiate these two groups later, see Proposition 2.3.10.

For d odd, the task would be more complicated. For instance, $G_{(i)}(\{0,1\}^*, \{0,1\}^*)$ and $G_{(i)}(\{1\}^*, \{1\}^*)$ have the same invariants when d is odd ($K = 2$, $\rho = 2^{d-1}$ and s is surjective). This occurs because Lemma 2.3.5 is no longer true in that case. We will not deal with the odd case in this text.

2.3.4 Labelled graphs associated to a (d_1, d_2)-group

Let us come back to a (d_1, d_2)-group Γ associated to some (d_1, d_2)-datum $(A, B, \varphi_A, \varphi_B, R)$. We assume that $d_2 \geq 6$ is even, that $H_2 = \overline{\mathrm{proj}_2(\Gamma)}$ is non-discrete and that $\underline{H_2}(v_2) \geq \mathrm{Alt}(d_2)$. As explained in §2.3.2, the non-discreteness of H_2 can be checked by computing the action of $H_2(v_2)$ on $B(v_2, 2)$. We now would like an efficient algorithm to determine which group from $\mathcal{G}'_{(i)}$ is isomorphic to H_2. In this section, we give a way to compute $K^{(2)}$ (the K associated to H_2) and $s^{(2)}(H_2(v_2))$ by associating a labelled graph $G_\Gamma^{(2)}$ to our (d_1, d_2)-group Γ. In view of the results of §2.3.3, this will reduce to 4 (or less) the number of groups in $\mathcal{G}'_{(i)}$ that could be isomorphic to H_2. Of course, everything we do here for H_2 can be translated for H_1.

Given $h \in H_2(v_2)$ and $k \in \mathbf{Z}_{\geq 0}$, we write $s_k^{(2)}(h) = \mathrm{Sgn}_{(i)}(h, S(v_2, k))$ where i is any legal coloring of T_2 as in §2.3.3. The invariant $K^{(2)}$ can be characterized as the smallest non-negative integer such that the map

$$s^{(2)} \colon H_2(v_2) \to (\mathbf{C}_2)^{K^{(2)}+1} \colon h \mapsto (s_0^{(2)}(h), \dots s_{K^{(2)}}^{(2)}(h))$$

is not surjective. This indeed follows from the definition of $K^{(2)}$ and from Lemmas 2.3.2 and 2.3.6. An efficient algorithm to compute $s_k^{(2)}(a_j)$ for each $j \in \{1, \dots, d_1\}$ and each $k \in \mathbf{Z}_{\geq 0}$ would thus be sufficient to determine $K^{(2)}$ as well as $s^{(2)}(H_2(v_2))$. (Note that a_j should actually be read as $\mathrm{proj}_2(a_j)$ here, but we will omit the projection.) This is where the graph $G_\Gamma^{(2)}$, defined hereafter, becomes useful.

Let us define the **labelled graph** $G_\Gamma^{(2)}$ associated to Γ. First, the vertex set $V(G_\Gamma^{(2)})$ is simply defined to be A. Then, we put an edge between $a \in A$ and $a' \in A$ if and only if $|R \cap (\{a\} \times B \times \{a'^{-1}\} \times B)|$ is odd. Note that $|R\cap(\{a\}\times B\times\{a'^{-1}\}\times B)| = |R\cap(\{a'\}\times B\times\{a^{-1}\}\times B)|$ because R is invariant under the action of $\mathbf{C}_2\times\mathbf{C}_2$, so the edge set is well-defined. We obtain an undirected graph that can possibly contain loops (edges from a vertex to itself). Finally, to each vertex a of $G^{(2)}_\Gamma$, we associate a label $\sigma(a) = \pm 1$ whose value depends on the signature of the permutation that the generator $a \in \Gamma$ induces on $E(v_2)$ (the set of d_2 edges adjacent to v_2 in T_2). This labelled graph has a non-trivial automorphism defined by $a \mapsto a^{-1}$ for each $a \in A$. Indeed, we clearly have $\sigma(a) = \sigma(a^{-1})$ for each $a \in A$, and there is an edge between a and a' if and only if there is an edge between a^{-1} and a'^{-1}. Once again this follows from the fact that R is invariant under the action of $\mathbf{C}_2 \times \mathbf{C}_2$.

The labelled graph $G^{(2)}_\Gamma$ can easily be drawn from the geometric squares that define Γ. Indeed, the vertex set corresponds to the set of white symbols (with orientation) that the horizontal edges can have. For each vertex a, the permutation induced by a on $E(v_2)$ can also be directly computed from the geometric squares, see §2.3.1. We thus obtain the labels associated to the vertices by taking the signatures. Then, given two vertices a and a', we can determine if there is an edge between a and a' by counting the number of $b \in B$ such that there is a geometric square that can be read as $(a, b, a'^{-1}, *)$. We put an edge if and only if there is an odd number of such $b \in B$.

In the graph $G^{(2)}_\Gamma$, a **non-repeating path** p is a finite sequence of vertices $x_0, x_1, \ldots, x_n$ where x_{i-1} and x_i are connected by an edge for each $i \in \{1,\ldots,n\}$ and $x_{i-1} \neq x_{i+1}$ for each $i \in \{1,\ldots,n-1\}$. In other words, the path cannot use a same edge twice consecutively. Such a path has **length** n, **origin** x_0 and **destination** $d(p) = x_n$. We write $\mathcal{P}_n(x_0)$ for the set of all non-repeating paths of length n whose origin is x_0.

The next result now shows that the non-repeating paths in $G^{(2)}_\Gamma$ can be helpful in order to compute the values $s^{(2)}_k(a_j)$ defined above. Note that this proposition is true for any (d_1, d_2)-group.

Proposition 2.3.7. *Let Γ be a (d_1, d_2)-group and fix $j \in \{1,\ldots,d_1\}$ and $k \in \mathbf{Z}_{\geq 0}$. Then we have*

$$s^{(2)}_k(a_j) = \prod_{p\in\mathcal{P}_k(a_j)} \sigma(d(p)).$$

Proof. Given a sequence of vertices $x_0, \dots, x_n$ of $G_\Gamma^{(2)}$, we are first interested in rectangles $1 \times n$ (made of n geometric squares) appearing in $T_1 \times T_2$ and whose $n+1$ white symbols from bottom to top exactly correspond to the $n+1$ vertices $x_0, \dots, x_n$ (with the good orientations). Let us denote by $\mathrm{Rect}(x_0, \dots, x_n)$ the set of all those rectangles.

Claim. $|\mathrm{Rect}(x_0, \dots, x_n)|$ *is odd if and only if* $(x_0, \dots, x_n)$ *is a non-repeating path in* $G_\Gamma^{(2)}$*.*

Proof of the claim: We prove the claim by induction. For $n = 1$ it follows from the definition of the edge set of $G_\Gamma^{(2)}$. Now let $n \geq 2$ and assume the claim is true for $n-1$. Observe that

$$|\mathrm{Rect}(x_0, \dots, x_n)| = |\mathrm{Rect}(x_0, \dots, x_{n-1})| \cdot (|\mathrm{Rect}(x_{n-1}, x_n)| - \delta_{x_{n-2}, x_n}),$$

where $\delta_{x_{n-2}, x_n} = 1$ if $x_{n-2} = x_n$ and 0 otherwise. Indeed, a rectangle $1 \times n$ in $\mathrm{Rect}(x_0, \dots, x_n)$ is made of a rectangle $1 \times (n-1)$ in $\mathrm{Rect}(x_0, \dots, x_{n-1})$ and a square in $\mathrm{Rect}(x_{n-1}, x_n)$. The term -1 when $x_{n-2} = x_n$ appears because the square between x_{n-1} and x_n cannot be the same as the one between x_{n-1} and x_{n-2}.

Now if $(x_0, \dots, x_n)$ is a non-repeating path then $(x_0, \dots, x_{n-1})$ is a non-repeating path, $x_n \neq x_{n-2}$ and (x_{n-1}, x_n) is a non-repeating path. So $|\mathrm{Rect}(x_0, \dots, x_{n-1})|$ and $|\mathrm{Rect}(x_{n-1}, x_n)|$ are odd by hypothesis and hence $|\mathrm{Rect}(x_0, \dots, x_n)|$ is odd by the above formula.

Conversely, assume that $|\mathrm{Rect}(x_0, \dots, x_n)|$ is odd. By the formula above, this already means that $|\mathrm{Rect}(x_0, \dots, x_{n-1})|$ is odd and thus that $(x_0, \dots, x_{n-1})$ is a non-repeating path. Then there are two possibilities:

- If $x_{n-2} \neq x_n$ then we also get that $|\mathrm{Rect}(x_{n-1}, x_n)|$ is odd, so (x_{n-1}, x_n) is a non-repeating path. Altogether, these affirmations imply that $(x_0, \dots, x_n)$ is a non-repeating path.

- If $x_{n-2} = x_n$, we get that $|\mathrm{Rect}(x_{n-1}, x_n)|$ is even, so (x_{n-1}, x_n) is not a non-repeating path, i.e. there is no edge between x_{n-1} and x_n. This situation is however impossible: we already know that there is an edge between x_{n-2} and x_{n-1}, and $x_{n-2} = x_n$. ∎

We now prove the proposition. Recall from Lemma 2.3.1 that

$$s_k^{(2)}(a_j) = \prod_{w \in S(v_2,k)} \operatorname{sgn}(\iota_{a_j(w)} \circ a_j \circ \iota_w^{-1})$$

for any $k \in \mathbf{Z}_{\geq 0}$ and $j \in \{1, \ldots, d_1\}$, where $\iota_w \colon S(w,1) \to \{1, \ldots, d_2\}$ is any bijection for each $w \in S(v_2,k)$. In our context, we have a canonical choice for the bijections ι_w: the edges of T_2 are labelled by the black symbols, and the d_2 edges adjacent to any vertex carry the d_2 different black symbols (considered with their orientation). So the bijections ι_w can simply be defined by identifying $S(w,1)$ with $E(w)$ and the set $\{1, \ldots, d_2\}$ with the set of black symbols with orientation.

Take some $w \in S(v_2,k)$ and let $h \in \langle b_1, \ldots, b_{d_2} \rangle$ be the element such that $h(v_2) = w$. Let us draw, as in §2.3.1, the rectangle $1 \times k$ made of k geometric squares such that the bottom symbol corresponds to a_j, and the k symbols on the right side correspond to h. After having filled in the rectangle with geometric squares, we obtain the equation $a_j h = h' a_{j'}$, where $a_{j'}$ is given by the top side and h' by the left side of the rectangle. From the equality $a_j h(v_2) = h' a_{j'}(v_2)$ we obtain that $a_j(w) = h'(v_2)$. Moreover, since h and h' preserve the symbols in T_2, we have that $\iota_w^{-1} = h \iota_{v_2}^{-1}$ and $\iota_{a_j(w)}^{-1} = h' \iota_{v_2}^{-1}$. Using these equalities, we get

$$\iota_{a_j(w)} a_j \iota_w^{-1} = \iota_{v_2} h'^{-1} a_j h \iota_{v_2}^{-1} = \iota_{v_2} a_{j'} \iota_{v_2}^{-1}.$$

This implies that

$$\begin{aligned} s_k^{(2)}(a_j) &= \prod_{w \in S(v_2,k)} \operatorname{sgn}(\iota_{a_j(w)} \circ a_j \circ \iota_w^{-1}) \\ &= \prod_{(a_j, x_1, \ldots, x_k) \in V(G_\Gamma^{(2)})^{k+1}} \sigma(x_k)^{|\mathrm{Rect}(a_j, x_1, \ldots, x_k)|}. \end{aligned}$$

But $|\mathrm{Rect}(a_j, x_1, \ldots, x_k)|$ is odd if and only if $(a_j, x_1, \ldots, x_k)$ is a non-repeating path in $G_\Gamma^{(2)}$ (by the claim), so this leads to the formula

$$s_k^{(2)}(a_j) = \prod_{(a_j, x_1, \ldots, x_k) \in \mathcal{P}_k(a_j)} \sigma(x_k). \qquad \square$$

The graph $G_\Gamma^{(2)}$ has d_1 vertices and is somewhat redundant as it has a non-trivial automorphism defined by $a \mapsto a^{-1}$ for each $a \in A$. In the particular case where $a \neq a^{-1}$ for each $a \in A$, i.e. when $\tau_1 = 0$ (as defined in §2.2.1), we can define the **simplified labelled graph** $\tilde{G}_\Gamma^{(2)}$ associated to Γ as follows. The vertex set $V(\tilde{G}_\Gamma^{(2)})$ corresponds to the set of all $\{a, a^{-1}\}$ with $a \in A$, so that there are $\frac{d_1}{2}$ vertices. The label of $\{a, a^{-1}\}$ in $\tilde{G}_\Gamma^{(2)}$ is equal to the label of a (or a^{-1}) in $G_\Gamma^{(2)}$. Then, we put an edge between $\{a, a^{-1}\}$ and $\{a', a'^{-1}\}$ if and only if exactly one of a' and a'^{-1} is connected to a by an edge in $G_\Gamma^{(2)}$. This amounts to saying that $|R \cap (\{a\} \times B \times \{a', a'^{-1}\} \times B)|$ is odd. The automorphism of $G_\Gamma^{(2)}$ ensures that this edge set is well-defined. A **non-repeating path** in $\tilde{G}_\Gamma^{(2)}$ is defined exactly as in $G_\Gamma^{(2)}$, and we write $\tilde{\mathcal{P}}_n(x)$ for the set of all non-repeating paths in $\tilde{G}_\Gamma^{(2)}$ with length n and origin x.

The next proposition then shows that the values $s_k^{(2)}(a_j)$ can be computed from the simplified labelled graph $\tilde{G}_\Gamma^{(2)}$ when $\tau_1 = 0$.

Proposition 2.3.8. *Let Γ be a (d_1, d_2)-group with $\tau_1 = 0$ and fix $j \in \{1, \ldots, d_1\}$ and $k \in \mathbf{Z}_{\geq 0}$. Then we have*

$$s_k^{(2)}(a_j) = \prod_{p \in \tilde{\mathcal{P}}_k(\{a_j, a_j^{-1}\})} \sigma(d(p)).$$

Proof. Recall that $\tau_1 = 0$ means that $a \neq a^{-1}$ for all $a \in A$. Let us first focus on the (non-simplified) labelled graph $G_\Gamma^{(2)}$. Given an edge (x, y) in $G_\Gamma^{(2)}$, we say that (x, y) is **stylish** if (x, y^{-1}) is also an edge in $G_\Gamma^{(2)}$. Note that this also means that (x^{-1}, y) and (x^{-1}, y^{-1}) are edges in $G_\Gamma^{(2)}$. Let us say that a non-repeating path $(x_0, \ldots, x_n)$ in $G_\Gamma^{(2)}$ is **redundant** if there exists $i \in \{0, \ldots, n-1\}$ such that (x_i, x_{i+1}) is stylish. Given such a redundant non-repeating path, we let $i \geq 1$ be the smallest number such that (x_i, x_{i+1}) is stylish, and $j \leq n$ be the greatest number such that all edges (x_i, x_{i+1}), (x_{i+1}, x_{i+2}), …, (x_{j-1}, x_j) are stylish. Then we define the **mirror** $m(p)$ of the path $p = (x_0, \ldots, x_n)$ to be

$$(x_0, \ldots, x_i, x_{i+1}^{-1}, x_{i+2}, x_{i+3}^{-1}, \ldots, x_{j-1}^{-1}, x_j, x_{j+1}, \ldots, x_n)$$

if $j \equiv i \bmod 2$ and

$$(x_0, \ldots, x_i, x_{i+1}^{-1}, x_{i+2}, x_{i+3}^{-1}, \ldots, x_{j-1}, x_j^{-1}, x_{j+1}^{-1}, \ldots, x_n^{-1})$$

if $j \not\equiv i \bmod 2$. The mirror of a redundant non-repeating path is still a redundant non-repeating path, and taking the mirror is an involution. This means that, if we look at the formula

$$s_k^{(2)}(a_j) = \prod_{p \in \mathcal{P}_k(a_j)} \sigma(d(p)).$$

given by Proposition 2.3.7, we can simply compute the product over the non-redundant non-repeating path in $\mathcal{P}_k(a_j)$. Indeed, the redundant ones come by pairs $(p, m(p))$, and $d(p) = d(m(p))$ or $d(m(p))^{-1}$ so that p and $m(p)$ give the same signs. In order to conclude, there remains to observe that the map

$$(a_j, x_1, \ldots, x_n) \mapsto (\{a_j, a_j^{-1}\}, \{x_1, x_1^{-1}\}, \ldots, \{x_n, x_n^{-1}\})$$

defines a bijection between the set of non-redundant non-repeating paths in $G_\Gamma^{(2)}$ with origin a_j and the set of non-repeating paths in $\tilde{G}_\Gamma^{(2)}$ with origin $\{a_j, a_j^{-1}\}$. This is a simple exercise. □

Thanks to Proposition 2.3.7 (or 2.3.8), computing the invariants $K^{(2)}$ and $s^{(2)}\colon H_2(v_2) \to (\mathbf{C}_2)^{K^{(2)}+1}$ from the geometric squares defining Γ is easy. For small d_1 and d_2 this can be done by hand, as illustrated in §2.3.6. We also know in advance, from Lemmas 2.3.2 and 2.3.6, that $s^{(2)}(H_2(v_2))$ must take the form $\{(s_0, \ldots, s_{K^{(2)}}) \in (\mathbf{C}_2)^{K^{(2)}+1} \mid \prod_{r \in X} s_r = 1\}$ for some $X \subset_f \mathbf{Z}_{\geq 0}$ with $\max X = K^{(2)}$. If $0 \in X$, then $X \notin \alpha(\{Y \subset_f \mathbf{Z}_{\geq 0}\})$ by Lemma 2.3.4 and thus the only possibility for H_2 is to be equal to $G_{(i)}(X, X)$ for some legal coloring i of T_2. On the other hand, if $0 \notin X$ then there exists a unique $Y \subset_f \mathbf{Z}_{\geq 0}$ such that $\alpha(Y) = X$ (once again by Lemma 2.3.4) and we conclude that H_2 is equal to one of the four groups $G_{(i)}(X, X)$, $G_{(i)}(Y^*, Y^*)$, $G_{(i)}(Y, Y)^*$ and $G'_{(i)}(Y, Y)^*$ for some legal coloring i of T_2. Then it is still possible to compute the invariant $\rho^{(2)}$: if $\rho^{(2)} = 2$ then $H_2 = G_{(i)}(X, X)$, if $\rho^{(2)} = 2^{d_2}$ then $H_2 = G_{(i)}(Y^*, Y^*)$, and if $\rho^{(2)} = 2^{d_2-1}$ then $H_2 = G_{(i)}(Y, Y)^*$ or

$G'_{(i)}(Y,Y)^*$. However, computing $\rho^{(2)}$ in general requires a computer, and even a computer is too slow if $K^{(2)}$ is big. In the next subsection, we see how to identify by hand which of the four groups is the good one.

2.3.5 Choosing among the four possible groups

Let us suppose we are in presence of a (d_1,d_2)-group as in §2.3.4 and such that $s^{(2)}(H_2(v_2)) = \{(s_0,\dots,s_{K^{(2)}}) \in (\mathbf{C}_2)^{K^{(2)}+1} \mid \prod_{r\in X} s_r = 1\}$ for some $X \subset_f \mathbf{Z}_{\geq 0}$ with $\max X = K^{(2)}$ and $0 \notin X$. Let $Y \subset_f \mathbf{Z}_{\geq 0}$ be such that $\alpha(Y) = X$. Our goal is now to give a method enabling us to determine which of the four groups $G_{(i)}(X,X)$, $G_{(i)}(Y^*,Y^*)$, $G_{(i)}(Y,Y)^*$ and $G'_{(i)}(Y,Y)^*$ is isomorphic to H_2.

We start with the following proposition which, in some sense, enables to compute the invariant $\rho^{(2)} \in \{1, 2^{d_2-1}, 2^{d_2}\}$.

Proposition 2.3.9. *Let* Γ *be a* (d_1,d_2)*-group with* $d_2 \geq 6$ *even and suppose that* $H_2 = \overline{\mathrm{proj}_2(\Gamma)}$ *is non-discrete and satisfies* $\underline{H_2}(v_2) \geq \mathrm{Alt}(d_2)$*. Let* $X \subset_f \mathbf{Z}_{\geq 0}$ *be such that* $\max X = K^{(2)}$ *and*

$$s^{(2)}(H_2(v_2)) = \left\{ (s_0,\dots,s_{K^{(2)}}) \in (\mathbf{C}_2)^{K^{(2)}+1} \;\middle|\; \prod_{r\in X} s_r = 1 \right\},$$

and assume that $0 \notin X$*. Let* $Y \subset_f \mathbf{Z}_{\geq 0}$ *be such that* $\alpha(Y) = X$*.*

For each $j \in \{1,\dots,d_1\}$*, define* $\Sigma_j = \prod_{r\in Y} s_r^{(2)}(a_j) \in \{-1,1\}$*. Also, for each* $j \in \{1,\dots,d_1\}$ *and each* $k \in \{1,\dots,d_2\}$*, define* $\mu_{j,k} \in \{1,\dots,d_1\}$ *and* $\nu_{j,k} \in \{1,\dots,d_2\}$ *so that* $a_j b_k = b_{\nu_{j,k}} a_{\mu_{j,k}}$*.*

Then exactly one of the following assertions holds.

(1) There exist $x_1,\dots,x_{d_2} \in \{-1,1\}$ *such that*

$$(*) \quad \begin{cases} x_1 x_{\nu_{1,1}} \Sigma_{\mu_{1,1}} = x_2 x_{\nu_{1,2}} \Sigma_{\mu_{1,2}} = \cdots = x_{d_2} x_{\nu_{1,d_2}} \Sigma_{\mu_{1,d_2}} = \Sigma_1 \\ x_1 x_{\nu_{2,1}} \Sigma_{\mu_{2,1}} = x_2 x_{\nu_{2,2}} \Sigma_{\mu_{2,2}} = \cdots = x_{d_2} x_{\nu_{2,d_2}} \Sigma_{\mu_{2,d_2}} = \Sigma_2 \\ \quad\vdots \\ x_1 x_{\nu_{d_1,1}} \Sigma_{\mu_{d_1,1}} = x_2 x_{\nu_{d_1,2}} \Sigma_{\mu_{d_1,2}} = \cdots = x_{d_2} x_{\nu_{d_1,d_2}} \Sigma_{\mu_{d_1,d_2}} = \Sigma_{d_1} \end{cases}$$

and $H_2 = G_{(i)}(Y,Y)^*$ *or* $G'_{(i)}(Y,Y)^*$ *for some legal coloring* i *of* T_2*.*

(2) There exist no $x_1, \ldots, x_{d_2} \in \{-1,1\}$ *satisfying* $(*)$ *but there exist* $x_1, \ldots, x_{d_2} \in \{-1,1\}$ *such that*

$$(**)\quad \begin{cases} x_1 x_{\nu_{1,1}} \Sigma_{\mu_{1,1}} = x_2 x_{\nu_{1,2}} \Sigma_{\mu_{1,2}} = \cdots = x_{d_2} x_{\nu_{1,d_2}} \Sigma_{\mu_{1,d_2}} \\ x_1 x_{\nu_{2,1}} \Sigma_{\mu_{2,1}} = x_2 x_{\nu_{2,2}} \Sigma_{\mu_{2,2}} = \cdots = x_{d_2} x_{\nu_{2,d_2}} \Sigma_{\mu_{2,d_2}} \\ \quad \vdots \\ x_1 x_{\nu_{d_1,1}} \Sigma_{\mu_{d_1,1}} = x_2 x_{\nu_{d_1,2}} \Sigma_{\mu_{d_1,2}} = \cdots = x_{d_2} x_{\nu_{d_1,d_2}} \Sigma_{\mu_{d_1,d_2}} \end{cases}$$

and $H_2 = G_{(i)}(Y^*, Y^*)$ *for some legal coloring* i *of* T_2*.*

(3) There exist no $x_1, \ldots, x_{d_2} \in \{-1,1\}$ *satisfying* $(*)$ *or* $(**)$*, and* $H_2 = G_{(i)}(X, X)$ *for some legal coloring* i *of* T_2*.*

Proof. For each $w \in V(T_2)$, we define $\iota_w \colon S(w,1) \to \{1, \ldots, d_2\}$ as before, i.e. so that the edge $e \in E(w)$ is labelled by the black symbol corresponding to $b_{\iota_w(z)}$, where z is the vertex of e different from w.

For i a legal coloring of T_2 and $k \in \{1, \ldots, d_2\}$, we define $x_k^{(i)} = \prod_{w \in S(b_k(v_2), Y)} \operatorname{sgn}(i \circ \iota_w^{-1}) \in \{-1,1\}$. It is clear that any element of $\{-1,1\}^{d_2}$ can be written as $(x_1^{(i)}, \ldots, x_{d_2}^{(i)})$ for some legal coloring i. Now for such a coloring, we write $i_w \colon S(w,1) \to \{1, \ldots, d_2\}$ for the restriction of i to $S(w,1)$, and compute

$$\begin{aligned}
&\prod_{w \in S(b_k(v_2), Y)} \operatorname{sgn}(i_{a_j(w)} \circ a_j \circ i_w^{-1}) \\
&= \prod_{w \in S(b_k(v_2), Y)} \operatorname{sgn}(i_{a_j(w)} \circ \iota_{a_j(w)}^{-1}) \cdot \operatorname{sgn}(\iota_{a_j(w)} \circ a_j \circ \iota_w^{-1}) \cdot \operatorname{sgn}(\iota_w \circ i_w^{-1}) \\
&= x_k^{(i)} x_{\nu_{j,k}}^{(i)} \prod_{w \in S(b_k(v_2), Y)} \operatorname{sgn}(\iota_{a_j(w)} \circ a_j \circ \iota_w^{-1}) \\
&= x_k^{(i)} x_{\nu_{j,k}}^{(i)} \prod_{w \in S(v_2, Y)} \operatorname{sgn}(\iota_{a_j b_k(w)} \circ a_j \circ \iota_{b_k(w)}^{-1}) \\
&= x_k^{(i)} x_{\nu_{j,k}}^{(i)} \prod_{w \in S(v_2, Y)} \operatorname{sgn}(\iota_{b_{\nu_{j,k}} a_{\mu_{j,k}}(w)} \circ b_{\nu_{j,k}} \circ a_{\mu_{j,k}} \circ b_k^{-1} \circ \iota_{b_k(w)}^{-1}) \\
&= x_k^{(i)} x_{\nu_{j,k}}^{(i)} \prod_{w \in S(v_2, Y)} \operatorname{sgn}(\iota_{a_{\mu_{j,k}}(w)} \circ a_{\mu_{j,k}} \circ \iota_w^{-1}) \\
&= x_k^{(i)} x_{\nu_{j,k}}^{(i)} \Sigma_{\mu_{j,k}}.
\end{aligned}$$

This implies that, if $H_2 = G_{(i)}(Y,Y)^*$ or $G'_{(i)}(Y,Y)^*$, then $(x_1^{(i)}, \ldots, x_{d_2}^{(i)})$

is a solution of $(*)$. Conversely, if $(x_1^{(i)}, \ldots, x_{d_2}^{(i)})$ is a solution of $(*)$ for some coloring i, then the equalities defining $G_{(i)}(Y,Y)^*$ are true in $B(v_2, \max Y+2)$ and we can deduce in particular that $\rho^{(2)} \geq 2^{d_2}$. In view of Lemmas 2.3.2 and 2.3.6, the only options for H_2 are then $G_{(i)}(Y,Y)^*$ and $G'_{(i)}(Y,Y)^*$ (for some coloring i that may be different).

Now if we assume that $(*)$ has no solution, H_2 is different from $G_{(i)}(Y,Y)^*$ and $G'_{(i)}(Y,Y)^*$. Then by the same argument we obtain that $(**)$ has a solution if and only if $H_2 = G_{(i)}(Y^*,Y^*)$. In the case where neither $(*)$ nor $(**)$ has a solution, the only remaining possibility is to have $H_2 = G_{(i)}(X,X)$. $\square$

The next proposition then explains how $G_{(i)}(Y,Y)^*$ and $G'_{(i)}(Y,Y)^*$ can be distinguished. As explained earlier, this requires observing an element exchanging the two types of vertices.

Proposition 2.3.10. *Let Γ be a (d_1,d_2)-group as in Proposition 2.3.9, and assume that $H_2 = G_{(i)}(Y,Y)^*$ or $G'_{(i)}(Y,Y)^*$ for some legal coloring i of T_2. Let $k \in \{1,\ldots,d_2\}$, $m \geq 0$ and $j_1,\ldots,j_m,j'_1,\ldots,j'_m \in \{1,\ldots,d_1\}$ be such that*

$$a_{j_1}\cdots a_{j_m} b_k = b_k^{-1} a'_{j'_1}\cdots a'_{j'_m}.$$

Then $H_2 = G_{(i)}(Y,Y)^*$ if and only if $\Sigma_{j_1}\cdots\Sigma_{j_m}\Sigma_{j'_1}\cdots\Sigma_{j'_m} = 1$, where $\Sigma_j = \prod_{r\in Y} s_r^{(2)}(a_j) \in \{-1,1\}$ for each $j \in \{1,\ldots,d_1\}$.

Proof. The element $\gamma = \mathrm{proj}_2(a_{j_1}\cdots a_{j_m} b_k) \in H_2$ sends $v_2 \in V(T_2)$ to one of its neighbors, say w. In particular, γ exchanges the types of vertices in T_2. Moreover, the hypothesis implies that

$$\gamma^2 = \mathrm{proj}_2(a_{j_1}\cdots a_{j_m} a'_{j'_1}\cdots a'_{j'_m}).$$

So γ^2 fixes v_2 (i.e. γ exchanges v_2 and w) and

$$\mathrm{Sgn}_{(i)}(\gamma^2, S_Y(v_2)) = \Sigma_{j_1}\cdots\Sigma_{j_m}\Sigma_{j'_1}\cdots\Sigma_{j'_m}.$$

The conclusion then follows from the definitions of the groups $G_{(i)}(Y,Y)^*$ and $G'_{(i)}(Y,Y)^*$. Indeed, if $\gamma \in G_{(i)}(Y,Y)^*$ then $\mathrm{Sgn}_{(i)}(\gamma, S_Y(v_2)) = \mathrm{Sgn}_{(i)}(\gamma, S_Y(w))$ and hence $\mathrm{Sgn}_{(i)}(\gamma^2, S_Y(v_2)) = 1$. On the contrary, if

$\gamma \in G'_{(i)}(Y,Y)^*$ then $\mathrm{Sgn}_{(i)}(\gamma, S_Y(v_2)) \neq \mathrm{Sgn}_{(i)}(\gamma, S_Y(w))$ and in that case $\mathrm{Sgn}_{(i)}(\gamma^2, S_Y(v_2)) = -1$. $\square$

Note that there always exist elements $k \in \{1,\dots,d_2\}$, $m \geq 0$ and $j_1,\dots,j_m, j'_1,\dots,j'_m \in \{1,\dots,d_1\}$ as in Proposition 2.3.10: they simply correspond to a rectangle $m \times 1$ in the square-complex $T_1 \times T_2$ with the property that the left and right edges of the rectangle correspond to b_k^{-1} and b_k for some $k \in \{1,\dots,d_2\}$. The existence of such a rectangle is a consequence of the transitivity of $\underline{H_2}(v_2) \geq \mathrm{Alt}(d_2)$ on $E(v_2)$.

All our previous considerations lead to Theorem 2.A.

Proof of Theorem 2.A. As explained in §2.3.2, it suffices to look at the image of $H_t(v_t)$ in $\mathrm{Aut}(B(v_t,2))$ to see if Γ is reducible or irreducible, and this can be done with the method explained in §2.3.1. So (i) is clear. For (ii), we saw in §2.3.4 that computing the labelled graph $G_\Gamma^{(t)}$ gives at most 4 possibilities for H_t, and in Propositions 2.3.9 and 2.3.10 that choosing among the four groups could be done by solving two systems whose unknowns belong to $\{-1,1\}$ and constructing a suitable $m \times 1$ or $1 \times m$ rectangle. It is not hard to implement those algorithms on a computer and they have a pretty good complexity. We do not go into a detailed analysis of the complexity, but the slowest part of the algorithm is probably to check the irreducibility of Γ by observing the action of $H_t(v_t)$ on $B(v_t,2)$ (which has $d_t(d_t-1)$ leaves). $\square$

We end this section with a particular case where Propositions 2.3.9 and 2.3.10 always give the same conclusion.

Corollary 2.3.11. *Let Γ be a (d_1,d_2)-group as in Proposition 2.3.9. If $\prod_{r\in Y} s_r^{(2)}(a_j) = -1$ for each $j \in \{1,\dots,d_1\}$, then $H_2 = G_{(i)}(Y,Y)^*$ for some legal coloring i of T_2.*

Proof. In Proposition 2.3.9, we have $\Sigma_j = -1$ for each $j \in \{1,\dots,d_1\}$ and thus $x_1 = \cdots = x_{d_2} = 1$ is a solution of $(*)$. Moreover, in Proposition 2.3.10 we directly get $\Sigma_{j_1}\cdots\Sigma_{j_m}\Sigma_{j'_1}\cdots\Sigma_{j'_m} = (-1)^{2m} = 1$ which ends the proof. $\square$

2.3.6 Illustration on an example

Let us illustrate the previous ideas on a concrete example. Let Γ be the torsion-free $(6,6)$-group corresponding to the 9 geometric squares drawn in Figure 2.4. Our goal is to explain how H_1 and H_2 can be computed in this particular case.

We start by computing the action of $H_2(v_2)$ on $B(v_2,1)$. The six vertices in $S(v_2,1)$ are $b_1(v_2)$, $b_2(v_2)$, $b_3(v_2)$, $b_3^{-1}(v_2)$, $b_2^{-1}(v_2)$ and $b_1^{-1}(v_2)$. The six edges from v_2 to these six vertices are labelled by the six black symbols (with orientation) corresponding to the six generators b_1, b_2, b_3, $b_4 = b_3^{-1}$, $b_5 = b_2^{-1}$ and $b_6 = b_1^{-1}$. From the geometric squares, we directly get the actions of a_1, a_2 and a_3 on these six edges. We denote them by the generators to which they correspond.

$$\begin{aligned} a_1 &: (b_1)(b_1^{-1})(b_2)(b_2^{-1})(b_3\ b_3^{-1}) \\ a_2 &: (b_1)(b_1^{-1}\ b_2\ b_3\ b_3^{-1}\ b_2^{-1}) \\ a_3 &: (b_1\ b_2^{-1}\ b_3^{-1}\ b_3\ b_2)(b_1^{-1}) \end{aligned}$$

One can easily check that these three permutations generate $\mathrm{Sym}(6)$. As explained in §2.3.2, it then suffices to compute the action of $H_2(v_2)$ on $B(v_2,2)$ and to use [BM00a, Proposition 3.3.2] to conclude that H_2 is non-discrete and thus that Γ is irreducible.

Let us now find out which group H_2 exactly is. As $\tau_1 = 0$, we can compute the simplified labelled graph $\tilde{G}_\Gamma^{(2)}$ and use Proposition 2.3.8. We also compute $G_\Gamma^{(2)}$ so as to illustrate Proposition 2.3.7 as well. The two graphs we obtain are given in Figure 2.5.

From $\tilde{G}_\Gamma^{(2)}$ (or $G_\Gamma^{(2)}$) we can compute the values of $s_k^{(2)}(a_j)$ for $j \in$

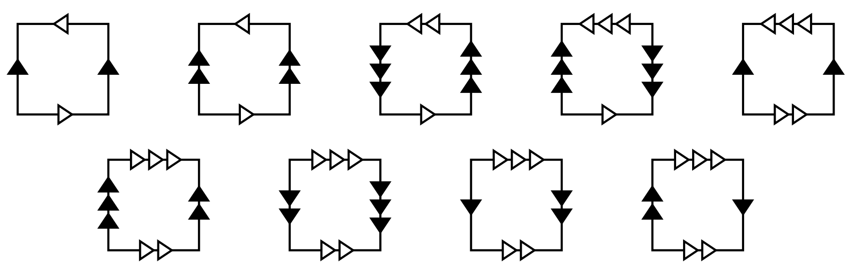


Figure 2.4: The geometric squares of a torsion-free $(6,6)$-group.

$\{1,2,3\}$ and $k \in \mathbf{Z}_{\geq 0}$. For $k \in \{0,1,2,3\}$ we obtain:

	$s_0^{(2)}$	$s_1^{(2)}$	$s_2^{(2)}$	$s_3^{(2)}$
a_1	-1	$+1$	$+1$	$+1$
a_2	$+1$	-1	-1	$+1$
a_3	$+1$	-1	-1	$+1$

The map $s^{(2)}\colon H_2(v_2) \to (\mathbf{C}_2)^3\colon h \mapsto (s_0^{(2)}(h), s_1^{(2)}(h), s_2^{(2)}(h))$ is not surjective, so $K^{(2)} = 2$. Moreover, we see that $s^{(2)}(H_2(v_2))$ is equal to $\left\{(s_0,s_1,s_2) \in (\mathbf{C}_2)^3 \;\middle|\; \prod_{r\in\{1,2\}} s_r = 1\right\}$. Since $\alpha(\{0,1\}) = \{1,2\}$ (by Lemma 2.3.3), these values for $K^{(2)}$ and $s^{(2)}(H_2(v_2))$ imply that H_2 is one of $G_{(i_2)}(\{1,2\},\{1,2\})$, $G_{(i_2)}(\{0,1\}^*,\{0,1\}^*)$, $G_{(i_2)}(\{0,1\},\{0,1\})^*$ and $G'_{(i_2)}(\{0,1\},\{0,1\})^*$ for some legal coloring i_2 of T_2. But for each $j \in \{1,2,3\}$ we have $\prod_{r\in\{0,1\}} s_r^{(2)}(a_j) = -1$, so Corollary 2.3.11 ensures that $H_2 = G_{(i_2)}(\{0,1\},\{0,1\})^*$.

Let us do the same work for H_1. The action of $H_1(v_1)$ on $B(v_1,1)$ is given by the following permutations: they also generate $\mathrm{Sym}(6)$.

$$\begin{aligned} b_1 &: (a_1\ a_1^{-1})(a_2\ a_3\ a_2^{-1}\ a_3^{-1}) \\ b_2 &: (a_1\ a_1^{-1})(a_2\ a_3)(a_2^{-1}\ a_3^{-1}) \\ b_3 &: (a_1\ a_2^{-1}\ a_3^{-1})(a_1^{-1}\ a_3\ a_2) \end{aligned}$$

As $\tau_2 = 0$, we can compute the simplified labelled graph $\tilde{G}_\Gamma^{(1)}$. It has no edge, and exactly one of the three vertices is labelled by -1: the one corresponding to $\{b_2, b_2^{-1}\}$. The values of $s_k^{(1)}(b_j)$ for $j \in \{1,2,3\}$ and

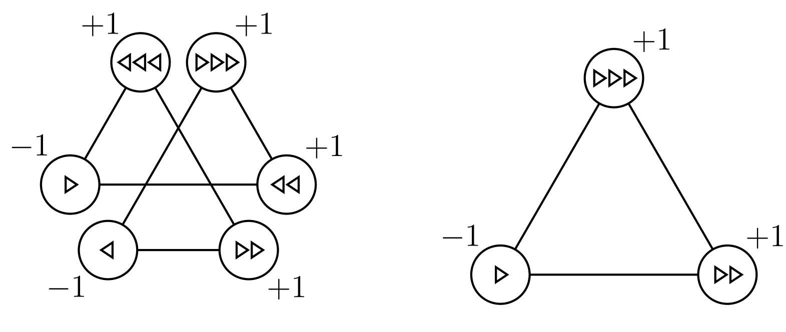


Figure 2.5: The labelled graphs $G_\Gamma^{(2)}$ and $\tilde{G}_\Gamma^{(2)}$.

$k \in \{0,1,2,3\}$ are therefore:

	$s_0^{(1)}$	$s_1^{(1)}$	$s_2^{(1)}$	$s_3^{(1)}$
b_1	$+1$	$+1$	$+1$	$+1$
b_2	-1	$+1$	$+1$	$+1$
b_3	$+1$	$+1$	$+1$	$+1$

The map $s^{(1)}\colon H_1(v_1) \to (\mathbf{C}_2)^2\colon h \mapsto (s_0^{(1)}(h), s_1^{(1)}(h))$ is not surjective, so $K^{(1)} = 1$. Also, $s^{(1)}(H_1(v_1)) = \left\{(s_0, s_1) \in (\mathbf{C}_2)^2 \,\middle|\, \prod_{r\in\{1\}} s_r = 1\right\}$. Since $\alpha(\{0\}) = \{1\}$, we obtain that H_1 must be one of $G_{(i_1)}(\{1\},\{1\})$, $G_{(i_1)}(\{0\}^*,\{0\}^*)$, $G_{(i_1)}(\{0\},\{0\})^*$ and $G'_{(i_1)}(\{0\},\{0\})^*$ for some legal coloring i_1 of T_1. This time we do not have $\prod_{r\in\{0\}} s_r^{(1)}(b_j) = -1$ for each $j \in \{1,2,3\}$, so Corollary 2.3.11 cannot be used. We therefore need Proposition 2.3.9. After looking carefully at the geometric squares, the system $(*)$ in Proposition 2.3.9 is

$$\begin{cases} x_1x_6 = -x_2x_3 = x_3x_5 = x_4x_2 = -x_5x_4 = x_6x_1 = 1 \\ -x_1x_6 = x_2x_3 = x_3x_2 = x_4x_5 = x_5x_4 = -x_6x_1 = -1 \\ x_1x_5 = x_2x_6 = -x_3x_2 = x_4x_1 = -x_5x_4 = x_6x_3 = 1 \end{cases}$$

From $x_1x_5 = x_4x_1$ it follows that $x_4 = x_5$, but this contradicts $x_4x_5 = -1$ so this system has no solution. Hence the groups $G_{(i_1)}(\{0\},\{0\})^*$ and $G'_{(i_1)}(\{0\},\{0\})^*$ can be excluded. The system $(**)$ in Proposition 2.3.9 is exactly the same, but without the last equality on each line:

$$\begin{cases} x_1x_6 = -x_2x_3 = x_3x_5 = x_4x_2 = -x_5x_4 = x_6x_1 \\ -x_1x_6 = x_2x_3 = x_3x_2 = x_4x_5 = x_5x_4 = -x_6x_1 \\ x_1x_5 = x_2x_6 = -x_3x_2 = x_4x_1 = -x_5x_4 = x_6x_3 \end{cases}$$

A solution to this system is given by $(1,1,1,-1,-1,-1)$, so we conclude that $H_1 = G_{(i_1)}(\{0\}^*,\{0\}^*)$.

We summarize our computations in the following lemma.

Lemma 2.3.12. *Let* $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ *be the torsion-free* $(6,6)$*-group defined by Figure 2.4. Then* $H_1 = G_{(i_1)}(\{0\}^*,\{0\}^*)$ *and* $H_2 = G_{(i_2)}(\{0,1\},\{0,1\})^*$ *for some legal colorings* i_1 *and* i_2 *of* T_1 *and* T_2.

Proof. See the discussion above. ∎

2.3.7 An inventory of possible projections

As already mentioned, we do not have any tool to check the irreducibility of a (d_1, d_2)-group in full generality. With a computer it can be quickly seen if, for some $t \in \{1, 2\}$ the fixator of $B(v_t, 2)$ in H_t is trivial. Indeed, since H_t is transitive on vertices of T_t, it suffices to see whether the fixator of $B(v_t, 2)$ also fixes $B(v_t, 3)$. We will therefore say in this section that some (d_1, d_2)-group Γ is **possibly irreducible** if the fixator of $B(v_t, 2)$ in H_t is non-trivial for each $t \in \{1, 2\}$.

We always consider (d_1, d_2)-groups up to equivalence (i.e. up to conjugation in $\mathrm{Aut}(T_1 \times T_2)$). For some values of d_1 and d_2, we could compute the total number of torsion-free (d_1, d_2)-groups and the number of (d_1, d_2)-groups with torsion (up to equivalence) by enumerating them all (thanks to the GAP system). Some of these groups can be seen to be reducible by simply showing that they are not possibly irreducible. Also, when $d_t = 6$ for some $t \in \{1, 2\}$, some (d_1, d_2)-groups can be proved to be irreducible, as explained previously, when $\underline{H_t}(v_t) \geq \mathrm{Alt}(d_t)$. The results we obtained are given in Table 2.1. The numbers in parentheses correspond to (d_1, d_2)-groups with torsion but with $\tau_1 = \tau_2 = 0$ (i.e. without generators of order 2). Indeed, for $(d_1, d_2) \in \{(4, 6), (6, 6)\}$ the number of (d_1, d_2)-groups is so big that we could not count them all up to equivalence. Note that we actually know a bit more that what is written in Table 2.1. For instance, we will see in §2.4.2 below that at least 60 of the 23839 possibly irreducible $(4, 5)$-groups are irreducible.

	Torsion-free				With torsion			
	Irred.	?	Red.	Total	Irred.	?	Red.	Total
$(3,3)$	-	-	-	-	0	4	56	60
$(3,4)$	-	-	-	-	0	59	664	723
$(3,5)$	-	-	-	-	0	457	1986	2443
$(3,6)$	-	-	-	-	204	3018	10529	13751
$(4,4)$	0	2	50	52	0	686	2992	3678
$(4,5)$	-	-	-	-	0	23839	34700	58539
$(4,6)$	16	95	890	1001	(111)	(433)	(1840)	(2384)
$(6,6)$	8227	5409	18426	32062	(83581)	(33565)	(76037)	(193083)

Table 2.1: (d_1, d_2)-groups up to equivalence.

In the remainder of this section, we give tables with the possible pairs of projections (H_1, H_2) that a possibly irreducible (d_1, d_2)-group can have, for some values of d_1 and d_2. The idea is, for each (equivalence class of) (d_1, d_2)-group, to check that it is possibly irreducible, to compute H_1 and H_2 (if possible) and to write in the table that there is a group with these two projections. However, we only saw in the previous sections how to determine H_t in the particular case where $d_t \geq 6$, $\underline{H_t}(v_t) \geq \mathrm{Alt}(d_t)$ and H_t is non-discrete. In all other cases, we therefore only take note of the local action $\underline{H_t}(v_t) \leq \mathrm{Sym}(d_t)$.

In order to make the rendering of the tables better, let us introduce some notation. For groups in $\mathcal{G}'_{(i)}$, we use the following abbreviations:

Notation	Group
X	$G_{(i)}(X, X)$
X^*	$G_{(i)}(X, X)^*$
X'^*	$G'_{(i)}(X, X)^*$
X^{**}	$G_{(i)}(X^*, X^*)$

Now for the local actions $\underline{H_t}(v_t) \leq \mathrm{Sym}(d_t)$, we need to give names to the conjugacy classes of subgroups of $\mathrm{Sym}(d_t)$. We will only give tables with $d_t \leq 6$, so we just need a notation for the conjugacy classes of subgroups of $\mathrm{Sym}(6)$. Indeed, each conjugacy class of subgroups of $\mathrm{Sym}(d)$ with $d < 6$ can also be seen as a conjugacy class in $\mathrm{Sym}(6)$ (by assuming that the $6 - d$ other points are fixed). It can be computed that $\mathrm{Sym}(6)$ has exactly 56 conjugacy classes of subgroups, and we give them names according to Table 2.2. The first part of the name of a conjugacy class is the order of a subgroup in that class, and we give for each one a set of generators of some representative subgroup. The classes of subgroups of $\mathrm{Sym}(3)$ (resp. $\mathrm{Sym}(4)$ and $\mathrm{Sym}(5)$) are marked with a Y in the first (resp. second and third) column of the table.

The results of our computations are given in Tables 2.3–2.15. Recall that, when $d_1 = d_2$, two (d_1, d_2)-groups can be conjugate by an element of $\mathrm{Aut}(T_1 \times T_2)$ exchanging T_1 and T_2. Hence, a group with projections (H_1, H_2) is also conjugate to a group with projections (H_2, H_1). For this reason, each equivalence class of (d_1, d_2)-groups appears once or twice in the table, depending on whether it is on the diagonal or not.

Remark that, when d_1 and d_2 are fixed with $d_2 \geq 6$ even, we could in advance give a co-finite subset of groups of $\mathcal{G}'_{(i)}$ that cannot appear as the closure of the projection of a (d_1, d_2)-group on T_2. Indeed, the labelled graph $G_\Gamma^{(2)}$ has d_1 vertices, it has a non-trivial automorphism as explained in §2.3.4, and we also know that the degree of each vertex is even (where a loop in a vertex is only counted once in the degree of that vertex). Hence, one can simply go through all labelled graphs satisfying these three properties and compute all groups $\mathcal{G}'_{(i)}$ that correspond to them. This gives a finite list of groups that covers all possible projections on T_2. If we are only interested in torsion-free (d_1, d_2)-groups, then the task is even shorter as we can use the simplified labelled graphs that have $\frac{d_1}{2}$ vertices. Moreover, the degree of each vertex is also even and loops cannot appear in that case.

Let us for instance consider the torsion-free $(6, 6)$-groups. We must

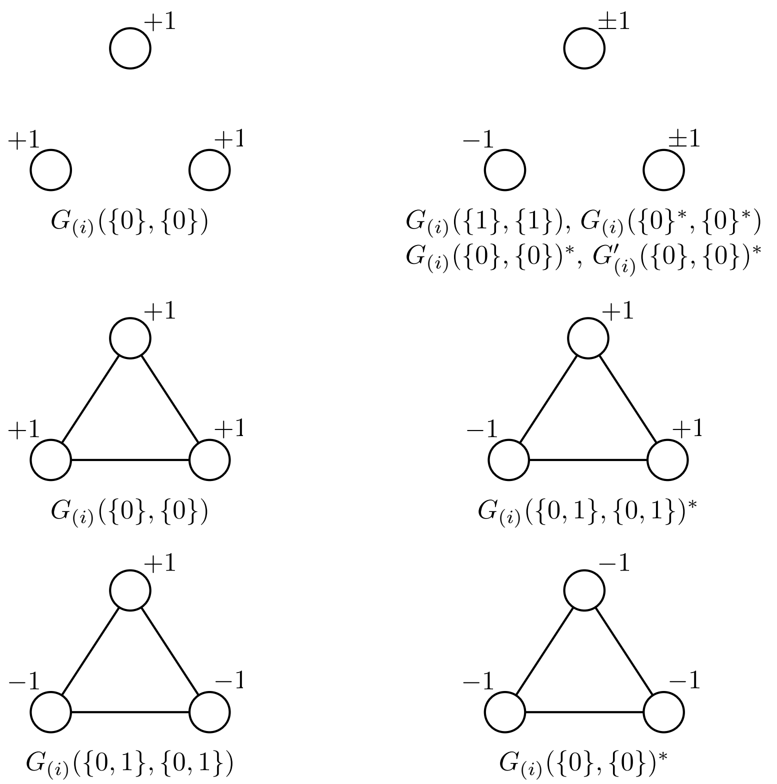


Figure 2.6: Possible projections for a torsion-free $(6, 6)$-group.

observe all labelled graphs with 3 vertices, without any loop and such that the degree of each vertex is even. If we do not consider the labels then there are only two such graphs: the one without any edge and the one with all three possible edges. For each of these two graphs we can put between zero and three labels -1, so at the end we get eight labelled graphs. The groups associated to those graphs are given in Figure 2.6. Note that we could exclude some groups by making use of Corollary 2.3.11. In total, we obtain only seven groups of $\mathcal{G}'_{(i)}$ that could possibly appear as a projection of a torsion-free $(6, 6)$-group. Our tables below (which were found with a computer) show that all these seven groups indeed arise, see Tables 2.12 and 2.13.

Proof of Theorem 2.B. See Tables 2.1, 2.12 and 2.13. $\square$

3?	4?	5?	Name	Generators	Isomorphic to
			720.1	(1,2,4,5)(3,6), (2,4), (1,2)(3,4)	Sym(6)
			360.1	(1,2)(3,4), (1,2,4,5)(3,6)	Alt(6)
			120.2	(1,3,2,4), (1,6,5,2,4)	$\mathrm{PGL}(2,5) \cong \mathrm{Sym}(5)$
		Y	120.1	(2,5), (1,2)(3,4,5)	Sym(5)
			72.1	(2,3)(4,6,5), (1,4)(2,6,3,5)	$(\mathrm{Sym}(3) \times \mathrm{Sym}(3)) \rtimes \mathbf{C}_2$
			60.2	(1,2)(3,4), (1,3,4)(2,5,6)	$\mathrm{PSL}(2,5) \cong \mathrm{Sym}(5)$
		Y	60.1	(1,2)(3,4), (2,3,5)	Alt(5)
			48.2	(1,2)(3,6)(4,5), (1,3,4,2,5,6)	$\mathbf{C}_2 \times \mathrm{Sym}(4)$
			48.1	(1,2)(3,6,5), (1,2)(3,6,4,5)	$\mathbf{C}_2 \times \mathrm{Sym}(4)$
			36.3	(1,3)(5,6), (1,5,2,6)(3,4)	$(\mathbf{C}_3 \times \mathbf{C}_3) \rtimes \mathbf{C}_4$
			36.2	(2,3)(5,6), (1,3,2)(4,6)	$\mathrm{Sym}(3) \times \mathrm{Sym}(3)$
			36.1	(1,3)(4,6), (1,6,3,5,2,4)	$\mathrm{Sym}(3) \times \mathrm{Sym}(3)$
			24.6	(4,5,6), (1,2)(3,4,6,5)	Sym(4)
			24.5	(1,3,6)(2,5,4), (1,2)(3,4,5,6)	Sym(4)
	Y	Y	24.4	(1,4,2), (1,2), (3,4)	Sym(4)
			24.3	(1,4,2,3), (1,6,2,5)	Sym(4)
			24.2	(1,5,4,2,6,3), (1,5,4)(2,6,3)	$\mathbf{C}_2 \times \mathrm{Alt}(4)$
			24.1	(3,6,4), (1,2)(3,6,5)	$\mathbf{C}_2 \times \mathrm{Alt}(4)$
		Y	20.1	(2,4,5,3), (1,5)(2,4)	$\mathrm{GA}(1,5) \cong \mathbf{C}_5 \rtimes \mathbf{C}_4$
			18.3	(5,6), (1,2,3)(4,5)	$\mathbf{C}_3 \times \mathrm{Sym}(3)$
			18.2	(1,4,2,5,3,6), (1,3,2)(4,5,6)	$\mathbf{C}_3 \times \mathrm{Sym}(3)$
			18.1	(1,2,3), (1,3)(5,6), (4,5,6)	$(\mathbf{C}_3 \times \mathbf{C}_3) \rtimes \mathbf{C}_2$
			16.1	(5,6), (3,6)(4,5), (1,2)(3,4)	$\mathbf{C}_2 \times \mathbf{D}_8$
		Y	12.4	(1,2)(3,4), (1,2)(3,5,4)	$\mathbf{C}_2 \times \mathrm{Sym}(3) \cong \mathbf{D}_{12}$
			12.3	(1,3)(4,6), (1,4)(2,6)(3,5)	$\mathbf{C}_2 \times \mathrm{Sym}(3) \cong \mathbf{D}_{12}$
			12.2	(1,5,3)(2,6,4), (1,2)(3,4)	Alt(4)
	Y	Y	12.1	(1,4,2), (1,2)(3,4)	Alt(4)
		Y	10.1	(1,5)(2,4), (1,3)(4,5)	$\mathbf{D}_{10}$
			9.1	(1,2,3), (4,5,6)	$\mathbf{C}_3 \times \mathbf{C}_3$
			8.7	(3,6)(4,5), (1,2)(3,4)	$\mathbf{D}_8$
			8.6	(1,2), (3,4,5,6)	$\mathbf{C}_2 \times \mathbf{C}_4$
	Y	Y	8.5	(1,2), (1,4,2,3)	$\mathbf{D}_8$
			8.4	(1,2)(3,4,5,6), (4,6)	$\mathbf{D}_8$
			8.3	(3,5,4,6), (1,2)(3,4)	$\mathbf{D}_8$
			8.2	(3,4)(5,6), (3,6)(4,5), (1,2)	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
			8.1	(1,2), (3,4), (5,6)	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
			6.6	(1,3)(4,6), (1,2,3)(4,5,6)	Sym(3)
		Y	6.5	(1,2), (3,4,5)	$\mathbf{C}_6$
		Y	6.4	(3,4,5), (1,2)(3,4)	Sym(3)
			6.3	(1,4,3,6,2,5)	$\mathbf{C}_6$
Y	Y	Y	6.2	(1,2,3), (2,3)	Sym(3)
			6.1	(1,2,3)(4,5,6), (1,6)(2,5)(3,4)	Sym(3)
		Y	5.1	(1,3,5,2,4)	$\mathbf{C}_5$
			4.7	(1,2)(3,4,5,6)	$\mathbf{C}_4$
	Y	Y	4.6	(1,3,2,4)	$\mathbf{C}_4$
	Y	Y	4.5	(3,4), (1,2)(3,4)	$\mathbf{C}_2 \times \mathbf{C}_2$
			4.4	(3,4)(5,6), (1,2)(3,6)(4,5)	$\mathbf{C}_2 \times \mathbf{C}_2$
			4.3	(3,4)(5,6), (1,2)	$\mathbf{C}_2 \times \mathbf{C}_2$
	Y	Y	4.2	(1,4)(2,3), (1,2)(3,4)	$\mathbf{C}_2 \times \mathbf{C}_2$
			4.1	(3,4)(5,6), (1,2)(3,4)	$\mathbf{C}_2 \times \mathbf{C}_2$
			3.2	(1,2,3)(4,5,6)	$\mathbf{C}_3$
Y	Y	Y	3.1	(1,2,3)	$\mathbf{C}_3$
	Y	Y	2.3	(1,2)(3,4)	$\mathbf{C}_2$
			2.2	(1,6)(2,5)(3,4)	$\mathbf{C}_2$
Y	Y	Y	2.1	(1,2)	$\mathbf{C}_2$
Y	Y	Y	1.1		1

Table 2.2: Conjugacy classes of subgroups of Sym(6).

	6.2	2.1
6.2	3	1
2.1	1	-

Table 2.3: Possibly irreducible $(3, 3)$-groups.

	6.2	2.1
24.4	9	9
12.1	4	2
8.5	2	1
6.2	14	2
4.5	10	-
2.1	6	-

Table 2.4: Possibly irreducible $(3, 4)$-groups.

	6.2	2.1
120.1	30	39
60.1	10	12
24.4	35	18
20.1	2	1
12.4	85	19
12.1	8	4
10.1	4	1
8.5	22	2
6.4	28	2
6.2	54	5
4.5	50	-
4.2	2	-
2.3	6	-
2.1	18	-

Table 2.5: Possibly irreducible $(3, 5)$-groups.

	6.2	2.1
$\{0,2\}^*$	7	-
$\{2\}$	4	-
$\{0,1\}^*$	22	-
$\{0,1\}$	3	-
$\{1\}^*$	4	22
$\{1\}$	1	7
$\{0\}^{**}$	2	-
$\{0\}^*$	19	45
$\{0\}$	35	33
120.2	41	31
120.1	154	78
72.1	9	30
60.2	30	19
60.1	37	24
48.2	33	32
48.1	215	121
36.3	-	4
36.2	58	14
36.1	4	8
24.6	60	36
24.5	2	5
24.4	141	45
24.3	12	7
24.2	1	1
24.1	64	26
20.1	4	2
18.3	4	-
18.2	4	4
18.1	9	-
16.1	93	12
12.4	364	38
12.3	10	7
12.1	20	10
10.1	18	2
8.7	23	2
8.6	3	-
8.5	108	5
8.4	13	5
8.3	54	2
8.2	15	-
8.1	124	-
6.6	23	3
6.4	88	4
6.2	136	8
4.5	210	-
4.4	9	-
4.3	106	-
4.2	12	-
4.1	36	-
2.3	36	-
2.1	50	-

Table 2.6: Possibly irreducible $(3,6)$-groups.

	24.4	12.1	8.5
24.4	1	-	-
12.1	-	-	1
8.5	-	1	-

Table 2.7: Possibly irreducible torsion-free $(4,4)$-groups.

	24.4	12.1	8.5	6.2	4.5	3.1	2.1
24.4	57	17	13	168	126	4	95
12.1	17	1	5	10	19	-	8
8.5	13	5	1	41	9	4	8
6.2	168	10	41	38	49	-	12
4.5	126	19	9	49	1	-	-
3.1	4	-	4	-	-	-	-
2.1	95	8	8	12	-	-	-

Table 2.8: Possibly irreducible $(4,4)$-groups with torsion.

	24.4	12.1	8.5	6.2	4.5	3.1	2.1
120.1	1833	64	81	1897	1436	-	1215
60.1	180	16	28	217	275	-	166
24.4	1739	49	301	781	679	4	190
20.1	28	2	2	13	21	-	9
12.4	2439	123	241	1431	613	-	192
12.1	122	2	48	20	84	-	16
10.1	36	2	2	30	18	-	7
8.5	480	38	109	269	50	4	16
6.5	66	1	16	-	4	-	-
6.4	291	12	38	109	48	-	12
6.2	1496	36	277	300	181	-	30
4.6	4	-	6	-	-	-	-
4.5	1879	80	117	255	3	-	-
4.2	31	5	10	14	-	-	-
3.1	30	-	32	-	-	-	-
2.3	128	9	6	12	-	-	-
2.1	592	20	35	36	-	-	-

Table 2.9: Possibly irreducible $(4,5)$-groups.

	24.4	12.1	8.5	4.5
$\{1\}$	3	2	-	1
$\{0\}^*$	2	-	3	1
$\{0\}$	2	-	2	-
120.2	5	-	2	1
72.1	1	-	-	-
60.2	4	-	3	1
48.2	5	2	1	4
48.1	2	-	-	-
36.3	-	1	2	1
36.2	3	1	3	1
24.6	2	-	-	-
24.5	-	-	-	1
24.4	5	-	-	-
24.3	3	1	1	2
24.1	-	-	5	-
12.3	-	1	-	1
12.1	-	-	6	-
9.1	2	-	2	-
8.7	1	1	-	-
8.5	3	3	-	-
8.4	-	2	-	-
8.3	1	1	-	-
6.6	4	2	1	1

Table 2.10: Possibly irreducible torsion-free $(4, 6)$-groups.

	24.4	12.1	8.5	4.5	2.1
$\{1\}^*$	18	-	8	-	-
$\{1\}$	11	2	4	-	-
$\{0\}^{**}$	1	1	-	-	-
$\{0\}^*$	13	2	10	15	6
$\{0\}$	9	3	3	3	2
120.2	18	2	11	11	4
72.1	13	2	10	5	2
60.2	5	-	3	-	-
48.2	11	-	5	-	-
48.1	20	4	6	-	-
36.3	15	-	7	6	2
36.1	2	-	-	-	-
24.6	21	3	12	-	-
24.5	2	-	1	-	-
24.4	15	3	12	-	-
24.3	3	1	2	-	-
24.2	-	-	2	-	-
24.1	17	-	11	-	-
18.2	6	1	6	-	-
12.3	8	-	2	-	-
12.2	-	-	1	-	-
12.1	8	-	4	-	-
8.7	15	3	2	-	-
8.6	7	-	1	-	-
8.5	19	3	6	-	-
8.4	24	4	6	-	-
8.3	17	3	2	-	-
4.7	4	-	-	-	-
4.6	4	-	2	-	-
4.4	2	-	-	-	-
4.3	2	-	-	-	-
4.1	2	-	-	-	-

Table 2.11: Possibly irreducible $(4, 6)$-groups with torsion and $\tau_1 = \tau_2 = 0$.

	$\{0,1\}^*$	$\{0,1\}$	$\{1\}$	$\{0\}^{**}$	$\{0\}'^*$	$\{0\}^*$	$\{0\}$	120.2	72.1	60.2	48.2	48.1	36.3	36.2	36.1	24.6	24.5	24.4	24.3	24.2	24.1
$\{0,1\}^*$	80	125	218	19	13	145	103	71	9	28	128	-	-	99	-	-	13	-	29	-	-
$\{0,1\}$	125	68	193	16	15	127	95	67	42	22	102	-	-	99	5	-	8	-	15	-	-
$\{1\}$	218	193	210	19	14	277	185	115	28	44	31	158	2	70	-	48	4	170	12	-	108
$\{0\}^{**}$	19	16	19	-	2	22	15	3	-	2	12	12	-	5	-	5	1	10	3	-	-
$\{0\}'^*$	13	15	14	2	2	7	6	9	5	3	6	-	2	5	-	-	-	-	3	-	-
$\{0\}^*$	145	127	277	22	7	90	122	65	24	32	125	297	6	97	-	61	10	291	28	-	35
$\{0\}$	103	95	185	15	6	122	52	68	54	28	79	94	-	91	4	27	9	85	24	-	13
120.2	71	67	115	3	9	65	68	18	29	12	60	103	2	49	-	21	8	86	18	-	3
72.1	9	42	28	-	5	24	54	29	-	21	13	16	-	29	-	4	5	20	9	13	4
60.2	28	22	44	2	3	32	28	12	21	2	43	45	-	19	1	13	2	22	9	-	6
48.2	128	102	31	12	6	125	79	60	13	43	101	203	-	106	-	42	17	212	46	-	125
48.1	-	-	158	12	-	297	94	103	16	45	203	37	-	26	2	19	23	70	36	2	-
36.3	-	-	2	-	2	6	-	2	-	-	-	-	-	-	-	-	-	-	-	-	12
36.2	99	99	70	5	5	97	91	49	29	19	106	26	-	16	4	10	16	12	24	20	12
36.1	-	5	-	-	-	-	4	-	-	1	-	2	-	4	-	-	-	4	1	-	2
24.6	-	-	48	5	-	61	27	21	4	13	42	19	-	10	-	3	9	11	8	2	-
24.5	13	8	4	1	-	10	9	8	5	2	17	23	-	16	-	9	2	26	7	-	16
24.4	-	-	170	10	-	291	85	86	20	22	212	70	-	12	4	11	26	19	32	-	-
24.3	29	15	12	3	3	28	24	18	9	9	46	36	-	24	1	8	7	32	8	-	15
24.2	-	-	-	-	-	-	-	-	13	-	-	2	-	20	-	2	-	-	-	-	2
24.1	-	-	108	-	-	35	13	3	4	6	125	-	12	12	2	-	16	-	15	2	-
18.3	-	-	10	2	2	1	1	1	-	1	-	-	1	-	-	-	-	-	-	-	-
18.2	-	4	9	-	3	-	9	2	-	-	7	4	-	6	-	-	2	8	2	3	-
18.1	21	18	54	2	2	25	21	6	5	2	32	16	-	6	-	4	3	8	6	-	4
16.1	-	-	67	-	-	283	78	35	11	20	145	29	8	31	1	5	10	30	25	8	54
12.3	-	6	2	-	2	1	12	5	-	6	11	6	-	9	-	-	2	12	5	-	16
12.2	-	-	-	-	-	-	-	-	4	-	-	-	-	2	-	-	-	-	-	-	-
12.1	-	-	68	-	-	28	12	2	8	2	130	-	4	4	2	-	14	-	16	-	-
9.1	10	10	6	1	1	11	9	3	4	3	26	10	-	4	-	2	5	4	7	9	-
8.7	-	-	14	-	-	81	24	7	5	7	39	9	4	11	-	4	7	6	5	2	13
8.6	-	-	18	-	-	36	6	11	4	4	44	-	8	8	-	-	7	-	6	3	-
8.5	-	-	68	-	-	429	139	55	33	19	254	30	8	25	2	18	35	18	41	-	48
8.4	-	-	-	-	-	54	57	19	-	15	7	-	8	12	-	-	2	-	4	-	22
8.3	-	-	35	-	-	96	30	12	2	6	43	7	4	11	-	3	8	6	5	2	10
8.2	-	-	2	-	-	19	8	3	2	-	5	-	2	2	-	-	-	-	-	1	-
8.1	-	-	68	-	-	55	22	11	-	11	93	-	8	8	1	-	17	-	14	1	-
6.6	39	40	10	2	2	40	38	17	10	7	66	30	-	9	1	10	11	12	16	-	20
4.7	-	-	-	-	-	4	4	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4.6	-	-	14	-	-	46	14	10	8	2	57	-	4	4	-	-	8	-	9	-	-
4.5	-	-	44	-	-	72	16	12	8	4	136	-	4	4	-	-	20	-	24	-	-
4.4	-	-	-	-	-	5	7	2	-	-	3	-	2	2	-	-	1	-	-	-	-
4.3	-	-	48	-	-	88	12	10	4	6	102	-	6	6	-	-	18	-	14	1	-
4.2	-	-	-	-	-	30	12	4	4	-	10	-	2	2	-	-	2	-	-	-	-
4.1	-	-	10	-	-	16	2	4	2	2	34	-	2	2	-	-	4	-	6	-	-
2.3	-	-	24	-	-	27	9	6	4	2	71	-	2	2	-	-	11	-	12	-	-

Table 2.12: Possibly irreducible torsion-free $(6,6)$-groups. (Part 1/2)

	18.3	18.2	18.1	16.1	12.3	12.2	12.1	9.1	8.7	8.6	8.5	8.4	8.3	8.2	8.1	6.6	4.7	4.6	4.5	4.4	4.3	4.2	4.1	2.3
$\{0,1\}^*$	-	-	21	-	-	-	-	10	-	-	-	-	-	-	-	39	-	-	-	-	-	-	-	-
$\{0,1\}$	-	4	18	-	6	-	-	10	-	-	-	-	-	-	-	40	-	-	-	-	-	-	-	-
$\{1\}$	10	9	54	67	2	-	68	6	14	18	68	-	35	2	68	10	-	14	44	-	48	-	10	24
$\{0\}^{**}$	2	-	2	-	-	-	-	1	-	-	-	-	-	-	-	2	-	-	-	-	-	-	-	-
$\{0\}'^{*}$	2	3	2	-	2	-	-	1	-	-	-	-	-	-	-	2	-	-	-	-	-	-	-	-
$\{0\}^*$	1	-	25	283	1	-	28	11	81	36	429	54	96	19	55	40	4	46	72	5	88	30	16	27
$\{0\}$	1	9	21	78	12	-	12	9	24	6	139	57	30	8	22	38	4	14	16	7	12	12	2	9
120.2	1	2	6	35	5	-	2	3	7	11	55	19	12	3	11	17	-	10	12	2	10	4	4	6
72.1	-	-	5	11	-	4	8	4	5	4	33	-	2	2	-	10	-	8	8	-	4	4	2	4
60.2	1	-	2	20	6	-	2	3	7	4	19	15	6	-	11	7	-	2	4	-	6	-	2	2
48.2	-	7	32	145	11	-	130	26	39	44	254	7	43	5	93	66	-	57	136	3	102	10	34	71
48.1	-	4	16	29	6	-	-	10	9	-	30	-	7	-	-	30	-	-	-	-	-	-	-	-
36.3	1	-	-	8	-	-	4	-	4	8	8	8	4	2	8	-	-	4	4	2	6	2	2	2
36.2	-	6	6	31	9	2	4	4	11	8	25	12	11	2	8	9	-	4	4	2	6	2	2	2
36.1	-	-	-	1	-	-	2	-	-	-	2	-	-	-	1	1	-	-	-	-	-	-	-	-
24.6	-	-	4	5	-	-	-	2	4	-	18	-	3	-	-	10	-	-	-	-	-	-	-	-
24.5	-	2	3	10	2	-	14	5	7	7	35	2	8	-	17	11	-	8	20	1	18	2	4	11
24.4	-	8	8	30	12	-	-	4	6	-	18	-	6	-	-	12	-	-	-	-	-	-	-	-
24.3	-	2	6	25	5	-	16	7	5	6	41	4	5	-	14	16	-	9	24	-	14	-	6	12
24.2	-	3	-	8	-	-	-	9	2	3	-	-	2	1	1	-	-	-	-	-	1	-	-	-
24.1	-	-	4	54	16	-	-	-	13	-	48	22	10	-	-	20	-	-	-	-	-	-	-	-
18.3	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
18.2	-	-	-	2	-	1	-	-	1	-	8	-	1	-	-	4	-	-	-	-	-	-	-	-
18.1	-	-	-	6	3	-	2	-	2	-	6	5	2	-	1	3	-	-	-	-	-	-	-	-
16.1	-	2	6	-	11	-	42	4	-	-	-	-	-	-	-	30	-	-	-	-	-	-	-	-
12.3	-	-	3	11	-	-	14	2	3	2	26	-	4	1	9	6	-	4	12	-	8	2	4	6
12.2	-	1	-	-	-	-	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
12.1	-	-	2	42	14	-	-	-	6	-	18	12	6	-	-	6	-	-	-	-	-	-	-	-
9.1	-	-	-	4	2	1	-	-	2	-	4	4	2	-	-	2	-	-	-	-	-	-	-	-
8.7	-	1	2	-	3	-	6	2	-	-	-	-	-	-	-	10	-	-	-	-	-	-	-	-
8.6	-	-	-	-	2	-	-	-	-	-	-	-	-	-	-	8	-	-	-	-	-	-	-	-
8.5	-	8	6	-	26	-	18	4	-	-	-	-	-	-	-	22	-	-	-	-	-	-	-	-
8.4	-	-	5	-	-	-	12	4	-	-	-	-	-	-	-	3	-	-	-	-	-	-	-	-
8.3	-	1	2	-	4	-	6	2	-	-	-	-	-	-	-	10	-	-	-	-	-	-	-	-
8.2	-	-	-	-	1	-	-	-	-	-	-	-	-	-	-	2	-	-	-	-	-	-	-	-
8.1	-	-	1	-	9	-	-	-	-	-	-	-	-	-	-	7	-	-	-	-	-	-	-	-
6.6	-	4	3	30	6	-	6	2	10	8	22	3	10	2	7	4	-	4	4	2	6	2	2	2
4.7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4.6	-	-	-	-	4	-	-	-	-	-	-	-	-	-	-	4	-	-	-	-	-	-	-	-
4.5	-	-	-	-	12	-	-	-	-	-	-	-	-	-	-	4	-	-	-	-	-	-	-	-
4.4	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2	-	-	-	-	-	-	-	-
4.3	-	-	-	-	8	-	-	-	-	-	-	-	-	-	-	6	-	-	-	-	-	-	-	-
4.2	-	-	-	-	2	-	-	-	-	-	-	-	-	-	-	2	-	-	-	-	-	-	-	-
4.1	-	-	-	-	4	-	-	-	-	-	-	-	-	-	-	2	-	-	-	-	-	-	-	-
2.3	-	-	-	-	6	-	-	-	-	-	-	-	-	-	-	2	-	-	-	-	-	-	-	-

Table 2.13: Possibly irreducible torsion-free $(6,6)$-groups. (Part 2/2)

	$\{0,2\}^*$	$\{0,2\}$	$\{2\}$	$\{0,1\}^*$	$\{0,1\}$	$\{1\}^{**}$	$\{1\}^*$	$\{1\}$	$\{0\}^{**}$	$\{0\}^*$	$\{0\}$	120.2	72.1	60.2	48.2	48.1	36.3	36.1	24.6	24.5	24.4
$\{0,2\}^*$	190	330	764	267	361	69	-	603	61	465	538	179	579	-	150	-	22	47	-	2	-
$\{0,2\}$	330	174	667	321	402	46	-	591	48	450	467	173	486	-	123	-	28	47	-	1	-
$\{2\}$	764	667	641	708	742	57	-	1167	82	898	723	344	859	-	28	-	289	53	-	2	-
$\{0,1\}^*$	267	321	708	191	377	62	-	596	70	564	479	250	507	29	352	-	76	45	-	26	-
$\{0,1\}$	361	402	742	377	266	44	-	670	69	569	645	274	733	29	422	-	64	56	-	31	-
$\{1\}^{**}$	69	46	57	62	44	5	-	47	4	48	71	13	61	-	-	-	22	-	-	-	-
$\{1\}^*$	-	-	-	-	-	-	76	69	9	181	32	17	85	-	5	4179	23	6	998	1	1424
$\{1\}$	603	591	1167	596	670	47	69	447	76	1004	777	374	581	92	134	2558	115	26	805	20	834
$\{0\}^{**}$	61	48	82	70	69	4	9	76	5	70	90	30	71	5	15	102	14	-	19	2	22
$\{0\}^*$	465	450	898	564	569	48	181	1004	70	521	705	347	972	28	308	6322	167	80	1694	31	1986
$\{0\}$	538	467	723	479	645	71	32	777	90	705	314	300	529	25	229	1632	110	39	407	14	620
120.2	179	173	344	250	274	13	17	374	30	347	300	70	358	12	142	1134	78	18	363	14	345
72.1	579	486	859	507	733	61	85	581	71	972	529	358	575	29	263	1183	193	53	397	20	409
60.2	-	-	-	29	29	-	-	92	5	28	25	12	29	4	11	133	6	2	30	2	33
48.2	150	123	28	352	422	-	5	134	15	308	229	142	263	11	106	1199	36	17	390	22	435
48.1	-	-	-	-	-	-	4179	2558	102	6322	1632	1134	1183	133	1199	2662	310	185	1136	114	1009
36.3	22	28	289	76	64	22	23	115	14	167	110	78	193	6	36	310	30	11	93	3	89
36.1	47	47	53	45	56	-	6	26	-	80	39	18	53	2	17	185	11	3	49	1	75
24.6	-	-	-	-	-	-	998	805	19	1694	407	363	397	30	390	1136	93	49	127	33	231
24.5	2	1	2	26	31	-	1	20	2	31	14	14	20	2	22	114	3	1	33	3	43
24.4	-	-	-	-	-	-	1424	834	22	1986	620	345	409	33	435	1009	89	75	231	43	107
24.3	3	2	2	40	55	-	1	32	5	48	31	27	25	2	24	156	3	1	51	6	74
24.2	-	-	-	-	-	-	-	-	-	-	-	-	56	-	-	12	8	-	12	-	117
24.1	-	-	-	-	-	-	-	928	42	751	200	141	130	-	190	602	15	23	112	25	126
18.2	60	41	-	19	26	-	2	68	5	95	48	50	94	2	41	96	22	2	34	11	27
16.1	-	-	-	-	-	-	1122	1269	66	2805	625	525	465	53	553	1954	147	53	464	75	542
12.3	86	50	-	40	102	-	1	38	5	93	36	37	58	1	45	182	8	3	60	4	60
12.2	-	-	-	-	-	-	-	-	-	-	-	-	3	-	-	2	1	-	2	-	7
12.1	-	-	-	-	-	-	-	206	10	186	56	32	28	-	36	114	-	5	22	4	24
8.7	-	-	-	-	-	-	270	276	25	749	169	122	105	14	156	369	37	16	82	23	102
8.6	-	-	-	-	-	-	161	224	26	477	120	139	119	8	134	258	48	3	26	23	48
8.5	-	-	-	-	-	-	704	560	40	1312	319	232	229	31	338	429	47	36	101	51	124
8.4	-	-	-	-	-	-	822	484	20	1787	303	316	330	25	186	504	61	38	126	15	150
8.3	-	-	-	-	-	-	284	446	25	851	194	138	127	15	156	443	36	18	91	20	118
8.2	-	-	-	-	-	-	-	106	10	121	34	31	20	2	34	72	10	-	8	5	16
8.1	-	-	-	-	-	-	-	227	4	1680	206	204	138	-	123	18	53	4	2	14	4
4.7	-	-	-	-	-	-	62	-	-	132	28	-	12	-	4	44	-	-	4	-	8
4.6	-	-	-	-	-	-	48	74	8	142	42	32	30	4	67	44	6	-	4	12	8
4.5	-	-	-	-	-	-	-	-	-	1598	202	170	120	-	-	-	44	-	-	-	-
4.4	-	-	-	-	-	-	-	102	12	128	22	30	23	2	24	36	5	2	4	3	8
4.3	-	-	-	-	-	-	-	142	8	1401	200	192	118	-	79	36	58	2	4	10	8
4.2	-	-	-	-	-	-	-	64	8	51	15	14	10	2	28	-	2	-	-	4	-
4.1	-	-	-	-	-	-	-	32	4	416	58	53	36	-	22	18	17	-	2	4	4
2.3	-	-	-	-	-	-	-	-	-	157	23	20	14	-	-	-	6	-	-	-	-
2.1	-	-	-	-	-	-	-	-	-	245	37	29	21	-	-	-	8	-	-	-	-

Table 2.14: Possibly irreducible $(6,6)$-groups with torsion and $\tau_1 = \tau_2 = 0$. (Part 1/2)

	24.3	24.2	24.1	18.2	16.1	12.3	12.2	12.1	8.7	8.6	8.5	8.4	8.3	8.2	8.1	4.7	4.6	4.5	4.4	4.3	4.2	4.1	2.3	2.1
$\{0,2\}^*$	3	-	-	60	-	86	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\{0,2\}$	2	-	-	41	-	50	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\{2\}$	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\{0,1\}^*$	40	-	-	19	-	40	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\{0,1\}$	55	-	-	26	-	102	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\{1\}^{**}$	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\{1\}^*$	1	-	-	2	1122	1	-	-	270	161	704	822	284	-	-	62	48	-	-	-	-	-	-	-
$\{1\}$	32	-	928	68	1269	38	-	206	276	224	560	484	446	106	227	-	74	-	102	142	64	32	-	-
$\{0\}^{**}$	5	-	42	5	66	5	-	10	25	26	40	20	25	10	4	-	8	-	12	8	8	4	-	-
$\{0\}^*$	48	-	751	95	2805	93	-	186	749	477	1312	1787	851	121	1680	132	142	1598	128	1401	51	416	157	245
$\{0\}$	31	-	200	48	625	36	-	56	169	120	319	303	194	34	206	28	42	202	22	200	15	58	23	37
120.2	27	-	141	50	525	37	-	32	122	139	232	316	138	31	204	-	32	170	30	192	14	53	20	29
72.1	25	56	130	94	465	58	3	28	105	119	229	330	127	20	138	12	30	120	23	118	10	36	14	21
60.2	2	-	-	2	53	1	-	-	14	8	31	25	15	2	-	-	4	-	2	-	2	-	-	-
48.2	24	-	190	41	553	45	-	36	156	134	338	186	156	34	123	4	67	-	24	79	28	22	-	-
48.1	156	12	602	96	1954	182	2	114	369	258	429	504	443	72	18	44	44	-	36	36	-	18	-	-
36.3	3	8	15	22	147	8	1	-	37	48	47	61	36	10	53	-	6	44	5	58	2	17	6	8
36.1	1	-	23	2	53	3	-	5	16	3	36	38	18	-	4	-	-	-	2	2	-	-	-	-
24.6	51	12	112	34	464	60	2	22	82	26	101	126	91	8	2	4	4	-	4	4	-	2	-	-
24.5	6	-	25	11	75	4	-	4	23	23	51	15	20	5	14	-	12	-	3	10	4	4	-	-
24.4	74	117	126	27	542	60	7	24	102	48	124	150	118	16	4	8	8	-	8	8	-	4	-	-
24.3	5	-	46	15	82	6	-	8	26	22	53	26	24	6	24	-	9	-	4	16	6	4	-	-
24.2	-	-	6	10	54	-	-	-	11	10	44	22	13	3	1	-	15	-	2	1	-	-	-	-
24.1	46	6	-	20	366	26	-	-	71	-	78	104	74	-	-	-	-	-	-	-	-	-	-	-
18.2	15	10	20	10	38	8	-	5	15	-	26	44	15	-	-	-	-	-	-	-	-	-	-	-
16.1	82	54	366	38	148	67	6	58	34	32	108	95	46	-	-	-	16	-	-	-	-	-	-	-
12.3	6	-	26	8	67	5	-	-	17	14	28	19	24	5	12	-	4	-	6	10	4	2	-	-
12.2	-	-	-	-	6	-	-	-	1	1	4	4	1	-	-	-	1	-	-	-	-	-	-	-
12.1	8	-	-	5	58	-	-	-	14	-	12	18	14	-	-	-	-	-	-	-	-	-	-	-
8.7	26	11	71	15	34	17	1	14	3	4	14	12	6	-	-	-	2	-	-	-	-	-	-	-
8.6	22	10	-	-	32	14	1	-	4	-	8	8	4	-	-	-	-	-	-	-	-	-	-	-
8.5	53	44	78	26	108	28	4	12	14	8	22	36	18	-	-	-	4	-	-	-	-	-	-	-
8.4	26	22	104	44	95	19	4	18	12	8	36	19	16	-	-	-	4	-	-	-	-	-	-	-
8.3	24	13	74	15	46	24	1	14	6	4	18	16	5	-	-	-	2	-	-	-	-	-	-	-
8.2	6	3	-	-	-	5	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
8.1	24	1	-	-	-	12	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4.7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4.6	9	15	-	-	16	4	1	-	2	-	4	4	2	-	-	-	-	-	-	-	-	-	-	-
4.5	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4.4	4	2	-	-	-	6	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4.3	16	1	-	-	-	10	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4.2	6	-	-	-	-	4	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4.1	4	-	-	-	-	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
2.3	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
2.1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

Table 2.15: Possibly irreducible $(6,6)$-groups with torsion and $\tau_1 = \tau_2 = 0$. (Part 2/2)

2.4 Virtually simple (d_1, d_2)-groups

In [BM00b] and [Rat04] the authors constructed virtually simple torsion-free (d_1, d_2)-groups for different values of (d_1, d_2), for instance $(6, 16)$ and $(10, 10)$. More recently and with the same ideas, Bondarenko and Kivva constructed two virtually simple torsion-free $(8, 8)$-groups in [BK17].

In this section we find a list of virtually simple $(6, 6)$-groups and $(4, 5)$-groups. We also give a virtually simple $(2n, 2n+1)$-group for each $n \geq 2$. We then end with virtually simple $(6, 4n)$-groups with $n \geq 2$ so that the projection on the 6-regular tree T_1 Chabauty converges to $\mathrm{Aut}(T_1)$ when n goes to infinity.

2.4.1 Virtually simple $(6, 6)$-groups

The idea for constructing virtually simple (d_1, d_2)-groups is to use the *Normal Subgroup Theorem* [BM00b, Theorem 4.1] due to Burger and Mozes, stating that if Γ is a (d_1, d_2)-group with H_t being 2-transitive on ∂T_t and $[H_t : H_t^{(\infty)}] < \infty$ for each $t \in \{1, 2\}$, then any non-trivial normal subgroup of Γ has finite index (i.e. Γ is **just-infinite**). Bader and Shalom later proved a generalization of that theorem in [BS06]. We give below a statement which is a consequence of their result. We call it the *Normal Subgroup Theorem* (NST) for future references. A tree is **thick** if each of its vertices has at least 3 neighbors.

Theorem (Normal Subgroup Theorem, Bader–Shalom)**.** *Let T_1 and T_2 be two locally finite thick trees and let $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ be a cocompact lattice such that $\overline{\mathrm{proj}_t(\Gamma)}$ is 2-transitive on ∂T_t for each $t \in \{1, 2\}$. Then Γ and all its finite index subgroups are just-infinite. In particular, Γ is either residually finite or virtually simple.*

Proof. By [BM00a, Proposition 3.1.2], all finite index subgroups of a closed subgroup of $\mathrm{Aut}(T_t)$ acting 2-transitively on ∂T_t also acts 2-transitively on ∂T_t. Up to replacing Γ by a finite index subgroup, we can therefore just show that Γ is just-infinite.

This is a consequence of [BS06, Theorem 1.1], modulo the fact that if H is a closed subgroup of $\mathrm{Aut}(T)$ acting 2-transitively on ∂T (with T being a locally finite thick tree), then H is just non-compact (i.e. it

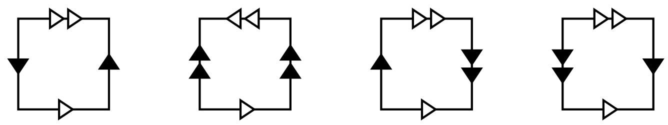


Figure 2.7: The torsion-free $(4,4)$-group $\Gamma_{4,4}$.

is non-compact and all its non-trivial normal subgroups are cocompact) and H does not contain any non-trivial abelian normal subgroup. This is an easy exercise: it suffices to remember that a non-trivial normal subgroup of a 2-transitive group is transitive, and to use the characterizations of the 2-transitivity on ∂T given in [BM00a, Lemma 3.1.1].

Let us now show that Γ is residually finite or virtually simple. As Γ is just-infinite, $\Gamma^{(\infty)}$ is trivial or has finite index in Γ. If $\Gamma^{(\infty)} = 1$ then Γ is residually finite. On the contrary, if $\Gamma^{(\infty)}$ has finite index in Γ then it is also just-infinite. So any non-trivial normal subgroup N of $\Gamma^{(\infty)}$ has finite index in $\Gamma^{(\infty)}$ and thus in Γ, hence $N = \Gamma^{(\infty)}$. This means that $\Gamma^{(\infty)}$ is simple. □

In this subsection, we first present a torsion-free $(4,4)$-group $\Gamma_{4,4}$ that is not residually finite. The NST does not directly apply to $\Gamma_{4,4}$, but the strategy is then to embed $\Gamma_{4,4}$ in some other (d_1, d_2)-group Γ on which the NST can be used. Then Γ cannot be residually finite because it contains $\Gamma_{4,4}$, and hence it must be virtually simple.

Let $\Gamma_{4,4}$ be the torsion-free $(4,4)$-group associated to the four squares in Figure 2.7. The local action of $\Gamma_{4,4}$ on T_1 (resp. T_2) is $\mathbf{D}_8$ (resp. $\mathrm{Alt}(4)$), and it is possibly irreducible: it appears in Table 2.7.

We show that $\Gamma_{4,4}$ is non-residually finite. This was already proved in [BK17, Theorem 15] and [CW17, Corollary 6.4] but we here prove it by finding an explicit non-trivial element $\gamma \in \Gamma_{4,4}^{(\infty)}$. In other words γ is a non-trivial element of $\Gamma_{4,4}$ such that $\varphi(\gamma) = 1$ for any finite quotient $\varphi\colon \Gamma_{4,4} \to Q$. The ideas of this proof are due to Caprace and are already written in [Cap17a, Remark 4.19] but we give here some additional details.

Proposition 2.4.1. *The group $\Gamma_{4,4}$ is irreducible and not residually finite. Moreover, $[a_1^3, a_2^4]$ and $[a_2^3, a_1^4]$ are non-trivial elements of $\Gamma_{4,4}^{(\infty)}$.*

Proof. Irreducibility has been proved in [JW09, Theorem 3], but it can also be established by using [Wei79, Theorem 1.1]. Indeed, in our situation the result of Weiss implies that, if $\Gamma_{4,4}$ was reducible, then $\mathrm{Fix}_{\mathrm{proj}_2(\Gamma_{4,4})}(B(v_2,4))$ would be trivial. So for proving irreducibility we just need to find some element in $\Gamma_{4,4}$ fixing $B(v_2,4)$ but not $B(v_2,5)$. We claim that $(a_1a_2)^{81}$ is such an element. First, we can compute that $(a_1a_2)^3$ fixes $B(v_2,1)$. Then, for each vertex w at distance 1 from v_2, $(a_1a_2)^3$ can only act trivially or as a 3-cycle on the three neighbors of w different from v_2 (because the local action is $\mathrm{Alt}(4)$). So $(a_1a_2)^9$ fixes $B(v_2,2)$. Continuing with the same argument, we obtain that $(a_1a_2)^{81}$ fixes $B(v_2,4)$. Finally, the fact that $(a_1a_2)^{81}$ does not fix $B(v_2,5)$ can be proved by drawing a 162×5 rectangle. This can be automatized with a computer, and we get for instance that $(a_1a_2)^{81}(b_1^5(v_2)) = b_1^4b_2^{-1}(v_2)$.

The strategy to find a non-trivial element in $\Gamma_{4,4}^{(\infty)}$ is to use that for any group G and any subgroup $H \leq G$, the inclusion $[C_G(H), \overline{H}] \subseteq G^{(\infty)}$ holds, where $\overline{H}$ is the profinite closure of H (see [Cap17a, Lemma 4.13]). Here we take $G = \Gamma_{4,4}$. For H we consider $B_{a_1} = \Gamma_{4,4}(v_1, a_1(v_1))$, i.e. the fixator of $a_1(v_1)$ in $B = \langle b_1, b_2\rangle = \Gamma_{4,4}(v_1)$. We claim that $a_1^3 \in C_{\Gamma_{4,4}}(B_{a_1})$ and $a_2^4 \in \overline{B_{a_1}}$. This will thus show that $[a_1^3, a_2^4] \in \Gamma_{4,4}^{(\infty)}$.

Since B acts transitively on the four vertices adjacent to v_1 in T_1, the subgroup B_{a_1} has index 4 in B. Using the Reidemeister–Schreier method, we could find the following set of generators for B_{a_1}:

$$B_{a_1} = \left\langle b_1b_2,\ b_1^{-1}b_2,\ b_2b_1b_2^2,\ b_2b_1^{-1}b_2^2,\ b_2^4 \right\rangle.$$

From the geometric squares, it can be checked that a_1^3 centralizes B_{a_1}, i.e. $a_1^3 \in C_{\Gamma_{4,4}}(B_{a_1})$. This indeed directly follows from the equalities

$$\begin{aligned} b_1a_1^3b_1^{-1} &= a_2^3,\\ b_1^{-1}a_1^3b_1 &= a_2^3,\\ b_2a_1^3b_2^{-1} &= a_2^3,\\ b_2^{-1}a_1^3b_2 &= a_2^{-3}. \end{aligned}$$

Note also that B_{a_1} is contained in $B^{(2)}$, the index 2 subgroup of B consisting of elements whose length is even (with respect to the generators b_1 and b_2). Our next goal is to show that $a_2^2 \in \overline{B^{(2)}}$, and it will then follow that $a_2^4 \in \overline{B_{a_1}}$ as wanted.

Consider a finite quotient $\varphi\colon \Gamma_{4,4} \to Q$. Since $\Gamma_{4,4}$ is irreducible, the projection $\mathrm{proj}_1(\Gamma_{4,4}(v_1))$ is infinite. Hence, the finite index subgroup $\mathrm{proj}_1(\mathrm{Fix}_{\Gamma_{4,4}}(B(v_1,1)) \cap B^{(2)} \cap \ker\varphi)$ is also infinite. Let γ be an element of $\mathrm{Fix}_{\Gamma_{4,4}}(B(v_1,1)) \cap B^{(2)} \cap \ker\varphi$ such that $\mathrm{proj}_1(\gamma)$ is non-trivial. In T_1, there is a vertex $w \neq v_1$ such that γ fixes the path from v_1 to w but does not fix some neighbor z of w: $\gamma(z) = z' \neq z$. Write $w = h(v_1)$ with $h \in \langle a_1, a_2\rangle$, $z = hx(v_1)$ with $x \in \{a_1, a_1^{-1}, a_2, a_2^{-1}\}$ and $z' = hx'(v_1)$ with $x' \in \{a_1, a_1^{-1}, a_2, a_2^{-1}\}$, see Figure 2.8. Recall that $\mathrm{proj}_1(\Gamma_{4,4}(v_1))$ acts on the four neighbors of v_1 as $\mathbf{D}_8$ acting on the four vertices of a square (where $a_1(v_1)$ and $a_1^{-1}(v_1)$ correspond to opposite vertices of the square). We thus have the same local action around w, and the fact that γ fixes w and some neighbor of w while not fixing z implies that $x' = x^{-1}$. On Figure 2.8 we see that $hx^{-1}\gamma' = \gamma hx$ for some $\gamma' \in B^{(2)}$. Using the fact that $\varphi(\gamma) = 1$, this implies that $\varphi(\gamma') = \varphi(x^2)$. We can summarize this by saying that, for each finite quotient $\varphi\colon \Gamma_{4,4} \to Q$, either $\varphi(a_1^2) \in \varphi(B^{(2)})$ or $\varphi(a_2^2) \in \varphi(B^{(2)})$ $(*)$. In fact, we can even say that there exists $k \in \{1,2\}$ such that $\varphi(a_k^2) \in \varphi(B^{(2)})$ for all finite quotients $\varphi\colon \Gamma_{4,4} \to Q$ $(**)$. Indeed, if $(**)$ was not true then we would have two finite quotients $\varphi_1\colon \Gamma_{4,4} \to Q_1$ and $\varphi_2\colon \Gamma_{4,4} \to Q_2$ with $\varphi_1(a_1^2) \notin \varphi_1(B^{(2)})$ and $\varphi_2(a_2^2) \notin \varphi_2(B^{(2)})$, and the new finite quotient $(\varphi_1 \times \varphi_2)\colon \Gamma_{4,4} \to Q_1 \times Q_2$ would give a contradiction with $(*)$. Now there suffices to remark that $\Gamma_{4,4}$ has an automorphism defined by $a_1 \mapsto a_2$, $a_2 \mapsto a_1$, $b_1 \mapsto b_1^{-1}$ and $b_2 \mapsto b_2^{-1}$. Therefore, $(**)$ even tells us that $\varphi(a_1^2)$

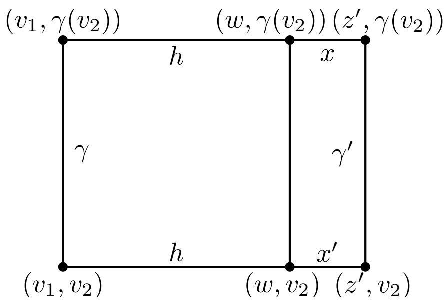


Figure 2.8: Illustration of Proposition 2.4.1.

and $\varphi(a_2^2)$ both belong to $\overline{\varphi(B^{(2)})}$ for all finite quotients $\varphi\colon \Gamma_{4,4} \to Q$. In particular, we have $a_2^2 \in \overline{B^{(2)}}$ as wanted.

Remark that, thanks to the automorphism of $\Gamma_{4,4}$ defined above, we also obtain $[a_2^3, a_1^4] \in \Gamma_{4,4}^{(\infty)}$. $\square$

Using GAP, we could search for (d_1, d_2)-groups Γ with $d_1, d_2 \geq 6$, containing $\Gamma_{4,4}$ (in the sense that the four geometric squares defining $\Gamma_{4,4}$ are part of the geometric squares defining Γ) and such that $\underline{H_1}(v_1) \geq \mathrm{Alt}(d_1)$ and $\underline{H_2}(v_2) \geq \mathrm{Alt}(d_2)$. We say that Γ satisfies $(*)$ if the above conditions are true. Since $\Gamma_{4,4}$ is irreducible, a group Γ satisfying $(*)$ is also irreducible and H_t is 2-transitive on ∂T_t for each $t \in \{1, 2\}$ (see [BM00a, Propositions 3.3.1 and 3.3.2]). (We even know by Corollary 1.E′ that H_t belongs to $\mathcal{G}'_{(i)}$ for some legal coloring i of T_t.) Thus Γ is virtually simple, by the NST.

We could find torsion-free $(6, 8)$-groups and torsion-free $(8, 6)$-groups satisfying $(*)$, by adding one (resp. two) horizontal generator(s), two (resp. one) vertical generator(s) and 8 geometric squares to the ones of $\Gamma_{4,4}$. We could also show that there does not exist any torsion-free $(6, 6)$-group satisfying $(*)$. However, there exist $(6, 6)$-groups (with torsion) with $(*)$. In total, there are 160 equivalence classes of such groups. We give all these groups in Tables 2.16–2.20, by giving the geometric squares that must be added to the four geometric squares $a_1b_1a_2^{-1}b_1$, $a_1b_2a_2b_2^{-1}$, $a_1b_2^{-1}a_2^{-1}b_1^{-1}$ and $a_1b_1^{-1}a_2^{-1}b_2$ defining $\Gamma_{4,4}$. We call them $\Gamma_{6,6,1}, \ldots, \Gamma_{6,6,160}$. Some remarks follow about these groups:

- The index of the simple subgroup $\Gamma_{6,6,k}^{(\infty)}$ of $\Gamma_{6,6,k}$ can be computed by using the fact that $[a_1^3, a_2^4] \in \Gamma_{4,4}^{(\infty)} \leq \Gamma_{6,6,k}^{(\infty)}$ (see Proposition 2.4.1). Indeed, let Q be the group obtained by adding the relator $[a_1^3, a_2^4]$ to the presentation of $\Gamma_{6,6,k}$. Then the kernel of the projection $\Gamma_{6,6,k} \to Q$ is the smallest normal subgroup of $\Gamma_{6,6,k}$ containing $[a_1^3, a_2^4]$, i.e. it must be $\Gamma_{6,6,k}^{(\infty)}$. Hence, we just need to compute (with GAP) the order of the finite group Q obtained as above and this gives us the index of $\Gamma_{6,6,k}^{(\infty)}$ in $\Gamma_{6,6,k}$. As written in the tables, for all groups $\Gamma_{6,6,k}$ with $k \in \{1, \ldots, 160\} \setminus \{104, 116\}$, we obtain that $|Q| = 4$. As $[\Gamma_{6,6,k} : \Gamma_{6,6,k}^{+}] = 4$, this implies that

$\Gamma_{6,6,k}^{(\infty)} = \Gamma_{6,6,k}^{+}$ i.e. $\Gamma_{6,6,k}^{+}$ is the simple subgroup of finite index in $\Gamma_{6,6,k}$. For $\Gamma_{6,6,104}$ and $\Gamma_{6,6,116}$, we get $|Q| = 12$. More precisely, $Q \cong (\mathbf{C}_2)^2 \times \mathbf{C}_3$. So $\Gamma_{6,6,k}^{(\infty)}$ is a subgroup of index 3 of $\Gamma_{6,6,k}^{+}$ when $k \in \{104, 116\}$.

- The groups $H_1 = \overline{\mathrm{proj}_1(\Gamma_{6,6,k})}$ and $H_2 = \overline{\mathrm{proj}_2(\Gamma_{6,6,k})}$ are given, using the notation of §2.3.7. As explained above, we could compute that $\Gamma_{6,6,k} \big/ \Gamma_{6,6,k}^{(\infty)} \cong (\mathbf{C}_2)^2$ or $(\mathbf{C}_2)^2 \times \mathbf{C}_3$ for each $k \in \{1, \dots, 160\}$. This explains why H_1 and H_2 never take the form X^{**} or X'^{*} for some $X \subset_f \mathbf{Z}_{\geq 0}$: recall that $[G_{(i)}(X^*, X^*) : G_{(i)}^{+}(X, X)] = 8$ and $G'_{(i)}(X, X)^* \big/ G_{(i)}^{+}(X, X) \cong \mathbf{C}_4$.
- For the last column, recall that $X_{\Gamma_{6,6,k}^{+}}$ is a (d_1, d_2)-complex, as defined in Definition 2.2.2. We write $\mathrm{Aut}(X_{\Gamma_{6,6,k}^{+}})$ for the set of automorphisms of that complex that do not exchange horizontal and vertical edges. We already know by hypothesis that $\mathrm{Aut}(X_{\Gamma_{6,6,k}^{+}}) \geq \mathbf{C}_2 \times \mathbf{C}_2$, and we computed the number of automorphisms of $X_{\Gamma_{6,6,k}^{+}}$ that fix the four vertices v_{00}, v_{10}, v_{11} and v_{01}. As written in the tables, for each $k \in \{1, \dots, 160\}$ we could observe that there is at most one non-trivial such automorphism, so that $\mathrm{Aut}(X_{\Gamma_{6,6,k}^{+}}) \cong \mathbf{C}_2 \times \mathbf{C}_2$ or $\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$.
  - If $\mathrm{Aut}(X_{\Gamma_{6,6,k}^{+}}) \cong \mathbf{C}_2 \times \mathbf{C}_2$, then there is exactly one good $\mathbf{C}_2 \times \mathbf{C}_2$-action on $X_{\Gamma_{6,6,k}^{+}}$, so this means that $\Gamma_{6,6,k}$ is the only $(6,6)$-group whose type-preserving subgroup is $\Gamma_{6,6,k}^{+}$.
  - If $\mathrm{Aut}(X_{\Gamma_{6,6,k}^{+}}) \cong \mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$, then there are four good $\mathbf{C}_2 \times \mathbf{C}_2$-actions on $X_{\Gamma_{6,6,k}^{+}}$. This leads to four $(6,6)$-groups whose type-preserving subgroup is $\Gamma_{6,6,k}^{+}$. For each such k we could compute the three new $(6,6)$-groups containing $\Gamma_{6,6,k}^{+}$, but it directly appears that they have much more torsion, i.e. their τ_1 and τ_2 satisfy $\tau_1 + \tau_2 \geq 4$. In particular, none of the new groups obtained in that way is equivalent to some $\Gamma_{6,6,k'}$.

From this discussion it follows that all $\Gamma_{6,6,k}^{+}$ are pairwise non-conjugate in $\mathrm{Aut}(T_1 \times T_2)$. By [BMZ09, Corollary 1.1.22], this also means that they are all pairwise non-isomorphic.

We summarize some of those results in the next theorem.

Theorem 2.4.2 (Theorem 2.C (i))**.** *Let* $\Gamma_{6,6,k}$ *(*$k \in \{1, \ldots, 160\}$*) be one of the* $(6,6)$*-groups given by Tables 2.16–2.20.*

- *If* $k \notin \{104, 116\}$*, then* $\Gamma^{+}_{6,6,k}$ *is simple.*
- *If* $k \in \{104, 116\}$*, then* $\Gamma^{+}_{6,6,k}$ *has a simple subgroup of index* 3*.*

Moreover, all simple groups $\Gamma^{(\infty)}_{6,6,k}$ *are pairwise non-isomorphic. In particular, the groups* $\Gamma_{6,6,k}$ *are pairwise non commensurable.*

Proof. See the discussion above. Note that $\Gamma^{(\infty)}_{6,6,k} \not\cong \Gamma^{(\infty)}_{6,6,k'}$ for each $k \in \{1, \ldots, 160\} \setminus \{104, 116\}$ and each $k' \in \{104, 116\}$, see [BMZ09, Theorem 1.4.1]. We also have $\Gamma^{(\infty)}_{6,6,104} \not\cong \Gamma^{(\infty)}_{6,6,116}$. Indeed, they are not conjugate in $\mathrm{Aut}(T_1 \times T_2)$ since $\overline{\mathrm{proj}_2(\Gamma^{(\infty)}_{6,6,104})} \not\cong \overline{\mathrm{proj}_t(\Gamma^{(\infty)}_{6,6,116})}$ for each $t \in \{1, 2\}$ (see Table 2.19). $\square$

Proof of Corollary 2.D (i). This group is $\Gamma_{6,6,2}$, see Table 2.16. $\square$

Name	τ_1	τ_2	Squares: $a_1b_1a_2^{-1}b_1,\ a_1b_2a_2b_2^{-1},\ a_1b_2^{-1}a_2^{-1}b_1^{-1},\ a_1b_1^{-1}a_2^{-1}b_2\ +$	H_1	H_2	$[\Gamma:\Gamma^{(\infty)}]$	$\mathrm{Aut}(X_{\Gamma^+})$
$\Gamma_{6,6,1}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3^{-1}b_1^{-1},\ a_3b_2a_3b_3,\ a_3b_2^{-1}a_3b_2^{-1}$	$\{0\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,2}$	0	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3^{-1}b_1^{-1},\ a_3b_2a_3b_3,\ a_3b_2^{-1}a_3b_2^{-1}$	$\{0\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,3}$	0	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3^{-1}b_1,\ a_3b_2a_3b_3,\ a_3b_2^{-1}a_3b_2^{-1}$	$\{0\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,4}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3^{-1}b_1,\ a_3b_2a_3b_3,\ a_3b_2^{-1}a_3b_2^{-1}$	$\{0\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,5}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3a_3b_3,\ a_3b_1a_3b_3^{-1},\ a_3b_2a_3^{-1}b_2^{-1},\ a_3b_1^{-1}a_3b_1^{-1}$	$\{0\}^*$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,6}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3a_3b_3,\ a_3b_1a_3b_1,\ a_3b_2a_3^{-1}b_2^{-1},\ a_3b_3^{-1}a_3b_1^{-1}$	$\{0\}^*$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,7}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ a_3b_1a_3b_3,\ a_3b_2a_3^{-1}b_2^{-1},\ a_3b_1^{-1}a_3b_1^{-1}$	$\{0\}^*$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,8}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ a_3b_1a_3b_1,\ a_3b_2a_3^{-1}b_2^{-1},\ a_3b_3a_3b_1^{-1}$	$\{0\}^*$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,9}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_3,\ a_3b_2a_3^{-1}b_2^{-1},\ a_3b_1^{-1}a_3b_1^{-1}$	$\{0\}^*$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,10}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_1,\ a_3b_2a_3^{-1}b_2^{-1},\ a_3b_3a_3b_1^{-1}$	$\{0\}^*$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,11}$	0	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_3,\ a_3b_2a_3^{-1}b_2^{-1},\ a_3b_1^{-1}a_3b_1^{-1}$	$\{0\}^*$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,12}$	0	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_1,\ a_3b_2a_3^{-1}b_2^{-1},\ a_3b_3a_3b_1^{-1}$	$\{0\}^*$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,13}$	0	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_3,\ a_3b_2a_3^{-1}b_2,\ a_3b_1^{-1}a_3b_1^{-1}$	$\{0\}^*$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,14}$	0	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_1,\ a_3b_2a_3^{-1}b_2,\ a_3b_3a_3b_1^{-1}$	$\{0\}^*$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,15}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ a_3b_1a_3b_3,\ a_3b_2a_3^{-1}b_2,\ a_3b_1^{-1}a_3b_1^{-1}$	$\{0\}^*$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,16}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ a_3b_1a_3b_1,\ a_3b_2a_3^{-1}b_2,\ a_3b_3a_3b_1^{-1}$	$\{0\}^*$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,17}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3a_3b_3,\ a_3b_1a_3b_3^{-1},\ a_3b_2a_3^{-1}b_2,\ a_3b_1^{-1}a_3b_1^{-1}$	$\{0\}^*$	$\{0,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,18}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3a_3b_3,\ a_3b_1a_3b_1,\ a_3b_2a_3^{-1}b_2,\ a_3b_3^{-1}a_3b_1^{-1}$	$\{0\}^*$	$\{0,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,19}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_3,\ a_3b_2a_3^{-1}b_2,\ a_3b_1^{-1}a_3b_1^{-1}$	$\{0\}^*$	$\{0,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,20}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_1,\ a_3b_2a_3^{-1}b_2,\ a_3b_3a_3b_1^{-1}$	$\{0\}^*$	$\{0,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,21}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_1^{-1},\ a_3b_2a_3b_3,\ a_3b_2^{-1}a_3b_2^{-1}$	$\{1\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,22}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3a_3b_3,\ a_3b_1a_3b_3^{-1},\ a_3b_2a_3b_2^{-1},\ a_3b_1^{-1}a_3b_1^{-1}$	$\{1\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,23}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3a_3b_3,\ a_3b_1a_3b_1,\ a_3b_2a_3b_2^{-1},\ a_3b_3^{-1}a_3b_1^{-1}$	$\{1\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,24}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ a_3b_1a_3b_3,\ a_3b_2a_3b_2^{-1},\ a_3b_1^{-1}a_3b_1^{-1}$	$\{1\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,25}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ a_3b_1a_3b_1,\ a_3b_2a_3b_2^{-1},\ a_3b_3a_3b_1^{-1}$	$\{1\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,26}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_3,\ a_3b_2a_3b_2^{-1},\ a_3b_1^{-1}a_3b_1^{-1}$	$\{1\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,27}$	0	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_1,\ a_3b_2a_3b_2^{-1},\ a_3b_3a_3b_1^{-1}$	$\{1\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,28}$	0	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_1^{-1},\ a_3b_2a_3b_3,\ a_3b_2^{-1}a_3b_2^{-1}$	$\{1\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,29}$	0	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_3,\ a_3b_2a_3b_2^{-1},\ a_3b_1^{-1}a_3b_1^{-1}$	$\{1\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,30}$	0	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_1,\ a_3b_2a_3b_2^{-1},\ a_3b_3a_3b_1^{-1}$	$\{1\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,31}$	0	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_1,\ a_3b_2a_3b_3,\ a_3b_2^{-1}a_3b_2^{-1},\ a_3b_1^{-1}a_3b_1^{-1}$	$\{1\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,32}$	0	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1},\ a_3b_1a_3b_3,\ a_3b_2a_3b_2,\ a_3b_2^{-1}a_3b_2^{-1},\ a_3b_1^{-1}a_3b_1^{-1}$	$\{1\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$

Table 2.16: Some virtually simple (6, 6)-groups. (Part 1/5)

Name	τ_1	τ_2	Squares: $a_1b_1a_2^{-1}b_1$, $a_1b_2a_2b_2^{-1}$, $a_1b_2^{-1}a_2^{-1}b_1^{-1}$, $a_1b_1^{-1}a_2^{-1}b_2$ +	H_1	H_2	$[\Gamma : \Gamma^{(\infty)}]$	$\mathrm{Aut}(X_{\Gamma+})$
$\Gamma_{6,6,33}$	0	0	$a_1b_3a_1^{-1}b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_2$, $a_3b_3a_3b_1^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{1\}$	$\{2\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,34}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{1\}$	$\{2\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,35}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_3b_3^{-1}$, $a_2b_3a_2^{-1}b_3$, $a_3b_1a_3b_3$, $a_3b_2a_3b_2$, $a_3b_2^{-1}a_3b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{1\}$	$\{2\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,36}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_3b_3^{-1}$, $a_2b_3a_2^{-1}b_3$, $a_3b_1a_3b_1$, $a_3b_2a_3b_2$, $a_3b_3a_3b_1^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{1\}$	$\{2\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,37}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2b_3^{-1}$, $a_2b_3a_3b_3$, $a_3b_1a_3b_3^{-1}$, $a_3b_2a_3b_2$, $a_3b_2^{-1}a_3b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{1\}$	$\{0, 2\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,38}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2b_3^{-1}$, $a_2b_3a_3b_3$, $a_3b_1a_3b_1$, $a_3b_2a_3b_2$, $a_3b_3^{-1}a_3b_1^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{1\}$	$\{0, 2\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,39}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2^{-1}b_3^{-1}$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3b_2$, $a_3b_2^{-1}a_3b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{1\}$	$\{0, 2\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,40}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2^{-1}b_3^{-1}$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_2$, $a_3b_3a_3b_1^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{1\}$	$\{0, 2\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,41}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2b_3^{-1}$, $a_2b_3a_3b_3$, $a_3b_1a_3b_2^{-1}$, $a_3b_2a_3b_3^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{0, 1\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,42}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2b_3^{-1}$, $a_2b_3a_3b_3$, $a_3b_1a_3b_1$, $a_3b_2a_3b_1^{-1}$, $a_3b_3^{-1}a_3b_2^{-1}$	$\{0, 1\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,43}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_3b_3^{-1}$, $a_2b_3a_2^{-1}b_3$, $a_3b_1a_3b_2^{-1}$, $a_3b_2a_3b_3$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{0, 1\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,44}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_3b_3^{-1}$, $a_2b_3a_2^{-1}b_3$, $a_3b_1a_3b_1$, $a_3b_2a_3b_1^{-1}$, $a_3b_3a_3b_2^{-1}$	$\{0, 1\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,45}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2^{-1}b_3^{-1}$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_2^{-1}$, $a_3b_2a_3b_3$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{0, 1\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,46}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2^{-1}b_3^{-1}$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_1^{-1}$, $a_3b_3a_3b_2^{-1}$	$\{0, 1\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,47}$	0	0	$a_1b_3a_1^{-1}b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_2^{-1}$, $a_3b_2a_3b_3$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{0, 1\}$	$\{1\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,48}$	0	0	$a_1b_3a_1^{-1}b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_1^{-1}$, $a_3b_3a_3b_2^{-1}$	$\{0, 1\}$	$\{1\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,49}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_2^{-1}$, $a_3b_2a_3b_2$, $a_3b_3a_3b_1^{-1}$	$\{0, 1\}^*$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,50}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3b_1^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{0, 1\}^*$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,51}$	0	0	$a_1b_3a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_2^{-1}$, $a_3b_2a_3b_2$, $a_3b_3a_3b_1^{-1}$	$\{0, 1\}^*$	$\{1\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,52}$	0	0	$a_1b_3a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3b_1^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{0, 1\}^*$	$\{1\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,53}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,54}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3^{-1}b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,55}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_2^{-1}$, $a_3b_3a_3b_1^{-1}$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,56}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3^{-1}b_2^{-1}$, $a_3b_3a_3b_1^{-1}$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,57}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2b_3^{-1}$, $a_2b_3a_3b_3$, $a_3b_1a_3b_1^{-1}$, $a_3b_2a_3b_3^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,58}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2b_3^{-1}$, $a_2b_3a_3b_3$, $a_3b_1a_3^{-1}b_1^{-1}$, $a_3b_2a_3b_3^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,59}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_3b_3^{-1}$, $a_2b_3a_2^{-1}b_3$, $a_3b_1a_3b_1^{-1}$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,60}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_3b_3^{-1}$, $a_2b_3a_2^{-1}b_3$, $a_3b_1a_3^{-1}b_1^{-1}$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,61}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2^{-1}b_3^{-1}$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1^{-1}$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,62}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2^{-1}b_3^{-1}$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3^{-1}b_1^{-1}$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,63}$	0	0	$a_1b_3a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{6,6,64}$	0	0	$a_1b_3a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3^{-1}b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2 \times \mathbf{C}_2 \times \mathbf{C}_2$

Table 2.17: Some virtually simple $(6, 6)$-groups. (Part 2/5)

Name	τ_1	τ_2	Squares: $a_1b_1a_2^{-1}b_1$, $a_1b_2a_2b_2^{-1}$, $a_1b_2^{-1}a_2^{-1}b_1^{-1}$, $a_1b_1^{-1}a_2^{-1}b_2$ +	H_1	H_2	$[\Gamma:\Gamma^{(\infty)}]$	$\mathrm{Aut}(X_{\Gamma+})$
$\Gamma_{6,6,65}$	0	0	$a_1b_3a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_2^{-1}$, $a_3b_3a_3b_1^{-1}$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,66}$	0	0	$a_1b_3a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3^{-1}b_2^{-1}$, $a_3b_3a_3b_1^{-1}$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,67}$	0	0	$a_1b_3a_1^{-1}b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1^{-1}$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,68}$	0	0	$a_1b_3a_1^{-1}b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3^{-1}b_1^{-1}$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,69}$	0	0	$a_1b_3a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3b_2$, $a_3b_2^{-1}a_3b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{2\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,70}$	0	0	$a_1b_3a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3^{-1}b_2$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{2\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,71}$	0	0	$a_1b_3a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_2$, $a_3b_3a_3b_1^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,72}$	0	0	$a_1b_3a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3^{-1}b_2$, $a_3b_3a_3b_1^{-1}$	$\{2\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,73}$	0	0	$a_1b_3a_1^{-1}b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{2\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,74}$	0	0	$a_1b_3a_1^{-1}b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3^{-1}b_1$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,75}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3b_2$, $a_3b_2^{-1}a_3b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{2\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,76}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3^{-1}b_2$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{2\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,77}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_2$, $a_3b_3a_3b_1^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,78}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3^{-1}b_2$, $a_3b_3a_3b_1^{-1}$	$\{2\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,79}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_3b_3^{-1}$, $a_2b_3a_2^{-1}b_3$, $a_3b_1a_3b_1$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{2\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,80}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_3b_3^{-1}$, $a_2b_3a_2^{-1}b_3$, $a_3b_1a_3^{-1}b_1$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,81}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2b_3^{-1}$, $a_2b_3a_3b_3$, $a_3b_1a_3b_1$, $a_3b_2a_3b_3^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{2\}$	$\{0,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,82}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2b_3^{-1}$, $a_2b_3a_3b_3$, $a_3b_1a_3^{-1}b_1$, $a_3b_2a_3b_3^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{0,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,83}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2^{-1}b_3^{-1}$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{2\}$	$\{0,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,84}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2^{-1}b_3^{-1}$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3^{-1}b_1$, $a_3b_2a_3b_3$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{2\}$	$\{0,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,85}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2b_3^{-1}$, $a_2b_3a_3b_3$, $a_3b_1a_3b_2^{-1}$, $a_3b_2a_3b_2$, $a_3b_3^{-1}a_3b_1^{-1}$	$\{0,2\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,86}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2b_3^{-1}$, $a_2b_3a_3b_3$, $a_3b_1a_3b_3^{-1}$, $a_3b_2a_3b_1^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{0,2\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,87}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_3b_3^{-1}$, $a_2b_3a_2^{-1}b_3$, $a_3b_1a_3b_2^{-1}$, $a_3b_2a_3b_2$, $a_3b_3a_3b_1^{-1}$	$\{0,2\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,88}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_3b_3^{-1}$, $a_2b_3a_2^{-1}b_3$, $a_3b_1a_3b_3$, $a_3b_2a_3b_1^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{0,2\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,89}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2^{-1}b_3^{-1}$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_2^{-1}$, $a_3b_2a_3b_2$, $a_3b_3a_3b_1^{-1}$	$\{0,2\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,90}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_2^{-1}b_3^{-1}$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3b_1^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{0,2\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,91}$	0	0	$a_1b_3a_1^{-1}b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_2^{-1}$, $a_3b_2a_3b_2$, $a_3b_3a_3b_1^{-1}$	$\{0,2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,92}$	0	0	$a_1b_3a_1^{-1}b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_3$, $a_3b_2a_3b_1^{-1}$, $a_3b_2^{-1}a_3b_2^{-1}$	$\{0,2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,93}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_2^{-1}$, $a_3b_2a_3b_3$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{0,2\}^*$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,94}$	0	0	$a_1b_3a_1b_3$, $a_1b_3^{-1}a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_1^{-1}$, $a_3b_3a_3b_2^{-1}$	$\{0,2\}^*$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,95}$	0	0	$a_1b_3a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_2^{-1}$, $a_3b_2a_3b_3$, $a_3b_1^{-1}a_3b_1^{-1}$	$\{0,2\}^*$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,96}$	0	0	$a_1b_3a_1b_3^{-1}$, $a_2b_3a_2b_3$, $a_2b_3^{-1}a_3b_3^{-1}$, $a_3b_1a_3b_1$, $a_3b_2a_3b_1^{-1}$, $a_3b_3a_3b_2^{-1}$	$\{0,2\}^*$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$

Table 2.18: Some virtually simple $(6,6)$-groups. (Part 3/5)

Name	τ_1	τ_2	Squares: $a_1b_1a_2^{-1}b_1,\ a_1b_2a_2b_2^{-1},\ a_1b_2^{-1}a_2^{-1}b_1^{-1},\ a_1b_1^{-1}a_2^{-1}b_2\ +$	H_1	H_2	$[\Gamma:\Gamma^{(\infty)}]$	$\mathrm{Aut}(X_{\Gamma^+})$
$\Gamma_{6,6,97}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3A_3b_3,\ A_3b_1A_3b_1,\ A_3b_2A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,98}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3A_3b_3,\ A_3b_1A_3b_1,\ A_3b_2A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,99}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}A_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ A_3b_1A_3b_1,\ A_3b_2A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,100}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}A_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ A_3b_1A_3b_1,\ A_3b_2A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,101}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,102}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,103}$	2	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,104}$	2	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{1\}$	12	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,105}$	2	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,106}$	2	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,107}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3A_3b_3,\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,108}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3A_3b_3,\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,109}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}A_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,110}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}A_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,111}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,112}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,113}$	2	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{0,1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,114}$	2	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{0,1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,115}$	2	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,116}$	2	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{1,2\}$	12	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,117}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}A_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,118}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}A_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,119}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}A_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ A_3b_1A_3b_1,\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,120}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}A_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ A_3b_1A_3b_1,\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,121}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3A_3b_3,\ A_3b_1A_3b_1,\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{0,1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,122}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3A_3b_3,\ A_3b_1A_3b_1,\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{0,1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,123}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{0,1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,124}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{0,1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,125}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3A_3b_3,\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{2,3\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,126}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2b_3^{-1},\ a_2b_3A_3b_3,\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{2,3\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,127}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{0\}$	$\{2,3\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,128}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_2^{-1}b_3^{-1},\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2,\ A_4b_1A_4b_3$	$\{0\}$	$\{2,3\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$

Table 2.19: Some virtually simple $(6,6)$-groups. (Part 4/5)

Name	τ_1	τ_2	Squares: $a_1b_1a_2^{-1}b_1,\ a_1b_2a_2b_2^{-1},\ a_1b_2^{-1}a_2^{-1}b_1^{-1},\ a_1b_1^{-1}a_2^{-1}b_2\ +$	H_1	H_2	$[\Gamma:\Gamma^{(\infty)}]$	$\mathrm{Aut}(X_{\Gamma+})$
$\Gamma_{6,6,129}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_4b_1,\ A_3b_2A_3b_2,\ A_4b_2A_4b_3^{-1}$	$\{0\}^*$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,130}$	2	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_4b_1^{-1},\ A_3b_2A_3b_2^{-1},\ A_3b_1^{-1}A_4b_1,\ A_4b_2A_4b_3^{-1}$	$\{0\}^*$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,131}$	2	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_4b_1,\ A_3b_2A_3b_2,\ A_4b_2A_4b_3^{-1}$	$\{0\}^*$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,132}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_4b_1^{-1},\ A_3b_2A_3b_2^{-1},\ A_3b_1^{-1}A_4b_1,\ A_4b_2A_4b_3^{-1}$	$\{0\}^*$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,133}$	2	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_4b_1^{-1},\ A_3b_2A_3b_2,\ A_3b_1^{-1}A_4b_1,\ A_4b_2A_4b_3^{-1}$	$\{0\}^*$	$\{0,1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,134}$	2	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_4b_1,\ A_3b_2A_3b_2^{-1},\ A_4b_2A_4b_3^{-1}$	$\{0\}^*$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,135}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_4b_1^{-1},\ A_3b_2A_3b_2,\ A_3b_1^{-1}A_4b_1,\ A_4b_2A_4b_3^{-1}$	$\{0\}^*$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,136}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_4b_1,\ A_3b_2A_3b_2^{-1},\ A_4b_2A_4b_3^{-1}$	$\{0\}^*$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,137}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,138}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2,\ A_4b_1A_4b_3$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,139}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}A_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ A_3b_1A_4b_1,\ A_3b_2A_3b_2,\ A_4b_2A_4b_3^{-1}$	$\{2\}$	$\{0\}^*$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,140}$	2	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,141}$	2	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,142}$	2	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,143}$	2	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2,\ A_4b_1A_4b_3$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,144}$	2	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_4b_1^{-1},\ A_3b_2A_3b_2^{-1},\ A_3b_1^{-1}A_4b_1,\ A_4b_2A_4b_3^{-1}$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,145}$	2	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_4b_1,\ A_3b_2A_3b_2,\ A_4b_2A_4b_3^{-1}$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,146}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,147}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,148}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}A_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ A_3b_1A_4b_1^{-1},\ A_3b_2A_3b_2^{-1},\ A_3b_1^{-1}A_4b_1,\ A_4b_2A_4b_3^{-1}$	$\{2\}$	$\{1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,149}$	2	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{2\}$	$\{0,1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,150}$	2	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3$	$\{2\}$	$\{0,1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,151}$	2	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_4b_1^{-1},\ A_3b_2A_3b_2,\ A_3b_1^{-1}A_4b_1,\ A_4b_2A_4b_3^{-1}$	$\{2\}$	$\{0,1\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,152}$	2	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{2\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,153}$	2	0	$a_1b_3a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2,\ A_4b_1A_4b_3$	$\{2\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,154}$	2	0	$a_1b_3a_1^{-1}b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_4b_1,\ A_3b_2A_3b_2^{-1},\ A_4b_2A_4b_3^{-1}$	$\{2\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,155}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{2\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,156}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1^{-1},\ A_3b_2A_4b_2,\ A_4b_1A_4b_3$	$\{2\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,157}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3^{-1}$	$\{2\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,158}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}A_3b_3^{-1},\ A_3b_1A_3b_1,\ A_3b_2A_4b_2^{-1},\ A_3b_2^{-1}A_4b_2,\ A_4b_1A_4b_3$	$\{2\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,159}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}A_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ A_3b_1A_4b_1^{-1},\ A_3b_2A_3b_2,\ A_3b_1^{-1}A_4b_1,\ A_4b_2A_4b_3^{-1}$	$\{2\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$
$\Gamma_{6,6,160}$	2	0	$a_1b_3a_1b_3,\ a_1b_3^{-1}A_3b_3^{-1},\ a_2b_3a_2^{-1}b_3,\ A_3b_1A_4b_1,\ A_3b_2A_3b_2^{-1},\ A_4b_2A_4b_3^{-1}$	$\{2\}$	$\{1,2\}$	4	$\mathbf{C}_2\times\mathbf{C}_2$

Table 2.20: Some virtually simple $(6,6)$-groups. (Part 5/5)

2.4.2 Virtually simple $(4,5)$-groups

In this subsection we use the same strategy as above so as to discover virtually simple $(4,5)$-groups. The NST however requires the closures of the projections to be boundary-2-transitive. In the previous section we were dealing with 6-regular trees, so [BM00a, Propositions 3.3.1 and 3.3.2] could be used to ensure the 2-transitivity on the boundary. For 4-regular and 5-regular trees, those results do not apply. We will therefore need the following theorem, due to Trofimov.

Theorem 2.4.3 (Trofimov)**.** *Let X be a connected $(q+1)$-regular graph with $q \geq 2$ a prime power and let $G \leq \mathrm{Aut}(X)$ be vertex-transitive. Let $v \in V(X)$ and suppose that $\underline{G}(v)$ contains $\mathrm{PSL}(2,q)$ (acting on the projective line over $\mathbf{F}_q$) as a normal subgroup. If G is non-discrete, then X is the $(q+1)$-regular tree and the closure $\overline{G} \leq \mathrm{Aut}(X)$ of G is 2-transitive on ∂X.*

Proof. See [Tro07, Proposition 3.1 and Example 3.2]. Note that the original statement only mentions that X is the $(q+1)$-regular tree. However, the proof consists in showing that G is transitive on paths of length ℓ of X for each $\ell \geq 1$. This assertion implies that X is a tree, but also that $\overline{G}$ is 2-transitive on ∂X. □

In §2.4.1 we started with a non-residually finite torsion-free $(4,4)$-group. This time we start with a non-residually finite $(3,3)$-group. Let $\Gamma_{3,3}$ be the $(3,3)$-group associated to the six squares in Figure 2.9. The local action of $\Gamma_{3,3}$ on T_1 (resp. T_2) is $\mathrm{Sym}(3)$ (resp. $\mathbf{C}_2$). In the next result, with the same ideas as for Proposition 2.4.1, we show that $\Gamma_{3,3}$ is irreducible and not residually finite. Note that we could have used [CW17, Corollary 6.4] for the non-residual finiteness, but once again we wanted an explicit non-trivial element of $\Gamma_{3,3}^{(\infty)}$.

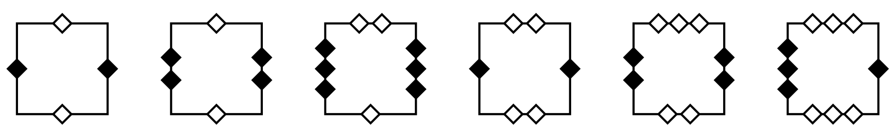


Figure 2.9: The $(3,3)$-group $\Gamma_{3,3}$.

Proposition 2.4.4. *The group* $\Gamma_{3,3}$ *is irreducible and not residually finite. Moreover,* $[B_2(B_1B_3)^2B_2, B_1B_3]$ *or* $[B_2(B_1B_3)^2B_2, B_1B_3A_2]$ *is a non-trivial element of* $\Gamma_{3,3}^{(\infty)}$.

Proof. By [Wei79, Theorems 1.1 and 1.4], $\Gamma_{3,3}$ is irreducible if and only if $\mathrm{Fix}_{\mathrm{proj}_1(\Gamma_{3,3})}(B(v_1,3))$ is non-trivial. We first remark that $(B_1B_2)^2$ fixes $B(v_1,1)$. Hence, $(B_1B_2)^4$ fixes $B(v_1,2)$ and $(B_1B_2)^8$ fixes $B(v_1,3)$. Moreover, $(B_1B_2)^8$ does not fix $B(v_1,4)$ as $(B_1B_2)^8(A_2A_3A_1A_3(v_1)) = A_2A_3A_1A_2(v_1)$ (this can be seen by drawing a 4×16 rectangle). So $\Gamma_{3,3}$ is irreducible.

We now want a non-trivial element in $\Gamma_{3,3}^{(\infty)}$. Recall from [Cap17a, Lemma 4.13] that $[C_G(H), \overline{H}] \subseteq G^{(\infty)}]$ for any subgroup H of a group G. We take $G = \Gamma_{3,3}$ and $H = \langle A_1, A_3, A_2A_1A_2, A_2A_3A_2\rangle$. Note that H is a subgroup of $\langle A_1, A_2, A_3\rangle = \Gamma_{3,3}(v_2)$. Actually, $\Gamma_{3,3}(v_2)$ acts simply transitively on the vertices of T_1 and H has two orbits of vertices in T_1, so its index in $\Gamma_{3,3}(v_2)$ is 2. It is also quick to check that $B_2(B_1B_3)^2B_2 \in C_{\Gamma_{3,3}}(H)$. We now claim that $B_1B_3 \in \overline{\Gamma_{3,3}(v_2)}$. As $A_2 \notin H$ and H is an index 2 subgroup of $\Gamma_{3,3}(v_2)$, it will follow that $B_1B_3 \in \overline{H}$ or $B_1B_3A_2 \in \overline{H}$.

We show that $B_1B_3 \in \overline{\Gamma_{3,3}(v_2)}$ by mimicking the proof of Proposition 2.4.1, which was illustrated on Figure 2.8. Consider a finite quotient $\varphi\colon \Gamma_{3,3} \to Q$. Since $\Gamma_{3,3}$ is irreducible, the projection $\mathrm{proj}_2(\Gamma_{3,3}(v_2))$ is infinite. Hence, its finite index subgroup $\mathrm{proj}_2(\mathrm{Fix}_{\Gamma_{3,3}}(B(v_2,1)) \cap \ker\varphi)$ is also infinite. Let γ be an element of $\mathrm{Fix}_{\Gamma_{3,3}}(B(v_2,1)) \cap \ker\varphi$ such that $\mathrm{proj}_2(\gamma)$ is non-trivial. In T_2, there is a vertex $w \neq v_2$ such that γ fixes the path from v_2 to w but does not fix some neighbor z of w: $\gamma(z) = z' \neq z$. Write $w = h(v_2)$ with $h \in \langle B_1, B_2, B_3\rangle$, $z = hx(v_2)$ with $x \in \{B_1, B_2, B_3\}$ and $z' = hx'(v_2)$ with $x' \in \{B_1, B_2, B_3\}$. Recall that $\mathrm{proj}_2(\Gamma_{3,3}(v_2))$ acts on the three neighbors of v_2 as $\mathbf{C}_2$: the only non-trivial permutation induced on these three vertices is the transposition $(B_1(v_2)\ B_3(v_2))$. We thus have the same local action around w, and the fact that γ fixes w and some neighbor of w while not fixing z implies that $\{x, x'\} = \{B_1, B_3\}$. Then we get that $\gamma hx = hx'\gamma'$ for some $\gamma' \in \Gamma_{3,3}(v_2)$. As $\varphi(\gamma) = 1$, this implies that $\varphi(x'^{-1}x) \in \varphi(\Gamma_{3,3}(v_2))$. Either $x'^{-1}x$ or its inverse is equal to B_1B_3, so $\varphi(B_1B_3) \in \varphi(\Gamma_{3,3}(v_2))$.

This is true for all finite quotients $\varphi\colon \Gamma_{3,3} \to Q$, so $B_1B_3 \in \overline{\Gamma_{3,3}(v_2)}$. $\square$

The same proof can actually show that $[B_2(B_1B_3)^2B_2, (B_1B_3)^2]$ is always a non-trivial element of $\Gamma_{3,3}^{(\infty)}$. But this element has length 20 and our computer could hardly deal with it. Instead, the elements $[B_2(B_1B_3)^2B_2, B_1B_3]$ and

$$[B_2(B_1B_3)^2B_2, B_1B_3A_2] = B_2(B_1B_3)^2B_2B_1B_3B_2(B_1B_3)^2B_2B_3B_1$$

given by Proposition 2.4.4 have length 16, which is slightly better.

Using GAP we could search for $(4,5)$-groups Γ containing $\Gamma_{3,3}$ or the mirror of $\Gamma_{3,3}$ (i.e. $\{(g_1, g_2) \in \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2) \mid (g_2, g_1) \in \Gamma_{3,3}\}$), and such that $\underline{H_1}(v_1) \trianglerighteq \mathrm{PSL}(2,3) \cong \mathrm{Alt}(4)$ and $\underline{H_2}(v_2) \trianglerighteq \mathrm{PSL}(2,4) \cong \mathrm{Alt}(5)$. We say that Γ satisfies $(**)$ if the above conditions are true. Since $\Gamma_{3,3}$ is irreducible, a group Γ satisfying $(**)$ is also irreducible and H_t is 2-transitive on ∂T_t for each $t \in \{1, 2\}$ by Theorem 2.4.3. Thus the NST applies and Γ is virtually simple.

There are 60 equivalence classes of $(4,5)$-groups satisfying $(**)$, all with $\tau_1 = 4$ and $\tau_2 = 5$ (i.e. with 9 generators, each of order 2). We give in Tables 2.21–2.22 a group in each class: call them $\Gamma_{4,5,1}, \ldots, \Gamma_{4,5,60}$. Note that $\Gamma_{4,5,1}, \ldots, \Gamma_{4,5,28}$ contain $\Gamma_{3,3}$ while $\Gamma_{4,5,29}, \ldots, \Gamma_{4,5,60}$ contain its mirror. We can make some remarks similar to those in §2.4.1:

- The index of the simple subgroup $\Gamma_{4,5,k}^{(\infty)}$ of $\Gamma_{4,5,k}$ can be computed by using the fact that $r_1 = [B_2(B_1B_3)^2B_2, B_1B_3]$ or $r_2 = [B_2(B_1B_3)^2B_2, B_1B_3A_2]$ belongs to $\Gamma_{3,3}^{(\infty)}$ (see Proposition 2.4.4). Indeed, if Q_1 (resp. Q_2) is the group obtained by adding the relator r_1 (resp. r_2) to the presentation of $\Gamma_{4,5,k}$, then $[\Gamma_{4,5,k} : \Gamma_{4,5,k}^{(\infty)}] = \max(|Q_1|, |Q_2|)$. (For $k \geq 29$ we must actually consider the mirrors of r_1 and r_2). The indices that we obtain are written in the tables. When the index is 4 we have $\Gamma_{4,5,k}^{(\infty)} = \Gamma_{4,5,k}^{+}$, and when it is 8 we have that $\Gamma_{4,5,k}^{(\infty)}$ is an index 2 subgroup of $\Gamma_{4,5,k}^{+}$.
- For each $k \in \{1, \ldots, 60\}$ we get $\mathrm{Aut}(X_{\Gamma_{4,5,k}^{+}}) \cong \mathbf{C}_2 \times \mathbf{C}_2$, so $\Gamma_{4,5,k}$ is the only $(4,5)$-group whose type-preserving subgroup is $\Gamma_{4,5,k}^{+}$. Therefore all $\Gamma_{4,5,k}^{+}$ are pairwise non-conjugate in $\mathrm{Aut}(T_1 \times T_2)$ (and thus pairwise non-isomorphic by [BMZ09, Corollary 1.1.22]).

Theorem 2.4.5 (Theorem 2.C (ii)). *Let $\Gamma_{4,5,k}$ ($k \in \{1, \ldots, 60\}$) be one of the* $(4,5)$*-groups given by Tables 2.21–2.22.*

- *If $k \in \{1, \ldots, 32\} \cup \{39, \ldots, 54\}$, then $\Gamma^+_{4,5,k}$ is simple.*
- *If $k \in \{33, \ldots, 38\} \cup \{55, \ldots, 60\}$, then $\Gamma^+_{4,5,k}$ has a simple subgroup of index* 2.

Moreover, all groups $\Gamma^+_{4,5,k}$ are pairwise non-isomorphic.

Proof. See the discussion above. □

Corollary 2.4.6. *For each $k \in \{1, \ldots, 32\} \cup \{39, \ldots, 54\}$, there exist two injections $F_{11} \hookrightarrow F_3$ of free groups such that the simple group $\Gamma^+_{4,5,k}$ is isomorphic to the amalgamated free product $F_3 *_{F_{11}} F_3$.*

Proof. Recall that $\Gamma_{4,5,k}(v_2) = \langle A_1, A_2, A_3, A_4 \rangle$, where A_1, A_2, A_3 and A_4 are order 2 elements sending v_1 to its four neighbours in T_1. Hence, the index 2 subgroup $G = \Gamma^+_{4,5,k}(v_2)$ is a free group of rank 3, on the 3 generators A_1A_2, A_1A_3 and A_1A_4. If v_2' is a vertex adjacent to v_2 in T_2, then $G' = \Gamma^+_{4,5,k}(v_2')$ is also isomorphic to F_3. Moreover, these two point stabilizers G and G' generate $\Gamma^+_{4,5,k}$ so that $\Gamma^+_{4,5,k} = G *_{G \cap G'} G'$ (see [Ser77, Theorem 6]). The subgroup $G \cap G'$ has index 5 in both G and G', so $G \cap G'$ is free of rank $1 + 5(3-1) = 11$ by the Nielsen–Schreier formula. □

Proof of Corollary 2.D (ii). This is the presentation of $\Gamma^+_{4,5,9}$ (see Figure 2.10). In order to find this presentation, we write $x_1 = A_1A_2$, $x_2 = A_1A_3$ and $x_3 = A_1A_4$ so that $G = \Gamma^+_{4,5,9}(v_2)$ is freely generated by

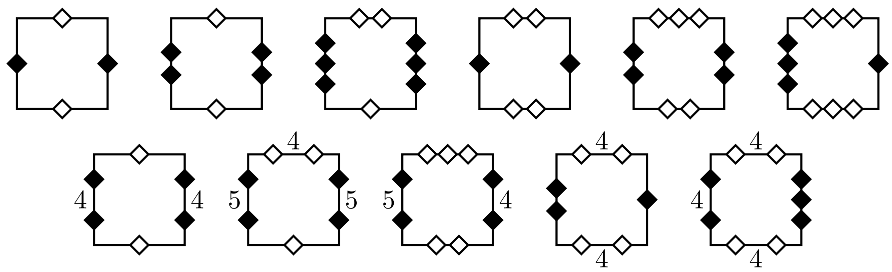


Figure 2.10: The $(4,5)$-group $\Gamma_{4,5,9}$.

x_1, x_2 and x_3. In the same manner, the stabilizer $G' = \Gamma^+_{4,5,9}(B_1(v_2))$ of the vertex $B_1(v_2)$ (which is adjacent to v_2 in T_2) is freely generated by $y_1 = B_1A_1A_2B_1$, $y_2 = B_1A_1A_3B_1$ and $y_3 = B_1A_1A_4B_1$. The subgroup $G \cap G'$ of G has index 5 in G, and the Reidemeister–Schreier method can be used to find 11 generators of this subgroup (which is isomorphic to F_{11}). After some computations we came up with the 11 elements of $G = \langle x_1, x_2, x_3\rangle$ written on the left-hand sides of the 11 relations of the presentation given in the statement. Those 11 elements also belong to $G' = \langle y_1, y_2, y_3\rangle$, and it then suffices to write them in terms of the y_i's. For instance, for $x_2^2 \in G$ we have

$$\begin{aligned} x_2^2 &= (A_1A_3)^2 \\ &= B_1(B_1(A_1A_3)^2B_1)B_1 \\ &= B_1(A_1A_3A_2A_3)B_1 \\ &= B_1(A_1A_3A_2A_1A_1A_3)B_1 \\ &= B_1(X_2X_1^{-1}X_2)B_1 \\ &= y_2y_1^{-1}y_2 \end{aligned}$$

The geometric squares defining $\Gamma_{4,5,9}$ have been used to find the equality $B_1(A_1A_3)^2B_1 = A_1A_3A_2A_3$. $\square$

Name	Squares: $A_1B_1A_1B_1$, $A_1B_2A_1B_2$, $A_1B_3A_2B_3$, $A_2B_1A_2B_1$, $A_2B_2A_3B_2$, $A_3B_1A_3B_3$ +	$\underline{H_1}(v_1)$	$\underline{H_2}(v_2)$	$[\Gamma : \Gamma^{(\infty)}]$	$\mathrm{Aut}(X_{\Gamma+})$
$\Gamma_{4,5,1}$	$A_1B_4A_1B_4$, $A_1B_5A_1B_5$, $A_2B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,2}$	$A_1B_4A_1B_4$, $A_1B_5A_1B_5$, $A_2B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,3}$	$A_1B_4A_1B_4$, $A_1B_5A_3B_5$, $A_2B_4A_2B_5$, $A_3B_4A_4B_4$, $A_4B_1A_4B_2$, $A_4B_3A_4B_5$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,4}$	$A_1B_4A_1B_4$, $A_1B_5A_3B_5$, $A_2B_4A_2B_5$, $A_3B_4A_4B_4$, $A_4B_1A_4B_5$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,5}$	$A_1B_4A_1B_4$, $A_1B_5A_4B_5$, $A_2B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_3B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,6}$	$A_1B_4A_1B_4$, $A_1B_5A_4B_5$, $A_2B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_3B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,7}$	$A_1B_4A_1B_4$, $A_1B_5A_4B_5$, $A_2B_4A_2B_5$, $A_3B_4A_3B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,8}$	$A_1B_4A_1B_4$, $A_1B_5A_4B_5$, $A_2B_4A_2B_5$, $A_3B_4A_3B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,9}$	$A_1B_4A_1B_4$, $A_1B_5A_4B_5$, $A_2B_4A_3B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,10}$	$A_1B_4A_1B_4$, $A_1B_5A_4B_5$, $A_2B_4A_3B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,11}$	$A_1B_4A_3B_4$, $A_1B_5A_4B_5$, $A_2B_4A_2B_5$, $A_3B_5A_3B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,12}$	$A_1B_4A_3B_4$, $A_1B_5A_4B_5$, $A_2B_4A_2B_5$, $A_3B_5A_3B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,13}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,14}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,15}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_3B_5$, $A_3B_4A_4B_4$, $A_4B_1A_4B_2$, $A_4B_3A_4B_5$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,16}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_3B_5$, $A_3B_4A_4B_4$, $A_4B_1A_4B_5$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,17}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_4B_5$, $A_3B_4A_3B_4$, $A_3B_5A_3B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,18}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_4B_5$, $A_3B_4A_3B_4$, $A_3B_5A_3B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,19}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_4B_5$, $A_3B_4A_3B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,20}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_4B_5$, $A_3B_4A_3B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,21}$	$A_1B_4A_1B_5$, $A_2B_4A_3B_4$, $A_2B_5A_4B_5$, $A_3B_5A_3B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,22}$	$A_1B_4A_1B_5$, $A_2B_4A_3B_4$, $A_2B_5A_4B_5$, $A_3B_5A_3B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,23}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,24}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,25}$	$A_1B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,26}$	$A_1B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,27}$	$A_1B_4A_3B_5$, $A_2B_4A_2B_4$, $A_2B_5A_4B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,28}$	$A_1B_4A_3B_5$, $A_2B_4A_2B_4$, $A_2B_5A_4B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$

Table 2.21: Some virtually simple $(4,5)$-groups containing $\Gamma_{3,3}$.

Name	Squares: $A_1B_1A_1B_1$, $A_1B_2A_1B_2$, $A_1B_3A_3B_3$, $A_2B_1A_2B_1$, $A_2B_2A_2B_3$, $A_3B_1A_3B_2$ +	$\underline{H_1}(v_1)$	$\underline{H_2}(v_2)$	$[\Gamma : \Gamma^{(\infty)}]$	$\mathrm{Aut}(X_{\Gamma^+})$
$\Gamma_{4,5,29}$	$A_1B_4A_1B_4$, $A_1B_5A_2B_5$, $A_2B_4A_4B_4$, $A_3B_4A_3B_5$, $A_4B_1A_4B_1$, $A_4B_2A_4B_2$, $A_4B_3A_4B_5$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,30}$	$A_1B_4A_1B_4$, $A_1B_5A_2B_5$, $A_2B_4A_4B_4$, $A_3B_4A_3B_5$, $A_4B_1A_4B_1$, $A_4B_2A_4B_5$, $A_4B_3A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,31}$	$A_1B_4A_1B_4$, $A_1B_5A_2B_5$, $A_2B_4A_4B_4$, $A_3B_4A_3B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_5$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,32}$	$A_1B_4A_1B_4$, $A_1B_5A_2B_5$, $A_2B_4A_4B_4$, $A_3B_4A_3B_5$, $A_4B_1A_4B_3$, $A_4B_2A_4B_5$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,33}$	$A_1B_4A_1B_4$, $A_1B_5A_4B_5$, $A_2B_4A_3B_5$, $A_4B_1A_4B_1$, $A_4B_2A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	8	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,34}$	$A_1B_4A_1B_4$, $A_1B_5A_4B_5$, $A_2B_4A_3B_5$, $A_4B_1A_4B_1$, $A_4B_2A_4B_4$, $A_4B_3A_4B_3$	Sym(4)	Sym(5)	8	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,35}$	$A_1B_4A_1B_4$, $A_1B_5A_4B_5$, $A_2B_4A_3B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Alt(5)	8	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,36}$	$A_1B_4A_1B_4$, $A_1B_5A_4B_5$, $A_2B_4A_3B_5$, $A_4B_1A_4B_3$, $A_4B_2A_4B_4$	Sym(4)	Alt(5)	8	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,37}$	$A_1B_4A_1B_4$, $A_1B_5A_4B_5$, $A_2B_4A_3B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_2$, $A_4B_3A_4B_3$	Sym(4)	Sym(5)	8	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,38}$	$A_1B_4A_1B_4$, $A_1B_5A_4B_5$, $A_2B_4A_3B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Alt(5)	8	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,39}$	$A_1B_4A_2B_4$, $A_1B_5A_4B_5$, $A_2B_5A_2B_5$, $A_3B_4A_3B_5$, $A_4B_1A_4B_1$, $A_4B_2A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,40}$	$A_1B_4A_2B_4$, $A_1B_5A_4B_5$, $A_2B_5A_2B_5$, $A_3B_4A_3B_5$, $A_4B_1A_4B_1$, $A_4B_2A_4B_4$, $A_4B_3A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,41}$	$A_1B_4A_2B_4$, $A_1B_5A_4B_5$, $A_2B_5A_2B_5$, $A_3B_4A_3B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,42}$	$A_1B_4A_2B_4$, $A_1B_5A_4B_5$, $A_2B_5A_2B_5$, $A_3B_4A_3B_5$, $A_4B_1A_4B_3$, $A_4B_2A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,43}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_3B_5$, $A_3B_4A_4B_4$, $A_4B_1A_4B_1$, $A_4B_2A_4B_2$, $A_4B_3A_4B_5$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,44}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_3B_5$, $A_3B_4A_4B_4$, $A_4B_1A_4B_1$, $A_4B_2A_4B_5$, $A_4B_3A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,45}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_3B_5$, $A_3B_4A_4B_4$, $A_4B_1A_4B_2$, $A_4B_3A_4B_5$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,46}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_3B_5$, $A_3B_4A_4B_4$, $A_4B_1A_4B_3$, $A_4B_2A_4B_5$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,47}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_3B_5$, $A_3B_4A_4B_4$, $A_4B_1A_4B_5$, $A_4B_2A_4B_2$, $A_4B_3A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,48}$	$A_1B_4A_1B_5$, $A_2B_4A_2B_4$, $A_2B_5A_3B_5$, $A_3B_4A_4B_4$, $A_4B_1A_4B_5$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,49}$	$A_1B_4A_1B_5$, $A_2B_4A_3B_4$, $A_2B_5A_4B_5$, $A_3B_5A_3B_5$, $A_4B_1A_4B_1$, $A_4B_2A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,50}$	$A_1B_4A_1B_5$, $A_2B_4A_3B_4$, $A_2B_5A_4B_5$, $A_3B_5A_3B_5$, $A_4B_1A_4B_1$, $A_4B_2A_4B_4$, $A_4B_3A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,51}$	$A_1B_4A_1B_5$, $A_2B_4A_3B_4$, $A_2B_5A_4B_5$, $A_3B_5A_3B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,52}$	$A_1B_4A_1B_5$, $A_2B_4A_3B_4$, $A_2B_5A_4B_5$, $A_3B_5A_3B_5$, $A_4B_1A_4B_3$, $A_4B_2A_4B_4$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,53}$	$A_1B_4A_1B_5$, $A_2B_4A_3B_4$, $A_2B_5A_4B_5$, $A_3B_5A_3B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_2$, $A_4B_3A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,54}$	$A_1B_4A_1B_5$, $A_2B_4A_3B_4$, $A_2B_5A_4B_5$, $A_3B_5A_3B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	4	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,55}$	$A_1B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_1$, $A_4B_2A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	8	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,56}$	$A_1B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_1$, $A_4B_2A_4B_4$, $A_4B_3A_4B_3$	Sym(4)	Sym(5)	8	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,57}$	$A_1B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_2$, $A_4B_3A_4B_4$	Sym(4)	Sym(5)	8	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,58}$	$A_1B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_3$, $A_4B_2A_4B_4$	Sym(4)	Sym(5)	8	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,59}$	$A_1B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_2$, $A_4B_3A_4B_3$	Sym(4)	Sym(5)	8	$\mathbf{C}_2 \times \mathbf{C}_2$
$\Gamma_{4,5,60}$	$A_1B_4A_2B_5$, $A_3B_4A_3B_4$, $A_3B_5A_4B_5$, $A_4B_1A_4B_4$, $A_4B_2A_4B_3$	Sym(4)	Sym(5)	8	$\mathbf{C}_2 \times \mathbf{C}_2$

Table 2.22: Some virtually simple $(4, 5)$-groups containing the mirror of $\Gamma_{3,3}$.

2.4.3 Virtually simple $(2n, 2n+1)$-groups $(n \geq 2)$

In §2.4.2 we gave a list of virtually simple $(4,5)$-groups $\Gamma_{4,5,k}$. We now construct for each $n \geq 2$ a virtually simple $(2n, 2n+1)$-group. For $n = 2$ we take $\Gamma_{4,5} = \Gamma_{4,5,9}$, see Table 2.21 and Figure 2.10. For $n \geq 3$ we define $\Gamma_{2n,2n+1}$ as the $(2n, 2n+1)$-group whose geometric squares are:

(1) the 11 geometric squares of $\Gamma_{4,5}$;

(2) the 3 geometric squares $A_{2r}B_{2r+1}A_1B_{2r}$, $A_{2r-1}B_{2r}A_{2r-1}B_1$ and $A_{2r-1}B_{2r+1}A_2B_{2r+1}$ for each $3 \leq r \leq n$, see Figure 2.11;

(3) all geometric squares $A_jB_kA_jB_k$ with $j \in \{1, \ldots, 2n\}$ and $k \in \{1, \ldots, 2n+1\}$ such that the corner (A_j, B_k) does not already appear in a geometric square of (1) or (2).

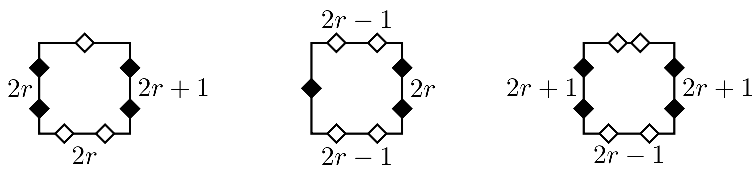


Figure 2.11: Additional squares in $\Gamma_{2n,2n+1}$.

Theorem 2.4.7 (Theorem 2.E)**.** *For each $n \geq 2$, $\Gamma_{2n,2n+1}$ is a virtually simple $(2n, 2n+1)$-group with $\underline{H_1}(v_1) \cong \mathrm{Sym}(2n)$ and $\underline{H_2}(v_2) \cong \mathrm{Sym}(2n+1)$ and such that $\Gamma_{2n,2n+1} \big/ \Gamma^{(\infty)}_{2n,2n+1} \cong (\mathbf{C}_2)^n$. Moreover, if $n \geq 3$, then there is a legal coloring i of T_1 such that*

- *$H_1 = G_{(i)}(\{4\}, \{4\})$ if n is even;*
- *$H_1 = G_{(i)}(\{0, 2, 3\}, \{0, 2, 3\})$ if n is odd.*

Proof. The group $\underline{H_1}(v_1)$ is generated by the following $2n+1$ permutations, which clearly generate $\mathrm{Sym}(2n)$:

$$\begin{aligned} B_1 &: () \\ B_2 &: (A_2\ A_3) \\ B_3 &: (A_1\ A_2) \end{aligned}$$

$$\begin{aligned} B_4 &: (A_2\ A_3) \\ B_5 &: (A_1\ A_4)(A_2\ A_3) \\ B_{2r}\ (3 \le r \le n) &: (A_1\ A_{2r}) \\ B_{2r+1}\ (3 \le r \le n) &: (A_1\ A_{2r})(A_2\ A_{2r-1}) \end{aligned}$$

For $\underline{H_2}(v_2)$ we have the $2n$ permutations

$$\begin{aligned} A_1 &: (B_6\ B_7)(B_8\ B_9)\dots(B_{2n}\ B_{2n+1}) \\ A_2 &: (B_4\ B_5) \\ A_3 &: (B_1\ B_3)(B_4\ B_5) \\ A_4 &: (B_1\ B_2)(B_3\ B_4) \\ A_{2r-1}\ (3 \le r \le n) &: (B_1\ B_{2r}) \\ A_{2r}\ (3 \le r \le n) &: (B_{2r}\ B_{2r+1}) \end{aligned}$$

The permutations A_2, A_3, A_4 generate $\mathrm{Sym}(5)$, and we then get by induction that the permutations $A_2, \dots, A_{2r}$ generate $\mathrm{Sym}(2r+1)$ for each $3 \le r \le n$. In particular, we have $\underline{H_2}(v_2) \cong \mathrm{Sym}(2n+1)$.

We already know that $\Gamma_{4,5}$ is virtually simple. For $n \ge 3$, by [BM00a, Proposition 3.3.2] the groups H_1 and H_2 are boundary-2-transitive. Moreover $\Gamma_{2n,2n+1}$ contains $\Gamma_{4,5}$ so it is irreducible and non-residually finite. The NST then implies that $\Gamma_{2n,2n+1}$ is virtually simple.

We now compute $\Gamma_{2n,2n+1} \big/ \Gamma^{(\infty)}_{2n,2n+1}$. Recall from Proposition 2.4.4 that $r_1 = [B_2(B_1B_3)^2B_2, B_1B_3]$ or $r_2 = [B_2(B_1B_3)^2B_2, B_1B_3A_2]$ belongs to $\Gamma^{(\infty)}_{3,3}$ (and thus to $\Gamma^{(\infty)}_{2n,2n+1}$). We have by Theorem 2.4.5 that $\Gamma_{4,5} \big/ \Gamma^{(\infty)}_{4,5} \cong (\mathbf{C}_2)^2$, so r_1 and r_2 both belong to $\Gamma^{(\infty)}_{4,5} \le \Gamma^{(\infty)}_{2n,2n+1}$. The quotient $\Gamma_{2n,2n+1} \big/ \Gamma^{(\infty)}_{2n,2n+1}$ is therefore isomorphic to the finite group Q_n obtained by adding the relator r_1 to the presentation of $\Gamma_{2n,2n+1}$. We write $\overline{A}_j$ (resp. $\overline{B}_k$) instead of A_j (resp. B_k) for the generators of Q_n. The relators of $Q_2 \cong (\mathbf{C}_2)^2$ all appear in the presentation of Q_n, so the subgroup

$$\langle \overline{A}_1, \overline{A}_2, \overline{A}_3, \overline{A}_4, \overline{B}_1, \overline{B}_2, \overline{B}_3, \overline{B}_4, \overline{B}_5 \rangle \le Q_n$$

is isomorphic to $(\mathbf{C}_2)^2$ (with $\overline{A}_1 = \overline{A}_2 = \overline{A}_3 = \overline{A}_4$ and $\overline{B}_1 = \overline{B}_2 =$

$\overline{B}_3 = \overline{B}_4 = \overline{B}_5$). Moreover, for each $5 \leq j \leq 2n$ there exists a geometric square in the definition of $\Gamma_{2n,2n+1}$ of the form $A_jB_kA_jB_k$ for some $1 \leq k \leq 5$. Similarly, for each $6 \leq k \leq 2n+1$ there is a geometric square of the form $A_jB_kA_jB_k$ for some $1 \leq j \leq 4$. We thus deduce that $\overline{B}_1$ commutes with $\overline{A}_j$ for each $1 \leq j \leq 2n$ and that $\overline{A}_1$ commutes with $\overline{B}_k$ for each $1 \leq k \leq 2n+1$. From the geometric square $A_{2r-1}B_{2r}A_{2r-1}B_1$ we therefore get that $\overline{B}_{2r} = \overline{B}_1$ for each $3 \leq r \leq n$, and from the geometric square $A_{2r-1}B_{2r+1}A_2B_{2r+1}$ we get that $\overline{A}_{2r-1} = \overline{A}_1$ for each $3 \leq r \leq n$. The relators of Q_n then directly give $\overline{A}_j\overline{B}_k = \overline{B}_k\overline{A}_j$ for all $1 \leq j \leq 2n$ and $1 \leq k \leq 2n+1$, except when $j \geq 6$ is even and $k = j+1$. In that particular case, we can however remark from the geometric square $A_{2r}B_{2r+1}A_1B_{2r}$ that $\overline{A}_{2r}\overline{B}_{2r+1} = \overline{B}_1\overline{A}_1$ for each $3 \leq r \leq n$. As $(\overline{A}_{2r}\overline{B}_{2r+1})^{-1} = \overline{B}_{2r+1}\overline{A}_{2r}$ and $(\overline{B}_1\overline{A}_1)^{-1} = \overline{A}_1\overline{B}_1 = \overline{B}_1\overline{A}_1$, we also obtain $\overline{A}_{2r}\overline{B}_{2r+1} = \overline{B}_{2r+1}\overline{A}_{2r}$. Using those equalities, the presentation of Q_n can be reduced to

$$\left\langle \begin{array}{c} \overline{A}_1, \overline{A}_6, \overline{A}_8, \ldots, \overline{A}_{2n}, \\ \overline{B}_1, \overline{B}_7, \overline{B}_9, \ldots, \overline{B}_{2n+1} \end{array} \;\middle|\; \begin{array}{l} \overline{A}_j^2 \text{ for all } j,\ \overline{B}_k^2 \text{ for all } k, \\ \overline{A}_j\overline{B}_k\overline{A}_j\overline{B}_k \text{ for all } j \text{ and } k, \\ \overline{A}_{2r}\overline{B}_{2r+1}\overline{A}_1\overline{B}_1 \text{ for each } 3 \leq r \leq n \end{array} \right\rangle$$

One easily checks from the relators that all $\overline{A}_j$ pairwise commute and all $\overline{B}_k$ pairwise commute, so that Q_n is an abelian finite group. Each relator $\overline{A}_{2r}\overline{B}_{2r+1}\overline{A}_1\overline{B}_1$ can then be used to erase $\overline{B}_{2r+1}$ from the presentation, and we are left with the generators $\overline{A}_1, \overline{A}_6, \overline{A}_8, \ldots, \overline{A}_{2n}, \overline{B}_1$ and no other relator than the commutators. So $Q_n \cong (\mathbf{C}_2)^n$ as wanted.

When $n \geq 3$, the group H_1 can be computed thanks to the algorithms developed in §2.3. We do not give the details here, but it can be seen when computing the graph $G^{(1)}_{\Gamma_{2n,2n+1}}$ that the result will only depend on the parity of n. So it suffices to proceed for $n = 3$ and $n = 4$. □

2.4.4 Virtually simple $(6, 4n)$-groups ($n \geq 2$)

Let T_1 be the 6-regular tree and $T_2^{(n)}$ be the $4n$-regular tree for $n \geq 2$. In this section we describe a sequence $(\Gamma_{6,4n})_{n\geq 2}$ of groups, with $\Gamma_{6,4n}$ being a $(6, 4n)$-group, such that $\overline{\mathrm{proj}_1(\Gamma_{6,4n})} \to \mathrm{Aut}(T_1)$ in the Chabauty topology of $\mathrm{Aut}(T_1)$ when $n \to \infty$.

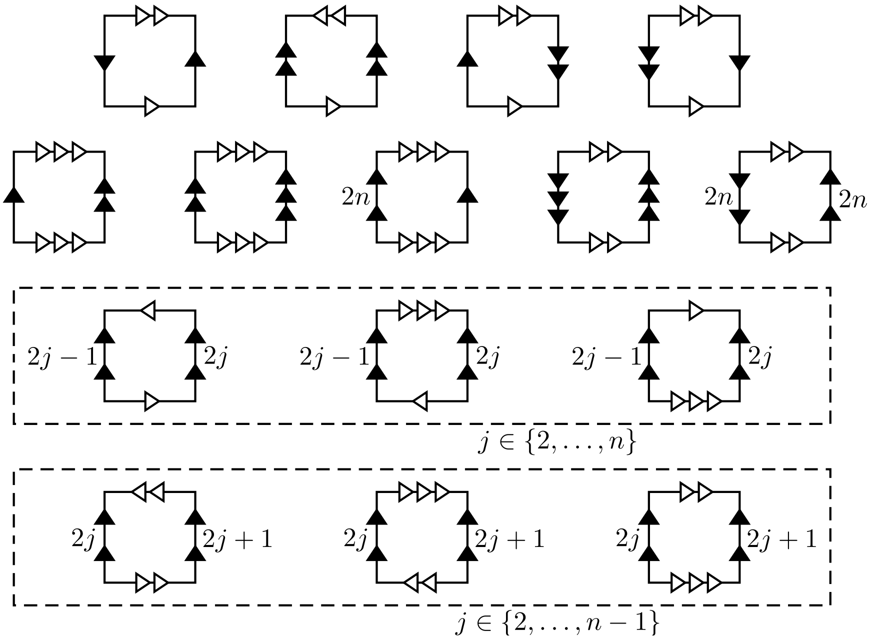


Figure 2.12: The torsion-free $(6, 4n)$-group $\Gamma_{6,4n}$.

Theorem 2.4.8 (Theorem 2.F)**.** *Let $n \geq 2$ be an integer and let $\Gamma_{6,4n}$ be the torsion-free $(6, 4n)$-group associated to the geometric squares in Figure 2.12. Then $\Gamma_{6,4n}$ is virtually simple, $\overline{\mathrm{proj}_1(\Gamma_{6,4n})} = G_{(i_1)}(\{n\}, \{n\})$ for some legal coloring i_1 of T_1 and $\overline{\mathrm{proj}_2(\Gamma_{6,4n})} = G_{(i_2)}(\{0\}, \{0\})$ for some legal coloring i_2 of $T_2^{(n)}$.*

Proof. One easily checks that the geometric squares given in Figure 2.12 indeed define a torsion-free $(6, 4n)$-group. The first four squares correspond to $\Gamma_{4,4}$ (see §2.4.1), so that $\Gamma_{4,4} \leq \Gamma_{6,4n}$. In particular, $\Gamma_{6,4n}$ is irreducible and non-residually finite. If we show that $\underline{H_1}(v_1) \geq \mathrm{Alt}(6)$ and $\underline{H_2}(v_2) \geq \mathrm{Alt}(4n)$ (where $H_t = \overline{\mathrm{proj}_t(\Gamma_{6,4n})}$), then it will follow from the NST and [BM00a, Propositions 3.3.1 and 3.3.2] that $\Gamma_{6,4n}$ is virtually simple.

The group $\underline{H_1}(v_1)$ is generated by the following permutations.

$$
\begin{aligned}
b_1 &: (a_1\ a_2)(a_1^{-1}\ a_2^{-1})(a_3)(a_3^{-1}) \\
b_2 &: (a_1\ a_2\ a_1^{-1}\ a_2^{-1})(a_3)(a_3^{-1}) \\
b_3 &: (a_1\ a_3\ a_1^{-1})(a_2)(a_2^{-1})(a_3^{-1})
\end{aligned}
$$

$$\begin{aligned}
b_{2j} &: (a_1\ a_1^{-1}\ a_3^{-1})(a_2\ a_3\ a_2^{-1}) && (j \in \{2,\dots,n-1\})\\
b_{2j+1} &: (a_1\ a_3\ a_1^{-1})(a_2\ a_2^{-1}\ a_3^{-1}) && (j \in \{2,\dots,n-1\})\\
b_{2n} &: (a_1\ a_1^{-1}\ a_3^{-1})(a_2)(a_2^{-1})(a_3)
\end{aligned}$$

The permutations induced by b_1, b_2, b_3 and b_{2n} generate $\mathrm{Sym}(6)$, so $\underline{H_1}(v_1) = \mathrm{Sym}(6)$. For $\underline{H_2}(v_2)$ we get:

$$\begin{aligned}
a_1 &: (b_1\ b_1^{-1}\ b_2^{-1})(b_2)(b_3\ b_4)(b_3^{-1}\ b_4^{-1})\dots(b_{2n-1}\ b_{2n})(b_{2n-1}^{-1}\ b_{2n}^{-1})\\
a_2 &: (b_1\ b_2\ b_1^{-1})(b_2^{-1})(b_3\ b_3^{-1})(b_{2n}\ b_{2n}^{-1})\\
&\qquad (b_4\ b_5)(b_4^{-1}\ b_5^{-1})\dots(b_{2n-2}\ b_{2n-1})(b_{2n-2}^{-1}\ b_{2n-1}^{-1})\\
a_3 &: (b_{2n}\ b_{2n-1}\ \dots\ b_2\ b_1)(b_{2n}^{-1}\ b_{2n-1}^{-1}\ \dots\ b_2^{-1}\ b_1^{-1})
\end{aligned}$$

We observe that a_1^2 and a_2^2 induce the permutations $(b_1\ b_2^{-1}\ b_1^{-1})$ and $(b_1\ b_1^{-1}\ b_2)$ respectively, which generate $\mathrm{Alt}(\{b_1, b_1^{-1}, b_2, b_2^{-1}\})$. Conjugating this alternating group by several powers of a_3, we obtain all $\mathrm{Alt}(\{b_j, b_j^{-1}, b_{j+1}, b_{j+1}^{-1}\})$ with $j \in \{1,\dots,2n-1\}$. These alternating groups together generate $\mathrm{Alt}(2n)$. As the permutations induced by a_1, a_2 and a_3 are all even, we get $\underline{H_2}(v_2) = \mathrm{Alt}(2n)$. This already implies that $H_2 = G_{(i_2)}(\{0\},\{0\})$ for some legal coloring i_2 of $T_2^{(n)}$.

There remains to compute H_1, using the algorithms developed in §2.3. The simplified labelled graph $\tilde{G}^{(1)}_{\Gamma_{6,4n}}$ is a cycle with only one label -1, see Figure 2.13. From this graph and via Proposition 2.3.8, we can compute the values of $s_k^{(1)}(b_j)$ for $j \in \{1,\dots,2n\}$ and $k \in \mathbf{Z}_{\geq 0}$:

	$s_0^{(1)}$	$s_1^{(1)}$	$s_2^{(1)}$	$\dots$	$s_{n-1}^{(1)}$	$s_n^{(1)}$
b_2	-1	$+1$	$+1$	$\dots$	$+1$	$+1$
b_3	$+1$	-1	$+1$	$\dots$	$+1$	$+1$
$\vdots$						
b_{n+1}	$+1$	$+1$	$+1$	$\dots$	-1	$+1$
b_{n+2}	$+1$	$+1$	$+1$	$\dots$	$+1$	$+1$
b_{n+3}	$+1$	$+1$	$+1$	$\dots$	-1	$+1$
$\vdots$						
b_{2n}	$+1$	$+1$	-1	$\dots$	$+1$	$+1$
b_1	$+1$	-1	$+1$	$\dots$	$+1$	$+1$

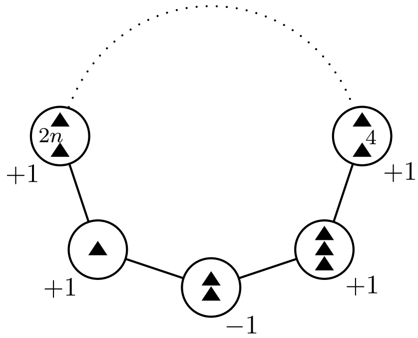


Figure 2.13: The simplified labelled graph $\tilde{G}^{(1)}_{\Gamma_{6,4n}}$.

From these values we deduce that $K^{(1)} = n$ and

$$s^{(1)}(H_1(v_1)) = \{(s_0, \ldots, s_n) \in (\mathbf{C}_2)^{n+1} \mid s_n = 1\}.$$

Hence, the four groups that can be isomorphic to H_1 are $G_{(i_1)}(\{n\}, \{n\})$, $G_{(i_1)}(Y, Y)^*$, $G'_{(i_1)}(Y, Y)^*$ and $G_{(i_1)}(Y^*, Y^*)$ where $\alpha(Y) = \{n\}$. We have $Y = \{1, 3, \ldots, n-1\}$ if n is even and $Y = \{0, 2, \ldots, n-1\}$ if n is odd.

Now let us see which of the four groups is the good one, thanks to Proposition 2.3.9. The very first equality in both systems $(*)$ and $(**)$ comes from the first and third geometric squares defining $\Gamma_{6,4n}$ and is $x_1 x_2 \Sigma_{4n} = x_2 x_1 \Sigma_{4n-1}$, where $\Sigma_{4n} = \prod_{r \in Y} s_r^{(1)}(b_1^{-1})$ and $\Sigma_{4n-1} = \prod_{r \in Y} s_r^{(1)}(b_2^{-1})$. But from the table above we can compute that $\Sigma_{4n} \neq \Sigma_{4n-1}$ in any case, so $(*)$ and $(**)$ have no solution. Hence $H_1 = G_{(i_1)}(\{n\}, \{n\})$ for some legal coloring i_1 of T_1. $\square$

Proof of Corollary 2.G. Define $\Gamma_{6,4n} \leq \mathrm{Aut}(T) \times \mathrm{Aut}(T_2^{(n)})$ for $n \geq 2$ as in Theorem 2.4.8. Given $v_2 \in V(T_2^{(n)})$, the group $F = \mathrm{proj}_1(\Gamma_{6,4n}(v_2)) \leq \mathrm{Aut}(T)$ is torsion-free and acts simply transitively on the vertices of $\mathrm{Aut}(T)$: it is thus conjugate to F_3 in $\mathrm{Aut}(T)$. Moreover, the full projection $\mathrm{proj}_1(\Gamma_{6,4n}) \leq \mathrm{Aut}(T)$ commensurates F. Indeed, if $\gamma \in \Gamma_{6,4n}$ then $\gamma\Gamma_{6,4n}(v_2)\gamma^{-1} = \Gamma_{6,4n}(\gamma(v_2))$ so $\Gamma_{6,4n}(v_2) \cap \gamma\Gamma_{6,4n}(v_2)\gamma^{-1}$ is nothing else than the fixator of v_2 and $\gamma(v_2)$ in $\Gamma_{6,4n}$. This is a finite index subgroup of $\Gamma_{6,4n}(v_2)$ and $\Gamma_{6,4n}(\gamma(v_2))$ as wanted. Hence, the closure of the commensurator of F_3 in $\mathrm{Aut}(T)$ contains $G_{(i^{(n)})}(\{n\}, \{n\})$ for some legal coloring $i^{(n)}$ of T (see Theorem 2.4.8). The conclusion follows from the fact that $\bigcup_{n\geq 2} G_{(i^{(n)})}(\{n\},\{n\})$ is dense in $\mathrm{Aut}(T)$. □

2.5 About products of three trees

We conclude the chapter with the proof of Theorem 2.H.

Proof of Theorem 2.H. Suppose that such a group Γ exists. Let us consider $\Gamma(v_3)$, the fixator of v_3 in T_3. The group $\Gamma' = \mathrm{proj}_{1,2}(\Gamma(v_3)) \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ acts simply transitively on the vertices of $T_1 \times T_2$, i.e. it is a $(6,6)$-group. By hypothesis, $\mathrm{proj}_{1,3}(\Gamma)$ is dense in $H_1 \times H_3$, so $\mathrm{proj}_{1,3}(\Gamma(v_3))$ is dense in $H_1 \times H_3(v_3)$ (because $H_3(v_3)$ is open in H_3). Taking images under the continuous map proj_1, we get that $\mathrm{proj}_1(\Gamma(v_3))$ is dense in H_1, i.e. $\overline{\mathrm{proj}_1(\Gamma')} = H_1$. Similarly, we have $\overline{\mathrm{proj}_2(\Gamma')} = H_2$. We deduce in particular that Γ' is an irreducible $(6,6)$-group whose local actions on T_1 and T_2 contain $\mathrm{Alt}(6)$. The last hypothesis also implies that the values τ_1 and τ_2 associated to Γ' are both equal to zero.

As can be read from Tables 2.12 and 2.14, there are 23225 equivalence classes of irreducible $(6,6)$-groups with $\tau_1 = \tau_2 = 0$ and $\underline{H_1}(v_1), \underline{H_2}(v_2) \geq \mathrm{Alt}(6)$. We indeed have 2240 such groups that are torsion-free and 20985 such groups with torsion.

There remains to prove that none of those 23225 groups can be equal to Γ'. Let γ be an element of $\Gamma(v_3)$. It induces a permutation of the six neighbors of v_3. Since all elements of $\mathrm{Sym}(6)$ have order ≤ 6, there exists $o \in \{4,5,6\}$ such that γ^o fixes $B(v_3,1)$ in T_3. If Q_{γ^o} is the group obtained by adding the relation $\gamma^o = 1$ to the presentation of $\Gamma(v_3) \cong \Gamma'$, then we have a natural surjection $Q_{\gamma^o} \to \underline{H_3}(v_3)$. Now recall that, if γ^o is non-trivial, then Q_{γ^o} is a finite group by the Normal Subgroup Theorem. Also, $\underline{H_3}(v_3)$ is isomorphic to $\mathrm{Alt}(6)$ or $\mathrm{Sym}(6)$ by hypothesis. Hence, for each of the 23225 groups mentioned above and for each generator $g \in \{a_1,a_2,a_3,b_1,b_2,b_3\}$ we can compute (with GAP) the groups Q_{g^o} for each $o \in \{4,5,6\}$ and check if one of these three finite groups surjects onto $\mathrm{Alt}(6)$ or $\mathrm{Sym}(6)$. If the answer is no for one of the six generators, then that group can be excluded.

We could check this condition on all 23225 groups, and the answer is clear: none of them satisfies the condition. □

Chapter 3

A lattice in a residually non-Desarguesian $\tilde{A}_2$-building

In Chapter 2 we studied groups acting simply transitively on the vertices of a product of two trees. Another related subject concerns groups acting simply transitively on the vertices of an $\tilde{A}_2$-building. There is a strong similarity between the lattices appearing in these two contexts. In fact both belong to the formal framework of *polygonal presentations* introduced in [Vdo02]. They do not, however, enjoy the same qualitative properties (e.g. QI-rigidity, Kazhdan's Property (T)). Groups acting simply transitively on the vertices of an $\tilde{A}_2$-building are also hard to construct in general. The main achievement of this chapter, whose content has been published in [Rad17b], is the construction of a locally exotic $\tilde{A}_2$-building admitting such a lattice.

3.1 Main results

A (thick) $\tilde{A}_2$-building is a simply connected simplicial complex of dimension 2 such that all simplicial spheres of radius 1 around vertices are isomorphic to the incidence graph of a projective plane. These projective planes are called the **residue planes** of the $\tilde{A}_2$-building.

The vertex set $V(\Delta)$ of an $\tilde{A}_2$-building Δ can be partitioned as $V(\Delta) = V_0(\Delta) \sqcup V_1(\Delta) \sqcup V_2(\Delta)$, so that each triangle of Δ has a vertex of each type (where a vertex of **type** t is a vertex in $V_t(\Delta)$). An element $g \in \mathrm{Aut}(\Delta)$ is called **type-rotating** if its induced action σ on the set of types $\{0,1,2\}$ satisfies $\sigma(t) = t + c \bmod 3$ for some $c \in \{0,1,2\}$.

Our main motivation is to prove the following result.

Theorem 3.A. *There exist an $\tilde{A}_2$-building Δ and a group $\Gamma \leq \mathrm{Aut}(\Delta)$ satisfying the following properties:*

(1) All residue planes of Δ are isomorphic to the Hughes plane of order 9.[1]

(2) The group Γ acts simply transitively on the set of vertices of Δ.

(3) All elements of Γ are type-rotating.

(4) The index 3 subgroup Γ^+ of Γ consisting of the type-preserving automorphisms is torsion-free.

(5) The derived subgroup $[\Gamma,\Gamma]$ of Γ is perfect and $\Gamma \big/ [\Gamma,\Gamma] \cong \mathbf{C}_2 \times \mathbf{C}_3$.

(6) There exists an infinite family $\{\Delta_0^n\}_n$ of disjoint isomorphic sub-buildings of Δ whose residue planes are isomorphic to $\mathrm{PG}(2,3)$ and such that each vertex of Δ is contained in one sub-building Δ_0^n.

(7) The stabilizer of a vertex in $\mathrm{Aut}(\Delta)$ has order 96, i.e. $[\mathrm{Aut}(\Delta) : \Gamma] = 96$. In particular, $\mathrm{Aut}(\Delta)$ equipped with the topology of pointwise convergence is discrete.

Moreover, as any unimodular locally compact group acting continuously, properly and cocompactly on an $\tilde{A}_2$-building, Γ satisfies Kazhdan's Property (T) (see [BdlHV08, Theorem 5.7.7]). Groups with Property (T) are deeply studied in [BdlHV08].

[1] The Hughes plane of order 9 was actually first constructed by O. Veblen and J. Maclagan-Wedderburn in 1907, see [VMW07]. This was the first discovered finite non-Desarguesian projective plane, and the role of D. Hughes in [Hug57] has been to generalize their construction to get an infinite family of finite non-Desarguesian planes (with order p^{2n} for p an odd prime).

The tools that we develop in the discussion toward Theorem 3.A also leads us to the following side result about groups acting simply transitively on the vertices of a thick $\tilde{A}_2$-building. Given a projective plane Π with point set P and line set L, a **correlation** (or **duality**) δ of Π is a pair of bijections $\delta_P\colon P \to L$ and $\delta_L\colon L \to P$ that preserve incidence, i.e. such that $p \in \ell$ if and only if $\delta_P(p) \ni \delta_L(\ell)$. It is then customary to also call $\delta_P\colon P \to L$ a correlation (and δ_L is uniquely determined by δ_P).

Proposition 3.B. *Let Δ be a thick $\tilde{A}_2$-building and let $\Gamma \leq \mathrm{Aut}(\Delta)$ be a group of type-rotating automorphisms of Δ acting simply transitively on $V(\Delta)$. Let P (resp. L) be the set of neighbors of type 1 (resp. 2) of a fixed vertex $v_0 \in V_0(\Delta)$, and denote by Π the residue plane at v_0 (with P and L as sets of points and lines). Let $\lambda\colon P \to L$ be the bijection such that, for each $x \in P$, the unique element of Γ sending v_0 to $x \in P$ sends $\lambda(x) \in L$ to v_0. Then λ is not a correlation of Π.*

3.2 Previous work on the subject

In [CMSZ93a], Cartwright, Mantero, Steger and Zappa were interested in groups acting simply transitively on the vertices of an $\tilde{A}_2$-building. We will make great use of their work and give in this section the essential definitions and results.

3.2.1 P-L correspondences and triangle presentations

We start with the following definition from [CMSZ93a].

Definition 3.2.1. Let P and L be the sets of points and lines respectively in a projective plane Π. A bijection $\lambda\colon P \to L$ is called a **P-L correspondence** in Π. A subset $\mathcal{T} \subseteq P^3$ is then called a **triangle presentation** compatible with λ if the two following conditions hold:

1. For all $x, y \in P$, there exists $z \in P$ such that $(x, y, z) \in \mathcal{T}$ if and only if $y \in \lambda(x)$ in Π. In this case, z is unique.

2. If $(x, y, z) \in \mathcal{T}$, then $(y, z, x) \in \mathcal{T}$.

Example 3.2.2. The projective plane $\mathrm{PG}(2,2)$ can be defined by $P = L = \mathbf{Z}/7\mathbf{Z}$ with line $x \in L$ being adjacent to the points $x+1$, $x+2$ and $x+4$ in P. Consider the P-L correspondence $\lambda: P \to L: x \in P \mapsto x \in L$ in Π. Then

$$\mathcal{T} := \{(x, x+1, x+3), (x+1, x+3, x), (x+3, x, x+1) \mid x \in P\}$$

is a triangle presentation compatible with λ. Indeed, (ii) is obviously satisfied and, for $x, y \in P$, it is apparent that there exists (a unique) $z \in P$ such that $(x, y, z) \in \mathcal{T}$ if and only if $y \in \{x+1, x+2, x+4\}$, which is exactly the set of points on the line $\lambda(x)$.

Now suppose we have a thick $\tilde{A}_2$-building Δ and a group $\Gamma \leq \mathrm{Aut}(\Delta)$ of type-rotating automorphisms of Δ acting simply transitively on $V(\Delta)$. In this context, the following theorem shows how one can associate to Γ a P-L correspondence and a triangle presentation compatible with it.

Theorem 3.2.3 (Cartwright–Mantero–Steger–Zappa)**.** *Let Δ be a thick $\tilde{A}_2$-building and let $\Gamma \leq \mathrm{Aut}(\Delta)$ be a group of type-rotating automorphisms of Δ acting simply transitively on $V(\Delta)$. Let P (resp. L) be the set of neighbors of type 1 (resp. 2) of a fixed vertex $v_0 \in V_0(\Delta)$, and denote by Π the residue plane at v_0 (with P and L as sets of points and lines). For each $x \in P$, let g_x be the unique element of Γ such that $g_x(v_0) = x$. Let $\lambda: P \to L$ be the P-L correspondence in Π defined by $\lambda(x) = g_x^{-1}(v_0)$ for each $x \in P$. Then there exists a triangle presentation $\mathcal{T}$ compatible with λ such that Γ has the following presentation:*

$$\Gamma = \langle \{g_x\}_{x \in P} \mid g_x g_y g_z = 1 \text{ for each } (x, y, z) \in \mathcal{T} \rangle.$$

Proof. See [CMSZ93a, Theorem 3.1]. □

What makes triangle presentations really interesting is the fact that a reciprocal result exists. Given a projective plane Π, a P-L correspondence $\lambda: P \to L$ in Π and a triangle presentation compatible with λ, one can construct an $\tilde{A}_2$-building Δ locally isomorphic to Π and a group acting simply transitively on $V(\Delta)$.

Theorem 3.2.4 (Cartwright–Mantero–Steger–Zappa)**.** *Let P and L be the sets of points and lines in a projective plane Π, let $\lambda\colon P \to L$ be a P-L correspondence in Π and let $\mathcal{T}$ be a triangle presentation compatible with λ. Define*

$$\Gamma_{\mathcal{T}} := \langle \{a_x\}_{x\in P} \mid a_x a_y a_z = 1 \text{ for each } (x,y,z) \in \mathcal{T}\rangle,$$

where $\{a_x\}_{x\in P}$ are distinct letters. Then there exists an $\tilde{A}_2$-building $\Delta_{\mathcal{T}}$ whose residue planes are isomorphic to Π and such that $\Gamma_{\mathcal{T}}$ acts simply transitively on $V(\Delta_{\mathcal{T}})$, by type-rotating automorphisms.

Proof. See [CMSZ93a, Theorem 3.4], or §3.2.2 below. □

Example 3.2.5. From Example 3.2.2 and Theorem 3.2.4, we get an $\tilde{A}_2$-building Δ whose residue planes are isomorphic to $\mathrm{PG}(2,2)$ and a group acting simply transitively on the set of vertices of Δ. The building Δ is actually the Bruhat–Tits building associated to $\mathrm{PGL}(3, \mathbf{F}_2(\!(X)\!))$ (see [CMSZ93b, §4] and [CMSZ93a, Theorem 4.1]).

3.2.2 Building associated to a triangle presentation

In [CMSZ93a, Theorem 3.4], the authors gave an explicit construction of the $\tilde{A}_2$-building $\Delta_{\mathcal{T}}$ associated to a triangle presentation $\mathcal{T}$ (see Theorem 3.2.4 above). In this section we show a geometric way to construct $\Delta_{\mathcal{T}}$ and $\Gamma_{\mathcal{T}}$. The following discussion can also be seen as an alternative proof of Theorem 3.2.4.

Following [Kan86], an $\tilde{A}_2$**-SCAB** is a connected chamber system of rank 3 whose residues of rank 2 are generalized 3-gons (i.e. incidence graphs of projective planes). We will always think of an $\tilde{A}_2$-SCAB as a set of triangles, representing the chambers, glued together so that two chambers are adjacent if and only if they share an edge.

Suppose we are given a P-L correspondence $\lambda\colon P \to L$ in a projective plane Π and a triangle presentation $\mathcal{T}$ compatible with λ. Let us first define a finite $\tilde{A}_2$-SCAB $\mathcal{C}_{\mathcal{T}}$ as follows. Consider three vertices v_1, v_2, v_3: those will be the only vertices of $\mathcal{C}_{\mathcal{T}}$. Then, for each $x \in P$, put an edge e_x between v_1 and v_2, an edge e'_x between v_2 and v_3 and an edge e''_x between v_3 and v_1. Finally, for each $(x,y,z) \in \mathcal{T}$, attach a triangle to the three edges e_x, e'_y and e''_z. One readily checks that the definition of a triangle presentation ensures that the three rank 2 residues of $\mathcal{C}_\mathcal{T}$ are incidence graphs of the projective plane Π, and hence that $\mathcal{C}_\mathcal{T}$ is indeed an $\tilde{A}_2$-SCAB. Note that $\mathcal{C}_\mathcal{T}$ is not a simplicial complex in the usual sense as all simplices of dimension 2 have the same three vertices.

We then consider the **universal covering $\tilde{A}_2$-SCAB** $\tilde{\mathcal{C}}_\mathcal{T}$ of $\mathcal{C}_\mathcal{T}$, as defined in [Kan86, Definition B.3.3, Proposition B.3.4]. This universal covering $\tilde{\mathcal{C}}_\mathcal{T}$ is a simply connected simplicial complex of dimension 2 whose simplicial spheres of radius 1 are isomorphic to the incidence graph of Π, so it is an $\tilde{A}_2$-building (see [Kan86, Theorem B.3.8] for a more rigorous proof of this fact). We therefore set $\Delta_\mathcal{T} := \tilde{\mathcal{C}}_\mathcal{T}$. Moreover, because of (2) in Definition 3.2.1, there is an automorphism $\alpha \in \mathrm{Aut}(\mathcal{C}_\mathcal{T})$ sending e_x to e'_x, e'_x to e''_x and e''_x to e_x for each $x \in P$. In other words, there is a natural action of the group $\mathbf{C}_3$ of order 3 on $\mathcal{C}_\mathcal{T}$. This automorphism group $\mathbf{C}_3$ then lifts to an automorphism group $\tilde{\mathbf{C}}_3$ of $\Delta_\mathcal{T}$ (see [Kan86, Corollary B.3.7]), and $\tilde{\mathbf{C}}_3$ acts simply transitively on the set of vertices of $\Delta_\mathcal{T}$ (and by type-rotating automorphisms). The group $\Gamma_\mathcal{T}$ can thus be taken to be $\tilde{\mathbf{C}}_3$. The presentation of $\Gamma_\mathcal{T}$ given in Theorem 3.2.4 can finally be found by observing that the 1-skeleton of $\Delta_\mathcal{T}$ is a Cayley graph for $\Gamma_\mathcal{T}$.

3.3 The strategy

A way to construct an $\tilde{A}_2$-building with non-Desarguesian residues and admitting a lattice is, in view of Theorem 3.2.4, to consider a non-Desarguesian projective plane Π and to find a P-L correspondence in Π and a triangle presentation compatible with it. The smallest non-Desarguesian projective planes are the Hughes plane of order 9, the Hall plane of order 9 and the dual of the Hall plane. The Hughes plane is self-dual, so there exist some natural P-L correspondences in it: the correlations. For this reason, we decided to work on the Hughes plane of order 9. It will appear later that correlations do not actually admit a triangle presentation (see Proposition 3.B), but they will still be helpful in our search for a suitable P-L correspondence.

For any Desarguesian projective plane, Cartwright–Mantero–Steger–

Zappa gave in [CMSZ93a, §4] an explicit formula for one P-L correspondence admitting a triangle presentation. Of course, they use the finite field from which the projective plane is constructed, and it is not clear how to find a similar formula for a particular non-Desarguesian projective plane.

Since we are searching for purely combinatorial objects, the use of a computer could be considered. In [CMSZ93b], the authors used a computer to find all triangle presentations in the projective planes of order 2 and 3. The number of points in these projective planes being not too large (i.e. 7 and 13), they could do a *brute-force* computation. However, already for order 3 they needed to use some symmetries of the problem so as to reduce the search space. Even if computers are now more powerful than in the 1990s, such a method would still be far too slow for a projective plane of order 9.

The key point is that we are not searching for all triangle presentations in the Hughes plane: we only want to find one. In this section, we describe our strategy in order to do so.

3.3.1 The graph associated to a P-L correspondence

In the context of triangle presentations, it is natural to associate a particular graph to each P-L correspondence $\lambda\colon P \to L$ of a projective plane Π.

Definition 3.3.1. Let $\lambda\colon P \to L$ be a P-L correspondence in a projective plane Π. The **graph** G_λ **associated to** λ is the directed graph with vertex set $V(G_\lambda) := P$ and edge set $E(G_\lambda) := \{(x, y) \in P^2 \mid y \in \lambda(x)\}$.

For λ, admitting a triangle presentation can now be rephrased as a condition on its associated graph G_λ. In order to state this reformulation, we first define what we will call a *triangle* in a directed graph.

Definition 3.3.2. Let G be a directed graph. A set $\{e_1, e_2, e_3\}$ of edges in G such that the destination vertex of e_1 (resp. e_2 and e_3) is the origin vertex of e_2 (resp. e_3 and e_1) is called a **triangle**. If two of the three edges e_1, e_2 and e_3 are equal, then they are all equal. In this case, the triangle contains only one edge and is also called a **loop**.

The next definition will also be convenient.

Definition 3.3.3. Let $\lambda\colon P \to L$ be a P-L correspondence in a projective plane Π. A triple $(x, y, z) \in P^3$ is called **λ-admissible** if $y \in \lambda(x)$, $z \in \lambda(y)$ and $x \in \lambda(z)$.

By definition, a triangle presentation compatible with λ only contains λ-admissible triples. Thanks to these definitions, there is now an obvious bijection between triangles of G_λ and (triples of) λ-admissible triples. Indeed, for $x, y, z \in P$, (x, y, z) is λ-admissible if and only if there is a triangle $\{e_1, e_2, e_3\}$ in G_λ with x, y and z being the origins of e_1, e_2 and e_3 respectively. Note that the triangle $\{e_1, e_2, e_3\}$ then corresponds to the three λ-admissible triples (x, y, z), (y, z, x) and (z, x, y) (which are equal when $x = y = z$, i.e. when $e_1 = e_2 = e_3$ or equivalently when the triangle is a loop).

This observation directly gives us the next result.

Lemma 3.3.4. *Let $\lambda\colon P \to L$ be a P-L correspondence in a projective plane Π. There exists a triangle presentation compatible with λ if and only if there exists a partition of $E(G_\lambda)$ into triangles.*

Proof. Via the above bijection, a partition of $E(G_\lambda)$ into triangles exactly corresponds to a triangle presentation compatible with λ. □

3.3.2 The score of a P-L correspondence

Most P-L correspondences λ in a projective plane do not admit a triangle presentation, i.e. the set of edges $E(G_\lambda)$ of the graph G_λ can generally not be partitioned into triangles. We would still like to measure if a correspondence λ is "far from admitting" a triangle presentation or not. We therefore introduce the notion of a *triangle partial presentation* compatible with λ.

Definition 3.3.5. Let $\lambda\colon P \to L$ be a P-L correspondence in a projective plane Π. A subset $\mathcal{T} \subseteq P^3$ is called a **triangle partial presentation** compatible with λ if the two following conditions hold:

(1) For all $x, y \in P$, if there exists $z \in P$ such that $(x, y, z) \in \mathcal{T}$ then $y \in \lambda(x)$ and z is unique.

(2) If $(x, y, z) \in \mathcal{T}$, then $(y, z, x) \in \mathcal{T}$.

We directly have the following.

Lemma 3.3.6. *Let $\lambda\colon P \to L$ be a P-L correspondence in a projective plane Π of order q. A subset $\mathcal{T} \subseteq P^3$ is a triangle presentation compatible with λ if and only if it is a triangle partial presentation compatible with λ and $|\mathcal{T}| = (q+1)(q^2+q+1)$.*

Proof. This is clear from the definitions, since there are exactly $(q+1)(q^2+q+1)$ pairs $(x,y) \in P^2$ with $y \in \lambda(x)$. $\square$

We now define the *score* of a P-L correspondence as follows.

Definition 3.3.7. Let $\lambda\colon P \to L$ be a P-L correspondence in a projective plane Π of order q. The **score** $S(\lambda)$ of λ is the greatest possible size of a triangle partial presentation compatible with λ.

Thanks to the bijection between triangles of G_λ and (triples of) λ-admissible triples (see §3.3.1), we can restate this definition in the following terms.

Definition 3.3.8. Let $\lambda\colon P \to L$ be a P-L correspondence in a projective plane Π of order q. The **score** $S(\lambda)$ of λ is the maximal number of edges of G_λ that can be covered with disjoint triangles.

A P-L correspondence then admits a triangle presentation if and only if its score reaches the maximal theoretical value $(q+1)(q^2+q+1)$.

Lemma 3.3.9. *Let $\lambda\colon P \to L$ be a P-L correspondence in a projective plane Π of order q. There exists a triangle presentation compatible with λ if and only if $S(\lambda) = (q+1)(q^2+q+1)$.*

Proof. This follows from Lemma 3.3.6. $\square$

3.3.3 Scores of correlations

When $\lambda\colon P \to L,\ L \to P$ is a correlation of a (self-dual) projective plane Π of order q, there is an explicit formula for the score of the P-L correspondence $\lambda\colon P \to L$.

Proposition 3.3.10. *Let* $\lambda\colon P \to L,\ L \to P$ *be a correlation in a projective plane* Π *of order* q*. Let* $a(\lambda)$ *be the number of points* $p \in P$ *such that* $\lambda^3(p) \ni p$ *and let* $b(\lambda)$ *be the number of points* $p \in P$ *such that* $\lambda^3(p) \ni p$ *and* $\lambda^6(p) = p$*. Then*

$$S(\lambda) = (q+1)(q^2+q+1) - (2q-3) \cdot a(\lambda) - b(\lambda).$$

Proof. For fixed $x, y \in P$ with $y \in \lambda(x)$ (i.e. (x, y) is an edge of G_λ), a point $z \in P$ is such that (x, y, z) is λ-admissible if and only if $z \in \lambda(y) \cap \lambda^{-1}(x)$. We call the edge (x, y) **unpopular** if $\lambda(y) \neq \lambda^{-1}(x)$ and **popular** if $\lambda(y) = \lambda^{-1}(x)$. This means that an unpopular edge of G_λ is contained in exactly one triangle while a popular edge is contained in exactly $(q+1)$ triangles.

(i) There are exactly $a(\lambda)$ popular edges in G_λ.
Proof: By definition, (x, y) is popular if $y = \lambda^{-2}(x)$, so a vertex $x \in P$ is the origin of a (unique) popular edge if and only if $\lambda^{-2}(x) \in \lambda(x)$, i.e. $x \in \lambda^3(x)$. There are exactly $a(\lambda)$ such x and hence $a(\lambda)$ popular edges.

(ii) There are exactly $(q+1)(q^2+q+1) + q \cdot a(\lambda)$ λ-admissible triples.
Proof: By (i), there are $(q+1)(q^2+q+1) - a(\lambda)$ unpopular edges and $a(\lambda)$ popular edges in G_λ. As each unpopular edge (resp. popular edge) is the beginning of one (resp. $(q+1)$) λ-admissible triple(s), we get

$$[(q+1)(q^2+q+1) - a(\lambda)] \cdot 1 + a(\lambda) \cdot (q+1)$$

λ-admissible triples.

(iii) There are exactly $(q+1) \cdot a(\lambda)$ λ-admissible triples (x, y, z) with (x, y) popular (resp. (y, z) popular, (z, x) popular).
Proof: There are $a(\lambda)$ popular edges by (i), each one being the beginning of $(q+1)$ λ-admissible triples.

(iv) There are exactly $a(\lambda)$ λ-admissible triples (x, y, z) with (x, y) and (y, z) popular (resp. (y, z) and (z, x) popular, (z, x) and (x, y) popular).

Proof: If (x, y, z) is λ-admissible with (x, y) and (y, z) popular, then $y = \lambda^{-2}(x)$, $z = \lambda^{-2}(y)$ and $x \in \lambda^3(x)$. Moreover, these conditions are sufficient to be λ-admissible with (x, y) and (y, z) popular. Since there are $a(\lambda)$ points x such that $x \in \lambda^3(x)$, there are exactly $a(\lambda)$ such triples.

(v) There are exactly $b(\lambda)$ λ-admissible triples (x, y, z) with (x, y), (y, z) and (z, x) popular.
Proof: Such triples satisfy $x \in \lambda^3(x)$, $y = \lambda^{-2}(x)$, $z = \lambda^{-2}(y)$ and $x = \lambda^{-2}(z)$, so in particular $x = \lambda^6(x)$. Moreover, if $x \in \lambda^3(x)$ and $x = \lambda^6(x)$, then $(x, \lambda^{-2}(x), \lambda^{-4}(x))$ is λ-admissible with three popular edges, so there are exactly $b(\lambda)$ such triples.

(vi) There are exactly $(q+1)(q^2+q+1) - 2q \cdot a(\lambda) - b(\lambda)$ λ-admissible triples (x, y, z) with (x, y), (y, z) and (z, x) unpopular.
Proof: By the inclusion-exclusion principle, the number of such triples is

$$[(q+1)(q^2+q+1) + q \cdot a(\lambda)] - 3(q+1) \cdot a(\lambda) + 3 \cdot a(\lambda) - b(\lambda).$$

We now prove that $S(\lambda) \leq (q+1)(q^2+q+1) - (2q-3) \cdot a(\lambda) - b(\lambda)$. Let $\mathcal{T}$ be a triangle partial presentation with $|\mathcal{T}| = S(\lambda)$, i.e. a set of disjoint triangles of G_λ covering $S(\lambda)$ edges. By maximality, all λ-admissible triples (i.e. triangles) (x, y, z) with (x, y), (y, z) and (z, x) unpopular are in $\mathcal{T}$ (because each of these 3 edges is only covered by this particular triangle). By (vi), this means we already have $(q+1)(q^2+q+1) - 2q \cdot a(\lambda) - b(\lambda)$ triples in $\mathcal{T}$. The other triangles in $\mathcal{T}$ all contain at least one popular edge. There are $a(\lambda)$ popular edges (by (i)), so we obtain

$$S(\lambda) \leq (q+1)(q^2+q+1) - 2q \cdot a(\lambda) - b(\lambda) + 3 \cdot a(\lambda).$$

Let us now show that $S(\lambda) \geq (q+1)(q^2+q+1) - (2q-3) \cdot a(\lambda) - b(\lambda)$, by covering that number of edges of G_λ with disjoint triangles. We first cover exactly $(q+1)(q^2+q+1) - 2q \cdot a(\lambda) - b(\lambda)$ edges of G_λ thanks to the triangles only containing unpopular edges. By definition of an unpopular edge, these triangles are all disjoint. Now, for each popular edge (x,y), there are $(q+1)$ values of z such that (x,y,z) is λ-admissible. Among these $(q+1) \geq 3$ values for z, choose z_0 different from $\lambda^{-2}(y)$ and $\lambda^2(x)$. In this way, (y,z_0) and (z_0,x) are unpopular. We then add the triangle (x,y,z_0) to our covering. This triangle is not a loop since (x,y) is popular and (y,z) is unpopular, so it covers three new edges. Doing so for each popular edge (x,y), we cover $3 \cdot a(\lambda)$ new edges and get

$$S(\lambda) \geq (q+1)(q^2+q+1) - 2q \cdot a(\lambda) - b(\lambda) + 3 \cdot a(\lambda). \qquad \square$$

It follows from Proposition 3.3.10 that a correlation λ admits a triangle presentation if and only if λ^3 sends no point to an adjacent line. However, the following elegant result of Devillers, Parkinson and Van Maldeghem shows that this never happens.

Theorem 3.3.11 (Devillers–Parkinson–Van Maldeghem)**.** *Let $\lambda\colon P \to L$, $L \to P$ be a correlation in a finite projective plane Π. Then there exists $p \in P$ such that $p \in \lambda(p)$.*

Proof. See [DPVM13, Proposition 5.4]. The case of polarities (i.e. correlations that are involutions) goes back to [Bae46]. $\square$

Corollary 3.3.12 (Proposition 3.B)**.** *Let $\lambda\colon P \to L$, $L \to P$ be a correlation in a finite projective plane Π. Then there is no triangle presentation compatible with $\lambda\colon P \to L$.*

Proof. Applying Theorem 3.3.11 to the correlation λ^3, we get $a(\lambda) > 0$ and hence $S(\lambda) < (q+1)(q^2+q+1)$ by Proposition 3.3.10. The conclusion then follows from Lemma 3.3.9. $\square$

Remark 3.3.13. In the semifield plane of order 16 and with kernel GF(4), we could observe a correlation λ such that $a(\lambda) = b(\lambda) = 1$. This means that there is exactly one point p of the plane such that $p \in \lambda^3(p)$. The score of this correlation λ is thus $S(\lambda) = 4611$, the maximal theoretical score being $(16+1)(16^2+16+1) = 4641$.

3.3.4 Estimated score for a general P-L correspondence

It does not seem possible to get a general formula for the score of all P-L correspondences. One can however obtain (good) lower bounds for the score, simply by trying to cover the most possible edges of G_λ with triangles. There are different algorithms that could be used. Our principal goal being to know whether $E(G_\lambda)$ admits a partition into triangles, we should design an algorithm that will find such a partition when it exists. The idea is simple: if an edge of G_λ is not yet covered and if there is only one triangle containing this edge and disjoint from the already chosen ones, then this triangle must be part of the (possible) partition. Our algorithm to cover as many edges as we can in G_λ is thus the following:

> While there exists $e \in E(G_\lambda)$ such that there is a unique triangle t in G_λ containing e, choose this triangle t, remove the edge(s) of t from G_λ and start again this procedure. If, at the end, there is no more triangles in G_λ, then we say that the score-algorithm **succeeds** and that the **estimated score** $s(\lambda)$ of λ is the number of edges that are covered by the chosen triangles. Otherwise, there still are triangles in G_λ but all edges are contained in 0 or at least 2 triangles. In this case, we say that the score-algorithm **fails**. For a pseudo-code, see Algorithm 1.

One should note that the value of $s(\lambda)$ (and whether the score-algorithm succeeds or not) may depend on the choice made for $e \in E(G_\lambda)$ at each step. We will still talk about *the* estimated score $s(\lambda)$ of λ, assuming that an order is fixed once and for all on the set $E(G_\lambda)$ for each λ.

Lemma 3.3.14. *Let $\lambda\colon P \to L$ be a P-L correspondence in a projective plane Π of order q. Assume that the score-algorithm succeeds. Then $s(\lambda) \leq S(\lambda)$ and, if $S(\lambda) = (q+1)(q^2+q+1)$, then $s(\lambda) = S(\lambda)$.*

Proof. See discussion above. □

When the score-algorithm fails, it cannot conclude whether there exists a partition of $E(G_\lambda)$ into triangles. Actually, we never encountered a P-L correspondence for which the algorithm fails for the Hughes plane of order 9. We therefore did not need to treat this particular case. Note however that, for a Desarguesian plane, we are aware of some P-L

Algorithm 1: Computing the estimated score $s(\lambda)$ of λ.

```
score ← 0;
edgesInOneTriangle ← true;
while edgesInOneTriangle do
    edgesInOneTriangle ← false;
    for e in E(G_λ) do
        if e is contained in exactly one triangle t of G_λ then
            edgesInOneTriangle ← true;
            remove the edge(s) of t from E(G_λ);
            if t is a loop then
                score ← score + 1;
            else
                score ← score + 3;
if there still are triangles in G_λ then
    return FAIL
else
    return score
```

correspondences for which the algorithm fails and that indeed admit a triangle presentation, so this case should not in general be forgotten.

3.3.5 Scores in the Hughes plane of order 9

By Corollary 3.3.12, we know that a correlation never reaches the score of $(q+1)(q^2+q+1)$. A naive approach to find a P-L correspondence of the Hughes plane of order 9 with a score of $(9+1)(9^2+9+1) = 910$ is to simply evaluate $s(\lambda)$ for a lot of random correspondences λ and to cross one's fingers. This idea is however not successful at all. Indeed, we computed the estimated score of 100000 random P-L correspondences and got, on average, an estimated score of 486.6 (with a standard deviation of 17.3). The best estimated score we could observe was only 561, very far from 910.

Compared with these pretty low values, the formula given by Proposition 3.3.10 for correlations seems to give better scores. In the Hughes plane of order 9, there are 33696 correlations. Their scores, computed thanks to Proposition 3.3.10, are given in Table 3.1. Note that, as soon as

# of concerned λ	$a(\lambda)$	$b(\lambda)$	$S(\lambda)$	$s(\lambda)$ (mean)
6318	4	4	846	846.00
4212	10	2	758	757.97
6318	10	10	750	750.00
4212	16	0	670	669.92
6318	16	16	654	654.00
6318	22	22	558	558.00

Table 3.1: Scores of the correlations of the Hughes plane of order 9.

two correlations λ and λ' are *conjugate* (in the sense that $\lambda = \alpha\lambda'\alpha^{-1}$ for some automorphism α of the plane), we have $a(\lambda) = a(\lambda')$, $b(\lambda) = b(\lambda')$ and $S(\lambda) = S(\lambda')$. (Actually, G_λ and $G_{\lambda'}$ are isomorphic.)

We also computed the estimated scores of all these correlations: they are also given in Table 3.1. They show that, at least for correlations, the estimated score is almost always equal to the real score. As expected, correlations have higher estimated scores than random P-L correspondences: they reach 846. This fact will be helpful for our final strategy to find a correspondence with score 910, described in the next subsection.

3.3.6 Improving a P-L correspondence

In order to find a P-L correspondence with a score greater than what we already obtained, it is natural to try to slightly modify a P-L correspondence with a high score. The smallest change we can make is to swap the images of two points. The next lemma shows that the score function is somewhat continuous.

Lemma 3.3.15. *Let $\lambda\colon P \to L$ be a P-L correspondence in a projective plane Π of order q and let $a, b \in P$. Define $\lambda_{a,b}\colon P \to L$ by $\lambda_{a,b}(x) := \lambda(x)$ for all $x \in P \setminus \{a, b\}$, $\lambda_{a,b}(a) := \lambda(b)$ and $\lambda_{a,b}(b) := \lambda(a)$. Then $|S(\lambda_{a,b}) - S(\lambda)| \leq 6(q+1)$.*

Proof. The graph $G_{\lambda_{a,b}}$ can be obtained from G_λ by deleting the edges having a or b as origin and replacing them by other edges. In total, $2(q+1)$ edges are deleted and $2(q+1)$ edges are added. Since a triangle contains at most 3 edges, we deduce that $|S(\lambda_{a,b}) - S(\lambda)| \leq 6(q+1)$. $\square$

In Lemma 3.3.15, it is even reasonable to think that $|S(\lambda_{a,b}) - S(\lambda)|$ will often be much smaller than $6(q+1)$. In other words, the score should not vary too much when replacing λ by $\lambda_{a,b}$, and we can in general hope to have $S(\lambda_{a,b}) > S(\lambda)$ for some $a, b \in P$.

Based on this observation, our idea is simple. Start with a correlation λ, whose score is known to be higher than for a random correspondence (see §3.3.5). For all distinct $a, b \in P$, consider $\lambda_{a,b}$ (as defined above) and compute its estimated score $s(\lambda_{a,b})$. Then choose $\tilde{a}, \tilde{b} \in P$ such that $s(\lambda_{\tilde{a},\tilde{b}}) = \max\{s(\lambda_{a,b}) \mid a, b \in P\}$. Now replace λ by $\lambda_{\tilde{a},\tilde{b}}$ and start this procedure again! We just need to keep track of the correspondences we already tried so as to avoid being blocked in a local maximum of the score function. This idea is explained in Algorithm 2. If after some time the algorithm does not seem able to produce a score of 910, then we stop it and start it again from another correlation.

This procedure is pretty slow: with our implementation, one step (i.e. computing $s(\lambda_{a,b})$ for all $a, b \in P$ so as to find $\tilde{a}$ and $\tilde{b}$) takes ~1.25 seconds. For this reason and because we could still not reach 910, we decided not to try all possible pairs $a, b \in P$. Instead, we can observe which points seem to be the *worst*, where the **badness** of $p \in P$ is the number of edges containing p in G_λ that were not covered by a triangle in Algorithm 1. Then, it is natural to only try the pairs $a, b \in P$ where

Algorithm 2: Finding a P-L correspondence λ with $s(\lambda) = 910$.

$\lambda \leftarrow$ some correlation of the Hughes plane;
while $s(\lambda) < 910$ **do**
    visited$[\lambda] \leftarrow$ true;
    bestA $\leftarrow -1$; *bestB* $\leftarrow -1$;
    bestScore $\leftarrow -1$;
    for a *in* P ***and*** b *in* P **do**
        if *visited*$[\lambda_{a,b}] =$ *false* ***and*** $s(\lambda_{a,b}) >$ *bestScore* **then**
            bestScore $\leftarrow s(\lambda_{a,b})$;
            bestA $\leftarrow a$;
            bestB $\leftarrow b$;
    $\lambda \leftarrow \lambda_{bestA,bestB}$;
return λ;

a is one of the worst points (for instance the 5 worst points) and b is arbitrary. Obviously, with this change Algorithm 2 does not visit the same correspondences as before, but it has the advantage that a step only takes $\sim$0.13 seconds.

After three weeks of slight changes in the algorithm (e.g. the definition of a *bad* point, the number of worst points we consider, the condition under which we stop and start with another correlation, etc), the computer eventually shouted (at least wrote) victory. The starting correlation had a score equal to 750, and the evolution of the estimated score until 910 is shown in Figure 3.1.

Remark 3.3.16. The last change we made to the algorithm before it could solve the problem was actually mistaken! Whereas we wanted to speed up the computation of the five worst points, we made an error in the implementation of that idea resulting in the fact that the five computed points were actually not the worst ones. This mistake still led us to the discovery of a (valid) P-L correspondence λ with a score of 910. The funny part of the story is that if we correct this implementation

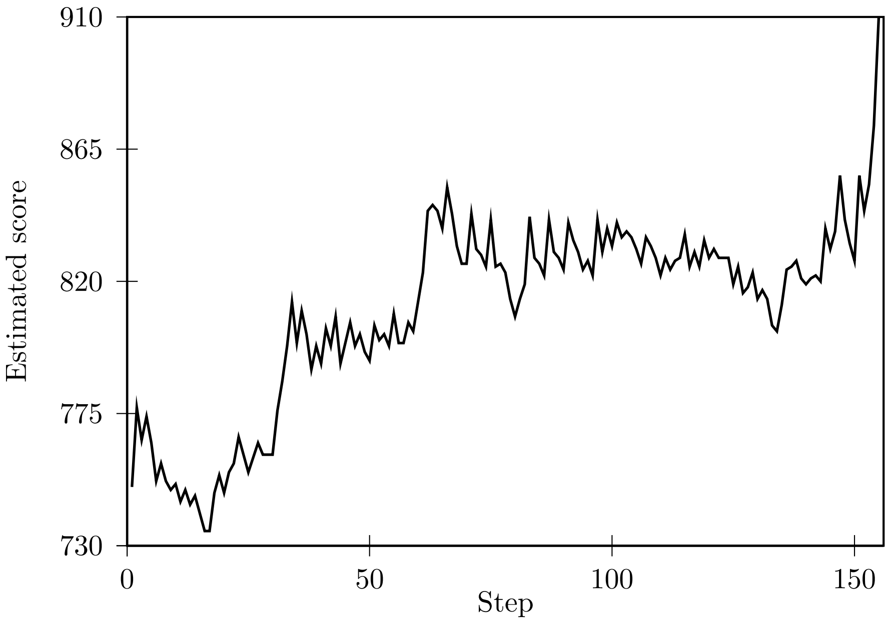


Figure 3.1: Evolution of the estimated score with Algorithm 2.

error and start the algorithm with the same correlation, then it misses the correspondence λ.

3.4 The building and its lattice

In this section, we first give the description of the building and the lattice that we discovered. We then give various properties of these objects (i.e. we prove (4), (5), (6) and (7) in Theorem 3.A).

3.4.1 Description

The structure of the Hughes plane of order 9 is given in Table 3.2 (in Appendix 3.I) and comes from [Moo]. Points and lines are numbered from 0 to 90 (let us call them $p_0, \ldots, p_{90}$ and $\ell_0, \ldots, \ell_{90}$), and the n^{th} row (with $0 \leq n \leq 90$) gives the indices of the 10 points incident to ℓ_n. The reader may be skeptical that the structure of incidence Π defined by these point-line incidences is indeed the Hughes plane, but this is not so hard to verify by analyzing its properties. It is at least really easy to implement a program checking that Π satisfies the axioms of a projective plane. Also, Π has a projective subplane that is isomorphic to the Fano plane: consider for instance the 7 points $\{p_0, p_2, p_9, p_{17}, p_{18}, p_{38}, p_{41}\}$ and 7 lines $\{\ell_0, \ell_1, \ell_2, \ell_{31}, \ell_{34}, \ell_{64}, \ell_{87}\}$. This implies that Π is non-Desarguesian, since the only Desarguesian projective planes containing the Fano plane are those whose order is a power of 2. Moreover, our computations show that Π is self-dual (since there exist correlations), so it can only be the Hughes plane (see, for instance, [LKT91]).

Relative to this numbering of points and lines, the P-L correspondence $\lambda\colon P \to L$ that we found is given in Table 3.3 (in Appendix 3.II). For the image of p_{10x+y} by λ, one should look at the intersection of rows $x\_$ and $\_y$. The triangle presentation $\mathcal{T}$ compatible with λ is then given in Table 3.4. In this table, the appearance of (x, y, z) means that (x, y, z), (y, z, x) and (z, x, y) all belong to $\mathcal{T}$. There are, in Table 3.4, 298 triples (x, y, z) with x, y, z not all equal and 16 triples (x, x, x), which means that $\mathcal{T}$ contains $298 \cdot 3 + 16 = 910$ elements as required. While a computer helped to find $\mathcal{T}$, it can once again be checked by hand (or with a trivial program) that $\mathcal{T}$ is indeed a triangle presentation compatible with λ. Indeed, one only needs to check that for each $(x, y, z) \in \mathcal{T}$, the line $\lambda(x)$ contains y and there exists no $z' \neq z$ such that $(x, y, z') \in \mathcal{T}$. This suffices to show that $\mathcal{T}$ is a triangle presentation compatible with λ, since $|\mathcal{T}| = 910$.

It follows from Theorem 3.2.4 that the building $\Delta_{\mathcal{T}}$ and the group $\Gamma_{\mathcal{T}} \leq \mathrm{Aut}(\Delta_{\mathcal{T}})$ satisfy (1), (2) and (3) in Theorem 3.A. In the next four subsections we prove (4), (5), (6) and (7).

3.4.2 Torsion in $\Gamma_{\mathcal{T}}$

The group $\Gamma_{\mathcal{T}}$ has elements of order 3: when $(x, x, x) \in \mathcal{T}$ for some $x \in P$, we have the relation $a_x^3 = 1$ in the presentation of $\Gamma_{\mathcal{T}}$. However, the subgroup $\Gamma_{\mathcal{T}}^+$ of $\Gamma_{\mathcal{T}}$ consisting of the type-preserving automorphisms is torsion-free. Indeed, let γ be a torsion element of $\Gamma_{\mathcal{T}}^+$, say of order n. If v_0 is a fixed vertex of $\Delta_{\mathcal{T}}$, then γ stabilizes the set $\{v_0, \gamma(v_0), \gamma^2(v_0), \ldots, \gamma^{n-1}(v_0)\}$. By [BH99, Corollary 2.8 (1)], γ must fix a point of $\Delta_{\mathcal{T}}$, i.e. it stabilizes a simplex of $\Delta_{\mathcal{T}}$. Since γ preserves the types, it fixes this simplex pointwise and thus fixes its vertices. But $\Gamma_{\mathcal{T}}^+$ acts freely on the set of vertices of $\Delta_{\mathcal{T}}$, so $\gamma = 1$.

3.4.3 A perfect subgroup of $\Gamma_{\mathcal{T}}$

Clearly, $\Gamma_{\mathcal{T}}^+$ is a normal subgroup of index 3 of $\Gamma_{\mathcal{T}}$. We find that $\Gamma_{\mathcal{T}}$ also has a subgroup of index 2. Indeed, if we define $A \subset P$ by $A = \ell_3 \cup \ell_{11} \cup \ell_{62} \cup \ell_{64} \cup \ell_{87}$, then one can check that for each $(x, y, z) \in \mathcal{T}$, either one or three of the points x, y, z belong to A. Equivalently, either none or two of the points x, y, z belong to $P \setminus A$. Hence, there is a well-defined group homomorphism $f \colon \Gamma_{\mathcal{T}} \to \mathbf{C}_2$ defined on the generators $\{a_x\}_{x \in P}$ by $f(a_x) := 0$ if $x \in A$ and $f(a_x) := 1$ if $x \notin A$. The kernel $\ker(f)$ of f is then a subgroup of index 2 of $\Gamma_{\mathcal{T}}$.

The intersection $\Gamma_{\mathcal{T}}^+ \cap \ker(f)$ of these two subgroups is thus a normal subgroup of index 6 of $\Gamma_{\mathcal{T}}$ (with $\Gamma_{\mathcal{T}} \big/ \Gamma_{\mathcal{T}}^+ \cap \ker(f) \cong \mathbf{C}_2 \times \mathbf{C}_3$). We checked using the GAP system that $\Gamma_{\mathcal{T}}^+ \cap \ker(f)$ is a perfect group, so that $[\Gamma_{\mathcal{T}}, \Gamma_{\mathcal{T}}] = \Gamma_{\mathcal{T}}^+ \cap \ker(f)$.

3.4.4 Partition of $\Delta_{\mathcal{T}}$ into sub-buildings

A **Baer subplane** of a projective plane Π is a proper projective subplane Π_0 of Π with the property that every point of Π is incident to at least one line of Π_0 and every line of Π is incident to at least one point of Π_0. Let us take for Π the Hughes plane of order 9. Then Π has a (Desarguesian) Baer subplane Π_0 of order 3, which has the property that all automorphisms and all correlations of Π preserve Π_0 (see [Dem68, 5.4.1]). With respect to our numbering of the points and lines of the Hughes plane (see Table 3.2), the sets of points and lines of Π_0 are

$$P_0 := \{p_n \mid n \in \{9, 17, 20, 33, 38, 42, 43, 46, 47, 56, 59, 64, 70\}\}$$

and

$$L_0 := \{\ell_n \mid n \in \{3, 11, 22, 34, 46, 53, 62, 64, 70, 79, 84, 87, 89\}\}$$

(see the numbers in bold in Table 3.2).

What is surprising is that our P-L correspondence λ also preserves Π_0. This is indeed clear from Table 3.3. Even better, if we call λ_0 the restriction of λ to P_0, then the triangle presentation $\mathcal{T}$ can be restricted to a triangle presentation $\mathcal{T}_0$ compatible with λ_0. In other terms, for each $(x, y, z) \in \mathcal{T}$, if $x \in P_0$ and $y \in P_0$ then $z \in P_0$. This can also be simply observed by inspecting Table 3.4. The author does not know any theoretical reason why these properties are true (and whether they must be true for any P-L correspondence admitting a triangle presentation).

This observation has different consequences. First, we have a P-L correspondence λ_0 in the Desarguesian projective plane Π_0 of order 3, and a triangle presentation $\mathcal{T}_0$ compatible with it. Theorem 3.2.4 thus gives an $\tilde{A}_2$-building $\Delta_{\mathcal{T}_0}$ whose projective plane at each vertex is isomorphic to Π_0 and a group $\Gamma_{\mathcal{T}_0}$ acting simply transitively on $V(\Delta_{\mathcal{T}_0})$. The triangle presentations in the projective plane of order 3 have all been given by Cartwright–Mantero–Steger–Zappa in [CMSZ93b], so $\mathcal{T}_0$ must be one of their list. It turns out that $\mathcal{T}_0$ is equivalent (as defined in [CMSZ93b, §2]) to their triangle presentation numbered 14.1 (see [CMSZ93b, Appendix B]; one such equivalence takes the p_n, in the order listed in the definition of P_0, to $12, 2, 5, 0, 8, 11, 10, 3, 1, 9, 6, 4$ and 7, respectively). In particular, this means by [CMSZ93b, §8] that $\Delta_{\mathcal{T}_0}$ is a non-linear building, i.e. is not the building of $\mathrm{PGL}(3, K)$ for some local field K.

The group $\Gamma_{\mathcal{T}_0}$ and the building $\Delta_{\mathcal{T}_0}$ also appear as subgroups and sub-buildings of $\Gamma_{\mathcal{T}}$ and $\Delta_{\mathcal{T}}$, respectively. With the notation of §3.2.2, there is a clear embedding $e\colon \mathcal{C}_{\mathcal{T}_0} \hookrightarrow \mathcal{C}_{\mathcal{T}}$. Now $\Delta_{\mathcal{T}_0}$ and $\Delta_{\mathcal{T}}$ are the universal coverings of $\mathcal{C}_{\mathcal{T}_0}$ and $\mathcal{C}_{\mathcal{T}}$ respectively, so by fixing some vertices $v_0 \in V(\Delta_{\mathcal{T}_0})$ and $v \in V(\Delta_{\mathcal{T}})$ such that $p(v) = e(p_0(v_0))$ (where $p\colon \Delta_{\mathcal{T}} \to \mathcal{C}_{\mathcal{T}}$ and $p_0\colon \Delta_{\mathcal{T}_0} \to \mathcal{C}_{\mathcal{T}_0}$ are the natural projections), we get an embedding $\tilde{e}\colon \Delta_{\mathcal{T}_0} \hookrightarrow \Delta_{\mathcal{T}}$ with $\tilde{e}(v_0) = v$. We can then see $\Gamma_{\mathcal{T}_0}$ as the subgroup of $\Gamma_{\mathcal{T}}$ such that $\Gamma_{\mathcal{T}_0}(v)$ is exactly the set of vertices of $\tilde{e}(\Delta_{\mathcal{T}_0})$. Moreover, for each $g \in \Gamma_{\mathcal{T}}$ the set $g\Gamma_{\mathcal{T}_0}(v) \subset V(\Delta_{\mathcal{T}})$ is also the 0-skeleton of a building isomorphic to $\Delta_{\mathcal{T}_0}$. This means that the vertices of $\Delta_{\mathcal{T}}$ are partitioned into sub-buildings isomorphic to $\Delta_{\mathcal{T}_0}$ (where each sub-building corresponds to a left coset of $\Gamma_{\mathcal{T}_0}$ in $\Gamma_{\mathcal{T}}$).

One should note that these sub-buildings that are isomorphic to $\Delta_{\mathcal{T}_0}$ cover all the vertices of $\Delta_{\mathcal{T}}$, but this is not true for edges and chambers (i.e. triangles): some edges (and chambers) of $\Delta_{\mathcal{T}}$ do not belong to any of the sub-buildings.

3.4.5 Automorphism group of $\Delta_{\mathcal{T}}$

The automorphism group $\mathrm{Aut}(\Delta_{\mathcal{T}})$ of $\Delta_{\mathcal{T}}$ contains $\Gamma_{\mathcal{T}}$, which acts simply transitively on the vertices of the building. In order to know whether $\mathrm{Aut}(\Delta_{\mathcal{T}})$ is substantially larger than $\Gamma_{\mathcal{T}}$, we should try to see what the stabilizer of a vertex in $\mathrm{Aut}(\Delta_{\mathcal{T}})$ looks like. This can be done by making use of the GAP system. I am very thankful to Tim Steger, who had done the same work for triangle presentations in the projective plane of order 3, and who gave me all his source codes and a great deal of advice.

Let v be a vertex of $\Delta_{\mathcal{T}}$. In the next discussion, $X_{\mathcal{T}}$ will denote the sub-building of $\Delta_{\mathcal{T}}$ containing v and isomorphic to $\Delta_{\mathcal{T}_0}$ (see §3.4.4). We have the following facts.

(i) Any automorphism of $\Delta_{\mathcal{T}}$ fixing v preserves the sub-building $X_{\mathcal{T}}$.
Explanation: For a vertex x contained in $X_{\mathcal{T}}$, there are $2 \cdot 91 = 182$

vertices adjacent to x in $\Delta_{\mathcal{T}}$, and exactly $2\cdot 13 = 26$ of them belong to $X_{\mathcal{T}}$. Those 26 vertices are characterized by the fact that, in the local Hughes plane Π associated to x, they correspond to the 13 points and 13 lines of the Baer subplane Π_0 of Π. Hence, if $\alpha \in \mathrm{Aut}(\Delta_{\mathcal{T}})$ is such that $\alpha(x) = y$ with x, y belonging to $X_{\mathcal{T}}$, then α must send the 26 neighbors of x in $X_{\mathcal{T}}$ to the 26 neighbors of y in $X_{\mathcal{T}}$ because all automorphisms and correlations of Π preserve Π_0. Starting with $x = v$, we obtain step by step that any automorphism of $\Delta_{\mathcal{T}}$ fixing v must stabilize $X_{\mathcal{T}}$.

(ii) There are 16 automorphisms of $X_{\mathcal{T}}$ stabilizing v.
Explanation: This was previously done by Steger with GAP.

(iii) For each $x \in V(\Delta_{\mathcal{T}})$, the only automorphism of the ball of radius 2 centered at x that pointwise stabilizes the ball of radius 1 is the trivial automorphism.
Explanation: This was proved with GAP. As was pointed out to me by H. Van Maldeghem, this can also be proved by hand. One just needs to see the ball of radius 2 as a projective Hjelmslev plane of level 2 (see §4.2) and to use the properties of the Hughes plane of order 9.

Point (iii) implies that the pointwise stabilizer of a ball of radius 1 in $\mathrm{Aut}(\Delta_{\mathcal{T}})$ is trivial, and hence that an automorphism of $\Delta_{\mathcal{T}}$ is completely determined by its action on the ball of radius 1 centered at v. In particular, the stabilizer of v in $\mathrm{Aut}(\Delta_{\mathcal{T}})$ is finite and $\mathrm{Aut}(\Delta_{\mathcal{T}})$ is discrete (for the topology of pointwise convergence).

(iv) There are 6 automorphisms of Π that pointwise stabilize Π_0.
Explanation: This can be checked with a computer, but one can also see [Lün76, Corollary 5] or [Ros58] for a more theoretical approach.

The first four points imply that there are at most $16 \cdot 6 = 96$ automorphisms of $\Delta_{\mathcal{T}}$ stabilizing v. Denote by G_1 the set of the 96 automorphisms of the ball of radius 1 centered at v that could maybe be extended to automorphisms of the whole building.

(v) Each automorphism in G_1 can be extended to an automorphism of the ball of radius 2 centered at v.
Explanation: This was proved with GAP.

Now denote by G_2 the set of these extended automorphisms.

(vi) Each automorphism in G_2 can be extended to $\Delta_{\mathcal{T}}$.
Explanation: This could be checked with a clever GAP program written by Steger.

These steps actually gave us the explicit description of the 96 automorphisms of $\Delta_{\mathcal{T}}$ fixing v. Six of them pointwise stabilize the subbuilding $X_{\mathcal{T}}$. The file describing these automorphisms is pretty big so we do not append it to this text.

3.I The Hughes plane of order 9

0 :	0	1	2	3	4	5	6	7	8	9
1 :	0	10	11	12	13	14	15	16	17	18
2 :	0	19	34	35	36	37	38	39	40	41
3 :	0	**20**	27	**42**	55	**56**	57	58	**59**	60
4 :	0	21	33	48	54	61	76	78	89	90
5 :	0	22	30	43	49	63	68	72	79	80
6 :	0	23	28	44	50	69	70	77	81	82
7 :	0	24	29	45	51	64	73	74	83	84
8 :	0	25	31	46	52	62	67	75	85	86
9 :	0	26	32	47	53	65	66	71	87	88
10 :	1	10	19	20	21	22	23	24	25	26
11 :	1	11	34	**42**	**43**	44	45	**46**	**47**	48
12 :	1	12	28	35	55	61	62	63	64	65
13 :	1	13	31	41	54	56	74	80	82	88
14 :	1	14	33	36	50	58	68	73	85	87
15 :	1	15	29	37	52	59	71	76	79	81
16 :	1	16	27	38	51	66	72	77	86	89
17 :	1	17	32	39	49	57	69	75	78	83
18 :	1	18	30	40	53	60	67	70	84	90
19 :	2	10	35	42	49	50	51	52	53	54
20 :	3	10	29	34	56	61	66	67	68	69
21 :	4	10	31	38	48	57	63	81	84	87
22 :	5	10	**33**	40	**47**	**59**	**64**	72	75	82
23 :	6	10	28	41	43	58	71	83	86	90
24 :	7	10	27	37	45	65	70	78	80	85
25 :	8	10	30	39	46	55	73	76	77	88
26 :	9	10	32	36	44	60	62	74	79	89
27 :	2	11	19	27	28	29	30	31	32	33
28 :	2	13	21	34	57	62	70	71	72	73
29 :	2	14	22	37	48	60	64	69	86	88
30 :	2	12	24	39	47	58	67	80	81	89
31 :	2	18	20	41	45	61	75	77	79	87
32 :	2	16	26	40	44	56	63	76	83	85
33 :	2	15	25	36	43	55	66	78	82	84
34 :	2	**17**	23	**38**	**46**	**59**	65	68	74	90
35 :	4	11	22	35	58	66	70	74	75	76
36 :	5	11	21	39	50	56	65	79	84	86
37 :	3	11	26	36	52	57	64	77	80	90
38 :	6	11	20	40	51	62	68	78	81	88
39 :	9	11	23	37	54	55	67	72	83	87
40 :	8	11	24	38	49	60	61	71	82	85
41 :	7	11	25	41	53	59	63	69	73	89
42 :	5	14	19	42	63	67	71	74	77	78
43 :	4	13	19	47	51	55	69	79	85	90
44 :	3	16	19	43	54	60	65	73	75	81
45 :	9	12	19	45	53	57	68	76	82	86
46 :	6	15	19	**46**	49	**56**	**64**	**70**	87	89
47 :	7	17	19	44	52	58	61	72	84	88
48 :	8	18	19	48	50	59	62	66	80	83
49 :	5	18	23	29	35	43	57	85	88	89
50 :	3	13	24	30	35	44	59	78	86	87
51 :	8	14	25	32	35	45	56	72	81	90
52 :	7	15	21	31	35	47	60	68	77	83
53 :	**9**	16	**20**	**33**	35	**46**	69	71	80	84
54 :	6	17	26	27	35	48	67	73	79	82
55 :	4	14	20	30	34	52	65	82	83	89
56 :	5	17	25	28	34	51	60	76	80	87
57 :	6	12	22	32	34	54	59	77	84	85
58 :	9	15	24	27	34	50	63	75	88	90
59 :	8	16	23	31	34	53	58	64	78	79
60 :	7	18	26	33	34	49	55	74	81	86
61 :	4	15	23	32	40	42	61	73	80	86
62 :	3	12	25	**33**	**38**	**42**	**70**	79	83	88
63 :	8	13	26	28	37	42	68	75	84	89
64 :	**9**	**17**	21	30	41	**42**	**64**	66	81	85
65 :	7	16	22	29	39	42	62	82	87	90
66 :	6	18	24	31	36	42	65	69	72	76
67 :	4	12	26	29	41	46	50	60	72	78
68 :	4	16	21	28	36	45	49	59	67	88
69 :	4	18	25	27	39	44	54	64	68	71
70 :	4	**17**	24	**33**	37	**43**	53	**56**	62	77
71 :	5	13	22	27	36	46	53	61	81	83
72 :	5	12	20	31	37	44	49	66	73	90
73 :	5	15	26	30	38	45	54	58	62	69
74 :	5	16	24	32	41	48	52	55	68	70
75 :	3	14	23	27	41	47	49	62	76	84
76 :	3	18	21	32	37	46	51	58	63	82
77 :	3	17	22	31	40	45	50	55	71	89
78 :	3	15	20	28	39	48	53	72	74	85
79 :	7	13	**20**	32	**38**	**43**	50	**64**	67	76
80 :	9	13	25	29	40	48	49	58	65	77
81 :	6	13	23	33	39	45	52	60	63	66
82 :	6	14	21	29	38	44	53	55	75	80
83 :	7	14	24	28	40	46	54	57	66	79
84 :	**9**	14	26	31	39	**43**	51	**59**	61	**70**
85 :	8	12	21	27	40	43	52	69	74	87
86 :	7	12	23	30	36	48	51	56	71	75
87 :	**9**	18	22	28	**38**	**47**	52	**56**	73	78
88 :	8	15	22	33	41	44	51	57	65	67
89 :	8	**17**	**20**	29	36	**47**	54	63	**70**	86
90 :	6	16	25	30	37	47	50	57	61	74

Table 3.2: Incidence relation of the Hughes plane of order 9, from [Moo].

3.II The triangle presentation

λ	_0	_1	_2	_3	_4	_5	_6	_7	_8	_9
0_	20	0	44	75	78	77	50	76	37	**3**
1_	54	39	30	8	88	68	18	**34**	65	57
2_	**70**	82	42	23	38	90	81	13	61	69
3_	73	4	83	**22**	58	28	59	55	**64**	60
4_	56	2	**87**	**84**	26	45	**53**	**11**	80	41
5_	25	14	63	72	7	32	**62**	86	51	**46**
6_	36	27	31	29	**79**	33	16	71	85	24
7_	**89**	35	17	19	5	47	67	10	66	43
8_	6	21	1	52	74	40	12	48	9	15
9_	49									

Table 3.3: P-L correspondence λ.

(0,3,41)	(0,10,82)	(0,29,54)	(0,34,9)	(0,56,88)	(0,61,31)	(0,66,1)
(0,67,13)	(0,68,74)	(0,69,80)	(1,1,1)	(1,2,16)	(1,3,47)	(1,4,72)
(1,5,89)	(1,6,86)	(1,7,51)	(1,8,77)	(1,9,27)	(2,3,62)	(2,19,12)
(2,43,61)	(2,54,73)	(2,60,65)	(2,65,55)	(2,73,35)	(2,75,17)	(2,81,63)
(3,3,3)	(3,14,8)	(3,23,6)	(3,27,56)	(3,49,7)	(3,76,4)	(3,84,5)
(4,15,28)	(4,20,37)	(4,28,15)	(4,39,81)	(4,48,29)	(4,53,20)	(4,74,79)
(4,85,71)	(5,17,90)	(5,22,67)	(5,31,33)	(5,40,60)	(5,45,53)	(5,50,30)
(5,55,40)	(5,71,22)	(6,13,25)	(6,24,78)	(6,30,26)	(6,35,21)	(6,44,10)
(6,59,19)	(6,78,24)	(6,87,59)	(7,18,39)	(7,21,75)	(7,32,57)	(7,37,83)
(7,46,69)	(7,58,32)	(7,63,64)	(7,82,18)	(8,11,87)	(8,26,52)	(8,36,58)
(8,52,68)	(8,57,36)	(8,64,50)	(8,80,70)	(8,90,85)	**(9,20,43)**	**(9,42,38)**
(9,55,44)	**(9,56,42)**	(9,57,48)	(9,58,45)	**(9,59,46)**	(9,60,11)	(10,17,23)
(10,26,33)	(10,27,82)	(10,35,73)	(10,48,77)	(10,67,81)	(10,73,50)	(10,79,69)
(11,11,11)	(11,23,71)	(11,37,82)	(11,54,24)	(11,55,85)	(11,67,61)	(11,72,49)
(11,83,47)	(12,12,12)	(12,24,68)	(12,39,86)	(12,47,45)	(12,58,56)	(12,67,53)
(12,80,82)	(12,81,57)	(12,89,76)	(13,31,48)	(13,46,35)	(13,52,26)	(13,62,79)
(13,67,27)	(13,75,52)	(13,85,82)	(13,86,64)	(14,15,21)	(14,22,63)	(14,33,82)
(14,41,37)	(14,44,32)	(14,51,58)	(14,57,51)	(14,65,43)	(14,67,22)	(15,16,30)
(15,36,34)	(15,45,82)	(15,49,59)	(15,59,89)	(15,67,83)	(15,88,65)	(16,18,82)
(16,40,25)	(16,53,66)	(16,60,84)	(16,67,46)	(16,70,36)	(16,84,55)	(16,90,18)
(17,17,38)	**(17,46,20)**	**(17,59,70)**	(17,65,82)	(17,68,40)	(17,74,72)	(18,22,42)
(18,29,39)	(18,42,78)	(18,62,87)	(18,87,62)	(18,90,29)	(19,22,77)	(19,32,79)
(19,34,75)	(19,54,45)	(19,59,87)	(19,77,22)	(19,84,41)	(19,85,61)	(20,24,62)
(20,33,64)	(20,53,37)	**(20,56,70)**	(20,62,77)	(20,77,24)	(21,21,38)	(21,29,68)
(21,44,60)	(21,53,31)	(21,55,83)	(21,80,77)	(22,71,74)	(22,74,63)	(22,78,42)
(23,28,80)	(23,41,36)	(23,43,26)	(23,58,90)	(23,83,77)	(23,86,28)	(23,90,57)
(24,40,34)	(24,51,85)	(24,81,84)	(24,88,32)	(25,25,25)	(25,30,58)	(25,37,65)
(25,47,48)	(25,50,77)	(25,57,56)	(25,61,29)	(25,74,49)	(26,39,55)	(26,45,76)
(26,60,39)	(26,63,88)	(26,66,77)	(27,31,61)	(27,41,34)	(27,54,29)	(27,74,68)
(27,80,69)	(27,88,66)	(28,32,40)	(28,40,80)	(28,42,52)	(28,61,32)	(28,73,42)
(28,86,61)	(29,44,89)	(29,64,76)	(29,71,70)	(30,30,38)	(30,45,57)	(30,54,74)
(30,62,61)	(30,69,37)	(31,54,83)	(31,76,78)	(31,78,36)	(31,89,81)	(31,90,43)
(32,46,84)	(32,54,64)	(32,66,72)	**(33,33,33)**	(33,40,51)	**(33,47,46)**	**(33,59,56)**
(33,72,39)	(33,75,61)	(34,34,34)	(34,50,39)	(34,63,37)	(34,88,47)	(34,90,35)
(35,57,71)	(35,62,41)	(35,70,86)	(35,71,58)	(35,72,83)	(36,53,44)	(36,64,67)
(36,78,65)	(36,79,51)	(37,52,89)	(37,89,52)	(38,41,41)	**(38,42,56)**	**(38,64,64)**
(38,66,66)	(38,81,81)	(38,85,85)	(39,49,41)	(39,74,43)	(40,76,41)	(40,87,48)
(42,47,47)	(42,73,52)	**(43,43,43)**	(43,51,68)	**(43,59,64)**	**(43,70,47)**	(44,44,44)
(44,62,75)	(44,74,80)	(44,79,47)	(45,45,45)	(45,68,69)	(45,86,62)	**(46,46,46)**
(46,71,76)	(46,80,50)	(48,48,48)	(48,49,63)	(48,58,81)	(48,65,84)	(49,53,73)
(49,69,85)	(49,73,53)	(49,89,59)	(50,55,76)	(50,73,51)	(50,76,60)	(50,88,87)
(51,73,54)	(51,87,66)	(52,75,84)	(52,84,68)	(53,90,88)	(54,84,70)	(55,56,79)
(55,63,86)	(56,83,60)	(57,75,72)	(58,72,75)	(60,79,85)	(60,86,63)	(63,69,70)
(65,66,86)	(65,78,69)	(66,89,71)	(68,87,83)	(69,78,72)	**(70,70,70)**	(71,75,88)
(72,78,76)	(79,79,79)	(79,90,89)	(80,81,87)	(83,83,83)	(88,88,88)	

Table 3.4: Triangle presentation $\mathcal{T}$ compatible with λ.

Chapter 4

A discreteness criterion for the automorphism group of an $\tilde{A}_2$-building

In Chapter 3 we built a residually non-Desarguesian $\tilde{A}_2$-building with a cocompact lattice. Thanks to a computer we could prove that the automorphism group of that building was discrete. Whether or not the automorphism group of a particular $\tilde{A}_2$-building is discrete is a subtle question in general. In this chapter, we give sufficient conditions on an $\tilde{A}_2$-building ensuring that its automorphism group is discrete.

4.1 Main results

Throughout this chapter, Δ is a locally finite thick $\tilde{A}_2$-building. The simplices of dimension 2 in Δ (i.e. triangles) are the **chambers** of Δ, and those of dimension 1 (i.e. edges) are the **panels** of Δ. As before a **vertex** of Δ is a simplex of dimension 0. Recall from the previous chapter that there are three types of vertices: $V(\Delta) = V_0(\Delta) \sqcup V_1(\Delta) \sqcup V_2(\Delta)$. We also define the **type** of a panel in Δ as $\{a, b\}$ where $a, b \in \{0, 1, 2\}$ are the types of the two vertices of the panel. Thus each chamber has one vertex of each type (0, 1 and 2) and one panel of each type ($\{0, 1\}$, $\{1, 2\}$ and $\{0, 2\}$). (Note that what we call *type* here is generally called *cotype* in the literature.)

As in the previous chapter, $\mathrm{Aut}(\Delta)$ denotes the full automorphism group of Δ (as a simplicial complex). We then write $\mathrm{Aut}(\Delta)^+$ for the subgroup of $\mathrm{Aut}(\Delta)$ consisting of the automorphisms that preserve the types. It is clear that $[\mathrm{Aut}(\Delta) : \mathrm{Aut}(\Delta)^+] \leq 6$, so that the locally compact group $\mathrm{Aut}(\Delta)$ (equipped with the topology of pointwise convergence) is non-discrete if and only if $\mathrm{Aut}(\Delta)^+$ is non-discrete.

The goal of this chapter is to provide sufficient conditions under which an **exotic** (i.e. non Bruhat–Tits) $\tilde{A}_2$-building has a discrete automorphism group. Our main result is the following.

Theorem 4.A. *Let Δ be a locally finite thick $\tilde{A}_2$-building and suppose that $\mathrm{Aut}(\Delta)^+$ is transitive on panels of each type. Then either:*

(a) Δ is Bruhat–Tits; or

(b) $\mathrm{Aut}(\Delta)$ is discrete.

In the text we actually state and prove Theorem 4.A′ which is a more precise version of Theorem 4.A. The same will be true for our other main results: alternative statements can be found in the text.

In the next result, panel-transitivity is replaced by the weaker hypothesis of vertex-transitivity. As explained in the introduction, that condition alone is not sufficient for the conclusion of Theorem 4.A to hold: additional assumptions are required.

Theorem 4.B. *Let Δ be a locally finite thick $\tilde{A}_2$-building. Suppose that $\mathrm{Aut}(\Delta)$ is transitive on vertices and unimodular, that $\mathrm{Aut}(\Delta)^+$ is transitive on vertices of each type, and that Δ has thickness $p+1$ for some prime p. Then either:*

(a) Δ is Bruhat–Tits; or

(b) $\mathrm{Aut}(\Delta)$ is discrete.

The following result is of different nature but is somewhat complementary to Theorems 4.A and 4.B. Indeed, it gives a local condition under which an $\tilde{A}_2$-building is ensured to be exotic.

Theorem 4.C. *Let Δ be a locally finite thick $\tilde{A}_2$-building, let x_0, x_1 be two adjacent vertices in Δ and let C be the set of chambers adjacent to both x_0 and x_1. For each $j \in \{0,1\}$, let $G_j \leq \mathrm{Sym}(C)$ be the image of $\mathrm{Aut}(\Pi_{x_j})(x_{1-j})$ in $\mathrm{Sym}(C)$, where Π_{x_j} is the projective plane at x_j. If $G_0 \neq G_1$, then Δ is exotic.*

We apologize that the condition is indeed that the groups G_0 and G_1 do not coincide as subgroups of $\mathrm{Sym}(C)$. In particular they might very well be isomorphic.

Our theorems can be applied in the context of *Singer cyclic lattices.* Recall that a **Singer cyclic lattice** is a group $\Gamma \leq \mathrm{Aut}(\Delta)$ acting simply transitively on the panels of each type of an $\tilde{A}_2$-building Δ and such that each vertex stabilizer in Γ is cyclic. It is called **exotic** if Δ is exotic, and the **parameter** of Γ is the order of the local projective planes in Δ.

Corollary 4.D. *For each $q \geq 2$, there are at most $\left(\frac{q(q^2-1)}{3}\right)^2$ isomorphism classes of non-exotic Singer cyclic lattices with parameter q.*

Combined with the fact that the total number of Singer cyclic lattices with parameter q grows super-exponentially (see [Wit17, Theorem B]), we obtain the following.

Corollary 4.E. *Almost all Singer cyclic lattices are exotic in the following sense:*

$$\lim_{q\to\infty} \frac{|\{\textit{exotic Singer cyclic lattices with parameter } q\}/\sim|}{|\{\textit{Singer cyclic lattices with parameter } q\}/\sim|} = 1,$$

where q ranges over prime powers and $\sim$ is the isomorphism relation.

4.2 Projective Hjelmslev planes

This section gives the definition and first properties of projective Hjelmslev planes, which will be of central importance in the whole chapter. It is largely inspired from the work of Van Maldeghem and Van Steen [VMVS98].

Given a vertex O in Δ and a natural number $n \geq 1$, we define the geometry ${}^nH(O)$ as follows. The geometry ${}^1H(O)$ is just the residue of O, which is a projective plane. So the points of ${}^1H(O)$ are certain vertices of Δ adjacent to O, and similarly for the lines of ${}^1H(O)$. Now for $n \geq 1$, the **points** (resp. **lines**) of ${}^nH(O)$ are the sequences $(v_1, \ldots, v_n)$ of vertices of Δ, where v_1 is a point (resp. a line) of ${}^1H(O)$ and $\{v_{i-1}, v_{i+1}\}$ is a pair of non-incident point and line in ${}^1H(v_i)$ (where $v_0 := O$). We will sometimes identify an element $(v_1, \ldots, v_n)$ of ${}^nH(O)$ with the vertex v_n of Δ (the other vertices $v_1, \ldots, v_{n-1}$ being uniquely determined by v_n). A point $(p_1, \ldots, p_n)$ of ${}^nH(O)$ is **incident** with a line $(\ell_1, \ldots, \ell_n)$ if all vertices $O, p_1, \ldots, p_n, \ell_1, \ldots, \ell_n$ are contained in a common apartment and if p_1 and ℓ_1 are adjacent in Δ. This geometry ${}^nH(O)$ is called a **projective Hjelmslev plane of level** n. When the vertex O has no real importance, we write ${}^nH$ instead of ${}^nH(O)$. The point set (resp. line set) of ${}^nH$ is then ${}^n\mathcal{P}$ (resp. ${}^n\mathcal{L}$), while incidence is denoted by ${}^nI$.

For $i \leq n$, the natural morphism from ${}^nH$ to ${}^iH$ is denoted by ${}^i\pi$. If $P, Q \in {}^n\mathcal{P}$ satisfy ${}^i\pi(P) = {}^i\pi(Q)$ for some $0 < i \leq n$, then we call P and Q i**-neighboring**. For $i = 1$ we just say that P and Q are **neighboring**. Similarly for lines. Also, if $P \in {}^n\mathcal{P}$ and $\ell \in {}^n\mathcal{L}$ are such that ${}^i\pi(P)\ {}^iI\ {}^i\pi(L)$ for some $0 < i \leq n$ then we say that P is i**-near** ℓ. Once again, P is **near** L when $i = 1$.

A **collineation** α of ${}^nH$ is, as usual, a bijection from ${}^n\mathcal{P}$ to itself and a bijection from ${}^n\mathcal{L}$ to itself that preserve ${}^nI$. It is not hard to see that all i-neighboring relations are determined by the geometry of ${}^nH$, so that every collineation α of ${}^nH$ induces in ${}^iH$ a unique collineation $\alpha^{\star_i}$. When acting on elements of ${}^iH$, $\alpha^{\star_i}$ will sometimes be replaced by α, so as to simplify the notation. For a fixed vertex O in Δ, the group of all collineations of ${}^nH(O)$ that are induced from an automorphism in $\mathrm{Aut}(\Delta)^+$ fixing O will be denoted by ${}^n\Psi(O)$. When $\alpha \in {}^n\Psi(O)$ is induced by $g \in \mathrm{Aut}(\Delta)^+$, it will be convenient to talk about the action of α (instead of g) on Δ.

Given $P \in {}^n\mathcal{P}$ and $\ell \in {}^n\mathcal{L}$ with $P\ {}^nI\ \ell$, an **elation** of ${}^nH$ with axis ℓ and center P is a collineation of ${}^nH$ fixing all points incident with ℓ and fixing all lines incident with P. As the next lemma shows, an elation also fixes additional points and lines.

Lemma 4.2.1. *Let α be an elation of ${}^nH$ $(n \geq 2)$ with axis ℓ and center P. Then α fixes all points (resp. lines) of ${}^nH$ that are $(n-1)$-neighboring P (resp. ℓ).*

Proof. See [VMVS98, Lemma 5]. $\square$

A **${}^1h$-collineation** α of ${}^nH$ is an elation of ${}^nH$ such that $\alpha^{\star_{n-1}}$ is trivial. (All elations of ${}^1H$ are ${}^1h$-collineations.) By definition, an elation α with axis ℓ and center P fixes all points incident with ℓ and all lines incident with P. The following lemma states that when α is a ${}^1h$-collineation, it also fixes the points near ℓ and the lines near P.

Lemma 4.2.2. *Let α be a ${}^1h$-collineation of ${}^nH$ with axis ℓ and center P. Then α fixes all points (resp. lines) of ${}^nH$ that are near ℓ (resp. P).*

Proof. See [VMVS98, Lemma 14]. $\square$

We then get the following result as a direct consequence.

Lemma 4.2.3. *Let α be a ${}^1h$-collineation of ${}^nH$ with axis ℓ and center P. Then for each $\ell' \in {}^n\mathcal{L}$ neighboring ℓ and each $P' \in {}^n\mathcal{P}$ neighboring P, α is also a ${}^1h$-collineation with axis ℓ' and center P'.*

Proof. By Lemma 4.2.2, α fixes all points (resp. lines) of ${}^nH$ that are near ℓ (resp. P). Given P' neighboring P and ℓ' neighboring ℓ, this is equivalent to saying that α fixes all points (resp. lines) that are near ℓ' (resp. P'). In particular, α fixes all points (resp. lines) incident with ℓ' (resp. P'), which means that α is an elation (and thus a ${}^1h$-collineation) with axis ℓ' and center P'. $\square$

The following lemma also comes from [VMVS98]. For $n = 1$, this is a particular case of a well-known result of Tits [Tit74, Theorem 4.1.1].

Lemma 4.2.4. *Let α be a non-trivial ${}^1h$-collineation of ${}^nH$ with axis ℓ and center P. Then α does not fix any point (resp. line) of ${}^nH$ that is not near ℓ (resp. P).*

Proof. See [VMVS98, Lemma 16 (iv)]. $\square$

From this lemma we can easily deduce the more general next result.

Lemma 4.2.5. *Let α be a non-trivial elation of ${}^nH$ with axis ℓ and center P. Then α does not fix any point (resp. line) of ${}^nH$ that is not near ℓ (resp. P). In particular, if $m \in {}^n\mathcal{L}$ is incident with P but not neighboring ℓ, then the group of all elations with axis ℓ and center P acts freely on the points incident with m but not neighboring P.*

Proof. Let us prove it by induction on n. For $n = 1$, this is equivalent to Lemma 4.2.4. Now assume the assertion is proved in ${}^{n-1}H$ and let α be a non-trivial elation of ${}^nH$ with axis ℓ and center P. It is clear that $\alpha^{\star_{n-1}}$ is an elation of ${}^{n-1}H$, with axis ${}^{n-1}\pi(\ell)$ and center ${}^{n-1}\pi(P)$. If $\alpha^{\star_{n-1}}$ is trivial then α is a ${}^1h$-collineation of ${}^nH$ and we can directly apply Lemma 4.2.4 to conclude. If on the contrary $\alpha^{\star_{n-1}}$ is not trivial then it is a non-trivial elation of ${}^{n-1}H$ and the result follows from the induction hypothesis. □

We now explain what it means for ${}^nH$ to be *Moufang*. First fix $P \in {}^n\mathcal{P}$ and $\ell \in {}^n\mathcal{L}$ with $P \; {}^nI \; \ell$. Given $m \in {}^n\mathcal{L}$ incident with P but not neighboring ℓ, we say that ${}^nH$ is (P, ℓ)**-transitive** if the group of all elations with axis ℓ and center P acts transitively on the points incident with m but not neighboring P. In view of Lemma 4.2.5, this condition does not depend on the choice for m and the action is then automatically simply transitive. When ${}^nH$ is (P, ℓ)-transitive for all $P \in {}^n\mathcal{P}$ and all $\ell \in {}^n\mathcal{L}$ with $P \; {}^nI \; \ell$, we say that ${}^nH$ is **Moufang**. For $n = 1$, this definition is equivalent to the definition of a Moufang projective plane.

Given $n \geq 1$ we say that Δ is n**-Moufang** if ${}^nH(O)$ is Moufang for each vertex O in Δ. Being n-Moufang is clearly weaker than being $(n+1)$-Moufang. As the next lemma shows, if Δ is n-Moufang for each $n \geq 1$ then the projective plane Δ^∞ at infinity of Δ is Moufang. This is equivalent to saying that Δ is Bruhat–Tits, see [Wei08, Chapter 28]. The proof of this lemma essentially comes from [VMVS99, §5].

Lemma 4.2.6. *Suppose that Δ is n-Moufang for each $n \geq 1$. Then the projective plane Δ^∞ is Moufang, i.e. Δ is Bruhat–Tits.*

Proof. Consider ℓ^∞ and m^∞ two lines in Δ^∞, and denote by P^∞ the point of Δ^∞ incident to ℓ^∞ and m^∞. We want to show that Δ^∞ is (P^∞, ℓ^∞)-transitive, i.e. that the group of all elations of Δ^∞ with axis

ℓ^∞ and center P^∞ acts transitively on the points incident with m^∞ but different from P^∞. Consider Q^∞ and R^∞ two points incident with m^∞, different from P^∞. Let A^∞ be some apartment in Δ^∞ containing ℓ^∞, P^∞, m^∞ and Q^∞. There exists an apartement A in Δ whose apartement at infinity is A^∞. Now choose a vertex O in A so that the two open rays from O to P^∞ and R^∞ are disjoint. For each $n \geq 1$, ${}^nH(O)$ is Moufang, so there exists an elation α_n of ${}^nH(O)$ with axis ${}^n\pi(\ell^\infty)$ and center ${}^n\pi(P^\infty)$, sending ${}^n\pi(Q^\infty)$ to ${}^n\pi(R^\infty)$ (where ${}^n\pi(x^\infty)$ is the point or line of ${}^nH(O)$ represented by the ray from O to x^∞). Lemma 4.2.5 implies that $\alpha_n^{\star_k} = \alpha_k$ for each $1 \leq k \leq n$. We can thus consider the inverse limit α of the sequence (α_n), which is an elation of Δ^∞ with axis ℓ^∞ and center P^∞, sending Q^∞ to R^∞. $\square$

Finally, for our future needs we give a name to some vertices of Δ. Given $P \in {}^n\mathcal{P}(O)$ and $\ell \in {}^n\mathcal{L}(O)$ with $P \; {}^nI \; \ell$ (where O is a vertex of Δ), the consecutive vertices of the geodesic from P to ℓ in Δ are denoted by $P = v_0(P,\ell), v_1(P,\ell), \dots, v_n(P,\ell) = \ell$.

4.3 Panel-transitive $\tilde{A}_2$-buildings

Given $n \geq 1$, we say that $\mathrm{Aut}(\Delta)$ (or $\mathrm{Aut}(\Delta)^+$) is **n-discrete** if there exists a vertex O in Δ such that the only element of $\mathrm{Aut}(\Delta)$ fixing O and acting trivially on ${}^nH(O)$ is the identity. Being n-discrete is clearly stronger than being $(n+1)$-discrete. Then $\mathrm{Aut}(\Delta)$ (or $\mathrm{Aut}(\Delta)^+$) is **non-n-discrete** if for each vertex O in Δ there exists a non-trivial element of $\mathrm{Aut}(\Delta)$ fixing O and acting trivially on ${}^nH(O)$. Remark that $\mathrm{Aut}(\Delta)$ is non-discrete if and only if it is non-n-discrete for all $n \geq 1$. In this section we prove Theorem 4.A′ which is thus a more precise version of Theorem 4.A (in view of Lemma 4.2.6).

Theorem 4.A′. *Let Δ be a locally finite thick $\tilde{A}_2$-building and suppose that $\mathrm{Aut}(\Delta)^+$ is transitive on panels of each type. Then for each $n \geq 1$, at least one of the following assertions holds:*

(a) Δ is n-Moufang, or

(b) $\mathrm{Aut}(\Delta)$ is $(6n+2)$-discrete.

In the proof, we assume that $\mathrm{Aut}(\Delta)^+$ is transitive on panels of each type and non-$(6n+2)$-discrete, and aim to show that ${}^nH(O)$ is Moufang for each vertex O in Δ.

4.3.1 Constructing ${}^1h$-collineations

In this first subsection, we observe that the non-$(n+3)$-discreteness of $\mathrm{Aut}(\Delta)$ together with its transitivity on vertices of each type implies the existence of non-trivial ${}^1h$-collineations in ${}^n\Psi(O)$ for each vertex O in Δ. We start with an easy result valid in any $\tilde{A}_2$-building Δ.

Lemma 4.3.1. *Let $v_0, \dots, v_k$ $(k \geq 1)$ be consecutive vertices of a wall of Δ. Consider vertices $w_0, \dots, w_{k-1}$ with w_i adjacent to v_i, v_{i+1} and w_{i-1} (if $i \geq 1$) for each $i \in \{0, \dots, k-1\}$. Similarly, consider vertices $w'_0, \dots, w'_{k-1}$ with w'_i adjacent to v_i, v_{i+1} and w'_{i-1} (if $i \geq 1$) for each $i \in \{0, \dots, k-1\}$. If $g \in \mathrm{Aut}(\Delta)^+$ fixes $v_0, \dots, v_k$ and if $g(w_0) = w'_0$, then $g(w_i) = w'_i$ for each $i \in \{0, \dots, k-1\}$.*

Proof. For each $i \in \{1, \dots, k-1\}$, the fact that g fixes v_{i-1}, v_i and v_{i+1} clearly implies that g sends w_{i-1} to w'_{i-1} if and only if g sends w_i to w'_i (see Figure 4.1 for an illustration). The conclusion then follows immediately. □

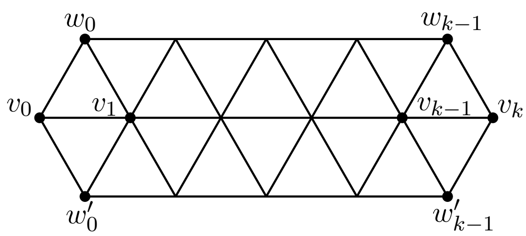


Figure 4.1: Illustration of Lemma 4.3.1.

This enables us to show the following.

Lemma 4.3.2. *Let O be a vertex of Δ and let $\alpha \in {}^n\Psi(O)$ $(n \geq 2)$ be a non-trivial collineation such that $\alpha^{\star_{n-1}}$ is trivial. Then there exists $P \in {}^n\mathcal{P}(O)$ and $\ell \in {}^n\mathcal{L}(O)$ with $P \; {}^nI \; \ell$ and such that α does not fix $v_1(P,\ell), v_2(P,\ell), \dots, v_{n-1}(P,\ell)$.*

Proof. For any $P \in {}^n\mathcal{P}(O)$ and $\ell \in {}^n\mathcal{L}(O)$ with $P \ {}^nI \ \ell$ we know by Lemma 4.3.1 that either all vertices $v_1(P,\ell), \dots, v_{n-1}(P,\ell)$ are fixed by α or none of them is fixed by α (because $\alpha^{\star_{n-1}}$ is trivial).

We therefore proceed by contradiction, assuming that for all such P and ℓ, all the vertices $v_1(P,\ell), \dots, v_{n-1}(P,\ell)$ are fixed by α. We show that, in this case, α is trivial (which gives the contradiction).

Consider any point $P \in {}^n\mathcal{P}(O)$ and choose two lines $\ell, \ell' \in {}^n\mathcal{L}(O)$ incident with P and such that ℓ and ℓ' are not neighboring. Then α fixes $v_1(P,\ell)$, $v_1(P,\ell')$ and ${}^{n-1}\pi(P)$, so it must fix P. This can be done for any choice of a point $P \in {}^n\mathcal{P}(O)$, and similarly for any choice of a line $\ell \in {}^n\mathcal{L}(O)$, so α is trivial. □

Proposition 4.3.3. *Let $n \geq 1$ and suppose that* $\mathrm{Aut}(\Delta)$ *is non-$(n+3)$-discrete and transitive on vertices of each type. Then for each vertex O in Δ, there exists a non-trivial ${}^1h$-collineation in ${}^n\Psi(O)$.*

Proof. In view of the transitivity of $\mathrm{Aut}(\Delta)$ on vertices of each type, it suffices to find three vertices O_0, O_1, O_2 of types 0, 1 and 2 such that ${}^n\Psi(O_i)$ contains a non-trivial ${}^1h$-collineation for each $i \in \{0,1,2\}$.

Fix some vertex O in Δ. The non-$(n+3)$-discreteness of $\mathrm{Aut}(\Delta)$ implies that there exists $N \geq n+4$ such that ${}^N\Psi(O)$ contains a non-trivial collineation α with $\alpha^{\star_{N-1}}$ trivial. By Lemma 4.3.2, there exists $P \in {}^N\mathcal{P}(O)$ and $\ell \in {}^N\mathcal{L}(O)$ with $P \ {}^NI \ \ell$ and such that none of the vertices $v_1(P,\ell), \dots, v_{N-1}(P,\ell)$ is fixed by α. Now write $X = {}^{N-n}\pi(P)$ and $Y = {}^{N-n}\pi(\ell)$ (see Figure 4.2). As $N - n \geq 4$, the geodesic from X to Y

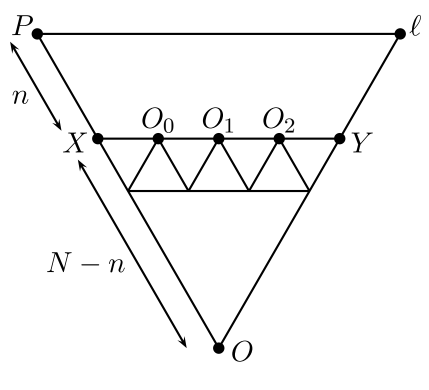


Figure 4.2: Illustration of Proposition 4.3.3.

in Δ contains at least three vertices different from X and Y. Since three consecutive vertices in such a configuration always have the three different types, there exist O_0, O_1 and O_2 with types 0, 1 and 2 and strictly between X and Y. For each $i \in \{0,1,2\}$, the action induced by α on ${}^nH(O_i)$ is non-trivial, because α acts non-trivially on $v_1(P,\ell),\ldots,v_{N-1}(P,\ell)$. There remains to check that it is a ${}^1h$-collineation of ${}^nH(O_i)$, but this is a consequence of the fact that $\alpha^{\star_{N-1}}$ is trivial. $\square$

The previous proposition shows the existence of a non-trivial ${}^1h$-collineation in ${}^n\Psi(O)$, in some circumstances. We already know some properties of such collineations (see Lemma 4.2.4), but the next lemma is more precise.

Lemma 4.3.4. *Let O be a vertex of Δ and consider $P \in {}^n\mathcal{P}(O)$ and $\ell \in {}^n\mathcal{L}(O)$ with $P\ {}^nI\ \ell$ ($n \geq 2$). Let also $Q \in {}^n\mathcal{P}(O)$ be a point not near ℓ and $o \in {}^n\mathcal{L}(O)$ be a line not near P, such that $Q\ {}^nI\ o$.*

(i) Let $\alpha \in {}^n\Psi(O)$ be a non-trivial ${}^1h$-collineation with axis ℓ and center P. Then α does not fix $v_i(Q,o)$, for any $i \in \{0,1,\ldots,n\}$.

(ii) Denote by $m \in {}^n\mathcal{L}(O)$ the line incident with P and Q. Suppose that the group G of all ${}^1h$-collineations in ${}^n\Psi(O)$ with axis ℓ and center P acts transitively on the set of points $(n-1)$-neighboring Q and incident with m. Then, for each $i \in \{0,\ldots,n-2\}$, G acts transitively on the set of chambers of Δ having vertices $v_i({}^{n-1}\pi(Q), {}^{n-1}\pi(o))$ and $v_{i+1}({}^{n-1}\pi(Q), {}^{n-1}\pi(o))$ but not $v_i({}^{n-2}\pi(Q), {}^{n-2}\pi(o))$.

Proof. Let $\alpha \in {}^n\Psi(O)$ be a ${}^1h$-collineation with axis ℓ and center P. Let also $m \in {}^n\mathcal{L}(O)$ be the line incident with P and Q (see Figure 4.3). We know by definition of an elation that α fixes m, and the fact that $\alpha^{\star_{n-1}}$ is trivial implies that it also fixes ${}^{n-1}\pi(Q)$. Hence, from Lemma 4.3.1 applied to the segment from ${}^{n-1}\pi(m)$ to ${}^{n-1}\pi(Q)$, we get that α fixes $v_1(Q,m)$. Assertions (i) and (ii) then follow thanks to another application of Lemma 4.3.1 to the segment with vertices $v_1(Q,m)$, ${}^{n-1}\pi(Q)$, $v_1({}^{n-1}\pi(Q), {}^{n-1}\pi(o)),\ldots,{}^{n-1}\pi(o)$. (Recall, for (i), that when α is non-trivial it does not fix Q nor o by Lemma 4.2.4.) $\square$

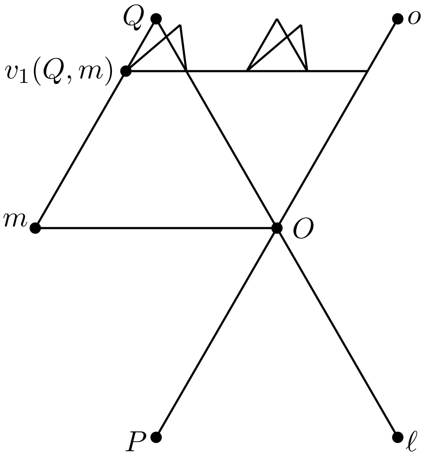


Figure 4.3: Illustration of Lemma 4.3.4.

4.3.2 From panel-transitivity to chamber-transitivity

In this subsection, we prove that if $\mathrm{Aut}(\Delta)^+$ is non-4-discrete and transitive on panels of each type, then $\mathrm{Aut}(\Delta)^+$ is transitive on chambers. We start by the following easy lemma, valid in any projective Hjelmslev plane of level 1 (i.e. any projective plane).

Lemma 4.3.5. *Let α be a non-trivial elation of ${}^1H$ with axis ℓ and center P. Let $m \in {}^1\mathcal{L}$ be incident with P but different from ℓ. Then the permutation induced by α on the set of q points incident with m but different from P is a product of $k \geq 1$ disjoint cycles of the same length $c \geq 2$, where $k \cdot c = q$. Moreover, k and c do not depend on m.*

Proof. Let σ be the permutation induced by α on this set of q points. By Lemma 4.2.4, σ has no fixed point. Now it suffices to prove that two cycles in the cycle decomposition of σ always have the same length. Suppose for a contradiction that there are two cycles of different lengths $c_1 < c_2$. Then α^{c_1} is an elation of ${}^1H$ which is non-trivial (because σ^{c_1} is non-trivial) and σ^{c_1} has fixed points, which contradicts Lemma 4.2.4. So all k cycles in the cycle decomposition must have the same length $c \geq 2$, and of course $k \cdot c = q$. Note that k and c do not depend on m, otherwise we would once again get a power of α that is non-trivial but has forbidden fixed points. □

Proposition 4.3.6. *Suppose that* $\mathrm{Aut}(\Delta)^+$ *is non-*4*-discrete and transitive on panels of each type. Then* $\mathrm{Aut}(\Delta)^+$ *is chamber-transitive.*

Proof. Let us assume for a contradiction that $\mathrm{Aut}(\Delta)^+$ is not chamber-transitive. Then we can color the chambers of Δ in blue and red so that each color is used at least once and two chambers with different colors do not belong to the same orbit. (For instance, color one orbit of chambers in blue and all other orbits in red.) For each type $t \in \{\{0,1\},\{1,2\},\{0,2\}\}$, the transitivity on t-panels implies that there exist b_t and r_t such that all t-panels are adjacent to b_t blue chambers and r_t red chambers. Note that $b_t \geq 1$ and $r_t \geq 1$, otherwise all chambers of Δ would be the same color. In Δ, all panels have the same number of chambers, say $1+q$, so $b_t + r_t = 1+q$ for each t.

We first claim that $b_{\{0,1\}} = b_{\{1,2\}} = b_{\{0,2\}}$ (and $r_{\{0,1\}} = r_{\{1,2\}} = r_{\{0,2\}}$). Indeed, take $t, t' \in \{\{0,1\},\{1,2\},\{0,2\}\}$ with $t \neq t'$ and consider a vertex v of type $t \cap t'$ in Δ. The number of blue chambers adjacent to v (i.e. in the residue corresponding to v) is equal to $p_t \cdot b_t$, where p_t is the number of t-panels adjacent to v. Since the residue associated to v is a projective plane of order q, we have $p_t = q^2+q+1$ and the number of blue chambers adjacent to v is $(q^2+q+1)\cdot b_t$. But for the same reason with t' instead of t, this number is also equal to $(q^2+q+1)\cdot b_{t'}$. So $b_t = b_{t'}$ and $r_t = r_{t'}$. In the following we therefore write $b = b_{\{0,1\}} = b_{\{1,2\}} = b_{\{0,2\}}$ and $r = r_{\{0,1\}} = r_{\{1,2\}} = r_{\{0,2\}}$. Recall that $b + r = 1 + q$.

Now consider a vertex O in Δ and a non-trivial elation α in ${}^1\Psi(O)$, whose existence is ensured by Proposition 4.3.3. Let $P \in {}^1\mathcal{P}(O)$ and $\ell \in {}^1\mathcal{L}(O)$ be the center and axis of the elation α. Consider $m \in {}^1\mathcal{L}(O)$ a line incident with P but different from ℓ. By Lemma 4.3.5, the permutation induced by α on the set of q points incident with m but different from P is a product of $k \geq 1$ cycles of length $c \geq 2$, with $k \cdot c = q$. If the chamber with vertices O, P and m is blue, then this implies that $b \equiv 1 \pmod{c}$ and $r \equiv 0 \pmod{c}$. If it is red, then $r \equiv 1 \pmod{c}$ and $b \equiv 0 \pmod{c}$. But c does not depend on m, so this reasoning is valid for any choice of m. As b cannot be congruent to both 0 and 1 modulo c (because $c \geq 2$), this means that all the chambers with vertices O, P and some $m \neq \ell$ have the same color. We can assume that this common color is blue, so that $b \equiv 1 \pmod{c}$ and $r \equiv 0 \pmod{c}$. In particular we have $r \geq c \geq 2$, but this is a contradiction with the fact that there is at most one red chamber adjacent to the panel defined by O and P. $\square$

Remark 4.3.7. The non-4-discreteness in Proposition 4.3.6 can be replaced by non-2-discreteness. Indeed, in the proof we only need a non-trivial elation in ${}^1\Psi(O)$ for some vertex O (of any type), and Proposition 4.3.3 with $n+1$ instead of $n+3$ indeed gives a non-trivial ${}^1h$-collineation in ${}^n\Psi(O)$ for a vertex O whose type is not controlled.

A similar remark can be done for many of our following results: we often assume that $\mathrm{Aut}(\Delta)$ is non-$f(n)$-discrete for some linear function f of n, but we never claim that our choice for f is optimal. In particular, the value $6n+2$ appearing in Theorem 4.A$'$ can certainly be replaced by a smaller value with some more effort.

4.3.3 From chamber-transitivity to 1-Moufangness

The following theorem is due to Kantor [Kan87] and concerns finite projective planes with a collineation group transitive on incident point-line pairs. It will be helpful to get local information about $\mathrm{Aut}(\Delta)$.

Theorem 4.3.8 (Kantor, 1987)**.** *Let Π be a projective plane of order q, and let F be a collineation group of Π transitive on incident point-line pairs. Then either*

(a) Π is Desarguesian and $F \geq \mathrm{PSL}(3,q)$, or

(b) F is a Frobenius group of odd order $(q^2+q+1)(q+1)$, and q^2+q+1 is prime.

Proof. See [Kan87, Theorem A]. $\square$

Corollary 4.3.9. *Suppose that $\mathrm{Aut}(\Delta)^+$ is non-4-discrete and chamber-transitive. Then for each vertex O in Δ, the projective plane ${}^1H(O)$ is Desarguesian and ${}^1\Psi(O) \geq \mathrm{PSL}(3,q)$, where $q+1$ is the number of chambers in each panel of Δ. In particular, ${}^1H(O)$ is Moufang and ${}^1\Psi(O)$ contains all elations of ${}^1H(O)$.*

Proof. For any vertex O in Δ, ${}^1H(O)$ is a projective plane of order q. The chamber-transitivity of Δ directly implies that ${}^1\Psi(O)$ is transitive on incident point-line pairs of ${}^1H(O)$. Hence, by Theorem 4.3.8, either ${}^1H(O)$ is Desarguesian and ${}^1\Psi(O) \geq \mathrm{PSL}(3,q)$, or $|{}^1\Psi(O)| = (q^2+q+1)(q+1)$. We only need to show that the latter is impossible. Note that there are exactly $(q^2+q+1)(q+1)$ incident point-line pairs in ${}^1H(O)$, so the equality $|{}^1\Psi(O)| = (q^2+q+1)(q+1)$ would imply that the action of ${}^1\Psi(O)$ on these point-line pairs is free. However, by Proposition 4.3.3, there exists a non-trivial elation in ${}^1\Psi(O)$. So the action is not free and the statement stands proved.

Note that, for a finite projective plane Π (say of order q), being Desarguesian is equivalent to being Moufang. Also, in this case, the group generated by all elations of Π is called the *little projective group* and is exactly $\mathrm{PSL}(3,q)$. $\square$

4.3.4 From chamber-transitivity to Bruhat–Titsness

We have seen with Corollary 4.3.9 that all ${}^1H(O)$ are Moufang when $\mathrm{Aut}(\Delta)^+$ is chamber-transitive and non-4-discrete. Our next goal is to show, for each $n \geq 2$, that all ${}^nH(O)$ are Moufang when $\mathrm{Aut}(\Delta)^+$ is chamber-transitive and non-$(6n+2)$-discrete.

We start with the next easy corollary of Proposition 4.3.3.

Lemma 4.3.10. *Let $n \geq 1$ and suppose that $\mathrm{Aut}(\Delta)^+$ is non-$(n+3)$-discrete and chamber-transitive. Then for each vertex O in Δ, each point $P \in {}^n\mathcal{P}(O)$ and each line $\ell \in {}^n\mathcal{L}(O)$ with $P \ {}^nI \ \ell$, there exists a non-trivial ${}^1h$-collineation in ${}^n\Psi(O)$ with axis ℓ and center P.*

Proof. By Proposition 4.3.3, there exists a non-trivial ${}^1h$-collineation $\alpha \in {}^n\Psi(O)$, say with axis $\ell' \in {}^n\mathcal{L}(O)$ and center $P' \in {}^n\mathcal{P}(O)$. Let c (resp. c') be the chamber of Δ with vertices O, ${}^1\pi(\ell)$ and ${}^1\pi(P)$ (resp. O, ${}^1\pi(\ell')$ and ${}^1\pi(P')$). Since $\mathrm{Aut}(\Delta)^+$ is chamber-transitive, there exists $g \in \mathrm{Aut}(\Delta)^+$ such that $g(c) = c'$. Then $g^{-1}\alpha g$ is a non-trivial ${}^1h$-collineation, and by Lemma 4.2.3 it has axis ℓ and center P. $\square$

Lemma 4.3.11. *Let $n \geq 2$ and let $1 \leq k < n$. In the following, O is a vertex of Δ, P is a point in ${}^n\mathcal{P}(O)$ and ℓ is a line in ${}^n\mathcal{L}(O)$ with $P \ {}^nI \ \ell$,*

Q is a point in ${}^n\mathcal{P}(O)$ not near ℓ, and $m \in {}^n\mathcal{L}(O)$ is the line incident with P and Q.

(i) Suppose that for any O, P and ℓ, there exists a non-trivial ${}^1h$-collineation in ${}^{2n+k}\Psi(O)$ with axis ℓ and center P. Then for any O, P and ℓ, there exists an elation $\alpha \in {}^n\Psi(O)$ with axis ℓ and center P such that $\alpha^{\star_{k-1}}$ is trivial but $\alpha^{\star_k}$ is non-trivial.

(ii) Suppose that for any O, P, ℓ and Q, the group of all ${}^1h$-collineations in ${}^{2n+k}\Psi(O)$ with axis ℓ and center P acts transitively on the set of points $(2n+k-1)$-neighboring Q and incident with m. Then for any O, P, ℓ and Q, the group of all elations $\alpha \in {}^n\Psi(O)$ with axis ℓ and center P and with $\alpha^{\star_{k-1}}$ trivial is transitive on the set of points in ${}^k\mathcal{P}(O)$ that are $(k-1)$-neighboring ${}^k\pi(Q)$ and incident with ${}^k\pi(m)$.

Proof. Fix O, ℓ and P and let A be an appartment of Δ containing them (seen as vertices of Δ). In A, we denote by O' the reflection of O over the line through ℓ and P (see Figure 4.4). Also, P' (resp. ℓ') is the vertex of A at distance $2n+k$ from O' such that O' lies on the segment from ℓ to P' (resp. from P to ℓ').

We first prove (i). By hypothesis, there exists a non-trivial ${}^1h$-collineation $\beta \in {}^{2n+k}\Psi(O')$ with axis ℓ' and center P'. We now consider the element $\alpha \in {}^n\Psi(O)$ induced by β. The fact that $\beta^{\star_{2n+k-1}}$ is trivial implies that $\alpha^{\star_{k-1}}$ is trivial. Also, it is clear from Lemma 4.3.4 (i) applied to β that $\alpha^{\star_k}$ is non-trivial. There remains to show that α is an elation of ${}^nH(O)$ with axis ℓ and center P, i.e. that α fixes all points incident with ℓ and all lines incident with P. This is actually also a consequence from the fact that $\beta^{\star_{2n+k-1}}$ (even $\beta^{\star_{2n}}$) is trivial. Indeed, all points incident with ℓ (and all lines incident with P) in ${}^nH(O)$ correspond to vertices of Δ that are contained in ${}^{2n}H(O')$ (more precisely in the convex hull of the vertices of Δ associated to ${}^{2n}\mathcal{P}(O')$ and ${}^{2n}\mathcal{L}(O')$).

The reasoning is the same for (ii). Take $Q \in {}^n\mathcal{P}(O)$ a point not near ℓ and denote by $m \in {}^n\mathcal{L}(O)$ the line incident with P and Q. Here also, we see Q and m as vertices of Δ and we can even assume that they belong to A. Let Q' be the vertex of A at distance $2n+k$ from O', in

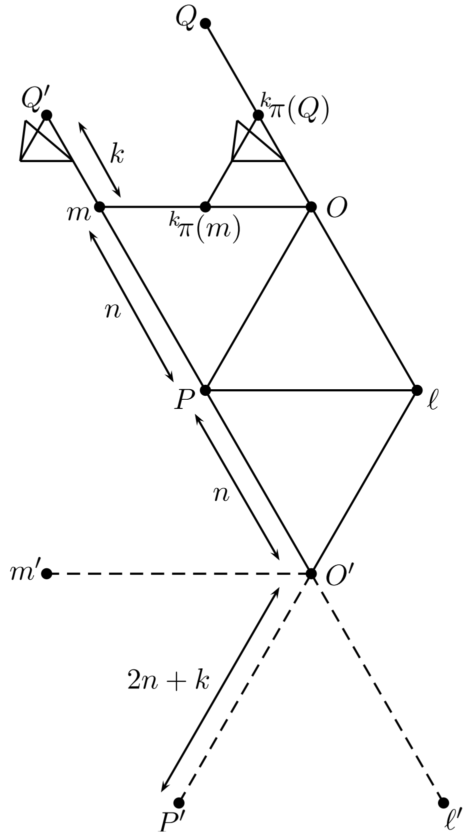


Figure 4.4: Illustration of Lemma 4.3.11.

the direction of P and m. If $m' \in {}^{2n+k}\mathcal{L}(O')$ is the line incident with P' and Q' in ${}^{2n+k}H(O')$, then the hypothesis states that the group of all ${}^1h$-collineations in ${}^{2n+k}\Psi(O')$ with axis ℓ' and center P' acts transitively on the set of points $(2n+k-1)$-neighboring Q' and incident with m'. Using Lemma 4.3.4 (ii) and as for (i), we obtain that the group of all elations $\alpha \in {}^n\Psi(O)$ with axis ℓ and center P with $\alpha^{\star_{k-1}}$ trivial is transitive on the set of points in ${}^k\mathcal{P}(O)$ that are $(k-1)$-neighboring ${}^k\pi(Q)$ and incident with ${}^k\pi(m)$. $\square$

The key result of this section is then the following.

Proposition 4.3.12. *Let $n \geq 1$ and suppose that* $\mathrm{Aut}(\Delta)^+$ *is non-$(2n+4)$-discrete and chamber-transitive. Let O be a vertex in Δ and consider a point $P \in {}^n\mathcal{P}(O)$ and a line $\ell \in {}^n\mathcal{L}(O)$ with $P \;{}^nI\; \ell$. Let*

$Q \in {}^n\mathcal{P}(O)$ be a point not near ℓ and denote by $m \in {}^n\mathcal{L}(O)$ the line incident with P and Q. Then the group of all ${}^1h$-collineations in ${}^n\Psi(O)$ with axis ℓ and center P acts transitively on the set of points $(n-1)$-neighboring Q and incident with m.

Proof. We introduce the three following assertions, all depending on $N \geq 1$ (actually $N \geq 2$ for (C_N)). Remark that (A_n) is exactly what we need to prove.

(A_N) Let O be a vertex in Δ. Let $P \in {}^N\mathcal{P}(O)$ and $\ell \in {}^N\mathcal{L}(O)$ be such that $P\ {}^NI\ \ell$, let $Q \in {}^N\mathcal{P}(O)$ be a point not near ℓ and denote by m the line incident with P and Q. The group of all ${}^1h$-collineations in ${}^N\Psi(O)$ with axis ℓ and center P acts transitively on the set of points $(N-1)$-neighboring Q and incident with m.

(B_N) Let i, j, k be the three types of panels in some order and let f be the word $ijkijkijk\ldots$ of length $2N$. Let $(c_0, c_1, \ldots, c_{2N})$ be a gallery of type f in Δ (i.e. for each $1 \leq s \leq 2N$, the chambers c_{s-1} and c_s are adjacent and their common panel has type given by the s^{th} letter of f). Then for any two chambers d and d' adjacent to both c_0 and c_1 (but different from them), there exists an automorphism of Δ fixing $c_0, c_1, \ldots, c_{2N}$ and sending d to d'.

(C_N) Let O be a vertex in Δ. Let $P \in {}^N\mathcal{P}(O)$ and $\ell \in {}^N\mathcal{L}(O)$ be such that $P\ {}^NI\ \ell$, let $Q \in {}^N\mathcal{P}(O)$ be a point near ℓ but not neighboring P, and let $m, o \in {}^N\mathcal{L}(O)$ be two lines near Q but not neighboring ℓ. There exist a point $P' \in {}^N\mathcal{P}(O)$ $(N-1)$-neighboring P, a line $\ell' \in {}^N\mathcal{L}(O)$ neighboring ℓ (with $P'\ {}^NI\ \ell'$) and an elation in ${}^N\Psi(O)$ with axis ℓ' and center P' sending ${}^1\pi(m)$ to ${}^1\pi(o)$.

Note that (A_1) is given by Corollary 4.3.9. It also follows from this corollary that (B_1) is true. Indeed, if O is the vertex of Δ adjacent to c_0, c_1 and c_2 (as defined in (B_1)), then having ${}^1\Psi(O) \geq \mathrm{PSL}(3, q)$ implies the existence of an automorphism fixing c_0, c_1, c_2 and sending d to d'. We now show three different relations between (A_N), (B_N) and (C_N).

Claim 1. $(B_{N-1}) + (C_N) \Rightarrow (A_N)$ *for each* $2 \leq N \leq n$.

Proof of the claim: Let O, P, ℓ, Q and m be as in (A_N). Let also $R \in {}^N\mathcal{P}(O)$ be a point $(N-1)$-neighboring Q and incident with m (see Figure 4.5). We want to prove that there exists some ${}^1h$-collineation in ${}^N\Psi(O)$ with axis ℓ and center P, sending Q to R.

From Lemma 4.3.10 we know that there exists a non-trivial ${}^1h$-collineation $\alpha \in {}^N\Psi(O)$ with axis ℓ and center P. (Note that $\mathrm{Aut}(\Delta)$ is non-$(N+3)$-discrete because $N+3 \leq n+3 \leq 2n+2$.) By Lemma 4.2.4, α sends Q to some $S \neq R$. We know from (B_{N-1}) that there exists $\beta \in \mathrm{Aut}(\Delta)^+$ fixing Q, O and ${}^1\pi(m)$ and sending S to R. Then $\beta\alpha\beta^{-1}$ sends Q to R (as desired) and is a ${}^1h$-collineation with axis ℓ' and center P', with ${}^1\pi(P')\ {}^1I\ {}^1\pi(m)$. Now there are two different cases:

- If ${}^1\pi(\ell') = {}^1\pi(\ell)$, then also ${}^1\pi(P') = {}^1\pi(P)$, and hence $\beta\alpha\beta^{-1}$ is a ${}^1h$-collineation with axis ℓ and center P in view of Lemma 4.2.3.
- If ${}^1\pi(\ell') \neq {}^1\pi(\ell)$, then denote by $T \in {}^N\mathcal{P}(O)$ the point incident with ℓ and ℓ' and by $o \in {}^N\mathcal{L}(O)$ the line incident with Q and T. By (C_N), there exist a point $Q' \in {}^N\mathcal{P}(O)$ $(N-1)$-neighboring Q, a line $o' \in {}^N\mathcal{L}(O)$ neighboring o (with $Q'\ {}^NI\ o'$) and an elation $\gamma \in {}^N\Psi(O)$ with axis o' and center Q' sending ${}^1\pi(\ell')$ to ${}^1\pi(\ell)$. Note that γ fixes Q and R by Lemma 4.2.1. Thus $\gamma(\beta\alpha\beta^{-1})\gamma^{-1}$ is a ${}^1h$-collineation with axis ℓ and center P, and it sends Q to R. (Remark that this argument is valid when ${}^1\pi(P') = {}^1\pi(P)$, even

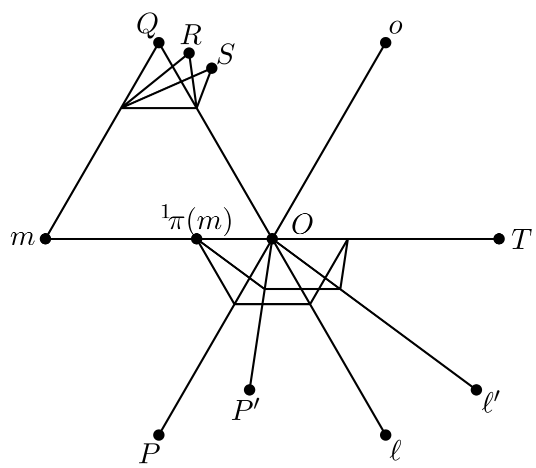


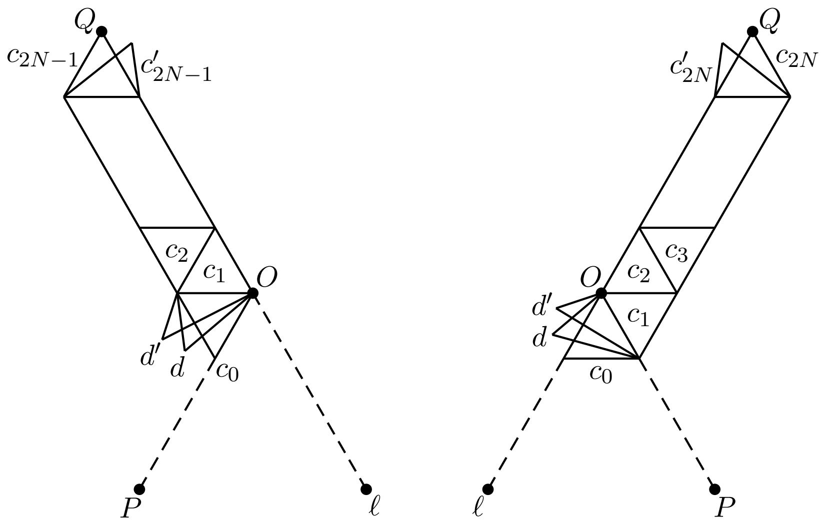
Figure 4.5: Illustration of Proposition 4.3.12, Claim 1.

though Figure 4.5 does not represent that case.) ■

Claim 2. $(B_{N-1}) + (A_N) \Rightarrow (B_N)$ *for each* $N \geq 2$.

Proof of the claim: Let i, j, k, w, $(c_0, c_1, \dots, c_{2N})$, d and d' be as in (B_N). We need an automorphism of Δ fixing $c_0, \dots, c_{2N}$ and sending d to d'.

By (B_{N-1}), we already have some $g \in \mathrm{Aut}(\Delta)^+$ fixing $c_0, \dots, c_{2N-2}$ and sending d to d'. Denote by c'_{2N-1} the image of c_{2N-1} by g. Now taking O, P, ℓ and Q as in Figure 4.6a, we can apply (A_N) to get an element $h \in \mathrm{Aut}(\Delta)^+$ fixing $c_0, \dots, c_{2N-2}$ as well as d and d' and sending c'_{2N-1} to c_{2N-1}. So hg sends d to d' and fixes $c_0, \dots, c_{2N-1}$. Now we can use the same method one step further: if c'_{2N} denotes the image of c_{2N} by hg, then we can find thanks to (A_N) (see Figure 4.6b) an element h' fixing $c_0, \dots, c_{2N-1}$, d and d' and sending c'_{2N} to c_{2N}. The element $h'hg$ then fixes $c_0, \dots, c_{2N}$ and sends d to d'. ■



(a) From c_{2N-1} to c'_{2N-1}. (b) From c_{2N} to c'_{2N}.

Figure 4.6: Illustration of Proposition 4.3.12, Claim 2.

Claim 3. $(B_{N-1}) \Rightarrow (C_N)$ *for each* $2 \leq N \leq n$.

Proof of the claim: Let O, P, ℓ, Q, m and o be as in (C_N). We must find an elation in ${}^N\Psi(O)$ sending ${}^1\pi(m)$ to ${}^1\pi(o)$, with axis ℓ' and center P' where ℓ' is neighboring ℓ and P' is $(N-1)$-neighboring P.

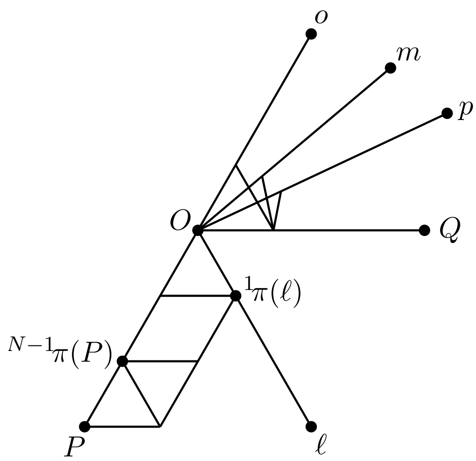


Figure 4.7: Illustration of Proposition 4.3.12, Claim 3.

By Lemma 4.3.10 (with $2N+1$) and Lemma 4.3.11 (i) (with N), there exists an elation $\alpha \in {}^N\Psi(O)$ with axis ℓ and center P and such that $\alpha^{\star_1}$ is non-trivial. In view of Lemma 4.2.4 (applied in ${}^1H(O)$), if p denotes the image of m by α, then ${}^1\pi(p) \neq {}^1\pi(m)$. By (B_{N-1}), there exists $g \in \mathrm{Aut}(\Delta)^+$ fixing ${}^{N-1}\pi(P)$, ${}^1\pi(\ell)$ and ${}^1\pi(m)$ and sending ${}^1\pi(p)$ to ${}^1\pi(o)$ (see Figure 4.7). Then $g\alpha g^{-1}$ is an elation with axis $g(\ell)$ and center $g(P)$ which sends ${}^1\pi(m)$ to ${}^1\pi(o)$. Since $g(\ell)$ is neighboring ℓ and $g(P)$ is $(N-1)$-neighboring P, we are done. ■

Claims 1 and 3 together imply that $(B_{N-1}) \Rightarrow (A_N)$ for each $2 \leq N \leq n$ $(*)$, so that Claim 2 then reads as $(B_{N-1}) \Rightarrow (B_N)$ for each $2 \leq N \leq n$. From (B_1) we therefore get (B_N) for all $1 \leq N \leq n$, and hence (A_N) is true for all $2 \leq N \leq n$ by $(*)$. (Remember that (A_1) was already true.) □

Proof of Theorem 4.A′. Suppose that $\mathrm{Aut}(\Delta)$ is non-$(6n+2)$-discrete. By Proposition 4.3.6, $\mathrm{Aut}(\Delta)^+$ is chamber-transitive (because $6n+2 \geq 4$). We want to prove that ${}^nH(O)$ is Moufang for each vertex O in Δ.

Consider $P \in {}^n\mathcal{P}(O)$ and $\ell \in {}^n\mathcal{L}(O)$ with $P \; {}^nI \; \ell$. We need to show that ${}^nH(O)$ is (P,ℓ)-transitive. Let $m \in {}^n\mathcal{L}(O)$ be incident with P but not neighboring ℓ and let $Q, R \in {}^n\mathcal{P}(O)$ be incident with m but not neighboring P. We must find an elation of ${}^nH(O)$ with axis ℓ and center P sending Q to R. We actually show by induction on k that, for each

$0 \leq k \leq n$, there exists an elation with axis ℓ and center P sending ${}^k\pi(Q)$ to ${}^k\pi(R)$. For $k = 0$ we can take the identity (because ${}^0\pi(Q) = {}^0\pi(R) = O$ by convention). Now consider $1 \leq k \leq n$ and assume that this is true for $k-1$. Thus there is an elation α with axis ℓ and center P such that $\alpha({}^{k-1}\pi(Q)) = {}^{k-1}\pi(R)$. Denote by Q' the image of Q by α. Then Q' is $(k-1)$-neighboring R and incident with m, and it suffices to find an elation with axis ℓ and center P sending ${}^k\pi(Q')$ to ${}^k\pi(R)$ in . For $k = n$, such an elation exists by Proposition 4.3.12, and for $k < n$ we need this same proposition (with $2n + k$) together with Lemma 4.3.11 (ii). (Note for Proposition 4.3.12 that $2(2n + k) + 4 \leq 6n + 2$ when $k \leq n + 1$.) $\square$

4.4 Vertex-transitive $\tilde{A}_2$-buildings

The goal of this section is to prove Theorem 4.B′ below.

Theorem 4.B′. *Let Δ be a locally finite thick $\tilde{A}_2$-building. Suppose that* $\mathrm{Aut}(\Delta)$ *is transitive on vertices and unimodular, that* $\mathrm{Aut}(\Delta)^+$ *is transitive on vertices of each type, and that Δ has thickness $p+1$ for some prime p. Then for each $n \geq 1$, at least one of the following assertions holds:*

(a) Δ is n-Moufang, or

(b) $\mathrm{Aut}(\Delta)$ *is $(6n+2)$-discrete.*

We will once again suppose that $\mathrm{Aut}(\Delta)$ is non-$(6n + 2)$-discrete and, under the hypotheses of Theorem 4.B′, prove that $\mathrm{Aut}(\Delta)^+$ must be transitive on panels of each type. The conclusion will then follow from Theorem 4.A′.

4.4.1 About finite projective planes

We begin with several lemmas concerning finite projective planes. They will become useful later in the section. The first lemma is classical.

Lemma 4.4.1. *Let Π be a finite projective plane and F be a collineation group of Π. Then F is transitive on points of Π if and only if F is transitive on lines of Π.*

Proof. It is actually true that, for any collineation group F of a finite projective plane Π, F has as many point orbits as line orbits, see [HP73, Theorem 13.4]. $\square$

The following lemma is also classical but, because of the lack in finding a suitable reference, we give its proof here.

Lemma 4.4.2. *Let Π be a finite projective plane of prime order and F be a collineation group of Π. Suppose that F contains a non-trivial elation. Then either F is transitive on points of Π or F fixes a point or a line of Π.*

Proof. We color the points of Π according to their orbit under the action of F. Let us suppose that F is not transitive on points of Π, i.e. that there are at least 2 colors. Let us denote by P and ℓ the center and axis of a non-trivial elation α in F. By Lemma 4.3.5 and since Π has prime order, for each line o incident to P and different from ℓ, the elation α is transitive on points incident to o and different from P. Thus, for each such o, all points incident to o and different from P have the same color $(*)$. Now let us distinguish several cases:

- If P has a color that no other point has, then P is fixed by F.
- Otherwise, and if the only points with the same color as P are incident to ℓ, then ℓ is fixed by F.

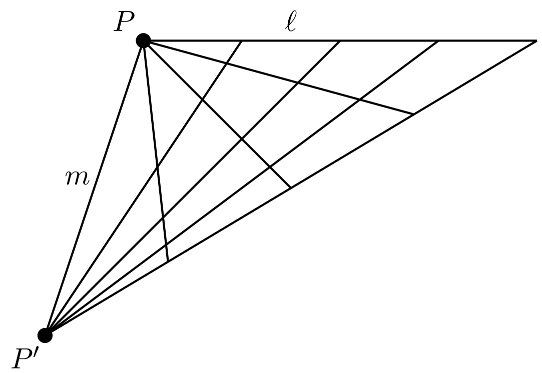


Figure 4.8: Illustration of Lemma 4.4.2.

- Now assume that there exists a point P' not incident to ℓ but with the same color as P. This means that there exists $\beta \in F$ with $\beta(P) = P'$. Denote by m the line through P and P', and write $\ell' = \beta(\ell)$. Note that, in view of $(*)$, for each line o' incident to P' and different from ℓ', all points incident to o' and different from P' have the same color $(**)$. See Figure 4.8 for an illustration.
  - If $\ell' = m$, we deduce from $(*)$, $(**)$ and the fact that $\beta(\ell) = \ell'$ that all points have the same color, which is a contradiction.
  - If $\ell' \neq m$, then we obtain from $(*)$ and $(**)$ that all points incident to m have the same color, say c_1, and that all points not incident to m but different from $Q = \ell \cap \ell'$ have the same color, say c_2. We write c_3 for the color of Q. If $c_3 \neq c_1, c_2$, then Q is the only point with color c_3 so it is fixed by F. If $c_3 = c_2$, then $c_1 \neq c_2$ (because there are at least two colors), and m is fixed by F. Finally, if $c_3 = c_1$, then $c_1 \neq c_2$ and there should exist $\gamma \in F$ with $\gamma(P) = Q$. But this gives a contradiction with the coloring. □

We conclude with a third lemma about finite projective planes of prime order which can be applied in some really precise situation.

Lemma 4.4.3. *Let* Π *be a finite projective plane of prime order and* F *be a collineation group of* Π. *Suppose that* F *contains a non-trivial elation and that* F *fixes exactly one point* Q *and one line* m, *with* Q *not incident to* m. *Then* F *is transitive on points incident to* m *and transitive on points not incident to* m *but different from* Q.

Proof. Let α be a non-trivial elation in F, say with axis ℓ and center P. From Lemma 4.2.4, we deduce that Q is incident to ℓ (and different from P) and that m is incident to P (and different from ℓ), see Figure 4.9. We color the points of Π according to their orbit (under the action of F). By Lemma 4.3.5 and since Π has prime order, for each line o incident to P and different from ℓ, all points incident to o and different from P have the same color $(*)$. Now by hypothesis, P is not fixed by F. Thus there exists $\beta \in F$ with $\beta(P) = P' \neq P$. Moreover, P' is incident to m since m is fixed by F (and hence by β). As Q is fixed by F, we also

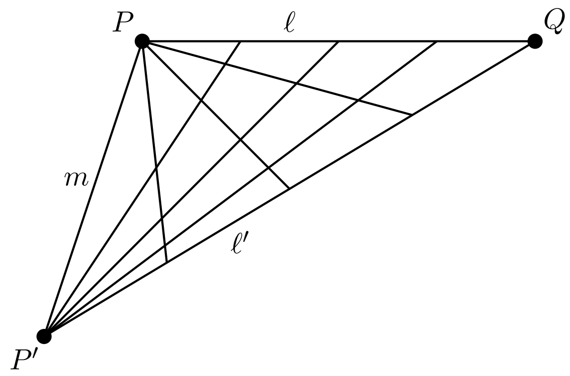


Figure 4.9: Illustration of Lemma 4.4.3.

have $\beta(\ell) = \ell'$ where ℓ' is the line incident to P' and Q. So by $(*)$, we get that for each line o' incident to P' and different from ℓ', all points incident to o' and different from P' have the same color $(**)$. From $(*)$ and $(**)$ we deduce that there are exactly three colors: one for Q, one for the points incident to m and one for all other points. $\square$

4.4.2 From vertex-transitivity to panel-transitivity

We start this subsection with two easy lemmas.

Lemma 4.4.4. *Suppose that* $\mathrm{Aut}(\Delta)^+$ *is transitive on vertices of each type but not transitive on panels of each type. Then for each vertex* O *in* Δ*,* ${}^1\Psi(O)$ *is not transitive on* ${}^1\mathcal{P}(O)$ *(resp.* ${}^1\mathcal{L}(O)$*).*

Proof. Suppose for a contradiction that ${}^1\Psi(O)$ is transitive on ${}^1\mathcal{P}(O)$ for some vertex O in Δ, say of type 1. By Lemma 4.4.1, ${}^1\Psi(O)$ is also transitive on ${}^1\mathcal{L}(O)$. Since $\mathrm{Aut}(\Delta)^+$ is transitive on vertices of each type, this implies that $\mathrm{Aut}(\Delta)^+$ is transitive on panels of type $\{0,1\}$ and of type $\{0,2\}$ of Δ. Now if we consider a vertex O' of type 1, then we know that the stabilizer of O' in $\mathrm{Aut}(\Delta)^+$ is transitive on panels of type $\{0,1\}$ adjacent to O'. By Lemma 4.4.1, it is also transitive on panels of type $\{1,2\}$ adjacent to O'. It follows that $\mathrm{Aut}(\Delta)^+$ is also transitive on panels of type $\{1,2\}$, which contradicts the hypothesis. $\square$

Lemma 4.4.5. *Suppose that* $\mathrm{Aut}(\Delta)$ *is transitive on vertices and unimodular. If* v *and* w *are two vertices in* Δ *such that the stabilizer*

$\mathrm{Aut}(\Delta)^+(v)$ *of* v *in* $\mathrm{Aut}(\Delta)^+$ *fixes* w, *then* $\mathrm{Aut}(\Delta)^+(v) = \mathrm{Aut}(\Delta)^+(w)$.

Proof. We have $\mathrm{Aut}(\Delta)^+(v) \subseteq \mathrm{Aut}(\Delta)^+(w)$ by hypothesis. Take $g \in \mathrm{Aut}(\Delta)$ such that $g(v) = w$. Since $\mathrm{Aut}(\Delta)$ is unimodular, the Haar measure μ of $\mathrm{Aut}(\Delta)$ satisfies $\mu(\mathrm{Aut}(\Delta)^+(v)) = \mu(g\,\mathrm{Aut}(\Delta)^+(v)g^{-1}) = \mu(\mathrm{Aut}(\Delta)^+(w))$. So $\mathrm{Aut}(\Delta)^+(v) = \mathrm{Aut}(\Delta)^+(w)$. $\square$

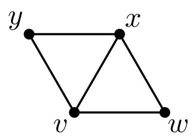
Proposition 4.4.6. *Suppose that* $\mathrm{Aut}(\Delta)$ *is transitive on vertices, non-*6*-discrete and unimodular, that* $\mathrm{Aut}(\Delta)^+$ *is transitive on vertices of each type, and that* Δ *has thickness* $p+1$ *for some prime* p. *Then* $\mathrm{Aut}(\Delta)^+$ *is transitive on panels of each type.*

Proof. Let us assume for a contradiction that $\mathrm{Aut}(\Delta)^+$ is not transitive on panels of each type. By Lemma 4.4.4, this implies that ${}^1\Psi(v)$ is not transitive on ${}^1\mathcal{P}(v)$ (and on ${}^1\mathcal{L}(v)$) for each vertex v in Δ. In view of Lemmas 4.4.2 and 4.4.5, for each such v there exists w adjacent to v in Δ such that $\mathrm{Aut}(\Delta)^+(v) = \mathrm{Aut}(\Delta)^+(w)$. From now on, we color in red all panels (i.e. edges) $[v, w]$ in Δ such that $\mathrm{Aut}(\Delta)^+(v) = \mathrm{Aut}(\Delta)^+(w)$. We have just seen that each vertex is adjacent to at least one red edge.

Claim 1. *Let* v, w, x, y *be vertices in* Δ, *placed as shown below.*

(i) If $[v, w]$ *and* $[v, x]$ *are red, then* $[w, x]$ *is red.*

(ii) If $[v, w]$ *and* $[v, y]$ *are red, then* $[v, x]$ *is red.*



Proof of the claim: The claim follows from the definition of a red edge:

(i) Having $[v, w]$ and $[v, x]$ red means that $\mathrm{Aut}(\Delta)^+(v) = \mathrm{Aut}(\Delta)^0(w)$ and $\mathrm{Aut}(\Delta)^+(v) = \mathrm{Aut}(\Delta)^+(x)$, so $\mathrm{Aut}(\Delta)^+(w) = \mathrm{Aut}(\Delta)^+(x)$ and $[w, x]$ is red.

(ii) Having $[v, w]$ and $[v, y]$ red means that $\mathrm{Aut}(\Delta)^+(v) = \mathrm{Aut}(\Delta)^+(w)$ and $\mathrm{Aut}(\Delta)^+(v) = \mathrm{Aut}(\Delta)^+(y)$. In particular, this implies that $\mathrm{Aut}(\Delta)^+(v)$ fixes x. By Lemma 4.4.5, this gives us $\mathrm{Aut}(\Delta)^+(v) = \mathrm{Aut}(\Delta)^+(x)$ so that $[v, x]$ is red. $\blacksquare$

Claim 2. *Let v be a vertex in Δ and let α be a non-trivial elation of ${}^1H(v)$, with axis ℓ and center P. Then all vertices w adjacent to v with $[v,w]$ red are incident to P or ℓ.*

Proof of the claim: This follows from Lemma 4.2.4. ∎

Claim 3. *For each vertex v in Δ, there exist two vertices w, x adjacent to v and opposite in ${}^1H(v)$ such that $[v,w]$ and $[v,x]$ are red.*

Proof of the claim: By Lemmas 4.4.2 and 4.4.5, there is at least one red edge adjacent to any vertex. Since $\mathrm{Aut}(\Delta)$ is transitive on vertices, each vertex is adjacent to the same number of red edges. This number cannot be exactly one, because then there would be an issue with the types of the red panels (because $\mathrm{Aut}(\Delta)^+$ is transitive on vertices of each type). So each vertex is adjacent to at least two red edges.

We want to show that, for each vertex v, there exists w, x adjacent to v and opposite in ${}^1H(v)$ such that $[v,w]$ and $[v,x]$ are red. If this situation occurs at one vertex v, then it occurs at any vertex v in view of the vertex-transitivity. So we assume for a contradiction that this situation does not appear anywhere.

First assume that, for some vertex v, there exist two vertices w, y adjacent to v, with the same type and such that $[v,w]$ and $[v,y]$ are red. Then the edge $[v,x]$ between w and y must also be red, as well as $[w,x]$

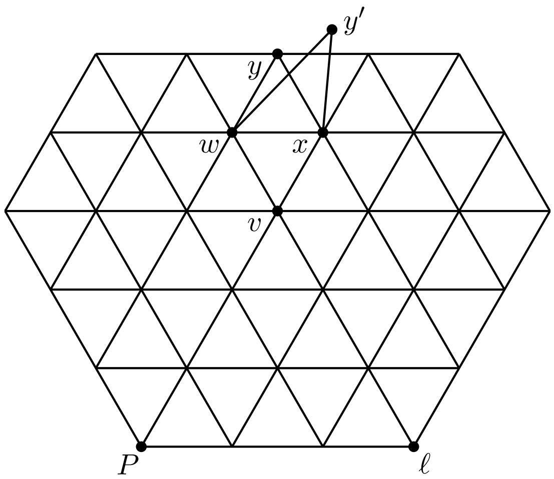


Figure 4.10: Illustration of Proposition 4.4.6, Claim 3.

and $[x, y]$ (by Claim 1). But there must also be two red edges of the same type adjacent to w. In all cases, we find (via Claim 1) two opposite red edges adjacent to a same vertex. So two such red edges $[v, w]$ and $[v, x]$ cannot exist, and the only remaining possibility is to have, for each vertex v in Δ, exactly two red edges adjacent to v, of different types and incident in ${}^1H(v)$ $(*)$.

We now show that this situation is impossible. Let us consider some non-trivial ${}^1h$-collineation α in ${}^3H(v)$, which exists by Proposition 4.3.3. Denote by P and ℓ its center and axis. Let w, x be two vertices adjacent to v in Δ, placed as in Figure 4.10. Now for each vertex y adjacent to both w and x but different from v, α induces an elation of ${}^1H(y)$ with axis x and center w. Observing $(*)$ and Claim 2 at y, we deduce that at least one of the edges $[y, w]$ and $[y, x]$ is red. This observation is true for any choice of y. If $p \geq 3$, there are at least three such vertices y and we get two red edges $[w, y]$ and $[w, y']$ (or $[x, y]$ and $[x, y']$) with y and y' of the same type, which contradicts $(*)$. In the particular case where $p = 2$, we can also get a contradiction. First, if we denote by y and y' the two vertices adjacent to w and x and different from v, then the only way to not have a contradiction is to have $[w, y]$ and $[x, y']$ red (or $[w, y']$ and $[x, y]$ red). Now consider x' a vertex adjacent to v and w, different from x and not adjacent to ${}^1\pi(P)$. Then with the same argument as above we get two vertices t and t' adjacent to w and x' and such that $[w, t]$ and $[x, t']$ are red. This gives a contradiction with $(*)$ at w: the two edges $[w, y]$ and $[w, t]$ are red but y and t have the same type. ■

Claim 4. *For each vertex v in Δ, there are exactly two red edges adjacent to v, and they are opposite in ${}^1H(v)$.*

Proof of the claim: For each vertex v in Δ, we have two red edges adjacent to v and opposite in ${}^1H(v)$, by Claim 3. Now assume that some (and hence any) vertex is adjacent to a third red edge.

For some vertex v, we consider some non-trivial ${}^1h$-collineation α in ${}^3H(v)$, with axis ℓ and center P. Let w, x be two vertices adjacent to v in Δ, placed as in Figure 4.11. Given a vertex y adjacent to both w and x but different from v, α induces an elation of ${}^1H(y)$ with axis x and center w. Applying Claims 2 and 3 at y, we obtain two red edges $[y, s]$

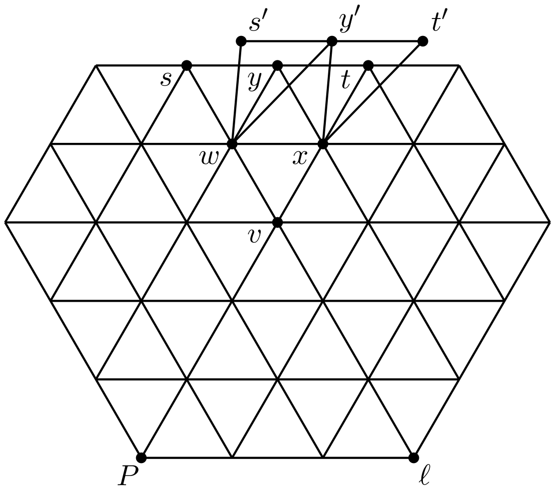


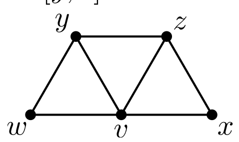
Figure 4.11: Illustration of Proposition 4.4.6, Claim 4.

and $[y, t]$, with s adjacent to w and t adjacent to x, see Figure 4.11. We assumed that there is a third red edge $[y, r]$ adjacent to y. By Claim 2, r must be adjacent to w or x. Via Claim 1, this implies that all edges $[y, w]$, $[y, x]$, $[s, w]$, $[w, x]$ and $[x, t]$ are red. Now we can do the same reasoning with another vertex y adjacent to w and x but different from v. This gives us two vertices s' and t' with $[y', s']$, $[y', t']$, $[y', w]$, $[y', x]$, $[s', w]$ and $[x, t']$ red. In particular, we get that the three edges $[w, s]$, $[w, x]$ and $[w, s']$ are red, with s, x and s' having the same type. In view of Claim 2, since there exists a non-trivial elation of ${}^1H(w)$, these three edges should be incident to a common edge. This is not the case, so we have our contradiction. ■

Claim 5. *For each vertex v in Δ, there is a red bi-infinite geodesic through v.*

Proof of the claim: This follows directly from Claim 4. ■

Claim 6. *Let v, w, x, y, z be vertices in Δ placed as shown below. If $[v, w]$ and $[v, x]$ are red, then $[y, z]$ is red.*



Proof of the claim: Consider some non-trivial ${}^1h$-collineation α of ${}^2H(v)$ given by Proposition 4.3.3 and denote by P and ℓ its center and axis. Assume without loss of generality that the vertex ${}^1\pi(P)$ (resp. ${}^1\pi(\ell)$) has the same type as x (resp. w). Recall from Claims 4 and 5 that there is a red bi-infinite geodesic through w, v and x. We deduce that w cannot be opposite to ${}^1\pi(P)$ in ${}^1H(v)$, because then α would fix a line not near P, contradicting Lemma 4.2.4. So w must be adjacent to ${}^1\pi(P)$. In the same way, we deduce that x must be adjacent to ${}^1\pi(\ell)$. Moreover, since $\mathrm{Aut}(\Delta)^+(v)$ fixes w and x and is transitive on points adjacent to v and w (by Lemma 4.4.3), we can assume without loss of generality that y and z are different from ${}^1\pi(P)$ and ${}^1\pi(\ell)$, as in Figure 4.12.

We now prove that $[y, z]$ is red. By the previous claims, there is a (unique) vertex s adjacent to y and with the same type as z such that $[y, s]$ is red. Our goal is to show that $s = z$. First observe that s cannot be opposite to v in ${}^1H(y)$ (as s_1 in Figure 4.12). Indeed, if this was the case, then it would mean that α fixes s, a point of ${}^2H(v)$ not near P. This is impossible by Lemma 4.2.4. So s is adjacent to v.

Of course we cannot have $s = w$ since $[w, v]$ and $[w, y]$ cannot be both red. In order to show that $s = z$, there remains to show that s is adjacent to x. We proceed by contradiction, assuming that s is not adjacent to x (as s_2 in Figure 4.12). We thus have a red edge $[y, s]$ with y and s adjacent to v, y adjacent to w but s not adjacent to x. In the case where $p \geq 3$, the contradiction will come from Lemma 4.4.3. Indeed, if we denote by Y the set of vertices adjacent to v and w, and by S the set

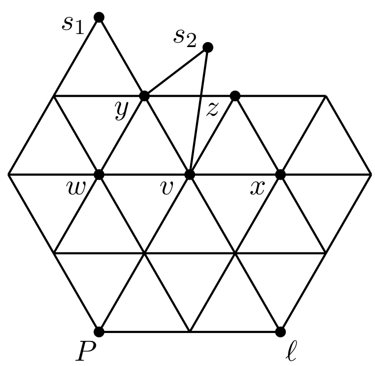


Figure 4.12: Illustration of Proposition 4.4.6, Claim 6.

of vertices with the same type as s, adjacent to v and not adjacent to x, then Lemma 4.4.3 tells us that $\mathrm{Aut}(\Delta)^+(v)$ is transitive on Y and on S. But $|Y| = p+1$ and $|S| = p^2 - 1$, so if $p \geq 3$ then having a red edge $[y, s]$ from a vertex in Y to a vertex in S implies that each vertex in Y has more than one red edge going to a vertex in S. This is impossible, as s is the only vertex of that type with $[y, s]$ red.

Let us now consider the last case $p = 2$. We continue our proof by contradiction, assuming that $s \neq z$. This time we have $|Y| = 3 = |S|$, and each vertex in Y is adjacent to a unique vertex in S. This gives us three red edges. If we do the same reasoning around z instead of y, then we denote by Z the set of vertices adjacent to v and x, by S' the set of vertices with the same type as y, adjacent to v and not adjacent to w, and we get three other red edges, each one connecting a vertex of Z and a vertex of S'. In total, we got six red edges connecting neighbors of v. Now since $\mathrm{Aut}(\Delta)$ is transitive on vertices, this whole situation around v also occurs around w. If we denote by a the vertex adjacent to w such that $[w, a]$ is red (with $a \neq v$), this means that $[y, b]$ is red, where b is the unique vertex adjacent to w and y, different from v and not adjacent to a (see Figure 4.13). But then, around y, we have $[y, b]$ and $[y, s]$ red, while $[w, v]$ is also red. This situation is different from the one around v, so we get our contradiction. ■

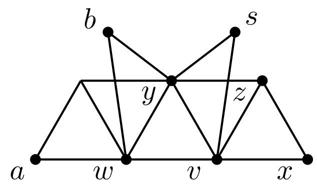


Figure 4.13: Illustration of Proposition 4.4.6, Claim 6.

We now find a new contradiction. This will show that our hypotheses were wrong since the beginning, i.e. that $\mathrm{Aut}(\Delta)^+$ must be transitive on panels of each type.

Fix a vertex v in Δ and consider a non-trivial ${}^1h$-collineation α of ${}^2H(v)$ given by Proposition 4.3.3, say with axis ℓ and center P. We choose a vertex w adjacent to v and ${}^1\pi(P)$ but different from ${}^1\pi(\ell)$ and a vertex x adjacent to w and v but different from ${}^1\pi(P)$, as shown in

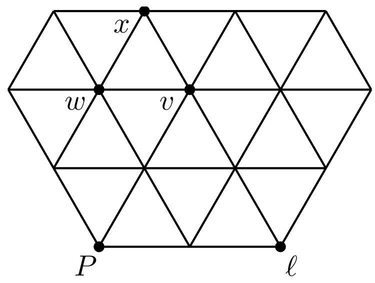


Figure 4.14: Illustration of Proposition 4.4.6.

Figure 4.14. The ${}^1h$-collineation α induces a non-trivial elation of ${}^1H(x)$ with axis v and center w. By Claim 2, this implies that the two red edges adjacent to x (given by Claim 4) are incident to w and v in ${}^1H(x)$. Hence, we conclude via Claim 6 that $[v, w]$ is also red. However, this reasoning could be done for any choice of w. So if w' is another vertex adjacent to v and ${}^1\pi(P)$ but different from ${}^1\pi(\ell)$, then we also get that $[v, w']$ is red. This gives a contradiction with Claim 4. $\square$

Theorem 4.B$'$ now follows immediately.

Proof of Theorem 4.B$'$. See Proposition 4.4.6 and Theorem 4.A$'$. $\square$

4.5 A sufficient condition for exoticity

In this section we prove Theorem 4.C$'$, which gives a sufficient condition under which an $\tilde{A}_2$-building is not 2-Moufang (and in particular exotic).

Theorem 4.C$'$. *Let Δ be a locally finite thick $\tilde{A}_2$-building, let x_0, x_1 be two adjacent vertices in Δ and let C be the set of chambers adjacent to both x_0 and x_1. For each $j \in \{0, 1\}$, let $G_j \leq \mathrm{Sym}(C)$ be the image of $\mathrm{Aut}({}^1H(x_j))(x_{1-j})$ in $\mathrm{Sym}(C)$. If $G_0 \neq G_1$, then Δ is not 2-Moufang.*

Proof. Say that Δ has thickness $q + 1$, i.e. $|C| = q + 1$. Then ${}^1H(x_0)$ and ${}^1H(x_1)$ are projective planes of order q. If one of them is non-Desarguesian then Δ is not 1-Moufang (in particular not 2-Moufang), so we can assume that q is a prime power and that they are both Desarguesian. The full automorphism group of the Desarguesian projective plane of order q is $\mathrm{P\Gamma L}(3, q)$, and the stabilizer of a line acts on the set of points incident to it as $\mathrm{P\Gamma L}(2,q)$ acting on the projective line over $\mathbf{F}_q$. So for each $t \in \{0,1\}$, the image $G_t \leq \mathrm{Sym}(C)$ of $\mathrm{Aut}({}^1H(x_j))(x_{1-j})$ in $\mathrm{Sym}(C)$ is conjugate to $\mathrm{P\Gamma L}(2,q)$ in $\mathrm{Sym}(C)$. Let us now suppose for a contradiction that Δ is 2-Moufang.

The subgroup $\mathcal{G}_0$ of $\mathrm{Aut}({}^1H(x_0))$ generated by all elations of ${}^1H(x_0)$ (i.e. the *little projective group* of ${}^1H(x_0)$) is isomorphic to $\mathrm{PSL}(3,q)$. The image $G'_0 \leq \mathrm{Sym}(C)$ of $\mathcal{G}_0(x_1)$ in $\mathrm{Sym}(C)$ is thus conjugate to $\mathrm{PGL}(2,q)$ (acting on the projective line over $\mathbf{F}_q$) in $\mathrm{Sym}(C)$. Now the fact that ${}^2H(x_0)$ is Moufang implies that each elation of ${}^1H(x_0)$ is the restriction of an elation of ${}^2H(x_0)$. We thus deduce that the image of $\mathrm{Aut}({}^2H(x_0))(x_1)$ in $\mathrm{Sym}(C)$ contains G'_0, while being contained in G_0 and G_1. But $G_0 \cong \mathrm{P\Gamma L}(2,q)$ has only one subgroup that is conjugate to $\mathrm{PGL}(2,q)$ in $\mathrm{Sym}(C)$, and it is the normalizer of that subgroup, so $G_0 = N_{\mathrm{Sym}(C)}(G'_0)$. The same is true for G_1, so $G_1 = N_{\mathrm{Sym}(C)}(G'_0) = G_0$. This contradicts the hypothesis. □

4.6 Singer cyclic lattices

Let us now focus on Singer cyclic lattices, i.e. groups $\Gamma \leq \mathrm{Aut}(\Delta)$ acting simply transitively on the panels of each type of an $\tilde{A}_2$-building Δ and with the additional property that vertex-stabilizers are cyclic. These lattices have been deeply studied by Essert and Witzel in [Ess13] and [Wit17]. The notion of a *difference matrix* was defined in the latter reference. For our purpose, we present another way of understanding the relation between difference matrices and Singer cyclic lattices.

A **difference set** with parameter q is a subset $D = \{d_1, \ldots, d_{q+1}\}$ of $\mathbf{Z}/(q^2+q+1)\,\mathbf{Z}$ such that, for each $x \in \mathbf{Z}/(q^2+q+1)\,\mathbf{Z}$ with $x \neq 0$, there exists a unique ordered pair $(d,d') \in D^2$ satisfying $x = d - d'$. Given such a difference set D with parameter q, we can construct a projective plane Π_D of order q as follows. The point set P and line set L of Π_D are simply $P = L = \mathbf{Z}/(q^2+q+1)\,\mathbf{Z}$, and the incidence relation $R \subseteq L \times P$ is given by

$$R = \{(x, x+d) \mid x \in L, d \in D\}.$$

It is an easy task to check that this defines a projective plane of order q.

Define a **difference vector** with parameter q as a vertical vector $v = (d_1, \ldots, d_{q+1})^T$ where $\{d_1, \ldots, d_{q+1}\}$ is a difference set. To such a difference vector v, we associate a *labelled projective plane* of order q. A **labelled projective plane** is a projective plane whose flags are labelled by $\{1, \ldots, q+1\}$, i.e. with a map $\ell \colon R \to \{1, \ldots, q+1\}$. Given a difference vector v, we take the projective plane Π_D associated to the difference set D inherent to v, and we label its flags by defining $\ell(x, x+d_j) = j$ for each $x \in L$ and each $j \in \{1, \ldots, q+1\}$. Note that we need a difference vector (and not only a difference set) for this map to be well defined. We call Π_v this labelled projective plane associated to v.

Now a **difference matrix** with parameter q is a matrix with $q+1$ lines and 3 columns, such that each of the three columns is a difference vector with parameter q. Let us write $M = (v_0, v_1, v_2)$ for such a matrix, where v_0, v_1 and v_2 are difference vectors. To a difference matrix M, we associate a **labelled $\tilde{A}_2$-building**, i.e. an $\tilde{A}_2$-building whose chambers are labelled by $\{1, \ldots, q+1\}$. Note that at each vertex of a labelled $\tilde{A}_2$-building, we see a labelled projective plane. (At a vertex of type $t \in \{0, 1, 2\}$, we consider vertices of type $t+1$ mod 3 as points and those of type $t+2$ mod 3 as lines). The labelled $\tilde{A}_2$-building Δ_M associated to the difference matrix $M = (v_0, v_1, v_2)$ is then defined as the unique one whose labelled projective plane at each vertex of type t is Π_{v_t} (for each $t \in \{0, 1, 2\}$). This building can be constructed recursively with the method of [Ron86]: the labellings of the projective planes exactly tells us how two adjacent projective planes must be glued in the building. Moreover, we can define $\Gamma_M \leq \mathrm{Aut}(\Delta_M)$ as the group of all type-preserving automorphisms of Δ_M preserving the labellings. It is a direct fact that Γ_M acts simply transitively on the panels of each type of Δ_M and that vertex stabilizers in Γ_M are cyclic (of order q^2+q+1). So Γ_M is a Singer cyclic lattice. Conversely, given a Singer cyclic lattice $\Gamma \leq \mathrm{Aut}(\Delta)$ we can label the chambers of Δ according to their orbit under the action of Γ and get a (not necessarily unique) difference matrix M such that $\Gamma = \Gamma_M$ and $\Delta = \Delta_M$.

Two Singer cyclic lattices $\Gamma \leq \mathrm{Aut}(\Delta)$ and $\Gamma' \leq \mathrm{Aut}(\Delta')$ are **isomorphic** if there exists an isomorphism from Δ to Δ' conjugating Γ to Γ'. This is actually equivalent to saying that Γ and Γ' are isomorphic as groups (see [Wit17, Proposition 3.7]). Two difference matrices M and M' are then said to be **equivalent** if $\Gamma_M \leq \mathrm{Aut}(\Delta_M)$ and $\Gamma_{M'} \leq \mathrm{Aut}(\Delta_{M'})$ are isomorphic. This equivalence relation on difference matrices was deeply studied in [Wit17]. In order to prove Corollary 4.D we do not need to really study the notion of equivalent difference matrices. We will however need the following basic results which can also be found in [Wit17]. A difference set D (resp. difference vector v) is called **Desarguesian** if Π_D (resp. Π_v) is Desarguesian. A difference matrix $M = (v_0, v_1, v_2)$ is called **Desarguesian** if v_0, v_1 and v_2 are Desarguesian. Note that there exist Desarguesian difference sets with parameter q for each prime power q, see [Sin38] or [Wit17, Theorem 2.2].

Lemma 4.6.1. *Let $q = p^\eta$, with p prime and $\eta \geq 1$.*

(i) Let M be a difference matrix with parameter q and let M' be a difference matrix obtained by permuting the $q+1$ lines of M. Then M and M' are equivalent.

(ii) Let $M = (v_0, v_1, v_2)$ be a difference matrix with parameter q, and let $g_0, g_1, g_2 \in \mathrm{AGL}(1, \mathbf{Z}/(q^2+q+1)\,\mathbf{Z})$. Then M is equivalent to $M' = (g_0(v_0), g_1(v_1), g_2(v_2))$, where g_t acts on the difference vector v_t componentwise.

(iii) Let D be a Desarguesian difference set with parameter q. The stabilizer of D in $\mathrm{AGL}(1, \mathbf{Z}/(q^2+q+1)\,\mathbf{Z})$ has order 3η.

(iv) Let D be a Desarguesian difference set and M be a Desarguesian difference matrix (both with parameter q). Then M is equivalent to a difference matrix whose columns are equal to D as a set.

Proof.

(i) Permuting the lines of a difference matrix $M = (v_0, v_1, v_2)$ simply permutes the labels in the three labelled projective planes Π_{v_0}, Π_{v_1} and Π_{v_2} simultaneously. So the labelled $\tilde{A}_2$-buildings Δ_M and $\Delta_{M'}$ are equal, up to permuting the labels. In particular, $\Gamma_M \leq \mathrm{Aut}(\Delta_M)$ and $\Gamma_{M'} \leq \mathrm{Aut}(\Delta_{M'})$ are isomorphic.

(ii) When $g \in \mathrm{AGL}(1, \mathbf{Z}/(q^2+q+1)\,\mathbf{Z})$ and v is a difference vector with parameter q, the labelled projective planes Π_v and $\Pi_{g(v)}$ are isomorphic. Replacing a column v_t by $g_t(v_t)$ thus does not change the Singer cyclic lattice.

(iii) See [Ber53] or [Wit17, Lemma 4.5].

(iv) This follows from (ii) and the fact that $\mathrm{AGL}(1, \mathbf{Z}/(q^2+q+1)\,\mathbf{Z})$ is transitive on the Desarguesian difference sets with parameter q, see [Ber53]. $\square$

We can now prove Corollary 4.D′ below.

Corollary 4.D′. *For each $q \geq 2$, there are at most $\left(\frac{q(q^2-1)}{3}\right)^2$ isomorphism classes of Singer cyclic lattices $\Gamma \leq \mathrm{Aut}(\Delta)$ with parameter q such that Δ is 2-Moufang.*

Proof. If q is not a prime power then the claim is obvious: an $\tilde{A}_2$-building with thickness $q+1$ is never 1-Moufang when q is not a prime. We now assume that $q = p^\eta$ and fix some Desarguesian difference set $D = \{d_1, \dots, d_{q+1}\}$ with parameter q. We need an upper bound on the number of equivalence classes of difference matrices M with parameter q such that Δ_M is 2-Moufang. Let M be such a difference matrix, in particular M is Desarguesian. Up to replacing M by an equivalent matrix, we can assume that each column of M is equal to D as a set (by Lemma 4.6.1 (iv)). Moreover, up to permuting the lines of M (see Lemma 4.6.1 (i)), we can assume that the first column of M is exactly $(d_1, \dots, d_{q+1})^T$. So we look at matrices in

$$\mathcal{M} = \left\{ M = \begin{pmatrix} d_1 & d_{\alpha_1(1)} & d_{\alpha_2(1)} \\ d_2 & d_{\alpha_1(2)} & d_{\alpha_2(2)} \\ \vdots & \vdots & \vdots \\ d_{q+1} & d_{\alpha_1(q+1)} & d_{\alpha_2(q+1)} \end{pmatrix} \;\middle|\; \begin{array}{c} \alpha_1, \alpha_2 \in \mathrm{Sym}(q+1), \\ \Delta_M \text{ is 2-Moufang} \end{array} \right\}.$$

Let $M \in \mathcal{M}$ and write $M = (v_0, v_1, v_2)$. In the Desarguesian projective plane Π_{v_t}, a point is incident to $q+1$ lines, and the $q+1$ flags they form have $q+1$ different labels. The action of the point stabilizer on these $q+1$ flags thus gives a subgroup G_t of $\mathrm{Sym}(q+1)$ which is conjugate to $\mathrm{P\Gamma L}(2,q)$. This subgroup $G_t \leq \mathrm{Sym}(q+1)$ does not depend on the chosen point because the subgroup of $\mathrm{Aut}(\Pi_{v_t})$ preserving the labels is transitive on points. The correlation of Π_{v_t} defined by $x \in P \mapsto -x \in L$, $x \in L \mapsto -x \in P$ also preserves the labels so the stabilizer of a line in Π_{v_t} also gives birth to the same group $G_t \leq \mathrm{Sym}(q+1)$. We can moreover observe that $G_1 = \alpha_1^{-1} G_0 \alpha_1$ and $G_2 = \alpha_2^{-1} G_0 \alpha_2$, where $\alpha_1, \alpha_2 \in \mathrm{Sym}(q+1)$ behave as in the definition of $\mathcal{M}$. In Δ_M, if x_t and $x_{t'}$ are two adjacent vertices of type t and t' respectively, then the chambers adjacent to x_t and $x_{t'}$ have the $q+1$ different labels, and Theorem 4.C$'$ exactly tells us that Δ_M is not 2-Moufang when $G_t \neq G_{t'}$. Here we suppose that Δ_M is 2-Moufang, so we deduce that $G_0 = G_1 = G_2$. As $\mathrm{P\Gamma L}(2,q)$ is its own normalizer in $\mathrm{Sym}(q+1)$, we obtain that $\alpha_1, \alpha_2 \in G_0$. In particular, we have $|\mathcal{M}| \leq |\mathrm{P\Gamma L}(2,q)|^2 = (q(q^2-1)\eta)^2$. But Lemma 4.6.1 (ii),(iii) implies that each matrix in $\mathcal{M}$ is equivalent to at least $(3\eta)^2$ matrices in $\mathcal{M}$, so $|\mathcal{M}/{\sim}| \leq \left(\frac{q(q^2-1)}{3}\right)^2$ (where $\sim$ denotes the equivalence relation). This concludes the proof. $\square$

Proof of Corollary 4.E. By [Wit17, Theorem B], the number of isomorphism classes of Singer cyclic lattices with parameter $q = p^\eta$ is bounded below by $A(q) = \frac{1}{162\eta^3}((q+1)!\,)^2$. Moreover, by Corollary 4.D at most $B(q) = \left(\frac{q(q^2-1)}{3}\right)^2$ of them are non-exotic. The conclusion follows from the fact that $\frac{B(q)}{A(q)} \to 0$ when q goes to infinity. $\square$

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